



Reductive covers of klt varieties

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Abstract. In this article, we study G -covers of klt varieties, where G is a reductive group. First, we exhibit an example of a klt singularity admitting a $\mathbb{PGL}_n(\mathbb{K})$ -cover that is not of klt type. Then, we restrict ourselves to G -quasi-torsors, a special class of G -covers that behave like G -torsors outside closed subsets of codimension two. Given a G -quasi-torsor $X \rightarrow Y$, where G is a finite extension of a torus \mathbb{T} , we show that X is of klt type if and only if Y is of klt type. We prove a structural theorem for \mathbb{T} -quasi-torsors over normal varieties in terms of Cox rings. As an application, we show that every sequence of \mathbb{T} -quasi-torsors over a variety with klt type singularities is eventually a sequence of \mathbb{T} -torsors. This is the torus version of a result due to Greb–Kebekus–Peternell regarding finite quasi-torsors of varieties with klt type singularities. On the contrary, we show that in any dimension there exists a sequence of finite quasi-torsors and \mathbb{T} -quasi-torsors over a klt type variety, such that infinitely many of them are not torsors. We show that every variety with klt type singularities is a quotient of a variety with canonical factorial singularities. We prove that a variety with Zariski locally toric singularities is indeed the quotient of a smooth variety by a solvable group. Finally, motivated by the work of Stibitz, we study the optimal class of singularities for which the previous results hold.

1. Introduction

In algebraic geometry, we often encounter singularities which are quotients of other singularities by algebraic groups. Orbifold singularities are finite quotients of smooth points, toric singularities are abelian quotients of smooth points [16], and terminal 3-fold singularities are finite quotients of hypersurface singularities [35]. Furthermore, many interesting factorial singularities are $\mathrm{SL}_n(\mathbb{K})$ -quotients of smooth points [8, 9]. In most cases, the respective group is reductive [32]. Indeed, the reductivity assumption is what ensures that the quotient is of finite type. Reductive quotients preserve normal singularities and rational singularities [7]. Recently, together with Greb and Langlois, the authors proved that reductive quotients preserve the singularities of the minimal model program [13], the so-called klt type singularities [24].

In this article, we study a central topic in algebraic geometry: how to improve the singularities of an algebraic variety by taking appropriate covers. We focus on the singularities of the minimal model program. To tackle this question, we need to comprehend

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what type of covers will indeed improve the singularities that we are studying. This will lead us to the concepts of G -covers and G -quasi-torsors.

1.1. G -covers of klt singularities

Let G be an algebraic group. A G -cover of a singularity $(X; x)$ is an algebraic singularity $(Y; y)$ endowed with a G -action fixing y so that X is isomorphic to the quotient $Y//G$ and x is the image of y (see Definition 2.6). In this setting, we say that $(Y; y)$ is a G -cover of $(X; x)$, and we say that $(X; x)$ is a G -quotient of $(Y; y)$. One can think about a G -cover of a singularity as a degenerate principal G -bundle over the singularity having maximal degeneration at the distinguished singular point. G -covers often occur in singularity theory; when replacing a singularity with its universal cover [10, 12, 28], when taking the index one cover with respect to a \mathbb{Q} -Cartier divisor [25], and when taking the Cox ring of the singularity [11, 14]. G -covers are often useful to compute invariants of singularities [30]. Thus, it is natural to ask whether a class of singularities is preserved by G -covers. Of course, the answer to this question depends on the choice of G . The first question that we settle on in this article is whether the class of klt type singularities is closed under reductive covers. Our first theorem is a negative answer to this question. We show the existence of a 3-dimensional toric singularity admitting a 5-dimensional $\mathbb{PGL}_3(\mathbb{K})$ -cover which is not of klt type.

Theorem 1.1. *There exists a 3-fold toric singularity $(X; x)$ that admits a $\mathbb{PGL}_3(\mathbb{K})$ -cover $(Y; y) \rightarrow (X; x)$ from a 5-dimensional singularity $(Y; y)$ which is not of klt type.*

In Proposition 3.2, we give further examples in this direction in which the group $\mathbb{PGL}_n(\mathbb{K})$ acts freely on an open subset Y . However, the singularities are of higher dimension in these cases.

The previous theorem is the local analog of the well-known fact that projective bundles over Fano type varieties may not be of Fano type. Indeed, there exist projective bundles over Fano type surfaces that are not Mori dream spaces [20]. However, it is known that split projective bundles over Fano type varieties are Fano type [14]. Furthermore, finite covers of Fano type varieties are again Fano type varieties, under a restrictive hypothesis in case there is ramification in codimension one (see, e.g., Lemma 3.18 in [31]). These two facts motivate the proof of the following theorem.

Theorem 1.2. *Let $(X; x)$ be a klt type singularity. Let G be a finite extension of a torus and let $Y \rightarrow X$ be a G -quasi-torsor. Then $(Y; y)$ is a klt type singularity.*

A G -quasi-torsor is a special kind of G -cover which behaves like a G -torsor outside codimension two subsets of X and Y . G -quasi-torsors are also called *almost principal fiber bundles* in the literature. Hence, the class of klt type singularities is preserved under reductive quotients and under G -quasi-torsors, whenever G is a finite extension of a torus. We emphasize that the condition on the ramification is necessary: even finite covers of a smooth point with codimension one ramification may not be of klt type (see Example 7.1). Note that G being a finite extension of a torus is equivalent to asking that the derived subgroup of its connected component is trivial [23]. It is an open problem to decide whether the previous statement holds for G a reductive group (see Question 7.8). With the previous theorem, we have found the right type of covers that can improve our klt type singularity:

finite quasi-torsors and torus quasi-torsors. Whenever these are torsors, i.e., finite étale covers and toric bundles, the class of singularities of our variety will not change. Thus, we are mostly interested in the finite quasi-torsors and \mathbb{T} -quasi-torsors that are not torsors. These are the covers for which the étale class of a singularity may change. Moreover, these are exactly the covers detected by the regional fundamental group of the singularity and by the local Cox ring of the singularity (see [10] and Definition 2.11).

1.2. Torus covers of klt varieties

As mentioned above, one way to improve the singularities of a variety is to produce \mathbb{T} -quasi-torsors. For instance, all toric varieties are quotients of a smooth affine variety by the action of an abelian linear algebraic group. In a similar vein, the local Cox ring of a singularity often simplifies the singularity. A natural way to obtain a \mathbb{T} -quasi-torsor of a variety is to mimic the Cox ring construction. For example, if we consider Weil divisors W_1, \dots, W_k in X spanning the subgroup N of $\text{WDiv}(X)$, then we can define the sheaf

$$\mathcal{R}(X)_N := \bigoplus_{(m_1, \dots, m_k) \in \mathbb{Z}^k} \mathcal{O}_X(m_1 W_1 + \dots + m_k W_k).$$

Then, the relative spectrum

$$Y := \text{Spec}_X(\mathcal{R}(X)_N) \rightarrow X,$$

admits a natural \mathbb{T} -cover structure over X which is a \mathbb{T} -quasi-torsor. Here, \mathbb{T} is a k -dimensional algebraic torus, and the action of \mathbb{T} on Y is induced by the \mathbb{Z}^k -grading of the sheaf $\mathcal{R}(X)_N$. Note that $Y \rightarrow X$ is a \mathbb{T} -torsor precisely at the points at which all the W_i 's are Cartier divisors. The variety Y will be called a *relative Cox space* of X . Indeed, this relative version of the Cox space locally behaves like the Cox space of the singularities of X .

Our next theorem states that every \mathbb{T} -quasi-torsor over a normal variety is equivariantly isomorphic to a relative Cox space.

Theorem 1.3. *Let X be a normal variety. Let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor. Then, we can find Weil divisors W_1, \dots, W_k on X for which there is a \mathbb{T} -equivariant isomorphism*

$$Y \simeq \text{Spec}_X \left(\bigoplus_{(m_1, \dots, m_k) \in \mathbb{Z}^k} \mathcal{O}_X(m_1 W_1 + \dots + m_k W_k) \right).$$

Theorem 1.2 implies that a \mathbb{T} -quasi-torsor over a klt type singularity is again of klt type. On the other hand, Theorem 1.3, implies that a relative Cox ring of a normal variety is equivariantly isomorphic to a \mathbb{T} -quasi-torsor. Combining these two results, we obtain that a relative Cox ring of a klt type variety also has klt type singularities.

Theorem 1.4. *Let X be a variety with klt type singularities. Let $Y \rightarrow X$ be a relative Cox ring. Then Y has klt type singularities.*

In summary, the class of \mathbb{T} -quasi-torsors of klt type varieties agrees with the class of relative Cox spaces. Furthermore, the Cox spaces again have klt type singularities. In

Example 7.3, we show that the singularities can indeed improve by taking the relative Cox ring.

In Theorem 1.1 of [21], the authors show that any sequence of finite quasi-torsors over a variety with klt type singularities is eventually a sequence of finite torsors. This means that all but finitely many of the finite quasi-torsors are torsors, i.e., finite Galois étale covers. It is natural to ask if a similar principle holds for \mathbb{T} -quasi-torsors. In this direction, we prove a torus version of the theorem due to Greb, Kebekus, and Peternell.

Theorem 1.5. *Let X be a variety with klt type singularities. Consider a sequence of morphisms*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

such that each $\phi_i : X_i \rightarrow X_{i-1}$ is a \mathbb{T} -quasi-torsor. Then, there exists j such that, for every $i \geq j$, the morphism ϕ_j is a \mathbb{T} -torsor.

1.3. Iteration of torus and finite covers

Our previous theorem states that any sequence of \mathbb{T} -quasi-torsors over a variety with klt type singularities is eventually a sequence of \mathbb{T} -torsors. It is natural to investigate what happens for sequences of \mathbb{T} -quasi-torsors and finite quasi-torsors, i.e., to study mixed sequences of torus and finite covers. In this direction, we will show that all but finitely many of the finite quasi-torsors are indeed torsors.

Theorem 1.6. *Let X be a variety with klt type singularities. Consider a sequence of morphisms*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{i+2} \dots$$

such that each ϕ_i is either a finite quasi-torsor or a torus quasi-torsor. Then, all but finitely many of the finite quasi-torsors are torsors, i.e., finite étale Galois morphisms.

Note that the previous theorem gives a generalization of Theorem 1.1 in [21]. Indeed, if we require that each ϕ_i is a finite quasi-torsor, then we recover this statement. It is natural to wonder whether in the context of the previous theorem we can further obtain that all but finitely many of the \mathbb{T} -quasi-torsors are torsors. First, notice that such a statement holds trivially for smooth varieties. Indeed, by the purity of the branch locus, every finite quasi-torsor over a smooth variety is a finite torsor. On the other hand, by Theorem 1.3, every \mathbb{T} -quasi-torsor over a smooth variety is a \mathbb{T} -torsor. Hence, in order to produce interesting sequences of quasi-torsors we need to consider singular varieties. Toric singularities are arguably the simplest kind of klt singularities because of their combinatorial nature. The following theorem shows that even for varieties with toric singularities, we may produce infinite sequences of finite quasi-torsors and \mathbb{T} -quasi-torsors so that infinitely many of the \mathbb{T} -quasi-torsors are not torsors.

