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Three-chromatic geometric hypergraphs

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Abstract. We prove that for any planar convex body C there is a positive integer m with the property that any finite point set P in the plane can be three-colored in such a way that no translate of C contains m points of P (or more), all of the same color. As a part of the proof, we show a strengthening of the Erdős–Sands–Sauer–Woodrow conjecture. Surprisingly, the proof also relies on the two-dimensional case of the Illumination Conjecture. The extended abstract of this paper already appeared in the proceedings of SoCG '22.

Keywords: discrete geometry, geometric hypergraph coloring, decomposition of multiple coverings.

1. Introduction

Our main result is the following.

Theorem 1. *For any planar convex body C there is a positive integer $m = m(C)$ such that any finite point set P in the plane can be three-colored in such a way that no translate of C contains m points of P (or more), all of the same color.*

This result closes a long line of research about coloring points with respect to planar range spaces that consist of translates of a fixed set, a problem initiated by Pach over forty years ago [22]. In general, a pair $(P, \mathcal{S})$, where P is a set of points in the plane and $\mathcal{S}$ is a family of subsets of the plane, called the *range space*, defines a *primal* hypergraph $\mathcal{H}(P, \mathcal{S})$ whose vertex set is P, and for each $S \in \mathcal{S}$ we add the edge $S \cap P$ to the hypergraph. Given any hypergraph $\mathcal{H}$, a planar realization of $\mathcal{H}$ is defined as a pair $(P, \mathcal{S})$ for which $\mathcal{H}(P, \mathcal{S})$ is isomorphic to $\mathcal{H}$. If $\mathcal{H}$ can be realized with some pair $(P, \mathcal{S})$ where $\mathcal{S}$ is from some family $\mathcal{F}$, then we say that $\mathcal{H}$ is *realizable* with $\mathcal{F}$. The hypergraph, where the elements of the range space $\mathcal{S}$ are the vertices and the points P define the edges such that $\{S \in \mathcal{S} \mid p \in S\}$ is an edge for every $p \in P$, is known as the *dual* hypergraph of

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$\mathcal{H}(P, \mathcal{S})$ and is denoted by $\mathcal{H}(\mathcal{S}, P)$. If $\mathcal{H} = \mathcal{H}(\mathcal{S}, P)$ where $\mathcal{S}$ is from some family $\mathcal{F}$, then we say that $\mathcal{H}$ has a dual realization with $\mathcal{F}$. Pach [22, 25] observed that if $\mathcal{F}$ is the family of translates of some set, then $\mathcal{H}$ has a dual realization with $\mathcal{F}$ if and only if $\mathcal{H}$ has a (primal) realization with $\mathcal{F}$.

Pach proposed studying the chromatic number of hypergraphs realizable with different geometric families $\mathcal{F}$. It is important to distinguish between two types of hypergraph colorings that we will use, *proper* coloring and *polychromatic* coloring.

Definition 2. A hypergraph is *properly k-colorable* if its vertices can be colored with k colors so that each edge contains points from at least two color classes. Such a coloring is called a *proper k-coloring*. If a hypergraph has a proper k-coloring but not a proper $(k-1)$-coloring, then it is called *k-chromatic*.

A hypergraph is *polychromatic k-colorable* if its vertices can be colored with k colors so that each edge contains points from each color class. Such a coloring is called a *polychromatic k-coloring*.

Note that for a polychromatic k-coloring to exist, each edge of the underlying hypergraph should have at least k vertices. More generally, we say that a hypergraph is *m-heavy* if each of its edges has at least m vertices.

The main question that Pach raised can be rephrased for translates as follows.

Question 3. *For which planar families $\mathcal{F}$ is there an $m_k = m(\mathcal{F}, k)$ such that any m_k-heavy hypergraph realizable with $\mathcal{F}$ has a proper/polychromatic k-coloring?*

Initially, this question has been mainly studied for polychromatic k-colorings (known in the case of a dual range space as the *cover-decomposition* problem), and it was shown that such an m_k exists if $\mathcal{F}$ is the family of translates of some convex polygon [23,29,34], or the family of all halfplanes [15, 33], or the homothetic[1] copies of a triangle [16] or of a square [2], while it was also shown that even m_2 does not exist if $\mathcal{F}$ is the family of translates of some appropriate concave polygon [27, 28] or any body[2] with a smooth boundary [24]. It was also shown that there is no m_k for proper k-colorings if $\mathcal{F}$ is the family of all lines [27] or all axis-parallel rectangles [11]; for these families, the same holds for dual realizations [26,27]. For homothets of convex polygons other than triangles, it is known that there is no m_2 for dual realizations [20], unlike for primal realizations. Higher dimensional variants [10, 16] and improved bounds for m_k have also been studied [4, 5, 8, 9, 14, 17]. For other results, see also the decade old survey [25], or the up-to-date website https://coge.elte.hu/cogezoo.html.

[1] A *homothetic copy*, or *homothet*, is a scaled and translated (but non-rotated) copy of a set. We always require the scaling factor to be positive. Note that this is sometimes called a positive homothet.

[2] By *body*, we always mean a compact subset of the plane with a non-empty interior, though our results (and most of the results mentioned) also hold for sets that are unbounded, or that contain an arbitrary part of their boundary, and are thus neither open nor closed. This is because a realization of a hypergraph can be perturbed slightly to move the points off the boundaries of the sets realizing the respective edges of the hypergraph.

If $\mathcal{F}$ is the translates or homothets of some planar convex body, it is an easy consequence of the properties of generalized Delaunay triangulations and the Four Color Theorem that any hypergraph realizable with $\mathcal{F}$ is proper four-colorable if every edge contains at least two vertices. We have recently shown that this cannot be improved for homothets.

Theorem 4 (Damásdi–Pálvölgyi [13]). *Let C be any convex body in the plane that has two parallel supporting lines such that C is strictly convex in some neighborhood of the two points of tangencies. For any positive integer m, there exists a four-chromatic m-uniform hypergraph that is realizable with homothets of C.*

For translates, we recall the following result.

Theorem 5 (Pach–Pálvölgyi [24]). *Let C be any convex body in the plane that has two*[3] *parallel supporting lines such that C is strictly convex in some neighborhood of the two points of tangencies. For any positive integer m, there exists a three-chromatic m-uniform hypergraph that is realizable with translates of C.*

This left only the following question open: Is it true for any planar convex body C that there is a positive integer m such that no 4-chromatic m-uniform hypergraph is realizable with translates of C? Our Theorem 1 answers this question affirmatively for all C by showing that all realizable m-heavy hypergraphs are three-colorable for some m. This has been hitherto known to hold only when C is a polygon (in which case two colors suffice [29], and three colors are known to be enough even for homothets [19]) and pseudodisk families that intersect in a common point [1] (which generalizes the case when C is unbounded, in which case two colors suffice [24]).

The proof of Theorem 1 relies on a surprising connection with two other famous results, the solution of the two-dimensional case of the Illumination Conjecture [21], and a recent solution of the Erdős–Sands–Sauer–Woodrow conjecture by Bousquet, Lochet and Thomassé [7]. In fact, we need a generalization of the latter result, which we prove with the addition of one more trick to their method; this can be of independent interest.

Note that the extended abstract of our first proof attempt appeared recently in the proceedings of EuroComb 2021 [12]. That proof did not use the above two results, however, it only worked when C was a disk, and while the generalization to other convex bodies with a smooth boundary seemed feasible, we saw no way to extend it to arbitrary convex bodies.

The rest of the paper is organized as follows. In Section 2 we present the three main ingredients of our proof:

- the Union Lemma (Section 2.1),
- the Erdős–Sands–Sauer–Woodrow conjecture (Section 2.2),
- the Illumination Conjecture (Section 2.3), which is a theorem of Levi in the plane.

[3] In fact, it can be shown that one such line is sufficient; we sketch this construction in Section 5.

In Section 3 we give the detailed proof of Theorem 1. In Section 4 we prove our generalization of the Bousquet–Lochet–Thomassé theorem. In Section 5 we show a generalization of Theorem 5. Finally, in Section 6, we pose some problems.

