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Schatten class composition operators on the Hardy space of Dirichlet series and a comparison-type principle

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Abstract. We give necessary and sufficient conditions for a composition operator with Dirichlet series symbol to belong to the Schatten classes S_p of the Hardy space $\mathcal{H}^2$ of Dirichlet series. For $p \geq 2$, these conditions lead to a characterization for the subclass of symbols with bounded imaginary parts. Finally, we establish a comparison-type principle for composition operators. Applying our techniques in conjunction with classical geometric function theory methods, we prove the analogue of the polygonal compactness theorem for $\mathcal{H}^2$ and we give examples of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p > 0$.

1. Introduction

The Hardy space $\mathcal{H}^2$ of Dirichlet series, which was first systematically studied in 1997 by H. Hedenmalm, P. Lindqvist, and K. Seip [13], is defined as

$$\mathcal{H}^2 = \Big\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|_{\mathcal{H}^2}^2 = \sum_{n\geq 1} |a_n|^2 < \infty \Big\}.$$

Gordon and Hedenmalm [12] determined the class $\mathfrak{G}$ of symbols which generate bounded composition operators on the Hardy space $\mathcal{H}^2$. The Gordon–Hedenmalm class $\mathfrak{G}$ consists of all functions $\psi(s) = c_0 s + \varphi(s)$, where c_0 is a nonnegative integer, called the characteristic of ψ, and φ is a Dirichlet series such that

(i) if $c_0 = 0$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$,

(ii) if $c_0 \geq 1$, then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\varphi \equiv i\tau$ for some $\tau \in \mathbb{R}$.

We denote by $\mathbb{C}_\theta$, $\theta \in \mathbb{R}$, the half-plane $\{s : \operatorname{Re} s > \theta\}$. We will also use the notation $\mathfrak{G}_0$ and $\mathfrak{G}_{\geq 1}$ for the subclasses of symbols that satisfy (i) and (ii), respectively.

In this paper, we are mostly interested in the case $\psi = \varphi \in \mathfrak{G}_0$. In that context, the compact operators $C_\varphi \colon \mathcal{H}^2 \to \mathcal{H}^2$ were characterized only very recently in [9], in terms

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of the behavior of the mean counting function

$$M_\varphi(w) = \lim_{\sigma\to 0^+}\lim_{T\to\infty}\frac{\pi}{T}\sum_{\substack{s\in\varphi^{-1}(\{w\})\\ |\operatorname{Im} s|<T\\ \sigma<\operatorname{Re} s<\infty}} \operatorname{Re} s, \quad w\in\mathbb{C}_{1/2}\setminus\{\varphi(+\infty)\}.$$

It turns out that C_φ is compact if and only if

$$\lim_{\operatorname{Re} w\to 1/2^+}\frac{M_\varphi(w)}{\operatorname{Re} w - 1/2} = 0. \tag{1.1}$$

The next step would be to characterize symbols $\varphi\in\mathfrak{G}_0$ such that C_φ belongs to the Schatten class S_p, $p>0$. In the disk setting, D. H. Luecking and K. Zhu [17] proved that a composition operator C_ϕ on the Hardy space $H^2(\mathbb{D})$ belongs to the Schatten class S_p, $p>0$, if and only if

$$\int_{\mathbb{D}}\frac{(N_\phi(z))^{p/2}}{(1-|z|^2)^{p/2+2}}\,dA(z) < +\infty, \tag{1.2}$$

where for $z=x+iy$, $dA(z)=dx\,dy$ is the area measure, ϕ is a holomorphic self-map of the unit disk, and N_ϕ is the associated Nevanlinna counting function [24].

Our first main result is that the analogue characterization holds in the Dirichlet series setting provided the symbol has bounded imaginary part.

Theorem 1.1. *Suppose that the symbol $\varphi\in\mathfrak{G}_0$ has bounded imaginary part and that $p\geq 1$. Then, the composition operator C_φ belongs to the class S_{2p} if and only if φ satisfies the condition*

$$\int_{\mathbb{C}_{1/2}}\frac{(M_\varphi(w))^p}{(\operatorname{Re} w-1/2)^{p+2}}\,dA(w) < +\infty. \tag{1.3}$$

For $p>0$, the above condition remains necessary, and if $p\geq 2$, then it is necessary for all symbols in $\mathfrak{G}_0$.

When $p=1$, namely if we want to know if C_φ is Hilbert–Schmidt, things are easier, and Hilbert–Schmidt composition operators with symbols φ in $\mathfrak{G}_0$ have already been characterized in [9]. This is equivalent to saying that

$$\int_{\mathbb{C}_{1/2}}\zeta''(2\operatorname{Re}(w))\,M_\varphi(w)\,dA(w) < +\infty.$$

We generalize this characterization for $C_\varphi\in S_{2m}$, $m\in\mathbb{N}$.

Theorem 1.2. *Let $\varphi\in\mathfrak{G}_0$ and $m\in\mathbb{N}$. Then C_φ belongs to S_{2m} if and only if*

$$\int_{\mathbb{C}_{1/2}}\cdots\int_{\mathbb{C}_{1/2}}\zeta''(\overline{w_1}+w_2)\cdots\zeta''(\overline{w_{m-1}}+w_m)\,\zeta''(\overline{w_m}+w_1)\prod_{k=1}^m M_\varphi(w_k)\,dA(w_k) < \infty. \tag{1.4}$$

Our next result is a comparison-type principle. Using the Lindelöf principle for Green's functions, we will be able to establish geometric conditions on the symbols that imply that the associated composition operator is compact or belongs to S_p. To our knowledge, this is the first example of a technique that gives geometric conditions that apply to all symbols $\varphi \in \mathfrak{G}_0$. To exemplify this, we focus on symbols whose range is contained in angular sectors.

Theorem 1.3. *Let $\varphi \in \mathfrak{G}_0$ and assume that $\varphi(\mathbb{C}_0) \subset \{s \in \mathbb{C}_{1/2} : |\arg(s) - 1/2| < \pi/(2\alpha)\}$ for some $\alpha > 1$. Then C_φ is compact. If we further assume that $\alpha \leq 2$, then $C_\varphi \in S_{2p}$ for any $p > 1/(\alpha - 1)$.*

We can strengthen the previous result proving that if the range of the symbol meets the boundary inside a finite union of angular sectors, then the induced composition operator is compact.

This geometric method also applies to continuity and compactness of composition operators acting on the other Hardy spaces of Dirichlet series $\mathcal{H}^p$, $p \neq 2$. Recall that for $0 < p < \infty$, the Hardy space $\mathcal{H}^p$ of Dirichlet series is defined as the completion of Dirichlet polynomials under the Besicovitch norm (or quasi-norm if $0 < p < 1$)

$$\|P\|_{\mathcal{H}^p} := \Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p \, dt\Big)^{1/p}.$$

The characterization of bounded composition operators with Dirichlet series symbols on $\mathcal{H}^p$, $p \notin 2\mathbb{N}$, is an open and challenging question. The condition $\varphi \in \mathfrak{G}_0$ is necessary but not sufficient [21], and there is no known sufficient conditions which may be applied to a large class of symbols whose range touches the boundary of $\mathbb{C}_0$. We provide such a sufficient condition under the assumption that the range of the symbol is contained in an angular sector.

