Rev. Mat. Iberoam. 40 (2024), no. 5, 1799–1832
DOI 10.4171/RMI/1479

L^p improving properties and maximal estimates for certain multilinear averaging operators

Chu-hee Cho, Jin Bong Lee and Kalachand Shuin

Abstract. In this article we focus on L^p estimates for two types of multilinear lacunary maximal averages over hypersurfaces with curvature conditions. Moreover, we give a different proof for the bilinear lacunary spherical maximal functions. To obtain our results, we make use of the L^1-improving estimates of multilinear averaging operators. We also obtain L^p-improving estimates for certain multilinear averages by means of the nonlinear Brascamp–Lieb inequality.

1. Introduction

Let $\mathcal{S}$ be a compact and smooth hypersurface contained in a unit ball $\mathbb{B}^d(0,1)$ with κ non-vanishing principal curvatures, and let $\Theta_1,\dots,\Theta_m$ be rotation matrices in $\mathbf{M}_{d,d}(\mathbb{R})$. We assume that $\{\Theta_j\}_{j=1}^m$ is mutually linearly independent. Then, for $f_1, f_2,\dots, f_m \in \mathcal{S}(\mathbb{R}^d)$, we define

$$\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})(x) := \int_{\mathcal{S}} \prod_{j=1}^{m} f_j(x+\Theta_j y)\, \mathrm{d}\sigma_{\mathcal{S}}(y), \tag{1.1}$$

where $\mathrm{F} = (f_1, f_2,\dots, f_m)$ and $\mathrm{d}\sigma_{\mathcal{S}}$ is the normalized surface measure on $\mathcal{S}$. We also consider another m-linear averaging operator defined by

$$A_{\Sigma}(\mathrm{F})(x) := \int_{\Sigma} \prod_{j=1}^{m} f_j(x+y_j)\, \mathrm{d}\sigma_{\Sigma}(y), \quad (y_1,\dots,y_m) = y \in \mathbb{R}^{md}, \tag{1.2}$$

where Σ is a compact $(md-1)$-dimensional smooth hypersurface contained in a unit ball $\mathbb{B}^{md}(0,1)$ with κ non-vanishing principal curvatures. Note that κ arising in (1.1) satisfies $1 \le \kappa \le d-1$, while κ in (1.2) satisfies $1 \le \kappa \le md-1$. Moreover, we are interested in the following lacunary maximal operators associated with (1.1) and (1.2):

$$\mathcal{M}_{\mathcal{S}}^{\Theta}(\mathrm{F})(x) = \sup_{\ell\in\mathbb{Z}} \Big| \int_{\mathcal{S}} \prod_{j=1}^{m} f_j(x-2^{\ell}\Theta_j y)\, \mathrm{d}\sigma_{\mathcal{S}}(y) \Big| \tag{1.3}$$

Mathematics Subject Classification 2020: 42B25 (primary); 47H60 (secondary).
Keywords: multilinear lacunary maximal operators, L^p improving, nonlinear Brascamp–Lieb inequality.

and

$$\mathfrak{M}_\Sigma(\mathrm{F})(x) = \sup_{\ell\in\mathbb{Z}} \Big| \int_\Sigma \prod_{j=1}^m f_j(x - 2^\ell y_j)\, \mathrm{d}\sigma_\Sigma(y) \Big|. \tag{1.4}$$

The purpose of this article is to prove L^p-improving estimates of the multilinear averaging operators defined by (1.1) and (1.2). Further, using these L^p-improving estimates, we show $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \to L^p$ boundedness, for $1/p = \sum_{j=1}^m 1/p_j$, of the multi-(sub)linear lacunary maximal functions $\mathcal{M}_{\mathcal{S}}^\Theta$ and $\mathfrak{M}_\Sigma$.

The averaging operators given in (1.1) and (1.2), and some related maximal operators, arise in many studies in multilinear harmonic analysis. Since Coifman and Meyer [13] opened the path of multilinear harmonic analysis in 1975, there have been significant developments in the area over the last few decades. Among those achievements, we mention the works [24, 25] of Lacey and Thiele, in which they proved L^p-boundedness of the bilinear Hilbert transform given as

$$\mathrm{BHT}_\alpha(f,g)(x) := \mathrm{p.v.} \int_{-\infty}^{\infty} f(x-t)g(x-\alpha t)\, \frac{\mathrm{d}t}{t}, \quad \alpha \neq 0, 1.$$

Their seminal work settled a long standing conjecture of Calderón. Later, Lacey [23] studied L^p-boundedness of the bilinear maximal operator

$$M_\alpha(f,g)(x) := \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} |f(x-y)g(x-\alpha y)|\, \mathrm{d}y, \quad \alpha \neq 0, 1,$$

which is related to the bilinear Hilbert transform. One may regard the averaging operator $\mathcal{A}_{\mathcal{S}}^\Theta$ as a generalization of M_α without the supremum, because the condition $\alpha \neq 0, 1$ corresponds to the linear independence condition of $\{\Theta_j\}$.

On the other hand, A_Σ (given in (1.2)) is a direct analogue, for $t = 1$, of the spherical averages $A_{\mathbb{S}^{d-1}}^t f(x)$ defined by

$$A_{\mathbb{S}^{d-1}}^t f(x) := \int_{\mathbb{S}^{d-1}} f(x - ty)\, \mathrm{d}\sigma(y).$$

Therefore, we write $A_{\mathbb{S}^{md-1}}(\mathrm{F})(x) = A_{\mathbb{S}^{md-1}}^1(f_1 \otimes \cdots \otimes f_m)(x, \ldots, x)$. For studies on $A_{\mathbb{S}^{md-1}}(\mathrm{F})$, we recommend [1, 14, 31, 35] and references therein. In the literature, $A_{\mathbb{S}^{d-1}}^t$ have been extensively studied in terms of maximal operators.

Consider the (sub)linear spherical maximal operator $M_{\mathbb{S}^{d-1}}^*$ defined by

$$M_{\mathbb{S}^{d-1}}^* f(x) = \sup_{t>0} |A_{\mathbb{S}^{d-1}}^t f(x)| := \sup_{t>0} \Big| \int_{\mathbb{S}^{d-1}} f(x-ty)\, \mathrm{d}\sigma(y) \Big|,$$

where $\mathrm{d}\sigma$ is the normalized surface measure on the sphere $\mathbb{S}^{d-1}$. In 1976, Stein [36] proved, for $d \geq 3$, that the spherical maximal operator $M_{\mathbb{S}^{d-1}}^*$ is bounded in L^p if and only if $p > d/(d-1)$. Later, Bourgain [10] obtained L^p boundedness of $M_{\mathbb{S}^1}^*$ for $p > 2$. Those restricted boundedness of $M_{\mathbb{S}^{d-1}}^*$ can be improved if one considers the lacunary spherical maximal operator

$$M_{\mathbb{S}^{d-1}} f(x) := \sup_{j\in\mathbb{Z}} \big| A_{\mathbb{S}^{d-1}}^{2^j} f(x) \big|.$$

Calderón [11] proved L^p estimates of the operator $M_{\mathbb{S}^{d-1}}$ for $1 < p \leq \infty$ and $d \geq 2$. After that, Seeger and Wright [34] showed L^p estimates of general lacunary maximal operators $M_{\mathcal{S}}$ for $1 < p \leq \infty$, when the Fourier transform of the surface measure σ of $\mathcal{S}$ satisfies $|\hat{\sigma}(\xi)| \lesssim |\xi|^{-\varepsilon}$, for any $\varepsilon > 0$. There are also L^p-L^q estimates for $p \leq q$ (we call these L^p-improving estimates) of the spherical average $A^1_{\mathbb{S}^{d-1}}$ [29, 37].

Lacey [22] used the L^p-improving estimates of spherical averages to prove sparse domination of the corresponding lacunary and full spherical maximal functions. It is well known that sparse domination of an operator implies vector valued boundedness and weighted boundedness of that operator with respect to Muckenhoupt A_p weights, see [27, 30]. This idea has been extensively used to obtain sparse domination of several linear and sub-linear operators in the field of harmonic analysis, see [3]. The idea of Lacey [22], together with L^p-improving estimates of certain bilinear averaging operators, can be used to study sparse domination of maximal operators associated with the bilinear operators. We recommend [9, 32, 33] and references therein, which contain results of bilinear spherical maximal operators, bilinear maximal triangle averaging operators and bilinear product-type spherical maximal operators, respectively.

Recently, Christ and Zhou [12] studied $L^{p_1} \times L^{p_2} \to L^p$ (with $1/p_1 + 1/p_2 = 1/p$) boundedness of bi-(sub)linear lacunary maximal functions defined on a class of singular curves, which might be understood in the sense of both (1.3) and (1.4):

$$\mathcal{M}(f_1, f_2)(x) := \sup_{\ell \in \mathbb{Z}} |B_{2^\ell}(f_1, f_2)(x)| = \sup_{\ell \in \mathbb{Z}} \left| \int_{\mathbb{R}^1} \prod_{j=1}^{2} f_j(x - 2^\ell \gamma_j(t))\, \eta(t)\, \mathrm{d}t \right|,$$

where $\gamma = (\gamma_1, \gamma_2) \colon (-1, 1) \to \mathbb{R}^2$ and $\eta \in C_0^\infty((-1, 1))$. In consequence, they have proved $L^{p_1} \times L^{p_2} \to L^p$ estimates for $1 < p_1, p_2 \leq \infty$, $1/p_1 + 1/p_2 = 1/p$, of the bi-(sub)linear lacunary spherical maximal operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$, for dimension $d = 1$:

$$\mathfrak{M}_{\mathbb{S}^1}(f_1, f_2)(x) := \sup_{\ell \in \mathbb{Z}} |A^{2^\ell}_{\mathbb{S}^1}(f_1, f_2)(x)| = \sup_{\ell \in \mathbb{Z}} \left| \int_{\mathbb{S}^1} \prod_{j=1,2} f_j(x - 2^\ell y_j)\, \mathrm{d}\sigma(y) \right|,$$

where $\mathrm{d}\sigma(y)$ is the normalized surface measure on the circle $\mathbb{S}^1$. For $d \geq 2$, the complete $(L^{p_1} \times L^{p_2} \to L^p)$-estimate of the operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$ was not known. However, there are some partial results of the operator $\mathfrak{M}_{\mathbb{S}^{2d-1}}$ in [9, 32], and very recently, Borges and Foster [8] have obtained almost sharp results including some endpoint estimates. In this paper, we give a different proof of the same $(L^{p_1} \times L^{p_2} \to L^p)$-estimate for $\mathfrak{M}_{\mathbb{S}^{2d-1}}$.

There is another important bi-(sub)linear maximal function

$$\mathfrak{M}^*_{\mathbb{S}^{2d-1}}(f_1, f_2)(x) := \sup_{t>0} |A^t_{\mathbb{S}^{2d-1}}(f_1, f_2)(x)|,$$

which is known as bilinear spherical maximal function. The study of this operator started in [2]. Later, in [21], Jeong and Lee proved almost complete $L^{p_1} \times L^{p_2} \to L^p$ estimates for $1/p_1 + 1/p_2 = 1/p$, $p_1, p_2 > 1$ and $p > d/(2d-1)$ when $d \geq 2$. The result was extended to $d = 1$ by Chirst and Zhou [12]. It would be interesting to study $L^{p_1} \times L^{p_2} \to L^p$ boundedness of $\mathfrak{M}^*_\Sigma$, where Σ is a compact smooth hypersurface with κ non-vanishing principal curvatures $(\kappa \leq 2d-1)$. For some specific hypersurfaces, the optimal (except few border line cases) $L^{p_1} \times L^{p_2} \to L^p$ boundedness is known, see [26].

However, for general hypersurfaces with non-vanishing Gaussian curvature, the estimate $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is only known for $d \geq 4$, see [15]. It would be interesting to study $L^{p_1} \times L^{p_2} \to L^p$ estimates of such full maximal averages for $p \leq 1$ in all dimensions and their multilinear analogues. However, multilinear estimates for m-linear full maximal operators with $m \geq 3$ have not been pursued, while $L^2 \times \cdots \times L^2 \to L^{2/m}$ bounds for lacunary maximal operators were studied by Grafakos, He, Honzík and Park [17]. In this paper, we focus on $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ bounds for the lacunary maximal functions for $1/p = 1/p_1 + \cdots + 1/p_m$ and $p < 2/m$. It will be our future goal to study m-linear estimates for the full maximal functions for $m \geq 3$.

We first state L^1-improving and quasi-Banach estimates of the m-linear averaging operators $\mathcal{A}_{\mathcal{S}}^{\Theta}$ and A_Σ. Note that the following two propositions are derived by simple Fourier analysis and multilinear interpolation, and we will give a proof of the propositions for self-containedness.

Proposition 1.1. *Let $\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})$ be given as in (1.1) and let $\mathcal{S}$ be a compact smooth hypersurface contained in $\mathbb{B}^d(0,1)$ with $\kappa \leq d-1$ nonvanishing principal curvatures. Let $\Theta = \{\Theta_j\}_{j=1}^m$ be a family of mutually linearly independent rotation matrices. Let also $\mathcal{V}_\kappa^{ij} = \{z = (z_1, \ldots, z_m) \in [0,1]^m : z_i = z_j = (\kappa+1)/(\kappa+2), z_l = 0, l \neq i, j\}$ and let $\mathrm{conv}(\mathcal{V}_\kappa)$ be its convex hull. Then, for $(1/p_1, \ldots, 1/p_m) \in \mathrm{conv}(\mathcal{V}_\kappa)$, we have*

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

whenever $1 \leq \frac{1}{p} \leq \frac{2(\kappa+1)}{\kappa+2} = \sum_{j=1}^m \frac{1}{p_j}$.

Proposition 1.2. *Let $d \geq 2$ and let $A_\Sigma(\mathrm{F})$ be an average given by (1.2) over a compact smooth hypersurface Σ with κ nonvanishing principal curvatures, with $(m-1)d < \kappa \leq md - 1$. Then, for $1 \leq p_j \leq 2$, $j = 1, 2, \ldots, m$ and $\frac{m+1}{2} \leq \sum_{j=1}^m \frac{1}{p_j} < \frac{2d+\kappa}{2d}$, the following L^1-improving estimates hold:*

$$\|A_\Sigma(\mathrm{F})\|_{L^1(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}. \tag{1.5}$$

Moreover, for $\frac{m+1}{2} \leq \frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} < \frac{2d+\kappa}{2d}$, we have

$$\|A_\Sigma(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}. \tag{1.6}$$

Let $1 \leq p, p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Then, for $f_1, \ldots, f_m$ with $\mathrm{supp}(\widehat{f_j}) \subset \mathbb{A}_{n_j} := \{\xi_j \in \mathbb{R}^d : 2^{n_j-1} \leq |\xi_j| \leq 2^{n_j+1}\}$, $n_j \in \mathbb{Z}$, $j = 1, \ldots, m$, we have

$$\|A_\Sigma(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}, \tag{1.7}$$

where $\delta = \delta(p, \kappa, m, d) > 0$ and $|n| = \sqrt{\sum_{j=1}^m n_j^2}$.

