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Local mean value estimates for Weyl sums

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Abstract. We obtain new estimates – both upper and lower bounds – on the mean values of the Weyl sums over a small box inside of the unit torus. In particular, we refine recent conjectures of C. Demeter and B. Langowski (2022), and improve some of their results.

1. Introduction

1.1. Background and motivation

The study of exponential sums occupies a central location in the analytic theory of numbers, as they are a crucial tool connecting the language of number theory with the language of Fourier analysis. In fact, many of the most celebrated results in number theory either are equivalent to or at least crucially depend on strong bounds on exponential sums, either in an average or a pointwise sense.

In this paper, we are interested in exponential sums of the shape

$$S_d(\mathbf{x}; N) = \sum_{n=1}^{N} \mathbf{e}(x_1 n + \cdots + x_d n^d), \quad \mathbf{x} = (x_1, \ldots, x_d) \in [0,1]^d,$$

where $\mathbf{e}(z) = \exp(2\pi i z)$. Such sums are the main protagonists in Vinogradov's mean value theorem. Thanks to the breakthrough results of Bourgain, Demeter and Guth [6], as well as Wooley [27, 28], we now have very good control over the average value of these sums as $\mathbf{x}$ ranges over the unit hypercube $[0,1)^d$. If we put

$$J_{s,d}(N) = \int_{[0,1]^d} |S_d(\mathbf{x}; N)|^{2s} \, d\mathbf{x}, \tag{1.1}$$

then [6, 27, 28] show that

$$J_{s,d}(N) \leqslant N^{s+o(1)} + N^{2s-d(d+1)/2+o(1)}, \tag{1.2}$$

which is optimal up to (at most) the $o(1)$. It should be noted that one can show as a consequence of (1.2) that $S_d(\mathbf{x}; N) \leqslant N^{1/2+o(1)}$ for almost all $\mathbf{x} \in [0,1)^d$, see Corollary 2.2

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in [11]. Thus, we have now a close to complete understanding of the size of exponential sums both on average and in an almost-all sense.

Unfortunately, neither of these results is apt to tell us much about the pointwise size of $S_d(\mathbf{x}; N)$ for any fixed point $\mathbf{x}$, and indeed our understanding of this problem is still far from the conjectured bounds. It is not hard to see that such pointwise bounds necessarily depend on the diophantine approximation properties of $\mathbf{x}$. Suppose that x_d has an approximant a_d/q with $\|qx_d\| \leqslant q^{-1}$. Then an argument going back to Vinogradov (see Theorem 5.2 in [26]) shows that the mean value bound (1.2) can be used to derive the pointwise estimate

$$|S_d(\mathbf{x}; N)| \leqslant N^{1+o(1)}(N^{-1} + q^{-1} + qN^{-d})^{1/d(d-1)}.$$

However, in order to make progress towards the bound

$$|S_d(\mathbf{x}; N)| \leqslant N^{1+o(1)}\,(q^{-1} + qN^{-d})^{1/d},$$

conjectured by Montgomery in Conjecture 1 in Chapter 3 of [25], one likely needs different methods – although we point out that in the case of one-dimensional exponential sums, a bound of at least comparable quality to the conjectured one, with the exponent $1/d$ replaced by $1/(2d-2)$, would follow from the conjectured mean value (Hua-type) bound for such sums, see Theorem 2.1 in [8].

The purpose of the manuscript at hand is to investigate $S_d(\mathbf{x}; N)$ and related exponential sums as $\mathbf{x}$ ranges over small boxes. This should rightly be viewed as an attempt to interpolate between our almost complete understanding of mean values of $S_d(\mathbf{x}; N)$ and our deficient understanding of the pointwise behaviour of these sums. Our work ties in with work by Demeter and Langowski [16], as well as some speculations of Wooley [29]. By introducing several new ideas, based partly on bounds for inhomogeneous Vinogradov systems as explored recently by Brandes and Hughes [7] as well as Wooley [29], and partly on the structure of large Weyl sums investigated in some depth by Baker (see, for example, [1,2]), we are able to extend and improve some of the results of [16]. We obtain a diverse zoo of bounds, which we describe and discuss in more detail in Section 3 below. These bounds have fairly different character depending on the size of the small box. In a sense, this is not unexpected, since the mean value of $S_d(\mathbf{x}; N)$ over a very small box located at the origin is dominated by the spike at $\mathbf{x} = \mathbf{0}$, whereas the behaviour comes to increasingly resemble that of mean values over the entire unit hypercube as the size of the box increases. What is not clear is how and at what scale(s) the transition between these two behaviours takes place. Taken collectively, our bounds hint that this transition may be more intricate than hitherto anticipated, and we hope that future research can provide a more accurate picture of these phenomena.

1.2. Set-up

For an integer $\nu \geqslant 1$, we denote by T_ν the ν-dimensional unit torus, which we also identify with the ν-dimensional unit cube, that is,

$$\mathsf{T}_\nu = (\mathbb{R}/\mathbb{Z})^\nu = [0, 1)^\nu.$$

For positive integers d and N, a sequence of complex weights $\boldsymbol{a} = (a_n)_{n=1}^N$, and a vector $\mathbf{x} \in \mathsf{T}_d$, we define the Weyl sums

$$S_d(\mathbf{x}; \boldsymbol{a}, N) = \sum_{n=1}^{N} a_n \, \mathbf{e}(x_1 n + \cdots + x_d n^d),$$

where $\mathbf{e}(z) = \exp(2\pi i z)$.

For a positive $\delta \leqslant 1$ and $\boldsymbol{\xi} \in \mathsf{T}_d$, we define

$$I_{s,d}(\delta, \boldsymbol{\xi}; \boldsymbol{a}, N) = \int_{\boldsymbol{\xi}+[0,\delta]^d} |S_d(\mathbf{x}; \boldsymbol{a}, N)|^{2s} \, d\mathbf{x}. \tag{1.3}$$

We note that the exponent s in (1.3) is not necessary integer, but can take arbitrary real positive values. The question of estimating $I_{s,d}(\delta, \boldsymbol{\xi}; \boldsymbol{a}, N)$ for suitable choices of $\boldsymbol{\xi}$ and $\boldsymbol{a}$ has recently received some attention, see, for example, [9, 12, 16, 29] for various bounds and applications. The case of boxes at the origin is especially interesting. In fact, it is easy to see that the question about the size of $I_{s,d}(\delta, \boldsymbol{\xi}; \boldsymbol{a}, N)$ can be reduced to $I_{s,d}(\delta, \mathbf{0}; \widetilde{\boldsymbol{a}}, N)$, with $\widetilde{a}_n = a_n \, \mathbf{e}(\xi_1 n + \cdots + \xi_d n^d)$ for $n = 1, \ldots, N$. We thus put

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) = I_{s,d}(\delta, \mathbf{0}; \boldsymbol{a}, N).$$

Hence, in the case of arbitrary weights, and without loss of generality, it suffices to study the quantity $I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N)$.

Meanwhile, arguably the most relevant choice of weights $\boldsymbol{a}$ is that in which $a_n = 1$ for $n \leqslant N$, so we consider this situation separately. Thus, in the case when $\boldsymbol{a} = \mathbf{1}$, we define

$$I_{s,d}^{(0)}(\delta; N) = I_{s,d}^{(0)}(\delta; \mathbf{1}, N),$$

as well as

$$I_{s,d}^{\sharp}(\delta; N) = \sup_{\boldsymbol{\xi} \in \mathsf{T}_d} I_{s,d}(\delta, \boldsymbol{\xi}; \mathbf{1}, N) \quad \text{and} \quad I_{s,d}^{\flat}(\delta; N) = \inf_{\boldsymbol{\xi} \in \mathsf{T}_d} I_{s,d}(\delta, \boldsymbol{\xi}; \mathbf{1}, N).$$

Note that since the unit torus $\mathsf{T}_d = (\mathbb{R}/\mathbb{Z})^d$ is compact as an additive group, the infimum and supremum here are actually attained as the exponential sum is continuous.

By the discussion following (1.3), it is easy to see that

$$I_{s,d}^{\sharp}(\delta; N) \leqslant \sup_{\|\boldsymbol{a}\|_\infty \leqslant 1} I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N), \tag{1.4}$$

where supremum is taken over all sequences of complex weights with $\|a_n\|_\infty \leqslant 1$.

1.3. Notation

Throughout the paper, we use the Landau and Vinogradov notations $U = O(V)$, $U \ll V$ and $V \gg U$ to express that $|U| \leqslant cV$ for some positive constant c, which throughout the paper may depend on the degree d and occasionally on the small real positive parameter ε and the arbitrary real parameter t. We also write $U \asymp V$ as an equivalent of $U \ll V \ll U$.

Moreover, for any quantity $V > 1$, we write $U = V^{o(1)}$ (as $V \to \infty$) to indicate a function of V which satisfies $V^{-\varepsilon} \leqslant |U| \leqslant V^{\varepsilon}$ for any $\varepsilon > 0$, provided V is large enough. One additional advantage of using $V^{o(1)}$ is that it absorbs $\log V$ and other similar quantities without changing the whole expression.

We also recall the definition of the ℓ^p-norm, which for a sequence of complex numbers $\boldsymbol{a} = (a_n)_{1\leqslant n\leqslant N}$ and a real number $p \geqslant 1$ is given by

$$\|\boldsymbol{a}\|_p = \Bigl(\sum_{n=1}^{N} |a_n|^p\Bigr)^{1/p}.$$

For $m \in \mathbb{N}$, we write $[m]$ to denote the set $\{0, 1, \dots, m-1\}$. We denote the cardinality of a finite set $\mathcal{S}$ by $\#\mathcal{S}$, and for a measurable set $\mathcal{T} \subseteq \mathsf{T}_\nu$ we write $\lambda(\mathcal{T})$ for the Lebesgue measure of the appropriate dimension ν.

We use the notation $\lfloor x\rfloor$ and $\lceil x\rceil$ for the largest integer no larger than x and the smallest integer no smaller than x, respectively. We then write $\{x\} = x - \lfloor x\rfloor \in [0, 1)$.

2. What we know and what we believe to be true

2.1. State of the art and previous conjectures

In order to get a better sense of what to expect, it is helpful to first record some known bounds that can serve as a benchmark for our ensuing considerations. On the one hand, when $\delta = 1$, the recent advances of Bourgain, Demeter and Guth [6] and Wooley [28] towards the optimal form of the Vinogradov mean value theorem yield the bound

$$I^{(0)}_{s,d}(1; \boldsymbol{a}, N) \leqslant \|\boldsymbol{a}\|_2^{2s}\, N^{o(1)}(1 + N^{s-s(d)}) \tag{2.1}$$

for all $s > 0$, where

$$s(d) = d(d+1)/2.$$

For general $\boldsymbol{a}$, this is essentially sharp, since for $\boldsymbol{a} = \mathbf{1}$ a standard argument shows that

$$I^{(0)}_{s,d}(1; N) = J_{s,d}(N) \gg N^s + N^{2s-s(d)}, \tag{2.2}$$

where $J_{s,d}(N)$ is given by (1.1). In fact, by adapting the argument of Lemma 3.1 in [9], one can show that $S_d(\mathbf{x}; N) \gg N^{1/2}$ for a positive proportion of $\mathbf{x} \in \mathsf{T}_d$.

On the other hand, for very small values of δ, we can bound the integral trivially and obtain

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^d\, \|\boldsymbol{a}\|_1^{2s}. \tag{2.3}$$

By a slightly more sophisticated argument, combining the bound of (2.1) with Hölder's inequality, we obtain the bound

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d-2s/(d+1)}\, \|\boldsymbol{a}\|_2^{2s} N^{o(1)}, \quad 0 \leqslant s \leqslant s(d), \tag{2.4}$$

see also equation (2.3) in [16]. Clearly, in the limit $\delta \to 1$, as expected, the bound (2.4) approaches the bound (2.1). At the same time, we see that for small δ, this is weaker than the trivial bound (2.3).

In the special case $s = 2$, a further example can be derived from Lemma 4.5 in [9], which implies that if $|a_n| \leqslant 1$ for $n = 1, \ldots, N$, then

$$I^{(0)}_{2,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^2 + \delta^{d-4} N^{1+o(1)}.$$

For lower bounds, for any N and $N^{-d} < \delta < 1$, by partitioning $[0, 1]^d$ into δ^{-d} boxes with side length δ and the definition of $I^{\sharp}_{s,d}(\delta; N)$, we obtain

$$I^{(0)}_{s,d}(1; N) \ll \delta^{-d} I^{\sharp}_{s,d}(\delta; N).$$

Upon combining this with the classical lower bound of (2.2), we thus conclude that

$$I^{\sharp}_{s,d}(\delta; N) \gg \delta^d (N^s + N^{2s-s(d)}).$$

Clearly, this suggests the question of whether this bound is sharp, and if so, in what ranges. A version of that conjecture has been proposed in recent work by Wooley, see Conjectures 8.1 and 8.2 in [29].

Conjecture 2.1 (Wooley [29]). *Suppose that*

$$s \geqslant \tfrac{1}{4} d(d + 1) + 1 \quad \textit{or} \quad \delta \geqslant N^{1/d-(d+1)/4}.$$

Then

$$I^{\sharp}_{s,d}(\delta; N) \leqslant \delta^d N^{s+o(1)} + N^{2s-s(d)+o(1)}.$$

In Wooley's setting [29], the bound on the number of variables is motivated by considerations concerning the convergence of the singular series; however, it seems not unreasonable that the validity of the bound in Conjecture 2.1 in the δ-aspect might extend below the proposed range. We also remark that Wooley allows for general measurable sets, whereas we restrict to axis-aligned hypercubes.

Another conjecture that is relevant to our work, and which permits arbitrary positive values of δ and s, has been fielded in recent work by Demeter and Langowski, see Conjecture 1.3 in [16].

Conjecture 2.2 (Demeter–Langowski [16]). *Let*

$$\rho(d) = \lceil 3d^2/4 \rceil - 1.$$

For any $s > 0$, we have

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^{(d+1)/2} \|\boldsymbol{a}\|_2^{2s} (1 + N^{s-\rho(d)/2}) N^{o(1)}. \tag{2.5}$$

By Theorem 2.4 in [16], we have (2.5) for $d = 2$ and $d = 3$ in the full range. Moreover, the authors establish bounds of a similar quality also for $d = 4$ and $d = 5$. We also remark that there is nothing intrinsically special about the power of δ occurring in (2.5) or the concomitant value $\rho(d)$. Rather, it seems that the precise formulation and choice of parameters of Conjecture 2.2 were chosen mostly in view of applications to the mean value of Weyl sums along curves, see Proposition 2.2 in [16].

A comparison of Conjectures 2.1 and 2.2 shows that neither is strictly stronger than the other; rather, they make different predictions for various ranges of s and various values of δ. It is apparent from the discussion preceding Conjecture 2.1 that it is sharp for small s and δ not too small. At the same time, we remark that Conjecture 2.2, if correct, is the best possible in the sense that the exponent $(d+1)/2$ cannot be increased if one wants a bound which holds for all $\delta \in (0,1)$. Evidence for this has been given in [16], after the formulation of Conjecture 1.3 in [16]. Moreover, for *extremely* small values of δ, the trivial bound (2.3) is both sharp and stronger than (2.5). It is therefore an interesting question to derive even a valid heuristic for the behaviour of $I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N)$ that reflects the true expected size of the quantity for all choices of δ and N.

