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Half-space theorems for recurrent minimal and H-surfaces of $\mathbb{R}^3$

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Abstract. We prove a version of the strong half-space theorem between the classes of recurrent minimal surfaces and complete minimal surfaces with bounded curvature of $\mathbb{R}^3$. The use of subsolutions in the barrier sense allow us to deal with non-proper minimal surfaces immersed with bounded curvature. We show that any minimal hypersurface immersed with bounded curvature in $M \times \mathbb{R}_+$ equals some $M \times \{s\}$ provided M is a complete, recurrent n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq 0$ and whose sectional curvatures are bounded from above. Furthermore, we prove a half-space theorem for the class of stochastically complete H-surfaces. We present a maximum principle at infinity assuming M has non-empty boundary. Finally, we present examples of a complete non-proper recurrent minimal surface with unbounded curvature.

1. Introduction

A classical theorem in the global theory of minimal surfaces by Xavier [54] states that the convex hull of a complete non-planar minimal surface of $\mathbb{R}^3$ with bounded curvature is the entire $\mathbb{R}^3$. This implies that the class of complete minimal surfaces with bounded curvature has the half-space property, meaning that any complete minimal surface with bounded curvature cannot lie in a half-space defined by some plane $\mathcal{P} \subset \mathbb{R}^3$ unless it is a plane parallel to $\mathcal{P}$. To show that the examples of complete minimal surfaces between two parallel planes constructed in [24, 49] were not properly immersed, Hoffman and Meeks [19] showed that the class of properly immersed minimal surfaces of $\mathbb{R}^3$ has the half-space property. This result, together with Theorem 8 and Corollary 1 in [34], yielded the Hoffman–Meeks strong half-space theorem, which states that two properly immersed minimal surfaces of $\mathbb{R}^3$ intersect unless they are parallel planes.

Likewise, Xavier's half-space theorem yielded a strong half-space theorem for minimal surfaces with bounded curvature, i.e., two complete minimal surfaces of $\mathbb{R}^3$ with bounded curvature must intersect unless they are parallel planes [3, 47]. The proof given in [3] yields, as a corollary, a strong half-space theorem between the classes of complete proper minimal surfaces and complete minimal surfaces with bounded curvature of $\mathbb{R}^3$, see Corollary 1.4 in [3].

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In this spirit, we add another piece in the puzzle proving a strong half-space theorem between the classes of complete minimal surfaces with bounded curvature and of recurrent minimal surfaces of $\mathbb{R}^3$.

Theorem 1.1. *Let M be a recurrent minimal surface and let N be a complete minimal surface with bounded curvature, immersed into $\mathbb{R}^3$. Then $M \cap N \neq \emptyset$ unless they are parallel flat surfaces.*[1]

A Riemannian manifold is said to be recurrent (parabolic) if the standard Brownian motion visits any open set at arbitrary large moments of time with probability one, and it is transient otherwise. It is known that the recurrence of a manifold, not necessarily geodesically complete, can be described in terms of various analytic, geometric and potential theoretic properties (see [16, 43, 44]), for instance, it is equivalent to the following Liouville property: any bounded solution of the differential inequality $\triangle u \geq 0$ is constant.

The class of recurrent immersed minimal surfaces of $\mathbb{R}^3$ contains the complete minimal immersions of the complex plane $\mathbb{C}$ into $\mathbb{R}^3$, the complete properly embedded minimal surfaces of $\mathbb{R}^3$ with finite genus [35], the complete minimal surfaces with quadratic volume growth, in particular, the complete surfaces with finite total curvature. In contrast, the first surface of Scherk is transient, see [20, 29]. We also have examples of complete recurrent, non-proper minimal surface with bounded curvature, see [1]. In Section 2, we present examples of recurrent non-proper minimal surfaces of $\mathbb{R}^3$ with unbounded curvature, showing that Theorem 1.1 is not contained in the results in [3, 19, 47].

Remark 1.2. We recall that every n-dimensional minimal hypersurface M immersed in $\mathbb{R}^{n+1}$, $n \geq 3$, is transient. Indeed, the isoperimetric inequality for a minimal hypersurface is given by $|\partial\Omega| \geq n|\mathbb{B}^n|^{1/n}|\Omega|^{(n-1)/n}$ for all $\Omega \Subset M$, see [5]. Then by Theorem 8.1 in [16], M must be transient.

In Theorem 1.2 of [48], Rosenberg, Schulze, and Spruck, capturing the stochastic nature of the Hoffmann–Meeks half-space theorem, proved a higher dimension half-space theorem for properly immersed minimal hypersurfaces of $M \times \mathbb{R}$, assuming that M was a complete recurrent n-manifold with bounded curvature (see also [31, 39]). Recently, Theorem 1.2 in [48] was extended by Colombo, Magliaro, Mari and Rigoli [7] to complete recurrent Riemannian n-manifolds with Ricci curvature bounded from below $\mathrm{Ric} \geq -(n-1)\Lambda^2$.

Theorem 1.3 ([7], Theorem 1.3 (ii)). *Let M be a complete recurrent Riemannian n-manifold with Ricci curvature $\mathrm{Ric} \geq -(n-1)\Lambda^2$ for some $\Lambda > 0$. Then any complete hypersurface minimally and properly immersed in $M \times \mathbb{R}_+$ is a slice $M \times \{s\}$.*

Our second result is a version of Theorem 1.2 in [48] and Theorem 1.3 (ii) in [7] for complete minimal hypersurfaces with bounded curvature.

Theorem 1.4. *Let M be a complete recurrent Riemannian n-manifold with non-negative Ricci curvature $\mathrm{Ric}_M \geq 0$, sectional curvature bounded from above $K_M \leq \Lambda^2$, and positive injectivity radius. Then, any complete hypersurface N with bounded sectional curvature minimally immersed in $M \times \mathbb{R}_+$ equals a slice $M \times \{s\}$.*

[1] M could be a plane minus a set of capacity zero parallel to a plane N.

Remark 1.5. B. White proved a strong half-space type theorem (Corollary 9.2 of [53]) in 3-manifolds Ω with non-negative Ricci curvature $\mathrm{Ric}_\Omega \geq 0$, implying the Hoffmann–Meeks strong half-space theorem when $\Omega = \mathbb{R}^3$. Other half-space theorems have been established in homogeneous spaces by Daniel, Meeks, and Rosenberg [9]. They proved half-space theorems for properly immersed minimal surfaces of Nil_3 and Sol_3, where the half-space was defined by some distinguished minimal surfaces of these spaces, see also [8, 38].

The intersection problem for surfaces of $\mathbb{R}^3$ with constant mean curvature $H > 0$, called H-surface for short, was addressed by Ros and Rosenberg in [46]. Recall that a properly embedded H-surfaces N separates $\mathbb{R}^3$ into two connected components, and the mean convex side is the connected component of $\mathbb{R}^3 \setminus N$ into which the mean curvature vector field of N points.

Theorem 1.6 (Ros–Rosenberg). *A properly embedded H-surface M of $\mathbb{R}^3$ cannot lie in the mean convex side of another properly embedded H-surface N.*

The Ros–Rosenberg half-space theorem for embedded H-surface was extended to other homogeneous three-spaces. For instance, Rodriguez and Rosenberg in [45] proved a half-space theorem for properly embedded 1-surfaces of $\mathbb{H}^3$, while Hauswirth, Rosenberg and Spruck [18], and Earp and Nelli in [40], proved half-space theorems for properly embedded 1/2-surfaces of $\mathbb{H}^2 \times \mathbb{R}$; see [33] for graphs with $0 < H < 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$. Using fairly general techniques, Mazet [31] unified the proof of various half-spaces theorems for generic three-spaces, in particular for Lie groups with left-invariant metric, see also [32].

In the second part of this article, we are going to consider half-space theorems for H-surfaces of $\mathbb{R}^3$ in the same vein of Theorems 1.1 and 1.6. Following ideas from [31], Section 4, we have the next definition.

Definition 1.7. Let N be a complete oriented properly immersed surface of $\mathbb{R}^3$. Assume that the mean curvature of N does not change sign, and choose the unit normal vector field ξ in such a way that $\vec{H}_N = H_N \xi$, with $H_N \geq 0$ on N. Let W be a connected component of $\mathbb{R}^3 \setminus N$.

(1) The mean curvature vector field of N at $z_0 \in N \cap \partial W$ is said to point into W if either $H_N(z_0) = 0$ or for any sequence $y_n \in W$ with $y_n \to z_0$, we have $y_n = \exp_z(t\xi(z))$ for some $0 < t < \varepsilon$ and z in a neighborhood $V \subset N$ of z_0.