When $p > 1$, one can obtain different L^p-improving estimates for $\mathcal{A}_{\mathcal{S}}^{\Theta}$ under a specific choice of $\{\Theta_j\}$ and $\mathcal{S}$. In this case, we do not need any curvature condition on $\mathcal{S}$ and only the dimension of surfaces matters. Let $\mathcal{S}^k$ be a k-dimensional C^2 surface in $\mathbb{R}^d$. We choose mutually linearly independent $\{\Theta_j\}$. Moreover, we assume that for any choice of $\{j_i\}_{i=1}^{\ell}$, with $2 \le \ell \le k+1 \le m$, the family $\{\Theta_j\}$ satisfies

$$\dim\big(\mathrm{span}_{1\le i\le \ell}(\{\Theta_{j_i}(y',0) \in \mathbb{R}^d : y' \in \mathbb{R}^k\})\big) \ge \min\{k-1+\ell, d\}, \tag{1.8}$$

$$\dim\Big(\bigcap_{i=1}^{\ell}\{\Theta_{j_i}(y',0) \in \mathbb{R}^d : y' \in \mathbb{R}^k\}\Big) \le k+1-\ell. \tag{1.9}$$

The assumption (1.9) yields that the dimension of intersection of any subset $\{\Theta_{j_i}\}_{i=1}^{k+1}$ of $\{\Theta_j\}_{j=1}^m$ is equal to zero. The following theorem is one of our main results.

Theorem 1.3. *Let $m \ge d \ge 2$ and let $\mathcal{S}^k$ be a k-dimensional C^2 surface in $\mathbb{B}^d(0,1)$. Suppose that $\{\Theta_j\}$ satisfies (1.8) and (1.9), and k is given so that*

$$\frac{m-d+k}{m} \ge \frac{d-k-1}{d}k, \tag{1.10}$$

$$\frac{m-1}{m} \ge \frac{(d-k)k}{d}. \tag{1.11}$$

Then $\mathcal{A}_{\mathcal{S}^k}^{\Theta}$ is of strong-type $(m,\dots,m,d/(d-k))$. That is, we have

$$\|\mathcal{A}_{\mathcal{S}^k}^{\Theta}(\mathrm{F})\|_{L^{d/(d-k)}(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^m(\mathbb{R}^d)}.$$

In our proof of Theorem 1.3, we mainly use the nonlinear Brascamp–Lieb inequality proved in [5]. We give details on the inequality and the proof of Theorem 1.3 in Section 3.

In Theorem 1.3, one can use $m \ge d$ to check that (1.10) and (1.11) are equivalent when $d = 2k+1$. Precisely, (1.10) implies (1.11) when $d \ge 2k+1$, and (1.11) implies (1.10) when $d \le 2k+1$. Moreover, if we assume $k = d-1$, then we only need (1.9) to guarantee the following result.

Corollary 1.4. *Let $m \ge d \ge 2$, let $\mathcal{S}^{d-1}$ be a C^2 hypersurface, and let $\{\Theta_j\}$ be mutually linearly independent that satisfy (1.9). Then $\mathcal{A}_{\mathcal{S}^{d-1}}^{\Theta}$ is of strong-type $(m,\dots,m,d)$.*

One can find similar results in Theorem 1.2 of [20], which yields restricted strong-type $(m,\dots,m,m)$ and $(m\frac{d+1}{d},\dots,m\frac{d+1}{d},d+1)$ estimates for $\mathcal{A}_{\mathcal{S}^{d-1}}^{\Theta}$ when $\mathcal{S}^{d-1}$ is a sphere. Note that in [20], the authors consider $m \le d$ cases with linearly independent $\{\Theta_j\}$, so it cannot be directly compared to Corollary 1.4 in which $m \ge d$ and (1.9) are considered. When $m = d$, however, Corollary 1.4 with $\mathcal{S}^{d-1} = \mathbb{S}^{d-1}$ gives strong-type $(m,\dots,m,m)$ estimates.

To study further how Theorem 1.2 of [20] and Corollary 1.4 are related, we introduce a quantity $\mathfrak{D}$ which is given, for each $(p_1,\dots,p_m,p)$-estimate, by

$$\mathfrak{D}(p_1,\dots,p_m;p) := \Big(\frac{1}{p_1}+\dots+\frac{1}{p_m}\Big) - \frac{1}{p}.$$

One can measure the extent of L^p-improving by means of the difference $\mathfrak{D}$. Then we have

$$\mathfrak{D}(m,\dots,m;m)=\frac{m-1}{m},\quad \mathfrak{D}\Big(m\frac{d+1}{d},\dots,m\frac{d+1}{d};d+1\Big)=\frac{d-1}{d+1},$$

where $m\le d$. On the other hand, Corollary 1.4 yields

$$\mathfrak{D}(m,\dots,m;d)=\frac{d-1}{d},\quad m\ge d.$$

Thus, Corollary 1.4 yields wider range of L^p-improving than $(m\frac{d+1}{d},\dots,m\frac{d+1}{d},d+1)$-estimate of Theorem 1.2 in [20] under a certain choice of $\{\Theta_j\}$. We also note that the difference $(d-1)/(d+1)$ is the best possible for linear spherical averages, since $\mathcal{A}_{\mathbb{S}^{d-1}}$ satisfies $L^{(d+1)/d}(\mathbb{R}^d)\to L^d(\mathbb{R}^d)$ boundedness. Even for L^1-improving estimates in Proposition 1.1, we obtain $\mathfrak{D}(p_1,\dots,p_m;1)=(d-1)/(d+1)$. Hence, one can say that the number $(d-1)/d$ only occurs for multilinear averaging operators with certain transversality of $\{\Theta_j\}$. Moreover, we only assume that a surface $\mathcal{S}$ is of class C^2 without any curvature condition, and it would be very interesting to study boundedness of maximal operators associated with $\mathcal{A}^{\Theta}_{\mathcal{S}^k}$.

By making use of the quasi-Banach space estimates, Propositions 1.1 and 1.2 together with Sobolev regularity estimates, we obtain multilinear estimates for the lacunary maximal operators $\mathcal{M}^{\Theta}_{\mathcal{S}}$ and $\mathfrak{M}_{\Sigma}$.

Theorem 1.5. *Let $1\le p_i^\circ\le\infty$ and let $\sum_{i=1}^m 1/p_i^\circ=1/p^\circ$, with $p^\circ\ge 1$ for $d\ge 2$. Suppose that $\mathcal{A}^{\Theta}_{\mathcal{S}}$ satisfies the following Sobolev regularity estimates:*

$$\|\mathcal{A}^{\Theta}_{\mathcal{S}}(\mathrm{F})\|_{L^{p^\circ}(\mathbb{R}^d)}\lesssim 2^{-\varepsilon|\mathbf{n}|}\prod_{j=1}^m\|f_j\|_{L^{p_j^\circ}(\mathbb{R}^d)},\tag{1.12}$$

where $f_1,\dots,f_m$ with $\operatorname{supp}(\widehat{f_j})\subset\mathbb{A}_{n_j}:=\{\xi_j\in\mathbb{R}^d:2^{n_j-1}\le|\xi_j|\le 2^{n_j+1}\}$, $j=1,\dots,m$, and $\varepsilon=\varepsilon(p,\kappa,m,d)>0$. Then the lacunary maximal function $\mathcal{M}^{\Theta}_{\mathcal{S}}$ maps $L^{p_1}(\mathbb{R}^d)\times\cdots\times L^{p_m}(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ for $(1/p_1,\dots,1/p_m)\in\operatorname{conv}(\mathcal{V}_\kappa^\circ)\cup\{(0,\dots,0)\}$ and $1/p=1/p_1+\cdots+1/p_m$, where $\operatorname{conv}(\mathcal{V}_\kappa^\circ)$ denotes an interior of the convex hull of $\operatorname{conv}(\mathcal{V}_\kappa)$ and the origin. In particular, if one considers a lacunary maximal operator associated with $\mathbb{S}^{d-1}$, then the range of p becomes $p>(d+1)/(2d)$.

Observe that the multilinear averaging operator (1.1) is an analogous multilinear averaging operator to the bilinear operator B_θ considered by Greenleaf et al. [19]:

$$B_\theta(f,g)(x)=\int_{\mathbb{S}^1}f(x-y)g(x-\theta y)\,\mathrm{d}\sigma(y),$$

where θ denotes a counter-clockwise rotation. Therefore, Theorem 1.5 (when $m=2$) yields boundedness of the lacunary maximal function corresponding to the averaging operator B_θ under the assumption on the Sobolev regularity estimates (1.12). Thus, one only need to show (1.12), but it is not accomplished in this paper.

On the other hand, one can actually obtain Sobolev regularity estimates for A_Σ, as in (1.7) of Proposition 1.2. Thus, another main result of this paper is the following lacunary maximal estimates for A_Σ.

Theorem 1.6. *Let*

$$\frac{m+1}{2} \le \frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j} < \frac{2d+\kappa}{2d} \quad \text{for } 1 \le p_j \le 2 \text{ and } \kappa > (m-1)d.$$

Then the lacunary maximal operator $\mathfrak{M}_\Sigma$ maps $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.

The $(L^{p_1} \times \cdots \times L^{p_m} \to L^p)$-estimates of Theorem 1.6 are easily extended to $1 \le p_j \le \infty$ via multilinear interpolation, since $\mathfrak{M}_\Sigma$ is bounded from $L^\infty \times \cdots \times L^\infty$ to L^∞.

Remark 1.7. What we will prove in Sections 4 and 5 is that multi-linear estimates of lacunary maximal operators will be derived from L^1-improving estimates and Sobolev regularity estimates of corresponding averaging operators. Specifically, if one obtains $L^{p_1^\circ} \times \cdots \times L^{p_m^\circ} \to L^1$ estimates of averaging operators with $\sum_{j=1}^m 1/p_j^\circ > 1$, then one also obtains $L^{p_1^\circ} \times \cdots \times L^{p_m^\circ} \to L^{p^\circ,\infty}$ estimates of the lacunary maximal operators for $\sum_{j=1}^m 1/p_j^\circ = 1/p^\circ$ together with certain polynomial growth, which is Lemma 4.2. The polynomial growth of Lemma 4.2 will be handled by interpolation with an exponential decay estimates of Lemma 4.3, which is originated by the Sobolev regularity estimates of averaging operators. As a result, we obtain $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ estimates for $\sum_{j=1}^m 1/p_j = 1/p$, with $p_1, \ldots, p_m \ge 1$ and $p > p^\circ$.

As a simple application of Remark 1.7, we obtain the following result.

Remark 1.8. Theorem 1.6 also yields the following boundedness of the bilinear lacunary spherical maximal function $\mathfrak{M}_{\mathbb{S}^{2d-1}}$. Let $d \ge 1$, $1 < p_1, p_2 \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. Then

$$\|\mathfrak{M}_{\mathbb{S}^{2d-1}}(f_1, f_2)\|_{L^p} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{1.13}$$

Note that we make use of the $L^1 \times L^1 \to L^{1/2}$ estimates of $A^1_{\mathbb{S}^{2d-1}}$ given by [20] and the machinery of Section 4 to obtain (1.13) for $p > 1/2$. This estimate is already given in [8] and we give a different proof at the end of this paper.

Remark 1.9. It is known that $\mathfrak{M}_\Sigma$ satisfies $(L^2 \times \cdots \times L^2 \to L^{2/m})$-estimates for certain κ, see [17]. One can check that even for the worst indices, our Theorem 1.6 is better than the $(L^2 \times \cdots \times L^2 \to L^{2/m})$-estimates in the sense that Theorem 1.6 holds for L^p spaces with lower indices, since $2/m > 2/(m+1) > 2d/(2d+\kappa)$. When $\kappa \le (m-1)d$, we do not know anything yet.

Notations and definitions

- For a cube Q or a ball B in $\mathbb{R}^d$, we define CQ and CB whose sidelength and radius are C times those of Q and B with the same centers, respectively. For a measurable set E, we denote by meas(E) the measure of E.
- Choose a Schwartz class function ϕ such that $\text{supp}(\widehat{\phi}) \subset B(0,2)$ and $\widehat{\phi}(\xi) = 1$ for $\xi \in B(0,1)$. Also consider $\widehat{\psi}(\xi) = \widehat{\phi}(\xi) - \widehat{\phi}(2\xi)$ so that $\text{supp}(\widehat{\psi}) \subset \{2^{-1} < |\xi| < 2\}$. By introducing the symbols $\widehat{\phi_\ell}(\xi) = \widehat{\phi}(2^{-\ell}\xi)$ and $\widehat{\psi_\ell}(\xi) = \widehat{\psi}(2^{-\ell}\xi)$, we define the frequency projection operators:

$$\widehat{P_{<\ell} f}(\xi) = \widehat{f}(\xi)\widehat{\phi_\ell}(\xi) \quad \text{and} \quad \widehat{P_\ell f}(\xi) = \widehat{f}(\xi)\widehat{\psi_\ell}(\xi). \tag{1.14}$$

2. Proofs of Propositions 1.1 and 1.2

2.1. Proof of Proposition 1.1

The proof of Proposition 1.1 follows from the following lemma and a standard technique from [18, 20].

Lemma 2.1. *The operator $\mathcal{A}_{\mathcal{S}}^{\Theta}$ is bounded from $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ for $(1/p_1, \ldots, 1/p_m) \in \operatorname{conv}(\mathcal{V}_\kappa)$. In particular, if $\kappa = d-1$, then one example of $\mathcal{A}_{\mathcal{S}}^{\Theta}$ is the spherical averaging operator $\mathcal{A}_{\mathbb{S}^{d-1}}^{\Theta}$.*

Let $p = \frac{k+2}{2(k+1)}$ and $(1/p_1, \ldots, 1/p_m) \in \operatorname{conv}(\mathcal{V}_\kappa)$. We begin with

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \Big| \int_{\mathcal{S}} \prod_{j=1}^m f_j(x + \Theta_j y)\, \mathrm{d}\sigma(y) \Big|^p \mathrm{d}x.$$

Decompose $\mathbb{R}^d$ into countable union of unit cubes $Q_{\mathbf{n}} = \mathbf{n} + [0,1)^d$, $\mathbf{n} \in \mathbb{Z}^d$. Using the compactness of $\mathcal{S}$, we have

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)}^p = \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{Q_{\mathbf{n}}} \Big| \int_{\mathcal{S}} \prod_{j=1}^m f_j(x + \Theta_j y)\, \mathrm{d}\sigma(y) \Big|^p \mathrm{d}x. \tag{2.1}$$

Now we apply Hölder's inequality to obtain

$$\begin{aligned} &\int_{Q_{\mathbf{n}}} \Big| \int_{\mathcal{S}} \prod_{j=1}^m f_j(x + \Theta_j y)\, \mathrm{d}\sigma(y) \Big|^p \mathrm{d}x \\ &\qquad\lesssim \Big(\int_{Q_{\mathbf{n}}} \int_{\mathcal{S}} \Big| \prod_{j=1}^m f_j(x + \Theta_j y) \Big| \mathrm{d}\sigma(y)\, \mathrm{d}x \Big)^p. \end{aligned} \tag{2.2}$$

Since $x \in Q_{\mathbf{n}}$ and $y \in \operatorname{supp}(\mathcal{S})$, we have the following equality:

$$f_j(x + \Theta_j y) = (f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}})(x + \Theta_j y), \tag{2.3}$$

where $\widetilde{Q}$ denotes a cube whose sidelength is 3 times that of Q with the same center. With the help of (2.3) and Lemma 2.1, we have

$$\begin{aligned} &\Big(\int_{Q_{\mathbf{n}}} \int_{\mathcal{S}} \Big| \prod_{j=1}^m f_j(x + \Theta_j y) \Big| \mathrm{d}\sigma(y)\, \mathrm{d}x \Big)^p \\ &\qquad= \Big(\int_{Q_{\mathbf{n}}} \int_{\mathcal{S}} \Big| \prod_{j=1}^m (f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}})(x + \Theta_j y) \Big| \mathrm{d}\sigma(y)\, \mathrm{d}x \Big)^p \\ &\qquad\lesssim \Big(\prod_{j=1}^m \|(f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}})\|_{L^{p_j}(\mathbb{R}^d)} \Big)^p = \prod_{j=1}^m \|f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}}\|_{L^{p_j}(\mathbb{R}^d)}^p \end{aligned} \tag{2.4}$$

whenever $(1/p_1, \ldots, 1/p_m)$ is in $\operatorname{conv}(\mathcal{V}_\kappa)$.