2.2. An upper bound for a small cube at the origin and some new conjectures

Before embarking on a precise discussion of our results, we remark on a general fact concerning the behaviour of mean values of the type considered in this paper. Typically, for fixed parameters d and δ, we endeavour to establish bounds of the shape

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d-\alpha} \|\boldsymbol{a}\|_2^{2s} (1 + N^{s-\sigma_0}) N^{o(1)}$$

for some $\alpha \in [0, d]$ and some $\sigma_0 \geqslant 1$ depending on d. In particular, if we can establish such a bound at the critical point $s = \sigma_0$, the corresponding results for the *subcritical* and *supercritical* ranges $s < \sigma_0$ and $s > \sigma_0$ follow by standard arguments. In this paper, we give bounds applicable to both the sub- and supercritical ranges.

Our first result shows some limitations of what one can prove for high moments of Weyl sums over a small cube at the origin. The proof, which is based on the continuity of Weyl sums $S_d(\mathbf{x}; N)$ as functions of $\mathbf{x}$, is rather straightforward. We then use this simple bound as a benchmark and a basis for several conjectured upper bounds. It also motivates our results in Section 3, which are based on a variety of new ideas.

We define

$$\sigma_d(\alpha) = \frac{\alpha(2d - \alpha + 1) - \{\alpha\}(1 - \{\alpha\})}{2}. \tag{2.6}$$

Theorem 2.3. *Let*

$$s_0(d,\alpha) = \sup\left\{s \geqslant 0 : \ I^{(0)}_{s,d}(\delta; N) \leqslant \delta^{d-\alpha} N^{s+o(1)}, \ \forall \delta \in [N^{-d}, 1], \ \textit{as } N \to \infty\right\}.$$

We then have

$$s_0(d,\alpha) \leqslant \sigma_d(\alpha).$$

To put this into context, we compare Theorem 2.3 with our preceding discussion. Consider first the case $\alpha = 0$, for which $\sigma_d(0) = 0$. Consequently, for any s, the bound (2.7) reduces to (2.3). Meanwhile, taking $\alpha = d$ we obtain $\sigma_d(d) = d(d+1)/2$, which we also know to be sharp when $\delta = 1$. Finally, the value $\alpha = (d-1)/2$ produces the bound

$$\sigma_d((d-1)/2) = \frac{3(d^2-1)}{8} - \frac{\{(d-1)/2\}(1 - \{(d-1)/2\})}{2} = \frac{1}{2}\left(\lceil 3d^2/4 \rceil - 1\right),$$

which recovers Conjecture 2.2 by Demeter and Langowski (Conjecture 1.3 in [16]). In this way, Theorem 2.3 suggests a natural extension of Conjecture 2.2.

Conjecture 2.4. *Fix $\alpha \in [0, d]$. For any sufficiently large N and any δ in the range $N^{-d} \leqslant \delta \leqslant 1$, the bound*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d-\alpha} \|\boldsymbol{a}\|_2^{2s} (1 + N^{s-\sigma_d(\alpha)}) N^{o(1)}$$

holds for all $s \geqslant 0$.

By our above discussion, the conclusion of Conjecture 2.4, if true, can be used to derive bounds on $I_{s,d}^{(0)}(\delta; N)$ for general values of s. In fact, for $s > \sigma_d(\alpha)$, we obtain

$$I_{s,d}^{(0)}(\delta; N) \leqslant \delta^{d-\alpha} N^{2s-\sigma_d(\alpha)+o(1)}. \tag{2.7}$$

Meanwhile, for $0 < s < s_0(d, \alpha)$, our Theorem 2.3 in combination with Hölder's inequality yields

$$I_{s,d}^{(0)}(\delta; N) \leqslant \delta^{d-\alpha s/\sigma_d(\alpha)} N^{s+o(1)}.$$

We note that we do not suggest that Conjecture 2.4 is always sharp, and there are situations where we do, in fact, obtain stronger upper bounds, as can be gleaned from Figures 1, 2 and 3 below. For $\delta < N^{-d}$, it is not hard to see that the trivial bound (2.3) gives a stronger result. We also note that a careful inspection of the proof of Theorem 3.5 shows that for any given $\alpha > 0$, Conjecture 2.4 is sharp at the point $\delta = N^{-\lfloor d-\alpha \rfloor - 1}$.

The presence of the additional parameter α in these considerations is somewhat irritating. One checks easily that

$$\sigma_d(\alpha) = \alpha d \quad \text{for all } \alpha \in (0, 1]. \tag{2.8}$$

For general values of α, one can show by a modicum of computation that $\sigma_d(\alpha)$ is continuous and strictly increasing in α for $\alpha \in [0, d]$. Indeed, we clearly have

$$\Big(\frac{1}{2}\alpha(2d - \alpha + 1)\Big)' = d - \alpha + 1/2,$$

while $\frac{1}{2}\{\alpha\}(1 - \{\alpha\})$ is the periodic continuation of the function $u(1-u)/2$ for $u \in [0, 1)$, and this latter function has derivative $-u + 1/2 \in [-1/2, 1/2)$, so that the whole function $\sigma_d(\alpha)$ is continuous and satisfies $\sigma_d'(\alpha) > 0$ for all non-integer $\alpha < d$.

For a fixed value s, denote by $\alpha_0(d, s)$ the unique α for which $\sigma_d(\alpha) = s$. In this notation, we can change perspective and propose a reformulation of the above conjecture in which we seek to determine the optimal value of α for any given set of parameters s and d.

Conjecture 2.5. *For any parameters d and $s \leqslant s(d)$, and for any sufficiently large N and any δ in the range $N^{-d} \leqslant \delta \leqslant 1$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d-\alpha_0(d,s)} \|\boldsymbol{a}\|_2^{2s} N^{o(1)}.$$

Unfortunately, the function $\alpha_0(d, s)$ is not straightforward to describe explicitly. However, we can give a rough indication of its size. Recalling (2.6), write

$$\sigma_d(\alpha) = \alpha(2d - \alpha + 1)/2 - \omega, \tag{2.9}$$

and note that $\omega = \{\alpha\}(1-\{\alpha\})/2 \in [0, 1/8]$. Upon solving (2.9) for α and substituting $\sigma_d(\alpha) = s$, we obtain that

$$\alpha_0(d,s) = d + 1/2 - \sqrt{d(d+1) - 2s + \nu},$$

where $\nu = 1/4 - 2\omega \in [0, 1/4]$. With these considerations, for $s < s(d)$, the bound in Conjecture 2.5 can be seen to be of the size

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{\sqrt{2s(d)-2s}-1/2+\eta(d,s)} \, \|\boldsymbol{a}\|_2^{2s} \, N^{o(1)},$$

where

$$\eta(d,s) \leqslant \frac{c}{\sqrt{(s(d)-s)}}$$

for some absolute constant $c > 0$.

Finally, we remark that Theorem 2.3 as well both Conjectures 2.4 and 2.5 address only the range $\delta \geqslant N^{-d}$. However, for smaller δ it is not hard to show that the bound (2.3) is sharp. We give some details on this fact after the proof of Theorem 2.3 below.

3. New bounds

3.1. Bounds on mean values with weights

We first present a family of bounds that can be obtained by combining Lemma 3.8 in [9] with a result of Wooley (Theorem 1.3 in [29]), which improves a previous result of Brandes and Hughes [7].

Theorem 3.1. *Suppose that $\|\boldsymbol{a}\|_\infty \leqslant 1$ and $0 < s \leqslant s(d)/2$. Suppose that $N^{-1} \geqslant \delta > N^{-d}$, and let k be the unique integer satisfying $N^{-k-1} < \delta \leqslant N^{-k}$. We then have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{(d+k)/2} \, N^{s+s(k)/2+o(1)}.$$

Meanwhile, for $\delta > N^{-1}$, we have the bounds

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \begin{cases} \delta^{d/2} \, N^{s+o(1)} & \text{for } N^{-1} < \delta < N^{-1/d}, \\ N^{s-1/2+o(1)} & \text{for } N^{-1/d} < \delta < N^{-1/(2d-1)}, \\ \delta^{d-1/2} \, N^{s+o(1)} & \text{for } N^{-1/(2d-1)} < \delta < 1. \end{cases}$$

We remark that for $\delta \leqslant N^{-d}$, the same methods yield the bound

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^d \, N^{s+s(d)/2+o(1)},$$

which is weaker than the trivial bound (2.3) by our assumption that $s \leqslant s(d)/2$. Since (2.3) is sharp for small δ, it is worth mentioning that the two bounds coincide at the point $s = s(d)/2$. The interested reader may also note that the range of validity of Theorem 3.1 covers values of s and δ for which Conjecture 2.1 does not apply.

For larger values of s, we have the following more complicated bound.

Theorem 3.2. *For any integer s in the range $s(d)/2 < s < s(d)$ and for any $\delta \geqslant N^{-1}$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant N^{s+o(1)}\Big(\delta^{d-1} + \sum_{j=1}^{d-1} \min\left\{\delta^{j-1}(N^{-1/2} + N^{-\eta_{s,d}(j)}), \delta^{(d+j-1)/2} N^{s-s(d)/2}\right\}\Big),$$

where

$$\eta_{s,d}(\ell) = (s(d) - s)\,\frac{d-\ell+1}{d+\ell+1} \quad (1 \leqslant \ell \leqslant d-1). \tag{3.1}$$

Unfortunately, the fairly general bound of Theorem 3.2 may be somewhat hard to parse. However, we note that by always taking the second term in the minimum, we obtain the following simple bound.

Corollary 3.3. *For any integer s in the range $s(d)/2 < s < s(d)$ and for any $\delta \geqslant N^{-1}$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d/2} N^{2s-s(d)/2+o(1)}.$$

Similarly, by using always the first expression in the minimum, one can show with a modicum of calculations that in the range $s(d)/2 < s < s(d)$ and for all $\delta \leqslant N^{-1/(2d-2)}$, one has

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant N^{s-1/2+o(1)}.$$

Clearly, the bound of Corollary 3.3 is not very strong in terms of δ, so for the convenience of the reader we state a further corollary to Theorem 3.2 concerning the range of δ in which the first term dominates. While by no means being deep, this consequence of our result needs some more notation to state.

For a function

$$f(x) = \frac{d+1-x}{(d+x+1)(d-x)}, \tag{3.2}$$

define the parameter $\vartheta(d)$ by putting

$$\vartheta(d) = \min\left\{f\left(d+1-\left\lfloor\sqrt{2(d+1)}\right\rfloor\right), f\left(d+1-\left\lceil\sqrt{2(d+1)}\right\rceil\right)\right\}. \tag{3.3}$$

In particular, we see that

$$\vartheta(d) \sim \frac{1}{2d} \quad (d \to \infty).$$

A list of explicit values of $\vartheta(d)$ for $2 \leqslant d \leqslant 10$ is given in Table 1.

d	2	3	4	5	6	7	8	9	10
$\vartheta(d)$	1/2	3/10	3/14	1/6	2/15	1/9	2/21	1/12	5/68

Table 1. Values of $\vartheta(d)$ for $d = 2, \ldots, 10$.

Corollary 3.4. *Let $d \geqslant 2$ and recall the definition of $\vartheta(d)$ from (3.3). Furthermore, fix some integer $s(d)/2 < s < s(d)$ and a sequence of weights satisfying $\|\boldsymbol{a}\|_\infty \leqslant 1$. Suppose that*

$$\delta > \max\left\{N^{-1/(2d-2)}, N^{-(s(d)-s)\vartheta(d)}\right\}.$$

Then

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{d-1} N^{s+o(1)}.$$

The proofs of Theorems 3.1 and 3.2 depend crucially on the existence of non-trivial bounds for certain inhomogeneous Vinogradov systems. For $\boldsymbol{h} = (h_1, \ldots, h_d) \in \mathbb{Z}^d$, let $J_{s,d}(\boldsymbol{h}; N)$ be the number of solutions to the system of d equations

$$\sum_{j=1}^{2s} (-1)^j n_j^i = h_i \quad (i = 1, \ldots, d), \tag{3.4}$$

in integer variables $1 \leqslant n_1, \ldots, n_{2s} \leqslant N$. By the triangle inequality, we trivially have

$$J_{s,d}(\boldsymbol{h}; N) \leqslant J_{s,d}(N) \leqslant N^{s+o(1)}, \tag{3.5}$$

where in the last step we have used the classical Vinogradov mean value bound of Theorem 1.1 in [6] in the subcritical range $s \leqslant s(d)$, see (1.2). For most choices of $\boldsymbol{h}$, recent results by Brandes and Hughes [7] and Wooley [29] give some slight improvement over this in the entire subcritical range. However, the bounds of their work are not expected to be sharp, and indeed one may be tempted to conjecture that for all integers s in some range $s \leqslant s_1(d)$, for $s_1(d) \leqslant s(d) - 1$, one has the stronger bound

$$\max_{\boldsymbol{h} \neq \boldsymbol{0}} J_{s,d}(\boldsymbol{h}; N) \leqslant N^{s-\nu+o(1)} \tag{3.6}$$

for some $\nu \in (0, 1]$. Clearly, the sharpest version of the conjecture in (3.6) is the one corresponding to the parameters $\nu = 1$ and $s_1(d) = s(d) - 1$. Note that for $\nu > 1$, the bound (3.6) is false even for small values of s, as can be seen by choosing n_1, n_2 and $\boldsymbol{h}$ such that $n_1^j - n_2^j = h_j$ for $1 \leqslant j \leqslant d$, thus reducing the system (3.4) to a homogeneous system in $2(s-1)$ variables which has $J_{s-1,d}(N) \gg N^{s-1}$ solutions. However, the set of possible choices for $\boldsymbol{h}$ for which the bound (3.6) is sharp with $\nu = 1$ is fairly small. Consequently, in many cases we obtain stronger results by averaging over the $\boldsymbol{h}$ (see Lemma 5.3 below).

Conditionally on (3.6) being known for $\nu = 1$, we have the following.

Theorem 3.5. *Let $d \geqslant 2$ and $\|\boldsymbol{a}\|_\infty \leqslant 1$. Assume that (3.6) holds with $\nu = 1$ for all s in some range $s \leqslant s_1(d)$. Let $1 \geqslant \delta > N^{-d}$, and let k be the unique integer satisfying $N^{-k-1} < \delta \leqslant N^{-k}$.*

(1) *Suppose that $0 < s \leqslant \min\{s(d)/2, s_1(d)\}$.*
   - *For $k \geqslant 1$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^{(d+k)/2} N^{s+s(k)/2+o(1)}.$$

- *For $k=0$, we have*

$$I_{s,d}^{(0)}(\delta;\boldsymbol{a},N) \leqslant \begin{cases} \delta^{d/2}N^{s+o(1)} & N^{-1}<\delta\leqslant N^{-2/d},\\ N^{s-1+o(1)} & N^{-2/d}<\delta\leqslant N^{-1/d},\\ \delta^{d}N^{s+o(1)} & N^{-1/d}<\delta\leqslant 1.\end{cases}$$

(2) *Suppose now that $s(d)/2<s\leqslant s_1(d)$. For $k\geqslant 0$, we have*

$$I_{s,d}^{(0)}(\delta;\boldsymbol{a},N) \leqslant N^{s+o(1)}\left(\delta^d+\min\left\{\delta^k N^{s(k)-1},\delta^{(k+d)/2}N^{s-(s(d)-s(k))/2}\right\}\right).$$

We remark that Wooley's range for s coincides with that in part (2) of Theorem 3.5 when $d\equiv 0$ or $d\equiv 3 \pmod 4$, while for $d\equiv 1$ and $d\equiv 2 \pmod 4$, the value $s=(s(d)+1)/2$ is not covered by Conjecture 8.1 in [29], whereas our result is applicable. This is in fact the situation in the (otherwise well-understood) case $d=s=2$, which we discuss below as an example.