(2) Let M be a surface immersed into $\mathbb{R}^3$. We say that N is well-oriented with respect to M if M lies in the connected component W of $\mathbb{R}^3 \setminus N$ into which the mean curvature vector field of N points.

Note that with such definition, every complete oriented properly immersed minimal surface N of $\mathbb{R}^3$ is well-oriented with respect to any surface immersed in $\mathbb{R}^3 \setminus N$. Our next result gives a stochastic version of Theorem 1.6.

Theorem 1.8. *Let M be an immersed surface of $\mathbb{R}^3$, and let N be an oriented complete surface properly immersed in $\mathbb{R}^3$ with bounded curvature. Then, unless M and N are parallel flat minimal surfaces, the surface N cannot be well-oriented with respect to M provided that*

(a) *either M is recurrent with mean curvature* $\sup_M |H_M| \leq \inf_N H_N$,

(b) *or M is stochastically complete with* $\sup_M |H_M| < \inf_N H_N$.

Remark 1.9. This result should be compared with the barrier principle proved in Theorem 1.1 of [14]. Indeed, since N is well-oriented, the connected component Ω of $\mathbb{R}^3 \setminus N$ that contains M, for which the mean curvature vector points to, satisfies $H_N \geq \inf_N H_N$ in the barrier sense. By Gauss equation, N has bounded second fundamental form, which implies that $\partial\Omega$ has locally bounded bending from outwards according to [14]. Therefore, Ω matches the assumptions in Theorem 1.1 of [14]. On the other hand, our requirements on M differ from those therein since the parabolicity and stochastic completeness of M improve conditions (1.7) and (1.4) in [14], respectively. Compare also with Theorem 7.3 in [53].

A Riemannian manifold M is stochastically complete if the diffusion process associated to the Laplacian $\triangle$ satisfies the conservation property

$$\int_M p(t,x,y)\,d\mu(y) = 1, \tag{1.1}$$

for some/every $x \in M$ and all $t > 0$. Here $p \in C^\infty((0,+\infty) \times M \times M$ is the heat kernel of M. The equation (1.1) has the following stochastic interpretation. The probability of the Brownian motion X_t emanating from x to be found in M is 1, see [16, 17]. The class of stochastically complete manifolds contains all complete manifolds with quadratic curvature decay or with quadratic exponential volume growth, as well as the properly immersed submanifolds of $\mathbb{R}^n$ with bounded mean curvature [16, 44].

Among the many equivalent characterizations for stochastic completeness, we are going to use the following Liouville property: for all $\lambda > 0$, any bounded, non-negative solution of the differential inequality $\triangle u \geq \lambda u$ is identically zero. It implies that recurrent manifolds are also stochastically complete. However, it is known that the converse statement is false, as Scherk's first surface is transient and has bounded curvature, thus stochastically complete, see [20, 29].

It is curious that the difference between the nature of intersection results for the class of minimal and H-surfaces is revealed by the threshold $\lambda = 0$ and $\lambda > 0$ in the Liouville properties, translated as the conditions (a) and (b). When M is a stochastically complete surface of $\mathbb{R}^3$ satisfying $\sup_M |H_M| = \inf_N H_N > 0$, we are able to prove only that $\mathrm{dist}(M,N) = 0$, which can be seen as a version of Theorem 5.1 in [39] for surfaces with positive mean curvature.

Theorem 1.10. *Let M be a stochastically complete surface, and let N be a complete proper surface with bounded curvature, immersed in $\mathbb{R}^3$. If N is well-oriented with respect to M and* $\sup_M |H_M| = \inf_N H_N > 0$, *then* $\mathrm{dist}(M,N) = 0$.

Theorem 1.10 can be restated in terms of relevant geometric conditions, sufficient for stochastic completeness as follow.

Corollary 1.11. *Let N be a complete embedded H-surface of $\mathbb{R}^3$ with bounded curvature, and let M be an H-surface of $\mathbb{R}^3$ immersed in the mean convex side of N. Then,* $\mathrm{dist}(M,N) = 0$ *provided that either*

(1) *M is properly immersed* (*cf. Proposition* 4 *in* [25]*, or* [44] *and references therein*),

(2) *M has curvature* $K_M(x) \geq -\rho^2(x)$, $\rho = \mathrm{dist}_M(x_o, x)$ (*see Theorem* 15.4 *in* [16] *and references therein*),

(3) *M has volume growth* $\mathrm{vol}(B_o(r) \cap M) \leq A\, e^{r^2}$, $A > 0$ (*see Theorem* 9.1 *in* [16] *and references therein*).

Finally, we notice that the techniques developed to prove our results can be adapted to prove the maximum principle at infinity between parabolic surfaces with non-empty boundary (possibly non-compact) and complete surfaces with bounded curvature immersed in $\mathbb{R}^3$. Several versions of the maximum principle at infinity were proved in [26,36,50] for minimal surfaces, and in [10,11] for H-surfaces, and later generalized in [14,37,46].

Theorem 1.12. *Let M and N be two disjoint immersed surfaces of a complete flat three-manifold P. Assume that M is parabolic with boundary $\partial M \neq \emptyset$, and that N is a complete surface with bounded curvature.*

(1) *If both are minimal surfaces, then* $\mathrm{dist}(M, N) = \mathrm{dist}(\partial M, N)$.

(2) *If* $\sup_M |H_M| \leq \inf_N H_N$, $\inf_N H_N > 0$, *N is two-sided, proper and well-oriented with respect to M, then* $\mathrm{dist}(M, N) = \mathrm{dist}(\partial M, N)$.

From the stochastic viewpoint, a surface M with boundary ∂M is said to be parabolic if the absorbed Brownian motion is recurrent, that is, if any Brownian path, starting from an interior point of M, reaches the boundary (and dies) in a finite time with probability 1 (see [41]). From a potential-theoretic point of view (cf. Proposition 10 in [42]), it is equivalent to the following Ahlfors maximum principle: every bounded solution $u \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(\mathrm{int}\, M)$ of the differential inequality $\triangle u \geq 0$ on $\mathrm{int}\, M$ must satisfy

$$\sup_M u = \sup_{\partial M} u.$$

This notion of parabolicity for surfaces with boundary is weaker than the natural definition for which the Brownian motion reflects at ∂M (see [21,42]).

Throughout this paper, all the surfaces are smooth, orientable, connected and with empty boundary, unless stated otherwise.

2. Preliminaries

Let $\varphi\colon M \to P$ be an isometric immersion of a complete manifold M into a complete Riemannian manifold P. The limit set $\lim \varphi$ of φ is defined as

$$\lim \varphi = \{y \in P : \exists \{x_k\}_{k=1}^{\infty} \subset M,\ x_k \to \infty,\ \varphi(x_k) \to y\}.$$

Observe that $\lim \varphi$ is closed , and notice that $\lim \varphi = \emptyset$ if and only if φ is proper.

In this section, we are going to sketch the local structure of $\lim \varphi$ when $\varphi\colon M \to P$ is a minimal hypersurface with bounded curvature immersed in a complete Riemmanian manifold P with bounded curvature. For $\lim \varphi \neq \emptyset$, we take $y \in \lim \varphi$ and a sequence $\{x_k\} \to \infty$ in M with $\varphi(x_k) \to y$ in P. By Gauss equations, the second fundamental form α of φ is uniformly bounded in the set $\Omega_y(1) = \varphi^{-1}(B_y(1))$, where $B_y(1)$ is the geodesic ball of P centred at y with radius 1. This implies that there are $r_0, r_1 > 0$, depending on $\sup_{\Omega_y}\|\alpha\|$ and on the injective radius $\mathrm{inj}_P(y)$ of P at y, such that for each $x_k\in\Omega_y(1)$, the ball $B_{\varphi(x_k)}(r_1)\subset\varphi(M)$ is graph of a smooth function $u_k\colon B_0(r_0)\subset T_{x_k}M\to\mathbb{R}$. Here $T_{x_k}M$ is the tangent space of M at x_k. This sequence of graphs converge, passing to a subsequence if necessary, to a minimal graph $y\in S_y\subset P$ with same curvature bounds of M containing a geodesic ball of radius $\varepsilon=\varepsilon(r_0,r_1)>0$ centred at y. This sketches the proof of the following lemma (more details can be found in [3,4,47]).