Theorem 1.4. *Let $k \in \mathbb{N}$ and $p \in (0, 2k]$. If the symbol $\varphi \in \mathfrak{G}_0$ maps the right half-plane into an angular sector of the form $\Omega = \{s \in \mathbb{C}_{1/2} : |\arg(s - 1/2)| < p\pi/(4k)\}$, then C_φ is bounded on $\mathcal{H}^p$. Furthermore, if $\max(1, p) < q \leq 2k$, then the composition operator is compact on $\mathcal{H}^q$.*

In the last section, we briefly discuss the case of Bergman spaces of Dirichlet series as well as some results on Carleson measures.

Notation. Throughout the article, we will be using the convention that C denotes a positive constant which may vary from line to line. We will write that $C = C(x)$ to indicate that the constant depends on a parameter x. If f and g are two real functions defined on the same set Ω, we will write $f \ll g$ if there exists $C > 0$ such that for all $x \in \Omega$, $f(x) \leq Cg(x)$, and $f \sim g$ if $f \ll g$ and $g \ll f$.

2. Background material

2.1. Schatten classes

A compact operator T acting on a separable Hilbert space H can be written as

$$T(x) = \sum_{n\geq 1} s_n \langle x, e_n\rangle \, h_n, \quad x \in H, \tag{2.1}$$

where $\{s_n\}_{n\geq 1}$ is the sequence of singular values and $\{e_n\}_{n\geq 1}$ and $\{h_n\}_{n\geq 1}$ are orthonormal sequences. In case T is self-adjoint, then $e_n = \pm h_n$ for all $n \geq 1$.

For $p > 0$, the S_p Schatten class of compact operators T on H is defined as

$$S_p = S_p(H) = \Big\{T \in \mathfrak{K}(H) : \|T\|_{S_p}^p := \sum_{n\geq 1} s_n^p < \infty\Big\}.$$

Equivalently (see [14]), for $p \geq 1$, a bounded linear operator $T \in \mathfrak{L}(H)$ belongs to S_p if and only if there exists a positive constant C such that

$$\sum_n |\langle Te_n, e_n\rangle|^p \leq C$$

for every orthonormal basis $\{e_n\}$. Furthermore, if T is self-adjoint,

$$\|T\|_{S_p}^p = \sup \sum_n |\langle Te_n, e_n\rangle|^p$$

the supremum being taken over all orthonormal basis of H.

For a compact and positive operator T on H, we define the power T^p, $p > 0$, as

$$T^p(x) = \sum_{n\geq 1} s_n^p \langle x, e_n\rangle\, e_n, \quad x \in H.$$

When $p = n \in \mathbb{N}$, the operator T^n is the n-th iteration of T. We observe that $T \in S_p$ if and only if $T^p \in S_1$. If T is not assumed to be positive, we can still use that $T \in S_p$ if and only if $|T|^p = (T^*T)^{p/2} \in S_1$, if and only if $T^*T \in S_{p/2}$.

For a unit vector $x \in H$ and a positive operator T, applying Hölder's inequality in (2.1) we obtain the following inequality:

$$\langle T^p(x), x\rangle \geq (\langle T(x), x\rangle)^p, \quad p \geq 1. \tag{2.2}$$

For $0 < p \leq 1$, the inequality is reversed.

2.2. The infinite polytorus and vertical limits

The infinite polytorus $\mathbb{T}^\infty$ is defined as the (countable) infinite Cartesian product of copies of the unit circle $\mathbb{T}$,

$$\mathbb{T}^\infty = \{\chi = (\chi_1, \chi_2, \dots) : \chi_j \in \mathbb{T},\ j \geq 1\}.$$

It is a compact abelian group with respect to coordinatewise multiplication. We can identify the Haar measure m_∞ of the infinite polytorus with the countable infinite product measure $m \times m \times \cdots$, where m is the normalized Lebesgue measure of the unit circle.

The polytorus $\mathbb{T}^\infty$ is isomorphic to the group of characters of $(\mathbb{Q}_+, \cdot)$. Given a point $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, the corresponding character $\chi\colon \mathbb{Q}_+ \to \mathbb{T}$ is the completely multiplicative function on $\mathbb{N}$ such that $\chi(p_j) = \chi_j$, where $\{p_j\}_{j\geq 1}$ is the increasing sequence of primes, extended to $\mathbb{Q}_+$ through the relation $\chi(n^{-1}) = \overline{\chi(n)}$.

Suppose $f(s) = \sum_{n\geq 1} a_n/n^s$ is a Dirichlet series and χ is a character. The vertical limit function f_χ is defined as

$$f_\chi(s) = \sum_{n\geq 1} \frac{a_n\, \chi(n)}{n^s}.$$

By Kronecker's theorem [6], for any $\varepsilon > 0$, there exists a sequence of real numbers $\{t_j\}_{j\geq 1}$ such that $f(s+it_j) \to f_\chi(s)$ uniformly on $\mathbb{C}_{\sigma_u(f)+\varepsilon}$, where $\sigma_u(f)$ denotes the abscissa of uniform convergence of f.

If $f \in \mathcal{H}^2$, then for almost every character $\chi \in \mathbb{T}^\infty$, the vertical limit function f_χ converges in the right half-plane and has boundary values $f_\chi(it) = \lim_{\sigma\to 0^+} f_\chi(\sigma+it)$ for almost every $t \in \mathbb{R}$, see [2]. For $\psi(s) = c_0 s + \varphi(s) \in \mathfrak{G}$, we set

$$\psi_\chi(s) = c_0 s + \varphi_\chi(s).$$

Then for every $\chi \in \mathbb{T}^\infty$, we have that

$$(C_\psi(f))_\chi = f_{\chi^{c_0}} \circ \psi_\chi. \tag{2.3}$$

The symbol ψ has boundary values $\psi_\chi(it) = \lim_{\sigma\to 0^+} \psi_\chi(\sigma+it)$ for almost every $\chi \in \mathbb{T}^\infty$ and for almost every $t \in \mathbb{R}$.

2.3. Composition operators on $\mathcal{H}^2$

O. F. Brevig and K–M. Perfekt (see Theorem 1.3 in [9]) proved the following analogue of Stanton's formula for the Hardy spaces of Dirichlet series:

$$\|C_\varphi(f)\|^2_{\mathcal{H}^2} = |f(\varphi(+\infty))|^2 + \frac{2}{\pi}\int_{\mathbb{C}_{1/2}} |f'(w)|^2\, M_\varphi(w)\, dA(w), \tag{2.4}$$

where $\varphi \in \mathfrak{G}_0$ and $f \in \mathcal{H}^2$. By $f(+\infty)$ we denote the first coefficient a_1 of the Dirichlet series $f(s) = \sum_{n\geq 1} a_n/n^s$. We apply the polarization identity in (2.4) yielding to

$$\langle C_\varphi(f), C_\varphi(g)\rangle = f(\varphi(+\infty))\,\overline{g(\varphi(+\infty))} + \frac{2}{\pi}\int_{\mathbb{C}_{1/2}} f'(w)\,\overline{g'(w)}\, M_\varphi(w)\, dA(w). \tag{2.5}$$

We will make use of two properties of the counting function $M_\varphi(w)$ proved in [9], the submean value property and a Littlewood-type inequality. Those respectively are

$$M_\varphi(w) \leq \frac{1}{|D(w,r)|}\int_{D(w,r)} M_\varphi(z)\, dA(z), \tag{2.6}$$

for every disk $D(w,r) \subset \mathbb{C}_{1/2}$ that does not contain $\varphi(+\infty)$, and

$$M_\varphi(w) \leq \log\left|\frac{\varphi(+\infty)+\overline{w}-1}{\varphi(+\infty)-w}\right|, \qquad w \in \mathbb{C}_{1/2}\setminus\{\varphi(+\infty)\}. \tag{2.7}$$