Unfortunately, proving (3.6) seems to be quite delicate in general even for non-optimal values of ν. In some special cases, however, suitable bounds are available. For instance, Dendrinos, Hughes and Vitturi, see Lemmas 5 and 6 in [17], showed that (3.6) holds with $\nu=1$ in the cases $d=s=2$ (which implies the statement for $(s,d)=(3,2)$) and $d=s=3$. Thus, after a comparison of all terms in Theorem 3.5, in combination also with (2.3), we obtain the following unconditional bounds.

Corollary 3.6. *Let $\|\boldsymbol{a}\|_\infty\leqslant 1$. For $s=d=2$, as well as $d=3$ and $s=2$ or $s=3$, the mean value $I_{s,d}(\delta;\boldsymbol{a},N)$ is bounded above as detailed in Table 2.*

δ	$(0,N^{-2}]$	$(N^{-2},N^{-1}]$	$(N^{-1},N^{-1/2}]$	$(N^{-1/2},1]$
$I_{2,2}^{(0)}$	$\delta^2N^{4+o(1)}$	$\delta N^{2+o(1)}$	$N^{1+o(1)}$	$\delta^2N^{2+o(1)}$

δ	$(0,N^{-3/2}]$	$(N^{-3/2},N^{-1}]$	$(N^{-1},N^{-2/3}]$	$(N^{-2/3},N^{-1/3}]$	$(N^{-1/3},1]$
$I_{2,3}^{(0)}$	$\delta^3N^{4+o(1)}$	$\delta^2N^{5/2+o(1)}$	$\delta^{3/2}N^{2+o(1)}$	$N^{1+o(1)}$	$\delta^3N^{2+o(1)}$

δ	$(0,N^{-3}]$	$(N^{-3},N^{-2}]$	$(N^{-2},N^{-1}]$	$(N^{-1},N^{-2/3}]$	$(N^{-2/3},N^{-1/3}]$	$(N^{-1/3},1]$
$I_{3,3}^{(0)}$	$\delta^3N^{6+o(1)}$	$\delta^{5/2}N^{9/2+o(1)}$	$\delta^2N^{7/2+o(1)}$	$\delta^{3/2}N^{3+o(1)}$	$N^{2+o(1)}$	$\delta^3N^{3+o(1)}$

Table 2. Upper bounds for $\sup_{\|\boldsymbol{a}\|_\infty\leqslant 1} I_{s,d}^{(0)}(\delta;\boldsymbol{a},N)$ for selected choices of s and d, with δ in corresponding intervals.

For comparison, in the special case $\boldsymbol{a}=\mathbf{1}$, the conjecture proposed by Wooley [29] (Conjecture 2.1) claims that

$$\begin{aligned} I_{2,2}^{\sharp}(\delta;N) &\leqslant \delta^2N^{2+o(1)} \quad \text{for } \delta\geqslant N^{-1/4},\\ I_{2,3}^{\sharp}(\delta;N) &\leqslant \delta^3N^{2+o(1)} \quad \text{for } \delta\geqslant N^{-2/3},\\ I_{3,3}^{\sharp}(\delta;N) &\leqslant \delta^3N^{3+o(1)} \quad \text{for } \delta\geqslant N^{-2/3}.\end{aligned}$$

Clearly, the range of applicability here is much smaller than that of our setting, and for $d=2$, Corollary 3.6 establishes the bound conjectured by Wooley in a much larger range than suggested in [29]. For $d=3$, we establish the bounds from Conjecture 2.1 in the range $N^{-1/3}\leqslant\delta\leqslant 1$, but fall short in the range $N^{-2/3}\leqslant\delta<N^{-1/3}$.

3.2. Bounds on mean values with shifts

When δ is not too small, we also have some results that stem from exploiting the structure of large Weyl sums.

Theorem 3.7. *For any $s > 0$ and any $\delta \geqslant N^{-3/(6+2s)}$, we have*

$$I^{\sharp}_{s,2}(\delta; N) \leqslant \delta^2 N^{2s(1-3/(6+2s))+o(1)}.$$

For $d \geqslant 3$, we put

$$D = \min\{2^{d-1}, 2d(d-1)\}.$$

We then have the following.

Theorem 3.8. *For any $s > (s(d)D - d^2 - 1)/2$ and $\delta \geqslant N^{-(d+1)/(2(2s+d^2+1))}$, we have*

$$I^{\sharp}_{s,d}(\delta; N) \leqslant \delta^d N^{2s(1-s(d)/(2s+d^2+1))+o(1)}.$$

For context, note that when δ assumes the smallest possible value, the upper bounds in Theorems 3.7 and 3.8 take the shape

$$I^{\sharp}_{s,2}(\delta; N) \leqslant N^{2s-3\left(1-\frac{4}{s+3}\right)+o(1)} \quad \text{and} \quad I^{\sharp}_{s,d}(\delta; N) \leqslant N^{2s-s(d)\left(1-\frac{d^2}{2s+d^2+1}\right)+o(1)},$$

respectively. Clearly, $\delta \to 1$ as $s \to \infty$, so it is no surprise that these expressions converge to the bound of (1.2) (and thus also Conjecture 2.1) as s tends to infinity.

Our upper bounds are complemented by the following general lower bounds.

Theorem 3.9. *Fix $s > 0$. There is an absolute constant $C > 0$ with the following properties.*

(1) *For $\delta \geqslant C/N$, we have*

$$I^{\flat}_{s,2}(\delta; N) \gg \delta^2 N^{s-1} \max\{1, (\delta N)^{s-2}\}.$$

(2) *If furthermore $\delta \geqslant C/\sqrt{N}$, we have the stronger bound*

$$I^{\flat}_{s,2}(\delta; N) \gg \delta^2 N^{3(s-1)/2}.$$

We observe that for $\delta \geqslant c_2/\sqrt{N}$, the second bound of Theorem 3.9 improves the first bound, which at the point $\delta = N^{-1/2}$ takes the form $\delta^2 N^{3s/2-2}$.

Our methods also give a bound for dimension $d \geqslant 3$. For $1 \leqslant k < d$, it is convenient to define

$$\nu(d,k) = \min\left\{\frac{1}{2k}, \frac{1}{2d-k}\right\}. \tag{3.7}$$

In that notation, our bound is as follows.

Theorem 3.10. *Fix any $s > 0, k \in \{1, \dots, d\}, d \geqslant 3$. There is a large constant $C > 0$ such that for any $\delta \geqslant CN^{-\nu(d,k)}(\log N)^{1+\nu(d,k)}$, we have*

$$I^{\flat}_{s,d}(\delta; N) \geqslant \delta^d N^{d+s-s(d)+o(1)} \max\left\{1, (\delta^{1/\nu(d,k)} N)^{s-d}\right\}.$$

In particular, for $s \leqslant d$, the bound of Theorem 3.10 simplifies as

$$I^{\flat}_{s,d}(\delta; N) \geqslant \delta^d N^{s+d-s(d)+o(1)},$$

which does not depend on k, and thus holds for $\delta \geqslant N^{-\mu(d)}$, where

$$\mu(d) = \max_{k=1,\dots,d} \nu(d,k).$$

We obviously have

$$\mu(d) \sim \frac{3}{4d} \quad (d \to \infty).$$

Moreover, a list of explicit values of $\mu(d)$ for $3 \leqslant d \leqslant 10$ is given in Table 3.

d	3	4	5	6	7	8	9	10
$\mu(d)$	1/4	1/6	1/7	1/8	1/10	1/11	1/12	1/14

Table 3. Values of $\mu(d)$ for $d = 3, \dots, 10$.

3.3. Discussion and comparison of our results

Here we compare the bounds proposed by Demeter and Langowski (see Conjecture 1.3 in [16]), as well as Wooley's Conjecture 8.2 in [29], with our Conjecture 2.4 as well as with our other upper bounds. It should be emphasised that we do this in the case of $s = 2, 3$, for which Conjecture 1.3 in [16] is actually established in Theorem 2.4 of [16].

To compare our various upper bounds, it is convenient to define

$$\kappa^{(0)}_{s,d}(\tau) = \limsup_{N\to\infty} \sup_{\|\boldsymbol{a}\|_\infty \leqslant 1} \frac{\log I^{(0)}_{s,d}(N^{-\tau}; \boldsymbol{a}, N)}{\log N},$$

$$\kappa^{\sharp}_{s,d}(\tau) = \limsup_{N\to\infty} \frac{\log I^{\sharp}_{s,d}(N^{-\tau}; N)}{\log N},$$

where in $\kappa^{(0)}_{s,d}(\tau)$, the inner supremum is taken over all sequences of complex weights with $\|\boldsymbol{a}\|_\infty \leqslant 1$. It follows from (1.4) that

$$\kappa^{\sharp}_{s,d}(\tau) \leqslant \kappa^{(0)}_{s,d}(\tau).$$

We now present some plots of $\kappa^{\sharp}_{s,d}(\tau)$ and $\kappa^{(0)}_{s,d}(\tau)$ for small values of d and s, which help to compare various bounds and conjectures.

Figure 1 compares the bounds proposed by Demeter and Langowski in Theorem 2.4 of [16] and by Wooley in Conjecture 8.2 of [29], as well as the upper bound of Corollary 3.6 and the lower bounds of Theorem 3.9, in the case $d = s = 2$. We note that the results and conjectures of [29] apply only to $I^{\sharp}_{s,d}(\delta; N)$, while ours apply to the more general quantity $I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N)$ for $\|a\|_\infty \leqslant 1$.

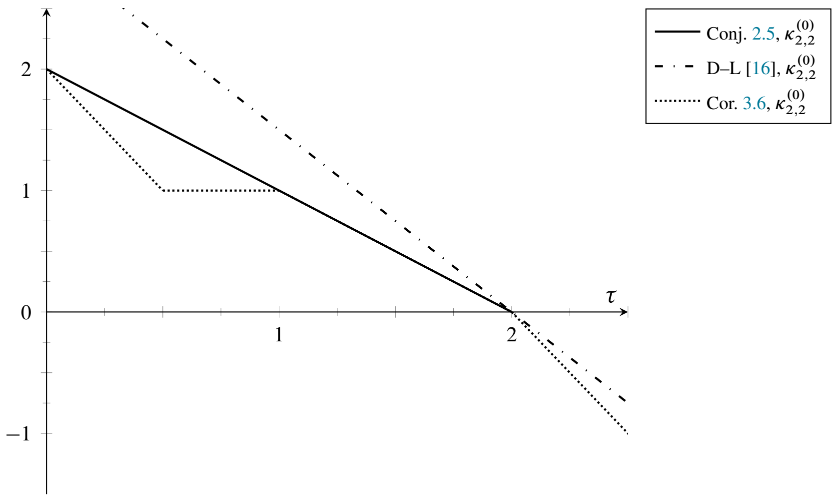

Figure 1. Comparison of upper bounds and conjectures on $\kappa^{(0)}_{2,2}(\tau)$ and $\kappa^{\sharp}_{2,2}(\tau)$ for various values of $\delta = N^{-\tau}$. Wooley's conjecture (Conjecture 2.1) is identical to our Corollary 3.6, but applies only in the range $\tau \leqslant 1/4$.

Observe that Demeter and Langowski [16] conjecture (and prove) diagonal behaviour up to the point $s = \rho(2)/2 = 1$, which puts our configuration of parameters into the supercritical range. In contrast, our more flexible formulation in Conjecture 2.5 allows us to choose parameters in such a way that the value $s = 2$ does correspond to the critical point. Indeed, from (2.8) we see that the choice of $\alpha = 1$ is optimal for our choice of parameters, and consequently our conjecture takes a stronger form than the result obtained by Demeter and Langowski [16]. Moreover, it is evident that at least for the choice of parameters at hand, our conjecture is fully established by the bounds of Corollary 3.6. We also note that our Corollary 3.6 coincides with the bound conjectured by Wooley in Conjecture 8.2 of [29] in the latter one's range of applicability, but is valid for a significantly larger range of δ.

In Figure 2, we present the proved and conjectured bounds for $\kappa^{(0)}_{3,3}(\tau)$ and $\kappa^{\sharp}_{3,3}(\tau)$. In this setting, Demeter and Langowski (Conjecture 1.3 in [16]) address the case $\alpha = (d-1)/2 = 1$, so in view of (2.8), the critical point of their conjecture coincides with that of our Conjecture 2.5, and consequently they anticipate the same bound.

Our Corollary 3.6 gives bounds which are actually stronger than those in Conjecture 1.3 of [16] and in Conjecture 2.5 for $\delta > N^{-1/2}$, but is not strong enough to establish them in the full range. It also establishes Wooley's conjecture (Conjecture 8.2 in [29]) for $\delta \geqslant N^{-1/3}$. Note that for $\delta < N^{-3}$, the trivial bound (2.3) is sharp.

Our final Figure 3 compares the bounds for $\kappa^{(0)}_{3,2}(\tau)$ and $\kappa^{\sharp}_{3,2}(\tau)$. Again, it is obvious from the graph that the theorem by Demeter and Langowski, optimised for a different set of parameters, fails to be sharp in this setting, and indeed, we obtain sharper bounds in our Corollary 3.6 for all $\delta < N^{-2}$ as well as $\delta > N^{-1/2}$. In our Conjecture 2.4, we are allowed to take $\alpha < 1$, and it follows from (2.8) that the value $\alpha = 2/3$ is optimal. As in the previous

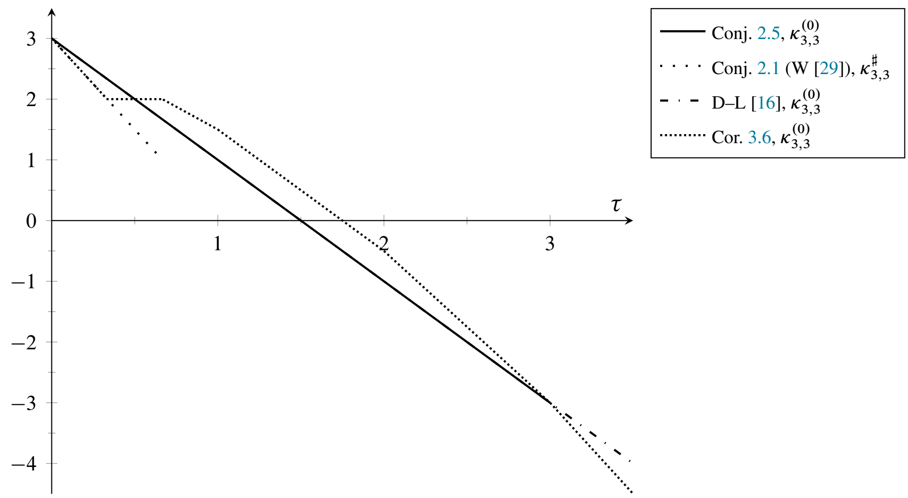

Figure 2. Comparison of upper bounds and conjectures on $\kappa^{(0)}_{3,3}(\tau)$ and $\kappa^{\sharp}_{3,3}(\tau)$ for various values of $\delta = N^{-\tau}$. Observe that in this situation, the bounds of our Conjecture 2.5 and the result of Demeter and Langowski [16] coincide. Wooley's conjecture [29] applies to $\tau \leqslant 2/3$.

setting, this conjecture is overfulfilled for $\delta > N^{-3/7}$, but open for $N^{-3/7} > \delta > N^{-3}$. We see again that our bounds establish Wooley's conjecture (Conjecture 8.2 in [29]) for $\delta \geqslant N^{-1/3}$, but fall short in the range $N^{-2/3} < \delta < N^{-1/3}$.