2. Tools

2.1. Union Lemma

Polychromatic colorability is a much stronger property than proper colorability. Any polychromatic k-colorable hypergraph is proper two-colorable. We generalize this trivial observation to the following statement about unions of polychromatic k-colorable hypergraphs.

Lemma 6 (Union Lemma). *Let $\mathcal{H}_1 = (V, E_1), \dots, \mathcal{H}_{k-1} = (V, E_{k-1})$ be hypergraphs on a common vertex set V. If $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ are polychromatic k-colorable, then the hypergraph*

$$\bigcup_{i=1}^{k-1} \mathcal{H}_i = \left(V, \bigcup_{i=1}^{k-1} E_i\right)$$

is proper k-colorable.

Proof. Let $c_i : V \to \{1, \dots, k\}$ be the polychromatic coloring of the i-th hypergraph. Using the c_i's we construct a proper coloring $c : V \to \{1, \dots, k\}$ for the union. Choose $c(v) \in \{1, \dots, k\}$ that differs from each $c_i(v)$. We claim that c is a proper k-coloring of $\bigcup_{i=1}^{k-1} \mathcal{H}_i$. To prove this, it is enough to show that for every edge $H \in \mathcal{H}_i$ and for every color $j \in \{1, \dots, k\}$, there is a $v \in H$ such that $c(v) \neq j$. Indeed, we can pick $v \in H$ for which $c_i(v) = j$. This finishes the proof. ∎

Lemma 6 is sharp in the sense that for every k there are $k-1$ hypergraphs such that each is polychromatic k-colorable, but their union is not properly $(k-1)$-colorable. Take for example the following $k-1$ hypergraphs. The vertex set V of each hypergraph is the grid $\{1, \dots, k\}^{k-1}$ and for each $i \in \{1, \dots, k-1\}$ the edge set E_i is those subsets of V where each one of $1, \dots, k$ appears in the i-th coordinate for some element of the subset. Clearly, (V, E_i) is polychromatic k-colorable: simply color the vertices according to their i-th coordinate. Suppose that their union has a proper $(k-1)$-coloring. The i-th color class cannot form an edge in E_i since that would be a monochromatic edge. Therefore, there must be a number $l_i \in \{1, \dots, k-1\}$ such that no vertex of the i-th color has l_i in its i-th coordinate. But this is a contradiction, because the vertex $(l_1, \dots, l_{k-1})$ has no color. This shows the sharpness of the lemma.

We will apply the Union Lemma combined with the theorem below. A *pseudoline arrangement* is a collection of simple curves, each of which splits $\mathbb{R}^2$ into two unbounded parts, such that any two curves intersect at most once. A *pseudohalfplane* is the region on one side of a pseudoline in such an arrangement. For hypergraphs realizible by pseudohalfplanes the following was proved, generalizing a result of Smorodinsky and Yuditsky [33] about halfplanes.

Theorem 7 (Keszegh–Pálvölgyi [18]). *Any $(2k-1)$-heavy hypergraph realizable by pseudohalfplanes is polychromatic k-colorable, i.e., given a finite set of points and a pseudohalfplane arrangement in the plane, the points can be k-colored in such a way that every pseudohalfplane that contains at least $2k-1$ of these points contains all k colors.*

Combining Theorem 7 with Lemma 6 for $k = 3$, we obtain the following.

Corollary 8. *Any 5-heavy hypergraph realizable by two pseudohalfplane families is proper three-colorable, i.e., given a finite set of points and two different pseudohalfplane arrangements in the plane, the points can be three-colored in such a way that every pseudohalfplane that contains at least five of these points contains two differently colored points.*

2.2. *Erdős–Sands–Sauer–Woodrow conjecture*

Given a quasi-order[4] $\prec$ on a set V, we interpret it as a digraph $D = (V, A)$, where the vertex set is V and a pair (x, y) defines an arc in A if $x \prec y$. The *closed in-neighborhood* of a vertex $x \in V$ is $N^-(x) = \{x\} \cup \{y \mid (y, x) \in A\}$. Similarly the *closed out-neighborhood* of a vertex x is $N^+(x) = \{x\} \cup \{y \mid (x, y) \in A\}$. We extend this to subsets $S \subset V$ as $N^-(S) = \bigcup_{x \in S} N^-(x)$ and $N^+(S) = \bigcup_{x \in S} N^+(x)$. A set S of vertices such that $N^+(S) = V$ is said to be *dominating*. For $A, B \subset V$ we will also say that *A dominates B* if $B \subset N^+(A)$.

A *complete multidigraph* is a digraph where parallel edges are allowed and in which there is at least one arc between each pair of distinct vertices. Let D be a complete multidigraph whose arcs are the disjoint union of k quasi-orders $\prec_1, \dots, \prec_k$ (parallel arcs are allowed). Define $N_i^-(x)$ (resp. $N_i^+(x)$) to be the closed in-neighborhood (resp. out-neighborhood) of the digraph induced by $\prec_i$.

Proving the conjecture of Erdős and of Sands, Sauer and Woodrow [32], Bousquet, Lochet and Thomassé recently showed the following.

Theorem 9 (Bousquet–Lochet–Thomassé [7]). *For every k, there exists an integer $f(k)$ such that if D is a complete multidigraph whose arcs are the union of k quasi-orders, then D has a dominating set of size at most $f(k)$.*

We show the following generalization of Theorem 9.

Theorem 10. *For every pair of positive integers k and l, there exists an integer $f(k,l)$ such that if $D = (V, A)$ is a complete multidigraph whose arcs are the union of k quasi-orders $\prec_1, \dots, \prec_k$, then V contains a family of pairwise disjoint subsets S_i^j for $i \in [k]$ and $j \in [l]$ with the following properties:*

- $|\bigcup_{i,j} S_i^j| \le f(k,l)$.
- *For each vertex $v \in V \setminus \bigcup_{i,j} S_i^j$ there is an $i \in [k]$ such that for each $j \in [l]$ there is an edge of $\prec_i$ from a vertex of S_i^j to v.*

[4] A *quasi-order* $\prec$ is a reflexive and transitive relation, but it is not required to be antisymmetric, so $p \prec q \prec p$ is allowed, unlike for partial orders.

Note that disjointness is the real difficulty here – without it the theorem would trivially hold by repeated application of Theorem 9. We saw no way to derive Theorem 10 from Theorem 9, but with an extra modification the proof of the latter goes through. The proof of Theorem 10 can be found in Section 4.

2.3. *Hadwiger's Illumination Conjecture and pseudolines*

Hadwiger's Illumination Conjecture has a number of equivalent formulations and names.[5] For a recent survey, see [6]. We will use the following.

Let $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. For a convex body C, let ∂C denote the boundary of C and let $\text{int}(C)$ denote its interior. A direction (light) $u \in \mathbb{S}^{d-1}$ *illuminates* $b \in \partial C$ if $\{b + \lambda u \mid \lambda > 0\} \cap \text{int}(C) \neq \emptyset$.

Conjecture 11. *The boundary of any convex body in $\mathbb{R}^d$ can be illuminated by 2^d or fewer directions. Furthermore, the 2^d lights are necessary if and only if the body is a parallelepiped.*

The conjecture is open in general. The $d = 2$ case was settled in the affirmative by Levi [21] in 1955. For $d = 3$ the best result is due to Prymak [31], who showed that 16 lights are enough, improving the earlier method of Papadoperakis [30] with the help of a computer program.

In the following part we make an interesting connection between the Illumination Conjecture for $d = 2$ and pseudolines. Roughly speaking, we show that the Illumination Conjecture implies that for any convex body in the plane the boundary can be broken into three parts such that the translates of each part behave similarly to pseudolines, i.e., we get three pseudoline arrangements from the translates of the three parts.