In Subsection 4, we will prove a weaker version of the Littlewood inequality (2.7) but sufficient for our purpose. The standard technique to prove such inequalities goes through regularity results for conformal maps [9, 15]. We shall use the following consequence of (2.7) (see Lemma 2.3 in [9]): for $\sigma_\infty > \mathrm{Re}(\varphi(+\infty))$, there exists $C > 0$ such that, for all $w \in \mathbb{C}_{\sigma_\infty}$,

$$M_\varphi(w) \leq C\,\frac{\mathrm{Re}(w)-1/2}{(1+|\mathrm{Im}(w)|)^2}. \tag{2.8}$$

2.4. Carleson measures

Let H be a Hilbert space of holomorphic functions on a domain Ω. A Borel measure μ in Ω is called a Carleson measure for H if there exists a constant $C > 0$ such that, for all $f \in H$,

$$\int_{\Omega} |f(w)|^2 \, d\mu(w) \leq C \|f\|_H^2.$$

We will denote by $C(\mu, H)$, or simply by $C(\mu)$, the infimum of such constants. For instance, Carleson measures on the Hardy space $H^2(\mathbb{C}_{1/2})$, that consist of holomorphic function f in $\mathbb{C}_{1/2}$ equipped with norm

$$\|f\|_{H^2(\mathbb{C}_{1/2})}^2 := \sup_{\sigma > 1/2} \int_{\mathbb{R}} |f(\sigma + it)|^2 \, dt < \infty,$$

are characterized as follows.

Theorem 2.1 ([10]). *A Borel measure μ on $\mathbb{C}_{1/2}$ is a Carleson measure for $H^2(\mathbb{C}_{1/2})$ if and only if there exists a constant $D > 0$ such that for every square Q with one side I on the line $\{\operatorname{Re} s = 1/2\}$,*

$$\mu(Q) \leq D|I|. \tag{2.9}$$

Moreover, there exist two absolute constants $a, b > 0$ such that, for all Borel measures μ on $\mathbb{C}_{1/2}$, denoting by $D(\mu)$ the infimum of the constants D satisfying (2.9), then $aD(\mu) \leq C(\mu) \leq bD(\mu)$.

2.5. Weighted Hilbert spaces of Dirichlet series

Our main strategy (inspired by [17]) to obtain the membership of C_φ to S_{2p} is to derive it from the membership to S_p of an associated Toeplitz operator defined on another space of Dirichlet series. We now introduce this class of spaces. For $a \leq 1$, we define the weighted Hilbert space $\mathcal{D}_a$ of Dirichlet series as

$$\mathcal{D}_a = \left\{ f(s) = \sum_{n \geq 1} \frac{a_n}{n^s} : \|f\|_a^2 = |a_1|^2 + \sum_{n \geq 2} |a_n|^2 (\log n)^a < \infty \right\}.$$

The reproducing kernel $k_{w,-a}$, $a \geq 0$, of $(\mathcal{D}_{-a})_0$ (space mod constants) at a point $w \in \mathbb{C}_{1/2}$ is given by

$$k_{w,-a}(s) = \sum_{n > 1} \frac{(\log n)^a}{n^{s+\overline{w}}} = \frac{\Gamma(1+a)}{(\overline{w} + s - 1)^{1+a}} + E_a(s + \overline{w}), \quad s \in \mathbb{C}_{1/2}, \tag{2.10}$$

where $E_a(\cdot)$ is a holomorphic function on $\mathbb{C}_0$, see Lemma 5.1 in [15]. Observe that

$$\|k_{w,-a}\|_{-a}^2 = k_{w,-a}(w) \sim_{\operatorname{Re}(w) \to 1/2} \frac{1}{(\operatorname{Re}(w) - 1/2)^{a+1}}. \tag{2.11}$$

For $a = -2$, we have $k_{w,-a}(s) = \zeta''(s + \overline{w})$ and $\zeta''(w) \sim_{\operatorname{Re}(w) \to 1/2} (\operatorname{Re}(w) - 1/2)^{-3}$. Recall also that for any orthonormal basis $\{f_n\}$ of $(\mathcal{D}_{-a})_0$, for any $w \in \mathbb{C}_{1/2}$,

$$\sum_n |f_n(w)|^2 = k_{w,-a}(w). \tag{2.12}$$

The local embedding theorem, [13], states that there exists an absolute constant $C > 0$ such that for every $f \in \mathcal{H}^2$,

$$\frac{1}{2T}\int_{-T}^{T}\left|f\left(\frac{1}{2}+it\right)\right|^2 dt \leq C\|f\|_{\mathcal{H}^2}^2, \quad T > 0. \tag{2.13}$$

A direct application of (2.13) is that for every $f(s) = \sum_{n\geq 2} a_n/n^s \in (\mathcal{D}_{-a})_0$, $a > 0$, we have the following embedding:

$$\begin{aligned}\frac{1}{2T}\int_{-T}^{T}\int_{1/2}^{\infty}|f(\sigma+it)|^2\left(\sigma-\frac{1}{2}\right)^{a-1} d\sigma\, dt &\leq C\sum_{n\geq 2}|a_n|^2\int_{1/2}^{\infty} n^{1-2\sigma}\left(\sigma-\frac{1}{2}\right)^{a-1} d\sigma\, dt\\ &\leq C\,\frac{\Gamma(a)}{2^a}\sum_{n\geq 2}|a_n|^2(\log n)^{-a} \leq C\,\frac{\Gamma(a)}{2^a}\,\|f\|_{-a}^2, \end{aligned}\tag{2.14}$$

where C is the constant appearing in (2.13). In particular, if B is a subset of $\mathbb{C}_{1/2}$ with bounded imaginary part, then $\mathbf{1}_B(\operatorname{Re}(\cdot)-1/2)^{a-1}\,dA$ is a Carleson measure for $(\mathcal{D}_{-a})_0$. More generally, if $\kappa\colon [0,+\infty) \to [0,+\infty)$ is integrable, bounded and decreasing then $\kappa(|\operatorname{Im}(\cdot)|)(\operatorname{Re}(\cdot)-1/2))^{a-1}\,dA$ is a Carleson measure for $(\mathcal{D}_{-a})_0$.

The differentiation operator $D(f) = f'$ is an isometry between $\mathcal{H}_0^2$ and $(\mathcal{D}_{-2})_0$. By (2.5), the composition operator C_φ belongs to $S_{2p}(\mathcal{H}^2)$, $p > 0$, if and only if the operator $(D\circ C_\varphi\circ D^{-1})^* D\circ C_\varphi\circ D^{-1}$ exists in $S_p((\mathcal{D}_{-2})_0)$ if and only if the operator $T_\varphi\colon (\mathcal{D}_{-2})_0 \to (\mathcal{D}_{-2})_0$ defined as

$$\langle T_\varphi(f), g\rangle = \int_{\mathbb{C}_{1/2}} f(w)\,\overline{g(w)}\,M_\varphi(w)\,dA(w) \tag{2.15}$$

belongs to $S_p((\mathcal{D}_{-2})_0)$.

3. Composition operators belonging to Schatten classes

3.1. Schatten class and Carleson measures

We shall divide the proof of Theorem 1.1 into several parts. We first handle the case $p \geq 1$ in a more general context by giving a necessary and a sufficient condition for C_φ to belong to S_{2p}. Both conditions involve M_φ and Carleson measures. At this stage, we do not assume anything on the imaginary part of φ.