Theorem 1.7. *For each $n \geq 2$, there exists an n -dimensional projective variety X^n with toric singularities and an infinite sequence of morphisms*

$$X^n = X_0^n \xleftarrow{\phi_1} X_1^n \xleftarrow{\phi_2} X_2^n \xleftarrow{\phi_3} X_3^n \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i^n \xleftarrow{\phi_{i+1}} X_{i+1}^n \xleftarrow{i+2} \dots$$

such that the following conditions hold:

- (1) each ϕ_i is either a finite quasi-torsor or a \mathbb{T} -quasi-torsor,
- (2) infinitely many of the ϕ_i 's are finite torsors, and
- (3) infinitely many of the ϕ_i 's are \mathbb{T} -quasi-torsors that are not torsors.

Note that (2) in the previous theorem is implied by Theorem 1.5. Thus, the importance relies on (3). It shows that a full generalization of Greb-Kebekus-Peternell to the case of \mathbb{T} -quasi-torsors and finite quasi-torsors is not feasible. Our final statement in this subsection says that this failure can be fixed if we restrict ourselves to a special class of \mathbb{T} -quasi-torsors. A \mathbb{T} -quasi-torsor $Y \rightarrow X$ is said to be *factorial* if the variety Y is factorial.

Theorem 1.8. *Let X be a variety with klt type singularities. Consider a sequence of morphisms*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

such that each ϕ_i is either a finite quasi-torsor or a factorial \mathbb{T} -quasi-torsor. Then, all but finitely many of the ϕ_i are torsors.

1.4. Factorial models

In this section, motivated by the previous statement, we study factorial covers of klt varieties. In Theorem 1.5 of [21], the authors prove that a variety X with klt type singularities admits a quasi-étale finite Galois cover $Y \rightarrow X$ for which $\pi_1(Y^{\text{reg}})$ is isomorphic to $\pi_1(Y)$. In particular, every étale cover of Y^{reg} extends to an étale cover of Y . Our next aim is to improve this result by considering both; finite quasi-torsors and torus quasi-torsors. By doing so, we can also improve the local class groups of the variety Y obtained by Greb, Kebekus, and Peternell. We show that any variety with klt type singularities is a G -quotient of a variety with canonical factorial singularities for which its étale fundamental group agrees with the étale fundamental group of its smooth locus.

Theorem 1.9. *Let X be a variety with klt type singularities. Then, there exists a variety Y satisfying the following conditions:*

- (1) the natural epimorphism $\hat{\pi}_1(Y^{\text{reg}}) \rightarrow \hat{\pi}_1(Y)$ of étale fundamental groups is an isomorphism,
- (2) for every finite quasi-étale morphism $Y' \rightarrow Y$, the variety Y' has canonical factorial singularities,
- (3) Y admits the action of a reductive group G ,
- (4) the group G is the extension of an algebraic torus by a finite solvable group, and
- (5) the isomorphism $X \simeq Y//G$ holds.

In particular, Y itself has canonical factorial singularities.

In general, this factorial variety is highly non-unique. The previous theorem can be regarded as a generalization of Theorem 1.5 in [21]. Part (1) of Theorem 1.9 follows from Theorem 1.1 in [21]. An equivalent statement of the latter does not hold for combinations

of finite and torus covers (cf. Theorem 1.7). However, the statement is still valid if we restrict ourselves to finite quasi-torsors and factorial torus quasi-torsors. Thus, we may still apply Theorem 1.8. We also observe that the singularities of the variety Y produced in the previous theorem cannot be improved by taking finite quasi-étale covers and relative Cox rings. Indeed, every finite quasi-torsor or torus quasi-torsor over Y is a torsor.

A classic topic in algebraic geometry is deciding when a variety which is locally a quotient is indeed globally a quotient. Fulton asked whether varieties with finite quotient singularities are finite quotients of smooth varieties. In [17], the authors prove that a variety with finite quotient singularities is the quotient of a smooth variety by a linear algebraic group. In [27], it is proved that a variety with finite quotient singularities admits a finite flat surjection from a smooth variety. In Theorem 1.2 of [18], the authors show that a variety with finite abelian quotient singularities that is globally the quotient of a smooth variety by a torus is globally the quotient of a smooth variety by a finite group. In this last paper, the language of stacks and Cox rings is used. In this direction, we prove the following positive result in the case of locally toric singularities.

Theorem 1.10. *Let X be a variety with locally toric singularities. Then, X admits a torus quasi-torsor which is a smooth variety. In particular, X is the quotient of a smooth variety by the action of a torus.*

In the previous theorem, a singularity $x \in X$ is said to be locally toric if there exists a toric variety T and a closed invariant point $t \in T$ such that $X_x \simeq T_t$. Here, X_x (respectively, T_t) is the spectrum of the local ring $\mathcal{O}_{X,x}$ (respectively, $\mathcal{O}_{T,t}$). The previous theorem can be regarded as a generalization of the fact that a toric variety is the quotient of an open subset of \mathbb{A}^n by a torus action. A point $x \in X$ is said to be *formally toric* if there exists a toric variety T and a closed invariant point $t \in T$ such that $\widehat{X}_x \simeq \widehat{T}_t$. The statement of the previous theorem does not hold if we replace the condition on locally toric singularities with formally toric singularities (see Example 7.4).

1.5. Normal singularities

Throughout the introduction, we focused on varieties with klt type singularities. In this last part, we discuss what class of singularities is the optimal class for which the previous theorems work. We recall the following theorem due to Stibitz (see Theorem 1 in [37]).

Theorem 1.11. *Let X be a normal variety. The following conditions are equivalent.*

- (1) *Every sequence of finite quasi-torsors*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

is eventually a sequence of torsors.

- (2) *For every point $x \in X$, the image of the homomorphism $\widehat{\pi}_1^{\text{reg}}(X; x) \rightarrow \widehat{\pi}_1^{\text{reg}}(X)$ is finite.*

The group $\widehat{\pi}_1^{\text{reg}}(X)$ is the étale fundamental group of the smooth locus of X . On the other hand, $\widehat{\pi}_1^{\text{reg}}(X; x)$, called *étale regional fundamental group*, is the profinite completion of the fundamental group of the smooth locus around the singularity (see, e.g., [11]).

The regional fundamental group of klt type singularities is finite, so the previous theorem recovers Theorem 1.1 in [21]. Motivated by the previous result, we prove the following theorem regarding \mathbb{T} -quasi-torsors and the Zariski-local class group $\text{Cl}(X; x)$ of X at x , cf. [2], p. 61.

Theorem 1.12. *Let X be a normal variety. The following conditions are equivalent.*

- (1) *Every sequence of \mathbb{T} -quasi-torsors*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

is eventually a sequence of torsors.

- (2) *For every point $x \in X$, the group $\text{Cl}(X; x)$ is finitely generated.*

Due to Theorem 1.7, we know that the similar statement for finite quasi-torsors and torus quasi-torsors fails, even for toric singularities. However, in view of Theorem 1.8, we can expect a similar statement to hold for finite quasi-torsors and factorial \mathbb{T} -quasi-torsors. In order to state the following theorem, we need to introduce the concept of *partial quasi-étale Henselizations*.

Definition 1.13. Let X be an algebraic variety and let $x \in X$ be a point. The *partial quasi-étale Henselization* of X at x , denoted by X_x^{ph} , is the spectrum of the colimit of all quasi-étale covers $\mathcal{O}_{X,x} \rightarrow R$ that extend to quasi-étale covers of X itself.

With the previous definition, we can state the theorem that describes the optimal class of singularities for which every sequence of finite quasi-torsors and factorial \mathbb{T} -quasi-torsors is eventually étale.

Theorem 1.14. *Let X be a normal variety. The following conditions are equivalent.*

- (1) *Every sequence of finite quasi-torsors and factorial \mathbb{T} -quasi-torsors*

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

is eventually a sequence of torsors.

- (2) *For every point $x \in X$, the following two conditions are satisfied:*

- (a) *the image $\hat{\pi}_1^{\text{reg}}(X; x) \rightarrow \hat{\pi}_1^{\text{reg}}(X)$ is finite, and*
- (b) *the class group $\text{Cl}(X_x^{\text{ph}})$ is finitely generated.*

The proofs of Theorem 1.12 and Theorem 1.14 are quite similar to those of the statements for klt type singularities. We will prove these statements in Subsection 6.1.

2. Preliminaries

In this section, we recall the definitions of the singularities of the minimal model program. We also recall the definition of G -quotients and G -quasi-torsors, and prove some preliminary results. We work over an algebraically closed field \mathbb{K} of characteristic zero. All the considered varieties are normal unless stated otherwise. A *reductive group* G is a linear algebraic group G for which the unipotent radical is trivial.

2.1. Singularities of the MMP

In this subsection, we recall the definitions of the singularities of the MMP.

Definition 2.1. A *log pair* (X, Δ) consists of the data of a quasi-projective variety X and an effective \mathbb{Q} -divisor Δ for which $K_X + \Delta$ is \mathbb{Q} -Cartier. The *standard approximation* of Δ is the largest effective divisor Δ_s with coefficients of the form $\text{coeff}_D(\Delta_s) = 1 - 1/n_D$ for positive integers n_D and satisfying $\Delta \geq \Delta_s$. Let $x \in X$ be a closed point. We write $(X, \Delta; x)$ for the log pair (X, Δ) around x . When we write statements about $(X, \Delta; x)$, we mean that such statement holds for (X, Δ) on a sufficiently small neighborhood of x .

Definition 2.2. Let (X, Δ) be a log pair. Let $\pi: Y \rightarrow X$ be a projective birational morphism from a normal quasi-projective variety Y . Let $E \subset Y$ be a prime divisor. We let Δ_Y be the strict transform of Δ on Y . We fix canonical divisors K_Y on Y and K_X on X for which $\pi_*K_Y = K_X$. The *log discrepancy* of (X, Δ) at E , denoted by $a_E(X, \Delta)$, is the rational number

$$1 + \text{coeff}_E(K_Y - \pi^*(K_X + \Delta)).$$

Hence, the following equality holds:

$$\pi^*(K_X + \Delta) = K_Y + \Delta_Y + (1 - a_E(X, \Delta))E.$$

The log discrepancy $a_E(X, \Delta)$ only depends on E and does not depend on Y . We say that (X, Δ) is a *Kawamata log terminal pair* (or *klt pair* for short) if the inequality

$$a_E(X, \Delta) > 0$$

holds for every prime divisor E over X . We say that the pair (X, Δ) is *log canonical* (or *lc* for short) if the inequality

$$a_E(X, \Delta) \geq 0$$

holds for every prime divisor E over X .

Definition 2.3. We say that (X, Δ_0) is of *klt type* if there exists a boundary $\Delta \geq \Delta_0$ on X for which the pair (X, Δ) is klt. We say that (X, Δ_0) is of *lc type* if there exists a boundary $\Delta \geq \Delta_0$ on X for which the pair (X, Δ) is lc.

The following proposition is proved in Section 4 of [13].

Proposition 2.4. *The klt type condition is an étale condition. More precisely, let X be an algebraic variety, if for every point $x \in X$ we can find an étale neighborhood $U_x \rightarrow X$ and a boundary Δ_x for which (U_x, Δ_x) is klt, then there exists a boundary Δ on X for which (X, Δ) is klt.*

2.2. G-quotients and G-quasi-torsors

In this section, we recall the definitions of G -quotients and G -quasi-torsors.