Remark 3.11. A common feature of Figures 1, 2 and 3 is that the bounds in the extreme ranges $\tau > d$ (corresponding to $\delta \leqslant N^{-d}$) and $\tau < 1/d$ (corresponding to $\delta > N^{-1/d}$) are represented by non-coinciding parallel lines. This is particularly intriguing since in both of these ranges the bounds are proven to be sharp, which raises the question of what the 'truth' looks like between these two ranges. Our result of Corollary 3.6 shows that the 'true' graph cannot be entirely convex or entirely concave, even in the otherwise well-understood case of small degrees and few variables. Instead, there we detect a noticeable plateau at the peak at the origin, and a lowland plain for the averages over larger boxes, but the shape of the slope connecting the two is unclear. This is an indication that the average behaviour of exponential sums over short intervals (and by extension their pointwise behaviour) is governed by phenomena that are poorly understood and deserving of more investigation.

Remark 3.12. We omitted to include our lower bounds in the graphs. The reason for this is that since our lower bounds are uniform in $\boldsymbol{\xi}$, that is, the location of the box within the unit torus. In contrast, our upper bounds either specifically discuss or at least accommodate the box located at the origin, where the exponential sum is known to have a spike. Thus, the lower bounds are of no representative value in the vicinity of the origin, where our upper bounds are known to be sharp. We have no evidence whether the lower bound might be sharp at some $\boldsymbol{\xi}$ away from the origin.

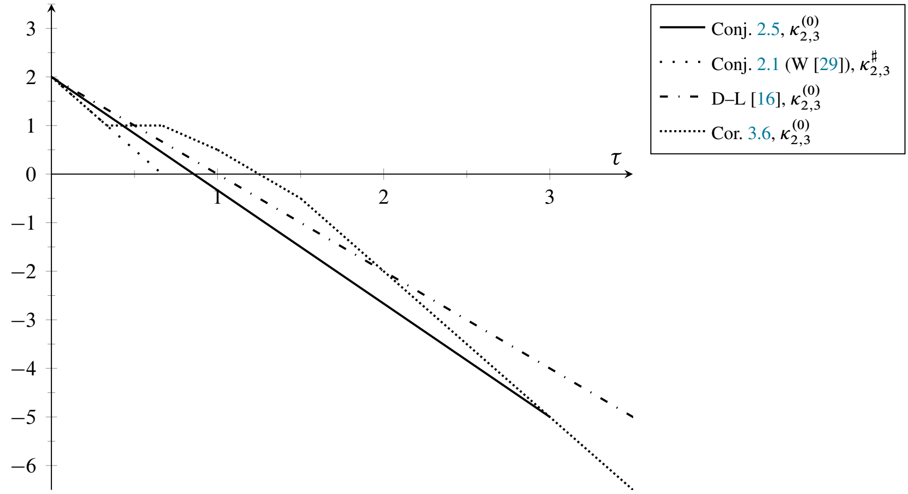

Figure 3. Comparison of upper bounds and conjectures on $\kappa_{2,3}^{(0)}(\tau)$ and $\kappa_{2,3}^{\sharp}(\tau)$ for various values of $\delta = N^{-\tau}$. Wooley's conjecture [29] applies to $\tau \leqslant 2/3$.

4. Proof of Theorem 2.3

Let $\delta \in [N^{-d}, 1]$ be fixed, and define $\mathscr{D} = [-\delta, \delta]^d$. Write further

$$\mathscr{C} = \prod_{j=1}^{d} [-cN^{-j}, cN^{-j}]$$

for some positive $c < 1/(8d)$. Clearly, for $\mathbf{x} \in \mathscr{C}$ we have $|x_1 n + \cdots + x_d n^d| \leqslant 1/8$, and hence

$$|S_d(\mathbf{x}; N)| \gg N.$$

Define $\kappa \in [0, d]$ by the relation $\delta^{-1} = N^{\kappa}$, and put $k = \lfloor \kappa \rfloor$ and $\tau = \kappa - k = \{\kappa\}$, so that we have the inequalities $N^{-(k+1)} < \delta \leqslant N^{-k}$. Since

$$\mathrm{vol}(\mathscr{C} \cap \mathscr{D}) \asymp \delta^k \prod_{j=k+1}^{d} N^{-j} \asymp (N^{-(k+\tau)})^k N^{-s(d)+s(k)} \asymp N^{-s(d)-k(k-1+2\tau)/2},$$

where by convention the empty product is taken to have value 1, we have

$$\int_{\mathscr{D}} |S_d(\mathbf{x}; N)|^{2s} dx \gg \mathrm{vol}(\mathscr{C} \cap \mathscr{D}) N^{2s} \asymp N^{2s-s(d)-k(k-1+2\tau)/2}.$$

From the definition of $s_0(d, \alpha)$, we also have the requirement that

$$I_{s,d}^{(0)}(\delta; N) \leqslant \delta^{d-\alpha} N^{s+o(1)}$$

for $s \leqslant s_0(d,\alpha)$. Thus we need that

$$N^{2s-s(d)-k(k-1+2\tau)/2} \leqslant (N^{-k-\tau})^{d-\alpha} N^{s+o(1)},$$

and in particular,

$$s \leqslant \frac{d(d+1)}{2} + \frac{k(k-1+2\tau)}{2} - (k+\tau)(d-\alpha).$$

Recall now that we aim for a statement that holds for all $\delta \in [N^{-d}, 1]$. We therefore want to minimise the expression

$$F(k,\tau) = \frac{d(d+1)}{2} + \frac{k(k-1+2\tau)}{2} - (k+\tau)(d-\alpha).$$

Observe that formally we have

$$F(k,1) = F(k+1,0), \tag{4.1}$$

as can be confirmed by a straightforward computation. Thus, we can extend the range of $\tau \in [0,1)$ by including the endpoint.

Suppose first that $\alpha \notin \mathbb{Z}$. Clearly, we have

$$\partial F(k,\tau)/\partial\tau = k - (d-\alpha). \tag{4.2}$$

Consequently, for any fixed value of k the function $F(k,\tau)$ is minimal for $\tau = 0$ when $k > d-\alpha$, and for $\tau = 1$ when $k < d-\alpha$.

Assume first that $k > d-\alpha$, so that we can assume that $\tau = 0$. In this case, we have

$$\frac{\partial F(k,0)}{\partial k} = k - (d-\alpha+1/2),$$

which is optimal when k is taken to be the integer that is closest to $d-\alpha+1/2$. Upon writing $d-\alpha+1/2 = \lfloor d-\alpha \rfloor + 1 + (\{d-\alpha\} - 1/2)$ and observing that $(\{d-\alpha\} - 1/2) \in (-1/2, 1/2)$, we see that this closest integer is given by $k = \lfloor d-\alpha \rfloor + 1$.

Similarly, if $k < d-\alpha$, we have $\tau = 1$ and thus

$$\frac{\partial F(k,1)}{\partial k} = k - (d-\alpha-1/2).$$

In this case, we have $d-\alpha-1/2 = \lfloor d-\alpha \rfloor + (\{d-\alpha\} - 1/2)$, where we note that $(\{d-\alpha\} - 1/2) \in (-1/2, 1/2)$, so that the optimal value for k in this setting is given by $k = \lfloor d-\alpha \rfloor$.

Consequently, the function $F(k,\tau)$ is minimised by either $k = \lfloor d-\alpha \rfloor + 1$ and $\tau = 0$, or for $k = \lfloor d-\alpha \rfloor$ and $\tau = 1$. Upon recalling (4.1), it is clear that these values coincide. It thus remains to compute the value of the minimum by inserting the values $k = \lfloor d-\alpha \rfloor + 1$ and $\tau = 0$. Upon writing $\lfloor d-\alpha \rfloor = d-\alpha-\{d-\alpha\}$ and noting that $\{d-\alpha\} = 1-\{\alpha\}$, we find that

$$\begin{aligned} s &\leqslant F(\lfloor d-\alpha\rfloor+1,0)\\ &= \frac{d(d+1)}{2}+\frac{\lfloor d-\alpha\rfloor(\lfloor d-\alpha\rfloor+1)}{2}-(\lfloor d-\alpha\rfloor+1)(d-\alpha)\\ &= \frac{d(d+1)}{2}-\frac{(d-\alpha)(d-\alpha+1)}{2}-\frac{\{d-\alpha\}}{2}+\frac{(\{d-\alpha\})^2}{2}\\ &= \frac{\alpha(2d-\alpha+1)}{2}-\frac{\{\alpha\}(1-\{\alpha\})}{2}. \end{aligned}$$

Finally, when $\alpha\in\mathbb{Z}$, we conclude from (4.2) that $F(k,\tau)$ is minimal for $\tau=0$ when $k>d-\alpha$, and for $\tau=1$ when $k<d-\alpha$, and that it is constant in τ when $k=d-\alpha$. In combination with the continuity property (4.1), it follows that F is minimised for $k=d-\alpha$ and on the entire interval $\tau\in[0,1]$, and we have the explicit value

$$F(d-\alpha+1,0)=\frac{d(d+1)}{2}-\frac{(d-\alpha)(d-\alpha+1)}{2}=\frac{\alpha(2d-\alpha+1)}{2}$$

as well. This completes the proof of Theorem 2.3.

Remark 4.1. It remains to comment on the situation when $\delta\leqslant N^{-d}$. Indeed, adapting the strategy of the above proof to this eventuality, we find that $\mathrm{vol}(\mathscr{C}\cap\mathscr{D})\asymp\delta^d$, and consequently $\delta^d N^{2s}\ll I_{s,d}^{(0)}(\delta;N)$, which matches the trivial bound (2.3).

5. Transition to inhomogeneous mean values

In the following, we denote by $\mathsf{J}_{s,d}(\delta;N)$ the number of solutions to the system of d inequalities

$$\Big|\sum_{j=1}^{2s}(-1)^j n_j^i\Big|\leqslant\delta^{-1},\qquad i=1,\dots,d,$$

in integer variables $1\leqslant n_1,\dots,n_{2s}\leqslant N$. We recall Lemma 3.8 in [9], in a form which is better suited for our applications.

Lemma 5.1. *If $|a_n|\leqslant 1$, $n=1,\dots,N$, then*

$$I_{s,d}^{(0)}(\delta;\boldsymbol{a},N)\ll\delta^d\,\mathsf{J}_{s,d}(\delta;N).$$

Recall the definition of $J_{s,d}(\boldsymbol{h};N)$ from the preamble of (3.5) above. The following is Theorem 1.3 in [29].

Lemma 5.2. *Suppose that $d\geqslant 2$ and $\boldsymbol{h}\neq\mathbf{0}$. Let ℓ be the smallest integer for which $h_\ell\neq 0$, and suppose that $\ell\leqslant d-1$. Then for any integer $s\leqslant d(d+1)/2$, we have*

$$J_{s,d}(\boldsymbol{h};N)\leqslant N^{s-1/2+o(1)}+N^{s-\eta_{s,d}(\ell)+o(1)},$$

where $\eta_{s,d}(j)$ is as given in (3.1).

We point out that we do not have any bound in the situation when $\ell = d$. Observe moreover that $J_{s,d}(\boldsymbol{h}; N) = 0$ trivially when $|h_j| > 2sN^j$ for any $j = 1, \ldots, d$.

Define

$$\mathcal{U} = [-\delta^{-1}, \delta^{-1}]^d \cap \mathbb{Z}^d \quad \text{and} \quad \mathcal{V} = \prod_{j=1}^{d} [-2sN^j, 2sN^j] \cap \mathbb{Z}^d.$$

Then for $1 \leqslant j \leqslant d$ put

$$\mathcal{H}_j = \{\boldsymbol{h} \in \mathcal{U} \cap \mathcal{V} : \ \boldsymbol{h} = (0, \ldots, 0, h_j, \ldots, h_d), \ h_j \neq 0\}.$$

In this notation, we have the obvious partition

$$\mathcal{U} \cap \mathcal{V} = \{\boldsymbol{0}\} \cup \bigcup_{j=1}^{d} \mathcal{H}_j,$$

so that

$$\mathsf{J}_{s,d}(\delta; N) = \sum_{\boldsymbol{h} \in \mathcal{U} \cap \mathcal{V}} J_{s,d}(\boldsymbol{h}; N) = J_{s,d}(N) + \sum_{j=1}^{d} \sum_{\boldsymbol{h} \in \mathcal{H}_j} J_{s,d}(\boldsymbol{h}; N). \tag{5.1}$$

Next we note that for each $j = 1, \ldots d$ we have

$$\#\mathcal{H}_j \asymp \prod_{i=j}^{d} \min\{\delta^{-1}, N^i\} = \delta^{-(d-j+1)} \prod_{i=j}^{d} \min\{1, N^i \delta\}.$$

In particular, if $\delta \in [N^{-k-1}, N^{-k})$ with some integer k we have

$$\#\mathcal{H}_j \asymp \delta^{-(d-j+1)} \prod_{\substack{i=j \\ i \leqslant k}}^{d} (N^i \delta),$$

where the empty product should be interpreted as having value 1. Consequently, we may write

$$\#\mathcal{H}_j \asymp \begin{cases} \delta^{-(d-j+1)} & \text{for } k < j, \\ \delta^{-(d-k)} N^{(k(k+1)-j(j-1))/2} & \text{for } j \leqslant k < d, \\ N^{(d(d+1)-j(j-1))/2} & \text{for } d \leqslant k. \end{cases} \tag{5.2}$$

For future reference we also record the obvious fact that

$$\#\mathcal{H}_1 \geqslant \ldots \geqslant \#\mathcal{H}_d,$$

as well as the bound

$$\#\mathcal{H}_1 \asymp \delta^{-d+k} N^{s(k)} \tag{5.3}$$

which is valid for $0 \leqslant k \leqslant d$.

Finally, we also record the following simple bound.