Theorem 3.1. *Let $p \geq 1$ and $\varphi \in \mathfrak{G}_0$.*

(a) *Assume that $C_\varphi \in S_{2p}$ and let μ be a Carleson measure for $(\mathcal{D}_{-2})_0$. Then*

$$\int_{\mathbb{C}_{1/2}} \frac{(M_\varphi(w))^p\,\zeta''(2\operatorname{Re}(w))}{(\operatorname{Re}(w)-1/2)^p}\,d\mu(w) < +\infty.$$

(b) *Assume that there exists $\rho\colon \varphi(\mathbb{C}_0) \to (0,+\infty)$ such that $\rho\,dA$ is a Carleson measure for $(\mathcal{D}_{-2})_0$ and that*

$$\int_{\varphi(\mathbb{C}_0)} \frac{(M_\varphi(w))^p\,\zeta''(2\operatorname{Re}(w))}{\rho(w)^{p-1}}\,dA(w) < +\infty. \tag{3.1}$$

Then $C_\varphi \in S_{2p}$.

Proof. We start by proving (a). Denote by I_μ the inclusion operator from $(\mathcal{D}_{-2})_0$ into $L^2(\mathbb{C}_{1/2},\mu)$, which is bounded since μ is Carleson. Moreover, assuming $C_\varphi\in S_{2p}$ or, equivalently, $T_\varphi\in S_p$, we get by the ideal property of Schatten classes that $I_\mu\circ T_\varphi^{p/2}\in S_2$. Let $\{f_n\}$ be any orthonormal sequence of $(\mathcal{D}_{-2})_0$. One can write

$$\begin{aligned}\infty > \|I_\mu\circ T_\varphi^{p/2}\|_{S_2}^2 &= \sum_{n\geq 1}\|T_\varphi^{p/2}(f_n)\|_{L^2(\mu)}^2=\sum_{n\geq 1}\int_{\mathbb{C}_{1/2}}|\langle T_\varphi^{p/2}(f_n),k_{w,-2}\rangle|^2\,d\mu(w)\\ &=\int_{\mathbb{C}_{1/2}}\|T_\varphi^{p/2}(k_{w,-2})\|_{(\mathcal{D}_{-2})_0}^2\,d\mu(w)=\int_{\mathbb{C}_{1/2}}\langle T_\varphi^{p}(k_{w,-2}),k_{w,-2}\rangle\,d\mu(w)\\ &\geq\int_{\mathbb{C}_{1/2}}(\langle T_\varphi(K_{w,-2}),K_{w,-2}\rangle)^p\,\|k_{w,-2}\|^2\,d\mu(w)\end{aligned}$$

by (2.2), where $K_{w,-2}$ is the normalized reproducing kernel of $(\mathcal{D}_{-2})_0$ at w. Observe that the exchange of integral and sum is justified by Tonelli's theorem. Fix $\sigma_\infty>\operatorname{Re}\varphi(+\infty)$. By (2.15),

$$\begin{aligned}\|I_\mu\circ T_\varphi^{p/2}\|_{S_2}^2 &\geq\int_{\mathbb{C}_{1/2}}\Big(\int_{\mathbb{C}_{1/2}}|K_{w,-2}(z)|^2\,M_\varphi(z)\,dA(z)\Big)^p\,\|k_{w,-2}\|^2\,d\mu(w)\\ &\geq\int_{\mathbb{C}_{1/2}\setminus\mathbb{C}_{\sigma_\infty}}\Big(\int_{D(w,\frac12(\operatorname{Re}(w)-1/2))}|K_{w,-2}(z)|^2\,M_\varphi(z)\,dA(z)\Big)^p\,\|k_{w,-2}\|^2\,d\mu(w).\end{aligned}$$

By (2.10), one can estimate the behaviour of $K_{w,-2}(z)$ in the disk $D(w,\frac12(\operatorname{Re}w-1/2))$, whenever $\operatorname{Re}w\leq\sigma_\infty$, to obtain

$$\begin{aligned}&\int_{D(w,\frac12(\operatorname{Re}(w)-1/2))}|K_{w,-2}(z)|^2\,M_\varphi(z)\,dA(z)\\ &\qquad\gg\int_{D(w,\frac12(\operatorname{Re}(w)-1/2))}\frac{M_\varphi(z)}{(\operatorname{Re}(w)-1/2)^3}\,dA(z)\gg\frac{M_\varphi(w)}{\operatorname{Re}(w)-1/2},\end{aligned}$$

where the last inequality follows from the submean value property of the mean counting function (2.6). Taking into account the value of $\|k_{w,-2}\|$, we get

$$\int_{\mathbb{C}_{1/2}\setminus\mathbb{C}_{\sigma_\infty}}\frac{(M_\varphi(w))^p\,\zeta''(2\operatorname{Re}(w))}{(\operatorname{Re}(w)-1/2)^p}\,d\mu(w)<+\infty.$$

Finally, (2.8) yields

$$\begin{aligned}\int_{\mathbb{C}_{\sigma_\infty}}\frac{(M_\varphi(w))^p\,\zeta''(2\operatorname{Re}(w))}{(\operatorname{Re}(w)-1/2)^p}\,d\mu(w)&\ll\int_{\mathbb{C}_{\sigma_\infty}}\zeta''(2\operatorname{Re}(w))\,d\mu(w)\\ &\ll\int_{\mathbb{C}_{\sigma_\infty}}|2^{-w}|^2\,d\mu(w)<+\infty.\end{aligned}$$

Conversely, assume that (3.1) holds and let q be the conjugate exponent of p. For $p=1$, the validity of (3.1) follows from the Hilbert–Schmidt characterization. Thus, we will also assume that $p>1$. Let $\{f_n\}$ be any orthonormal basis of $(\mathcal{D}_{-2})_0$. Then

$$\begin{aligned}\sum_{n\geq 1}\langle T_\varphi(f_n),f_n\rangle^p &= \sum_{n\geq 1}\Big(\int_{\mathbb{C}_{1/2}}|f_n(w)|^2\,M_\varphi(w)\,dA(w)\Big)^p\\ &= \sum_{n\geq 1}\Big(\int_{\varphi(\mathbb{C}_0)}\frac{|f_n(w)|^{2/p}\,M_\varphi(w)}{\rho(w)^{1/q}}\,|f_n(w)|^{2/q}\rho(w)^{1/q}\,dA(w)\Big)^p\\ &\leq \sum_{n\geq 1}\Big(\int_{\varphi(\mathbb{C}_0)}\frac{|f_n(w)|^2\,M_\varphi(w)^p}{\rho(w)^{p/q}}\,dA(w)\Big)\Big(\int_{\varphi(\mathbb{C}_0)}|f_n(w)|^2\,\rho(w)\,dA(w)\Big)^{p/q}.\end{aligned}$$

Since $\rho\,dA$ is a Carleson measure and since $\sum_n|f_n(w)|^2=k_{w,-2}(w)$ for any orthonormal basis of $(\mathcal{D}_{-2})_0$, we get

$$\begin{aligned}\sum_{n\geq 1}\langle T_\varphi(f_n),f_n\rangle^p &\ll \int_{\varphi(\mathbb{C}_0)}\frac{(M_\varphi(w))^p\,k_{w,-2}(w)}{\rho(w)^{p-1}}\,dA(w)\\ &\ll \int_{\varphi(\mathbb{C}_0)}\frac{(M_\varphi(w))^p\,\zeta''(2\operatorname{Re}(w))}{\rho(w)^{p-1}}\,dA(w).\end{aligned}$$

Hence, T_φ belongs to S_p. ■

In view of the above theorem, the ideal case would be to choose a function $\rho:\varphi(\mathbb{C}_0)\to(0,+\infty)$ such that $\rho\,dA$ is a Carleson measure for $(\mathcal{D}_{-2})_0$ and

$$\frac{1}{\rho(w)^{p-1}}=\frac{\rho(w)}{(\operatorname{Re}(w)-1/2)^p},\quad w\in\varphi(\mathbb{C}_0).$$

This yields $\rho(w)=\operatorname{Re}(w)-1/2$. Now if φ has bounded imaginary part, then the embedding inequality implies that $\mathbf{1}_{\varphi(\mathbb{C}_0)}(\operatorname{Re}(w)-1/2)\,dA$ is a Carleson measure for $(\mathcal{D}_{-2})_0$. This gives the way to the case $p>1$ of Theorem 1.1.

