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Short incompressible graphs and 2-free groups

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Abstract. Consider a finite connected 2-complex X endowed with a piecewise Riemannian metric, and whose fundamental group is freely indecomposable, of rank at least 3, and in which every 2-generated subgroup is free. In this paper, we show that we can always find a connected graph $\Gamma \subset X$ such that $\pi_1 \Gamma \simeq \mathbb{F}_2 \hookrightarrow \pi_1 X$ (in short, a 2-incompressible graph) whose length satisfies the following curvature-free inequality: $\ell(\Gamma) \leq 4\sqrt{2\,\mathrm{Area}(X)}$. This generalizes a previous inequality proved by Gromov for closed Riemannian surfaces with negative Euler characteristic. As a consequence, we obtain that the volume entropy of such 2-complexes with unit area is always bounded away from zero.

1. Introduction

We are interested in the geometry of 2-free groups. Recall that a finitely presented group G is said to be k-*free* for some $k \geq 1$ if any of its subgroups generated by k elements is free (possibly of rank $\leq k$). A 1-free group is just a group without torsion, and a k-free group is always p-free for any $p \leq k$. Obviously, the free group $\mathbb{F}_n$ with $n \geq 1$ generators is k-free for any positive k, and prime non-trivial examples of such groups are surface groups of genus $g \geq 2$ which are $(2g-1)$-free. Also, observe that the only 2-free groups with rank at most 2 are the free groups with one or two generators. According to [2], the subclass of 2-free groups is generic among groups with 3 generators, which makes this class particularly relevant.

In order to capture this algebraic property geometrically, we first consider the various topological realizations of a group as the fundamental group of some finite 2-complex, and then study the possible geometries that can be put on these complexes. More precisely, fix a 2-free finitely presented group G with rank at least 3 and any finite connected 2-complex X endowed with a piecewise Riemannian metric such that $\pi_1 X = G$. An embedded connected graph $i\colon \Gamma \hookrightarrow X$ is said to be 2-*incompressible* if (1) $\pi_1 \Gamma \simeq \mathbb{F}_2$, and (2) the induced map $i_*\colon \pi_1 \Gamma \to \pi_1 X$ is injective. It is worth saying that we do not require the graph to lie in the 1-skeleton of X, and that we can always find 2-incompressible graphs since loops lying in the 1-skeleton generate the fundamental group. We then define

$$L_2(X) := \inf_{\Gamma} \ell(\Gamma),$$

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where the infimum is taken over all 2-incompressible graphs Γ, and $\ell(\Gamma)$ denotes the total length of Γ for the length metric induced by X. This is a metric invariant closely related to the *Margulis constant* $\mu(X)$, which is by definition the largest number L such that at any point x, the subgroup of $\pi_1 X$ generated by loops based at x with length less than L is cyclic, see Definition 4.1 in [13]. In fact, it can be easily checked that

$$\mu(X) \leq L_2(X) \leq 2\mu(X). \tag{1.1}$$

The natural metric invariant L_2 belongs to a larger family of invariants defined as follows. For any finite connected 2-complex X endowed with a piecewise Riemannian metric, define the increasing sequence of positive numbers $\{L_k(X)\}_{k\geq 1}$ by setting $L_k(X) := \inf_\Gamma \ell(\Gamma)$, where the infimum is taken over graphs which are k-incompressible (that is, such that $\pi_1\Gamma \simeq \mathbb{F}_k \hookrightarrow \pi_1 X$). These numbers are well defined without any particular assumption on the fundamental group of X by setting $L_k(X) = \infty$ if X does not admit any k-incompressible graph. Observe that $L_1(X)$ is nothing but the *systole* of X (the shortest length of a non-contractible loop) in the case where the fundamental group of X is 1-free. So the higher invariants $L_k(X)$ can be thought of as a generalization of the systole. In this context, it is natural to define for any finitely presented group G its *k-free systolic area* by the formula