Definition 2.5. Let (X, Δ) be a pair. Let G be a reductive group acting on (X, Δ) . Assume that the quotient $Y := X//G$ exists. Then, we say that Y is a *G-quotient* of X . We also say that X is a *G-cover* of the variety Y .

Definition 2.6. Let $(X, \Delta; x)$ be a singularity of a pair. Assume that X is an affine variety. Let G be a reductive group acting on (X, Δ) and fixing x , i.e., we have that $g^* \Delta = \Delta$ and $g(x) = x$ for each $g \in G$. Let $(X, \Delta; x) \rightarrow (Y; y)$ be the quotient morphism where y is the image of x . We say that $(X, \Delta; x) \rightarrow (Y; y)$ is a G -quotient around x . The morphism $X \rightarrow Y$ will be called a G -quotient. We say that Y is the G -quotient of X and that X is the G -cover of Y .

Now, we turn to define better behaved quotients. We introduce the concept of G -quasi-torsors.

Definition 2.7. Let (Y, Δ_Y) be a log pair. Let X be a variety with the action of G reductive for which $Y \simeq X//G$. We say that the quotient morphism $\phi: X \rightarrow Y$ is a G -quasi-torsor for (Y, Δ_Y) if the following conditions are satisfied:

- (1) there are codimension two open subsets $U_Y \subset Y$ and $U_X = \phi^{-1}(U_Y) \subset X$ for which

$$\phi|_{U_X} : U_X \rightarrow U_Y$$

is a G -torsor, and

- (2) the global invertible homogeneous functions on X descend to Y via the induced homomorphism $\mathcal{O}(X)^G \simeq \mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$.

In general, the G -quotient Y does not come with a naturally defined boundary. However, in some cases, it is possible to introduce such boundary and compare the log discrepancies on X with those on Y . The following lemma is well known to the experts (see, e.g., Proposition 2.11 in [29]).

Lemma 2.8. *Let $(X; x)$ be a klt type singularity. Then, the following statements hold.*

- (1) *Let G be a finite group acting on $(X; x)$. The G -quotient $(Y; y)$ is of klt type.*
- (2) *Let G be a finite group and let $(Y; y) \rightarrow (X; x)$ be a G -quasi-torsor. Then, $(Y; y)$ is of klt type.*

Definition 2.9. We say that a G -quasi-torsor is an *abelian quasi-torsor* if G is an abelian group. We say that a G -quasi-torsor is a *torus quasi-torsor* if G is a torus. In this case, we also write \mathbb{T} -quasi-torsor or \mathbb{T} -torsor. A quasi-torsor $Y \rightarrow X$ is said to be a *factorial quasi-torsor* if Y is factorial.

The next lemma follows from the definitions.

Lemma 2.10. *Let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor and let $\mathbb{T}_0 \leq \mathbb{T}$ be a sub-torus. Let $Y \rightarrow Y'$ be the quotient of Y by \mathbb{T}_0 , and let $Y' \rightarrow X$ be the induced morphism. Then, both $Y \rightarrow Y'$ and $Y' \rightarrow X$ are torus quasi-torsors.*

2.3. Cox rings

In this subsection, we recall some statements about Cox rings for singularities and pairs. First, we define the concept of affine local Cox rings.

Definition 2.11. Let $(X; x)$ be a singularity. Recall that by $\text{Cl}(X; x)$ we denote the Zariski-local class group of X at x . Assume that $\text{Cl}(X; x)$ is finitely generated. Let $N \leq \text{WDiv}(X)$

be a free finitely generated subgroup surjecting onto $\text{Cl}(X; x)$ and N^0 be the kernel of the surjection $\pi: N \rightarrow \text{Cl}(X; x)$. Consider a group homomorphism $\chi: N^0 \rightarrow \mathbb{K}(X)^*$ for which

$$\text{div}(\chi(E)) = E$$

for all $E \in N^0$. We call such χ a *character*. Let \mathcal{S} be the sheaf of divisorial algebras associated to N and \mathcal{I} be the ideal subsheaf generated by sections $1 - \chi(E)$, where $E \in N^0$. Then, we define the *affine local Cox ring* of (X, Δ) at x to be

$$\text{Cox}(X; x)_{N, \chi}^{\text{aff}} := \bigoplus_{[D] \in \text{Cl}(X; x)} \frac{\bigoplus_{D' \in \pi^{-1}([D])} \mathcal{S}_{D'}(X)}{\mathcal{I}(X)}.$$

Now, we define the concept of relative Cox ring for a log pair (X, Δ) . Recall from Definition 3.1 in [14] that the group $\text{WDiv}(X, \Delta)$ of *orbifold Weil divisors* is the free abelian group generated by \mathbb{Q} -divisors D on X so that for the standard approximation $\Delta_s = \sum(1 - 1/n_P)P$ and every prime divisor $P \subset X$, the denominator of $\text{coeff}_P(D)$ divides n_P .

Definition 2.12. Let (X, Δ) be a log pair. Let W_1, \dots, W_k be orbifold Weil divisors on (X, Δ) . Let N be the subgroup of $\text{WDiv}(X)$ spanned by W_1, \dots, W_k . We define the sheaf

$$\mathcal{R}(X)_N := \bigoplus_{D \in N} \mathcal{O}_X(D) \simeq \bigoplus_{(m_1, \dots, m_k) \in \mathbb{Z}^k} \mathcal{O}_X(m_1 W_1 + \dots + m_k W_k).$$

The ring $\mathcal{R}(X)_N$ is called a *relative Cox ring* of X . The relative spectrum

$$Y := \text{Spec}_X(\mathcal{R}(X)_N) \rightarrow X,$$

is called a *relative Cox space* of X . We may also call $\mathcal{R}(X)_N$ the relative Cox ring associated to N and Y the *relative Cox space* associated to N .

Note that in the definition of the local Cox ring, we quotient by a certain ideal $\mathcal{I}(X)$ which comes from a character χ . However, in our definition of the relative Cox ring we do not perform such a quotient. Example 7.2 shows some pathology that would happen otherwise. The definition of the relative Cox ring does not depend on the choice of W_i in its linear equivalence class.

Lemma 2.13. *Let X be an algebraic variety. Let W_1, \dots, W_k be Weil divisors on X spanning N in $\text{WDiv}(X)$. For each $i \in \{1, \dots, k\}$, let $W'_i \sim W_i$. Let N' be the subgroup in $\text{WDiv}(X)$ spanned by the Weil divisors W'_i . Then, we have a \mathbb{T} -equivariant isomorphism*

$$\mathcal{R}(X)_N \simeq \mathcal{R}(X)_{N'}.$$

The following is proved in Proposition 4.10 of [14] for the case of klt type singularities. The general case follows from the theory of polyhedral divisors [1]. It states that in the affine setting a torus quasi-torsor is the same as a relative Cox space.

Lemma 2.14. *Let X be a normal affine variety and $x \in X$ a closed point. Let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor over X . Then, up to shrinking X around x , we can find a finitely generated subgroup $N \leq \text{WDiv}(X)$ for which the following isomorphism holds:*

$$Y \simeq \text{Spec}\left(\bigoplus_{D \in N} H^0(X, \mathcal{O}_X(D))\right).$$

Furthermore, the Cox ring in the local setting has klt type singularities (see, e.g., Theorem 3.23 in [14]).

Lemma 2.15. *Let X be an affine variety and let $(X; x)$ be a klt type singularity. Let $N \leq \text{WDiv}(X, \Delta)$ be a free finitely generated subgroup and let $N^0 := \ker(N \rightarrow \text{Cl}(X; x))$. Let $\chi: N^0 \rightarrow \mathbb{K}(X)^*$ be a character. Then, the spectrum of the affine local Cox ring,*

$$\text{Cox}(X; x)_{N, \chi}^{\text{aff}},$$

is an affine variety with klt type singularities.

Remark 2.16. Usually, when defining a Cox ring over some variety X , finite generation of $\text{Cl}(X)$ is required. On the other hand, even if $\text{Cl}(X)$ is not finitely generated, as long as the group N/N^0 is so, we can still investigate finite generation properties of Cox rings $\text{Cox}(X; x)_{N, \chi}^{\text{aff}}$. In other words, finite generation of $\text{Cl}(X)$ and finite generation of Cox rings over X associated to *finitely generated* groups of Weil divisors is a priori unrelated. This can be seen in the finite generation statement (Corollary 1.1.9 in [4]), which makes no assumption on the class group and is formulated relatively for a pushforward of a Cox sheaf.

This very statement leads to the well-known finite generation property of the Cox ring of a Fano variety, see Corollary 1.3.2 in [4]. The original statement is again relative over an affine base U , and when U is chosen to be a point, then X becomes a Fano variety and the pushforward Cox sheaf from Corollary 1.1.9 in [4] becomes the Cox ring. A central observation from [11], further developed in [14], is that if one considers the other extreme for the structure morphism $X \rightarrow U$, namely *the identity*, then one achieves *finite generation of Cox rings* – associated to finitely generated groups of Weil divisors – *over affine klt type varieties*.

Apart from the finite generation claim, Lemma 2.15 makes a statement about klt-ness of the Cox ring. Similar to the question of finite generation, this is analogous to the projective case, where [15, 19] shows that the Cox ring of a Fano variety has klt singularities. Indeed, the local and the global statement are equivalent. While [19] uses reduction to characteristic p , the proof in [15] reduces the question iteratively to line bundles. In [14], see Lemma 3.22, Theorem 3.23 and Corollary 3.24, the statement is proven with a flavor similar to [15], but in the relative and, in particular, in the affine setting. The reason that the global Cox ring of a klt but non-Fano variety may be non-klt, is that it captures global geometry of X in the non-klt points. These points are in the vanishing locus of the so-called *irrelevant ideal*, so they are not in the relative spectrum of the Cox sheaf (which is klt), and consequently do *not* map to some point of X .

The next lemma will be used in the comparison of quasi-torsors and relative Cox rings.

Lemma 2.17. *Let W and W' be two Weil divisors on a normal variety X . Denote*

$$\mathcal{X}_W := \text{Spec}_X \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mW) \right) \quad \text{and} \quad \mathcal{X}_{W'} := \text{Spec}_X \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mW') \right).$$

Then, the following are equivalent:

- (1) *There is a \mathbb{G}_m -equivariant isomorphism $\mathcal{X}_W \simeq \mathcal{X}_{W'}$.*
- (2) *We have that $W \sim W'$ on X .*

Proof. We first prove that item (1) follows from item (2). This follows verbatim from the proof of Construction 1.4.1.1 in [2]. Since the conditions there are different from ours (but lead to the same conclusion), we recall the argument. Let $W - W' = \text{div}(f)$ and define a homomorphism

$$\eta : \langle W_{\mathbb{Z}} \rangle \rightarrow \mathbb{K}(X)^* \quad \text{and} \quad kW \mapsto f^k.$$

Then we obtain an equivariant isomorphism between the sheaves $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mW)$ and $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mW')$ by mapping $f \in \mathcal{O}_X(kW)$ to $\eta(kW) \cdot f \in \mathcal{O}_X(kW')$.