Corollary 3.2. *Let $p\geq 1$ and let $\varphi\in\mathfrak{G}_0$ with bounded imaginary part. Then C_φ belongs to S_{2p} if and only if*

$$\int_{\mathbb{C}_{1/2}}\frac{(M_\varphi(w))^p}{(\operatorname{Re}w-1/2)^{p+2}}\,dA(w)<+\infty.$$

Proof. Our discussion actually shows that, under the assumptions of the corollary, we have $C_\varphi\in S_{2p}$ if and only if

$$\int_{\mathbb{C}_{1/2}}\frac{(M_\varphi(w)t)^p}{(\operatorname{Re}w-1/2)^{p-1}}\,\zeta''(2\operatorname{Re}(w))\,dA(w)<+\infty.$$

It remains to show that this is equivalent to (1.3). Let $\sigma_\infty=2\operatorname{Re}\varphi(+\infty)$. Then for $w\in\mathbb{C}_{1/2}\setminus\mathbb{C}_{\sigma_\infty}$,

$$\frac{1}{(\operatorname{Re}(w)-1/2)^3}\ll\zeta''(2\operatorname{Re}(w))\ll\frac{1}{(\operatorname{Re}(w)-1/2)^3}.$$

We may conclude if we are able to prove that for any $\varphi \in \mathfrak{G}_0$,

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p-1}}\,\zeta''(2\operatorname{Re}(w))\,dA(w) < +\infty,$$

and

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p+2}}\,dA(w) < +\infty.$$

Both of these properties follow from (2.8). ∎

When φ does not have bounded imaginary part, there are still interesting Carleson measures for $(\mathcal{D}_{-2})_0$, for instance $(\operatorname{Re}(w)-1/2)/(1+|\operatorname{Im}(w)|)^a\,dA$ for any $a>1$. This yields to the following result.

Corollary 3.3. *Let $p>1$, let $\varphi \in \mathfrak{G}_0$ and let $a>1$.*

(a) *If C_φ belongs to S_{2p}, then*

$$\int_{\mathbb{C}_{1/2}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p+2}(1+|\operatorname{Im}(w)|)^a}\,dA(w) < +\infty.$$

(b) *Assume that*

$$\int_{\mathbb{C}_{1/2}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p+2}}\,(1+|\operatorname{Im}(w)|)^{a(p-1)}\,dA(w) < +\infty.$$

Then $C_\varphi \in S_{2p}$.

Proof. This follows from Theorem 3.1 with $\rho(w)=(\operatorname{Re}(w)-1/2)/(1+|\operatorname{Im}(w)|)^a$ and $d\mu=\rho\,dA$. Again we can replace everywhere $\zeta''(2\operatorname{Re}(w))$ by $(\operatorname{Re}(w)-1/2)^{-3}$ since for (a),

$$\int_{\mathbb{C}_{\sigma_\infty}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p+2}\,(1+|\operatorname{Im}(w)|)^a}\,dA(w) < +\infty$$

and for (b), $\zeta''(2\operatorname{Re}(w)) \ll (\operatorname{Re}(w)-1/2)^{-3}$ is valid throughout $\mathbb{C}_{1/2}$. ∎

We now prove that (1.3) remains necessary for $p\geq 2$ without any assumption on φ.

Theorem 3.4. *Let $p\geq 2$ and $\varphi \in \mathfrak{G}_0$. Assume that $C_\varphi \in S_{2p}$. Then*

$$\int_{\mathbb{C}_{1/2}} \frac{(M_\varphi(w))^p}{(\operatorname{Re} w-1/2)^{p+2}}\,dA(w) < +\infty.$$

Proof. For the positive operator T_φ belonging to S_p, denoting by $\{f_n\}$ an orthonormal basis of eigenvectors of T_φ,

$$\begin{aligned}\infty > \|T_\varphi^p\|_{S_1} &= \sum_{n\geq 1}\langle T_\varphi^p(f_n), f_n\rangle = \sum_{n\geq 1}\int_{\mathbb{C}_{1/2}} T_\varphi^{p-1}(f_n)(w)\,\overline{f_n(w)}\,M_\varphi(w)\,dA(w)\\ &= \sum_{n\geq 1}\int_{\mathbb{C}_{1/2}} \langle T_\varphi^{p-1}(f_n), k_{w,-2}\rangle\,\overline{f_n(w)}\,M_\varphi(w)\,dA(w).\end{aligned}$$

The quantity under the integral sign is nonnegative since

$$\langle T_\varphi^{p-1}(f_n), k_{w,-2}\rangle\, \overline{f_n(w)} = s_n^{p-1}\langle f_n, k_{w,-2}\rangle\, \overline{f_n(w)} = s_n^{p-1}\,|f_n(w)|^2.$$

An application of Tonelli's theorem yields

$$\begin{aligned}\infty > \|T_\varphi^p\|_{S_1} &= \int_{\mathbb{C}_{1/2}} \sum_{n\geq 1}\langle f_n, T_\varphi^{p-1}(k_{w,-2})\rangle\, \overline{f_n(w)}\, M_\varphi(w)\, dA(w)\\ &= \int_{\mathbb{C}_{1/2}} \langle T_\varphi^{p-1}(k_{w,-2}), k_{w,-2}\rangle\, M_\varphi(w)\, dA(w)\\ &= \int_{\mathbb{C}_{1/2}} \langle T_\varphi^{p-1}(K_{w,-2}), K_{w,-2}\rangle\, \|k_{w,-2}\|^2\, M_\varphi(w)\, dA(w).\end{aligned}$$

We now use (2.2):

$$\begin{aligned}\infty > \|T_\varphi^p\|_{S_1} &\geq \int_{\mathbb{C}_{1/2}} \langle T_\varphi(K_{w,-2}), K_{w,-2}\rangle^{p-1}\, \|k_{w,-2}\|^2\, M_\varphi(w)\, dA(w)\\ &\geq \int_{\mathbb{C}_{1/2}} \Big(\int_{\mathbb{C}_{1/2}} |K_{w,-2}(z)|^2\, M_\varphi(z)\, dA(z)\Big)^{p-1}\, \|k_{w,-2}\|^2\, M_\varphi(w)\, dA(w).\end{aligned}$$

We conclude as above using the submean value property of the counting function (2.6) to deduce that (1.3) holds true. ∎

We end up the proof of Theorem 1.1 by considering the case $p \in (0,1)$.