To put this into precise terms, we need some technical definitions and statements. Fix a body C and an injective parametrization of ∂C, $\gamma : [0, 1] \to \partial C$, that follows ∂C counterclockwise. For each p of ∂C there is a set of possible tangents touching at p. Let $g(p) \subset \mathbb{S}^1$ denote the Gauss image of p, i.e., $g(p)$ is the set of unit outer normals of the tangent lines touching at p. Note that $g(p)$ is an arc of $\mathbb{S}^1$ and $g(p)$ is a proper subset of $\mathbb{S}^1$.

Let $g_+ : \partial C \to \mathbb{S}^1$ be the function that assigns to p the counterclockwise last element of $g(p)$ (see Figure 1, left.) Similarly let g_- be the function that assigns to p the clockwise last element of $g(p)$. Thus, $g(p)$ is the arc of $\mathbb{S}^1$ from $g_-(p)$ to $g_+(p)$. Let $|g(p)|$ denote the length of $g(p)$.

Observation 12. *$g_+ \circ \gamma$ is right continuous and $g_- \circ \gamma$ is left continuous.*

For $t_1 < t_2$ let $\gamma_{[t_1,t_2]}$ denote the restriction of γ to the interval $[t_1, t_2]$. For $t_1 > t_2$ let $\gamma_{[t_1,t_2]}$ denote the concatenation of $\gamma_{[t_1,1]}$ and $\gamma_{[0,t_2]}$. When it leads to no confusion, we

[5] These include names such as Levi–Hadwiger Conjecture, Gohberg–Markus Covering Conjecture, Hadwiger Covering Conjecture, Boltyanski–Hadwiger Illumination Conjecture.

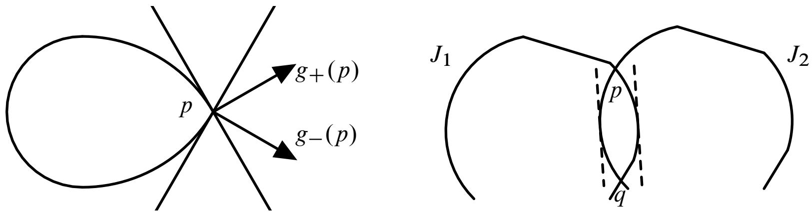

Fig. 1. Extremal tangents at a boundary point (left) and parallel tangents on two intersecting translates (right).

identify $\gamma_{[t_1,t_2]}$ with its image, which is a closed connected part of the boundary ∂C. For such a $J = \gamma_{[t_1,t_2]}$, let $g(J) = \bigcup_{p\in J} g(p)$. Clearly, $g(J)$ is an arc of $\mathbb{S}^1$ from $g_-(t_1)$ to $g_+(t_2)$; let $|g(J)|$ denote its length.

Lemma 13. *Let C be a convex body and assume that J is a closed connected part of ∂C such that $|g(J)| < \pi$. Then no two translates of J intersect in more than one point.*

Proof. Suppose J has two translates J_1 and J_2 that intersect in two points, p and q. Now both J_1 and J_2 have a tangent that is parallel to the segment pq, but since they lie on different sides of the pq line, they have opposite outer normal vectors (see Figure 1, right). This shows that J has two different tangents parallel to pq and therefore $|g(J)| \geq \pi$. ■

Lemma 14. *For a convex body C which is not a parallelogram, and an injective parametrization γ of ∂C, we can pick $0 \leq t_1 < t_2 < t_3 \leq 1$ such that*

$$|g(\gamma_{[t_1,t_2]})|, |g(\gamma_{[t_2,t_3]})|, |g(\gamma_{[t_3,t_1]})| < \pi.$$

Proof. We use the two-dimensional case of the Illumination Conjecture (proved by Levi [21]). If C is not a parallelogram, we can pick three directions, u_1, u_2 and u_3, that illuminate C. Pick t_1 such that $\gamma(t_1)$ is illuminated by both u_1 and u_2. To see why this is possible, suppose that the parts illuminated by u_1 and u_2 are disjoint. Each light illuminates a continuous open-ended part of the boundary. So in this case there are two disjoint parts of the boundary that are not illuminated. If u_3 illuminates both, then it illuminates everything that is illuminated by u_1 or everything that is illuminated by u_2. This would mean that two lights illuminate the whole boundary, which is impossible for any convex body. Indeed, suppose that two lights u and v illuminate the whole body. Then there is a halfplane H through the origin that contains both u and v. Take a translate of H that touches C. Clearly the touching point is not illuminated by either u or v, a contradiction.

Using the same argument, pick t_2 and t_3 such that $\gamma(t_2)$ is illuminated by both u_2 and u_3 and $\gamma(t_3)$ is illuminated by both u_3 and u_1.

Note that u_1 illuminates exactly those points for which $g_+(p) < u_1 + \pi/2$ and $g_-(p) > u_1 - \pi/2$. Therefore, $|g(\gamma_{[t_1,t_3]})| < u_1 + \pi/2 - (u_1 - \pi/2) = \pi$. Similarly $|g(\gamma_{[t_1,t_2]})| < \pi$ and $|g(\gamma_{[t_2,t_3]})| < \pi$. ■

Observation 12 and Lemma 14 immediately imply the following statement.

Lemma 15. *For a convex body C which is not a parallelogram, and an injective parametrization γ of ∂C, we can pick $0 \le t_1 < t_2 < t_3 \le 1$ and $\varepsilon > 0$ such that*

$$|g(\gamma_{[t_1-\varepsilon, t_2+\varepsilon]})|, |g(\gamma_{[t_2-\varepsilon, t_3+\varepsilon]})|, |g(\gamma_{[t_3-\varepsilon, t_1+\varepsilon]})| < \pi.$$

3. Proof of Theorem 1

3.1. Quasi-orders on planar point sets

Cones provide a natural way to define quasi-orders on point sets (see [34] for an example where this idea was used). A *cone* is a closed region in the plane that is bounded by two rays that emanate from the origin. For a cone K let $-K$ denote the cone that is the reflection of K across the origin, and let $q + K$ denote the translate of K by q.

Observation 16. *For any $p, q \in \mathbb{R}^2$ and cone K, the following are equivalent (see Fig. 2):*

- $p \in q + K$.
- $q \in p + (-K)$.
- $p + K \subseteq q + K$.

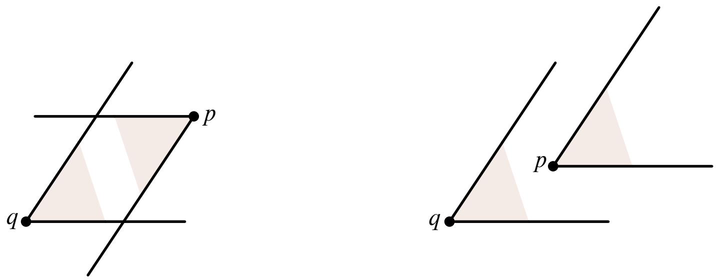

Fig. 2. Basic properties of cones.

For a cone K let $\prec_K$ denote the relation between points of the plane where $p \prec_K q$ if and only if $p + K$ contains q. By Observation 16, this relation is transitive so it is a quasi-order. Recall that when $\prec_K$ is interpreted as a digraph, qp is an edge if and only if $q \prec_K p$.

Suppose the cones K_1, K_2, K_3 are the translates of the three corners of a triangle so that all their apexes are at the origin; in other words, the cones $K_1, -K_3, K_2, -K_1, K_3, -K_2$ partition the plane around the origin in this order. Then we will say that K_1, K_2, K_3 is a *set of tri-partition* cones. In this case the intersection of any translates of K_1, K_2, K_3 forms a (possibly degenerate) triangle.

Observation 17. *Let K_1, K_2, K_3 be a set of tri-partition cones and let P be a planar point set. Then any two distinct points of P are comparable in either $\prec_{K_1}$, $\prec_{K_2}$ or $\prec_{K_3}$ (see Figure 3).*

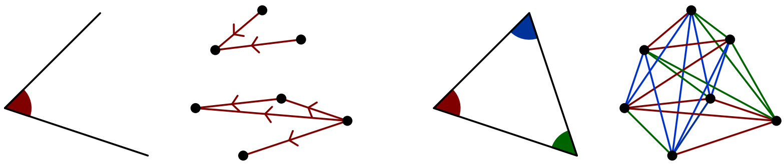

Fig. 3. Quasi-order on a point set.