Lemma 2.1. *Let $\varphi\colon M\to N$ be a complete non-proper minimal hypersurface with bounded curvature immersed into a complete Riemannian manifold P with bounded curvature. For each $y\in\lim\varphi$, there exist $r_0,r_1>0$, depending on φ and y, and a sequence of balls $B_{x_k}(r_1)\subset M$, graphs of $u_k\colon B_0(r_0)\subset T_{x_k}M\to\mathbb{R}$, converging uniformly in the C^∞-topology to a minimal hypersurface $S_\infty\subset\lim\varphi$ containing a geodesic ball of radius $\varepsilon=\varepsilon(r_0,r_1)>0$ centred at y. Moreover, if P has positive injective radius, then S_∞ can be extended to a complete minimal surface lying in $\lim\varphi$ with bounded curvature.*

With the aid of Lemma 2.1, it was proved in Theorem 1.2 of [3] that the limit set of a complete non-proper minimal surface $\varphi\colon M\to\mathbb{R}^3$ immersed with bounded curvature in a mean convex region $\Omega\subsetneq\mathbb{R}^3$ is a union of parallel planes lying in the interior of a slab or in a half-space inside Ω. Describing the structure of the limit set of a complete non-proper minimal hypersurfaces of $\mathbb{R}^n$ is an intriguing problem. For instance, what is the Hausdorff dimension of the limit set of a non-proper minimal hypersurface $\varphi\colon M\to\mathbb{R}^n$? When M has bounded curvature, then $\dim_{\mathcal{H}}(\lim\varphi)\geq 2$. In [30], Martin and Nadirashvili constructed a complete minimal immersion of the disk $\varphi\colon\mathbb{D}\to\mathbb{R}^3$ whose limit set is a non-rectifiable Jordan curve β of Hausdorff dimension 1.

In the following, we present two examples of recurrent minimal surfaces that are non-proper and have unbounded curvature. The first example is a complete, non-proper minimal immersion of $\mathbb{C}$ into $\mathbb{R}^3$ whose sectional curvature decay to zero along the lines $t\to te^{\theta}$, $\theta\neq k\pi/4$, $k=1,3,5,7$, and decay quadratically to $-\infty$ along the lines $\theta=k\pi/4$, $k=1,3,5,7$, with limit set having four points only. The second example, following ideas from Andrade [1], is a non-trivial geodesically incomplete minimal immersion of $\mathbb{C}$ into $\mathbb{R}^3$ with unbounded curvature whose closure in $\mathbb{R}^3$ has non-empty interior.

Example 2.2. Let $f,g\colon\mathbb{C}\to\mathbb{C}$ be the entire functions given by

$$f(z)=\frac{2}{\sqrt{\pi}}e^{r_1z^2}\quad\text{and}\quad g(z)=e^{-r_2z^2},$$

with the constants r_1 and r_2 satisfying either $r_2>r_1>0$ or $2r_2>r_1>r_2$. These functions define a minimal immersion $\chi\colon\mathbb{C}\to\mathbb{R}^3$ by $\chi(z)=(x_1(z),x_2(z),x_3(z))$ (cf. [2]), where

$$\begin{aligned}
x_1(z)&=\mathrm{Re}\int^z\frac{1}{2}(1-g^2)f\,dz=\frac{1}{\sqrt{\pi}}\mathrm{Re}\int^z\left(e^{r_1z^2}-e^{(r_1-2r_2)z^2}\right)dz,\\
x_2(z)&=\mathrm{Re}\int^z\frac{i}{2}(1+g^2)f\,dz=-\frac{1}{\sqrt{\pi}}\mathrm{Im}\int^z\left(e^{r_1z^2}+e^{(r_1-2r_2)z^2}\right)dz,\\
x_3(z)&=\mathrm{Re}\int^z gf\,dz=\frac{2}{\sqrt{\pi}}\mathrm{Re}\int^z e^{(r_1-r_2)z^2}\,dz.
\end{aligned}$$

For simplicity, we will consider $r_2 = 5$ and $r_1 = 1$. In this case, the above integrals defining $\chi\colon \mathbb{C} \to \mathbb{C} \times \mathbb{R}$ are given explicitly in terms of the error function and the imaginary error function:

$$\chi(z) = \Big(\frac{\operatorname{erfi}(\bar{z})}{2} - \frac{\operatorname{erf}(3z)}{6}, \frac{\operatorname{Re}[\operatorname{erf}(2z)]}{2}\Big), \tag{2.1}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \qquad \text{and} \qquad \operatorname{erfi}(z) = -\frac{2i}{\sqrt{\pi}} \int_0^{iz} e^{-t^2} dt.$$

The minimal surface χ is geodesically complete since its induced metric satisfies

$$ds = \frac{1}{2} |f| (1 + |g|^2) |dz| \geq \frac{1}{\sqrt{\pi}} |dz|.$$

The Gaussian curvature of χ is given by

$$K(z) = -\Big[\frac{20\sqrt{\pi}\, |z|}{(e^{3\operatorname{Re}(z^2)} + e^{-7\operatorname{Re}(z^2)})^2}\Big]^2.$$

For a small $\varepsilon > 0$, set

$$\mathbb{C}(\varepsilon) = \bigcup_{i=1}^{4} \Big\{ z = |z|e^{i\theta} \in \mathbb{C} : \theta \in \Big(\frac{(2i-1)\pi}{4} - \varepsilon, \frac{(2i-1)\pi}{4} + \varepsilon\Big)\Big\}.$$

The curvature of χ at $z = te^{i\theta}$, for $\theta \notin \mathbb{C}(\varepsilon)$, is bounded by

$$-\frac{A t^2}{e^{D(\varepsilon) t^2}} \geq K(\chi(z)) \geq -\frac{A t^2}{e^{B(\varepsilon) t^2}},$$

with $A = 400\pi$, $B(\varepsilon) = 6\cos(\pi/2 - 2\varepsilon)$ and $D(\varepsilon) = 14\cos(\pi/2 - 2\varepsilon)$. However, along $\chi(te^{j\pi/4})$, $j = 1, 3, 5, 7$, the curvature is unbounded,

$$K(\chi(te^{j\pi/4})) = -100\pi t^2.$$

Regarding the non-properness of χ, one can show that (see Appendix A)

$$\{q_{\pi/4}, q_{3\pi/4}, q_{5\pi/4}, q_{7\pi/4}\} \subset \lim \chi,$$

where

$$q_{\pi/4} = \Big(-\frac{1}{6}, -\frac{1}{2}, \frac{1}{2}\Big), \qquad q_{3\pi/4} = \Big(\frac{1}{6}, -\frac{1}{2}, -\frac{1}{2}\Big),$$
$$q_{5\pi/4} = \Big(\frac{1}{6}, \frac{1}{2}, -\frac{1}{2}\Big), \qquad q_{7\pi/4} = \Big(-\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\Big).$$

We plot below a piece of the curve $t \to \chi(\gamma_\theta(t))$ for $\theta = \pi/4,\ 5\pi/4$.

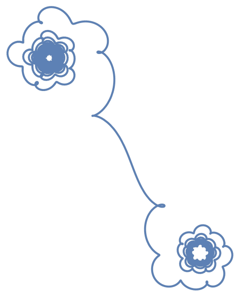

Figure 1. Curve $z(x) = x + ix$ in a surface with $r_1 = 1$ and $r_2 = 5$.

Example 2.3. Consider an Enneper immersion $\chi\colon \mathbb{C} \to \mathbb{C} \times \mathbb{R}$ given by

$$\chi(z) = (L(z) - \overline{H}(z), h(z)),$$

where L and H are holomorphic functions defined by

$$L(z) = (r_1 - r_2)\, e^z \quad \text{and} \quad H(z) = -d\, e^{(r_1/r_2-1)\,z},$$

and h is a harmonic function defined as follows:

$$h(z) = -4\Big(\frac{d}{r_2}\Big)^{1/2} \Big|\frac{r_2}{r_1}\Big|\, |r_1 - r_2|\, \mathrm{Re}\big(i\, e^{\frac{r_1}{2r_2}\, z}\big).$$

We assume some non-degenerate assumptions for the parameters $r_1, r_2, d \in \mathbb{R}$, namely, $r_1 \neq r_2$ and $r_1 r_2 d \neq 0$, as well as some extra technical conditions:

$$0 < r_1 < 4r_2 < 3r_1, \quad \frac{r_1}{r_2} \notin \mathbb{Q} \quad \text{and} \quad d = r_1 - r_2 > 0.$$

Following the same reasoning as in [1], it is possible to show that the immersion $\chi(u+iv)$ is dense in an open subset of $\mathbb{R}^3$, its Gaussian curvature $K(u+iv) \to -\infty$, and $ds^2 = \lambda^2(u+iv)|dz|^2 \to 0$ as $u \to -\infty$.