By (2.1), (2.2) and (2.4), we have

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)}^p \lesssim \sum_{\mathbf{n}\in\mathbb{Z}^d} \prod_{j=1}^m \|f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}}\|_{L^{p_j}(\mathbb{R}^d)}^p. \tag{2.5}$$

We make use of Hölder's inequality on (2.5) to obtain

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)}^p \lesssim \prod_{j=1}^m \Big(\sum_{\mathbf{n}\in\mathbb{Z}^d} \|f_j \mathbb{1}_{\widetilde{Q}_{\mathbf{n}}}\|_{L^{p_j}(\mathbb{R}^d)}^{p_j}\Big)^{p/p_j}.$$

Note that $\{\widetilde{Q}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a finitely overlapping cover of $\mathbb{R}^d$. Therefore, we have

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

where

$$\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} \quad \text{and} \quad \Big(\frac{1}{p_1}, \dots, \frac{1}{p_m}\Big) \in \operatorname{conv}(\mathcal{V}_\kappa).$$

Thus, by showing Lemma 2.1, we complete the proof of Proposition 1.1.

Proof of Lemma 2.1. Let $\mathfrak{a}$ be a symbol satisfying $|\mathfrak{a}(\xi)| \lesssim (1+|\xi|)^{-\rho}$ for some $\rho > 0$. Then, for $m = 1$, it is well known [29, 37] that $T_{\mathfrak{a}}(f) = (\mathfrak{a}\widehat{f}\,)^{\vee}$ is bounded from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ for $1/p + 1/p' = 1$, $p \in [1,2]$, and $1/p - 1/2 \le \frac{1}{2}(\frac{\rho}{\rho+1})$. Let $\mathcal{S}$ be a hypersurface with κ nonvanishing principal curvatures. For the bilinear case $m = 2$, by change of variables, we have

$$\begin{aligned}\|\mathcal{A}_{\mathcal{S}}^{\Theta}(f,g)\|_1 &\le \int_{\mathbb{R}^d}\int_{\mathcal{S}} |f(x+\Theta_1 y) g(x+\Theta_2 y)|\, d\sigma(y)\, dx \\ &= \int_{\mathbb{R}^d} |f(x)| \int_{\mathcal{S}} |g(x + (\Theta_2 - \Theta_1)y)|\, d\sigma(y)\, dx \le \|f\|_p \|g\|_p,\end{aligned}$$

where the last inequality follows from Hölder's inequality and $1/p = (\kappa+1)/(\kappa+2)$. Thus, for the m-linear case, it follows that

$$\begin{aligned}\|\mathcal{A}_{\mathcal{S}}^{\Theta}(\mathrm{F})\|_1 &\le \int_{\mathbb{R}^d}\int_{\mathcal{S}} \Big|\prod_{j=1}^m f_j(x+\Theta_j y)\Big|\, d\sigma(y)\, dx \\ &\le \int_{\mathbb{R}^d}\int_{\mathcal{S}} |f_1(x+\Theta_1 y) f_2(x+\Theta_2 y)|\, d\sigma(y)\, dx \times \prod_{3\le j\le m} \|f_j\|_\infty \\ &\le \|f_1\|_p \|f_2\|_p \prod_{3\le j\le m} \|f_j\|_\infty,\end{aligned}$$

where $1/p = (\kappa+1)/(\kappa+2)$. Similarly, interchanging the role of the functions and invoking multilinear interpolation we get the desired estimate. ∎

2.2. Proof of Proposition 1.2

2.2.1. L^1-improving estimates (1.5). By translation $x \to x + y_m$, we reduce the L^1-norm of A_Σ into L^∞-norm of the following $(m-1)$-linear operator:

$$\int_\Sigma \prod_{j=1}^{m-1} |f_j(x + y_m - y_j)| \, d\sigma_\Sigma(y). \tag{2.6}$$

By using the Fourier transform, we rewrite (2.6) as

$$\int_{\mathbb{R}^{(m-1)d}} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_{m-1})} \, d\widehat{\sigma}_\Sigma(\xi', -\xi_1 - \cdots - \xi_{m-1}) \prod_{j=1}^{m-1} |\widehat{f}_j|(\xi_j) \, d\xi', \tag{2.7}$$

where $\xi' = (\xi_1, \ldots, \xi_{m-1}) \in \mathbb{R}^{(m-1)d}$.

Since the hypersurface Σ has κ nonvanishing principal curvatures, using the result of Littman [28], we get $|d\widehat{\sigma}_\Sigma(\xi)| \lesssim (1 + |\xi|)^{-\kappa/2}$ for $\xi \in \mathbb{R}^{md}$. This implies that the symbol of (2.7) satisfies

$$|d\widehat{\sigma}_\Sigma(\xi', -\xi_1 - \cdots - \xi_{m-1})| \lesssim (1 + |\xi'|)^{-\kappa/2}.$$

By applying Hölder's inequality to the expression (2.7), we deduce that it is bounded above by

$$\Big(\int_{\mathbb{R}^{(m-1)d}} \prod_{j=1}^{m-1} ||\widehat{f}_j|(\xi_j)|^{p'} \, d\xi'\Big)^{1/p'} \times \Big(\int_{\mathbb{R}^{(m-1)d}} (1 + |\xi'|)^{-\kappa p/2} \, d\xi'\Big)^{1/p},$$

and the last term is finite if $p > 2d(m-1)/\kappa$. Thus, for $2d(m-1)/\kappa < p < 2$, we have

$$\Big|\int_\Sigma \prod_{j=1}^{m-1} f(x + y_m - y_j) \, d\sigma_\Sigma(y)\Big| \lesssim \prod_{j=1}^{m-1} \||\widehat{f}_j|\|_{L^{p'}(\mathbb{R}^d)}.$$

Together with the L^1-norm of f_m, for $2d(m-1)/\kappa < p \le 2$, we have

$$\|A_\Sigma(\mathrm{F})\|_{L^1(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m-1} \|f_j\|_{L^p(\mathbb{R}^d)} \times \|f_m\|_{L^1(\mathbb{R}^d)}. \tag{2.8}$$

The symmetry of estimates (2.8) and multilinear interpolation yield that

$$\|A_\Sigma(\mathrm{F})\|_{L^1(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}, \tag{2.9}$$

where $\frac{m+1}{2} \le \sum_{j=1}^m \frac{1}{p_j} < \frac{2d+\kappa}{2d}$ and $1 \le p_j \le 2$.

2.2.2. Quasi-Banach space estimates (1.6). Since we obtain L^1-improving estimates for A_Σ, one can apply the argument of Section 2.1 to show that A_Σ satisfies a Hölder-type multilinear estimates on $L^p(\mathbb{R}^d)$ for $1/p = \sum_{j=1}^m 1/p_j$, with p_j in (2.9). That is, we have

$$\|A_\Sigma(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

where $\frac{m+1}{2} \leq \sum_{j=1}^m \frac{1}{p_j} < \frac{2d+\kappa}{2d}$ and $1 \leq p_j \leq 2$. This proves the quasi-Banach space estimates.

2.2.3. Smoothing estimates (1.7). For the Sobolev regularity estimates, note that A_Σ is written in terms of Fourier multipliers:

$$A_\Sigma(\mathrm{F})(x) = \int_{\mathbb{R}^{md}} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\, \mathrm{d}\widehat{\sigma}_\Sigma(\vec{\xi}\,) \prod_{j=1}^m \widehat{f}_j(\xi_j)\, \mathrm{d}\vec{\xi}.$$

Moreover, we consider $f_1,\ldots,f_m$ whose Fourier transforms are supported in the sets $\{\xi \in \mathbb{R}^d : 2^{n_j-1} \leq |\xi| \leq 2^{n_j+1}\}$, with positive integers n_j, $j = 1,\ldots,m$, respectively. Since $\mathrm{d}\widehat{\sigma}_\Sigma$ satisfies the following limited decay condition,

$$|\partial^\alpha \mathrm{d}\widehat{\sigma}_\Sigma(\vec{\xi}\,)| \lesssim (1+|\vec{\xi}\,|)^{-\kappa/2} \quad \text{for any multi-indices } \alpha, \tag{2.10}$$

we are going to make use of one of main results of [16] initial estimates.

Theorem 2.2 (Theorem 1.1 in [16]). *Let m be a positive number such that $m \geq 2$ and $1 < q < 2m/(m-1)$. Set M_q to be a positive integer satisfying*

$$M_q > \frac{m(m-1)d}{2m-(m-1)q}.$$

Suppose that $\mathfrak{m} \in L^q(\mathbb{R}^{md}) \cap C^{M_q}(\mathbb{R}^{md})$ with

$$\|\partial^\alpha \mathfrak{m}\|_{L^\infty(\mathbb{R}^{md})} \leq D_0 \quad \textit{for } |\alpha| \leq M_q.$$

Then we have

$$\|T_{\mathfrak{m}}(f_1,\ldots,f_m)\|_{L^{2/m}(\mathbb{R}^d)} \lesssim D_0^{1-(m-1)q/(2m)} \|\mathfrak{m}\|_{L^q(\mathbb{R}^{md})}^{(m-1)q/(2m)} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^d)}.$$

Note that $T_{\mathfrak{m}}(f_1,\ldots,f_m)$ is a multilinear operator whose Fourier multiplier is $\mathfrak{m}$. Then, by putting (2.10) into Theorem 2.2, we have

$$\mathfrak{m}(\xi) = \mathrm{d}\widehat{\sigma}(\xi) \prod_{j=1}^m \widehat{\psi}_{n_j}(\xi_j), \quad D_0 \simeq 1, \quad \|\mathfrak{m}(\xi)\|_{L^q(\mathbb{R}^{md})} \lesssim 2^{-|\mathbf{n}|\kappa/2}\, 2^{|\mathbf{n}|md/q}.$$

Since $q \in (1, 2m/(m-1))$, for $f_1,\ldots,f_m$, whose Fourier transforms are supported in $\mathbb{A}_{n_j} = \{\xi_j \in \mathbb{R}^d : 2^{n_j-1} \leq |\xi_j| \leq 2^{n_j+1}\}$, we have

$$\|A_\Sigma(\mathrm{F})\|_{L^{2/m}(\mathbb{R}^d)} \lesssim 2^{-|\mathbf{n}|(\kappa/2-(m-1)d/2)} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^d)}. \tag{2.11}$$

Note that, for $1 \leq p, p_1, \ldots, p_m \leq \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, we have trivial estimates

$$\|A_\Sigma(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}. \tag{2.12}$$

By interpolating (2.11) and (2.12), for any $1 \le p, p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, there is a $\delta = \delta(p, \kappa, m, d) > 0$ such that

$$\|A_\Sigma(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)},$$

when $\widehat{f_j}$ is supported in $\mathbb{A}_{n_j}$ for $j = 1, \dots, m$. This proves (1.7).

3. A nonlinear Brascamp–Lieb inequality approach to L^p-improving estimates for $\mathcal{A}_{\mathcal{S}}^{\Theta}$

3.1. Nonlinear Brascamp–Lieb inequality

Let f_j be nonnegative integrable functions, let $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ be linear surjections, and let $c_j \in [0, 1]$ for $j = 1, \dots, m$. We also identify a finite-dimensional Hilbert space H and a Euclidean space $\mathbb{R}^n$, for instance, we let $H = \mathbb{R}^d$ and $H_j = \mathbb{R}^{d_j}$. Then we can consider the linear Brascamp–Lieb inequality

$$\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j(L_j x))^{c_j} \,\mathrm{d}x \le \mathrm{BL}(\mathbf{L}, \mathbf{c}) \prod_{j=1}^m \Big(\int_{\mathbb{R}^{d_j}} f_j(x_j) \,\mathrm{d}x_j \Big)^{c_j}, \tag{3.1}$$

where $\mathbf{L} = (H, \{H_j\}_{1 \le j \le m}, \{L_j\}_{1 \le j \le m})$, $\mathbf{c} = (c_1, \dots, c_m)$, and $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ is the smallest such constant. Here, we call $(\mathbf{L}, \mathbf{c})$ a Brascamp–Lieb datum, and $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ a Brascamp–Lieb constant. There have been studies on nonlinear generalizations of the Brascamp–Lieb inequality. Bennett, Carbery and Wright [7] showed that (3.1) holds for $d_j = d - 1$ and $c_j = 1/(m-1)$ when the L_j's are smooth submersions supported in a sufficiently small neighborhood. They also proved that the L_j's could be C^3 mappings under certain transversality conditions on the submersions. Later, Bennett and Bez [4] extended the results of [7] to general d_j and $C^{1,\beta}$ mappings. Recently, Bennett, Bez, Buschenhenke, Cowling, and Flock [5] proved the following nonlinear Brascamp–Lieb inequality.