Now, we prove that item (2) follows from item (1). This is essentially contained in the proof of Proposition 1.6.4.5 in [2]. Denote the multiplicative groups of homogeneous rational functions on \mathcal{X}_W and $\mathcal{X}_{W'}$ by $E(\mathcal{X}_W)$ and $E(\mathcal{X}_{W'})$, respectively. Denote by $q^* : \text{WDiv}(\mathcal{X}_W)^{\mathbb{G}_m} \rightarrow \text{WDiv}(X)$ the pushforward from invariant Weil divisors on \mathcal{X}_W to Weil divisors on X sending prime divisors to the closure of their images if it is of codimension 1 and to 0 otherwise. By abuse of notation, we denote the respective pushforward for $\mathcal{X}_{W'}$ by q^* as well. Then we have homomorphisms

$$\delta_W : E(\mathcal{X}_W) \rightarrow \text{WDiv}(X) \quad \text{and} \quad \delta_{W'} : E(\mathcal{X}_{W'}) \rightarrow \text{WDiv}(X)$$

mapping a rational function f to $q^*\text{div}(f)$. As the proof of Proposition 1.6.4.5 in [2] shows, these induce monomorphisms $\mathbb{Z} \rightarrow \text{Cl}(X)$ from the respective grading groups mapping $m \in \mathbb{Z}$ to $[\delta_W(f)] = [mW]$ (or $[\delta_{W'}(f)] = [mW']$, respectively) for some $f \in \mathcal{O}_X(mW)$ (or $f \in \mathcal{O}_X(mW')$, respectively). The claim follows since the \mathbb{G}_m -equivariant isomorphism $\mathcal{X}_W \simeq \mathcal{X}_{W'}$ induces an isomorphism of the grading groups. ■

3. G -covers of klt type singularities

In this section, we study G -covers of klt type singularities. As seen in Example 7.1, we need to focus on those G -covers that are unramified over codimension one points. First, we will show that semisimple covers of klt type singularities may not be of klt type. The following is a generalization of Theorem 1.1 to higher-dimensional toric singularities.

Theorem 3.1. *For any $n \geq 2$, there exists a $(n + 1)$ -dimensional toric singularity $(X; x)$ that admits a $\mathbb{PGL}_r(\mathbb{K})$ -cover $Y \rightarrow X$, satisfying the following conditions:*

- (1) *we have $r = 3$ if $n \in \{2, 3\}$ and $r = n$ otherwise,*
- (2) *the singularity $(Y; y)$ has dimension $(n + r - 1)$, and*
- (3) *the singularity $(Y; y)$ is not of klt type.*

Proof. First, we choose an appropriate projective toric variety, depending on the value of n . If $n = 2$, we choose a smooth projective toric surface T of Picard rank 4 and fix $r = 3$. If $n = 3$, we choose a smooth projective toric threefold T of Picard rank 3 and fix $r = 3$. For $n \geq 4$, we choose a smooth projective toric $(n - 1)$ -fold T of Picard rank n and we fix $r = n$. In any of the previous cases, by Theorem 1.1 in [20], we can find a vector bundle \mathcal{E} of rank r over T such that the Cox ring of $\mathbb{P}(\mathcal{E})$ is not finitely generated. We fix $G := \mathbb{PGL}_r(\mathbb{K})$ to be the projective linear group acting on $\mathbb{P}(\mathcal{E})$. Note that $\pi : \mathbb{P}(\mathcal{E}) \rightarrow T$ is a quotient for the G -action. Let A_T be an ample toric divisor on T . Let m be a positive integer. Then

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\pi^* A_T/m)$$

is an ample \mathbb{Q} -line bundle on $\mathbb{P}(\mathcal{E})$, for m large enough, which is G -invariant. Thus, the affine variety

$$Y = \text{Spec}\left(\bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n\pi^*A_T/m))\right)$$

admits a G -action which fixes the vertex $y \in Y$ of the \mathbb{G}_m -action induced by the \mathbb{Z} -grading. Observe that an element

$$f \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n\pi^*A_T/m))$$

is preserved by the action of G if and only if it is constant along the fibers of $\mathbb{P}(\mathcal{E}) \rightarrow T$. In other words, the G -invariant elements must have the form $f = \pi^*g$, for some $g \in H^0(T, \mathcal{O}_T(nA_T/m))$. We conclude that there is an isomorphism

$$Y//G \simeq \text{Spec}\left(\bigoplus_{n \in \mathbb{Z}} H^0(T, \mathcal{O}_T(nA_T/m))\right) = X.$$

Thus, the quotient Y/G is isomorphic to the cone over a \mathbb{Q} -ample toric divisor on a smooth projective toric variety. Hence, $(X; x)$ is a toric singularity of dimension n . It suffices to check that $(Y; y)$ is not of klt type.

We proceed by contradiction. Assume that $(Y; y)$ is of klt type. Let $\tilde{Y} \rightarrow Y$ be the blow-up of Y at the maximal ideal of y . Then, the exceptional divisor E of $\phi: \tilde{Y} \rightarrow Y$ is isomorphic to $\mathbb{P}(\mathcal{E})$. Since $\mathbb{P}(\mathcal{E})$ is smooth, we conclude that \tilde{Y} has \mathbb{Q} -factorial singularities. Let Δ_Y be the effective divisor through y for which $(Y, \Delta_Y; y)$ has klt singularities. Let $\Delta_{\tilde{Y}}$ be the strict transform of Δ_Y on \tilde{Y} . We write

$$\phi^*(K_Y + \Delta_Y) = K_{\tilde{Y}} + \Delta_{\tilde{Y}} + (1 - a)E,$$

for some positive number a . Note that $\Delta_{\tilde{Y}}$ is ample over Y as $\rho(\tilde{Y}/Y) = 1$. We conclude that $K_{\tilde{Y}} + (1 - a)E$ is antiample over Y , so $K_Y + E$ is antiample over Y as well. Since E is smooth, we conclude that the pair (Y, E) is plt. Thus, the pair $K_E + \Delta_E = (K_Y + E)|_E$, obtained by performing adjunction to E , is log Fano. In particular, the projective variety $\mathbb{P}(\mathcal{E}) \simeq E$ is of Fano type. Thus, the Cox ring of $\mathbb{P}(\mathcal{E})$ is finitely generated, by Corollary 1.9 in [4]. This leads to a contradiction. We conclude that $(Y; y)$ is not a klt type singularity. ■

In the previous theorem, the action is not free outside the point $y \in Y$. We show that this can be improved in the following statement.

Proposition 3.2. *For any $n \geq 2$, there exists a $(n + 1)$ -dimensional toric singularity $(X; x)$ that admits a $\mathbb{PGL}_r(\mathbb{K})$ -cover $Y \rightarrow X$, satisfying the following conditions:*

- (1) *we have that $r = 3$ if $n \in \{2, 3\}$ and $r = n$ otherwise,*
- (2) *the germ $(Y; y)$ has dimension $n + r^2$,*
- (3) *the action of $\mathbb{PGL}_r(\mathbb{K})$ on Y is free on a dense open set, and*
- (4) *the singularity $(Y; y)$ is not of klt type.*

Proof. Let T be a n -dimensional smooth projective toric variety. Let $\mathbb{P}(\mathcal{E})$ be the rank r vector bundle over T considered in the proof of Theorem 3.1. Hence, the variety $\mathbb{P}(\mathcal{E})$ is not a Mori dream space. Let $Y_0 \rightarrow T$ be the associated principal $\mathbb{PGL}_r(\mathbb{K})$ -bundle. Con-

sider a $\mathbb{PGL}_r(\mathbb{K})$ -equivariant projectivization $Y_0 \hookrightarrow \bar{Y}$ with a relatively ample line bundle $\mathcal{O}_{\bar{Y}}(1)$ over T . Observe that the action of $\mathbb{PGL}_r(\mathbb{K})$ on \bar{Y} is free on the open subset Y_0 . We claim that \bar{Y} is not a Mori dream space. Let $\rho: \mathbb{PGL}_r(\mathbb{K}) \rightarrow \text{Aut}(\mathbb{P}^r)$ be the standard representation. Then $\bar{Y} \times \mathbb{P}^r$ admits the action of G given by $g \cdot (u, v) = (g^{-1}u, \rho(g)v)$. The quotient of $\bar{Y} \times \mathbb{P}^r$ by G is isomorphic $\mathbb{P}(\mathcal{E})$. If \bar{Y} is a Mori dream space, then $\mathbb{P}(\mathcal{E})$ is also a Mori dream space, by Theorem 1.1 in [33]. This leads to a contradiction. We conclude that \bar{Y} is not a Mori dream space. The rest of the proof proceeds as in Theorem 3.1, by replacing $\mathbb{P}(\mathcal{E})$ with \bar{Y} . ■

Now, we turn to prove that G -covers of klt type singularities are again of klt type, provided that G is a finite extension of a torus. The following is a generalization of Theorem 1.2 which allows ramification over codimension one points.

Theorem 3.3. *Let $(X, \Delta; x)$ be a klt type singularity. Let G be a finite extension of a torus. Let $\pi: Y \rightarrow X$ be a G -quasi-torsor. Then, the variety Y is of klt type.*

Proof. By Lemma 2.8, we know that klt type singularities are preserved under finite covers and finite quotients. Hence, we may assume that $G \simeq \mathbb{G}_m^k$ for some k . By Lemma 2.14, we know that there exists a finitely generated subgroup $N \leq \text{WDiv}(X, \Delta)$ such that the isomorphism

$$Y \simeq \text{Spec}\left(\bigoplus_{D \in N} H^0(X, \mathcal{O}_X(D))\right)$$

holds. Let $\pi: N \rightarrow \text{Cl}(X, \Delta; x)$ be the induced homomorphism. Let N_0 be the kernel of π . We can choose a homomorphism $\chi: N_0 \rightarrow \mathbb{K}(X)^*$ so that $\text{div}(\chi(E)) = E$ for every $E \in N_0$. Then, we can define the local-affine Cox ring of $(X, \Delta; x)$ associated to the data N and χ as in Definition 2.11. We denote this ring by

$$\text{Cox}(X, \Delta; x)_{N, \chi}^{\text{aff}}$$

and we denote by Y_0 its spectrum. By Lemma 2.15, we know that Y_0 has klt type singularities. Applying Lemma 2.14 to the torus cover $Y \rightarrow Y_0$, we can find a free finitely generated subgroup $N_1 \leq \text{CaDiv}(Y_0)$ for which the isomorphism

$$Y \simeq \text{Spec}\left(\bigoplus_{D \in N_1} H^0(Y_0, \mathcal{O}_{Y_0}(D))\right)$$

holds. Observe that we can choose the divisors of N_1 to be Cartier on Y_0 . Indeed, these divisors correspond to the divisors of N , which become Cartier on Y_0 . Thus, we conclude that the torus quotient $Y \rightarrow Y_0$ is a principal torus cover. Hence, the variety Y has klt type singularities, since the klt type property is locally étale by Proposition 2.4. ■

4. Weil divisors modulo Cartier divisors

In this section, we study the group of Weil divisors modulo Cartier divisors, which is called the *local class group* in [6]. In general, the group $\text{WDiv}(X)/\text{CaDiv}(X)$ is not finitely generated. This group is trivial if and only if X is locally factorial. In [5], the authors

prove that the \mathbb{Q} -factorial and factorial locus of an algebraic variety are open. In Section 14 of [26], the author studies the non- \mathbb{Q} -Cartier loci of Weil divisors. We recall the following proposition due to Kollár (see Proposition 138 in [26]).