3. Proof of Theorem 1.1

We first observe that if N is a complete flat minimal surface, thus a plane, such that $M \cap N = \emptyset$, the associated height function restricted to M is harmonic and bounded on one side, so constant by recurrence. Hence M is a flat surface contained in a parallel plane to N.

In what follows, we are going to consider the case where N is not flat. We will split the proof in two steps. In the first we address the case where N is embedded and in the second step we improve the argument in the Step 1 to treat the general case. It should be remarked that differently from [14] and [53] we do not assume that the submanifold is contained in a region whose boundary offers a natural barrier.

Step 1. The case N embedded.

If N is embedded then there is a tubular ε-neighborhood $U(\varepsilon) = T_\varepsilon(N)$ which is embedded for every $0 < \varepsilon \leq (\sqrt{3}-1)/2|\Lambda|$, where $K_N \geq -\Lambda^2$ (see Theorem 2 in [51]).

For our purposes, we are going to consider $0 < \varepsilon < 1/2|\Lambda|$. Since N is embedded in $\mathbb{R}^3$, it is two-sided. Hence, we can choose a smooth normal vector field η to N and decompose the tubular neighborhood $U(\varepsilon) = U_-(\varepsilon) \cup N \cup U_+(\varepsilon)$, where η points toward the connected component $U_+(\varepsilon)$. Let $t\colon U(\varepsilon) \to \mathbb{R}$ be the signed distance function defined by

$$t(y) = \begin{cases} \operatorname{dist}(y, N) & \text{if } y \in U_+(\varepsilon), \\ -\operatorname{dist}(y, N) & \text{if } y \in U_-(\varepsilon). \end{cases}$$

Since recurrence is invariant by isometries, up to an Euclidean translation, we may assume that $\operatorname{dist}_{\mathbb{R}^3}(M, N) = 0$, and $M \cap U_+(\varepsilon/4) \neq \emptyset$, replacing η by $-\eta$ if necessary.

Define $F\colon U_+(\varepsilon) \subset \mathbb{R}^3 \to \mathbb{R}$ by

$$F(y) = \log\Big(\frac{2 + \varepsilon c}{2 + 4ct(y)}\Big), \tag{3.1}$$

with $c \doteq |\Lambda| = \sup \kappa > 0$, where $\kappa \doteq \kappa_2 \geq 0$ is the non-negative principal curvature of N, and the principal curvatures κ_i are computed with respect to the direction pointing towards $U_+(\varepsilon)$. Let $u\colon \varphi^{-1}(U_+(\varepsilon)) \to \mathbb{R}$ be given by $u = F \circ \varphi$, where $\varphi\colon M \to \mathbb{R}^3$ is the isometric minimal immersion of M into $\mathbb{R}^3$. Clearly u is smooth and bounded. Moreover, u is non-constant. For if we would have that M is contained in a parallel surface N^t of N, and by the evolution of the principal curvatures of N (see Corollary 3.5 in [15]),

$$(\kappa_i^t)' = (\kappa_i^t)^2 \quad \text{and} \quad (H^t)' \geq (H^t)^2,$$

we would have that $H^t \equiv 0$, and consequently $\kappa_i^t \equiv 0$. But this contradicts the fact that N is not flat. Therefore u is non-constant and $u \equiv 0$ on $\varphi^{-1}(\partial U_+(\varepsilon/4))$. We claim that

$$\triangle_M u \geq 0 \quad \text{on } \varphi^{-1}(U_+(\varepsilon/2)).$$

Indeed, consider the foliation N^t by parallel surfaces to N for $t \in (0, \varepsilon)$. For each $y \in N^t \cap U_+(\varepsilon/2) \cap M$ with coordinates $(x, t) \in N \times (0, \varepsilon/2)$, there exists an orthonormal basis $\{E_1, E_2\} \subset T_y N^t$ such that $\{E_1, E_2, \nabla t\}$ diagonalize the Hessian $\operatorname{Hess}_{\mathbb{R}^3} F$. An easy computation yields

$$\nabla_{\mathbb{R}^3} F = -\frac{2c}{1 + 2ct} \nabla t \quad \text{and} \quad \operatorname{Hess}_{\mathbb{R}^3} F = \frac{4c^2}{(1 + 2ct)^2} \nabla t \otimes \nabla t - \frac{2c}{1 + 2ct} \nabla^2 t.$$

Then, with respect to the splitting $\mathbb{R}\nabla t \oplus T_y N^t$, the eigenvalues of $\operatorname{Hess}_{\mathbb{R}^3} F$ are

$$\mu_1 = -\frac{2c}{1 + 2ct} \frac{\kappa}{1 + t\kappa}, \quad \mu_2 = \frac{2c}{1 + 2ct} \frac{\kappa}{1 - t\kappa} \quad \text{and} \quad \mu_3 = \frac{4c^2}{(1 + 2ct)^2}.$$

Since $0 < 2t \leq \varepsilon < 1/2|\Lambda| = 1/2c$, the monotonicity $\mu_1 \leq \mu_2 < \mu_3$ holds. Therefore, applying Lemma 2.3 in [23] for the 2-dimensional subspace $W \doteq T_y M$ of $\mathbb{R}^3$, we have

$$\triangle_M u = \operatorname{Tr}_{TM} \operatorname{Hess}_{\mathbb{R}^3} F_{|W} \geq \mu_1 + \mu_2 = \frac{2c}{1 + 2ct} \frac{2t\kappa^2}{1 - t^2\kappa^2} \geq 0.$$

Observe that $M \cap U_+(\varepsilon/4) \neq \emptyset$ and

$$u|\varphi^{-1}(U_+(\varepsilon/4)) > 0, \quad u|\varphi^{-1}(\partial U_+(\varepsilon/4)) \equiv 0, \quad u|\varphi^{-1}(U_+(\varepsilon/2) \setminus U_+(\varepsilon/4)) < 0.$$

Therefore, extending u negatively outside of $\varphi^{-1}(U_+(\varepsilon/2))$ and defining $\bar{u}: M \to \mathbb{R}$ by $\bar{u} = \max\{u, 0\}$, we may have a continuous, bounded subharmonic function in the sense of distributions on a recurrent manifold. However, by the Liouville property, it must be constant (cf. Theorem 5.1 in [16]). A contradiction.

Step 2. The case N immersed.

As we have seen in the embedded case, we need to construct a bounded weak solution $u \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ to the differential inequality $\triangle u \geq 0$ on M. In view of Step 1, it is natural to consider $u = F \circ \varphi: \varphi^{-1}(U(\varepsilon/2)) \to \mathbb{R}$, where $\varphi: M \to \mathbb{R}^3$ is an isometric immersion and $F: \mathbb{R}^3 \to \mathbb{R}$ is given as in (3.1) by a composition $F = g \circ t_N$ of the distance function from N, namely t_N, and a smooth real valued function g. However, the distance function to N is only Lipschitz continuous in general. The non-smoothness may occur when the set of self intersections $\Gamma \subset \mathbb{R}^3$ of N is non-empty, and the function $P: U(\varepsilon) \to \overline{N}$ given by

$$P(y) = \{z \in \overline{N} : \mathrm{dist}_{\mathbb{R}^3}(y, z) = \mathrm{dist}_{\mathbb{R}^3}(y, N)\}$$

is a multivalued function. Also, focal points are points at which t_N is not twice differentiable. We will search for solutions to the differential inequality $\triangle u \geq 0$ in the barrier sense. Recall that a function u is said to satisfy $\triangle u \geq 0$ at a point q in the barrier sense if, for any $\delta > 0$, there exists a smooth function ϕ_δ around q such that

$$\begin{cases} \phi_\delta = u & \text{at } q, \\ \phi_\delta \leq u & \text{near } q, \end{cases} \quad \text{and} \quad \triangle\phi_\delta(q) > -\delta.$$

To overcome these regularity issues, we introduce the following lemma, which could be of independent interest and should be compared to Lemma 2.2 in [6] and Lemma 1.3 in [13].

Lemma 3.1. *Let N be a complete minimal surface immersed in $\mathbb{R}^3$ with bounded curvature. For any immersed minimal surface $\varphi: M \to \mathbb{R}^3$ such that $M \cap N = \emptyset$, there exist $\varepsilon > 0$ and a bounded function $u: M \to \mathbb{R}$, $u = u(t_N)$ decreasing on t_N, satisfying*

$$\triangle u \geq 0 \quad \text{in the barrier sense}$$

on the subset $\Omega_\varepsilon = \{q \in M : 0 \leq 2t_N(\varphi(q)) < \varepsilon\}$, where t_N denotes the distance function from N.