Theorem 3.1 (Theorem 1.1 in [5]). *Let* $(\mathbf{L}, \mathbf{c})$ *be a Brascamp–Lieb datum. Suppose that* $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ *are* C^2 *submersions in a neighborhood of a point* x_0 *and* $\mathrm{d}B_j(x_0) = L_j$, $j = 1, 2, \dots, m$. *Then, for each* $\varepsilon > 0$, *there exists a neighborhood* U *of* x_0 *such that*

$$\int_U \prod_{j=1}^m (f_j(B_j(x)))^{c_j} \,\mathrm{d}x \le (1+\varepsilon)\,\mathrm{BL}(\mathbf{L}, \mathbf{c}) \prod_{j=1}^m \Big(\int_{\mathbb{R}^{d_j}} f_j(x_j) \,\mathrm{d}x_j \Big)^{c_j}.$$

Although Theorem 3.1 is stated with C^2 submersions, the proof of [5] guarantees that the theorem still holds if one takes $C^{1+\theta}$ submersions for any $\theta > 0$. It is known [6] that $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ is finite if and only if the following conditions hold:

$$\dim(V) \le \sum_{j=1}^m c_j \dim(L_j V) \quad \text{for all subspaces } V \text{ of } \mathbb{R}^d, \tag{3.2}$$

$$d = \sum_{j=1}^m c_j d_j. \tag{3.3}$$

These conditions are called the transversality condition and the scaling condition, respectively. We also present necessary conditions for finiteness of $\mathrm{BL}(\mathbf{L}, \mathbf{c})$:

$$\bigcap_{j=1}^{m} \ker(L_j) = \{0\}, \quad \sum_{j=1}^{m} c_j \geq 1.$$

But, it is not simple to check (3.2) for a given Brascamp–Lieb datum. The following lemma may be useful in such verification. First, we say a proper subspace V_c of $\mathbb{R}^d$ is a critical subspace if it satisfies

$$\dim(V_c) = \sum_{j=1}^{m} c_j \dim(L_j V_c).$$

For a given subspace V_c, we split the Brascamp–Lieb datum into two parts, $(\mathbf{L}_{V_c}, \mathbf{c})$ and $(\mathbf{L}_{V_c^\perp}, \mathbf{c})$, as follows:

$$\begin{aligned} \mathbf{L}_{V_c} &= (V_c, \{L_j V_c\}_{1\leq j\leq m}, \{L_{j,V_c}\}_{1\leq j\leq m}), \\ \mathbf{L}_{V_c^\perp} &= (H/V_c, \{H_j/(L_j V_c)\}_{1\leq j\leq m}, \{L_{j,H/V_c}\}_{1\leq j\leq m}), \end{aligned}$$

where $H/V_c = V_c^\perp$ and

$$\begin{aligned} L_{j,V_c} &: V_c \to L_j V_c, \\ L_{j,H/V_c} &: H/V_c \to H_j/(L_j V_c). \end{aligned}$$

In this paper, we choose $H = \mathbb{R}^d \times \mathbb{R}^k$ and $H_j = \mathbb{R}^d$.

Lemma 3.2 (Lemma 4.6 in [6]). *Let V_c be a critical subspace. Then* $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ *is finite if and only if* $(\mathbf{L}_{V_c}, \mathbf{c})$ *and* $(\mathbf{L}_{V_c^\perp}, \mathbf{c})$ *satisfy* (3.2) *and* (3.3) *for any subspace V of V_c and $V_c^\perp$, respectively.*

Now we will prove Theorem 1.3. We first decompose $\mathcal{S}^k$ into a finite cover $\{\mathcal{S}^k_\tau\}$ for which $\mathcal{A}_{\mathcal{S}^k}(\mathrm{F})(x)$ can be written as a finite summation of the following operators:

$$\mathcal{A}^{\Theta}_{\mathcal{S}^k_\tau}(\mathrm{F})(x) = \int_{\mathbb{R}^k} \prod_{j=1}^{m} f_j(x + \Theta_j(y', \Phi_\tau(y')))\chi_\tau(y')\,\mathrm{d}y',$$

where $\Phi_\tau\colon \mathbb{R}^k \to \mathbb{R}^{d-k}$ is a C^2-submersion and χ_τ is a smooth cut-off function.

To simplify our proof, we consider a more general m-linear operator $T^{\vec{B}}_K$. Suppose $B_j\colon \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$ are C^2 submersions and $L_j = \mathrm{d}B_j(0,0)$, with $j = 1, \dots, m$. Then $T^{\vec{B}}_K(\mathrm{F})$ is given by

$$T^{\vec{B}}_K(\mathrm{F})(x) = \int_{\mathbb{R}^k} \prod_{j=1}^{m} f_j(B_j(x,y))\, K(y)\,\mathrm{d}y, \quad y \in \mathbb{R}^k, x \in \mathbb{R}^d,$$

where K is a nonnegative bounded function supported in a ball $B(0,\varepsilon) \subset \mathbb{R}^k$. Note that $\mathcal{A}^{\Theta}_{\mathcal{S}^k_\tau}(\mathrm{F})$ is an example of $T^{\vec{B}}_K$ for $K = \chi_\tau$ and $B_j(x, y') = x + \Theta_j(y', \Phi_\tau(y'))$. Also, we take $c_j = 1/p_j$ for $j = 1, \dots, m$ and $c_{m+1} = 1/p'$, with $1/p = 1/p_1 + \cdots + 1/p_m - k/d$. Then we prove the following proposition.

Proposition 3.3. *Let $p_1, \dots, p_m \in [1, \infty)$ satisfy $\sum_{j=1}^m 1/p_j \geq 1$. Let $1/p = \sum_{j=1}^m 1/p_j - k/d$, with $1 \leq p \leq d/(d-k)$. Suppose $(\mathbf{L}, \mathbf{p})$ is a Brascamp–Lieb datum for*

$$\mathbf{L} = (\mathbb{R}^d \times \mathbb{R}^k, \{\mathbb{R}^d\}_{j=1}^{m+1}, \{L_j\}_{j=1}^{m+1}),$$

with $L_j = \mathrm{d}B_j(0,0)$, $j = 1, \dots, m$, $L_{m+1} = \mathrm{d}\pi_{\mathbb{R}^d}$, and

$$\mathbf{p} = \Big(\frac{1}{p_1}, \dots, \frac{1}{p_m}, \frac{1}{p'}\Big).$$

Then we have

$$\|T_K^{\vec{B}}(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim (1+\varepsilon)\,\mathrm{BL}(\mathbf{L}, \mathbf{p}) \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}.$$

Proposition 3.3 states that the Brascamp–Lieb inequality implies an L^p-improving estimate.

Proof. Since $p \geq 1$, by making use of duality, we get

$$\|T_K^{\vec{B}}(\mathrm{F})\|_{L^p(\mathbb{R}^d)} = \sup_{\|g\|_{p'} \leq 1} \int_{\mathbb{R}^d} T_K^{\vec{B}}(\mathrm{F})(x)\, g(x)\, \mathrm{d}x.$$

Now we choose $g \in L^{p'}(\mathbb{R}^d)$ such that $\|g\|_{L^{p'}(\mathbb{R}^d)} \leq 1$. As in Section 2, we decompose $\mathbb{R}^d$ into countable union of cubes $Q_{\mathbf{n}}(\varepsilon)$, where $Q(\varepsilon)$ is a cube centered at the origin with side-length ε, and $Q_{\mathbf{n}}(\varepsilon)$ denotes $\varepsilon\mathbf{n}$ translation of $Q(\varepsilon)$ for $\mathbf{n} \in \mathbb{Z}^d$. Then it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} T_K^{\vec{B}}(\mathrm{F})(x) g(x)\, \mathrm{d}x &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{Q_{\mathbf{n}}(\varepsilon)} \int_{\mathbb{R}^k} \Big(\prod_{j=1}^m f_j(B_j(x,y))\Big) g(x) K(y)\, \mathrm{d}y\, \mathrm{d}x \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{[-\varepsilon/2,\varepsilon/2)^d} \int_{\mathbb{R}^k} \Big(\prod_{j=1}^m f_j(B_j(x+\varepsilon\mathbf{n}, y))\Big) g(x+\varepsilon\mathbf{n}) K(y)\, \mathrm{d}y\, \mathrm{d}x \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} \int_{[-\varepsilon/2,\varepsilon/2)^d} \int_{\mathbb{R}^k} \Big(\prod_{j=1}^m \tau_{\varepsilon\mathbf{n}}[f_j](B_j(x,y))\Big) \tau_{\varepsilon\mathbf{n}}[g](x) K(y)\, \mathrm{d}y\, \mathrm{d}x,
\end{aligned}$$

where $\tau_{\varepsilon\mathbf{n}}[f](x) = f(x+\varepsilon\mathbf{n})$. Then we apply Theorem 3.1 to $\tau_{\varepsilon\mathbf{n}}[f_j]^{p_j}, \tau_{\varepsilon\mathbf{n}}[g]^{p'}$, together with the additional mapping $L_{m+1} = \mathrm{d}\pi_{\mathbb{R}^d}$, which yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} T_K^{\vec{B}}(\mathrm{F})(x) g(x)\, \mathrm{d}x \\
&\qquad \leq (1+\varepsilon)\,\mathrm{BL}(\mathbf{L}, \mathbf{p}) \sum_{\mathbf{n}\in\mathbb{Z}^d} \Big(\prod_{j=1}^m \|\tau_{\varepsilon\mathbf{n}}[f_j]\|_{L^{p_j}(\tilde{Q}(\varepsilon))}\Big) \|\tau_{\varepsilon\mathbf{n}}[g]\|_{L^{p'}(\tilde{Q}(\varepsilon))},
\end{aligned} \tag{3.4}$$

whenever

$$\dim(V) \leq \sum_{j=1}^m \frac{\dim(dB_j(0_x, 0_y)(V))}{p_j} + \frac{\dim(\mathrm{d}\pi_{\mathbb{R}^d}(V))}{p'}$$

for every subspace V of $\mathbb{R}^d \times \mathbb{R}^k$, together with

$$\frac{k}{d} + \frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}.$$

Note that ε in (3.4) is uniform in $\mathbf{n}$ due to $B_j(x + \varepsilon\mathbf{n}, y) = B_j(x, y) + \varepsilon\mathbf{n}$ and also that $\tilde{Q}$ denotes a cube whose side-length is 3 times that of Q with the same center.

We choose g so that $\|g\|_{p'} \leq 1$, so we ignore $\|g\|_{L^{p'}(\widetilde{Q}_{\mathbf{n}})}$, and that $\|\tau_{\varepsilon\mathbf{n}}[f_j]\|_{L^{p_j}(\widetilde{Q}(\varepsilon))} = \|f_j\|_{L^{p_j}(\widetilde{Q}_{\mathbf{n}}(\varepsilon))}$. Thus, by Hölder's inequality, we have

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\widetilde{Q}_{\mathbf{n}}(\varepsilon))} \lesssim \prod_{j=1}^{m} \Big\| \|f_j\|_{L^{p_j}(\widetilde{Q}_{\mathbf{n}}(\varepsilon))} \Big\|_{\ell^{r_j}(\mathbb{Z}^d)}$$

for $\sum_{j=1}^{m} 1/r_j = 1$ with $1 \leq r_j \leq \infty$. From $\sum_{j=1}^{m} 1/p_j \geq 1$, one can choose r_j's such that $1/r_j \leq 1/p_j$ for each $j = 1, \dots, m$, then we use the $\ell^{p_j} \hookrightarrow \ell^{r_j}$ embedding to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} T_K^{\vec{B}}(\mathrm{F})(x) g(x)\,\mathrm{d}x &\leq (1+\varepsilon)\,\mathrm{BL}(\mathbf{L},\mathbf{p}) \prod_{j=1}^{m} \Big\| \|f_j\|_{L^{p_j}(\widetilde{Q}_{\mathbf{n}}(\varepsilon))} \Big\|_{\ell^{r_j}(\mathbb{Z}^d)} \\ &\leq (1+\varepsilon)\,\mathrm{BL}(\mathbf{L},\mathbf{p}) \prod_{j=1}^{m} \Big\| \|f_j\|_{L^{p_j}(\widetilde{Q}_{\mathbf{n}}(\varepsilon))} \Big\|_{\ell^{p_j}(\mathbb{Z}^d)}. \end{aligned} \tag{3.5}$$

Since $\widetilde{Q}_{\mathbf{n}}(\varepsilon)$ are finitely overlapped, taking the supremum over $\|g\|_{p'} \leq 1$ in (3.5) gives

$$\|T_K^{\vec{B}}(\mathrm{F})\|_{L^p(\mathbb{R}^d)} \lesssim (1+\varepsilon)\,\mathrm{BL}(\mathbf{L},\mathbf{p}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^d)}$$

for the desired $p, p_1, \dots, p_m$. ■

Now we present the proof of Theorem 1.3.

3.2. Proof of Theorem 1.3

There is a C^2 mapping $\Phi\colon \mathbb{R}^k \to \mathbb{R}^{d_c}$ for $d_c = d - k$ such that $\mathcal{S}^k$ is locally a graph $\{(y', \Phi(y')) \in \mathbb{R}^d\}$. Then, in Proposition 3.3, we let $B_j(x, y') = x + \Theta_j(y', \Phi(y'))$ for $\Phi = (\phi^1, \dots, \phi^{d_c})$, $j = 1, \dots, m$. Now, for $j = m+1$, we let $B_{m+1} = \pi_{\mathbb{R}^d}$, where $\pi_{\mathbb{R}^d}\colon \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$ is a projection onto x-variable in $\mathbb{R}^d$. For $j = 1, \dots, m$, we define $L_j := \mathrm{d}B_j(0,0)$, which is given by

$$\begin{bmatrix} I_d & \Theta_j \nabla|_{y'=0} \begin{bmatrix} y_1' \\ \vdots \\ y_k' \\ \phi^1(y') \\ \vdots \\ \phi^{d-k}(y') \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_d & \Theta_j \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \phi^1_{y_1'}(0) & \phi^1_{y_2'}(0) & \dots & \phi^1_{y_k'}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{d_c}_{y_1'}(0) & \phi^{d_c}_{y_2'}(0) & \dots & \phi^{d_c}_{y_k'}(0) \end{bmatrix} \end{bmatrix},$$

where I_d denotes the $d \times d$ identity matrix. If there is no confusion, we simply write

$$L_j = \begin{bmatrix} I_d & \Theta_j \begin{bmatrix} I_k \\ \mathrm{d}\Phi(0) \end{bmatrix} \end{bmatrix}, \quad \mathrm{d}\Phi = (\mathrm{d}\phi^1, \ldots, \mathrm{d}\phi^{d_c}).$$

Without loss of generality, we assume that $\mathrm{d}\Phi(0)$ is a $d_c \times k$ zero matrix. That is, for $j = 1, \ldots, m$, we have

$$L_j = \begin{bmatrix} I_d & \Theta_j^1 \end{bmatrix}, \quad \Theta_j = \begin{bmatrix} \Theta_j^1 & \Theta_j^2 \end{bmatrix},$$

where Θ_j^1 and Θ_j^2 are $d \times k$ and $d \times d_c$ matrices, respectively. Since Θ_j^1 has k linearly independent columns, its rank is k. In the case of $j = m+1$, we have

$$\mathrm{d}\pi_{\mathbb{R}^d} = \begin{bmatrix} I_d & Z_d \end{bmatrix},$$

where Z_d means all $d \times d$ elements are zero. We show that $\mathbf{L} = (L_1, \ldots, L_m, \mathrm{d}\pi_{\mathbb{R}^d})$ and $\mathbf{p} = (1/m, \ldots, 1/m, k/d)$ are the Brascamp–Lieb data by making use of Lemma 3.2.