Proposition 4.1. *Let X be a normal proper variety. Let $Z \subset X$ be an irreducible variety. There exists a dense open subset $Z^0 \subset Z$ such that the following holds. Let D be a Weil divisor that is Cartier at the generic point η_Z of Z . Then, the divisor D is Cartier at every closed point of Z^0 .*

Due to the previous proposition, we can prove the following theorem using Noetherian induction.

Theorem 4.2. *Let X be a normal variety. There are finitely many closed points $x_1, \dots, x_r \in X$ such that the homomorphism*

$$\text{WDiv}(X)/\text{CaDiv}(X) \rightarrow \bigoplus_{i=1}^r \text{Cl}(X; x_i),$$

is a monomorphism.

Proof. Let U_1, \dots, U_s be an affine open cover of X . Observe that the homomorphism

$$\text{WDiv}(X)/\text{CaDiv}(X) \rightarrow \bigoplus_{i=1}^s \text{WDiv}(U_i)/\text{CaDiv}(U_i),$$

induced by restricting $D \mapsto (D|_{U_1}, \dots, D|_{U_s})$, is a monomorphism. Hence, it suffices to prove the statement for an affine variety.

Without loss of generality, we may assume that X is affine. Let \bar{X} be its closure in a projective space. By Proposition 4.1, there exists an open set $\bar{X}^0 \subset \bar{X}$ so that every Weil divisor on \bar{X} is Cartier at every closed point of \bar{X}^0 . Let Z_1, \dots, Z_k be the irreducible components of $\bar{X} \setminus \bar{X}^0$. For each $i \in \{1, \dots, k\}$, we choose Z_i^0 as in the statement of Proposition 4.1. Then, we proceed inductively with the irreducible components of each $Z_i \setminus Z_i^0$.

We obtain a finite set of irreducible subvarieties $Z_1, \dots, Z_{r_0} \subset \bar{X}$ and dense open subsets $Z_i^0 \subset Z_i$ so that the following set-theoretic equality holds:

$$\bar{X} = \bigcup_{i=1}^{r_0} Z_i^0.$$

We may assume that there exists $r \leq r_0$ for which

$$(4.1) \quad X = \bigcup_{i=1}^r Z_i^0 \cap X,$$

and each intersection $Z_i^0 \cap X$ is non-empty for $i \in \{1, \dots, r\}$. For each $i \in \{1, \dots, r\}$, we choose a closed point $x_i \in Z_i^0 \cap X$. The homomorphism

$$(4.2) \quad \text{WDiv}(X)/\text{CaDiv}(X) \rightarrow \bigoplus_{i=1}^r \text{Cl}(X; x_i),$$

is well defined. Indeed, Cartier divisors are mapped to the zero element on the right-hand side.

It suffices to prove that (4.2) is a monomorphism. Let D be a Weil divisor on X . Let \bar{D} be the closure of D on \bar{X} . Assume that $[D_{x_i}] = 0 \in \text{Cl}(X_{x_i})$ for every $i \in \{1, \dots, r\}$. Then, \bar{D} is Cartier at the generic point η_{Z_i} of Z_i for every $i \in \{1, \dots, r\}$. By Proposition 4.1, we conclude that \bar{D} is Cartier at every closed point of Z_i^0 for every $i \in \{1, \dots, r\}$. In particular, $D = \bar{D} \cap X$ is Cartier at every closed point of $Z_i^0 \cap X$. By equality (4.1), we conclude that D is Cartier at every closed point of X . This means that

$$[D] = 0 \in \text{WDiv}(X)/\text{CaDiv}(X).$$

This finishes the proof of the theorem. ■

We conclude that the group $\text{WDiv}(X)/\text{CaDiv}(X)$ is finitely generated if the variety has rational singularities. In particular, we have the following statement.

Theorem 4.3. *Let X be a normal variety. Then, the group $\text{WDiv}(X)/\text{CaDiv}(X)$ is finitely generated if X has klt type singularities.*

Proof. Let X be a variety with klt type singularities. Let $x_1, \dots, x_r \in X$ be closed points. By Theorem 3.27 in [14], we know that $\bigoplus_{i=1}^r \text{Cl}(X; x_i)$ is a finitely generated abelian group. By Theorem 4.2, we conclude that $\text{WDiv}(X)/\text{CaDiv}(X)$ is a finitely generated abelian group. ■

We have the following corollary from the previous theorem.

Corollary 4.4. *Let (X, Δ) be a klt type pair. Then, the group $\text{WDiv}(X, \Delta)/\text{CaDiv}(X)$ is finitely generated.*

5. Torus covers of klt type varieties

In this section, we study torus covers. We establish a characterization theorem for torus quasi-torsors over varieties with klt type singularities. We will start with the following lemma, that will be used in this section.

Lemma 5.1. *Let X be a variety that admits a \mathbb{T} -action. Let W be a Weil divisor on X . We can find a \mathbb{T} -invariant Weil divisor W' on X for which $W \sim W'$.*

Proof. By Sumihiro’s equivariant completion [38], we may assume that X is an affine \mathbb{T} -variety, where \mathbb{T} is an n -dimensional torus. By Proposition 1.6 in [1], we can find \mathbb{T} -invariant divisors D_1, \dots, D_k such that $X \setminus \bigcup_{i=1}^k D_i \simeq \mathbb{T} \times U$ for some variety U . We may assume that U is smooth. Let $U \hookrightarrow Y$ be a smooth projectivization. By Exercise 12.6 (b) in [22], we know that $\text{Cl}(Y \times \mathbb{P}^n) \simeq \text{Cl}(Y) \times \mathbb{Z}$. The class group of $Y \times \mathbb{P}^n$ is generated by $Y \times H$ and divisors of the form $\Gamma_1 \times \mathbb{P}^n, \dots, \Gamma_r \times \mathbb{P}^n$, where $\Gamma_i \subset Y$ are prime divisors and $H \subset \mathbb{P}^n$ is a hyperplane. Hence, the class group of $\mathbb{T} \times U$ is generated by torus invariant divisors. This implies that the class group of X is generated by torus invariant divisors. ■

Now, we can prove that every torus quasi-torsor is a relative Cox ring.

Proof of Theorem 1.3. Let X be a normal variety. Let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor. We will prove the statement by induction on k , the dimension of \mathbb{T} . First, we will show that the statement holds for $k = 1$.

Let $\pi: Y \rightarrow X$ be a \mathbb{G}_m -quasi-torsor. Let $U \subset X$ be the largest open subset of X for which there is a \mathbb{G}_m -equivariant isomorphism

$$(5.1) \quad \pi^{-1}(U) \simeq \text{Spec}_U \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_U(mW) \right)$$

for a certain Weil divisor W on U . By Lemma 2.14, we know that U is not empty. We claim that $U = X$. By contradiction, assume that $U \subsetneq X$. Let $x \in X$ be a closed point contained in the complement of U . By definition, we can find a Weil divisor W on X for which the isomorphism (5.1) holds. By Lemma 2.14, we may find an affine neighborhood V of x and a Weil divisor W' on V for which there is a \mathbb{G}_m -equivariant isomorphism

$$(5.2) \quad \pi^{-1}(V) \simeq \text{Spec}_V \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_V(W') \right).$$

In particular, there is a \mathbb{G}_m -equivariant isomorphism

$$\text{Spec}_{U \cap V} \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{U \cap V}(mW|_{U \cap V}) \right) \simeq \text{Spec}_{U \cap V} \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{U \cap V}(mW'|_{U \cap V}) \right)$$

over $U \cap V$. By Lemma 2.17, we conclude that $W|_{U \cap V} \sim W'|_{U \cap V}$ holds in $U \cap V$. Write

$$W|_{U \cap V} - W'|_{U \cap V} = \text{div}(f)|_{U \cap V}$$

for some $f \in \mathbb{K}(X)$. We can replace W' with $W' + \text{div}(f)$. By Lemma 2.13, this replacement preserves the equivariant isomorphism (5.2). Thus, we may assume that $W|_{U \cap V} = W'|_{U \cap V}$. Hence, we can find a Weil divisor W'' on $U \cup V$ for which there is a \mathbb{G}_m -equivariant isomorphism

$$\pi^{-1}(U \cup V) \simeq \text{Spec}_X \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_{U \cup V}(W'') \right).$$

This contradicts the maximality of U . We conclude that $U = X$. Thus, the statement of the theorem holds for $k = 1$.

Now, let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor. Let $Y \rightarrow Y_0$ be the quotient by a sub-torus $\mathbb{T}_0 \leq \mathbb{T}$ of dimension $k - 1$. By Lemma 2.10, we conclude that $\pi_1: Y \rightarrow Y_0$ is a \mathbb{T}_0 -quasi-torsor and $\pi_0: Y_0 \rightarrow X$ is a \mathbb{G}_m -quasi-torsor. By induction on the dimension, we can find Weil divisors W_1, \dots, W_{k-1} on Y_0 for which

$$Y \simeq \text{Spec}_{Y_0} \left(\bigoplus_{(m_1, \dots, m_{k-1}) \in \mathbb{Z}^{k-1}} \mathcal{O}_{Y_0}(m_1 W_1 + \dots + m_{k-1} W_{k-1}) \right),$$

and W on X for which

$$Y_0 \simeq \text{Spec}_X \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_X(mW) \right).$$

Both isomorphisms are torus equivariant. By Lemmas 5.1 and 2.13, we may assume that each W_i , with $i \in \{1, \dots, k - 1\}$ is torus invariant. Since $Y_0 \rightarrow X$ contains no horizontal \mathbb{G}_m -invariant divisors, we conclude that $W_i = \pi_0^* W_{i,X}$ for some Weil divisors $W_{i,X}$. Here, the pull-back is defined by restricting to the smooth locus. We set

$$Y' := \text{Spec}_X \left(\bigoplus_{(m_1, \dots, m_k) \in \mathbb{Z}^k} (m_1 W_{X,1} + \dots + m_{k-1} W_{X,k-1} + m_k W_k) \right).$$

Note that Y' has a \mathbb{T}_0 -quotient Y'_0 obtained by considering the graded subring given by $m_i = 0$ for every $i \in \{1, \dots, k - 1\}$. This quotient is isomorphic to Y_0 . Hence, we have a commutative diagram

$$\begin{array}{ccc} Y & & Y' \\ \pi_1 \downarrow & & \downarrow \pi'_1 \\ Y_0 & \xrightarrow{\phi} & Y'_0 \\ \pi_0 \downarrow & \swarrow \pi'_0 & \\ X & & \end{array}$$

By construction, we have that

$$W_i = \pi_0^* W_{X,i} = \phi^* \pi'_0{}^* W_{X,i}$$

holds for every $i \in \{1, \dots, k - 1\}$. We conclude that Y' is \mathbb{T} -equivariantly isomorphic to Y . This finishes the proof. ■

Proof of Theorem 1.4. This statement is local. Hence, it follows from Theorem 1.2 and Theorem 1.3. ■

In order to prove Theorem 1.5, we will need the following lemma.