Proof. Since N is a complete minimal surface immersed in $\mathbb{R}^3$ with bounded curvature, by Lemma 2.1, for each $x \in \overline{N}$, there exists a complete minimal surface $x \in L_x \subset \overline{N}$ with bounded curvature $K_{L_x} \geq -\Lambda^2$. We will split the proof in two cases. First, assume that $\overline{N} \cap M = \emptyset$, that is, $t_N(\varphi(q)) > 0$ for all $q \in M$. For a fixed $y \in U(\varepsilon/2)$ and for each $z \in P(y)$, there exists a locally embedded neighborhood $V_z \subset L_z \subset \overline{N}$ of z that is graph over an open ball $W_z \subset T_z L_z$ with radius uniformly bounded from below. Along each neighborhood V_z, we can consider a regular tubular neighborhood

$$C_z(\varepsilon) = \{\exp^{\perp}(tv) : v \in TV_z^{\perp}, t \in (-\varepsilon, \varepsilon)\} \tag{3.2}$$

with radius $\varepsilon > 0$ and define the oriented distance function to V_z, $t_z: C_z(\varepsilon) \to \mathbb{R}$, such that $t_z(y) > 0$. This yields the following split: $C_z(\varepsilon) = C_z^+(\varepsilon) \cup V_z \cup C_z^-(\varepsilon)$. In order to construct a smooth support function, we may select one neighborhood V_z for some $z \in P(y)$. To avoid the analysis of the non-focal points of the cut locus of the boundary ∂V_z of the surface V_z, and as in Section 2 of [14], we will consider a supporting surface S_z for $C_z^+(\varepsilon)$ at $z \in V_z$, that is, a smooth surface such that $z \in S_z$ and $C_z^+(\varepsilon) \cap S_z = \emptyset$. Following the agreement in [14], modifying S_z outside of a tiny neighborhood of z, we may assume that S_z is the boundary of a small topological ball $B_{S_z} \subset C_z^-(\varepsilon)$. For any given $\mu > 0$, we can find a supporting surface S_z^μ for $C_z^+(\varepsilon)$ at $z \in V_z$, with $B_{S_z^\mu} \subset B_{S_z}$, and whose mean curvature H^μ satisfies

$$H^\mu(z) > -\mu,$$

see Lemma 2.1 in [14]. This supporting surface can be constructed by deforming smoothly the boundary of a small ball $B \subset C_z^-(\varepsilon)$ touching V_z at z. Thus, since N has bounded curvature, the principal curvatures κ_1 and κ_2 of S_z^μ at z can be chosen to be uniformly bounded for all $z \in P(y)$ and all $y \in N$, $0 < \mu < 1$. We denote by $0 < c = \max\{\kappa_1, \kappa_2\} < \infty$, and for $0 < \varepsilon < 1/2c$, we define the function $u \colon \varphi^{-1}(U(\varepsilon/2)) \to \mathbb{R}$ by $u = F \circ \varphi$, where $F = g \circ t_N$ and

$$g(t) = \log\Big(\frac{2 + \varepsilon c}{2 + 4ct}\Big).$$

The following lemma says that we can choose S_z^μ so that $y \notin \mathrm{cut}(S_z^\mu)$.

Lemma 3.2 (Lemma 2.1 in [14]). *Fix $y \in U(\varepsilon/2)$ and a nearest point $z \in V_z$ to y. For a supporting surface S_z at z, there exists S_z', close to S_z in the C^∞ topology in a neighborhood of z, still supporting surface at z, and such that $y \notin cut(S_z')$.*

We pick a point $q \in M$ such that $y = \varphi(q) \in U(\varepsilon/2)$, $z \in P(y)$ and select a neighborhood V_z as described above. Given $\delta > 0$, we will consider as a support function to u at q, the function $\phi_\delta \doteq F_z^\mu \circ \varphi$, where $F_z^\mu = g \circ t_z^\mu$, and t_z^μ is the oriented distance function to S_z^μ with $t_z^\mu(y) > 0$. Moreover, by Lemma 3.2, we may assume that S_z^μ is a supporting surface for which $y \notin \mathrm{cut}(S_z^\mu)$ for some $\mu = \mu(\delta)$ to be chosen later. To show that t_z^μ supports t_N, one should guarantee that points near y can be joined to S_z^μ by at least one segment crossing V_z, but this is the case since we are in the Euclidean space. Hence, the support function ϕ_δ is smooth in a small neighborhood of q. Furthermore, $\phi_\delta(q) = u(q)$, and taking a small ball $B_\eta(y) \subset U(\varepsilon/2)$ centred at y and with radius $\eta > 0$, for every $\zeta \in B_\eta(y)$ it holds that

$$t_N(\zeta) \le t_z^\mu(\zeta).$$

The decreasing property of g in t asserts the inequality $\phi_\delta \le u$ near to q. In order to show that u satisfies $\triangle_M u(q) \ge 0$ in the barrier sense, we are going to prove

$$\triangle_M \phi_\delta(q) > -\delta.$$

Since $0 < 2t < \varepsilon \le 1/2c$, following up computations from Step 1, we have

$$\mathrm{Hess}_{\mathbb{R}^3} F_z^\mu = \frac{4c^2}{(1 + 2ct_z^\mu)^2} \nabla t_z^\mu \otimes \nabla t_z^\mu - \frac{2c}{1 + 2ct_z^\mu} \nabla^2 t_z^\mu,$$

whose eigenvalues are given by

$$\mu_1 = \frac{2c}{1 + 2ct_z^\mu} \kappa_1^t, \quad \mu_2 = \frac{2c}{1 + 2ct_z^\mu} \kappa_2^t \quad \text{and} \quad \mu_3 = \frac{4c^2}{(1 + 2ct_z^\mu)^2},$$

where
$$\kappa_1^t = \frac{\kappa_1}{1 - t_z^\mu \kappa_1} \quad \text{and} \quad \kappa_2^t = \frac{\kappa_2}{1 - t_z^\mu \kappa_2}$$
are the principal curvatures of the parallel surfaces to S_z^μ at y. We first observe that, independently of the sign of κ_i ($i = 1, 2$), it holds

$$\mu_i \geq \frac{2c}{1 + 2ct_z^\mu} \kappa_i, \quad \text{for } i = 1, 2. \tag{3.3}$$

The restriction on $\varepsilon \leq 1/2c$ and the inequality above give us the monotonicity $\mu_1 \leq \mu_2 < \mu_3$. Again, applying Lemma 2.3 in [23] and inequality $H_z^\mu > -\mu$, we can write

$$\triangle_M \phi_\delta \geq \mu_1 + \mu_2 \geq \frac{2c}{1 + 2ct_z^\mu} H_z^\mu \geq -\frac{2c\mu}{1 + 2ct_z^\mu}.$$

Then choose $\mu \doteq \delta/2c$ to conclude that $\triangle \phi_\delta > -\delta$.

It remains to consider the case $\overline{N} \cap M \neq \emptyset$. In this case, there are points $q \in M$ such that $t_N(\varphi(q)) = 0$ and $\varphi(q) \in \mathcal{L}_{\varphi(q)} \subset \lim N$ for some minimal leaf. However, by the maximum principle for minimal surfaces, M and $\mathcal{L}_{\varphi(q)}$ must coincide locally along an open disk $D_q \subset M$. Therefore, for every $p \in D_q$, we have $t_N(\varphi(p)) = 0$, and the function u must be constant on D_q. Therefore, the function $\phi(x) = \log(1 + \varepsilon c/2)$ is a barrier function at every point $q \in M$ such that $t_N(\varphi(p)) = 0$, and the proof is finished. ∎

To conclude the proof in the immersed case, we first assume that $\overline{N} \cap M = \emptyset$. In this case, we apply Lemma 3.1 to show that the function u is bounded and subharmonic in the barrier sense, therefore in the viscosity sense in $\varphi^{-1}(U(\varepsilon/2))$. As in Step 1, we extend u and recall that $u > 0$ on $\varphi^{-1}(U(\varepsilon/4))$ to define $\bar{u} = \max\{u, 0\}$ in the whole M. By Theorem 1 in [22], $\bar{u}$ is a non-negative subharmonic in the sense of distributions and $\bar{u} \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$. We achieve the same contradiction as in the embedded case from the Liouville property, see Theorem 5.1 in [16]; see also [27, 28] for a direct proof for viscosity solutions. If $\overline{N} \cap M \neq \emptyset$, then either $M \cap N \neq \emptyset$, or M is contained in some complete minimal leaf $\mathcal{L}$ of $\lim N$. Note that $\mathcal{L}$ has also bounded curvature, so by Theorem 1.1 in [3], N must intersect $\mathcal{L}$. Since M has empty boundary, $\mathcal{L} \setminus M$ is an union of points, and it is not possible that $M \cap N = \emptyset$.