Lemma 5.3. *Suppose that $d \geqslant 2$. For any finite set $\mathcal{H} \subseteq \mathbb{Z}^d$ we have*

$$\sum_{\boldsymbol{h}\in\mathcal{H}} J_{s,d}(\boldsymbol{h}; N) \leqslant \left(\#\mathcal{H}\, J_{2s,d}(N)\right)^{1/2}.$$

Proof. By Cauchy's inequality, we have

$$\Big(\sum_{\boldsymbol{h}\in\mathcal{H}} J_{s,d}(\boldsymbol{h}; N)\Big)^2 \leqslant \#\mathcal{H} \sum_{\boldsymbol{h}\in\mathcal{H}} J_{s,d}(\boldsymbol{h}; N)^2 \leqslant \#\mathcal{H} \sum_{\boldsymbol{h}\in\mathbb{Z}^d} J_{s,d}(\boldsymbol{h}; N)^2 = \#\mathcal{H}\, J_{2s,d}(N),$$

and the result follows. ∎

6. Proofs of Theorems 3.1, 3.2 and 3.5, and Corollary 3.4

6.1. General upper bounds for weighted Weyl sums over small boxes

We are now ready to establish our most general upper bound for Weyl sums, which implies Theorems 3.1 and 3.2 as well as Theorem 3.5 as special cases. The following result serves as a starting point for all ensuing deliberations.

Proposition 6.1. *Suppose that $\|\boldsymbol{a}\|_\infty \leqslant 1$.*

(1) *For any $s \leqslant s(d)$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^{s+o(1)} \Big(\min\left\{\#\mathcal{H}_d, (\#\mathcal{H}_d)^{1/2}(1 + N^{s-s(d)/2})\right\} + \sum_{j=1}^{d-1} \min\left\{\#\mathcal{H}_j(N^{-1/2} + N^{-\eta_{s,d}(j)}), (\#\mathcal{H}_j)^{1/2}(1+N^{s-s(d)/2})\right\}\Big),$$

where $\eta_{s,d}(j)$ is as given in (3.1).

(2) *Suppose now that (3.6) is known for some ν and some $s_1(d)$. For all integers $s \leqslant s_1(d)$ and any $\delta \in (0, 1]$, we have the potentially stronger bound*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^{s+o(1)} \left(1 + \min\left\{\#\mathcal{H}_1 N^{-\nu}, (\#\mathcal{H}_1)^{1/2}(1 + N^{s-s(d)/2})\right\}\right).$$

Proof. Our starting point is the decomposition (5.1). First we observe that we can apply Lemma 5.2 on the first $d-1$ of the inner summands. Furthermore, for $j = d$ we use the bound

$$\sum_{\boldsymbol{h}\in\mathcal{H}_d} J_{s,d}(\boldsymbol{h}; N) \ll \min\left\{\#\mathcal{H}_d\, J_{s,d}(N), (\#\mathcal{H}_d)^{1/2} J_{2s,d}(N)^{1/2}\right\},$$

which combines (3.5) and Lemma 5.3. Recalling (1.2), we obtain

$$\sum_{\boldsymbol{h}\in\mathcal{H}_d} J_{s,d}(\boldsymbol{h}; N) \ll N^{s+o(1)} \min\left\{\#\mathcal{H}_d, (\#\mathcal{H}_d)^{1/2}(1 + N^{s-s(d)/2})\right\}.$$

Similarly, for $1 \leqslant j \leqslant d-1$ we have

$$\sum_{\boldsymbol{h} \in \mathcal{H}_j} J_{s,d}(\boldsymbol{h}; N) \ll N^{s+o(1)} \min\big\{\#\mathcal{H}_j (N^{-1/2} + N^{-\eta_{s,d}(j)}), (\#\mathcal{H}_j)^{1/2}(1 + N^{s-s(d)/2})\big\}.$$

Combining both of these bounds with the result of Lemma 5.1 leads to the desired conclusion in the unconditional case.

For the conditional setting, we only need to make some minor modifications to the above argument. Again starting from (5.1), we can now use (3.6) inside all of the inner summands. Thus, for $1 \leqslant j \leqslant d$ we have

$$\sum_{\boldsymbol{h} \in \mathcal{H}_j} J_{s,d}(\boldsymbol{h}; N) \ll N^{s+o(1)} \min\big\{\#\mathcal{H}_j N^{-\nu}, (\#\mathcal{H}_j)^{1/2}(1 + N^{s-s(d)/2})\big\}.$$

Substituting this back into (5.1) and invoking Lemma 5.1 yields

$$\begin{aligned} I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) &\leqslant \delta^d N^{s+o(1)} \Big(1 + \sum_{j=1}^{d} \min\big\{\#\mathcal{H}_j N^{-\nu}, (\#\mathcal{H}_j)^{1/2}(1 + N^{s-s(d)/2})\big\}\Big) \\ &\leqslant N^{s+o(1)} \big(1 + \min\big\{\#\mathcal{H}_1 N^{-\nu}, (\#\mathcal{H}_1)^{1/2}(1 + N^{s-s(d)/2})\big\}\big), \end{aligned}$$

where in the last step we have used that $\#\mathcal{H}_1 = \max_j \#\mathcal{H}_j$. ■

6.2. Proofs of Theorems 3.1 and 3.5

We now specialise to the case $s \leqslant s(d)/2$. In that situation, the conclusion of Proposition 6.1 (1) can be simplified significantly.

Lemma 6.2. *For any integer $s \leqslant s(d)/2$ and any $\delta \in (0, 1]$, we have*

$$I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^{s+o(1)} \big(\min\big\{\#\mathcal{H}_1 N^{-1/2}, (\#\mathcal{H}_1)^{1/2}\big\} + (\#\mathcal{H}_d)^{1/2}\big).$$

Proof. Recall Proposition 6.1 (1). Clearly, under the assumptions of the lemma we have $N^{s-s(d)/2} \ll 1$. Moreover, for s in the admissible range we have

$$\min_{1 \leqslant j \leqslant d-1} \eta_{s,d}(j) \geqslant \frac{s(d)}{2} \cdot \min_{1 \leqslant j \leqslant d-1} \frac{d-j+1}{d+j+1} \geqslant \frac{s(d)}{2} \cdot \frac{2}{2d} = \frac{d+1}{4} > 1/2$$

for all $d \geqslant 2$. Consequently, the conclusion of Proposition 6.1 simplifies to

$$\begin{aligned} I_{s,d}^{(0)}(\delta; \boldsymbol{a}, N) &\leqslant \delta^d N^{s+o(1)} \Big(1 + \sum_{j=1}^{d-1} \min\big\{\#\mathcal{H}_j N^{-1/2}, (\#\mathcal{H}_j)^{1/2}\big\} + (\#\mathcal{H}_d)^{1/2}\Big) \\ &\leqslant N^{s+o(1)} \big(\min\big\{\#\mathcal{H}_1 N^{-1/2}, (\#\mathcal{H}_1)^{1/2}\big\} + (\#\mathcal{H}_d)^{1/2}\big), \end{aligned}$$

where in the last step we used that $\#\mathcal{H}_1 = \max_j \#\mathcal{H}_j$. This concludes the proof. ■

The derivation of Theorems 3.1 and 3.5 is now straightforward. We note from (5.2) that $\#\mathcal{H}_1 \gg N$ for all $\delta < N^{-1/d}$. This is obvious for $\delta < N^{-1}$, and can be checked in a straightforward manner for $N^{-1} < \delta < N^{-1/d}$. In those situations, we have

$$\min\left\{\#\mathcal{H}_1 N^{-1/2}, (\#\mathcal{H}_1)^{1/2}\right\} = (\#\mathcal{H}_1)^{1/2} \geqslant (\#\mathcal{H}_d)^{1/2},$$

and the bound becomes

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^{s+o(1)} (\#\mathcal{H}_1)^{1/2}.$$

Finally, if $\delta > N^{-1/d}$ we see from (5.2) that

$$\min\left\{\#\mathcal{H}_1 N^{-1/2}, (\#\mathcal{H}_1)^{1/2}\right\} = \#\mathcal{H}_1 N^{-1/2} \asymp \delta^{-d} N^{-1/2},$$

so that we obtain

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N) \leqslant \delta^d N^{s+o(1)} (\delta^{-d} N^{-1/2} + \delta^{-1/2}).$$

When $\delta > N^{-1/(2d-1)}$, the first term prevails. The proof of Theorem 3.1 is complete upon combining both of these bounds with (5.2).

We now pivot to the proof of Theorem 3.5, where we suppose that (3.6) is known with $\nu = 1$. At this point, the bound in part (2) of the theorem is immediate from Proposition 6.1 (2) upon inserting (5.3).

To establish the bounds of part (1), we begin by noting that $\#\mathcal{H}_1 \gg N^2$ for all $\delta < N^{-2/d}$. This is again immediate from (5.3) for $\delta \leqslant N^{-2}$, and straightforward to check in the intervals $N^{-2} < \delta \leqslant N^{-1}$ and $N^{-1} < \delta \leqslant N^{-2/d}$, respectively. Consequently, in this range of δ we find that

$$\min\left\{\#\mathcal{H}_1 N^{-1}, (\#\mathcal{H}_1)^{1/2}\right\} = \#\mathcal{H}_1^{1/2} > 1.$$

Finally, for $N^{-2/d} < \delta < 1$ we obtain

$$\min\left\{\#\mathcal{H}_1 N^{-1}, (\#\mathcal{H}_1)^{1/2}\right\} = \#\mathcal{H}_1 N^{-1} \begin{cases} \ll 1 & \text{if } N^{-2/d} < \delta < N^{-1/d}, \\ \gg 1 & \text{if } N^{-1/d} \leqslant \delta \leqslant 1. \end{cases}$$

The conclusion of Theorem 3.5 (1) is now complete upon using these bounds within Proposition 6.1 (2) and inserting the values of (5.3).

6.3. Proofs of Theorem 3.2 and Corollary 3.4

We now investigate the situation when $s(d)/2 < s < s(d)$ and $\delta > N^{-1}$. In that situation, we have $\#\mathcal{H}_j \asymp \delta^{-d+j-1}$ for $1 \leqslant j \leqslant d$. Moreover, since $\delta \geqslant N^{-1} \geqslant N^{s(d)-2s}$, we clearly have

$$\min\left\{\delta^{-1}, \delta^{-1/2} N^{s-s(d)/2}\right\} = \delta^{-1}.$$

Thus, under these conditions, the conclusion of Proposition 6.1 reads

$$I^{(0)}_{s,d}(\delta; \boldsymbol{a}, N)$$
$$\leqslant \delta^d N^{s+o(1)} \Big(\delta^{-1} + \sum_{j=1}^{d-1} \min\{\delta^{-d+j-1}(N^{-1/2} + N^{-\eta_{s,d}(j)}), \delta^{-(d-j+1)/2} N^{s-s(d)/2}\}\Big).$$

This completes the proof of Theorem 3.2.

To finish the proof of Corollary 3.4, we begin by noting that

$$\delta^{-1} < \delta^{-(d-1+j)/2} N^{s-s(d)/2} \quad \text{for all } j \text{ and all } \delta \leqslant 1.$$

Consequently, it is sufficient to check in what range of δ one has

$$\begin{aligned}\delta^{-1} &\gg \max_{1\leqslant j\leqslant d-1} \delta^{-d+j-1}(N^{-1/2} + N^{-\eta_{s,d}(j)}) \\ &\asymp \delta^{-d} N^{-1/2} + \max_{1\leqslant j\leqslant d-1} \delta^{-d+j-1} N^{-\eta_{s,d}(j)}.\end{aligned}$$

On comparing these terms and recalling the definition of $\eta_{s,d}(j)$ from (3.1), it is enough to choose

$$\delta \geqslant \max\left\{N^{-(s(d)-s)\vartheta(d)}, N^{-1/(2d-2)}\right\},$$

with

$$\vartheta(d) = \min_{j=1,\dots,d-1} f(j),$$

where the function f is defined by (3.2). The proof is thus complete if we can show that this definition of $\vartheta(d)$ coincides with the one given in (3.3).

Since the denominator of $f(x)$ vanishes at $x = d$ and at $x = -d-1$, neither of which lie in the interval $[1, d-1]$, we see that f is continuous inside said interval. Moreover, simple but somewhat tedious calculus shows that

$$f'(x) = -\frac{d^2 - 1 - 2(d+1)x + x^2}{(d^2 + d - x^2 - x)^2}.$$

This expression has two roots at

$$x_{\pm} = d + 1 \pm \sqrt{2(d+1)},$$

of which we can disregard the larger one since it is clearly outside the interval $[1, d-1]$. Since f' has a sign change from negative to positive at $x = x_-$, that root corresponds to a minimum. In order to compute the value, note that for $d \geqslant 3$ we have $d + 1 - \sqrt{2(d+1)} \in [1, d-1]$, so that both $d + 1 - \lfloor\sqrt{2(d+1)}\rfloor$ and $d + 1 - \lceil\sqrt{2(d+1)}\rceil$ lie in the set $\{1, \dots, d-1\}$. Thus, the values for $\vartheta(d)$ certainly coincide for $d \geqslant 3$. Finally, for $d = 2$ the identity is straightforward to check explicitly. This completes the proof of Corollary 3.4.

7. Approach via the structure of large Weyl sums

In what follows, it is be convenient to define

$$D = \min\{2^{d-1}, 2d(d-1)\}.$$

We begin our analysis with a description of the structure of large Weyl sums (for $d = 2$, these sums are also called Gauss sums):

$$G(x_1, x_2; N) = S_2((x_1, x_2); N) = \sum_{n=1}^{N} \mathbf{e}(x_1 n + x_2 n^2). \tag{7.1}$$

The following is Lemma 5.1 in [3], which in turn follows from a result (Theorem 3 in [1]) of Baker (see also Theorem 4 in [2]).

Lemma 7.1. *We fix some $\varepsilon > 0$, and suppose that for a real $A > N^{1/2+\varepsilon}$, we have the lower bound $|G(x_1, x_2; N)| \geqslant A$ for some $(x_1, x_2) \in \mathbb{R}^2$. Then there exist integers q, a_1 and a_2 such that*

$$1 \leqslant q \leqslant (NA^{-1})^2 N^{o(1)},$$

and for $i = 1, 2$, we have

$$\left| x_i - \frac{a_i}{q} \right| \leqslant (NA^{-1})^2 q^{-1} N^{-i+o(1)}.$$

For $d \geqslant 3$, we use the following result from [4], which is based on a combination of results of Baker (Theorem 3 in [1] and Theorem 4 in [2]) with bounds of complete rational sums, see, for example, [14]. Namely, by Lemma 2.7 in [4], we have the following.