Let $K_\pi = \ker(\pi_{\mathbb{R}^d}) = \{(0, y') \in \mathbb{R}^d \times \mathbb{R}^k : y' \in \mathbb{R}^k\}$, which is k-dimensional. Then it is clear that $K_\pi^\perp = \{(x, 0) \in \mathbb{R}^d \times \mathbb{R}^k : x \in \mathbb{R}^d\}$. For K_π, we have

$$\sum_{j=1}^m \frac{\dim(L_j K_\pi)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K_\pi)}{p'} = \sum_{j=1}^m \frac{k}{p_j} = k = \dim(K_\pi).$$

Since $\dim(L_j K)$ equals $\dim(K)$ for any subspace K of K_π, with $j = 1, \ldots, m$, we also have

$$\sum_{j=1}^m \frac{\dim(L_j K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} = \sum_{j=1}^m \frac{\dim(K)}{p_j} = \dim(K).$$

Thus, K_π is a critical subspace and $(\mathbf{L}_{K_\pi}, \mathbf{p})$ is a Brascamp–Lieb datum.

On the other hand, for $K_\pi^\perp = \{(x, 0) \in \mathbb{R}^d \times \mathbb{R}^k : x \in \mathbb{R}^d\}$, we consider $(\mathbf{L}_{K_\pi^\perp}, \mathbf{p})$:

$$\mathbf{L}_{K_\pi^\perp} = \big(K_\pi^\perp, \{\mathbb{R}^d/(L_j K_\pi)\}_{1 \le j \le m+1}, \{L_{j,K_\pi^\perp}\}_{1 \le j \le m+1}\big),$$
$$\mathbf{p} = \Big(\frac{1}{p_1}, \ldots, \frac{1}{p_m}, \frac{1}{p'}\Big).$$

Note that $\pi_{\mathbb{R}^d, K_\pi^\perp} = \pi_{\mathbb{R}^d}$. Then we have

$$\begin{aligned} \sum_{j=1}^m \frac{\dim(L_{j,K_\pi^\perp} K_\pi^\perp)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K_\pi^\perp)}{p'} &= \sum_{j=1}^m \frac{d-k}{p_j} + \frac{d}{p'} \\ &= d - k + \frac{dk}{d} = d = \dim(K_\pi^\perp). \end{aligned}$$

It remains to verify (3.2) for any proper subspace of $K_\pi^\perp$. In order to show this, we consider a subspace K of $K_\pi^\perp$ whose dimension d_K satisfies $d > d_K \ge k$ or $k > d_K \ge 1$. Note that it is important to check the dimension of $L_j K / L_j K_\pi$, but it suffices to consider K instead of $L_j K$ because every element of K is given by $(x, 0) \in \mathbb{R}^d \times \mathbb{R}^k$ and $L_j(x, 0) = x$.

3.2.1. The case $d > d_K > k$. Let K be a d_K-dimensional subspace of $K_\pi^\perp$ and observe that

$$L_\mu(0, y') = \Theta_\mu^1 y' \quad \text{for all } (0, y') \in K_\pi.$$

Then we define $K_j := L_j K_\pi = \{(\Theta_j^1 y', 0) \in \mathbb{R}^d \times \mathbb{R}^k y' \in \mathbb{R}^k\}$, which is a k-dimensional subspace of $K_\pi^\perp$ due to the full rank of Θ_j^1. It is possible that for some $\mu = 1, \dots, m$, $L_\mu K \cap K_\mu = K \cap K_\mu$ is k-dimensional. Thus, in general, we have $\dim(L_{\mu, K_\pi^\perp} K) = \dim(K/K_\mu) \geq d_K - k$. Note that our choice of $\{\Theta_j\}$ satisfying (1.8) allows us to have

$$\dim(\operatorname{span}\{K_\mu, K_{j_1}, \dots, K_{j_\ell}\}) \geq k + \ell, \quad j_i \neq \mu.$$

That is, there are at most $\ell = d_K - k$ j's such that K_j is a subspace of K. Therefore, we have $\dim(K/K_j) \geq d_K - k$ for $\ell + 1 = d_K - k + 1$ of the j's. Otherwise, for the rest $m - \ell - 1$ j's, we have $\dim(K/K_j) \geq d_K - k + 1$. Hence, it follows that

$$\begin{aligned} &\sum_{j=1}^m \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} \\ &\quad \geq \frac{(d_K - k)(\ell + 1)}{m} + \frac{(d_K - k + 1)(m - \ell - 1)}{m} + \frac{k}{d} d_K \\ &\quad = (d_K - k) + \frac{m - \ell - 1}{m} + \frac{k}{d} d_K = d_K - k + \frac{m - d_K + k - 1}{m} + \frac{k}{d} d_K. \end{aligned} \tag{3.6}$$

Here we choose $p_j = m$ for all $j = 1, \dots, m$ in order to minimize the loss, that is, to maximize the lower bound of (3.6). Thus, we fix $\mathbf{p}$ as $(1/p_1, \dots, 1/p_m, 1/p') = (1/m, \dots, 1/m, k/d)$. Note that the last expression of (3.6) is greater than or equal to d_K whenever

$$-k + \frac{m - d_K + k - 1}{m} + \frac{k}{d} d_K \geq 0. \tag{3.7}$$

Since $d > d_K > k$, it follows that the left-hand side of (3.7) is larger than

$$-k + \frac{m - d + 1 + k - 1}{m} + \frac{k(k+1)}{d}.$$

Thus, (3.7) holds whenever

$$\frac{m - d + k}{m} \geq \frac{d - k - 1}{d} k.$$

3.2.2. The case $k \geq d_K \geq 1$.

The case $d_K = k$.

Let K be not equal to any K_j for $j = 1, \dots, m$ so that $\dim(K \cap K_j) \leq k - 1$. Thus, we have $\dim(L_{j,K_\pi^\perp} K) = \dim(K/K_j) \geq 1$ for $j = 1, \dots, m$, so it follows that

$$\begin{aligned} \sum_{j=1}^m \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} &\geq \sum_{j=1}^m \frac{1}{m} + \frac{k^2}{d} \\ &= 1 + \frac{k^2}{d} = k + 1 - \frac{d-k}{d} k. \end{aligned}$$

The last expression is greater than or equal to $\dim(K) = k$ if

$$1 \geq \frac{d-k}{d}\, k. \tag{3.8}$$

On the other hand, let $K = K_\mu$ for some $\mu = 1, \ldots, m$. By using (1.9), we have that $\dim(K \cap K_j) \leq k - 1$ for $j \neq \mu$. Thus, it follows that

$$\begin{aligned}\sum_{j=1}^{m} \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} &\geq \sum_{j \neq \mu} \frac{1}{m} + \frac{k^2}{d} \\ &= \frac{m-1}{m} + \frac{k^2}{d} = k + \frac{m-1}{m} - \frac{d-k}{d}\, k.\end{aligned}$$

The last expression is greater than or equal to $\dim(K) = k$ if

$$\frac{m-1}{m} \geq \frac{d-k}{d}\, k. \tag{3.9}$$

Note that (3.8) is implied by (3.9).

The case $d_K = k - 1$.

For this case, we consider a subspace K which is not contained in K_j for all $j = 1, \ldots, m$. Then for some $\mu \in \{1, \ldots, m\}$ the worst case verifying (3.2) is that $K \cap K_\mu$ is $(k-2)$-dimensional, since $\dim(L_{j,K_\pi^\perp} K)$ gets lower as $\dim(K \cap K_\mu)$ gets larger. Thus, we have $\dim(K/K_\mu) = 1$, and this may happen for any $j = 1, \ldots, m$. It follows that

$$\begin{aligned}\sum_{j=1}^{m} \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} &\geq \sum_{j=1}^{m} \frac{1}{m} + \frac{k}{d}\,(k-1) \\ &= 1 + \frac{k}{d}\,(k-1) = (k-1) + 1 - \frac{d-k}{d}\,(k-1).\end{aligned}$$

The last expression is greater than or equal to $\dim(K) = k - 1$ if

$$1 \geq \frac{d-k}{d}\,(k-1). \tag{3.10}$$

Now, let K be a $(k-1)$-dimensional subspace of K_μ for some $\mu \in \{1, \ldots, m\}$. Then the worst case is when K is given by the intersection of K_μ and K_ν for some $\nu \neq \mu$. Thus, we have $\dim(K/K_\mu) = \dim(K/K_\nu) = 0$. However, if we choose any other $j \neq \mu, \nu$, then we have $\dim(K/K_j) \geq 1$ because $\dim(K_\mu \cap K_\nu \cap K_j) \leq k - 2$ due to (1.9). Without loss of generality, say $\mu = 1$ and $\nu = 2$, so that by (1.9) one can check

$$\begin{aligned}\sum_{j=1}^{m} \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} &\geq \sum_{j=3}^{m} \frac{1}{m} + \frac{k}{d}\,(k-1) = \frac{m-2}{m} + \frac{k}{d}\,(k-1) \\ &= (k-1) + \frac{m-2}{m} - \frac{d-k}{d}\,(k-1).\end{aligned}$$

The last expression is greater than or equal to $k - 1$ whenever

$$\frac{m-2}{m} \geq \frac{d-k}{d}\,(k-1). \tag{3.11}$$

Note that (3.11) implies (3.10).

The case $d_K = k - n$.

Similar to the k, $(k-1)$-dimensional cases of K, for an arbitrary $(k-n)$-dimensional subspace K, one can check that the worst case happens when K is contained in K_{j_i} for $j_1, \ldots, j_n$. Thus, we have

$$\begin{aligned}\sum_{j=1}^{m} \frac{\dim(L_{j,K_\pi^\perp} K)}{p_j} + \frac{\dim(\pi_{\mathbb{R}^d} K)}{p'} &\geq \sum_{j \neq \mu, j_1, \ldots, j_n} \frac{1}{m} + \frac{k}{d}(k-n) \\ &= \frac{(m-n-1)}{m} + \frac{k}{d}(k-n) \\ &= (k-n) + \frac{(m-n-1)}{m} - \frac{d-k}{d}(k-n).\end{aligned}$$

Then the last line is greater than or equal to $k-n$ whenever

$$\frac{(m-n-1)}{m} \geq \frac{d-k}{d}(k-n),$$

which leads to $m \geq d$ when $k = d-1$ or $k = n+1$.

Together with (3.7), one can conclude that $\mathrm{BL}(\mathbf{L}, \mathbf{p})$ is finite for given data $(\mathbf{L}, \mathbf{p})$ whenever $(m-n-1)/m \geq \frac{d-k}{d}(k-n)$ for all $0 \leq n \leq k-1$. Note that we can rewrite the condition as

$$\frac{m-1}{m} \geq \frac{d-k}{d} k - \left(\frac{d-k}{d} - \frac{1}{m}\right) n, \quad 0 \leq n \leq k-1. \tag{3.12}$$

Note that (3.12) is reduced to

$$\frac{m-1}{m} \geq \frac{d-k}{d} k$$

for all $0 \leq n \leq k-1$ when $m \geq d$. Thus, $(\mathbf{L}, \mathbf{p})$ is a Brascamp–Lieb datum. Hence, by Proposition 3.3, we end the proof of Theorem 1.3.

4. Proof of Theorem 1.5

Recall that the lacunary maximal function $\mathcal{M}_{\mathcal{S}}^{\Theta}$ is defined by

$$\mathcal{M}_{\mathcal{S}}^{\Theta}(\mathrm{F})(x) = \sup_{\ell \in \mathbb{Z}} \left| \int_{\mathcal{S}} \prod_{j=1}^{m} f_j(x - 2^{-\ell} \Theta_j y) \, d\sigma(y) \right|,$$

where $\mathcal{S}$ has κ-nonvanishing principal curvatures and $\Theta = \{\Theta_j\}$ is a family of mutually linearly independent rotation matrices.

Observe that for any fixed $\ell \in \mathbb{Z}$, we can write the identity operator I as follows:

$$I = P_{<\ell} + \sum_{n=0}^{\infty} P_{\ell+n} = P_{<\ell} + P_{\ell\leq}. \tag{4.1}$$

Then we have

$$\text{(4.2)} \qquad \prod_{j=1}^{m} f_j = \prod_{j=1}^{m}(P_{<\ell} f_j + P_{\ell\leq} f_j)$$
$$= \Big(\prod_{j=1}^{m} P_{<\ell} f_j\Big) + \Big(\prod_{j=1}^{m} P_{\ell\leq} f_j\Big)$$
$$+ \sum_{\alpha=1}^{m-1} \frac{1}{\alpha!\,(m-\alpha)!} \sum_{\tau\in S_m} \Big(\prod_{i=1}^{\alpha} P_{\ell\leq} f_{\tau(i)}\Big)\Big(\prod_{i=\alpha+1}^{m} P_{<\ell} f_{\tau(i)}\Big),$$

where the second summation runs over the symmetric group S_m over $\{1,\ldots,m\}$. For $\mathbf{n} = (n_1,\ldots,n_m) \in \mathbb{N}_0^m = (\mathbb{N}\cup\{0\})^m$, we define

$$\text{(4.3)} \qquad \mathcal{A}_\ell^{\alpha,\tau}(\mathrm{F})(x) := \int_{\mathcal{S}} \Big(\prod_{i=1}^{\alpha} P_{<\ell} f_{\tau(i)}(x - 2^{-\ell}\,\Theta_{\tau(i)} y)\Big) \times \Big(\prod_{i=\alpha+1}^{m} f_{\tau(i)}(x - 2^{-\ell}\,\Theta_{\tau(i)} y)\Big)\, \mathrm{d}\sigma(y),$$

$$\text{(4.4)} \qquad \widetilde{\mathcal{A}}_\ell^{\alpha,\tau}(\mathrm{F})(x) := \int_{\mathcal{S}} \Big(\prod_{i=1}^{\alpha} f_{\tau(i)}(x - 2^{-\ell}\,\Theta_{\tau(i)} y)\Big) \times \Big(\prod_{i=\alpha+1}^{m} P_{<\ell} f_{\tau(i)}(x - 2^{-\ell}\,\Theta_{\tau(i)} y)\Big)\, \mathrm{d}\sigma(y),$$

$$\text{(4.5)} \qquad \mathcal{M}_{\mathbf{n}}(\mathrm{F}) := \sup_{\ell\in\mathbb{Z}} \Big| \int_{\mathcal{S}} \prod_{j=1}^{m} P_{\ell+n_j} f_j(x - 2^{-\ell}\,\Theta_j y)\, \mathrm{d}\sigma(y) \Big|,$$

$$\text{(4.6)} \qquad \mathcal{S}_{\mathbf{n}}(\mathrm{F}) := \sum_{\ell\in\mathbb{Z}} \Big| \int_{\mathcal{S}} \prod_{j=1}^{m} P_{\ell+n_j} f_j(x - 2^{-\ell}\,\Theta_j y)\, \mathrm{d}\sigma(y) \Big|.$$

Note that $\mathcal{M}_{\mathbf{n}}(\mathrm{F})$ corresponds to $\alpha = 0$ in (4.2). Thus, the lacunary maximal function $\mathcal{M}_{\mathcal{S}}^{\Theta}$ can be controlled by a constant multiple of

$$\sum_{\alpha=1}^{m} \sum_{\tau\in S_m} \sup_{\ell\in\mathbb{Z}} (|\mathcal{A}_\ell^{\alpha,\tau}(\mathrm{F})| + |\widetilde{\mathcal{A}}_\ell^{\alpha,\tau}(\mathrm{F})|) + \sum_{\mathbf{n}\in\mathbb{N}_0^m} \mathcal{M}_{\mathbf{n}}(\mathrm{F}).$$

By the similarity of $\mathcal{A}_\ell^{\alpha,\tau}(\mathrm{F})$ and $\widetilde{\mathcal{A}}_\ell^{\alpha,\tau}(\mathrm{F})$ together with the symmetry on $\tau \in S_m$, instead of the first summation it suffices to consider estimates for $\mathcal{A}_\ell^{\alpha}(\mathrm{F})$, given by

$$\mathcal{A}_\ell^{\alpha}(\mathrm{F})(x) := \int_{\mathcal{S}} \Big(\prod_{j=1}^{\alpha} P_{<\ell} f_j(x - 2^{-\ell}\,\Theta_j y)\Big)\Big(\prod_{j=\alpha+1}^{m} f_j(x - 2^{-\ell}\,\Theta_j y)\Big)\, \mathrm{d}\sigma(y).$$

Then the proof will be completed by combination of the following lemmas and induction on m-linearity.