4. Proof of Theorem 1.4

In the proof of Theorem 1.4, we intend to make explicit how the geometry of the ambient space influences this kind of intersection problem for minimal hypersurfaces. We will follow the same strategy applied in the proof of Theorem 1.1. The main difference appears to be in the way to compute the Laplacian of the selected function, where we will use the ideas from [31].

Recall that our assumptions on the sectional and Ricci curvatures of M imply a uniform bound for the sectional curvature of the product ambient space $M \times \mathbb{R}$. Since N is a minimal hypersurface with bounded sectional curvature, the Gauss equation gives us a uniform bound for the second fundamental form of N. As a consequence of the extended

Rauch theorem (see Corollary 4.2 in [52]), there exists a real value $\varepsilon > 0$ such that, for every normal geodesic σ issuing from a point $\sigma(0) \in N$, there is no focal points on $\sigma_{|[0,\varepsilon)}$. By Proposition 4.4 in Chapter X of [12], this means that the restriction of the exponential map $\exp^{\perp}\colon (TN)^{\perp} \to M \times \mathbb{R}$ has no critical values in the tubular neighborhood $U(\varepsilon)$.

Following ideas from Lemma 3.1, we notice that for each point $y \in U(\varepsilon)$, the projection points that realize the distance from y to N are contained in N or in the limit set of N. If a projection point z lies on N, it is contained in an geodesic disk $V_z \subset N$ which is embedded and has radius uniformly bounded from below. On the other hand, if the projection point z lies on the limit set of N, since M has a uniform bound from below for the injective radius, we can apply Lemma 2.1 to guarantee the existence of a minimal geodesic disk V_z with radius uniformly bounded from below, and contained in the limit set of N. For each V_z, we associate the regular tubular neighborhood $C_z(\varepsilon) = C_z^+(\varepsilon) \cup V_z \cup C_z^-(\varepsilon)$ as defined in (3.2).

Let us consider a slice $M \times \{s\}$, still called M, such that $\operatorname{dist}(M, N) = 0$. Define a function $F\colon U(\varepsilon) \to \mathbb{R}$ given by $F = g \circ t_N$, where

$$g(t) = \log\left(\frac{2 + \varepsilon c}{2 + 4ct}\right), \tag{4.1}$$

$t_N\colon U(\varepsilon) \to \mathbb{R}$ denotes the distance function to N, and $c > 0$ is a constant which will depend on ε and the principal curvatures of N. Consider the function $u\colon \varphi^{-1}(U(\varepsilon/2)) \to \mathbb{R}$ given by $u = F \circ \imath$, with $\imath\colon M \to M \times \mathbb{R}$, $\imath(x) = (x, 0)$, being the inclusion isometric immersion. Again, our main claim is that u is a subharmonic function in the barrier sense on the subset $\varphi^{-1}(U(\varepsilon/2))$. Here is where the assumption on the injectivity radius enters into play. Up to reducing $\varepsilon > 0$, we can consider that 2ε is less than the injectivity radius of $M \times \mathbb{R}$.

Take a point $q \in M$ such that $y = \imath(q) \in C_z^+(\varepsilon/2)$, that is, we assume $t_N(y) > 0$, and select a projection point $z \in \overline{N}$ with a neighborhood V_z constructed as above. For any $\delta > 0$ given, we will take $\mu = \mu(\delta) > 0$, to be chosen later, and a supporting surface $S_z^\mu \subset C_z^-(\varepsilon/2)$ as in Lemma 3.2 satisfying

$$y \notin \operatorname{cut}(S_z^\mu) \quad \text{and} \quad H_z^\mu(z) > -\mu.$$

Since $y \in C_z^+(\varepsilon/2)$, there exists a minimizing geodesic $\gamma \subset C_z^+(\varepsilon/2)$ with end points y and z. We claim that there is a supporting surface $S_z^\mu \subset C_z^-(\varepsilon/2)$ such that the distance function to S_z^μ, namely t_z^μ, is a smooth supporting function from above to t_N. Indeed, assume the contrary. Then there exist balls $B_y(r_j)$, with $r_j \to 0$, points $y_j \in B_y(r_j)$, and support surfaces $S_{z,j}^\mu$ such that there is a minimizing geodesic σ_j from y_j to $S_{z,j}^\mu$ that does not intersect N. However, passing to a subsequence of σ_j, if necessary, we will find another minimizing geodesic from y to z different from γ, which contradicts our choice of ε being less than the injectivity radius of M.

Thus, a support function to $u = F \circ \imath$ at q is given by $\phi_\delta \doteq F_z^\mu \circ \imath$, where $F_z^\mu = g \circ t_z^\mu$ and $\mu = \mu(\delta)$.

It remains to compute the Laplace operator of ϕ_δ. For this, we take along M an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}\}$ of $TM \times \mathbb{R}_+$ such that $e_{n+1} = \partial/\partial t$. Using this basis, and since M is totally geodesic in $M \times \mathbb{R}_+$, we can write

$$\triangle_M \phi_\delta = g'(t_z^\mu) \sum_{i=1}^{n} \mathrm{Hess}_{M\times\mathbb{R}_+} t_z^\mu(e_i, e_i) + g''(t_z^\mu) \sum_{i=1}^{n} \langle \nabla_{M\times\mathbb{R}_+} t_z^\mu, e_i\rangle^2. \tag{4.2}$$

Let S_t be the parallel surface to S_z^μ given by the image of a exponential map at time t, and denote by $\kappa_1^t, \dots, \kappa_n^t$ its associated principal curvatures in direction ∇t_z^μ. Let $\{a_1^t, \dots, a_n^t\}$ be an orthonormal basis of TS_t which diagonalizes the shape operator of S_t, and set $a_{n+1}^t = \eta$, where η is the normal vector field along S_t pointing towards M. The matrix of change of basis from e_i to a_i^t has the elements $(\lambda_{ij})_{1\le i,j\le n+1}$ defined by

$$e_i = \sum_{j=1}^{n+1} \lambda_{ij}\, a_j^t.$$

With the above notation, we can rewrite (4.2) as

$$\triangle_M \phi_\delta = g'(t_z^\mu)\,(-\kappa_1^t(1-\lambda_{n+1,1}^2) - \cdots - \kappa_n^t(1-\lambda_{n+1,n}^2)) + g''(t_z^\mu)\,(1-\lambda_{n+1,n+1}^2).$$

A main fact we shall use is the monotonicity of the mean curvature of the parallel surfaces along normal geodesics issuing from S_z^μ, see (3.3). A sufficient condition for this monotonicity to hold is given by non-negativeness of the Ricci curvature of the ambient space, which in our case is guaranteed by the hypothesis $\mathrm{Ric}_M \ge 0$. In fact, from the evolution equation of the principal curvatures (cf. Corollary 3.5 in [15]), we have

$$(\kappa_i^t)^2 - \Lambda^2 \le (\kappa_i^t)' = (\kappa_i^t)^2 + K(\eta, a_i^t) \le (\kappa_i^t)^2 + \Lambda^2$$

and

$$(H^t)' \ge (H^t)^2 + \mathrm{Ric}(\eta, a_i^t) \ge 0,$$

where $K(\eta, a_i^t)$ is the sectional curvature of $M \times \mathbb{R}$ on the 2-plane spanned by $\{a_i^t, \eta\} \subset TS_t$, and $\Lambda^2 = \sup |K(\eta, a_i^t)|$ for all $0 < t < \varepsilon$. The monotonicity for H^t follows directly from the latter inequality, and the former guarantees that we can define the constant $c > 0$ used in (4.1) as

$$c \doteq \sup_{[0,\varepsilon]} \max\{|\kappa_1^t|, \dots, |\kappa_n^t|\} + 1 < \infty. \tag{4.3}$$