Lemma 7.2. *We fix $d \geqslant 3$, some $\varepsilon > 0$, and suppose that for a real number A satisfying $A > N^{1-1/D+\varepsilon}$ we have $|S_d(\mathbf{x}; N)| \geqslant A$ for some $\mathbf{x} \in \mathsf{T}_d$. Then there exist positive integers $q_2, \ldots, q_d$, with $\gcd(q_i, q_j) = 1$ for $2 \leqslant i < j \leqslant d$, such that*

(i) *q_2 is cube-free,*

(ii) *q_i is i-th power-full but $(i+1)$-th power-free when $3 \leqslant i \leqslant d-1$,*

(iii) *q_d is d-th power-full,*

and

$$\prod_{i=2}^{d} q_i^{1/i} \leqslant N^{1+o(1)} A^{-1},$$

and integers $b_1, \ldots, b_d$ with

$$\gcd(q_2 \cdots q_d, b_1, \ldots, b_d) = 1$$

such that

$$\left| x_j - \frac{b_j}{q_2 \cdots q_d} \right| \leqslant (NA^{-1})^d N^{-j+o(1)} \prod_{i=2}^{d} q_i^{-d/i}, \quad j = 1, \ldots, d.$$

Remark 7.3. For errors of the approximations to x_1 and x_2 of Lemma 7.1, by the condition of $A > N^{1/2+\varepsilon}$, we have

$$(NA^{-1})^2 q^{-1} N^{-i+o(1)} \leqslant q^{-1} N^{-2\varepsilon+o(1)}, \quad i = 1, 2. \tag{7.2}$$

Similarly, for errors of Lemma 7.2, we have

$$(NA^{-1})^d N^{-j+o(1)} \prod_{i=2}^{d} q_i^{-d/i} \leqslant N^{-d\varepsilon+o(1)} \prod_{i=2}^{d} q_i^{-1}, \quad j = 1, \ldots, d. \tag{7.3}$$

For a real $A > 0$, we define the level set

$$\mathscr{F}_{d,A} = \{\mathbf{x} \in \mathsf{T}_d : |S_d(\mathbf{x}; N)| \geqslant A\}. \tag{7.4}$$

Further, for a box $\mathfrak{B}(\boldsymbol{\xi}, \delta) = \boldsymbol{\xi} + [0, \delta]^d \subseteq \mathsf{T}_d$, denote

$$\lambda_{d,\boldsymbol{\xi}}(\delta, A; N) = \lambda(\mathfrak{B}(\boldsymbol{\xi}, \delta) \cap \mathscr{F}_{d,A}). \tag{7.5}$$

Lemma 7.4. *Suppose that $A > N^{1/2+\varepsilon}$ for some fixed $\varepsilon > 0$. Then for any $\delta \geqslant AN^{-1}$, we have*

$$\lambda_{2,\boldsymbol{\xi}}(\delta, A; N) \leqslant \delta^2 N^{3+o(1)} A^{-6}.$$

Proof. Let

$$Q = (NA^{-1})^2 N^{\eta},$$

for some small $\eta > 0$. For $q \in \mathbb{N}$ and $\mathbf{b} = (b_1, b_2) \in [q]^2$, define the rectangular box

$$R_q(\mathbf{b}) = B(b_1/q, Qq^{-1}N^{-1}) \times B(b_2/q, Qq^{-1}N^{-2}),$$

where $B(x, r) \subseteq \mathbb{R}$ denotes the interval with center x and radius r. Clearly, each such box has area

$$\lambda(R_q(\mathbf{b})) \asymp Q^2/(q^2 N^3).$$

By Lemma 7.1, for all sufficiently large N we obtain

$$\mathscr{F}_{2,A} \subseteq \bigcup_{q \leqslant Q} \bigcup_{(b_1,b_2)\in[q]^2} R_q(\mathbf{b}).$$

It is an easy consequence of (7.2) that the boxes $R_q(\mathbf{b})$ are disjoint for all $q \in \mathbb{N}$. It follows that any box $\mathfrak{B}(\boldsymbol{\xi}, \delta)$ intersects with at most $O(1 + (q\delta)^2)$ boxes $R_q(\mathbf{b})$. Consequently, recalling (7.5), we derive

$$\begin{aligned}
\lambda_{2,\boldsymbol{\xi}}(\delta, A; N) = \lambda(\mathfrak{B}(\boldsymbol{\xi}, \delta) \cap \mathscr{F}_{2,A}) &\ll \sum_{q \leqslant Q} \sum_{(b_1,b_2)\in[q]^2} \lambda(R_q(\mathbf{b}) \cap \mathfrak{B}(\boldsymbol{\xi}, \delta)) \\
&\ll \sum_{q \leqslant Q} \sum_{\substack{\mathbf{b}\in[q]^2 \\ R_q(\mathbf{b})\cap\mathfrak{B}(\boldsymbol{\xi},\delta)\neq\emptyset}} \lambda(R_q(\mathbf{b})) \ll \sum_{q \leqslant Q} (1 + (q\delta)^2) \frac{Q^2}{q^2 N^3} \\
&\ll \frac{Q^2}{N^3} + \frac{Q^3}{N^3}\delta^2.
\end{aligned}$$

Therefore, using $\delta \geqslant AN^{-1} \geqslant Q^{-1/2}$, we derive

$$\lambda_{2,\boldsymbol{\xi}}(\delta, A; N) \ll \frac{Q^2}{N^3} + \frac{Q^3}{N^3}\delta^2 \ll \frac{Q^3}{N^3}\delta^2 = N^{3+3\eta} A^{-6} \delta^2.$$

Since $\eta > 0$ is arbitrary, we obtain the desired bound. ■

For $d \geqslant 3$, we mimic the proof of Lemma 2.9 in [4] in order to obtain a level set estimation with restriction to some small box. Formally, taking $k = d$ in Lemma 2.9 of [4] and adding a factor of δ^d there, we have the following bound.

Lemma 7.5. *Suppose that $d \geqslant 3$ and $A > N^{1-1/D+\varepsilon}$ for some fixed $\varepsilon > 0$. Then for any $\delta \geqslant (AN^{-1})^{1/d}$, we have*

$$\lambda_{d,\boldsymbol{\xi}}(\delta, A; N) \leqslant N^{d^2+1-s(d)+o(1)} A^{-d^2-1} \delta^d.$$

Proof. Let

$$Q = (NA^{-1})^d N^\eta \tag{7.6}$$

for some small number $\eta > 0$. For any $q_2, \ldots, q_d \in \mathbb{N}$ and $b_1, \ldots, b_d \in \mathbb{Z}$, define the box

$$R_{q_2,\ldots,q_d}(\mathbf{b}) = \left\{\mathbf{x} \in \mathsf{T}_d \ : \ \left|x_j - \frac{b_j}{q_2 \ldots q_d}\right| \leqslant QN^{-j} \prod_{i=2}^d q_i^{-d/i}, \ j = 1, \ldots, d\right\}.$$

Again, we note that

$$\lambda(R_{q_2,\ldots,q_d}(\mathbf{b})) \asymp \prod_{j=1}^d \left(QN^{-j} \prod_{i=2}^d q_i^{-d/i}\right) \asymp Q^d N^{-s(d)} \prod_{i=2}^d q_i^{-d^2/i}. \tag{7.7}$$

Moreover, by (7.3), these boxes are pairwise disjoint. Thus, for any fixed d-tuple $(q_2, \ldots, q_d)$, the number of boxes $R_{q_2,\ldots,q_d}(\mathbf{b})$ intersecting the box $\mathfrak{B}(\boldsymbol{\xi}, \delta)$ nontrivially is given by

$$\#\left\{\mathbf{b} \in [q_2 \cdots q_d]^d \ : \ R_{q_2,\ldots,q_d}(\mathbf{b}) \cap \mathfrak{B}(\boldsymbol{\xi}, \delta) \neq \emptyset\right\} = O\left(1 + (\delta q_2 \cdots q_d)^d\right). \tag{7.8}$$

For any integer $i \geqslant 2$, it is convenient to denote

$$\mathcal{F}_i = \{n \in \mathbb{N} \ : \ n \text{ is } i\text{-th power full}\} \quad \text{and} \quad \mathcal{F}_i(x) = \mathcal{F}_i \cap [1, x],$$

so that an easy counting shows that

$$\#\mathcal{F}_i(x) \ll x^{1/i}, \tag{7.9}$$

and to put

$$\Omega = \left\{(q_2, \ldots, q_d) \in \mathbb{N}^{d-1} \ : \ q_i \in \mathcal{F}_i \quad (3 \leqslant i \leqslant d), \quad \prod_{i=2}^d q_i^{d/i} \leqslant Q\right\}.$$

Thus, recalling the definition (7.4), we clearly have

$$\mathscr{F}_{d,A} \subseteq \bigcup_{(q_2,\ldots,q_d)\in\Omega} \ \bigcup_{\mathbf{b}\in[q_2\cdots q_d]^d} R_{q_2,\ldots,q_d}(\mathbf{b}).$$

Combining this with (7.7) and (7.8), and recalling (7.5), we can estimate

$$\begin{aligned}
\lambda_{d,\boldsymbol{\xi}}(\delta, A; N) &= \lambda(\mathfrak{B}(\boldsymbol{\xi}, \delta) \cap \mathscr{F}_{d,A}) \\
&\leqslant \sum_{(q_2,\ldots,q_d)\in\Omega} \ \sum_{\substack{\mathbf{b}\in[q_2\cdots q_d]^d \\ R_{q_2,\ldots,q_d}(\mathbf{b})\cap\mathfrak{B}(\boldsymbol{\xi},\delta)\neq\emptyset}} \lambda(R_{q_2,\ldots,q_d}(\mathbf{b})) \\
&\ll \sum_{(q_2,\ldots,q_d)\in\Omega} \left(1 + (\delta q_2 \cdots q_d)^d\right) Q^d N^{-s(d)} \prod_{i=2}^d q_i^{-d^2/i}.
\end{aligned} \tag{7.10}$$

Write

$$U_1 = \sum_{(q_2,\ldots,q_d)\in\Omega} \prod_{i=2}^{d} q_i^{-d^2/i} \quad \text{and} \quad U_2 = \sum_{(q_2,\ldots,q_d)\in\Omega} \prod_{i=2}^{d} q_i^{d-d^2/i}.$$

Then (7.10) can be bounded by

$$\lambda_{d,\xi}(\delta, A; N) \ll Q^d N^{-s(d)} U_1 + \delta^d Q^d N^{-s(d)} U_2. \tag{7.11}$$

Clearly,

$$U_1 \leqslant \sum_{(q_2,\ldots,q_d)\in\mathbb{N}^d} \prod_{i=2}^{d} q_i^{-d^2/i} \ll 1. \tag{7.12}$$

We now turn to the estimation of U_2. For $Q_2, \ldots, Q_d$ and $\delta > 0$, denote

$$\Omega(Q_2, \ldots, Q_d) = \{(q_2, \ldots, q_d) \in \Omega : Q_i/2 < q_i \leqslant Q_i,\ i = 2, \ldots, d\},$$

and write

$$U_2(Q_2, \ldots, Q_d) = \sum_{(q_2,\ldots,q_d)\in\Omega(Q_2,\ldots,Q_d)} \prod_{i=2}^{d} q_i^{d-d^2/i}.$$

Thus, covering Ω by $O((\log N)^d)$ dyadic boxes, we see that

$$U_2 \ll \max\left\{U_2(Q_2, \ldots, Q_d) :\ Q_2, \ldots, Q_d \geqslant 1,\ \prod_{i=2}^{d} Q_i^{d/i} \leqslant Q\right\}(\log N)^d. \tag{7.13}$$

By (7.9), this yields

$$U_2(Q_2, \ldots, Q_d) \ll \sum_{Q_2/2<q_2\leqslant Q_2} q_2^{d-d^2/2} \prod_{i=3}^{d} \left(Q_i^{d-d^2/i} \#\mathcal{F}_i(Q_i)\right) \ll \prod_{i=2}^{d} Q_i^{\alpha_i}, \tag{7.14}$$

where

$$\alpha_2 = d - d^2/2 + 1 \quad \text{and} \quad \alpha_i = d - (d^2 - 1)/i \quad (i = 3, \ldots, d).$$

Observe that for every $i = 2, \ldots, d$, we have $\alpha_i \leqslant 1/i$. Combining this with the condition on $Q_2, \ldots, Q_d$ in (7.13), we derive from (7.13) and (7.14) that

$$U_2 \ll Q^{1/d}(\log N)^d. \tag{7.15}$$

Finally, we can combine the bounds of (7.11), (7.12) and (7.15). Thus, and recalling the condition $\delta \geqslant (AN^{-1})^{1/d}$, as well as the definition of Q from (7.6) together with arbitrary choice of $\eta > 0$, we obtain

$$\begin{aligned}\lambda_{d,\xi}(\delta, A; N) &\ll Q^d N^{-s(d)} + Q^{d+1/d} N^{-s(d)} \delta^d (\log N)^d \\ &\leqslant (NA^{-1})^{d^2+1} N^{-s(d)+o(1)} \delta^d,\end{aligned}$$

which finishes the proof. ∎

8. Proofs of Theorems 3.7 and 3.8

8.1. Proof of Theorem 3.7

Let $\boldsymbol{\xi} \in \mathsf{T}_2$ and

$$A = N^{1/2+s/(6+2s)}.$$

Next, we divide the set $\mathfrak{B}(\boldsymbol{\xi}, \delta) = \boldsymbol{\xi} + [0, \delta]^2$ into two parts, depending on whether $|S_2(\mathbf{x}; N)| \geqslant A$ or not. Thus combining with Lemma 7.4, which applies for the above choice of A, we derive

$$\begin{aligned} I^{\sharp}_{s,2}(\delta; N) &\leqslant A^{2s}\delta^2 + N^{2s} \sup_{\boldsymbol{\xi} \in \mathsf{T}_2} \lambda(\{\mathbf{x} \in \mathfrak{B}(\boldsymbol{\xi}, \delta) : |S_2(\mathbf{x}; N)| \geqslant A\}) \\ &\leqslant A^{2s}\delta^2 + N^{2s+3+o(1)} A^{-6}\delta^2, \end{aligned}$$

which yields the desired bound.

8.2. Proof of Theorem 3.8

Let $\boldsymbol{\xi} \in \mathsf{T}_d$ and let

$$A = N^{1-s(d)/(2s+d^2+1)},$$

noting that the hypothesis $s > (s(d)D - d^2 - 1)/2$ ensures that $A > N^{1-1/D+\varepsilon}$, so that Lemma 7.5 is applicable. Divide the box $\mathfrak{B}(\boldsymbol{\xi}, \delta) = \boldsymbol{\xi} + [0, \delta]^d$ into two parts depending on whether $|S_d(\mathbf{x}; N)| \geqslant A$ or not. Thus, applying Lemma 7.5, we obtain

$$\begin{aligned} I^{\sharp}_{s,d}(\delta; N) &\leqslant A^{2s}\delta^d + N^{2s} \sup_{\boldsymbol{\xi} \in \mathsf{T}_d} \lambda(\{\mathbf{x} \in \mathfrak{B}(\boldsymbol{\xi}, \delta) : |S_d(\mathbf{x}; N)| \geqslant A\}) \\ &\leqslant A^{2s}\delta^d + N^{2s+d^2+1-s(d)+o(1)} A^{-d^2-1}\delta^d, \end{aligned}$$

which yields the desired bound.

9. Rational exponential sums

9.1. Gauss sums

Recall the definition (7.1) of Gauss sums. We also record their explicit evaluation, which is classical (see, for example, equation (1.55) in [20]). For convenience of notation, denote $\mathbf{e}_p(z) = \mathbf{e}(z/p)$.