Lemma 4.1. *For $m = 2$ and $\alpha = 1$, we have*

$$\mathcal{A}_\ell^\alpha(\mathrm{F})(x) \le M_{\mathrm{HL}}(f_1)(x) \times M_{\mathcal{S}}^{\Theta_2}(f_2)(x),$$

where $\mathrm{F} = (f_1, f_2)$ *and*

$$\begin{aligned} M_{\mathcal{S}}^{\Theta_2}(f_2)(x) &= \sup_{\ell\in\mathbb{Z}} \Big| \int_{\mathcal{S}} f_2(x - 2^{-\ell}\,\Theta_2 y)\,\mathrm{d}\sigma(y) \Big| \\ &= \sup_{\ell\in\mathbb{Z}} \Big| \int_{\mathcal{S}} f_2(\Theta_2(\Theta_2^{-1}x - 2^{-\ell} y))\,\mathrm{d}\sigma(y) \Big|. \end{aligned}$$

Proof. For $m = 2$, we have

$$\mathcal{A}_\ell^\alpha(\mathrm{F})(x) = \Big| \int_{\mathcal{S}} P_{<\ell} f_1(x - 2^{-\ell}\,\Theta_1 y) \times f_2(x - 2^{-\ell}\,\Theta_2 y)\,\mathrm{d}\sigma(y) \Big|.$$

It suffices to show $\sup_{y\in\mathcal{S}} |P_{<\ell} f(x - 2^{-\ell} y)| \lesssim M_{\mathrm{HL}}(f)(x)$, where M_{HL} denotes the Hardy–Littlewood maximal function. Since $\phi_\ell(x) = 2^{\ell d}\phi(2^\ell x)$, we have

$$\begin{aligned} P_{<\ell} f(x - 2^{-\ell} y) &= \int_{\mathbb{R}^d} f(z)\,2^{\ell d}\,\phi(2^\ell(x - 2^{-\ell} y - z))\,\mathrm{d}z \\ &= \int_{\mathbb{R}^d} f(x + 2^{-\ell} z)\,\phi(y - z)\,\mathrm{d}z. \end{aligned}$$

Since y is contained in a compact surface $\mathcal{S}$, for any $N > 0$, we have

$$|P_{<\ell} f(x - 2^{-\ell} y)| \lesssim \int_{\mathbb{R}^d} |f(x + 2^{-\ell} z)|\,\frac{C_N}{(1 + |z|)^N}\,\mathrm{d}z \le M_{\mathrm{HL}}(f)(x). \qquad \blacksquare$$

Since $M_{\mathrm{HL}}, M_{\mathcal{S}}^{\Theta_2}$ are bounded on L^p for $p \in (1, \infty]$, we need to handle the summation of $\mathcal{M}_{\mathbf{n}}$ over $\mathbf{n} \in \mathbb{N}_0^m$. Note that for $\alpha = 2$, we have

$$\mathcal{A}_\ell^\alpha(\mathrm{F})(x) \lesssim M_{\mathrm{HL}}(f_1)(x) \times M_{\mathrm{HL}}(f_2)(x).$$

Lemma 4.2. *Let $\mathbf{n} \in \mathbb{N}^m$ and let $1/p = 2(\kappa + 1)/(\kappa + 2)$. Then, for $(1/p_1, \ldots, 1/p_m) \in \mathrm{conv}(\mathcal{V}_\kappa)$, we have*

$$\|\mathcal{M}_{\mathbf{n}}(\mathrm{F})\|_{L^{p,\infty}} \le C(1 + |\mathbf{n}|^m) \prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

In particular, we have $1/p = 2d/(d + 1)$ when we consider averages over $\mathcal{S} = \mathbb{S}^{d-1}$.

Lemma 4.3. *Let $\mathbf{n} \in \mathbb{N}^m$ and $1 = \sum_{j=1}^m 1/r_j$ for some $r_1, \ldots, r_m \in (1, \infty)$. Then we have*

$$\|\mathcal{S}_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|f_j\|_{L^{r_j}}.$$

The proofs of Lemmas 4.2 and 4.3 will be given in Section 5, and note that Lemma 4.3 is an easy consequence of the assumption (1.12). Since $\mathcal{M}_{\mathbf{n}} \le \mathcal{S}_{\mathbf{n}}$, by definition, it follows from interpolation between Lemmas 4.2 and 4.3 that

$$\|\mathcal{M}_{\mathbf{n}}(\mathrm{F})\|_{L^p} \lesssim 2^{-\delta'|\mathbf{n}|} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}}, \quad \delta' > 0,$$

whenever $1/p < 2(\kappa+1)/(\kappa+2)$ and $(1/p_1, \ldots, 1/p_m)$ is in the interior of the convex hull of $\mathrm{conv}(\mathcal{V}_\kappa)$ and $(1/r_1, \ldots, 1/r_m)$. Since $2^{-\delta'|\mathbf{n}|}$ is summable over $\mathbf{n} \in \mathbb{N}_{\mathbf{0}}^m$, this proves the theorem for $m = 2$ inside of the convex hull. Then, together with interpolation with trivial $L^\infty \times \cdots \times L^\infty \to L^\infty$ estimates, we prove the theorem for $m = 2$.

For the induction, we assume that Theorem 1.5 holds for N-linear operators, where $N = 2, \ldots, m-1$. Note that we already showed that the case $m = 2$ holds. By the assumption, we have the following lemma.

Lemma 4.4. *For $\alpha = 1, \ldots, m$, we have*

$$\mathcal{A}_\ell^\alpha(\mathrm{F})(x) \lesssim \prod_{\mu=1}^{\alpha} M_{\mathrm{HL}}(f_\mu)(x) \times \sup_{\ell\in\mathbb{Z}} \int_{\mathcal{S}} \Big| \prod_{\nu=\alpha+1}^{m} f_\nu(x - 2^{-\ell}\Theta_\nu y)\Big| \, d\sigma(y).$$

In addition, if we assume that Theorem 1.5 holds for N-linear operators, where $N = 2, 3, \ldots, m-1$, then it follows that

$$\sup_{\ell\in\mathbb{Z}} \int_{\mathcal{S}} \Big| \prod_{\nu=\alpha+1}^{m} f_\nu(x - 2^{-\ell}\Theta_\nu y)\Big| \, d\sigma(y)$$

satisfies the multilinear estimates of Theorem 1.5.

Proof. The first assertion of the lemma follows directly from the proof of Lemma 4.1. For the second assertion, it is just an $(m-\alpha)$-sublinear average, hence the conclusion follows directly by the assumption. ∎

We assume that Theorem 1.5 is true for N-linear operators, with $N = 2, \ldots, m-1$, and prove the case $N = 2$. For general m, by Lemma 4.4, we have

$$\begin{aligned}
(4.7)\quad & \|\mathcal{A}_\ell^\alpha(\mathrm{F})(x)\|_{L^p(\mathbb{R}^d)} \\
&\lesssim \Big\| \prod_{\mu=1}^{\alpha} M_{\mathrm{HL}}(f_\mu)(x) \times \sup_{\ell\in\mathbb{Z}} \int_{\Sigma} \Big| \prod_{\nu=\alpha+1}^{m} f_\nu(\cdot - 2^{-\ell} y_\nu)\Big| \, d\sigma(y)\Big\|_{L^p(\mathbb{R}^d)} \\
&\le \Big\| \prod_{\mu=1}^{\alpha} M_{\mathrm{HL}}(f_\mu)\Big\|_{L^{1/\alpha_1}(\mathbb{R}^d)} \times \Big\| \sup_{\ell\in\mathbb{Z}} \int_{\Sigma} \Big| \prod_{\nu=\alpha+1}^{m} f_\nu(\cdot - 2^{-\ell} y_\nu)\Big| \, d\sigma(y)\Big\|_{L^{1/\alpha_2}(\mathbb{R}^d)} \\
&\lesssim \prod_{\mu=1}^{\alpha} \|f_\mu\|_{L^{p_\mu}(\mathbb{R}^d)} \times \prod_{\nu=\alpha+1}^{m} \|f_\nu\|_{L^{p_\nu}(\mathbb{R}^d)},
\end{aligned}$$

where $\alpha_1 = 1/p_1 + \cdots + 1/p_\alpha$ and $\alpha_2 = 1/p_{\alpha+1} + \cdots + 1/p_m$.

Since we have already proved Lemmas 4.2 and 4.3 for general m, together with (4.7) we show that Theorem 1.5 holds for m-linear lacunary maximal averages under the assumption that N cases hold for $N = 2, \dots, m-1$. This closes the induction hence proves the theorem.

5. Proofs of Lemmas 4.2 and 4.3

5.1. Proof of Lemma 4.2

For the $L^{p,\infty}$-estimate of Lemma 4.2, we assume that $\| f_j \|_{p_j} = 1$ and we will show the following inequality:

$$\text{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(\mathrm{F})(x) > \lambda\}) \lesssim |\mathbf{n}|^m \lambda^{-p}. \tag{5.1}$$

To obtain (5.1), we exploit the approach of Chirst and Zhou [12], which is based on the multilinear Calderón–Zygmund decomposition. We now apply the Calderón–Zygmund decomposition at height $C\lambda^{p/p_j}$ to each f_j, $j = 1, \dots, m$, for some $C > 0$ so that for each j, we have $f_j = g_j + b_j$ such that

$$\|g_j\|_\infty \le C\lambda^{p/p_j}, \tag{5.2}$$

$$b_j = \sum_\gamma b_{j,\gamma}, \quad \text{supp}(b_{j,\gamma}) \subset Q_{j,\gamma}, \tag{5.3}$$

$$\|b_{j,\gamma}\|_{L^{p_j}}^{p_j} \lesssim \lambda^p \text{meas}(Q_{j,\gamma}), \quad \sum_\gamma \text{meas}(Q_{j,\gamma}) \lesssim \lambda^{-p}, \tag{5.4}$$

$$\int b_{j,\gamma} = 0. \tag{5.5}$$

Note that $Q_{j,\gamma}$ denotes a dyadic cube. Then we have

$$\begin{aligned}\text{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(\mathrm{F})(x) > \lambda\}) \lesssim\ & \text{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(g_1, \dots, g_m)(x) > 2^{-m}\lambda\}) \\ &+ \text{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(g_1, \dots, g_{m-1}, b_m)(x) > 2^{-m}\lambda\}) \\ &+ \dots + \text{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)(x) > 2^{-m}\lambda\}).\end{aligned}$$

For $C_{\mathcal{S}} = 5\max(1, \text{diam}(\mathcal{S}))$, we define $\mathcal{E} = \bigcup_{j=1}^m \bigcup_\gamma C_{\mathcal{S}} Q_{j,\gamma}$ so that $\text{meas}(\mathcal{E}) \lesssim \lambda^{-p}$. Note that $C_{\mathcal{S}} Q$ is a cube whose side-length is $C_{\mathcal{S}}$ times that of Q with the same center as Q. Thus, we estimate each level set for $x \in \mathbb{R}^d \setminus \mathcal{E}$.

5.1.1. Estimates for $\mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)$. Let $b_j^i = \sum_{\gamma: s(Q_{j,\gamma}) = 2^{-i}} b_{j,\gamma}$, where $s(Q)$ denotes the side-length of Q. Then $|\mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)|^p$, with $p = \frac{\kappa+2}{2(\kappa+1)}$, is bounded by

$$\sum_{i_1, \dots, i_m \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} |\mathcal{A}_\ell^{\mathbf{n}}(b_1^{i_1}, \dots, b_m^{i_m})|^p,$$

where

$$\mathcal{A}_\ell^{\mathbf{n}}(f_1, \dots, f_m)(x) = \int_{\mathcal{S}} \prod_{j=1}^m P_{n_j+\ell} f_j(x - 2^{-\ell}\Theta_j y)\, \mathrm{d}\sigma(y).$$

To proceed further, we need two lemmas, whose proofs will be given in the last part of this subsection.

Lemma 5.1. *For* $(1/p_1, \dots, 1/p_m) \in \operatorname{conv}(\mathcal{V}_\kappa)$ *with* $1/p = \sum_{j=1}^m 1/p_j$, *we have*

$$\|\mathcal{A}_\ell^{\mathbf{n}}(b_1^{i_1}, \dots, b_m^{i_m})\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}^p \lesssim \min_{j=1,\dots,m} \min\big(1, 2^{(n_j+\ell-i_j)(1+d/p_j')}, 2^{i_j-\ell}\big) \prod_{j=1}^m \|b_j^{i_j}\|_{p_j}^p.$$

For $m = 2$ and $p = 1/2$, Lemma 5.1 is given in [12]. The proof for general $m \geq 2$ and the case $p = \frac{\kappa+2}{2(\kappa+1)}$ is given in a similar manner.