In other words, when $\prec_{K_1}$, $\prec_{K_2}$ and $\prec_{K_3}$ are interpreted as digraphs, their union forms a complete multidigraph on P. As a warm up for the proof of Theorem 1, we show the following theorem.

Theorem 18. *There exists a positive integer m such that for any point set P, and any set K_1, K_2, K_3 of tri-partition cones, we can three-color P in such a way that no translate of K_1, K_2 or K_3 that contains m points of P (or more) is monochromatic.*

Proof. We set m to be $f(3,2)+13$ with f the function of Theorem 10. Consider the three quasi-orders $\prec_{K_1}$, $\prec_{K_2}$ or $\prec_{K_3}$. Their union gives a complete multidigraph on P, hence we can apply Theorem 10 with $k=3$ and $l=2$, resulting in subsets S_i^j for $i \in [3]$, $j \in [2]$. Let $S = \bigcup_{i\in[3],\, j\in[2]} S_i^j$. For each $p \in P \setminus S$ there is an i such that $\prec_{K_i}$ has an edge from a vertex of S_i^1 and S_i^2 to p. Let P_1, P_2, P_3 be the partition of $P \setminus S$ according to this i value.

We start by coloring the points of S. Color the points of $S_1^1 \cup S_2^1 \cup S_3^1$ with the first color and the points of $S_1^2 \cup S_2^2 \cup S_3^2$ with the second color.

Any translate of K_1, K_2 or K_3 that contains $f(3,2)+13$ points of P must contain five points from either P_1, P_2 or P_3 by the pigeonhole principle. (Note that the cone might contain all points of S.) Therefore, it is enough to show that for each $i \in [3]$ the points of P_i can be three-colored in such a way that no translate of K_1, K_2, or K_3 that contains five points of P_i (or more) is monochromatic.

Consider P_1; the proof for P_2 and P_3 is the same. Take a translate of K_1 and suppose that it contains a point p of P_1. By Theorem 10, there is an edge of $\prec_{K_1}$ from a vertex of S_1^1 to p and another edge from a vertex of S_1^2 to p. Thus any such translate contains a point from S_1^1 and another point from S_1^2, and hence it cannot be monochromatic.

Therefore, we only have to consider the translates of K_2 and K_3. Two translates of a cone intersect at most once on their boundary. Hence, the translates of K_2 form a pseudo-halfplane arrangement, and so do the translates of K_3. Therefore, by Corollary 8, there is a proper three-coloring for the translates of K_2 and K_3 together. ∎

Remark 19. From Theorem 18, it follows using standard methods (see Section 3.2) that Theorem 1 holds for triangles. This was of course known before, even for two-colorings of homothetic copies of triangles. Our proof cannot be modified for homothets, but a two-coloring would follow if instead of Corollary 8 we applied a more careful analysis for the two cones.

3.2. *Proof of Theorem 1*

If C is a parallelogram, then our proof method fails. Luckily, translates of parallelograms (and other symmetric polygons) were the first for which it was shown that even two colors are enough [23]; in fact, by now we know that two colors are enough even for homothets of parallelograms [2]. So from now on we assume that C is not a parallelogram.

The proof of Theorem 1 relies on the same ideas as for Theorem 18. We partition P into several parts, and for each part P_i, we divide the translates of C into three families such that two of the families each form a pseudohalfplane arrangement over P_i, while the third family will only contain translates that are automatically nonmonochromatic. Then Corollary 8 gives us a proper three-coloring. As in the proof of Theorem 18, this is not done directly. First, we divide the plane using a grid, and then in each small square we will use Theorem 10 to discard some of the translates of C at the cost of a bounded number of points.

Now we start the proof of Theorem 1. The first step is a classical divide and conquer idea [23]. We choose a constant $r = r(C)$ depending only on C and divide the plane into a grid of squares of side length r. Since each translate of C intersects a bounded number of squares, by the pigeonhole principle we can find for any positive integer m another integer m' such that the following holds: each translate $\hat{C}$ of C that contains at least m' points intersects a square Q such that $\hat{C} \cap Q$ contains at least m points. For example, we can choose $m' = m(\mathrm{diam}(C)/r + 2)^2$, where $\mathrm{diam}(C)$ denotes the diameter of C. Therefore, it is enough to show the following localized version of Theorem 1, since applying it separately for the points in each square of the grid provides a proper three-coloring of the whole point set.

Theorem 20. *There is a positive integer m such that for any convex body C there is a positive real r such that any finite point set P in the plane that lies in a square of side length r can be three-colored in such a way that no translate of C contains m points of P (or more), all of the same color.*

We will show that m can be chosen to be $f(3,2) + 13$ with the function of Theorem 10, independently of C.

Proof. We pick r the following way. First we fix an injective parametrization γ of ∂C and then fix t_1, t_2, t_3 and ε according to Lemma 15. Let ℓ_1, ℓ_2, ℓ_3 be the tangents of C touching at $\gamma(t_1), \gamma(t_2)$ and $\gamma(t_3)$. Let $K_{1,2}$, $K_{2,3}$, $K_{3,1}$ be the set of tri-partition cones bordered by ℓ_1, ℓ_2, ℓ_3 such that $K_{i,i+1}$ is bordered by ℓ_i on its counterclockwise side, and by ℓ_{i+1} on its clockwise side (see Figure 4 (left) and note that we always treat $3 + 1$ as 1 in the subscript).

For a translate $\hat{C}$ of C we will denote by $\hat{\gamma}$ the translated parametrization of $\partial\hat{C}$, i.e.,

$$\hat{\gamma}(t) = \gamma(t) + v$$

if $\hat{C}$ was translated by v. Our aim is to choose r small enough to satisfy the following two properties for each $i \in [3]$.

(A) Let $\hat{C}$ be a translate of C, and Q be a square of side length r such that $\partial\hat{C} \cap Q \subset \hat{\gamma}_{[t_i+\varepsilon/2, t_{i+1}-\varepsilon/2]}$ (see Figure 4, right). Then for any translate K of $K_{i,i+1}$ whose apex is in $Q \cap \hat{C}$, we have $K \cap Q \subset \hat{C}$ (i.e., r is small with respect to C).

(B) Let $\hat{C}$ be a translate of C, and Q be a square of side length r such that the curve $\hat{\gamma}_{[t_i-\varepsilon/2, t_{i+1}+\varepsilon/2]}$ intersects Q. Then $\partial\hat{C} \cap Q \subset \hat{\gamma}_{[t_i-\varepsilon, t_{i+1}+\varepsilon]}$ (i.e., r is small compared to ε.)

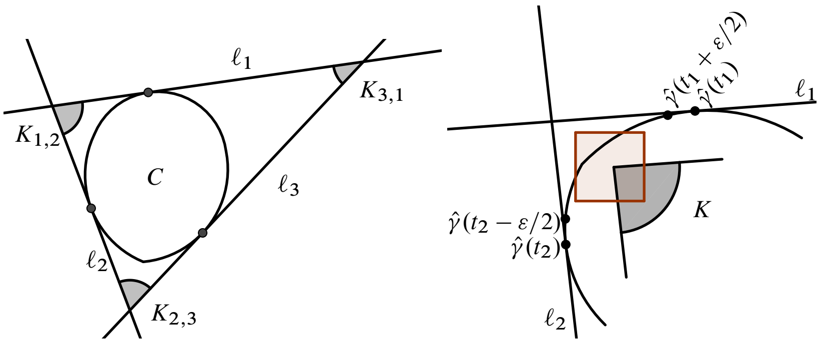

Fig. 4. Selecting the cones (left) and property (A) (right).

We show that an r satisfying properties (A) and (B) can be found for $i = 1$. The argument is the same for $i = 2$ and $i = 3$, and we can take the smallest of the three resulting values of r.