Lemma 9.1. *Let $p \geqslant 3$ be a prime number and let $a, b \in \mathbb{F}_p$, with $b \neq 0$. Then*

$$\left| \sum_{n=0}^{p-1} \mathbf{e}_p(an + bn^2) \right| = p^{1/2}.$$

We also recall a classical result of Fiedler, Jurkat and Körner, see Lemma 4 in [18].

Lemma 9.2. *For any prime p and any $a, b \in \mathbb{F}_p$ with $b \neq 0$, we have*

$$\max_{1 \leqslant M,N \leqslant p} \Big| \sum_{M+1 \leqslant n \leqslant M+N} \mathbf{e}_p(an + bn^2) \Big| \ll p^{1/2}.$$

Lemma 9.3. *Let p be a prime and let N be an integer with $N \geqslant Cp$ for some positive constant C. Suppose that the pair $(x_1, x_2) \in \mathsf{T}_2$ has a rational approximation of the shape*

$$|x_1 - a/p| \leqslant c/N \quad \text{and} \quad |x_2 - b/p| \leqslant c/N^2$$

for some positive constant c, where $\gcd(b, p) = 1$. *Then we have*

$$|G(x_1, x_2; N)| \gg Np^{-1/2}.$$

Proof. Combining Lemma 9.2 with Corollary 2.6 in [13], we obtain a continuity property of Gauss sums:

$$G(x_1, x_2; N) - G(a/p, b/p; N) \ll \sqrt{p}\,\big(1 + Np^{-1}\big(|x_1 - a/p|N + |x_2 - b/p|N^2\big)\big).$$

Since by Lemmas 9.1 and 9.2 we have

$$|G(a/p, b/p; N)| = \lfloor N/p \rfloor\, p^{1/2} + O(p^{1/2}) = Np^{-1/2} + O(p^{1/2}),$$

for an appropriate choice of C, we obtain the desired result. ∎

9.2. Rational sums with arbitrary polynomials

For $d \geqslant 3$, we do not have an analogue of Lemma 9.1. For an arbitrary box $\boldsymbol{\xi} + [0, \delta]^d \in \mathsf{T}_d$, we follow the same strategy as in [10] on the distribution of large complete rational sums. In fact, we need a more refined version of the argument presented in Lemma 2.6 of [10], that provides quantitative estimates on the number of large sums inside any given small box. Then using a method similar to those employed in the treatment of the case $d = 2$, we obtain some nontrivial lower bounds.

Let p be a prime and let $\mathbb{F}_p$ denote the finite field of p elements. For a vector $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{F}_p^d$, we consider the rational exponential sum

$$T_{d,p}(\mathbf{u}) = S_d(\mathbf{u}/p; p) = \sum_{n=1}^{p} \mathbf{e}_p(u_1 n + \cdots + u_d n^d).$$

We also consider discrete cubic boxes

$$\mathfrak{B} = \mathcal{I}_1 \times \cdots \times \mathcal{I}_d \subseteq \mathbb{F}_p^d \tag{9.1}$$

with side-length L, where for each $j = 1, \ldots, d$, the set $\mathcal{I}_j = \{k_j + 1, \ldots, k_j + L\}$ is a set of $L \leqslant p$ consecutive integers, reduced modulo p if $k_j + L \geqslant p$.

Our goal is to establish a quantitative version of Lemma 2.6 in [10]. As in [10], we start with recalling that by a result of Knizhnerman and Sokolinskii (Theorem 1 in [22], see also [21]), we have the following.

Lemma 9.4. *For every integer $d \geqslant 2$, there are some positive constants c_d and γ_d having the property that there exists a set $\mathcal{L}_p \subseteq \mathbb{F}_p^d$, of cardinality $\#\mathcal{L}_p \geqslant c_d\, p^d$, such that for all $\boldsymbol{a} \in \mathcal{L}_p$ one has*

$$|T_{d,p}(\boldsymbol{a})| \geqslant \gamma_d \sqrt{p}.$$

We also need a result on the distribution of monomial curves. The following result is Lemma 2.5 in [10], which we augment by also including the (trivial) case $k = 1$.

Lemma 9.5. *Let $(a_1, \dots, a_k) \in (\mathbb{F}_p^*)^k$. Then there exists a positive constant C, which depends only on k, such that for any box $\mathfrak{B}$ as in (9.1) with sidelength $L \geqslant Cp^{1-1/2k} \log p$ for $k \geqslant 2$ and $L \geqslant 1$ for $k = 1$, we have*

$$\#\{\lambda \in \mathbb{F}_p^* : (a_1\lambda, \dots, a_k\lambda^k) \in \mathfrak{B}\} \geqslant \frac{1}{2} L^k p^{1-k}.$$

We are now ready to establish our main result of this section. Recall the definition of $\nu(d,k)$ from (3.7). Then we have the following level-set result.

Lemma 9.6. *For any $d \geqslant 3$ and $1 \leqslant k < d$, there exist constants $\gamma_d, \Gamma_d > 0$, such that for any box $\mathfrak{B}$ as in (9.1) with side-length $L \geqslant \Gamma_d\, p^{1-\nu(d,k)} \log p$, we have*

$$\#\{\mathbf{u} \in \mathfrak{B} : |T_{d,p}(\mathbf{u})| \geqslant \gamma_d\, p^{1/2}\} \gg L^d.$$

Proof. Adjusting Γ_d if necessary, we can assume that p is large enough. By Lemma 9.4, there are a constant γ_d and a set $\mathcal{L}_p \subseteq \mathbb{F}_p^d$ of cardinality

$$\#\mathcal{L}_p \geqslant c_d\, p^d \tag{9.2}$$

for some suitable constant c_d, and having the property that $|T_{d,p}(\boldsymbol{a})| \geqslant \gamma_d \sqrt{p}$ for all elements $\boldsymbol{a} \in \mathcal{L}_p$. Clearly, if $(a_1, \dots, a_d) \in \mathcal{L}_p$, then for any $\lambda \in \mathbb{F}_p^*$ we also have $(a_1\lambda, \dots, a_d\lambda^d) \in \mathcal{L}_p$.

Denote by $\mathcal{A}_k \subseteq \mathbb{F}_p^k$ the set of all $(a_1, \dots, a_k) \in \mathbb{F}_p^k$ for which

$$\#\big(\mathcal{L}_p \cap \big((a_1, \dots a_k) \times \mathbb{F}_p^{d-k}\big)\big) \geqslant \frac{1}{2} c_d\, p^{d-k}, \tag{9.3}$$

where c_d is the constant of Lemma 9.4. Then by decomposing

$$\mathbb{F}_p^k = \mathcal{A}_k \cup (\mathbb{F}_p^k \setminus \mathcal{A})$$

and using (9.3) (in the contrapositive form) within the second term, we have

$$\begin{aligned}
\#\mathcal{L}_p &= \sum_{(a_1,\dots,a_k)\in\mathcal{A}_k} \sum_{\substack{(a_{k+1},\dots,a_d)\in\mathbb{F}_p^{d-k} \\ (a_1,\dots,a_k)\in\mathcal{L}_p}} 1 + \sum_{(a_1,\dots,a_k)\in\mathbb{F}_p^k\setminus\mathcal{A}_k} \sum_{\substack{(a_{k+1},\dots,a_d)\in\mathbb{F}_p^{d-k} \\ (a_1,\dots,a_d)\in\mathcal{L}_p}} 1 \\
&\leqslant \#\mathcal{A}_k\, p^{d-k} + \sum_{(a_1,\dots,a_k)\in\mathbb{F}_p^k\setminus\mathcal{A}_k} \frac{1}{2} c_d\, p^{d-k} \\
&\leqslant \#\mathcal{A}_k\, p^{d-k} + \frac{1}{2} c_d\, p^d.
\end{aligned} \tag{9.4}$$

On combining the bounds (9.2) and (9.4), we find that

$$c_d\, p^d \leqslant \#\mathcal{A}_k\, p^{d-k} + \frac{1}{2}\, c_d\, p^d$$

which rearranges to

$$\#\mathcal{A}_k \geqslant \frac{c_d}{2}\, p^k.$$

Put now

$$\mathcal{A}_k^* = \mathcal{A}_k \cap (\mathbb{F}_p^*)^k.$$

Thus we clearly have

$$\#\mathcal{A}_k^* \gg p^k. \tag{9.5}$$

We now fix $\boldsymbol{a}^* = (a_1, \ldots, a_k) \in \mathcal{A}_k^*$ and consider the set

$$\mathcal{L}_{p,k}(\boldsymbol{a}^*) = \mathcal{L}_p \cap \big(\{a_1, \ldots, a_k\} \times \mathbb{F}_p^{d-k}\big).$$

Clearly, from the definition (9.3) of the set $\mathcal{A}_k$, we have

$$\#\mathcal{L}_{p,k}(\boldsymbol{a}^*) \gg p^{d-k}. \tag{9.6}$$

Given a box $\mathfrak{B} \subseteq \mathbb{F}_p^d$ of the form (9.1), we decompose it in a natural way as

$$\mathfrak{B} = \mathfrak{B}_1 \times \mathfrak{B}_2 \subseteq \mathbb{F}_p^k \times \mathbb{F}_p^{d-k}.$$

Note that we have

$$\#\mathfrak{B}_1 = L^k.$$

Let further

$$\Lambda_k(\boldsymbol{a}^*) = \{\lambda \in \mathbb{F}_p^* :\ (\lambda a_1, \ldots, \lambda^k a_k) \in \mathfrak{B}_1\}.$$

Then Lemma 9.5 implies that

$$\#\Lambda_k(\boldsymbol{a}^*) \geqslant \frac{1}{2}\, L^k p^{1-k}, \tag{9.7}$$

provided that the condition

$$L \geqslant \Gamma_d\, p^{1-1/2k} \log p \tag{9.8}$$

is satisfied with a sufficiently large Γ_d if $k \geqslant 2$, or for any L if $k = 1$.

Let $R(\boldsymbol{a}^*)$ be the number of vectors of the form

$$(\boldsymbol{a}^*, a_{k+1}, \ldots a_d, \lambda) = (a_1, \ldots, a_k, a_{k+1}, \ldots a_d, \lambda) \in \mathcal{L}_{p,k}(\boldsymbol{a}^*) \times \Lambda_k$$

such that

$$(\lambda^{k+1} a_{k+1}, \ldots, \lambda^d a_d) \in \mathfrak{B}_2.$$

It is shown in the proof of Lemma 2.6 in [10] that

$$\begin{aligned} |R(\boldsymbol{a}^*) - \#\mathcal{L}_{p,k}(\boldsymbol{a}^*)\,\#\Lambda_k(\boldsymbol{a}^*)\,(L/p)^{d-k}| \qquad\qquad \\ \leqslant C_d\,\#\mathcal{L}_{p,k}(\boldsymbol{a}^*)\,(\#\Lambda_k(\boldsymbol{a}^*))^{1/2}(\log p)^{d-k} \end{aligned} \tag{9.9}$$

for some constant C_d depending only on d. Suppose now that

$$C_d\,(\log p)^{d-k} \leqslant \frac{1}{2}\,(L/p)^{d-k}\,(\#\Lambda_k(\boldsymbol{a}^*))^{1/2}. \tag{9.10}$$

Then the quantity $R(\boldsymbol{a}^*)$ from (9.9) can be bounded below by

$$\begin{aligned} R(\boldsymbol{a}^*) &\geqslant \frac{1}{2}\,\#\mathcal{L}_{p,k}(\boldsymbol{a}^*)\,\#\Lambda_k(\boldsymbol{a}^*)\,(L/p)^{d-k} \\ &\gg p^{d-k}L^k\,p^{1-k}\,(L/p)^{d-k} \gg L^d\,p^{1-k}, \end{aligned} \tag{9.11}$$

where we used (9.6) and (9.7).

On the other hand, (9.7) implies that the condition (9.10) is certainly satisfied when

$$\frac{1}{2\sqrt{2}}\,(L/p)^{d-k}\,(L^k\,p^{1-k})^{1/2} \geqslant C_d\,(\log p)^{d-k},$$

which can be rearranged to

$$L \geqslant \widetilde{C}_d\,p^{1-1/(2d-k)}\,(\log p)^{(d-k)/(d-k/2)}, \tag{9.12}$$

where

$$\widetilde{C}_d = (2\sqrt{2}C_d)^{1/(d-k/2)}.$$

Note that since (9.7) is true for all $k \geqslant 1$, so is the last bound.

Combining the conditions (9.8) and (9.12), recalling the definition of $\nu(d,k)$ in (3.7) and increasing Γ_d if necessary, we see that the inequality

$$L \geqslant \Gamma_d\,p^{1-\nu(d,k)}\log p$$

is sufficient to guarantee that (9.11) holds for any $\boldsymbol{a}^* \in \mathcal{A}_k^*$.

Clearly, each vector of $\mathbf{u} \in \mathbb{F}_p^d$ has at most p representations as

$$\mathbf{u} = (\lambda a_1, \ldots, \lambda^d a_d),$$

with $(\alpha_1, \ldots, a_d) \in \mathbb{F}_p^d$ and $\lambda \in \mathbb{F}_p^*$. Therefore, we derive from (9.11) that

$$\#\{\mathbf{u} \in \mathfrak{B} : |T_{d,p}(\mathbf{u})| \geqslant \gamma_d\,p^{1/2}\} \geqslant \frac{1}{p}\sum_{\boldsymbol{a}^* \in \mathcal{A}_k^*} R(\boldsymbol{a}^*) \gg L^d\,p^{-k}\,\#\mathcal{A}_k^*,$$

and recalling (9.5), we conclude the proof. ∎

9.3. Approximation of Weyl sums by rational sums

Let $\mathcal{Z}_d$ be the set of vectors $\mathbf{u} \in \mathbb{F}_p^d$ which are not of the form $\mathbf{u} = (u_1, 0, \dots, 0)$. We also recall that the classical *Weil bound* (see, for example, Theorem 3 in Chapter 6 of [23], or Theorem 5.38 in [24]), together with the completing technique described for instance in Section 12.2 of [20], implies that if $\mathbf{u} \in \mathcal{Z}_d$, then for any $N \leqslant p$ we have

$$\sum_{n=1}^{N} \mathbf{e}_p(u_1 n + \cdots + u_d n^d) \ll p^{1/2} \log p. \tag{9.13}$$

Using (9.13), adapting the proof of Lemma 2.9 in [10], and noticing that the condition $p \mid N$ in Lemma 2.9 of [10] is not necessary (see also Corollary 2.6 in [13]), we obtain the following continuity property for Weyl sums.

Lemma 9.7. *Let $\mathbf{u} \in \mathbb{F}_p^d$ and $\mathbf{x} \in \mathsf{T}_d$. Then we have*

$$|S_d(\mathbf{x}; N) - S_d(p^{-1}\mathbf{u}; N)| \ll \sqrt{p} \log p \left(1 + \frac{N}{p} \sum_{j=1}^{d} \left| x_j - \frac{u_j}{p} \right| N^j \right).$$

Lemma 9.7 immediately implies the following.