Lemma 5.2. *Under the same conditions of Lemma* 5.1, *we have*

$$\sum_{\ell\in\mathbb{Z}} \min_{i_1,\dots,i_m} \min(1, 2^{(n_j+\ell-i_j)(1+d/p_j')}, 2^{i_j-\ell}) \lesssim |\mathbf{n}| \prod_{j,j',j\neq j'} \min(1, 2^{|\mathbf{n}|-|i_j-i_{j'}|})^{1/[m(m-1)]}.$$

By using Lemmas 5.1 and 5.2, we have

$$\begin{aligned}\|\mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}^p &\lesssim \sum_{i_1\in\mathbb{Z}} \cdots \sum_{i_m\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \|\mathcal{A}_\ell^{\mathbf{n}}(b_1^{i_1}, \dots, b_m^{i_m})\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}^p \\ &\lesssim \sum_{i_1,\dots,i_m} |\mathbf{n}| \prod_{j,j',j\neq j'} \min(1, 2^{|\mathbf{n}|-|i_j-i_{j'}|})^{\frac{p}{m(m-1)}} \prod_{j=1}^m \|b_j^{i_j}\|_{p_j}^p.\end{aligned}$$

We apply Hölder's inequality to the last line and obtain

$$\|\mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}^p \lesssim |\mathbf{n}| \prod_{j=1}^m \Big(\sum_{i_1,\dots,i_m} \prod_{l\neq j} \min(1, 2^{|\mathbf{n}|-|i_j-i_l|})^{\frac{p_j}{m(m-1)}} \|b_j^{i_j}\|_{p_j}^{p_j} \Big)^{p/p_j}.$$

Observe that the summation over $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_m$ yields

$$\sum_{i_j\in\mathbb{Z}} |\mathbf{n}|^{m-1} \|b_j^{i_j}\|_{p_j}^{p_j}.$$

This is because we have

$$\sum_{i_l} \min(1, 2^{|\mathbf{n}|-|i_j-i_l|})^{\frac{p_j}{m(m-1)}} \lesssim |\mathbf{n}|,$$

since

$$\min(1, 2^{|\mathbf{n}|-|i_j-i_l|}) = \begin{cases} 2^{|\mathbf{n}|+i_j-i_l}, & i_l > i_j + |\mathbf{n}|, \\ 1, & i_j - |\mathbf{n}| \leq i_l \leq i_j + |\mathbf{n}|, \\ 2^{|\mathbf{n}|-i_j+i_l}, & i_l < i_j - |\mathbf{n}|. \end{cases}$$

Therefore, $\|\mathcal{M}_{\mathbf{n}}(b_1, \dots, b_m)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}$ is bounded by a constant multiple of

$$|\mathbf{n}|^{m/p} \prod_{j=1}^m \Big(\sum_{i_j\in\mathbb{Z}} \|b_j^{i_j}\|_{p_j}^{p_j} \Big)^{1/p_j} = |\mathbf{n}|^{m/p} \prod_{j=1}^m \|b_j\|_{p_j}. \tag{5.6}$$

With help of (5.6), we finally estimate the level set of $\mathcal{M}_{\mathbf{n}}(b_1,\dots,b_m)$:

$$\begin{aligned}\operatorname{meas}(\{x\colon \mathcal{M}_{\mathbf{n}}(b_1,\dots,b_m)(x) > 2^{-m}\lambda\}) &\lesssim \lambda^{-p}\|\mathcal{M}_{\mathbf{n}}(b_1,\dots,b_m)\|^p_{L^p(\mathbb{R}^d\setminus\mathcal{E})}\\ &\le \lambda^{-p}|\mathbf{n}|^m\prod_{j=1}^m\|b_j\|^p_{p_j}.\end{aligned}$$

Since $\|b_j\|_{p_j}\lesssim 1$, we obtain (5.1) for $b_1,\dots,b_m$.

5.1.2. Estimates for other terms. The cases $\mathcal{M}_{\mathbf{n}}(g_1,\dots,g_m)$, $\mathcal{M}_{\mathbf{n}}(g_1,\dots,g_{m-1},b_m)$, $\dots$, $\mathcal{M}_{\mathbf{n}}(b_1,\dots,b_{m-1},g_m)$ follow from simplified arguments given in Section 5.1.1. We first consider all cases except $\mathcal{M}_{\mathbf{n}}(g_1,\dots,g_m)$. Thus, we define, for $\alpha+\beta=m$ and $1\le\alpha,\beta\le m-1$,

$$\mathcal{M}_{\mathbf{n}}(\mathbf{g}^\alpha,\mathbf{b}^\beta)=\mathcal{M}_{\mathbf{n}}(g_1,\dots,g_\alpha,b_{\alpha+1},\dots,b_m).$$

Note that one can modify the proofs of Lemmas 5.1 and 5.2 to obtain $\mathbf{b}^\beta$-analogue. Let $(0,\dots,0,1/r_{\alpha+1},\dots,1/r_m)\in\operatorname{conv}(\mathcal{V}_\kappa)$ with $1/p=\sum_{\nu=\alpha+1}^m 1/r_\nu$. Then the proof of Lemma 5.1 yields that

$$\begin{aligned}(5.7)\quad &\|\mathcal{A}^{\mathbf{n}}_\ell(\mathbf{g}^\alpha,\mathbf{b}^\beta)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}\\ &\lesssim\prod_{\mu=1}^\alpha\|g_\mu\|_{L^\infty}\min_{\nu=\alpha+1,\dots,m}\min(1,2^{(n_\nu+\ell-i_\nu)(1+d/r_\nu')},2^{i_\nu-\ell})\prod_{\nu=\alpha+1}^m\|b_\nu^{i_\nu}\|_{r_\nu}.\end{aligned}$$

We have

$$\begin{aligned}(5.8)\quad &\sum_{\ell\in\mathbb{Z}}\min_{i_{\alpha+1},\dots,i_m}\min(1,2^{(n_\nu+\ell-i_\nu)(1+d/r_\nu')},2^{i_\nu-\ell})\\ &\lesssim|\mathbf{n}|\prod_{\alpha+1\le\nu,\nu'\le m,j\ne j'}\min(1,2^{|\mathbf{n}|-|i_\nu-i_{\nu'}|})^{\frac{1}{\beta(\beta-1)}}.\end{aligned}$$

With help of (5.7) and (5.8), we estimate $\|\mathcal{M}_{\mathbf{n}}(\mathbf{g}^\alpha,\mathbf{b}^\beta)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}$ as follows:

$$(5.9)\quad \|\mathcal{M}_{\mathbf{n}}(\mathbf{g}^\alpha,\mathbf{b}^\beta)\|_{L^p(\mathbb{R}^d\setminus\mathcal{E})}\lesssim|\mathbf{n}|^{\beta/p}\prod_{\mu=1}^\alpha\|g_\mu\|_{L^\infty}\times\prod_{\nu=\alpha+1}^m\|b_\nu\|_{L^{r_\nu}}.$$

Since $\operatorname{supp}(b_\nu)\subset\bigcup_\gamma Q_{\nu,\gamma}$ and $\sum_\gamma\operatorname{meas}(Q_{\nu,\gamma})\lesssim\lambda^{-p}$, due to (5.3) and (5.4), the right-hand side of (5.9) is bounded by a constant multiple of

$$(5.10)\quad |\mathbf{n}|^{\beta/p}\lambda^{\sum_{\mu=1}^\alpha p/p_j}\times\lambda^{-p(\sum_{\nu=\alpha+1}^m 1/r_\nu-1/p_\nu)}=|\mathbf{n}|^{\beta/p}.$$

Here the left-hand side of (5.10) is a consequence of Hölder's inequality on $\|b_\nu\|_{L^{r_\nu}}$. Finally, by making use of (5.9) and (5.10), we have

$$\operatorname{meas}(\{x\colon\mathcal{M}_{\mathbf{n}}(\mathbf{g}^\alpha,\mathbf{b}^\beta)(x)>2^{-m}\lambda\})\lesssim\lambda^{-p}\|\mathcal{M}_{\mathbf{n}}(\mathbf{g}^\alpha,\mathbf{b}^\beta)\|^p_{L^p(\mathbb{R}^d\setminus\mathcal{E})}\le\lambda^{-p}|\mathbf{n}|^\beta.$$

For $\mathcal{M}_n(g_1, \dots, g_m)$, we simply choose $C < 2^{-1}$ in (5.2) so that

$$|\mathcal{M}_n(g_1, \dots, g_m)| \le \prod_{j=1}^{m} \|g_j\|_{L^\infty} \le C^m \lambda < 2^{-m}\lambda.$$

Thus, we have

$$\operatorname{meas}(\{x: \mathcal{M}_{\mathbf{n}}(g_1, \dots, g_m)(x) > 2^{-m}\lambda\}) = 0.$$

5.1.3. Proof of Lemma 5.1. For simplicity, let $n_j + \ell = \tau$ and $b_j^{i_j} = b = \sum_Q b_Q$, where $Q = Q_{j,\gamma}$ whose sidelength is 2^{-i}. Then, thanks to Proposition 1.1, it suffices for the first and second term in the minimum to show that

$$\|P_\tau b_Q\|_p \lesssim \min(1, (2^\tau s(Q))^{1+d/p'})\|b_Q\|_p.$$

The first term, 1, is directly given by the fact that $\|\psi_\tau\|_1 = 1$ and Young's inequality.

For the second term, we make use of the vanishing property of b_Q. Let c_Q be the center of Q. Then

$$\begin{aligned} P_\tau b_Q(x) &= \int_{\mathbb{R}^d} (\psi_\tau(x-y) - \psi_\tau(x - c_Q)) b_Q(y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^d} \int_0^1 \langle \nabla_y(\psi_\tau)(x - c_Q - t(y - c_Q)), y - c_Q\rangle\, \mathrm{d}t\, b_Q(y)\, \mathrm{d}y. \end{aligned}$$

Since ψ is of Schwartz class, it follows that

$$\frac{1}{|y - c_Q|} \int_0^1 \langle \nabla_y(\psi_\tau)(x - c_Q - t(y - c_Q)), y - c_Q\rangle\, \mathrm{d}t$$

is bounded by a constant multiple of

$$\frac{2^{\tau(d+1)}}{(1 + 2^\tau|x - c_Q - t(y - c_Q)|)^N},$$

for any $N > 0$. Thus, we apply Minkowski's integral inequality to obtain

$$\begin{aligned} \|P_\tau b_Q\|_{L^p(\mathbb{R}^d)} &\lesssim \Big(\int_{\mathbb{R}^d} \frac{2^{\tau(d+1)p}}{(1 + 2^\tau|x|)^{pN}}\, \mathrm{d}x\Big)^{1/p} \times \int_Q |y - c_Q||b_Q(y)|\, \mathrm{d}y \\ &\le 2^{\tau(d+1)}\, 2^{-\tau d/p} s(Q) s(Q)^{d-d/p} \|b_Q\|_p. \end{aligned}$$

This establishes

$$\|P_\tau b_Q\|_{L^p(\mathbb{R}^d)} \lesssim 2^{\tau(1+d/p')} s(Q)^{1+d/p'} \|b_Q\|_p.$$

Therefore, we have

$$\begin{aligned} \text{(5.11)} \qquad \|P_\tau b\|_{L^p(\mathbb{R}^d)} &= \Big(\sum_Q \|P_\tau b_Q\|_p^p\Big)^{1/p} \\ &\lesssim (2^\tau s(Q))^{1+d/p'} \Big(\sum_Q \|b_Q\|_p^p\Big)^{1/p} = (2^\tau s(Q))^{1+d/p'} \|b\|_p. \end{aligned}$$

The first and the last equalities follow from the disjointness of Q's. This gives a decay estimate when $n_j + \ell < i_j$.

Lastly, we assume that $\ell > i_j$, so that for $x \in (C_{\mathcal{S}} Q)^{\mathrm{c}}$ and $z \in Q$, we have that $\operatorname{dist}(x - 2^{-\ell} y, z) \geq s(Q)$ uniformly in $y \in \mathcal{S}$, because we choose $C_{\mathcal{S}} = 5\max(1, \operatorname{diam}(\mathcal{S}))$. Thus, it follows that

$$P_{n_j+\ell}\, b^{i_j}_{j,\gamma}(x - 2^{-\ell}\Theta_j y) = \widetilde{P}_{n_j+\ell}\, b^{i_j}_{j,\gamma}(x - 2^{-\ell}\Theta_j y),$$

where the kernel of $\widetilde{P}_\tau$ is given by

$$\psi_\tau(y)\, \mathbb{1}_{|y| \geq s(Q)}(y).$$

Therefore, with help of (5.11), we obtain

$$\|\mathcal{A}^{\mathbf{n}}_{\ell}(b^{i_1}_1, \ldots, b^{i_m}_m)\|_{L^p(\mathbb{R}^d \setminus \mathcal{E})} \lesssim \min_{j=1,\ldots,m} \|\widetilde{P}_{n_j+\ell}\|_{p \to p} \prod_{j=1}^{m} \|b^{i_j}_j\|_{p_j}.$$

Observe that, for any $N > 0$,

$$\|\widetilde{P}_{n_j+\ell}\|_{p \to p} \leq \int_{\mathbb{R}^d} |\psi_{n_j+\ell-i_j}(x)|\, \mathbb{1}_{|x| \geq 1}(x)\, \mathrm{d}x \lesssim 2^{-N(n_j+\ell-i_j)}.$$

Thus, we have $\|\widetilde{P}_{n_j+\ell}\|_{p \to p} \leq 2^{-\ell+i_j}$ regardless of $n_j \geq 0$. This proves the lemma.

5.1.4. Proof of Lemma 5.2. It suffices to show that for any i_1, i_2,

$$\sum_{\ell \in \mathbb{Z}} \min_{i_1, i_2} \min(1, 2^{(n_j+\ell-i_j)(1+d/p'_j)}, 2^{i_j-\ell}) \lesssim |\mathbf{n}| \min(1, 2^{|\mathbf{n}|-|i_1-i_2|}). \tag{5.12}$$

Note that

$$\min(1, 2^{(n_j+\ell-i_j)(1+d/p'_j)}, 2^{i_j-\ell}) \leq \begin{cases} 2^{i_j-\ell}, & i_j < \ell, \\ 1, & i_j - |\mathbf{n}| \leq \ell \leq i_j, \\ 2^{(|\mathbf{n}|-i_j+\ell)(1+d/p'_j)}, & \ell < i_j - |\mathbf{n}|. \end{cases}$$

When $i_1 \sim i_2$, the left-hand side of (5.12) is bounded by a constant multiple of $|\mathbf{n}|$. Thus, we consider the case of i_2 being greater than $i_1 + |\mathbf{n}|$. Since $i_2 > i_1 + |\mathbf{n}|$, it follows that for $i_1 - |\mathbf{n}| \leq \ell \leq i_1$,

$$\min_{i_1, i_2} \min(1, 2^{(n_j+\ell-i_j)(1+d/p'_j)}, 2^{i_j-\ell}) \leq 2^{(|\mathbf{n}|-i_2+\ell)(1+d/p'_j)} \leq 2^{(|\mathbf{n}|-|i_1-i_2|)(1+d/p'_j)}. \tag{5.13}$$

Similarly, for $i_2 - |\mathbf{n}| \leq \ell \leq i_2$, we have

$$\min_{i_1, i_2} \min(1, 2^{(n_j+\ell-i_j)(1+d/p'_j)}, 2^{i_j-\ell}) \leq 2^{i_1-\ell} \leq 2^{|\mathbf{n}|-|i_1-i_2|}. \tag{5.14}$$

One can obtain the same bounds when $i_1 > i_2 + |\mathbf{n}|$ by changing roles of i_1, i_2 in (5.13) and (5.14). Therefore, we conclude that

$$\sum_{\ell \in \mathbb{Z}} \min_{i_1, i_2} \min(1, 2^{(n_j+\ell-i_j)(1+d/p'_j)}, 2^{i_j-\ell}) \lesssim |\mathbf{n}| \min(1, 2^{|\mathbf{n}|-|i_1-i_2|}).$$