First, consider property (A). Since the sides of K are parallel to ℓ_1 and ℓ_2, the portion of K that lies "above" the segment $\overline{\hat{\gamma}(t_1)\hat{\gamma}(t_2)}$ is in $\hat{C}$. Hence, if we choose r small enough so that Q cannot intersect $\overline{\hat{\gamma}(t_1)\hat{\gamma}(t_2)}$, then property (A) is satisfied. We can choose r to be smaller than $\frac{1}{\sqrt{2}}$ times the distance between the segments $\overline{\hat{\gamma}(t_1)\hat{\gamma}(t_2)}$ and $\overline{\hat{\gamma}(t_1+\varepsilon/2)\hat{\gamma}(t_2-\varepsilon/2)}$.

Since γ is a continuous function on a compact set, we can pick r such that property (B) is satisfied. Therefore, there is an r satisfying properties (A) and (B).

The next step is a subdivision of the point set P using Theorem 10, like we did in the proof of Theorem 18. The beginning of our argument is exactly the same.

Apply Theorem 10 for the graph given by the union of $\prec_{K_{1,2}}$, $\prec_{K_{2,3}}$ and $\prec_{K_{3,1}}$. By Observation 16, this is indeed a complete multidigraph on P.

We apply Theorem 10 with $k = 3$ and $l = 2$, resulting in subsets S_i^j for $i \in [3], j \in [2]$. Let $S = \bigcup_{i\in[3], j\in[2]} S_i^j$. For each point $p \in P \setminus S$ there is an i such that $\prec_{K_{i,i+1}}$ has an edge from a vertex of S_i^1 and S_i^2 to p. Let P_1, P_2, P_3 be the partition of $P \setminus S$ according to this i value.

We start by coloring the points of S. Color the points of $S_1^1 \cup S_2^1 \cup S_3^1$ with the first color and color the points of $S_1^2 \cup S_2^2 \cup S_3^2$ with the second color.

Note that m is at least $f(3, 2) + 13$. Any translate of C that contains $f(3, 2) + 13$ points of P must contain five points from either P_1, P_2 or P_3. (Note that the cone might contain all points of S.) Thus, it is enough to show that for each $i \in [3]$ the points of P_i can be three-colored in such a way that no translate of C that contains five points of P_i (or more) is monochromatic.

Consider P_1; the proof for P_2 and P_3 is the same. We divide the translates of C that intersect Q into four (not necessarily disjoint) groups. Let $\mathcal{C}_0$ denote the translates where $\hat{C} \cap Q = Q$. Let $\mathcal{C}_1$ denote the translates for which $\partial\hat{C} \cap Q \subset \hat{\gamma}_{[t_1+\varepsilon/2,t_2-\varepsilon/2]}$. Let $\mathcal{C}_2$ denote the translates for which $\partial\hat{C} \cap Q \cap \hat{\gamma}_{[t_2-\varepsilon/2,t_3]} \neq \emptyset$. Let $\mathcal{C}_3$ denote the remaining translates for which $\partial\hat{C} \cap Q \cap \hat{\gamma}_{[t_3,t_1+\varepsilon/2]} \neq \emptyset$.

We do not need to worry about the translates in $\mathcal{C}_0$, as Q itself will not be monochromatic.

Take a translate $\hat{C}$ from $\mathcal{C}_1$ and suppose that it contains a point $p \in P_1$. By Theorem 10, there is an edge of $\prec_{K_{1,2}}$ from a vertex of S_1^1 to p and another edge from a vertex of S_1^2 to p. That is, the cone $p + K_{1,2}$ contains a point from S_1^1 and another point from S_1^2, and hence it is not monochromatic. From property (A) we know that every point in $(p + K_{1,2}) \cap P$ is also in $\hat{C}$. Therefore, $\hat{C}$ is not monochromatic.

Now consider the translates in $\mathcal{C}_2$. From property (B) we know that for these translates we have $\partial\hat{C} \cap Q \subset \hat{\gamma}_{[t_2-\varepsilon,t_3+\varepsilon]}$. By the definition of t_1, t_2 and t_3, this implies that any two translates from $\mathcal{C}_2$ intersect at most once on their boundary within Q, i.e., they behave like pseudohalfplanes. To turn the translates in $\mathcal{C}_2$ into a pseudohalfplane arrangement as defined earlier, we can proceed as follows. For a translate $\hat{C}$, replace it with the convex set whose boundary is $\hat{\gamma}_{[t_2-\varepsilon,t_3+\varepsilon]}$ extended from its endpoints with two rays orthogonal to the segment $\overline{\hat{\gamma}(t_2-\varepsilon)\hat{\gamma}(t_3+\varepsilon)}$. This new family provides the same intersection pattern in Q and forms a pseudohalfplane arrangement. We can do the same with the translates in $\mathcal{C}_3$. Therefore, by Corollary 8 there is a proper three-coloring for the translates in $\mathcal{C}_2 \cup \mathcal{C}_3$. ∎

4. Proof of Theorem 10

Let us quickly recap the main steps from the proof of Theorem 9 from [7]. For a function $w : V \to \mathbb{R}$ and $S \subset V$ let $w(S) = \sum_{x \in S} f(x)$. We say that $w : V \to [0, 1]$ is a *probability distribution* on V if $w(V) = 1$. First the authors of [7] carefully define a partition of the vertex set. Then for each part P of the partition, a probability distribution w_P is defined which is concentrated on P. Each part of the partition is dominated independently of the other parts using a probabilistic argument. Namely, they show that we can pick some points according to w_P and these points will dominate P with positive probability. Also, each part is dominated using edges of just one $\prec_i$. The main reason that the proof does not immediately work for Theorem 10 is that the dominating sets for the different parts might intersect.

Our proof of Theorem 10 follows a very similar path. We will also define a partition of the vertex set (see Figure 5) and the corresponding probability distributions. Then we will apply the probabilistic argument for the parts simultaneously to ensure that the dominating sets of the parts are disjoint. We will apply the probabilistic argument l times for each part. To be able to ensure disjointness of the S_i^j's, the partition and the distributions are created a bit more carefully; we ensure that a vertex cannot have too much weight in any of the probability distributions. This way the probability of picking any vertex in two different S_i^j's will be sufficiently small.

We start with a number of useful lemmas. The following variant of LP duality was stated in this context in [3, Lemma 5].

Lemma 21 (Alon et al. [3]). *If $D = (V, A)$ is a complete multidigraph, then there exists a probability distribution w on V such that $w(N^-(x)) \geq 1/2$ for each $x \in V$.*

We prove the following modification of Lemma 21 to obtain a w whose min-entropy is large.

Lemma 22. *Let $D = (V, A)$ be a complete multidigraph and let $0 < \delta < 1$ be a fixed number. If $|V| > 1/\delta$, then there exists a probability distribution w on V such that for each $x \in V$ one of the following holds:*

- *$w(x) \leq 2\delta$ and $w(N^-(x)) \geq 1/2$,*
- *$\delta \leq w(x) \leq 4\delta$.*

Note that the second condition holds for at most $1/\delta$ vertices.

Proof. We define a sequence of probability distributions $w_1, \dots, w_{\lfloor 1/\delta \rfloor}$ and a sequence of subsets of V called $R_1, \dots, R_{\lfloor 1/\delta \rfloor}$. Let w_1 be a probability distribution given by Lemma 21, that is, $w_1(N^-(x)) \geq 1/2$ for all $x \in V$. Let $R_1 = \emptyset$ if $w_1(x) < 1$ for every $x \in V$ and let $R_1 = \{x\}$ if w_1 is concentrated on a single vertex $x \in V$.

For $i > 1$ we obtain w_i by applying Lemma 21 for the induced complete multidigraph $D[V \setminus R_{i-1}]$. That is, w_i is a probability distribution such that $w_i(N^-(x)) \geq 1/2$ for all $x \in V \setminus R_{i-1}$ and $w_i(x) = 0$ for $x \in R_{i-1}$. Let $R_i = \{x \in V \mid \sum_{j=1}^{i} w_j(x) \geq 1\}$.