Therefore, using this monotonicity property and recalling that $g'(t_z^\mu) < 0$, we can estimate

$$\triangle_M \phi_\delta \ge g'(t_z^\mu)\,(\mu + \kappa_1^t\,\lambda_{n+1,1}^2 + \cdots + \kappa_n^t\,\lambda_{n+1,n}^2) + g''(t_z^\mu)\,(1-\lambda_{n+1,n+1}^2).$$

We can represent the unitary vector $(\lambda_{n+1,1}, \lambda_{n+1,2}, \dots, \lambda_{n+1,n+1})$ using the n-dimensional spherical coordinates $(\theta_1, \theta_2, \dots, \theta_n) \in [0,\pi] \times \cdots \times [0,\pi] \times [0, 2\pi]$ as follows:

$$\begin{aligned}
\lambda_{n+1,n+1} &= \cos\theta_1,\\
\lambda_{n+1,n} &= \sin\theta_1 \cos\theta_2,\\
\lambda_{n+1,n-1} &= \sin\theta_1 \sin\theta_2 \cos\theta_3,\\
&\;\;\vdots\\
\lambda_{n+1,2} &= \sin\theta_1 \cdots \sin\theta_{n-1} \cos\theta_n,\\
\lambda_{n+1,1} &= \sin\theta_1 \cdots \sin\theta_{n-1} \sin\theta_n.
\end{aligned}$$

Applying these coordinates in the above estimate for the Laplacian, together with (4.3), we get

$$\begin{aligned}\triangle_M \phi_\delta &\geq g'(t_z^\mu)\big(\mu + \kappa_1^t(\sin\theta_1\cdots\sin\theta_{n-1}\sin\theta_n)^2 + \cdots + \kappa_n^t(\sin\theta_1\cos\theta_2)^2\big)\\ &\quad + g''(t_z^\mu)(1-\cos^2\theta_1)\\ &\geq c\,g'(t_z^\mu)\big((\sin\theta_2\cdots\sin\theta_{n-1}\sin\theta_n)^2 + \cdots + (\sin\theta_2\cos\theta_2)^2\big)\sin^2\theta_1\\ &\quad + \mu g'(t_z^\mu) + g''(t_z^\mu)\sin^2\theta_1\\ &\geq \mu g'(t_z^\mu) + (g''(t_z^\mu) + c\,g'(t_z^\mu))\sin^2\theta_1 \quad \text{on } \varphi^{-1}(U(\varepsilon/2)).\end{aligned}$$

As before, reducing ε if necessary, we can take $0 < 2t < \varepsilon \leq 1/2c$ and $\mu = \delta/2$. These choices lead us to conclude that

$$\triangle_M \phi_\delta > -\delta.$$

Therefore, once established that u is a bounded subharmonic function in the barrier sense on the subset $\varphi^{-1}(U(\varepsilon/2))$, we proceed extending u outside $\varphi^{-1}(U_+(\varepsilon/4))$ by zero, and showing that the function $\bar{u} = \max\{u, 0\}$ will satisfy $\bar{u} \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ and $\triangle_M \bar{u} \geq 0$ in the weak sense. Again, $\bar{u}$ will contradict the Liouville property.

5. Proof of Theorems 1.8 and 1.10

The proof of Theorems 1.8 and 1.10 follows the same strategy used in the proof of Theorem 1.1 in the immersed case. Let N be a complete proper surface immersed in $\mathbb{R}^3$ with bounded curvature, and let M be an immersed surface of $\mathbb{R}^3$. Suppose that N is well-oriented with respect to M. Definition 1.7 says that M lies in an open connected component W of $\mathbb{R}^3 \setminus N$ for which the mean curvature vector field $\vec{H}_N = H_N \xi$ along $\partial W \subset N$ points into W. We recall that the boundary of W is given as a union of smooth pieces of N with non-negative mean curvature H_N in the inward direction, and whose inner angles are not bigger than π along an intersection set Γ.

Similarly to the minimal case, there exists a regular tubular neighborhood $U_+(\varepsilon) \subset W$ with uniform radius $\varepsilon > 0$ depending only on the lower bound for the curvature and the norm of the second fundamental form of N. Let $t_N\colon U_+(\varepsilon) \to \mathbb{R}$ be the distance function to N, which is a positive Lipschitz function. For any point $y \in U_+(\varepsilon)$, it is not hard to see that the nearest points to y on ∂W cannot be on the part of Γ where the inner angle is less than to π, otherwise the minimizing segment connecting y to ∂W will be normal to two different tangent planes. Therefore, for any point $z \in \partial W$ that minimizes the distance to $y \in U_+(\varepsilon)$, and any $\mu > 0$, we can deform one smooth piece of N passing through z to obtain a smooth supporting surface S_z^μ for $U_+(\varepsilon)$ at $z \in \partial W$ with mean curvature $H_z^\mu(z) > H_N(z) - \mu$. Moreover, using Lemma 3.2, we can assume the oriented distance function to S_z^μ, here called t_z, is smooth around y and touches t_N from above at y.

Again, we set $c > 0$ to be the maximum norm of the principal curvatures of N and consider the function $F\colon U_+(\varepsilon) \subset \mathbb{R}^3 \to \mathbb{R}$ defined as $F = g \circ t_N$, where

$$g(t) = \log\Big(\frac{2+\varepsilon c}{2+4ct}\Big).$$

We also follow the convention $0 < \varepsilon \leq 1/2c$. Let $\varphi\colon M \to \mathbb{R}^3$ be the isometric immersion of M, and define the function $u = F \circ \varphi$ on $\varphi^{-1}(U_+(\varepsilon/2))$. Observe that, up to an isometry of $\mathbb{R}^3$, we may assume that $\varphi(M) \cap U_+(\varepsilon/4) \neq \emptyset$. We are going to prove that u is a solution, in the barrier sense, of the differential inequality

$$\triangle_M u \geq \frac{2c}{1+2ct_z}\Big(\inf_N H_N - \sup_M |H_M|\Big) \quad \text{on } \varphi^{-1}(U_+(\varepsilon/2)). \tag{5.1}$$

For any $x \in \varphi^{-1}(U_+(\varepsilon/2))$ and $\delta > 0$, let us consider $\phi_\delta \doteq F_z^\delta \circ \varphi$, where $F_z^\delta = g \circ t_z$ and t_z is the oriented distance function to $S_z^{\delta/2c}$ with $\varphi(x) \notin \operatorname{cut}(S_z^{\delta/2c})$. Then, the function ϕ_δ is a test function for u at x. Arguing along similar lines from the proof of Theorem 1.1, we see that

$$\operatorname{Hess}_{\mathbb{R}^3} F_z^\delta = \frac{4c^2}{(1+2ct_z)^2}\,\nabla t_z \otimes \nabla t_z - \frac{2c}{1+2ct_z}\,\nabla^2 t_z,$$

whose eigenvalues are

$$\mu_1 = \frac{2c}{1+2ct_z}\,\frac{\kappa_1}{1-t_z\kappa_1}, \quad \mu_2 = \frac{2c}{1+2ct_z}\,\frac{\kappa_2}{1-t_z\kappa_2} \quad \text{and} \quad \mu_3 = \frac{4c^2}{(1+2ct_z)^2},$$

where $\kappa_1 \leq \kappa_2$ are the ordered principal curvatures of $S_z^{\delta/2c}$. The monotonicity $\mu_1 \leq \mu_2 < \mu_3$ holds because $2\varepsilon c \leq 1$, as well as the inequality

$$\mu_i = \frac{2c}{1+2ct_z}\,\kappa_i^t \geq \frac{2c}{1+2ct_z}\,\kappa_i, \quad \text{for } i = 1, 2.$$

Applying Lemma 2.3 in [23], we get

$$\begin{aligned}\triangle_M \phi_\delta &= \operatorname{Tr}_{TM}\operatorname{Hess}_{\mathbb{R}^3} F_z^\delta + \langle \nabla_{\mathbb{R}^3} F_z^\delta, H_M\rangle \geq \mu_1 + \mu_2 - \frac{2c}{1+2c\,t_z}\,\sup_M |H_M| \\ &\geq \frac{2c}{1+2ct_z}\Big(H_N - \frac{\delta}{2c} - \sup_M |H_M|\Big) \geq \frac{2c}{1+2ct_z}\Big(\inf_N H_N - \sup_M |H_M|\Big) - \delta.\end{aligned}$$

Therefore, u is a solution of (5.1) in the barrier sense.