5.2. Proof of Lemma 4.3

By the assumption (1.12), for $1 \le r_1, \dots, r_m < \infty$ with $1 = \sum_{j=1}^m 1/r_j$, we have

$$\|\mathcal{A}_{\mathcal{S}}^{\Theta}(P_{n_1} f_1, \dots, P_{n_m} f_m)\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|P_{n_j} f_j\|_{L^{r_j}}, \quad \delta > 0.$$

We make use of the following scaling invariance of $\mathcal{A}_{\mathcal{S}}^{\Theta}$:

$$\Big\| \int_{\mathcal{S}} \prod_{j=1}^m P_{\ell+n_j} f_j(x - 2^{-\ell}\Theta_j y)\, \mathrm{d}\sigma(y) \Big\|_{L^1(\mathrm{d}x)} = 2^{-\ell d} \|\mathcal{A}_{\mathcal{S}}^{\Theta}(P_{n_1} f_{1,-\ell}, \dots, P_{n_m} f_{m,-\ell})\|_{L^1},$$

where $f_{j,-\ell}(x) = f_j(2^{-\ell}x)$. Then it follows that

$$\begin{aligned}
\|\mathcal{S}_{\mathbf{n}}(\mathrm{F})\|_{L^1} &\le \sum_{\ell\in\mathbb{Z}} 2^{-\ell d} \|\mathcal{A}_{\mathcal{S}}^{\Theta}(P_{n_1} f_{1,-\ell}, \dots, P_{n_m} f_{m,-\ell})\|_{L^1} \\
&\lesssim \sum_{\ell\in\mathbb{Z}} 2^{-\ell d}\, 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|P_{n_j} f_{j,-\ell}\|_{L^{r_j}} \\
&= \sum_{\ell\in\mathbb{Z}} 2^{-\ell d}\, 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m 2^{\ell d/r_j} \|P_{n_j+\ell} f_j\|_{L^{r_j}} \\
&= \sum_{\ell\in\mathbb{Z}} 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|P_{n_j+\ell} f_j\|_{L^{r_j}}.
\end{aligned}$$

We apply Hölder's inequality to the last line to obtain

$$\|\mathcal{S}_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \Big(\sum_{\ell\in\mathbb{Z}} \|P_{n_j+\ell} f_j\|_{L^{r_j}}^{r_j} \Big)^{1/r_j}.$$

Note that $(\sum_j \|P_j f\|_p^p)^{1/p} \lesssim \|f\|_p$ for $p \ge 2$, which gives

$$\|\mathcal{S}_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|f_j\|_{L^{r_j}}.$$

This proves the lemma.

6. Proof of Theorem 1.6

Recall that for an $(md-1)$-dimensional hypersurface Σ in $\mathbb{R}^{md}$ with κ non-vanishing principal curvatures and $\kappa > (m-1)d$, we define $A_\Sigma(\mathrm{F})(x)$ as follows:

$$\int_\Sigma \prod_{j=1}^m f_j(x - y_j)\, \mathrm{d}\sigma_\Sigma(y), \quad y = (y_1, \dots, y_m) \in \mathbb{R}^{md}.$$

By making use of the dyadic decomposition of Section 4 satisfying (1.14), (4.1), and (4.2), we define the following quantities, similar to (4.3)–(4.6):

$$A_\ell^{\alpha,\tau}(\mathrm{F})(x) := \int_\Sigma \Big(\prod_{i=1}^{\alpha} P_{<\ell} f_{\tau(i)}(x - 2^{-\ell} y_{\tau(i)})\Big) \times \Big(\prod_{i=\alpha+1}^{m} f_{\tau(i)}(x - 2^{-\ell} y_{\tau(i)})\Big)\, \mathrm{d}\sigma(y), \tag{6.1}$$

$$\widetilde{A}_\ell^{\alpha,\tau}(\mathrm{F})(x) := \int_\Sigma \Big(\prod_{i=1}^{\alpha} f_{\tau(i)}(x - 2^{-\ell} y_{\tau(i)})\Big) \times \Big(\prod_{i=\alpha+1}^{m} P_{<\ell} f_{\tau(i)}(x - 2^{-\ell} y_{\tau(i)})\Big)\, \mathrm{d}\sigma(y), \tag{6.2}$$

$$M_{\mathbf{n}}(\mathrm{F}) := \sup_{\ell\in\mathbb{Z}} \Big| \int_\Sigma \prod_{j=1}^{m} P_{\ell+n_j} f_j(x - 2^{-\ell} y_j)\, \mathrm{d}\sigma(y) \Big|, \tag{6.3}$$

$$S_{\mathbf{n}}(\mathrm{F}) := \sum_{\ell\in\mathbb{Z}} \Big| \int_\Sigma \prod_{j=1}^{m} P_{\ell+n_j} f_j(x - 2^{-\ell} y_j)\, \mathrm{d}\sigma(y) \Big|. \tag{6.4}$$

Therefore, the lacunary maximal operator $\mathfrak{M}_\Sigma$ is bounded by a constant multiple of

$$\sum_{\alpha=1}^{m} \sum_{\tau\in S_m} \sup_{\ell\in\mathbb{Z}} (|A_\ell^{\alpha,\tau}(\mathrm{F})| + |\widetilde{A}_\ell^{\alpha,\tau}(\mathrm{F})|) + \sum_{\mathbf{n}\in\mathbb{N}_0^m} M_{\mathbf{n}}(\mathrm{F}), \quad \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{6.5}$$

As in the previous section, instead of the first summation in (6.5), it suffices to consider estimates for $A_\ell^\alpha(\mathrm{F})$, given by

$$A_\ell^\alpha(\mathrm{F})(x) := \int_\Sigma \Big(\prod_{j=1}^{\alpha} P_{<\ell} f_j(x - 2^{-\ell} y_j)\Big)\Big(\prod_{j=\alpha+1}^{m} f_j(x - 2^{-\ell} y_j)\Big)\, \mathrm{d}\sigma(y).$$

Then the proof will be completed by a combination of the following lemmas and an induction argument which is slightly different from the argument in Section 4.

Lemma 6.1. *Let $F = (f_1, f_2, 1, \dots, 1)$ and $\alpha = 1$. Then we have*

$$A_\ell^\alpha(\mathrm{F})(x) \le M_{\mathrm{HL}}(f_1)(x) \times M_\Sigma(f_2)(x),$$

where

$$M_\Sigma(f)(x) = \sup_{\ell\in\mathbb{Z}} \Big| \int_\Sigma f(x - 2^{-\ell} y_2)\, \mathrm{d}\sigma(y) \Big|.$$

The proof of Lemma 6.1 is the same as that of Lemma 4.1, so we omit it. Note that M_{HL} and M_Σ are bounded on L^p for $p \in (1, \infty]$, hence we need the boundedness of the second term in (6.5).

Lemma 6.2. *Let* $\mathbf{n} \in \mathbb{N}^m$ *and* $(m+1)/2 \le 1/p < (2d+\kappa)/(2d)$. *For* $p_j \in [1,2]$, $j = 1, \dots, m$, *with* $\sum_{j=1}^m 1/p_j = 1/p$, *we have*

$$\|M_{\mathbf{n}}(\mathrm{F})\|_{L^{p,\infty}} \le C(1+|\mathbf{n}|^m) \prod_{j=1}^m \|f_j\|_{L^{p_j}}.$$

The proof of Lemma 6.2 is similar to the proof of Lemma 4.2. The only difference occurs in showing Lemma 5.1 in terms of $A_\ell^{\mathbf{n}}$ which corresponds to $\mathcal{A}_\ell^{\mathbf{n}}$, since $A_\ell^{\mathbf{n}}$ is an average over Σ, which in turn is $(md-1)$-dimensional and each f_j depends on $x - y_j$ not $x - \Theta_j y$. This difference is harmless, however, because only the compactness of $\mathcal{S}$ does matter in the proof of Lemma 5.1 and Σ is a compact hypersurface. On the other hand, the range $(m+1)/2 \le 1/p < (2d+\kappa)/(2d)$ follows from Proposition 1.2.

Lemma 6.3. *Let* $\mathbf{n} \in \mathbb{N}^m$ *and* $1 = \sum_{j=1}^m 1/r_j$. *Then*

$$\|S_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^m \|f_j\|_{L^{r_j}}.$$

The proof of Lemma 6.3 is the same with that of Lemma 4.3 together with (1.7), so we omit it.

Due to $M_{\mathbf{n}} \le S_{\mathbf{n}}$, by the definitions in (6.3) and (6.4), it follows from interpolation between Lemmas 6.2 and 6.3 that

$$\|M_{\mathbf{n}}(\mathrm{F})\|_{L^p} \lesssim 2^{-\delta'|\mathbf{n}|} \prod_{j=1} \|f_j\|_{L^{p_j}}, \quad \delta' > 0, \tag{6.6}$$

whenever $1/p_1 + \cdots + 1/p_m = 1/p < (2d+\kappa)/(2d)$. It should be noted that Lemmas 6.2 and 6.3 are still valid for $F = (f_1, \dots, f_N, 1, \dots, 1)$ with m replaced by N and taking L^∞ norms for 1's. That is, for $\mathrm{F} = (f_1, \dots, f_N, 1, \dots, 1)$, we have

$$\|M_{\mathbf{n}}(\mathrm{F})\|_{L^{p,\infty}} \le C(1+|\mathbf{n}|^N) \prod_{j=1}^N \|f_j\|_{L^{p_j}}, \qquad \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N},$$

$$\|S_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\tilde{\delta}|\mathbf{n}|} \prod_{j=1}^N \|f_j\|_{L^{r_j}}, \qquad 1 = \frac{1}{r_1} + \cdots + \frac{1}{r_N},$$

for some $\tilde{\delta} > 0$. Thus, instead of (6.6), we have, for $F = (f_1, \dots, f_N, 1, \dots, 1)$,

$$\|M_{\mathbf{n}}(\mathrm{F})\|_{L^p} \lesssim 2^{-\tilde{\delta}'|\mathbf{n}|} \prod_{j=1}^N \|f_j\|_{L^{p_j}} \quad \text{for some } \tilde{\delta}' > 0.$$

Since $2^{-\tilde{\delta}'|\mathbf{n}|}$ is summable over $\mathbf{n} \in \mathbb{N}_0^m$, this, together with Lemma 6.1, proves the theorem for $F = (f_1, f_2, 1, \dots, 1)$.

For the induction, we assume that Theorem 1.6 holds for $F = (f_1, \dots, f_N, 1, \dots, 1)$ for $N = 2, \dots, m-1$ with $1/p = 1/p_1 + \cdots + 1/p_N$. Note that we showed the case $N = 2$. By the assumption, we have the following lemma.

Lemma 6.4. *For* $\alpha = 1, \dots, m$, *we have*

$$A_\ell^\alpha(\mathrm{F})(x) \lesssim \prod_{\mu=1}^{\alpha} M_{\mathrm{HL}}(f_\mu)(x) \times \sup_{\ell\in\mathbb{Z}} \Big| \int_\Sigma \prod_{\nu=\alpha+1}^{m} f_\nu(x - 2^{-\ell} y_\nu)\, \mathrm{d}\sigma_\Sigma(y) \Big|.$$

Moreover, if we assume that Theorem 1.6 is true for $F = (f_1, \dots, f_N, 1, \dots, 1)$ *with* $N = 2, \dots, m-1$ *and* $1/p = 1/p_1 + \dots + 1/p_N$, *then it follows that*

$$\sup_{\ell\in\mathbb{Z}} \Big| \int_\Sigma \prod_{\nu=\alpha+1}^{m} f_\nu(x - 2^{-\ell} y_\nu)\, \mathrm{d}\sigma_\Sigma(y) \Big|$$

satisfies multilinear estimates of Theorem 1.6 for $(m-\alpha)$*-linear operators.*

Proof. The first assertion of the lemma follows directly by the proof of Lemma 4.1. For the second assertion, recall that $|\mathrm{d}\hat{\sigma}_\Sigma(\xi)| \lesssim (1+|\xi|)^{-\kappa/2}$ for $(m-1)d < \kappa \le md - 1$. Theorem 1.6 holds for $(m-\alpha)$-linear maximal averages when $\kappa > (m-\alpha-1)d$, which is already affirmative. Thus, the assertion holds from the assumption that Theorem 1.6 is true for $F = (f_1, \dots, f_N, 1, \dots, 1)$ with $N = 2, \dots, m-1$ and $1/p = 1/p_1 + \dots + 1/p_N$. ∎

Since we have already proved Lemmas 4.2 and 4.3 for general m, Theorem 1.6 for m-linear operators holds under the assumption that $N = 2, \dots, m-1$ cases hold. This closes the induction hence proves the theorem.

We end this section by suggesting the proof of Remark 1.8.

Proof of Remark 1.8. Note that, for dimension $d = 1$, the proof of this remark is already given in [12]. Although the proof for the case $d \ge 2$ is given in [8], we present a different proof by exploiting ideas of [12]. In fact, the proof follows from Theorem 1.6 with minor modifications in Lemmas 6.2 and 6.3. Indeed, note that in [20] the authors proved the $L^1 \times L^1 \to L^{1/2}$ estimate of the bilinear spherical average $A^1_{\mathbb{S}^{2d-1}}$. Using this estimate in Lemma 6.2, we get

$$\|M_{\mathbf{n}}(\mathrm{F})\|_{L^{1/2,\infty}} \le C(1+|\mathbf{n}|^2) \prod_{j=1}^{2} \|f_j\|_{L^1}.$$

Further, using the estimate in Lemma 6.3 with $\Sigma = \mathbb{S}^{2d-1}$, we get

$$\|S_{\mathbf{n}}(\mathrm{F})\|_{L^1} \lesssim 2^{-\delta|\mathbf{n}|} \prod_{j=1}^{m} \|f_j\|_{L^2} \quad \text{for some } \delta > 0.$$

The rest of the proof follows by imitating the machinery of Theorem 1.6. ∎

Acknowledgments. The authors are sincerely grateful to referees for their comments which have made this article more readable.

Funding. C. Cho, J. B. Lee and K. Shuin have been partially supported by NRF grant no. 2022R1A4A1018904 funded by the Korea government (MSIT). They are supported individually by NRF no. RS-2023-00239774 (C. Cho), no. 2021R1C1C2008252 (J. B. Lee), and BK21 Postdoctoral fellowship of Seoul National University (K. Shuin).

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Received June 6, 2023; revised January 22, 2024.

Chu-hee Cho
Research Institute of Mathematics, Seoul National University
Gwanak-ro 1, Seoul 08826, Republic of Korea;
akilus@snu.ac.kr

Jin Bong Lee
Research Institute of Mathematics, Seoul National University
Gwanak-ro 1, Seoul 08826, Republic of Korea;
jinblee@snu.ac.kr

Kalachand Shuin
Department of Mathematical Sciences, Seoul National University
Gwanak-ro 1, Seoul 08826, Republic of Korea;
kcshuin21@snu.ac.kr