Theorem 3.5. *Let $p \in (0,1)$ and $\varphi \in \mathfrak{G}_0$. Assume that φ has bounded imaginary part and that $C_\varphi \in S_{2p}$. Then*

$$\int_{\mathbb{C}_{1/2}} \frac{(M_\varphi(w))^p}{(\operatorname{Re} w - 1/2)^{p+2}}\, dA(w) < +\infty.$$

Proof. We still denote by $\{f_n\}$ an orthonormal basis of eigenvectors of T_φ. We now write

$$\begin{aligned}\|T_\varphi\|_{S_p}^p &= \sum_{n\geq 1}\big(\langle T_\varphi(f_n), f_n\rangle\big)^p = \sum_{n\geq 1}\Big(\int_{\mathbb{C}_{1/2}} |f_n(w)|^2 M_\varphi(w)\, dA(w)\Big)^p\\ &= \sum_{n\geq 1}\Big(\int_{\varphi(\mathbb{C}_0)} \frac{M_\varphi(w)}{\operatorname{Re}(w)-1/2}\, |f_n(w)|^2\, (\operatorname{Re}(w)-1/2)\, dA(w)\Big)^p.\end{aligned}$$

Now by (2.14), the measures $\mathbf{1}_{\varphi(\mathbb{C}_0)}|f_n(\cdot)|^2(\operatorname{Re}(\cdot)-1/2)\, dA$ are finite measures on $\mathbb{C}_{1/2}$ with uniformly bounded mass. It follows from Hölder's inequality and (2.12) that

$$\begin{aligned}\|T_\varphi\|_{S_p} &\gg \int_{\mathbb{C}_{1/2}} \sum_{n\geq 1} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^p}\, |f_n(w)|^2\, (\operatorname{Re}(w)-1/2)\, dA(w)\\ &\gg \int_{\mathbb{C}_{1/2}} \frac{M_\varphi(w)^p}{(\operatorname{Re}(w)-1/2)^{p-1}}\, \zeta''(2\operatorname{Re}(w))\, dA(w).\end{aligned}$$ ∎

3.2. The case of even integers

We now prove the $2m$-Schatten class characterization (1.4).

Proof of Theorem 1.2. We first prove that (1.4) implies that C_φ is compact. If this were not the case, then we could find $\delta > 0$ and a sequence $\{w(k)\} \subset \mathbb{C}_{1/2}$ with real part going to $1/2$ such that for every $\varepsilon \in (0,1)$, the rectangles

$$R_k = \Big(\frac{\operatorname{Re} w(k) - 1/2}{2}, \frac{3(\operatorname{Re} w(k) - 1/2)}{2}\Big) \\ \times \Big(\operatorname{Im} w(k) - \varepsilon\Big(\operatorname{Re} w(k) - \frac{1}{2}\Big), \operatorname{Im} w(k) + \varepsilon\Big(\operatorname{Re} w(k) - \frac{1}{2}\Big)\Big)$$

are pairwise disjoint and for all $k \geq 1$,

$$\frac{M_\varphi(w(k))}{\operatorname{Re}(w(k)) - 1/2} \geq \delta.$$

Let $A_k = \prod_{j=1}^m R_k$. We recall that $\zeta''(s)$ has a pole of order 3 at $s = 1$, thus we can choose $\varepsilon > 0$ close to zero such that

$$\operatorname{Re}\big(\zeta''(\overline{w_1} + w_2)\cdots\zeta''(\overline{w_{m-1}} + w_m)\,\zeta''(\overline{w_m} + w_1)\big) \gg \Big(\operatorname{Re} w(k) - \frac{1}{2}\Big)^{-3m},$$

for every $w = (w_1, \ldots, w_m) \in A_k$. Using the mean-value property of the counting function as well as the estimate above, we obtain that

$$\int_{A_k} \zeta''(\overline{w_1} + w_2)\cdots\zeta''(\overline{w_{m-1}} + w_m)\,\zeta''(\overline{w_m} + w_1) \prod_{j=1}^m M_\varphi(w_j)\,dA(w_j) \\ \gg \prod_{j=1}^m \frac{M_\varphi(w_k)}{\operatorname{Re}(w(k)) - 1/2} \geq \delta^m.$$

Since the sets A_k are pairwise disjoint, this would contradict (1.4).

Hence, for both implications of Theorem 1.2, we may assume that C_φ hence T_φ is compact. Let us consider the canonical decomposition of T_φ, $T_\varphi(f) = \sum_{n\geq 1} s_n \langle f, f_n\rangle f_n$. We know that $C_\varphi \in S_{2m}$ if and only if $T_\varphi^m \in S_1$, if and only if

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n\rangle < \infty.$$

We observe that

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n\rangle = \sum_{n\geq 1} \int_{\mathbb{C}_{1/2}} \langle T_\varphi^{m-1}(f_n), k_{w_1,-2}\rangle\,\overline{f_n(w_1)}\,M_\varphi(w_1)\,dA(w_1).$$

Arguing as in the proof of Theorem 3.4, we may use Tonelli's theorem to get

$$\sum_{n\geq 1} \langle T_\varphi^m(f_n), f_n\rangle = \int_{\mathbb{C}_{1/2}} \langle T_\varphi^{m-1}(k_{w_1,-2}), k_{w_1,-2}\rangle\,M_\varphi(w_1)\,dA(w_1) \\ = \int_{\mathbb{C}_{1/2}} \int_{\mathbb{C}_{1/2}} \langle T_\varphi^{m-2}(k_{w_1,-2}), k_{w_2,-2}\rangle \zeta''(w_1 + \overline{w_2})\,M_\varphi(w_2)\,M_\varphi(w_1)\,dA(w_2)\,dA(w_1).$$

By induction one obtains

$$\|T_\varphi^m\|_{S_1} = \int_{\mathbb{C}_{1/2}} \cdots \int_{\mathbb{C}_{1/2}} \zeta''(\overline{w_1}+w_2)\cdots\zeta''(\overline{w_{m-1}}+w_m)\,\zeta''(\overline{w_m}+w_1) \times \prod_{k=1}^{m} M_\varphi(w_k)\,dA(w_k). \qquad \blacksquare$$

We now intend to give a similar characterization involving the boundary values of $\varphi \in \mathfrak{G}_0$. For every $\chi \in \mathbb{T}^\infty$, φ_χ belongs to $\mathfrak{G}_0$ and for almost every χ, the generalized boundary value $\varphi(\chi) = \lim_{\sigma\to 0^+}\varphi_\chi(\sigma)$ does exist (see Section 2 in [8] or Corollary 3.3 in [9]). Of course, $\mathrm{Re}(\varphi(\chi)) \geq 1/2$ for almost every $\chi \in \mathbb{T}^\infty$. We first show that when C_φ is compact, this inequality is strict for almost every $\chi \in \mathbb{T}^\infty$.