$$\mathfrak{S}_k(G) := \inf_{\pi_1 X = G} \operatorname{Area}(X)/L_k^2(X),$$

where the infimum is taken over the set of finite connected 2-complexes X with given fundamental group G and endowed with a piecewise Riemannian metric. Note that taking the supremum over the space of all piecewise flat metrics on X would yield the same value, see [1] and Section 3 of [4]. Obviously, $\mathfrak{S}_k(G) = 0$ for any $k \geq 1$ if G is free. For a 1-free group G, the invariant $\mathfrak{S}_1(G)$ coincides with the notion of systolic area as defined in [7], p. 337. According to Theorem 6.7.A in [6], any 1-free group G which is not free satisfies the following inequality:

$$\mathfrak{S}_1(G) \geq 1/100.$$

The current best lower bound known is $\pi/16$, see [12]. The main purpose of this article is to prove the following analog for 2-free groups.

Theorem 1. *Any* 2*-free group G which is freely indecomposable and of rank at least* 3 *satisfies the following inequality*:

$$\mathfrak{S}_2(G) \geq 1/32.$$

Therefore, the new invariant $\mathfrak{S}_2$ is non-trivial for a large natural class of groups.

Theorem 1 can be restated as follows: *any finite connected* 2*-complex X endowed with a piecewise Riemannian metric whose fundamental group is* 2*-free and freely indecomposable, but not cyclic, satisfies the following estimate*:

$$L_2(X) \leq 4\sqrt{2\operatorname{Area}(X)}.$$

This generalizes the result (see Theorem 5.4.A in [6]) that any Riemannian closed orientable surface S of genus at least 2 satisfies $L_2(S) \leq 2\sqrt{2\operatorname{Area}(S)}$. Observe that here the assumption on the genus ensures that the fundamental group $\pi_1 S$ is 2-free. See also Theorem 6.6.C in [6] for a higher dimensional generalization of this last inequality. Combined with inequality (1.1), Theorem 1 also provides an analog in the context of 2-complexes of a curvature-free inequality between the volume and the Margulis constant obtained for Riemannian manifolds whose fundamental group is 2-free, see Theorem 4.5(1) in [13].

Presently, we do not see how to adapt our strategy to prove an analog of Theorem 1 for $k > 2$, but it seems reasonable to conjecture that for each such k, the invariant $\mathfrak{S}_k$ is uniformly bounded from below for any k-free group freely indecomposable with rank at least $k+1$. Also, we do not know how to extend our current proof to encompass the freely decomposable groups: a 2-complex X with decomposable fundamental group $\pi_1 X = G_1 * G_2$ does not have to split in any meaningful way in pieces corresponding to the subgroups G_1 and G_2.

Lastly, Theorem 1 implies the following curvature-free inequality relating the volume entropy and the area. Recall that the volume entropy $h(Y)$ of a finite connected complex Y (of any dimension) endowed with a piecewise Riemannian metric is defined as the exponential growth rate of the number of homotopy classes with length at most L, namely

$$h(Y) = \lim_{L \to \infty} \frac{1}{L} \cdot \log(\operatorname{card}\{[\gamma] \in \pi_1 Y \mid \gamma \text{ based loop of length at most } L\}).$$

This definition does not depend on the chosen point where loops are based. As a consequence of Theorem 1, we get the following.

Corollary 2. *Any finite connected* 2*-complex X endowed with a piecewise Riemannian metric whose fundamental group is* 2*-free, freely indecomposable and of rank at least* 3*, satisfies the following estimate*:

$$h(X) \cdot \sqrt{\operatorname{Area}(X)} \geq 3 \log 2/(4\sqrt{2}).$$

There is no reason for the above constant to be optimal, but this result generalizes the following (sharp) estimate [9] that for S an orientable closed surface whose fundamental group is 2-free, the inequality $h(S) \cdot \sqrt{\operatorname{Area}(S)} \geq 2\sqrt{\pi}$ is always satisfied. This corollary also improves a previous result, due to Babenko and privately communicated to the authors, proving an analog lower bound with a worst constant but valid without the freely indecomposable assumption.