To use Lemma 21, we need to check that $V \setminus R_{i-1}$ is nonempty. If $V \setminus R_{i-1}$ is empty, then by the definition of R_{i-1} we have $\sum_{j=1}^{i-1} w_j(x) \geq 1$ for each $x \in V$. Hence

$$i - 1 = \sum_{j=1}^{i-1} w_j(V) = \sum_{x \in V} \sum_{j=1}^{i-1} w_j(x) \geq |V| \cdot 1.$$

Since $i - 1 < 1/\delta$, this contradicts the $|V| > 1/\delta$ assumption of the lemma.

Let $w = \frac{1}{\lfloor 1/\delta \rfloor} \sum_{i=1}^{\lfloor 1/\delta \rfloor} w_i$. Clearly, this is a probability distribution on V. We have to check the correctness of w. The first condition holds for vertices not in $R_{\lfloor 1/\delta \rfloor}$ and the second one for the vertices of $R_{\lfloor 1/\delta \rfloor}$.

If $x \in V \setminus R_{\lfloor 1/\delta \rfloor}$, then

$$w(N^-(x)) = \frac{1}{\lfloor 1/\delta \rfloor} \sum_{i=1}^{\lfloor 1/\delta \rfloor} w_i(N^-(x)) \geq \frac{1}{\lfloor 1/\delta \rfloor} \lfloor 1/\delta \rfloor \frac{1}{2} = \frac{1}{2}.$$

Since $x \notin R_{\lfloor 1/\delta \rfloor}$ and $0 < \delta < 1$, we have $w(x) \leq \frac{1}{\lfloor 1/\delta \rfloor} \cdot 1 \leq 2\delta$.

In the other case, when $x \in R_{\lfloor 1/\delta \rfloor}$, by definition we have

$$w(x) \geq \frac{1}{\lfloor 1/\delta \rfloor} \sum_{i=1}^{\lfloor 1/\delta \rfloor} w_i(x) \geq \frac{1}{\lfloor 1/\delta \rfloor} \geq \delta.$$

On the other hand, if j is the smallest number for which $x \in R_j$, then $\sum_{i=1}^{j-1} w_i(x) \leq 1$ and $w_k(x) = 0$ for $k > j$. Therefore,

$$w(x) = \frac{1}{\lfloor 1/\delta \rfloor} \sum_{i=1}^{\lfloor 1/\delta \rfloor} w_i(x) = \frac{1}{\lfloor 1/\delta \rfloor} \sum_{i=1}^{j-1} w_i(x) + \frac{1}{\lfloor 1/\delta \rfloor} w_j(x) \leq 2\delta + 2\delta = 4\delta. \quad \blacksquare$$

An easy consequence of Lemma 22 is the following.

Lemma 23. *Let $0 < \delta < 1$ be fixed and let $D = (V, A)$ be a complete multidigraph on at least $1/\delta$ vertices whose arc set is the union of k quasi-orders. Then there exists a probability distribution w on V and a partition of V into sets $T_1, \ldots, T_k, R$ such that for every $i \in [k]$ and $x \in T_i$, we have $w(x) \leq 2\delta$ and $w(N_i^-(x)) \geq 1/(2k)$ and for every $x \in R$ we have $\delta \leq w(x) \leq 4\delta$.*

Proof. Take w according to Lemma 22. For every $i \in [k]$, let $T_i' = \{x \in V \mid w(N_i^-(x)) \geq 1/(2k)\}$ and let R be the rest of the vertices. As $\sum_{i=1}^{k} w(N_i^-(x)) \geq w(N^-(x))$, the sets T_i' cover those vertices that satisfy the first property in Lemma 22. By making the T_i's disjoint, we can obtain a partition with the required properties. $\blacksquare$

Proof of Theorem 10. Fix $0 < \delta < 1$; we will choose its value later. Let $\mathcal{I}$ denote the set of sequences of length at most $k + 1$ whose terms are from $[k]$, including the empty set. In other words, $\mathcal{I} = \{\emptyset\} \cup \bigcup_{l=1}^{k+1} [k]^l$.

For a complete multidigraph $D = (V, A)$ we are going to define two systems of subsets, $\{T_{j_1,\ldots,j_i}\}_{(j_1,\ldots,j_i) \in \mathrm{Ind}_1}$ and $\{R_{j_1,\ldots,j_i}\}_{(j_1,\ldots,j_i) \in \mathrm{Ind}_2}$, of the vertex set for some index sets $\mathrm{Ind}_1, \mathrm{Ind}_2 \subset \mathcal{I}$. The process that defines these subsets is similar to a tree traversal algorithm. The root node corresponds to V and the children of a given node in the tree will correspond to a partition of the node. Therefore in the end the leaves of the tree provide a partition of the vertex set. For each $T_{j_1,\ldots,j_i}$ we also define a probability distribution $w_{j_1,\ldots,j_i}$ which is concentrated on $T_{j_1,\ldots,j_i}$.

We start by setting $T_\emptyset = V$ and we apply Lemma 23 for $T_\emptyset$ and our fixed δ to obtain $T_1, \ldots, T_k$ and $R_\emptyset$ together with a probability distribution $w_\emptyset$.

Then, as long as possible, we do the following step. We pick a $T_{j_1,\ldots,j_i}$ from the already defined ones, not selected previously and such that $j_1, \ldots, j_i$ are pairwise distinct and $|T_{j_1,\ldots,j_i}| > 1/\delta$. For such a $T_{j_1,\ldots,j_i}$ we apply Lemma 23 to obtain the partition

$$T_{j_1,\ldots,j_i,1}, \ldots, T_{j_1,\ldots,j_i,k}, R_{j_1,\ldots,j_i}$$

and a probability distribution $w_{j_1,\ldots,j_i}$. This process terminates in (strictly) less than $|\mathcal{I}| \leq k^{k+2}$ steps as no index sequence can be longer than $k + 1$. Let Ind_1 and Ind_2 denote the index sets for the $T_{j_1,\ldots,j_i}$'s and $R_{j_1,\ldots,j_i}$'s. From Lemma 23 we know the following:

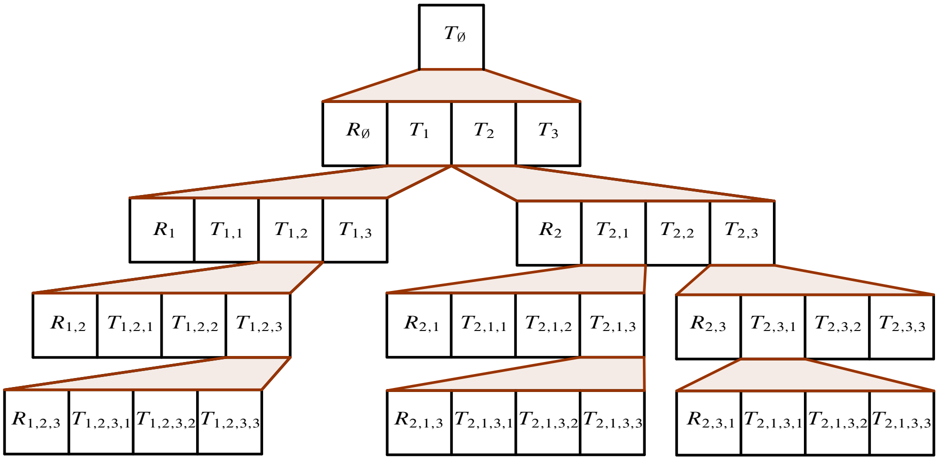

Fig. 5. The outcome of the partition process for $k = 3$ assuming that the size of T_3 and $T_{1,3}$ is less than $1/\delta$.