Theorem 3.6. *Let $\varphi \in \mathfrak{G}_0$ be such that C_φ induces a compact operator on $\mathcal{H}^2$. Then* $\mathrm{Re}(\varphi(\chi)) > 1/2$, *for almost every $\chi \in \mathbb{T}^\infty$.*

Proof. The norm of the image of a function $f \in \mathcal{H}^2$ under C_φ can be written as

$$\|C_\varphi(f)\|^2_{\mathcal{H}^2} = \int_{\mathbb{T}^\infty} |f\circ\varphi(\chi)|^2\,dm_\infty(\chi) = \int_{\overline{\mathbb{C}_{1/2}}} |f(w)|^2\,d\mu_\varphi(w),$$

where μ_φ is the push-forward measure of m_∞ by $\varphi(\chi)$, see [8]. Since C_φ is compact and the reproducing kernel $\zeta(\cdot+\overline{w})$ of $\mathcal{H}^2$ at w satisfies

$$\zeta(s+\overline{w}) = \frac{1}{\overline{w}+s-1} + O(1),$$

we can argue like in the proof of Theorem 3 in [19] to deduce that

$$\mu_\varphi(Q) = o(|I|), \quad \text{as } |I|\to 0, \tag{3.2}$$

where Q is a (Carleson) square in $\mathbb{C}_{1/2}$ with one side I on the vertical line $\{\mathrm{Re}\, s = 1/2\}$. This means that μ_φ is a vanishing Carleson measure for $H^2(\mathbb{C}_{1/2})$ and this implies that $\mu_\varphi|_{\{\mathrm{Re}\, s=1/2\}}$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}$. Following a standard argument, see for example [11], Chapter 3, we will prove that $\mu_\varphi|_{\{\mathrm{Re}\, s=1/2\}}$ is equal to 0. By the Lebesgue–Radon–Nikodym theorem, there exists a positive function $f \in L^1(\mathbb{R})$ such that

$$d\mu_\varphi|_{\{\mathrm{Re}\, s=1/2\}} = f(t)\,dt.$$

The set $E = \{\chi : \mathrm{Re}\,\varphi(\chi) > 1/2\}$ is of full measure if and only if $f \equiv 0$. Let us assume that there exists $\varepsilon > 0$ such that $|f^{-1}((\varepsilon,\infty))| > 0$. Let $F \subset f^{-1}((\varepsilon,+\infty))$ with positive and finite measure, and let $\delta > 0$ be such that

$$\mu_\varphi(Q) \leq \frac{\varepsilon}{4}|I|$$

for every Carleson square in $\mathbb{C}_{1/2}$ with length $|I| \leq \delta$. We can cover F by a sequence of intervals $\{I_n\}$ such that $|I_n| \leq \delta$ and

$$\sum_n |I_n| \leq 2|F|.$$

Now,

$$\varepsilon|F| \leq \mu_\varphi|_{\{\operatorname{Re} s=1/2\}}(F) \leq \frac{\varepsilon}{4}\sum_n |I_n| \leq \frac{\varepsilon}{2}|F|,$$

which is a contradiction with $|F| > 0$. Thus $\operatorname{Re}(\varphi(\chi)) > 1/2$ for a.e. $\chi \in \mathbb{T}^\infty$. ■

We are now ready to give an analogue of Theorem 1.2 involving the symbol directly.

Theorem 3.7. *Suppose that the symbol $\varphi \in \mathfrak{G}_0$ induces a compact operator and let $m \in \mathbb{N}$. Then, C_φ belongs to S_{2m}, if and only if*

$$\begin{aligned}\int_{\mathbb{T}^\infty}\cdots\int_{\mathbb{T}^\infty} \zeta(\overline{\varphi(\chi_1)}+\varphi(\chi_2))\cdots\zeta(\overline{\varphi(\chi_{m-1})}+\varphi(\chi_m))\,\zeta(\overline{\varphi(\chi_m)}+\varphi(\chi_1))\\ \times\prod_{k=1}^m dm_\infty(\chi_k) < \infty.\end{aligned} \tag{3.3}$$

Proof. Let $T = C_\varphi^* \circ C_\varphi$, and let us consider its canonical decomposition

$$T(f) = \sum_{n\geq 1} s_n \langle f, f_n\rangle f_n.$$

We know that $C_\varphi \in S_{2m}$ if and only if $T^m \in S_1$, and that

$$\langle T(f), g\rangle = \int_{\mathbb{T}^\infty} f(\varphi(\chi))\,\overline{g(\varphi(\chi))}\,dm_\infty(\chi).$$

Then

$$\begin{aligned}\sum_{n\geq 1}\langle T^m(f_n), f_n\rangle &= \sum_{n\geq 1}\langle T(f_n), T^{m-1}(f_n)\rangle\\ &= \sum_{n\geq 1}\int_{\mathbb{T}^\infty} f_n(\varphi(\chi_1))\,\overline{\langle T^{m-1}(f_n), \zeta(\cdot+\overline{\varphi(\chi_1)})\rangle}\,dm_\infty(\chi_1).\end{aligned}$$

As in the proof of Theorem 1.2, the quantity inside the integral is nonnegative which allows us to use Tonelli's theorem. Hence

$$\begin{aligned}\sum_{n\geq 1}\langle T^m(f_n), f_n\rangle &= \int_{\mathbb{T}^\infty}\langle T^{m-1}(k_{\varphi(\chi_1),0}), k_{\varphi(\chi_1),0}\rangle\,dm_\infty(\chi_1)\\ &= \int_{\mathbb{T}^\infty}\int_{\mathbb{T}^\infty}\langle T^{m-2}(k_{\varphi(\chi_1),0}), k_{\varphi(\chi_2),0}\rangle\,\zeta(\varphi(\chi_1)+\overline{\varphi(\chi_2)})\,dm_\infty(\chi_1)\,dm_\infty(\chi_2).\end{aligned}$$

By induction, one finally obtains

$$\begin{aligned}\|T^m\|_{S_1} = \int_{\mathbb{T}^\infty}\cdots\int_{\mathbb{T}^\infty} \zeta(\overline{\varphi(\chi_1)}+\varphi(\chi_2))\cdots\zeta(\overline{\varphi(\chi_{m-1})}+\varphi(\chi_m))\,\zeta(\overline{\varphi(\chi_m)}+\varphi(\chi_1))\\ \times\prod_{k=1}^m dm_\infty(\chi_k).\end{aligned}$$

■

4. A comparison-type principle

4.1. The Lindelöf principle and the Littlewood inequality

In this section, we will use the Lindelöf principle for Green's functions to give a simple proof of the non-contractive Littlewood inequality (2.7). Similar techniques have been used in the disk setting, [5].

We recall (see for instance [22]) that a Green function for a domain $\Omega \subset \mathbb{C}$ is a function $g_\Omega\colon \Omega \times \Omega \to (-\infty, +\infty]$ such that, for all $w \in \Omega$, $g(\cdot, w)$ is harmonic in $\Omega \setminus \{w\}$, $g_\Omega(z, w) \to 0$ n.e as $z \to \partial\Omega$ and $g_\Omega(\cdot, w) + \log|\cdot - w|$ is harmonic in a neighbourhood of w. If a domain admits a Green function then it is necessarily unique. For instance, the Green function on the disk $g_{\mathbb{D}}\colon \mathbb{D} \times \mathbb{D} \mapsto (0, +\infty]$ has the form

$$g_{\mathbb{D}}(z, w) = \log\left|\frac{1 - z\overline{w}}{z - w}\right|.$$

By conformal invariance, we can easily define Green's function on every simply connected subdomain of the complex plane, for example,

$$g_{\mathbb{C}_0}(z, w) = \log\left|\frac{z + \overline{w}}{z - w}\right|, \quad z, w \in \mathbb{C}_0.$$

The class of domains D possessing a Green function g_D is much larger than the simply connected domains, see Chapter 4 of [22]. The Lindelöf principle for Green's functions (see for instance [4]) states that if f is a holomorphic function mapping D_1 to D_2, where both of those domains possess Green's function, then for $z_0 \in D_1$ and $w \in D_2 \setminus \{f(z_0)\}$,

$$\sum_{z \in f^{-1}(\{w\})} g_{D_1}(z, z_0) \le g_{D_2}(w, f(z_0)). \tag{4.1}$$

Let us first show how to deduce, up to a multiplicative constant, the Littlewood inequality (2.7) and also a corresponding inequality for a symbol in $\mathfrak{G}_{\ge 1}$ (such an inequality was used in [3] to obtain a sufficient condition for composition operators with symbols in $\mathfrak{G}_{\ge 1}$ to be compact on $\mathscr{H}^2$). Recall that for $\psi \in \mathfrak{G}_{\ge 1}$, its restricted Nevanlinna counting function is defined by

$$N_\psi(w) = \sum_{\substack{s \in \psi_\chi^{-1}(\{w\}) \\ |\operatorname{Im} s| \le 1}} \operatorname{Re} s.$$