Lemma 9.8. *Let p be a prime, and let $\mathbf{u} = (u_1, \dots, u_d) \in \mathcal{Z}_d$ such that*

$$|T_{d,p}(\mathbf{u})| \geqslant \gamma_d p^{1/2}$$

for some $\gamma_d > 0$. Then there are constants $c_d, C_d > 0$ such that for all $N \geqslant C_d p \log p$ and all $\mathbf{x} = (x_1, \dots, x_d) \in \mathsf{T}_d$ satisfying

$$\left| x_j - \frac{u_j}{p} \right| \leqslant \frac{c_d}{N^j \log p}, \qquad j = 1, \dots, d, \tag{9.14}$$

we have

$$|S_d(\mathbf{x}; N)| \gg N p^{-1/2}.$$

10. Proof of Theorems 3.9 and 3.10

10.1. Proof of Theorem 3.9

Let $N \in \mathbb{N}$, and let and c and C be the constants of Lemma 9.3, noting that without loss of generality, we may assume that $c < C/2$. Suppose first that $\delta \geqslant 2C/N$, so that the interval $[\delta^{-1}, N/C]$ contains both the interval $[N/(2C), N/C]$ and $[\delta^{-1}, 2\delta^{-1}]$. Then for sufficiently large N, there is at least one prime number in the range

$$N/C \geqslant p \geqslant 2/\delta. \tag{10.1}$$

Now fix a point $\boldsymbol{\xi} \in \mathsf{T}_2$ and a $\delta > 0$, and let $\widetilde{R}_p(\mathbf{b})$ be the domain of admissible values of $(x_1, x_2) \in \mathfrak{B}(\boldsymbol{\xi}, \delta)$ having a rational approximation of the shape

$$|x_1 - b_1/p| \leqslant cN^{-1} \quad \text{and} \quad |x_2 - b_2/p| \leqslant cN^{-2}, \tag{10.2}$$

where p is a prime, $b_1, b_2 \in [p]$ and $0 < c < 1/2$ is some small number. This notation is reminiscent of that employed in our arguments in Section 7, but we stress that we have different conditions imposed on the exponential sums here than we had there. Write further

$$\mathfrak{U}_p(\boldsymbol{\xi}, \delta) = \bigcup_{\substack{\mathbf{b} \in [p]^2 \\ \widetilde{R}_p(\mathbf{b}) \cap \mathfrak{B}(\boldsymbol{\xi}, \delta) \neq \emptyset}} \widetilde{R}_p(\mathbf{b}),$$

noting that for all p in the range (10.1), we have $1/p > 2c/N$, and consequently the sets $\widetilde{R}_p(\mathbf{b})$ are pairwise disjoint by our initial assumptions.

Since the number of pairs $\mathbf{b} \in [p]$ for which $\widetilde{R}_p(\mathbf{b})$ intersects $\mathfrak{B}(\boldsymbol{\xi}, \delta)$ non-trivially is at least

$$(\delta p - 1)^2 \geqslant (\delta p/4)^2,$$

and each individual box has volume

$$\lambda(\widetilde{R}_p(\mathbf{b})) = (2c)^2 N^{-3},$$

it follows that

$$\lambda(\mathfrak{U}_p(\boldsymbol{\xi}, \delta)) \geqslant (c\delta p/2)^2 N^{-3}.$$

Then, applying Lemma 9.3, we derive

$$\begin{aligned} I^{\flat}_{s,2}(\delta; N) &\geqslant \inf_{\boldsymbol{\xi} \in \mathsf{T}_2} \left(\lambda(\mathfrak{U}_p(\boldsymbol{\xi}, \delta)) \inf_{(x_1, x_2) \in \mathfrak{U}_p(\boldsymbol{\xi}, \delta)} |G(x_1, x_2; N)|^{2s} \right) \\ &\gg (\delta p)^2 N^{-3} (N p^{-1/2})^{2s} = \delta^2 N^{2s-3} p^{2-s}. \end{aligned}$$

By the prime number theorem, for $s \leqslant 2$ we can choose $p \in [N/(2C), N/C]$, while for $s > 2$ we take $p \in [\delta^{-1}, 2\delta^{-1}]$. Hence

$$I^{\flat}_{s,2}(\delta; N) \gg \delta^2 N^{s-1} \max\{1, (\delta N)^{s-2}\},$$

which gives the desired lower bound in the case $\delta \gg 1/N$.

To treat the case when $\delta \geqslant c/\sqrt{N}$, for some constant $c > 0$, we first observe that for any distinct fractions a/q and b/r with coprime $q, r \in [\sqrt{N}, 2\sqrt{N}]$, we have

$$\left| \frac{a}{q} - \frac{b}{r} \right| \geqslant \frac{1}{qr} \geqslant \frac{1}{N}.$$

Combining this with (10.2), for any distinct primes $p_1, p_2 \in [\sqrt{N}, 2\sqrt{N}]$ and for $\mathbf{b}_1 \in [p_1]^2$ and $\mathbf{b}_2 \in [p_2]^2$, we obtain that

$$\widetilde{R}_{p_1}(\mathbf{b}_1) \cap \widetilde{R}_{p_2}(\mathbf{b}_2) = \emptyset.$$

Therefore, for any $\boldsymbol{\xi} \in \mathsf{T}_2$,

$$\mathfrak{U}_{p_1}(\boldsymbol{\xi}, \delta) \cap \mathfrak{U}_{p_2}(\boldsymbol{\xi}, \delta) = \emptyset, \tag{10.3}$$

allowing us to enhance our previous arguments by summing over all primes in the interval $[\sqrt{N}, 2\sqrt{N}]$. Then, proceeding in a similar way to before and applying Lemma 9.3 and (10.3), we derive the lower bound

$$\begin{aligned} I^{\flat}_{s,2}(\delta; N) &\geqslant \inf_{\boldsymbol{\xi} \in \mathsf{T}_2} \sum_{\substack{\sqrt{N} \leqslant p \leqslant 2\sqrt{N} \\ p \text{ is prime}}} \int_{\mathfrak{U}_p(\boldsymbol{\xi},\delta)} |G(x, y; N)|^{2s} \, dx \, dy \\ &\gg \sum_{\substack{\sqrt{N} \leqslant p \leqslant 2\sqrt{N} \\ p \text{ is prime}}} (\delta p)^2 N^{-3} (N p^{-1/2})^{2s} \gg \delta^2 N^{3(s-1)/2} (\log N)^{-1}, \end{aligned}$$

where the last inequality holds by the prime number theorem.

10.2. Proof of Theorem 3.10

Recalling the definition (3.7), suppose that

$$\delta > 2\Gamma_d \, (2C_d)^{\nu(d,k)} N^{-\nu(d,k)} (\log N)^{1+\nu(d,k)} \tag{10.4}$$

for some k, where Γ_d and C_d are the constants of Lemmas 9.6 and 9.8, respectively. This choice of δ implies that the interval

$$\left[(\delta^{-1} 2\,\Gamma_d \log N)^{1/\nu(d,k)}, \frac{N}{C_d \log N} \right]$$

fully encompasses the interval

$$\left[\frac{N}{2C_d \log N}, \frac{N}{C_d \log N} \right],$$

and thus contains at least one prime. We therefore can assume that there is a prime p satisfying

$$\delta \geqslant 2\Gamma_d \, p^{-\nu(d,k)} \log p \quad \text{ and } \quad N \geqslant C_d \, p \log p. \tag{10.5}$$

Consider now a box $\mathfrak{B}(\boldsymbol{\xi}, \delta) \subseteq \mathsf{T}_d$. Clearly, the set of $\mathbf{u} \in \mathbb{F}_p^d$ for which $\mathbf{u}/p \in \mathfrak{B}(\boldsymbol{\xi}, \delta)$ forms a box $\mathfrak{C}_p(\boldsymbol{\xi}, \delta) \subseteq \mathbb{F}_p^d$ with side-length

$$L \geqslant \lfloor p\delta \rfloor \geqslant \Gamma_d \, p^{1-\nu(d,k)} \log p.$$

Let

$$U_p(\boldsymbol{\xi}, \delta) = \#\{\mathbf{u} \in \mathfrak{C}_p(\boldsymbol{\xi}, \delta) \cap \mathcal{Z}_d : |T_{d,p}(\mathbf{u})| \geqslant \gamma_d \, p^{1/2}\},$$

where γ_d is as in Lemma 9.6. From that lemma, we obtain in a straightforward manner the bound

$$\begin{aligned} &U_p(\boldsymbol{\xi}, \delta) \\ &\quad \geqslant \#\{\mathbf{u} \in \mathfrak{C}_p(\boldsymbol{\xi}, \delta) : \ |T_{d,p}(\mathbf{u})| \geqslant \gamma_d \, p^{1/2}\} - \#\{u_1 \in \mathbb{F}_p : \ (u_1, 0, \ldots, 0) \in \mathfrak{C}_p(\boldsymbol{\xi}, \delta)\} \\ &\quad \gg L^d - L \gg (p\delta)^d. \end{aligned}$$

Therefore, if $\mathcal{N}_p(\boldsymbol{\xi}, \delta)$ denotes the set of all $(x_1, \dots x_d) \in \mathsf{T}_d$ having a diophantine approximation as in (9.14) with numerator $\mathbf{u}$ counted by $U_p(\boldsymbol{\xi}, \delta)$, we have

$$\lambda(\mathcal{N}_p(\boldsymbol{\xi}, \delta)) \gg \delta^d p^d \prod_{j=1}^{d} (N^j \log p)^{-1} = \delta^d p^d N^{-s(d)} (\log p)^{-d},$$

and thus for any prime p satisfying the conditions (10.5), we have

$$\begin{aligned} I^{\flat}_{s,d}(\delta; N) &\gg \inf_{\boldsymbol{\xi} \in T^d} \left(\lambda(\mathcal{N}_p(\boldsymbol{\xi}, \delta)) \inf_{\mathbf{x} \in \mathcal{N}_p(\boldsymbol{\xi}, \delta)} |S_d(\mathbf{x}; N)|^{2s} \right) \\ &\gg \delta^d p^d N^{-s(d)} (N p^{-1/2})^{2s} (\log p)^{-d} \gg \delta^d p^{d-s} N^{2s-s(d)} (\log N)^{-d}. \end{aligned}$$

Recall now that by our assumption (10.4), for a sufficiently large N, we can always find a prime p satisfying (10.5) with

$$p \ll \delta^{-1/\nu(d,k)} (\log N)^{1/\nu(d,k)}$$

as well as a prime p (also satisfying (10.5)) with

$$p \gg N/\log N.$$

Hence, under the condition (10.4), we have

$$\begin{aligned} I^{\flat}_{s,d}(\delta; N) &\gg \delta^d p^{d-s} N^{2s-s(d)} (\log N)^{-d} \\ &\geqslant \max\left\{\delta^d N^{s+d-s(d)}, \delta^{d-(d-s)/\nu(d,k)} N^{2s-s(d)}\right\} N^{o(1)}, \end{aligned}$$

which finishes the proof.

11. Further comments

11.1. Mean values over more general sets

Our setting involving multidimensional mean values opens up a certain degree of flexibility in terms of the shape of the underlying domain, and Wooley's conjecture (Conjecture 2.1) admits for arbitrary measurable sets. Arguably, boxes of variable sidelength that reflects the distinct powers in the exponential sum might be better suited to understand the local behaviour of Weyl sums. Another approach is to investigate local behaviour only with respect to the coordinate corresponding to the highest degree, which contributes most of the oscillations of exponential sums. The case of boxes of the shape $[0, 1)^{d-1} \times [0, \delta]$ has been studied in some detail in work by Demeter, Guth and Wang [15] as well as Guth and Maldague [19] on small cap decouplings, extending previous work by Bourgain [5]. Even though in the work at hand we restricted our attention to hypercubes, our methods can be extended without serious problems to other axis-aligned boxes as well.

11.2. Applications to the Schrödinger equation

Our results have consequences for solutions of Schrödinger equations over short intervals. The Schrödinger equation

$$2\pi u_t + i u_{xx} = 0$$

models the behaviour of quantum mechanical particles. We denote by $\rho(t, \mathcal{I})$ the probability that the particle belongs to the interval $\mathcal{I}$ at time t. When $u(x,t)$ is a solution to the Schrödinger equation, then this probability is given by

$$\rho(t, \mathcal{I}) = \int_{\mathcal{I}} |u(x,t)|^2 \, dx. \tag{11.1}$$

In the case when the boundary condition is periodic of the shape

$$u(x,0) = \sum_{n=1}^{N} a_n \, \mathbf{e}(xn),$$

the solutions of the Schrödinger equation are trigonometric polynomials with quadratic amplitudes of the shape

$$u(x,t) = \sum_{n=1}^{N} a_n \, \mathbf{e}(xn + tn^2).$$

For a fixed $t \in \mathsf{T}$, our results do not yield any estimate for the value (11.1). However, from our results we can deduce various upper and lower bounds on the above probability $\rho(t, \mathcal{I})$ for any short interval $\mathcal{I}$ and for some time in yet another short interval.

For example, in the case of the constant coefficients $a_n = 1, n \in \mathbb{N}$, by Theorems 3.7 and 3.9, we have the following.

Corollary 11.1. *Let $N \in \mathbb{N}$ be a large number, and let $(x_0, t_0) \in \mathsf{T}_2$. Then*

(1) *for $\delta \geqslant N^{-3/8}$, there exists $t \in [t_0, t_0 + \delta]$ such that*

$$\int_{x_0}^{x_0+\delta} \Big| \sum_{n=1}^{N} \mathbf{e}(xn + tn^2) \Big|^2 dx \leqslant \delta N^{5/4+o(1)};$$

(2) *if $\delta \geqslant c/N$ for some small $c > 0$, there exists $t \in [t_0, t_0 + \delta]$ such that*

$$\int_{x_0}^{x_0+\delta} \Big| \sum_{n=1}^{N} \mathbf{e}(xn + tn^2) \Big|^2 dx \gg \delta.$$

Proof. Clearly, we have

$$\begin{aligned} \delta \min_{t \in [t_0, t_0+\delta]} \int_{x_0}^{x_0+\delta} \Big| \sum_{n=1}^{N} \mathbf{e}(xn + tn^2) \Big|^2 dx &\leqslant \int_{t_0}^{t_0+\delta} \int_{x_0}^{x_0+\delta} \Big| \sum_{n=1}^{N} \mathbf{e}(xn + tn^2) \Big|^2 dx \\ &\leqslant I_{2,1}^{\sharp}(\delta; N). \end{aligned}$$

It thus suffices to observe that for the first statement, Theorem 3.7 with parameters $s = 1$ and any $\delta \geqslant N^{-3/8}$ yields the bound

$$I_{2,1}^{\sharp}(\delta; N) \leqslant \delta^2 N^{2(1-3/8)+o(1)} = \delta^2 N^{5/4+o(1)},$$

which proves the claim (1). The second statement (2) is established similarly by combining the bound

$$\delta \max_{t\in[t_0,t_0+\delta]} \int_{x_0}^{x_0+\delta} \Big|\sum_{n=1}^{N} \mathbf{e}(xn + tn^2)\Big|^2 dx \geqslant I_{2,1}^{\sharp}(\delta; N)$$

with the bound

$$I_{2,1}^{\sharp}(\delta; N) \gg \delta^2$$

from Theorem 3.9 (1). ∎

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