Proof of Theorem 1.8, item (a). Following the same arguments employed on the previous proofs, we may conclude from $\inf_N H_N \geq \sup_M |H_M|$ that t_N is constant, and thus M lies in a parallel surface of N. Since N is proper, M cannot be at distance zero to N, and the evolution of the mean curvature along the parallel surfaces (cf. Corollary 3.5 in [15]) gives that $\inf_N H_N = \sup_M |H_M|$, and consequently $H_N = H_M = 0$ everywhere. Thus, in this case, M and N must be parallel flat surfaces. ∎

Proof of Theorem 1.8, item (b). We notice that under the restriction on ε, setting $\lambda = \inf_N H_N - \sup_M |H_M| > 0$ and using the inequality $s - 1 - \log s \geq 0$ for $s > 0$, the function u satisfies

$$\begin{aligned}\triangle_M u &\geq \lambda \log\Big(\frac{2c+1+2ct_z}{1+2ct_z}\Big) - \delta \geq \lambda \log\Big(\frac{2+2\varepsilon}{2+4ct_z}\Big) - \delta \\ &= \lambda u - \delta \quad \text{on } \varphi^{-1}(U_+(\varepsilon/2)).\end{aligned} \tag{5.2}$$

Thus $\triangle_M u \geq \lambda u$ in the barrier sense on $\varphi^{-1}(U_+(\varepsilon/2))$. Since u vanishes only at the set $\varphi^{-1}(\partial U_+(\varepsilon/4))$ and it is subharmonic in the open set $\varphi^{-1}(U_+(\varepsilon/2))$, defining $\bar{u}\colon M \to \mathbb{R}$ as $\bar{u} = \max\{u, 0\}$, we have that $\bar{u}$ is a bounded solution for the differential inequality $\triangle_M \bar{u} \geq \lambda \bar{u}$ on M in the barrier sense, hence in the viscosity and weak sense (cf. [22]), such that $\sup_M u > 0$. Again this contradicts the Liouville property for stochastic completeness, see Theorem 5.1 in [16]. ■

Proof of Theorem 1.10. We shall assume by contradiction that $t_z \geq t_N \geq 2\gamma$ for a small constant γ such that $0 < 2\gamma H < 1$, where $H = \sup_M |H_M| = \inf_N H_N$. We recall that the principal curvatures κ_i^t of the parallel surfaces to $S_z^{\delta/2c}$ are given by

$$\kappa_i^t = \frac{\kappa_i}{1-\kappa_i t_z} \geq \kappa_i \quad \text{for } i = 1, 2.$$

Denote by H_z the mean curvature of $S_z^{\delta/2c}$ and assume $H_z > H - \delta/2c > H/2$. Therefore, using that $\kappa_1 \leq H_z/2 \leq \kappa_2$, we have

$$\begin{aligned}\triangle_M \phi_\delta &\geq \frac{2c}{1+2ct_z}\Big(\frac{\kappa_1}{1-\kappa_1 t_z} + \frac{\kappa_2}{1-\kappa_2 t_z} - H\Big)\\ &\geq \frac{2c}{1+2ct_z}\Big(\kappa_1 + \frac{\kappa_2}{1-\gamma H_z} - H_z - \frac{\delta}{2c}\Big)\\ &\geq \frac{2c}{1+2ct_z}\,\frac{\gamma H_z^2}{2-2\gamma H_z} - \frac{\delta}{1+2ct_z} \geq \frac{2c}{1+2ct_z}\,\frac{\gamma H^2}{4(2-\gamma H)} - \delta.\end{aligned}$$

Taking

$$\lambda = \frac{\gamma H^2}{4(2-\gamma H)} > 0$$

and recalling inequality (5.2), we will conclude that $\triangle_M u \geq \lambda u$ on $\varphi^{-1}(U_+(\varepsilon/2))$ in the barrier sense. The result can be finished by extending the function u outside $\varphi^{-1}(U_+(\varepsilon/4))$ by zero and using the Liouville property for stochastic completeness, see Theorem 5.1 in [16]. ■

6. Sketch of the proof of Theorem 1.12

We shall first prove that the theorem holds true in the Euclidean space $\mathbb{R}^3$. As have seen before, up to translation, we can assume that $\operatorname{dist}(M, N) = 0$. We just observe that the selected function u used in the proof of all theorems is also a bounded solution of $\triangle u \geq 0$ on $\varphi^{-1}(U(\varepsilon/2)) \cap \operatorname{int} M$, where φ denotes the usual isometric immersion of $(M, \partial M)$ into $\mathbb{R}^3$. Furthermore, $\bar{u} = \max\{u, 0\}$ belongs to $C^0(M) \cap W^{1,2}_{\mathrm{loc}}(\operatorname{int} M)$, and thus, it is a weak bounded subharmonic function on $\operatorname{int} M$. Since M is assumed to be parabolic, the Ahlfors maximum principle (Proposition 10 in [42]) says that

$$\sup_M u = \sup_{\partial M} u.$$

To conclude, we only recall that $u(x) \to \sup_M u$ if and only if $\operatorname{dist}(\varphi(x), N) \to 0$.

For the general case where the surfaces M and N are immersed in a complete flat three-manifold P, we may apply the arguments used in Corollary 5.2 of [37] to reduce to the case where M and N are immersed in $\mathbb{R}^3$. We notice that the lifting to the universal covering preserves parabolicity (see Proposition 3.3 in [37]), as well as the boundedness of curvature.

A. Appendix

In this appendix, we are going to show that the limit set of the minimal surface $\chi\colon\mathbb{C}\to\mathbb{R}^3$ described in Example 2.2 has at least four points. We recall from (2.1) that

$$\chi(z)=\Big(\frac{\operatorname{erfi}(\bar z)}{2}-\frac{\operatorname{erf}(3z)}{6},\frac{\operatorname{Re}[\operatorname{erf}(2z)]}{2}\Big).$$

Let us consider a curve $\gamma_\theta(t)=te^{i\theta}$ in $\mathbb{C}$. Note that $\gamma_\theta^2(t)=it^2$ if $\theta=\pi/4, 5\pi/4$, and that $\gamma_\theta^2(t)=-it^2$ if $\theta=3\pi/4, 7\pi/4$. Thus, for $\theta=\pi/4$,

$$\begin{aligned}
x_1(\gamma_{\pi/4}(t)) &= \frac{1}{\sqrt{\pi}}\operatorname{Re}\int_0^t (e^{it^2}-e^{-9it^2})\,e^{i\pi/4}\,dt\\
&= \frac{\sqrt{2}}{2\sqrt{\pi}}\Big(\int_0^t[\cos(t^2)-\sin(t^2)]\,dt-\int_0^t[\cos(9t^2)+\sin(9t^2)]\,dt\Big)\\
&= \frac{1}{6}\Big[3C\Big(\sqrt{\frac{2}{\pi}}\,t\Big)-3S\Big(\sqrt{\frac{2}{\pi}}\,t\Big)-C\Big(\sqrt{\frac{18}{\pi}}\,t\Big)-S\Big(\sqrt{\frac{18}{\pi}}\,t\Big)\Big].
\end{aligned}$$

Here $C(t)$ and $S(t)$ are, respectively, the FresnelC and the FresnelS functions. It is known that $\lim_{t\to\infty}(C(at)-S(at))=0$ and $\lim_{t\to\infty}(C(at)+S(at))=1$ for all $a>0$. Therefore,

$$\lim_{t\to\infty}x_1(\gamma_{\pi/4}(t))=-\frac{1}{6}.$$

Likewise,

$$x_2(\gamma_{\pi/4}(t))=-\frac{1}{6}\Big[3C\Big(\sqrt{\frac{2}{\pi}}\,t\Big)+3S\Big(\sqrt{\frac{2}{\pi}}\,t\Big)-C\Big(\sqrt{\frac{18}{\pi}}\,t\Big)+S\Big(\sqrt{\frac{18}{\pi}}\,t\Big)\Big]$$

and

$$x_3(\gamma_{\pi/4}(t))=\frac{1}{2}\Big[C\Big(\sqrt{\frac{8}{\pi}}t\Big)+S\Big(\sqrt{\frac{8}{\pi}}t\Big)\Big].$$

Thus

$$\lim_{t\to\infty}x_2(\gamma_{\pi/4}(t))=-\frac{1}{2}\quad\text{and}\quad\lim_{t\to\infty}x_3(\gamma_{\pi/4}(t))=\frac{1}{2}.$$

We conclude that the point

$$q_{\pi/4}=\Big(-\frac{1}{6},-\frac{1}{2},\frac{1}{2}\Big)\in\lim\chi.$$

The other three points $\{q_{3\pi/4},q_{5\pi/4},q_{7\pi/4}\}\subset\lim\chi$ can be found similarly.

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