- $|R_{j_1,\dots,j_i}| \le 1/\delta$ for each $(j_1, \dots, j_i) \in \mathrm{Ind}_2$.
- For every $r \in [k]$ and $x \in T_{j_1,\dots,j_i,r}$ we have $w_{j_1,\dots,j_i}(N_r^-(x)) \ge 1/(2k)$.
- $w_{j_1,\dots,j_i}(x) \le 4\delta$ for every $x \in V$ and every $(j_1, \dots, j_i) \in \mathrm{Ind}_2$.

When the process halts, we define a partition of V by taking every set corresponding to a leaf, i.e., every set of the following three kinds:

(1) $R_{j_1,\dots,j_i}$'s,

(2) $T_{j_1,\dots,j_i}$'s that have fewer elements than $1/\delta$,

(3) $T_{j_1,\dots,j_i}$'s where $j_i = j_r$ for some $r < i$.

These sets are clearly disjoint, and they cover the vertex set, since the process halted. Let R denote the union of the $R_{j_1,\dots,j_i}$'s, let T_{small} denote the union of the $T_{j_1,\dots,j_i}$'s that have fewer elements than $1/\delta$, and finally let T_{rep} denote the union of the $T_{j_1,\dots,j_i}$'s where $j_i = j_r$ for some $r < i$.

We dominate each part of the partition. Since each $R_{j_1,\dots,j_i}$ has fewer than $1/\delta$ elements, we have $|R| \le \frac{1}{\delta}|\mathrm{Ind}_2| \le \frac{1}{\delta}k^{k+2}$. Similarly $|T_{\mathrm{small}}| \le \frac{1}{\delta}k^{k+2}$. We will simply put the vertices of $R \cup T_{\mathrm{small}}$ into S_1^1 but we will not use them to dominate any point. Later, when we pick δ, we will see that $\frac{2}{\delta}k^{k+2}$ is upper bounded by a function of k and l, so we have not added too many vertices here.

It remains to dominate the vertices in T_{rep}. Let $I_{\mathrm{rep}} \subset \mathrm{Ind}_1$ be the index set of those $T_{j_1,\dots,j_i}$'s where $j_i = j_r$ for some $r < i$.

Fix $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$ and $o \in [l]$. First we explain how to choose a set $V^o_{j_1,\dots,j_i} \subset V$ that dominates $T_{j_1,\dots,j_i}$ and then we will argue that this can be done simultaneously for all $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$ and $o \in [l]$ in such a way that the $V^o_{j_1,\dots,j_i}$'s are disjoint.

For a fixed $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$, consider $T_{j_1,\dots,j_i} \subset T_{j_1,\dots,j_{i-1}} \subset T_{j_1,\dots,j_{r-1}}$ such that $j_i = j_r$. The idea is that if we appropriately pick $V^o_{j_1,\dots,j_i} \subset T_{j_1,\dots,j_{r-1}}$ that dominates a large fraction of $T_{j_1,\dots,j_{i-1}}$ through the edges of $\prec_{j_r}$, then that large fraction dominates entirely $T_{j_1,\dots,j_i}$, and since $j_r = j_i$, this happens through the edges of $\prec_{j_i}$ again. By transitivity, this implies that $T_{j_1,\dots,j_i}$ is dominated by $V^o_{j_1,\dots,j_i}$. To put this idea into precise terms note that for each $x \in T_{j_1,\dots,j_i}$ we have $w_{j_1,\dots,j_{i-1}}(N^-_{j_i}(x)) \geq \frac{1}{2k}$. Hence, if $w_{j_1,\dots,j_{i-1}}(N^+_{j_i}(V^o_{j_1,\dots,j_i})) > 1 - \frac{1}{2k}$, then $T_{j_1,\dots,j_i}$ is dominated by $V^o_{j_1,\dots,j_i}$.

Let $0 < \varepsilon < 1$ be fixed; we will choose its value later. Let

$$g(\varepsilon) = \left\lfloor \frac{\ln(\varepsilon)}{\ln\left(1 - \frac{1}{2k}\right)} \right\rfloor + 1$$

and let $V^o_{j_1,\dots,j_i}$ be a multiset of $g(\varepsilon)$ elements picked independently at random from $T_{j_1,\dots,j_{r-1}}$ according to the distribution $w_{j_1,\dots,j_{r-1}}$. For every vertex $x \in T_{j_1,\dots,j_{i-1}}$,

$$\mathbb{P}(x \in N^+_{j_i}(V^o_{j_1,\dots,j_i})) \geq 1 - \left(1 - \frac{1}{2k}\right)^{g(\varepsilon)} \geq 1 - \varepsilon.$$

Therefore, by linearity of expectation,

$$\mathbb{E}(w_{j_1,\dots,j_{i-1}}(N^+_{j_i}(V^o_{j_1,\dots,j_i}))) \geq \sum_{x \in T_{j_1,\dots,j_{i-1}}} w_{j_1,\dots,j_{i-1}}(x) \cdot (1 - \varepsilon) \geq 1 - \varepsilon.$$

This already shows that if $\varepsilon \leq 1/(2k)$, there is a positive probability that $V^o_{j_1,\dots,j_i}$ dominates $T_{j_1,\dots,j_i}$. But we will need a smaller ε for disjointness.

Let X be the random variable

$$\sum_{(j_1,\dots,j_i) \in I_{\mathrm{rep}},\, o \in [l]} w_{j_1,\dots,j_{i-1}}(N^+_{j_i}(V^o_{j_1,\dots,j_i})).$$

By our reasoning above, $\mathbb{E}(X) \geq l|I_{\mathrm{rep}}|(1 - \varepsilon)$.

Let Y be the indicator variable of the event that the sets $V^o_{j_1,\dots,j_i}$ for $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$ and $o \in [l]$ are pairwise disjoint. Consider the random variable $X \cdot Y$. Suppose that $\mathbb{P}(X \cdot Y > l|I_{\mathrm{rep}}| - \frac{1}{2k}) > 0$. This would imply that $Y = 1$, i.e., the $V^o_{j_1,\dots,j_i}$'s are pairwise disjoint. Since $w_{j_1,\dots,j_{i-1}}(N^+_{j_i}(V^o_{j_1,\dots,j_i})) \leq 1$ for each $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$, it would also imply that $w_{j_1,\dots,j_{i-1}}(N^+_{j_i}(V^o_{j_1,\dots,j_i})) > 1 - \frac{1}{2k}$ for each $(j_1, \dots, j_i) \in I_{\mathrm{rep}}$ and $o \in [l]$. From this it will be easy to construct the required sets S^j_i.

We will show a lower bound on $\mathbb{E}(X \cdot Y)$. Since Y is an indicator variable, we have

$$\mathbb{E}(X \cdot Y) \geq \mathbb{E}(X) - \max(X) \cdot \mathbb{P}(Y = 0).$$

Clearly $\max(X) \leq l|I_{\mathrm{rep}}|$. To bound $\mathbb{P}(Y = 0)$, note that in total we have picked at most $N = l|I_{\mathrm{rep}}|g(\epsilon) \leq lk^{k+2}g(\epsilon)$ elements. Every time we picked an element, the probability distribution was smaller than 4δ on each element of V. Hence

$$\mathbb{P}(Y = 0) \leq \binom{N}{2} 4\delta \leq N^2 2\delta$$

Therefore,

$$\begin{aligned}\mathbb{E}(X \cdot Y) &\geq \mathbb{E}(X) - \max(X) \cdot \mathbb{P}(Y = 0) \\ &\geq l|I_{\text{rep}}|(1-\varepsilon) - l|I_{\text{rep}}|N^2 2\delta = l|I_{\text{rep}}|(1-\varepsilon-N^2 2\delta).\end{aligned}$$

Pick $\varepsilon = \frac{1}{4lk^{k+3}}$ and $\delta = \frac{1}{8l^3k^{3k+7}g(\varepsilon)^2}$. With this choice

$$N^2 2\delta \leq \frac{2l^2k^{2k+4}g(\varepsilon)^2}{8l^3k^{3k+7}g(\varepsilon)^2} = \frac{1}{4lk^{k+3}}.$$

Therefore,

$$\begin{aligned}\mathbb{E}(X \cdot Y) &\geq l|I_{\text{rep}}|(1-\varepsilon-N^2 2\delta) \geq l|I_{\text{rep}}|\left(1 - \frac{1}{4lk^{k+3}} - \frac{1}{4lk^{k+3}}\right) \\ &\geq l|I_{\text{rep}}|\left(1 - \frac{1}{2kl|I_{\text{rep}}|}\right) = l|I_{\text{rep}}| - \frac{1}{2k}.\end{aligned}$$

Hence, it is possible that $X \cdot Y \geq l|I_{\text{rep}}| - \frac{1}{2k}$. In this case the $V^o_{j_1,\dots,j_i}$'s are pairwise disjoint and each $T_{j_1,\dots,j_i}$ is dominated by $V^o_{j_1,\dots,j_i}$ through the edges of $\prec_{j_i}$.