2. Topology of small balls in piecewise flat 2-complexes

Consider a finite connected 2-complex X endowed with a piecewise flat metric, and fix a point x in X. In this section, we focus on the topology of closed balls

$$B(x,r) := \{y \in X \mid d(y,x) \leq r\}$$

and their boundary spheres

$$\partial B(x,r) := \{y \in X \mid d(y,x) = r\}$$

for relatively small radius $r > 0$.

Our starting point is the following result, proved in Corollary 6.8 of [10], for which it is crucial that the metric is piecewise flat and not just piecewise smooth.

Proposition 3. *For any $r > 0$, the triangulation of X can be refined in such a way that both $B(x,r)$ and $\partial B(x,r)$ are CW-subcomplexes of X.*

As a direct consequence, we find the following.

Proposition 4. *For any $r > 0$ and any $x \in X$, the fundamental group of $B(x,r)$ is finitely presented.*

Proof. According to Proposition 3, choose a refinement of the triangulation of X such that $B(x,r)$ is a CW-subspace of X. Since X is compact, any triangulation contains finitely many simplices, as does the triangulation of the closed ball $B(x,r)$. Hence its fundamental group is finitely presented. ∎

We now turn to the boundary spheres and show that they generically admit trivial tubular neighborhoods.

Proposition 5. *For all but finitely many values of $r > 0$, the boundary sphere $\partial B(x,r)$ is a finite graph, and for each connected component C of $\partial B(x,r)$, there exists an open neighborhood of C in X homeomorphic to $C \times]0,1[$.*

Proof. Denote by $f = d(x,\cdot) : X \to \mathbb{R}_+$ the function *distance to the point x*. Recall that the Reeb space $R(f)$ is the quotient of X by the relation that identifies two points y_0 and y_1 if and only if $d(x,y_0) = d(x,y_1)$ and both points belong to the same connected component of the level set $f^{-1}(f(y_0))$. The space $R(f)$ admits a length structure induced from X. By construction, we have a canonical projection map $p\colon X \to R(f)$ which is 1-Lipschitz. We argue as in Section 4 of [10]: the function f is a semi-algebraic function, and then standard arguments show that $R(f)$ is a finite graph and that $R(f)$ admits a finite subdivision such that the natural map p yields a trivial bundle over the interior of each edge. For all distances r but the finitely many ones corresponding to the vertices of the subdivision, if C is a connected component of $f^{-1}(r)$, then by triviality of the bundle, the connected component of $p^{-1}(]r-\varepsilon, r+\varepsilon[)$ containing C is an open neighborhood of C of the desired form provided ε is small enough. More precisely, ε has to be chosen at most equal to the shortest distance from $p(C)$ to one of the two ends of the edge containing it. ∎

In the last part of this section, we focus on the image in X of the fundamental group of small metric balls. Consider the map $i_*\colon \pi_1(B(x,r),x) \to \pi_1(X,x)$ induced by the inclusion $B(x,r) \subset X$.

According to Proposition 3.2 in [12] (see also [10]), when $\pi_1 X$ is 1-free, $\operatorname{Im} i_*$ is trivial if the radius r satisfies $r < L_1(X)/2$. The last result of this section describes how $\operatorname{Im} i_*$ remains simple under a similar assumption on the radius.

Proposition 6. *Suppose that $\pi_1 X$ is a 2-free group and fix $r \in (0, L_2(X)/4)$.*

Then the image of the map $i_*\colon \pi_1(B(x,r),x) \to \pi_1(X,x)$ induced by the inclusion $B(x,r) \subset X$ is either trivial, or isomorphic to $\mathbb{Z}$.

Proof. Suppose that $\operatorname{Im} i_*$ is not trivial. We first prove that $\operatorname{Im} i_*$ is locally cyclic, that is, that every pair of elements in the group generates a cyclic group.