Lemma 5.2. *Let $\phi: Y \rightarrow X$ be a \mathbb{T}_ϕ -quasi-torsor and let $\psi: Z \rightarrow Y$ be a \mathbb{T}_ψ -quasi-torsor. The following statements hold:*

- (1) *the composition $\phi \circ \psi: Z \rightarrow X$ is a torus quasi-torsor,*
- (2) *if ϕ corresponds to the subgroup $N_Y \leq \text{WDiv}(X)$ and $Z \rightarrow X$ corresponds to the subgroup $N_Z \leq \text{WDiv}(X)$, then $N_Y \leq N_Z$, and*
- (3) *the torus quasi-torsor ψ is a torsor if and only if for every closed point $x \in X$, the images $N_Y \rightarrow \text{Cl}(X; x)$ and $N_Z \rightarrow \text{Cl}(X; x)$ agree. In particular, if the images of N_Y and N_Z agree on*

$$\text{WDiv}(X)/\text{CaDiv}(X),$$

then ψ is a torsor.

Proof. We start by showing that if $\phi: Y \rightarrow X$ and $\psi: Z \rightarrow Y$ are two torus quasi-torsors with acting tori \mathbb{T}_ϕ and \mathbb{T}_ψ respectively, then $\phi \circ \psi: Z \rightarrow X$ is a torus quasi-torsor as well. By Theorem 1.3, ψ corresponds to a relative Cox ring, i.e., to a sheaf of graded algebras as in Section 4.2.3 of [2]. Thus, we can lift the action of \mathbb{T}_ϕ on Y to Z by Proposition 4.2.3.6 in [2]. The quotient by the action of $\mathbb{T}_\phi \times \mathbb{T}_\psi$ is $\phi \circ \psi$. This is a quasi-torsor, since ϕ and ψ are so. Again by Theorem 1.3, $\phi \circ \psi$ corresponds to a relative Cox ring with respect to a subgroup $N \leq \text{WDiv}(X)$. This shows (1). The previous construction also shows (2).

For (3), note that ψ is a torsor if and only if every element of $\phi^* N_Z$ is Cartier in Y . Since $N_Y \leq N_Z$, this happens if and only if the image of N_Z equals the image of N_Y in every local class group of X . ■

Proof of Theorem 1.5. Let X be a variety with klt type singularities. Consider a sequence of morphisms

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

such that each $\phi_i: X_i \rightarrow X_{i-1}$ is a \mathbb{T} -quasi-torsor. We write $\psi_i = \phi_i \circ \dots \circ \phi_1$. By Lemma 5.2 (1), we know that each ψ_i is a \mathbb{T} -quasi-torsor corresponding to a subgroup $N_i \leq \text{WDiv}(X)$. By Lemma 5.2 (2), we know that there is a sequence of subgroups

$$N_1 \leq N_2 \leq \dots \leq N_i \leq \dots$$

By Theorem 4.3, we know that $\text{WDiv}(X)/\text{CaDiv}(X)$ is a finitely generated abelian group. In particular, for some i_0 , we have that the image of every N_i , with $i \geq i_0$, stabilizes in $\text{WDiv}(X)/\text{CaDiv}(X)$. By Lemma 5.2 (3), we conclude that each ϕ_i , with $i \geq i_0$, is a torus torsor. This finishes the proof of the theorem. ■

6. Iteration of torus and finite covers

In this section, we study the iteration of quasi-étale G -covers, where G is either finite or a torus.

Proof of Theorem 1.6. We consider a sequence

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

as in the statement of Theorem 1.6. We claim that if ϕ_i is a \mathbb{T} -cover and ϕ_{i+1} is a finite G_{i+1} -cover (for any $i \geq 1$), then we have a variety X'_i , a commutative diagram

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{\phi_{i+1}} & X_i \\ \phi'_{i+1} \downarrow & & \downarrow \phi_i \\ X'_i & \xrightarrow{\phi'_i} & X_{i-1}, \end{array}$$

where ϕ'_i (respectively, ϕ'_{i+1}) is a G_{i+1} -cover (respectively, \mathbb{T} -cover), which is étale if and only if ϕ_{i+1} (respectively, ϕ_i) is étale. This claim holds since, by the proof of Proposition 5.1 in [14], we have an exact sequence

$$\mathbb{Z}^{\dim(\mathbb{T})} \longrightarrow \pi_1(X_i^{\text{reg}}) \xrightarrow{\varphi} \pi_1(X_{i-1}^{\text{reg}}) \longrightarrow 1,$$

where \mathbb{T} is the general fiber of ϕ_i . Thus if the finite cover ϕ_{i+1} corresponds to the normal subgroup $N \leq \pi_1(X_i^{\text{reg}})$, then we get ϕ'_i as the finite cover of X_{i-1} corresponding to

the image of N in $\pi_1(X_i^{\text{reg}})$ under the above homomorphism φ . By the fiber product $X_{i+1} = X_i \times_{X_{i-1}} X'_i$ (which preserves étaleness, finiteness and GIT-quotients), we get the commutative diagram.

Thus, if for all $j \in \mathbb{N}$, there exists a $k \geq j$ such that ϕ_k is a finite quasi-étale but not étale cover, by reordering of the ϕ_i according to the claim just proven, we can construct an infinite sequence

$$X' = X_0 \xleftarrow{\phi'_1} X'_1 \xleftarrow{\phi'_2} X'_2 \xleftarrow{\phi'_3} X'_3 \xleftarrow{\phi'_4} \dots \xleftarrow{\phi'_i} X'_i \xleftarrow{\phi'_{i+1}} X'_{i+1} \xleftarrow{\phi'_{i+2}} \dots$$

where the ϕ'_i are finite Galois quasi-étale but not étale covers. But this is a contradiction to Theorem 1.1 in [21]. Thus, there are only finitely many finite Galois quasi-étale and not étale covers in the sequence, i.e., there are only finitely many finite quasi-torsors in this sequence that are not finite torsors. ■

In what follows, we turn to prove Theorem 1.7. To do so, we will use the following lemmata.

Lemma 6.1. *Let $\phi: Y \rightarrow X$ be a \mathbb{T} -quasi-torsor of a klt type variety corresponding to the subgroup N of $\text{WDiv}(X)$. Let $\pi: X' \rightarrow X$ be a finite torsor. Let $Y' = Y \times_X X'$ and $Y' \rightarrow X'$ be the associated \mathbb{T} -quasi-torsors so that we have a commutative diagram*

$$\begin{array}{ccc} Y & \longleftarrow & Y' \\ \phi \downarrow & & \downarrow \phi' \\ X & \xleftarrow{\pi} & X'. \end{array}$$

Then, ϕ' is the \mathbb{T} -quasi-torsor associated to the subgroup π^*N of $\text{WDiv}(X')$.

Proof. We consider the dual diagram

$$\begin{array}{ccc} \bigoplus_{D \in N} \mathcal{O}_X(D) & \longrightarrow & (\bigoplus_{D \in N} \mathcal{O}_X(D)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \\ \uparrow & & \uparrow \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'}. \end{array}$$

The top right entry equals $\bigoplus_{D \in N} (\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})$. By definition of the pull-back, we have $\mathcal{O}_{X'}(\pi^*D) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$. So the claim follows. ■

Lemma 6.2. *Let $Y \rightarrow X$ be a torus quasi-torsor corresponding to the subgroup N_Y of $\text{WDiv}(X)$. Let $Z \rightarrow X$ be a torus quasi-torsor corresponding to the subgroup N_Z of $\text{WDiv}(X)$. If $N_Z \supseteq N_Y$ and N_Z/N_Y is torsion free, then there is an induced torus quasi-torsor $Z \rightarrow Y$.*

Proof. The condition that N_Z/N_Y is torsion free means that we have a direct product representation $N_Z = N_Y \oplus N'$ with a subgroup N' of N_Z isomorphic to N_Z/N_Y . The downgrading of \mathcal{O}_Z from N_Z to N' gives an action of a subtorus $\mathbb{T}_{N'} \leq \mathbb{T}_{N_Z}$ on Z . By construction, $Z // \mathbb{T}_{N'} = Y$. Now the statement follows from Lemma 2.10. ■

Proof of Theorem 1.7. The proof of the theorem will consist of three steps. We briefly explain the steps here. In the first step, we will produce a singular variety with a special toric divisor. In the second step, we will produce an infinite sequence of finite torsors for such a singular variety. We show that the rank of the group of Weil divisors modulo Cartier divisors diverges in this sequence. Finally, we will use this divergence property to produce the infinite sequence of \mathbb{T} -quasi-torsors that are not torsors.

Step 1. For each $n \geq 2$, we construct a n -dimensional projective variety with a single isolated toric singularity and infinite étale fundamental group.

Let Z^n be a smooth projective variety with infinite étale fundamental group. Let $z \in Z^n$ be a smooth point. In local coordinates around $z \in Z$, the formal completion $\hat{\mathcal{O}}_{Z,z}$ corresponds to the standard fan $\langle e_1, \dots, e_n \rangle \subset \mathbb{R}^n$. For each $n \geq 2$, we consider the blow-up given by the fan decomposition

$$\Sigma_n := \{ \langle \bar{e}_1, e_2, e_3, \dots, e_n, v \rangle, \langle e_1, \bar{e}_2, e_2, e_3, \dots, v \rangle, \dots, \langle e_1, \dots, e_{n-1}, \bar{e}_n, v \rangle \},$$

where $v = 2e_1 + e_2 + e_3 + \dots + e_n$. We let $Y^n \rightarrow Z^n$ to be the corresponding blow-up. Observe that Y^n has a unique isolated toric singularity. We let $y^n \in Y^n$ be such isolated toric singularity. Note that the local class group of Y^n at y^n is \mathbb{Z}_2 . For $n = 2$, this point is a rational double point. By construction, there is a divisor $T^n \subset Y^n$ which is a normal projective toric variety and y^n is contained in T^n . Indeed, this toric variety corresponds to the primitive lattice generator $v \in \Sigma_n(1)$. Since Y^n has klt singularities and Z^n is smooth, we conclude that $\pi_1(Y^n) \simeq \pi_1(Z^n)$. In particular, the étale fundamental group of Y^n is infinite.

Step 2. We construct a sequence of finite étale Galois covers of Y^n and study their groups of Weil divisors modulo Cartier divisors.

Let

$$Y^n = Y_0^n \xleftarrow{f_1} Y_1^n \xleftarrow{f_2} Y_2^n \xleftarrow{f_3} Y_3^n \xleftarrow{f_4} \dots \xleftarrow{f_i} Y_i^n \xleftarrow{f_{i+1}} Y_{i+1}^n \xleftarrow{f_{i+2}} \dots$$

be an infinite sequence of finite étale Galois covers. Let $k_i := \deg(Y_{i+1}^n \rightarrow Y_i^n)$ be the degree of the cover. Then, the variety Y_i^n is n -dimensional and it has $k_0 \cdots k_{i-1}$ isolated singularities. We denote these singularities as

$$y_{i,(m_0, \dots, m_{i-1})}^n \in Y_i^n,$$

where $1 \leq m_j \leq k_j$ for each j . We can order the singularities in such a way that

$$f_i^{-1}(y_{i-1,(m_0, \dots, m_{i-1})}) = \bigcup_{m=1}^{k_i} y_{i,(m_0, \dots, m_{i-1}, m)}^n.$$

Since T^n has trivial fundamental group, we conclude that $f_i^* \dots f_1^* T^n$ is the disjoint union of $k_0 \cdots k_{i-1}$ toric varieties isomorphic to T^n . We write $T_{i,(m_0, \dots, m_{i-1})}^n$ with $1 \leq m_j \leq k_j$ for such toric divisors. By construction, the toric divisor $T_{i,(m_0, \dots, m_{i-1})}^n$ contains the singular point $y_{i,(m_0, \dots, m_{i-1})}^n$. We claim that

$$(6.1) \quad \text{WDiv}(Y_i^n)/\text{CaDiv}(Y_i^n) \simeq \bigoplus_{i=1}^{k_0 \cdots k_{i-1}} \mathbb{Z}_2.$$

First, observe that $T^n_{i,(m_0,\dots,m_{i-1})}$ is not a Cartier divisor. Indeed, if this was the case, then $T^n_{i,(m_0,\dots,m_{i-1})}$ would be analytically Cartier around $y^n_{i,(m_0,\dots,m_{i-1})}$. This implies that T^n is analytically Cartier around y^n , leading to a contradiction.