Finally, let $S_i^j = \bigcup_{(j_1,\dots,j_r,i)\in I_{\text{rep}}} V^j_{j_1,\dots,j_r,i}$ (be careful that i denotes something else than in the paragraph above). The S_i^j's are also pairwise disjoint. For any vertex $v \in V \setminus \bigcup_{i,j} S_i^j \subset T_{\text{rep}}$ there is a $T_{j_1,\dots,j_r,i}$ containing it for some $(j_1,\dots,j_r,i) \in I_{\text{rep}}$ (recall that we added $R \cup T_{\text{small}}$ to S_1^1), hence there is an edge of $\prec_i$ from v to a vertex of S_i^j for each $j \in [l]$.

To wrap up the proof, we calculate the size of $\bigcup_{i,j} S_i^j$. We have placed $R \cup T_{\text{small}}$ into S_1^1 and we have at most $g(\varepsilon)$ elements in each $V^o_{j_1,\dots,j_i}$. Therefore, we have at most

$$\begin{aligned}\frac{2}{\delta}k^{k+2} + lk^{k+2}g(\varepsilon) &\leq 2 \cdot 8l^3k^{3k+7}\left(\left\lfloor \frac{\ln\left(\frac{1}{4lk^{k+3}}\right)}{\ln\left(1-\frac{1}{2k}\right)} \right\rfloor + 1\right)^2 k^{k+2} \\ &\quad + lk^{k+2}\left(\left\lfloor \frac{\ln\left(\frac{1}{4lk^{k+3}}\right)}{\ln\left(1-\frac{1}{2k}\right)} \right\rfloor + 1\right)\end{aligned}$$

elements in $\bigcup_{i,j} S_i^j$. ■

5. Sketch of generalization of Theorem 5

Here we give a sketch of the construction that proves the following generalization of Theorem 5.

Theorem 24. *Let C be any convex body in the plane whose boundary has a point with a neighborhood where the boundary is smooth and not a straight line segment. For any positive integer m, there exists a three-chromatic m-uniform hypergraph that is realizable with translates of C.*

First we describe the abstract three-chromatic m-uniform hypergraph, first used in [28], that we realize with the translates of C; we briefly recall its definition from [24].

For any positive integers k and l the abstract hypergraph $\mathcal{H}(k,l)$ with vertex set $V(k,l)$ and edge set $E(k,l)$ is defined recursively. The edge set $E(k,l)$ is the disjoint union of two sets, $E(k,l) = E_R(k,l) \cup E_B(k,l)$, where the subscripts R and B stand for red and blue. All edges belonging to $E_R(k,l)$ are of size k, all edges belonging to $E_B(k,l)$ are of size l. In other words, $\mathcal{H}(k,l)$ is the union of a k*-uniform* and an l*-uniform* hypergraph. If $k = l = m$, we get an m-uniform hypergraph.

Definition 25. Let k and l be positive integers.

(1) For $k = 1$, let $V(1,l)$ be an l-element set. Set

$$E_R(1,l) := V(1,l) \quad \text{and} \quad E_B(1,l) := \{V(1,l)\}.$$

(2) For $l = 1$, let $V(k,1)$ be a k-element set. Set

$$E_R(k,1) := \{V(k,1)\} \quad \text{and} \quad E_B(k,1) := V(k,1).$$

(3) For any $k, l > 1$, we pick a new vertex p, called the *root*, and let

$$\begin{aligned} V(k,l) &:= V(k-1,l) \cup V(k,l-1) \cup \{p\}, \\ E_R(k,l) &:= \{e \cup \{p\} \mid e \in E_R(k-1,l)\} \cup E_R(k,l-1), \\ E_B(k,l) &:= E_B(k-1,l) \cup \{e \cup \{p\} \mid e \in E_B(k,l-1)\}. \end{aligned}$$

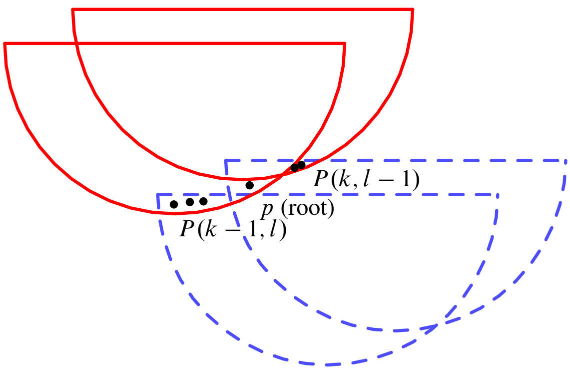

Fig. 6. The recursive step (3) of the construction with unit halfdisks.

It was shown in [28] that for any positive integers k, l, the hypergraph $\mathcal{H}(k,l)$ is not two-colorable. Moreover, for every coloring of $V(k,l)$ with red and blue, there is an edge in $E_R(k,l)$ such that all of its k vertices are red or an edge in $E_B(k,l)$ such that all of its l vertices are blue.

To prove Theorem 24, it is sufficient to show that (1)–(3) in Definition 25 can be realized with the translates of C. Our realization will be such that all the translates of C

realizing edges from $E_R(k,l)$ are very close to each other, and similarly all the translates realizing edges from $E_B(k,l)$ are also very close to each other. Moreover, a translate realizing an edge from $E_R(k,l)$ is always intersected, or almost intersected, by a translate realizing an edge from $E_B(k,l)$ very close to the smooth neighborhood on the boundary of C.

For (1) and (2), such a realization is trivial to find. For (3), the recursive step, it is depicted for the unit halfdisk in Figure 6 how the two point sets $P(k-1,l)$ and $P(k,l-1)$, representing the vertices $V(k-1,l)$ and $V(k,l-1)$, can be put together to form the point set $P(k,l)$ with the addition of a new point, p (the root), representing the vertices of $V(k,l)$. For other C a similar construction works, but we do not go into details about how the precise direction of translation has to be chosen, close to the supporting line of the smooth neighborhood guaranteed by the condition of Theorem 24. This finishes the description of the construction.

6. Open questions

It is a natural question whether there is a universal m that works for all convex bodies in Theorem 1, like in Theorem 20. This would follow if we could choose r to be a universal constant. While the r given by our algorithm can depend on C, we can apply an appropriate affine transformation to C before choosing r; this does not change the hypergraphs that can be realized with the range space determined by the translates of C. To ensure that properties (A) and (B) are satisfied would require a further study of the Illumination Conjecture.

Our bound for m is quite large, even for the unit disk, in both Theorems 1 and 20, which is mainly due to the fact that $f(3,2)$ given by Theorem 10 is huge. It has been conjectured that in Theorem 9 the optimal value is $f(3)=3$, and a similarly small number seems realistic for $f(3,2)$ as well.

While Theorem 1 closed the last main question left open for primal hypergraphs realizable by translates of planar convex bodies, the problem is still open in higher dimensions. While it is not hard to show that some hypergraphs with high chromatic number often used in constructions can be easily realized by unit balls in $\mathbb{R}^5$, we do not know whether the chromatic number is bounded or not in $\mathbb{R}^3$. From our Union Lemma (Lemma 6) it follows that to establish boundedness, it would be enough to find a polychromatic k-coloring for pseudohalfspaces, whatever this word means.

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