For this, let γ_1 and γ_2 be two non-contractible loops of X contained in $B(x,r)$ and based at x. As $\pi_1(X,x)$ is 2-free, these loops span in $\pi_1(X,x)$ a free subgroup $H(\gamma_1,\gamma_2)$

of rank at most 2. Fix $\delta > 0$ such that $2r + \delta < L_2(X)/2$. We first decompose each γ_i into segments of length at most δ. Then, for $i = 1, 2$, write γ_i as a concatenation of loops $c_{i,1} * \cdots * c_{i,n_i}$ based at x, where each $c_{i,k}$ is made of the union of one of these small segments together with two shortest paths from its extremal points to x. Any of these loops $c_{i,k}$ based at x lies by construction in $B(x,r)$ and has length at most $2r + \delta < L_2(X)/2$. So a graph made of the union of any two of these loops is of total length $< L_2(X)$, hence the subgroup in $\pi_1(X,x)$ generated by any of these pairs of loops is cyclic (if not zero). Then the subgroup $H(\{c_{i,j}\})$ generated by all the homotopy classes of the loops $\{c_{i,j}\}$ is abelian, as its generators pairwise commute. In particular, there exists some positive k such that $H(\{c_{i,j}\}) \simeq \mathbb{Z}^k$, as $\pi_1 X$ is torsion-free. But $\pi_1 X$ is also 2-free, so that $k = 1$. This implies that $H(\gamma_1, \gamma_2) = \mathbb{Z}$, and hence $\operatorname{Im} i_*$ is locally cyclic.

As $\operatorname{Im} i_*$ is also finitely generated, thanks to Proposition 4, we deduce that it is cyclic. Furthermore, as $\operatorname{Im} i_*$ has no torsion, since $\pi_1 X$ is torsion-free, it is isomorphic to $\mathbb{Z}$. ∎

3. Geometry of small balls in piecewise flat 2-complexes

In this section, we prove the central technical result of this paper.

Consider a finite connected 2-complex X endowed with a piecewise flat metric and whose fundamental group is 2-free, freely indecomposable and of rank at least 3. Fix $\varepsilon > 0$ and let Γ be a 2-incompressible graph whose length satisfies $\ell(\Gamma) \leq L_2(X) + \varepsilon$. Observe that Γ may be chosen with no vertex of degree 1. Let x be any point on Γ.

Theorem 7. *For all but finitely many values of $r \in (\varepsilon, L_2(X)/4)$, the following holds*:

$$\ell(\partial B(x,r)) \geq r - \varepsilon.$$

In particular, using the coarea formula (see Theorem 3.2.11 in [5]), we derive the lower bound

$$\operatorname{Area}(B(x, L_2(X)/4)) \geq (L_2(X) - \varepsilon)^2/32.$$

This implies that

$$\operatorname{Area}(X) \geq L_2(X)^2/32,$$

which still holds true for piecewise smooth Riemannian metrics by approximation (see [1] and Section 3 of [4]), and implies Theorem 1.

Proof. Fix $r \in (\varepsilon, L_2(X)/4)$ so that Proposition 5 applies, and set $B := B(x,r)$. Denote by $X_1, \ldots, X_k$ the path connected components of $X \setminus \operatorname{int}(B)$ with non-empty interior, and by $C_1, \ldots, C_n$ the connected components of ∂B. According to Proposition 3, each C_i is a connected finite graph, and there exists an open neighborhood U of $C_1 \sqcup \cdots \sqcup C_n$ in X such that

$$U \overset{\text{hom}}{\simeq} (C_1 \times]0,1[) \sqcup \cdots \sqcup (C_n \times]0,1[).$$

According to Proposition 6, the inclusion $i \colon B \hookrightarrow X$ induces a homomorphism of fundamental groups whose image is either trivial or isomorphic to $\mathbb{Z}$. So each graph C_i satisfies either $i_*(\pi_1 C_i) = 0$ or $i_*(\pi_1 C_i) = \mathbb{Z}$. Furthermore, if $\operatorname{rank} i_*(\pi_1 C_i) = \operatorname{rank} i_*(\pi_1 C_j) = 1$, then the subgroup generated by both these subgroups is a subgroup of $i_*(\pi(B)) = \mathbb{Z}$, and hence is again isomorphic to $\mathbb{Z}$. In particular, elements in $i_*(\pi_1 C_i)$ commute with those in $i_*(\pi_1 C_j)$.