On the other hand, $2T^n_{i,(m_0,\dots,m_{i-1})}$ is Cartier in Y^n_i as it is the pull-back of $2T^n$ on a neighborhood of the only singular point that it contains. We conclude that each $T^n_{i,(m_0,\dots,m_{i-1})}$ is 2-torsion in the abelian group $\text{WDiv}(Y^n)/\text{CaDiv}(Y^n)$. Let $J \subset ([1, k_0] \cap \mathbb{Z}) \times \dots \times ([1, k_{i-1}] \cap \mathbb{Z})$ be a subset. Assume that we have a relation of the form

$$\sum_{j \in J} T^n_{i,j} = 0 \in \text{WDiv}(Y^n)/\text{CaDiv}(Y^n).$$

This means that the divisor $\sum_{j \in J} T^n_{i,j}$ is Cartier in Y^n_i . Let $j_0 \in J$ be a fixed element. For each $j_k \neq j_0$ in J , we have that T^n_{i,j_k} is Cartier at y^n_{i,j_0} . We conclude that T^n_{i,j_0} is Cartier at y^n_{i,j_0} . Hence, it is a Cartier divisor. This leads to a contradiction. Then, the isomorphism (6.1) holds.

Step 3. We construct an infinite sequence of finite torsors and torus quasi-torsors of Y^n .

For each $i \geq 1$, we denote by N_i the group of $\text{WDiv}(Y^n_i)$ generated by

$$\{T^n_{i,(m_0,\dots,m_{i-1})} \mid 1 \leq m_0 \leq k_0, \dots, 1 \leq m_{i-2} \leq k_{i-2}, \text{ and } 1 \leq m_{i-1} \leq k_{i-1} - 1\}.$$

For each $i \geq 0$, we define X^n_{2i} to be the relative Cox ring of Y^n_i with respect to N_i . We define X^n_{2i+1} to be $X^n_{2i} \times_{Y^n_i} Y^n_{i+1}$. We define $\phi_{2i+1}: X^n_{2i+1} \rightarrow X^n_{2i}$ to be the induced morphism. Thus, we have a commutative diagram as follows:

$$\begin{array}{ccc} X^n_{2i} & \xleftarrow{\phi_{2i+1}} & X^n_{2i+1} \\ \downarrow & & \downarrow \\ Y^n_i & \xleftarrow{f_{i+1}} & Y^n_{i+1}. \end{array}$$

By Lemma 6.1, the torus quasi-torsor $X^n_{2i+1} \rightarrow Y^n_{i+1}$ is induced by the subgroup $f_{i+1}^* N_i \leq \text{WDiv}(Y^n_{i+1})$. By construction, we have that $f_{i+1}^* N_i \leq N_{i+1}$. By Lemma 6.2 there is a corresponding quasi-torsor $\phi_{2(i+1)}: X^n_{2(i+1)} \rightarrow X^n_{2i+1}$.

We claim that $\phi_{2(i+1)}$ is not a torus torsor. Let C be the class group of Y_{i+1} at the point $y_{i+1,(k_0,\dots,k_{i-1},1)}$. By the isomorphism (6.1), we know that $C \simeq \mathbb{Z}_2$. Note that the image of N_{i+1} in C is isomorphic to \mathbb{Z}_2 . Indeed, the image of the divisor $T^n_{i+1,(k_0,\dots,k_{i-1},1)}$ generates C . On the other hand, the image of $f_{i+1}^* N_i$ in C is trivial since no divisor among the generators of N_i pass through $y_{i,(k_0,\dots,k_{i-1})}$. By Lemma 5.2, we conclude that $\phi_{2(i+1)}$ is a torus quasi-torsor which is not a torsor. We deduce that there exists an infinite sequence

$$X^n = X^n_0 \xleftarrow{\phi_1} X^n_1 \xleftarrow{\phi_2} X^n_2 \xleftarrow{\phi_3} X^n_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X^n_i \xleftarrow{\phi_{i+1}} X^n_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

satisfying the following conditions:

- X^n is a n -dimensional projective variety with a single isolated toric singularity,
- each ϕ_i , with i odd, is a finite torsor, and
- each ϕ_i , with $i \geq 2$ even, is a \mathbb{T} -quasi-torsor which is not a torsor.

This finishes the proof. ■

Now, we turn to prove Theorem 1.8. We will need the following two lemmata.

Lemma 6.3. *Let $f : X' \rightarrow X$ be a finite G -torsor. Let $x \in X$ and $x' \in f^{-1}(x)$ be two closed points. Then, the induced homomorphism $f^* : \text{Cl}(X; x) \rightarrow \text{Cl}(X'; x')$ is a monomorphism.*

Proof. Let W be a Weil divisor through $x \in X$ such that f^*W is principal near x' . We can even assume that there is a G -invariant open around x' where $f^*W = \text{div}(h)$. By Theorem II.3.1 in [36], h is a semiinvariant, i.e., $g^*h = \chi(g)h$, where $\chi(g) \in \mathbb{K}^*$, for every g in G . But the induced action via $\chi : G \rightarrow \mathbb{K}^* \subseteq \mathbb{K}[X]^*$ on X is ramified if it is nontrivial. This cannot happen since f is étale. Therefore, h is G -invariant and defines $W = \text{div}_U(h)$ on some $x \in U \subseteq X$. The claim follows. ■

Lemma 6.4. *Let $Y \rightarrow X$ be the \mathbb{T} -quasi-torsor associated to the group $N \leq \text{WDiv}(X)$. The variety Y is locally factorial if and only if $N \rightarrow \text{Cl}(X; x)$ is surjective for every closed point $x \in X$. In particular, if $N \rightarrow \text{WDiv}(X)/\text{CaDiv}(X)$ is surjective, then $Y \rightarrow X$ is a factorial \mathbb{T} -quasi-torsor.*

Proof. The statement follows from Theorem 1.3.3.3 in [2] applied to the spectrum X_x of the local ring $\mathcal{O}_{X,x}$, where we view N as a subgroup of $\text{WDiv}(X_x)$ by restriction. Then the aforementioned theorem says that the stalk $\mathcal{R}(X)_{N,x}$ is factorial if and only if $N \rightarrow \text{Cl}(X; x)$ is surjective. In fact, Theorem 1.3.3.3 in [2] only states one direction, but it directly follows from applying Theorem 1.3.3.1 in [2] to the smooth locus, which gives an equivalence. It is clear that Y is locally factorial if and only if all the stalks $\mathcal{R}(X)_{N,x}$ are factorial. ■

Proof of Theorem 1.8. Consider a sequence

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

as in the statement of the theorem. This means that every ϕ_i is either a factorial \mathbb{T} -quasi-torsor or a finite quasi-torsor. By Theorem 1.6, we may assume, after possibly truncating our sequence, that every finite quasi-torsor in this sequence is a finite torsor, i.e., a finite Galois étale cover. Proceeding as in the proof of Theorem 1.6, we obtain a sequence of finite torsors

$$X' = X'_0 \xleftarrow{\phi'_1} X'_1 \xleftarrow{\phi'_2} X'_2 \xleftarrow{\phi'_3} X'_3 \xleftarrow{\phi'_4} \dots \xleftarrow{\phi'_i} X'_i \xleftarrow{\phi'_{i+1}} X'_{i+1} \xleftarrow{\phi'_{i+2}} \dots$$

so that each $X_i \rightarrow X'_i$ is a relative Cox ring with respect to the subgroup $N_i \leq \text{WDiv}(X'_i)$. By Lemma 5.2, we know that every \mathbb{T} -quasi-torsor over a factorial variety is indeed a torsor. We write $\psi_i = \phi_i \circ \dots \circ \phi_1$. By Theorem 3.4 (1) in [3], there exists a locally closed decomposition

$$X' = \bigsqcup_{j \in J} Y'_j$$

such that the class group of X'_i at x is independent of $x \in \psi_i^{-1}(Y'_j)$. Applying Lemma 6.3 to closed points of Y'_j , we conclude that there exists $i_0 \in \mathbb{Z}_{>0}$ such that

$$(6.2) \quad \phi'_i{}^* : \text{Cl}(X'_i; y) \rightarrow \text{Cl}(X'_{i+1}; x)$$

is an isomorphism for every $x \in X'_i$, every $y \in \phi_i^{-1}(x)$, and $i \geq i_0$. It suffices to show that whenever X_i is factorial and $X_{i+1} \rightarrow X_i$ is a finite étale Galois cover in our sequence, the variety X_{i+1} is again factorial for every $i \geq i_0$. In this case, we have a commutative diagram

$$\begin{CD} X_i @<\phi_i<< X_{i+1} \\ @V\pi_iVV @VV\pi_{i+1}V \\ X'_i @<\phi'_i<< X_{i+1}, \end{CD}$$

where π_i is the relative Cox ring over X'_i with respect to the subgroup $N_i \leq \text{WDiv}(X'_i)$. By Lemma 6.1, the \mathbb{T} -quasi-torsor $X_{i+1} \rightarrow X_i$ is induced by $\phi_i^* N_i \leq \text{WDiv}(X_{i+1})$. By Lemma 6.4, we know that for every closed point $x \in X'_i$, the induced homomorphism

$$(6.3) \quad N_i \rightarrow \text{Cl}(X'_i; x)$$

is surjective. By isomorphism (6.2) and surjectivity (6.3), we conclude that for every closed point $x \in X'_{i+1}$, we have that

$$\phi_i^* N_i \rightarrow \text{Cl}(X'_{i+1}; x)$$

is surjective. By Lemma 6.4, we conclude that X_{i+1} is a factorial variety. This finishes the proof. ■

Proof of Theorem 1.9. Due to Theorem 1.8, we may find a variety Y satisfying the following properties:

- (i) every finite quasi-torsor over Y is a finite torsor,
- (ii) every factorial \mathbb{T} -quasi-torsor over a finite torsor of Y is a \mathbb{T} -torsor,
- (iii) Y admits the action of a reductive group G ,
- (iv) the group G is an extension of an algebraic torus by a finite solvable group, and
- (v) the isomorphism $X \simeq Y // G$ holds.

Note that condition (i) implies that the natural epimorphism

$$\hat{\pi}_1(Y^{\text{reg}}) \rightarrow \hat{\pi}_1(Y)$$

is an isomorphism. Otherwise, we could find a finite Galois quasi-étale cover of Y that ramifies over the singular locus. This shows that (1) in the statement of the theorem holds.