Let $Y = (X_1 \sqcup \cdots \sqcup X_k)/\sim$, where $x \sim y$ if and only if x and y belong to the same connected component C_i for some $i \in \{1, \ldots, n\}$. Denote by $a_1, \ldots, a_n$ the points in Y that are images of the boundary graphs $C_1, \ldots, C_n$ under the projection map

$$f : X_1 \sqcup \cdots \sqcup X_k \to Y.$$

The space Y decomposes into a disjoint union

$$Y_1 \sqcup \cdots \sqcup Y_k$$

of path-connected components $Y_1, \ldots, Y_k$ such that $X_j = f^{-1}(Y_j)$. Define, for each $j = 1, \ldots, k$, the subset $I_j \subset \{1, \ldots, n\}$ such that $a_l \in Y_j \Leftrightarrow l \in I_j$. Therefore, $\{1, \ldots, n\} = I_1 \sqcup \cdots \sqcup I_k$ and

$$B \cap X_j = \bigsqcup_{l \in I_j} C_l.$$

If $k = n$, we may assume, up to reindexing the boundary graphs, that $a_j \in Y_j$ for each $j = 1, \ldots, n$ (or equivalently, that $I_j = \{j\}$).

If $k < n$, then $|I_j| \geq 2$ for some $j \in \{1, \ldots, k\}$, and the following holds true.

Lemma 8. *Assume that $|I_j| \geq 2$. Then $i_*(\pi_1 C_l) = \mathbb{Z}$ for all $l \in I_j$.*

Proof. By contradiction, let $l \in I_j$ be such that $i_*(\pi_1 C_l) = 0$, and fix a neighborhood U_l of C_l such that $U_l \simeq C_l \times]0,1[$. By construction, U_l is connected, $X = U_l \cup (X \setminus C_l)$, and because $|I_j| \geq 2$, the open set $X \setminus C_l$ is also connected. Observe that $A_l := U_l \cap (X \setminus C_l)$ has exactly two connected components, and choose a point x_1 and x_2 in each one of them. Fix a path β in U_l and a path γ in $X \setminus C_l$ both going from x_1 to x_2. We denote by $\varphi_1 : \pi_1(A_l, x_1) \to \pi_1(U_l, x_1)$ and $\psi_1 : \pi_1(A_l, x_1) \to \pi_1(X \setminus C_l, x_1)$ the homomorphisms induced by the respective inclusion maps, and we define two homomorphisms $\varphi_2 : \pi_1(A_l, x_2) \to \pi_1(U_l, x_1)$ and $\psi_2 : \pi_1(A_l, x_2) \to \pi_1(X \setminus C_l, x_1)$ by setting

$$\varphi_2(\alpha) = \beta\alpha\beta^{-1} \quad \text{and} \quad \psi_2(\alpha) = \gamma\alpha\gamma^{-1}.$$

We also define a homomorphism $\mu : \mathbb{Z} \simeq \langle a \rangle \to \pi_1(X, x_1)$ by setting

$$\mu(a) = \beta\gamma^{-1}.$$

By the Van Kampen theorem, see Proposition 2 on p. 422 of [3], there exists a unique surjective homomorphism

$$M : \pi_1(U_l, x_1) * \pi_1(X \setminus C_l, x_1) * \mathbb{Z} \to \pi_1(X, x_1)$$

which coincides with μ on the factor $\mathbb{Z}$ and with the homomorphisms induced by the respective natural inclusions on the two factors $\pi_1(U_l, x_1)$ and $\pi_1(X \setminus C_l, x_1)$, and whose kernel is normally generated by the elements of the form

(1) $\varphi_2(v) a \psi_2(v)^{-1} a^{-1}$ for $v \in \pi_1(A_l, x_2)$;

(2) $\varphi_1(v)\psi_1(v)^{-1}$ for $v \in \pi_1(A_l, x_1)$.

As the image of $\pi_1 C_l \simeq \pi_1(U_l, x_1)$ is trivial in $\pi_1(X, x_1)$, the homomorphisms $M \circ \varphi_1$ and $M \circ \varphi_2$ are trivial, and consequently, the surjective homomorphism M factorizes as

$$M : \pi_1(X \setminus C_l, x_1) * \mathbb{Z} \to \pi_1(X, x_1),$$

with kernel normally generated by the elements of the form

(1) $\psi_2(v)$ for $v \in \pi_1(A_l, x_2)$;

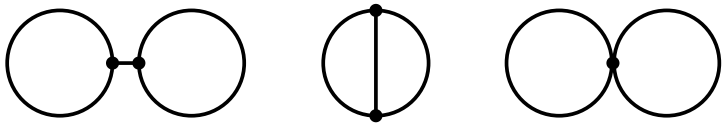
(2) $\psi_1(v)$ for $v \in \pi_1(A_l, x_1)$.

By definition, all these relations are written in the group $\pi_1(X \setminus C_l, x_1)$. So if we denote by H the quotient of $\pi_1(X \setminus C_l, x_1)$ by these relations, M induces an isomorphism

$$\overline{M} : H * \mathbb{Z} \to \pi_1(X, x_1),$$

contradicting the fact that the fundamental group of X is freely indecomposable and of rank at least 3. ∎

We may assume that Γ is transverse to $C_1 \sqcup \cdots \sqcup C_n$. Because it has no vertex of degree 1, Γ is one of the three following graphs with first Betti number equal to 2:



As the graph Γ is 2-incompressible, the subgraph $\Gamma \cap B$ has cyclic number at most 1 according to Proposition 6, and the graph Γ escapes from B and so necessarily intersects the boundary $C_1 \sqcup \cdots \sqcup C_n$. Set $\Gamma_j := \Gamma \cap X_j$ and observe that some of these graphs may be empty (but not all). Furthermore, let $\Gamma_0 = \Gamma \cap B$ be the remaining part of the graph Γ which completes the decomposition as follows:

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_k.$$

Now construct a new graph $\overline{\Gamma}$ starting from Γ, and obtained by deleting Γ_0 and pasting all the boundary graphs as follows:

$$\overline{\Gamma} := (\Gamma \setminus \Gamma_0) \cup (C_1 \cup \cdots \cup C_n).$$

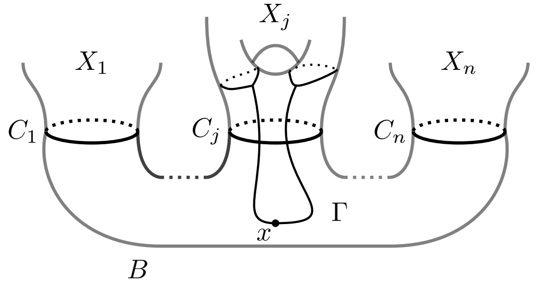
We shall see that we can always extract from $\overline{\Gamma}$ a 2-incompressible subgraph Γ', and this implies the desired lower bound. Indeed, the 2-incompressible subgraph Γ' will satisfy $\ell(\Gamma') \geq L_2(X)$, as well as $\ell(\Gamma') \leq \ell(\Gamma) - r + \sum_{j=1}^{n} \ell(C_j)$ as $\ell(\Gamma_0) \geq r$. Given that $\ell(\Gamma) \leq L_2(X) + \varepsilon$, we get the announced lower bound

$$\ell(\partial B) \geq \sum_{j=1}^{n} \ell(C_j) \geq r - \varepsilon.$$

To extract the 2-incompressible subgraph Γ' from $\overline{\Gamma}$, we argue as follows.