Let $Y' \rightarrow Y$ be a finite quasi-étale morphism. By condition (i) this morphism is indeed a finite étale morphism. Assume that Y' is not factorial at the point y' . By Theorem 3.7 in [21], there exists a finite étale Galois morphism $Y'' \rightarrow Y$ such that Y'' admits a finite étale Galois morphism to Y' . By Lemma 6.3, we conclude that Y'' is not factorial. Thus, Y'' admits a factorial \mathbb{T} -quasi-torsor that is not a \mathbb{T} -torsor. Indeed, we can take the relative Cox ring of Y'' with respect to a subgroup N of $\text{WDiv}(Y'')$ that surjects onto $\text{WDiv}(Y'')/\text{CaDiv}(Y'')$. This contradicts condition (ii). We conclude that (2) in the statement of the theorem holds. Note that (iii)–(v) are the same than (3)–(5) in the statement of the theorem. This finishes the proof. ■

Proof of Theorem 1.10. Locally toric singularities are klt type singularities. Then, we can apply Theorem 1.6 to deduce that X admits a torus quasi-torsor Y that is factorial. By Theorem 1.4, the variety Y has klt type singularities, hence canonical factorial singularities. The local Cox ring of a locally toric singularity is a locally toric singularity. Hence, the variety Y has factorial locally toric singularities. However, a factorial toric singularity is smooth. We conclude that Y is a smooth variety. ■

6.1. Normal singularities

In this subsection, we show that some of the proofs explained above naturally generalize to normal singularities with some minor considerations.

Proof of Theorem 1.12. If each class group $\text{Cl}(X; x)$ is finitely generated, then by Theorem 4.2, we know that $\text{WDiv}(X)/\text{CaDiv}(X)$ is finitely generated. Then, the proof is verbatim from the proof of Theorem 1.5. This shows that (2) implies (1).

Now, we turn to prove that (1) implies (2). On the other hand, assume that some local class group $\text{Cl}(X; x_0)$ is not finitely generated. We consider an infinite sequence of divisors $\{W_i\}_{i \in \mathbb{N}}$ in $\text{WDiv}(X)$ such that the image of $N_k := \langle W_1, \dots, W_k \rangle$ in $\text{Cl}(X; x_0)$ strictly contains N_{k-1} . Let $X_i \rightarrow X$ be the \mathbb{T} -quasi-torsor associated to N_i . By construction, the quotients N_{i+1}/N_i are free. Then, by Lemma 6.2, we have associated torus quasi-torsors

$$X = X_0 \xleftarrow{\phi_1} X_1 \xleftarrow{\phi_2} X_2 \xleftarrow{\phi_3} X_3 \xleftarrow{\phi_4} \dots \xleftarrow{\phi_i} X_i \xleftarrow{\phi_{i+1}} X_{i+1} \xleftarrow{\phi_{i+2}} \dots$$

By Lemma 5.2 (3), no ϕ_i is a \mathbb{T} -torsor. ■

Now, we turn to give a proof of Theorem 1.14 that discusses optimal normal singularities for which sequences of quasi-torsors are eventually torsors. We will use the following lemma.

Lemma 6.5. *Let $f: X' \rightarrow X$ be a quasi-étale finite Galois morphism. Let $x \in X$ and $x' \in f^{-1}(x)$ be finite points. Then, if $\text{Cl}(X'; x')$ is finitely generated, then $\text{Cl}(X; x)$ is finitely generated.*

Proof. Let W be a Weil divisor through $x \in X$ such that f^*W is Cartier. Then, passing to a G -invariant affine, we may assume $f^*W = \text{div}(h)$. The regular function $h^{|G|}$ is G -invariant. Hence $mW \sim 0$ around x . We conclude that the kernel of $f^*: \text{Cl}(X; x) \rightarrow \text{Cl}(X'; x')$ is torsion. So, if $\text{Cl}(X; x)$ is not finitely generated, then $f^*\text{Cl}(X; x)$ is not finitely generated and the claim follows. ■

Proof of Theorem 1.14. First, assume that conditions (a) and (b) are satisfied. In the proof of Theorem 1.8, we used the argument of Theorem 1.6 to deduce that the finite quasi-torsors are eventually torsors. The same argument goes through in the present case by replacing Theorem 1.1 in [21] with Theorem 1 in [37].

On the other hand, in the proof of Theorem 1.8, we used the constructibility of the functor Cl in the étale topology. We argue that this still holds in the present setting. By Lemma 6.5, every local class group $\text{Cl}(X; x)$ is finitely generated. Let $f: X' \rightarrow X$ be a

resolution of singularities. In this case, we have that $R^1 f_*(\mathcal{O}_{X'}) = 0$ as the local class groups are finitely generated. Then, we can apply Theorem 3.1 (4) in [21] to conclude that the class group functor is constructible in the étale topology. Now, the proof is verbatim from the proof of Theorem 1.8.

Now, assume that condition (1) is not satisfied. By Theorem 1 in [37], there is an infinite sequence of finite quasi-torsors of X that are not torsors. On the other hand, assume that condition (2) is not satisfied. Then, there exists a finite quasi-étale cover $X' \rightarrow X$ and a point x' for which $\text{Cl}(X'; x')$ is not finitely generated. Hence, proceeding as in the proof of Theorem 1.12, we conclude that there exists an infinite sequence of \mathbb{T} -quasi-torsors over X' that are not torsors. ■

7. Examples and questions

In this section, we collect some examples related to the theorems of the article and some questions that lead to further research.

Example 7.1. In this example, we show that finite covers of klt singularities ramified over codimension one points may not be klt. Let

$$D = \{(y, z) \mid y^3 + z^m = 0\} \subset \mathbb{A}_{y,z}^2.$$

Then the singularity

$$X := \{(x, y, z) \mid x^2 + y^3 + z^m = 0\}$$

is a double cover of $\mathbb{A}_{y,z}^2$ ramified along D . The singularity $(X; (0, 0, 0))$ is Du Val for $m \in \mathbb{N}_{\leq 5}$. Otherwise, it is not a klt surface singularity.

Example 7.2. In the construction of the local Cox ring of a singularity $(X; x)$, we choose a homomorphism $N \rightarrow \text{Cl}(X; x)$ with kernel N_0 and a character $\chi: N^0 \rightarrow \mathbb{K}(X)^*$ for which $\text{div}(\chi(E)) = E$ for all $E \in N^0$. If X is a projective variety for which $\text{WDiv}(X)/\text{CaDiv}(X)$ is torsion, then we can consider a surjective homomorphism $N \rightarrow \text{WDiv}(X)/\text{CaDiv}(X)$ with kernel N^0 and a character as before $\chi: N^0 \rightarrow \mathbb{K}(X)^*$. If \mathcal{I} is the ideal sheaf generated by $1 - \chi(E)$ with $E \in N^0$, then the morphism

$$Y := \text{Spec}_X(\mathcal{R}(X)_N/\mathcal{I}) \rightarrow X$$

is finite and ramifies over codimension one points. A priori, it is not clear how to control the divisors over which the previous morphism ramifies. Hence, it is not clear whether Y has klt type singularities provided that X has klt type singularities. This means that the concept of relative Cox ring with quotients induced by characters is not well behaved from the singularities perspective.

Example 7.3. The local Cox ring of a klt type singularity $(X; x)$ is non-trivial whenever $\text{Cl}(X; x)$ is non-trivial. If $(Y; y) \rightarrow (X; x)$ is the spectrum of the local Cox ring of $(X; x)$, then we expect that the equations defining $(Y; y)$ are somewhat simpler than the equations defining $(X; x)$. Although, whenever $\text{Cl}(X; x)$ has non-trivial free part, the dimension of $(Y; y)$ is larger than the dimension of $(X; x)$. For instance, if $(X; x)$ is a toric singularity, then $\text{Cox}(X; x)$ is a smooth point of dimension $\dim X + \rho(X_x)$.

Example 7.4. Consider the affine variety

$$X = \{(x, y, z, w) \mid xy + zw + z^3 + w^3 = 0\}.$$

The variety X is canonical with isolated singularities. Furthermore, X is factorial at $x := (0, 0, 0, 0)$. However, X is not analytically factorial at x . Let $Y \rightarrow X$ be a \mathbb{T} -quasi-torsor. Then, $\pi: Y \rightarrow X$ is a \mathbb{T} -torsor on an affine neighborhood of $x \in X$, by Lemma 5.2. Hence, Y is singular along $\pi^{-1}(x)$. This leads to a contradiction.

Example 7.5. Over a smooth point, every finite quasi-torsor is a torsor and every torus quasi-torsor is a torsor. In this example, we show the existence of a $SL_n(\mathbb{K})$ -quasi-torsor over a smooth germ that is not a torsor. We refer the reader to [34] for more examples in this direction.

In what follows, we let $n \geq 2$. Let W be the space of linear transformations from \mathbb{C}^{n+1} to \mathbb{C}^n . Note that W has dimension $n^2 + n$. Let $SL_n(\mathbb{C})$ act on W by acting on the range of the linear function. The action is free exactly at all the points corresponding to surjective linear transformations. The closure of the orbit of an element contains $0 \in W$ if and only if the corresponding linear transformation does not have full rank. Let $U \subset W$ be the open set consisting of surjective linear transformations. Note that the $SL_n(\mathbb{C})$ -action naturally extends to a $GL_n(\mathbb{C})$ -action. The space of surjective linear transformations, up to the $GL_n(\mathbb{C})$ -action, is parametrized by their kernels. Hence, the quotient $U//GL_n(\mathbb{C}) \simeq \mathbb{P}^n$. Furthermore, we have that $U//SL_n(\mathbb{C}) \simeq \mathbb{A}^{n+1} - \{0\}$. It follows that $W//SL_n(\mathbb{C}) \simeq \mathbb{A}^n$. As explained above, the action is free on U , so $W \rightarrow \mathbb{A}^n$ is a $SL_n(\mathbb{C})$ -torsor over $\mathbb{A}^n - \{0\}$. On the other hand, the fiber over $\{0\}$ is given by the vanishing of at least n minors, so its codimension in W is at least 2. Thus, $W \rightarrow \mathbb{A}^n$ gives a $SL_n(\mathbb{C})$ -quasi-torsor which is not a torsor.

Remark 7.6. One of the reasons that makes the understanding of finite quasi-torsors of a singularity $(X; x)$ easier than other kinds of quasi-torsors is the existence of an algebraic object that detects them. The same holds for torus quasi-torsors.

The previous example might point in the direction that this is not true anymore for arbitrary reductive groups, which might render the study of their (quasi-)torsors a lot more complicated.

Question 7.7. In Theorem 1.1, we showed that there exists a 3-fold toric singularity $(T; t)$ that admits a $\mathbb{P}GL_3(\mathbb{K})$ -cover from a 5-dimensional singularity which is not of klt type. This cover is unramified over codimension points over T so the pathology in Example 7.1 does not happen. This naturally leads to the following question. Is there a G -cover over a surface klt singularity that is unramified over codimension one points and is not of klt type?

Question 7.8. In Theorem 1.2, we showed that if $(X; x)$ is a klt type singularity and G is a finite extension of a torus, then a G -quasi-torsor over $(X; x)$ is of klt type. Does this statement still hold if we only assume that G is a reductive group? We expect that there are counter-examples for this statement if G is a unipotent group.

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