Suppose first that the inclusion $B \subset X$ induces the zero morphism: $i_*(\pi_1 B) = 0$.

In particular, any boundary component C satisfies $i_*(\pi_1 C) = 0$, as its fundamental group factors through $i_*(\pi_1 B)$. Thus Lemma 8 implies that $k = n$. The key point is that there exists a unique $j \in \{1, \ldots, n\}$ such that $i_*(\pi_1 X_j) \neq 0$. Indeed, given that $i_*(\pi_1 B) = 0$ and applying the Van Kampen theorem to the covering $\{B, X_1, \ldots, X_n\}$ of X, we get that $\pi_1 X \simeq \pi_1 X_1 * \cdots * \pi_1 X_n$. As $\pi_1 X$ is freely indecomposable, only one of these free factors is non-trivial.



So the 2-incompressible graph Γ, which has cyclic number 2, must intersect the boundary graph C_j of the non-trivial piece X_j. Fix two homotopically independent loops c_1 and c_2 of Γ based at the same point, say p, of the boundary graph C_j. By homotopically independent we mean that the two loops generate a free subrgoup of rank 2 of the fundamental group. If they are not entirely contained in X_j, and as $\pi_1(B \cup (\cup_{l \neq j} X_l)) = 0$, we can for each of the c_i's homotope each of their subarcs lying outside X_j into a subarc of C_j without moving their respective endpoints. Therefore we can homotope c_1 and c_2 into two new homotopically independent loops still based at p and lying in $\Gamma_j \cup C_j \subset \overline{\Gamma}$. Therefore, as wanted, we can extract a 2-incompressible subgraph from $\overline{\Gamma}$.

Suppose now that the inclusion $B \subset X$ induces a morphism of rank 1: $i_*(\pi_1 B) = \mathbb{Z}$.

Fix an element a of $\pi_1 B$ that generates $i_*(\pi_1 B) = \mathbb{Z}$ and a closed curve c of Γ based at x and homotopically independent from a. The loop c necessarily escapes from B. Denote by $p_1, \ldots, p_N$ the intersection points along c with ∂B (it may happen that $p_i = p_{i+1}$ for some i). Denote by δ_1 the subpath of c that goes from x to p_1, by δ_N the subpath of c going backwards from x to p_N, and fix for $i = 2, \ldots, N-1$ any path δ_i contained in B from x to p_i. We can decompose the loop c into a concatenation of loops c_i based at x, each one being made by first following δ_i, then the portion denoted by η_i of c from p_i to the next intersection point p_{i+1}, and then going back to x using δ_{i+1}^{-1}. One of these loops must be homotopically independent from the generator a of $\pi_1 B$: the loop c does not homotopically commute with a, and thus at least one of the c_i's does not homotopically commute with a either. Again, this loop, that we denote simply by c_i, necessarily escapes from B and the corresponding portion η_i lies outside $\mathrm{int}(B)$. Let X_j be the path connected component of $X \setminus \mathrm{int}(B)$ that contains η_i.

If X_j has more than one boundary component, then by Lemma 8 all boundary components are homotopically non-trivial in B and in X, and we argue as follows.

Suppose first that the endpoints of η_i belong to two distinct boundary graphs C_l and $C_{l'}$ for some $l \neq l'$. First observe that l and l' both necessarily belong to the same subset I_j, as $\eta_i \subset X_j$. Moreover, $i_*(\pi_1 C_l) = \mathbb{Z}$ and $i_*(\pi_1 C_{l'}) = \mathbb{Z}$, as already observed. Fix two non-trivial loops $b_l \in C_l$ and $b_{l'} \in C_{l'}$ respectively based at p_i and p_{i+1}. Set $\delta = \delta_i^{-1} * \delta_{i+1}$. Observe that the homotopy classes of $\eta_i * b_{l'} * \eta_i^{-1}$ and $c_i * (\delta * b_{l'} * \delta^{-1}) * c_i^{-1}$ (where c_i is viewed as a loop based at p_i) coincide. If the loop $\eta_i * b_{l'} * \eta_i^{-1}$ was not homotopically independent with b_l, then we would have that $[c_i] \cdot a^n \cdot [c_i^{-1}] = a^m$ for some $m, n \in \mathbb{Z} \setminus \{0\}$, as both loops $\delta * b_{l'} * \delta^{-1}$ and b_l induce homotopy classes in $\pi_1 B = \langle a \rangle$. But this is impossible, as c_i was chosen homotopically independent from the class a. So the two loops $\eta_i * b_{l'} * \eta_i^{-1}$ and b_l based at p_i are homotopically independent and are both contained in $\Gamma_j \cup C_l \cup C_{l'} \subset \overline{\Gamma}$. So their union forms a 2-incompressible graph $\Gamma' \subset \overline{\Gamma}$.

Now suppose that both endpoints of η_i belong to the same connected boundary component C_l, and fix some subarc α in C_l from p_i to p_{i+1}. The closed curve c_i (viewed as a loop based at p_i) is homotopic to the concatenation of the loop $\eta_i * \alpha^{-1}$ with the loop $\alpha * \delta_{i+1}^{-1} * \delta_i$. The second loop is included in B, and therefore its homotopy class $[\alpha * \delta_{i+1}^{-1} * \delta_i]$ is equal to a^k for some $k \in \mathbb{Z}$. Hence the first loop $\eta_i * \alpha^{-1}$ is homotopically independent from a. Now define Γ'' to be a subgraph of C_l that contains α, such that $\pi_1\Gamma'' \simeq \mathbb{Z}$ and $\pi_1\Gamma'' \to \pi_1 X$ is injective. Then $\Gamma' = \Gamma'' \cup \eta_i$ is the desired 2-incompressible subgraph of Γ.

If X_j has a unique boundary component C_l, observe that $i_*(\pi_1 C_l) \neq 0$. For if it is trivial, by applying the Van Kampen theorem to the covering of X by the open set $X \setminus X_j$ and its complement X_j slightly enlarged so that these two open sets overlap along a half-tubular neighborhood $U \simeq C_l \times]0,1[$ of C_l, we would get a non-trivial free decomposition $\pi_1 X \simeq \pi_1 X_j * \pi_1(X \setminus X_j)$, where both pieces are non-trivial: a contradiction. Finally, because the loop c_i is homotopically independent from the class a, we can extract a 2-incompressible subgraph from $C_l \cup \eta_i \subset \overline{\Gamma}$. ∎

4. A universal bound for the volume entropy

We conclude by explaining how to derive Corollary 2 from Theorem 1.

Proof of Corollary 2. Let X be a finite connected 2-complex X endowed with a piecewise Riemannian metric whose fundamental group is 2-free, freely indecomposable and of rank at least 3. According to Theorem 1, we can find a 2-incompressible graph $\Gamma \hookrightarrow X$ with induced length at most $4\sqrt{2}\sqrt{\mathrm{Area}(X)}$. The fact that $\pi_1\Gamma \simeq \mathbb{F}_2$ implies by [8] (see also [11]) that

$$\ell(\Gamma) \cdot h(\Gamma) \geq 3 \log 2,$$

where $h(\Gamma)$ denotes the volume entropy of the finite connected 1-dimensional complex Γ for the piecewise Riemannian metric induced by X. The injection $\pi_1\Gamma \hookrightarrow \pi_1 X$ ensures that $h(X) \geq h(\Gamma)$, from which we derive the desired lowerbound:

$$h(X) \cdot \sqrt{\mathrm{Area}(X)} \geq \frac{1}{4\sqrt{2}} \cdot h(\Gamma) \cdot \ell(\Gamma) \geq \frac{3 \log 2}{4\sqrt{2}}. \qquad ∎$$

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