Theorem 4.1. *The following statements hold.*

(a) *Let $\varphi \in \mathfrak{G}_0$. Then for all $w \in \mathbb{C}_{1/2} \setminus \{\varphi(+\infty)\}$,*

$$M_\varphi(w) \le \pi \log\left|\frac{\varphi(+\infty) + \overline{w} - 1}{\varphi(+\infty) - w}\right|.$$

(b) *Let $\psi \in \mathfrak{G}_{\ge 1}$. There exists $C > 0$ such that, for all $\chi \in \mathbb{T}^\infty$ and for all $w \in \mathbb{C}_0$ with* $\operatorname{Re} w \le c_0$,

$$N_{\psi_\chi}(w) \le C \frac{\operatorname{Re} w}{1 + (\operatorname{Im} w)^2}.$$

Proof. (a) Let $\varphi \in \mathfrak{G}_0$ and $w \in \mathbb{C}_{1/2} \setminus \{\varphi(+\infty)\}$. For $T > 0$ sufficiently large, $w \neq \varphi(T)$, so that the Lindelöf principle implies

$$\sum_{s \in \varphi^{-1}(\{w\})} \log \left| \frac{T + \bar{s}}{T - s} \right| \leq \log \left| \frac{\varphi(T) + \overline{w} - 1}{\varphi(T) - w} \right|.$$

On the other hand, using the elementary inequality $\log(x) \geq \frac{1}{2}(1 - x^{-2})$, valid for $x > 0$,

$$\sum_{s \in \varphi^{-1}(\{w\})} \log \left| \frac{T + \bar{s}}{T - s} \right| \geq \sum_{s \in \varphi^{-1}(\{w\})} \frac{2T \operatorname{Re} s}{|T + s|^2}.$$

Observe that $\{\operatorname{Re}(s) : s \in \varphi^{-1}(\{w\})\}$ is bounded. Therefore, for all $\varepsilon \in (0, 1)$, we can choose T large enough so that

$$\sum_{s \in \varphi^{-1}(\{w\})} \log \left| \frac{T + \bar{s}}{T - s} \right| \geq \frac{(1 - \varepsilon)}{T} \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| \leq T}} \operatorname{Re} s.$$

We can conclude by letting T to $+\infty$ and ε to 0.

Regarding (b), let $\chi \in \mathbb{T}^\infty$ and $w \in \mathbb{C}_0$ with $\operatorname{Re} w \leq c_0$. The Lindelöf principle says that

$$\sum_{s \in \psi_\chi^{-1}(\{w\})} \log \left| \frac{\bar{s} + 2}{s - 2} \right| \leq \log \left| \frac{\overline{w} + \psi_\chi(2)}{w - \psi_\chi(2)} \right|.$$

Now, when $\psi_\chi(s) = w$, then $0 < \operatorname{Re} s = (\operatorname{Re} w - \operatorname{Re} \varphi(s))/c_0 \leq 1$, since $\operatorname{Re} w \leq c_0$. We apply again the inequality $\log(x) \geq \frac{1}{2}(1 - x^{-2})$, $x > 0$, yielding to

$$N_{\psi_\chi}(w) \leq C \sum_{s \in \psi_\chi^{-1}(\{w\})} \log \left| \frac{\bar{s} + 2}{s - 2} \right|.$$

Finally, it was shown in [3] that

$$\log \left| \frac{\overline{w} + \psi_\chi(2)}{w - \psi_\chi(2)} \right| \leq C \frac{\operatorname{Re} w}{1 + (\operatorname{Im} w)^2},$$

where C does not depend neither on $\chi \in \mathbb{T}^\infty$ nor on w with $\operatorname{Re} w \leq c_0$. ∎

4.2. A comparison-type principle and a polygonal compactness theorem

We shall now apply the idea of the previous subsection when $\varphi \in \mathfrak{G}_0$ maps $\mathbb{C}_0$ into a subdomain D of $\mathbb{C}_{1/2}$. The Lindelöf principle helps us to find better estimates on M_φ. Indeed, provided D admits a Green function, the proof of Theorem 4.1 shows that

$$M_\varphi(w) \ll g_D(w, \varphi(+\infty)), \quad w \in \mathbb{C}_{1/2} \setminus \{\varphi(+\infty)\}. \tag{4.2}$$

We deduce the following comparison principle. Under similar conditions, a norm-comparison principle appeared in [8].

Theorem 4.2. *Let $\varphi \in \mathfrak{G}_0$ be such that $\varphi(\mathbb{C}_0) \subset D \subset \mathbb{C}_{1/2}$, where D is a simply connected domain. Let R_D be the Riemann map from $\mathbb{D}$ onto D such that $R_D(0) = \varphi(+\infty)$, and let $\varphi_D = R_D(2^{-s})$. Assume that C_{φ_D} is compact. Then C_φ is compact. Moreover, if $D \subset \{|\operatorname{Im}(s)| \leq C\}$ for some $C > 0$ and if C_{φ_D} belongs to S_{2p}, $p \geq 1$, then C_φ belongs to S_{2p}.*

Proof. Let $p = 2\pi/\log(2)$. By the ip-periodicity of φ_D, we have that for all $T > 0$,

$$\left\lfloor \frac{2T}{p} \right\rfloor \sum_{\substack{s \in \varphi_D^{-1}(\{w\}) \\ 0 \leq \operatorname{Im} s < p \\ \sigma < \operatorname{Re} s < \infty}} \operatorname{Re} s \leq \sum_{\substack{s \in \varphi_D^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} \operatorname{Re} s \leq \left(\left\lfloor \frac{2T}{p} \right\rfloor + 1 \right) \sum_{\substack{s \in \varphi_D^{-1}(\{w\}) \\ 0 \leq \operatorname{Im} s < p \\ \sigma < \operatorname{Re} s < \infty}} \operatorname{Re} s,$$

where $\lfloor x \rfloor$ is the integer part of the real number x, see Example 4.6 in [15]. Thus

$$M_{\varphi_D}(w) \sim \sum_{\substack{0 \leq \operatorname{Im} s < p \\ s \in \varphi_D^{-1}(\{w\})}} \operatorname{Re} s \sim \sum_{\substack{z \in \mathbb{D} \\ z \in R_D^{-1}(\{w\})}} \log\left(\frac{1}{|z|}\right)$$

since

$$\log 2 \sum_{\substack{s \in \varphi_D^{-1}(\{w\}) \\ |\operatorname{Im} s| < p \\ \operatorname{Re} s > 0}} \operatorname{Re} s = \sum_{\substack{z \in R_D^{-1}(\{w\}) \\ |z| < 1}} \log \frac{1}{|z|}.$$

By the conformal invariance of the Green function,

$$M_{\varphi_D}(w) \sim g_D(w, \varphi_D(+\infty)).$$

Hence our assumption on C_{φ_D} gives an estimate on M_{φ_D} which transfers to M_φ thanks to (4.2), which itself gives the corresponding result on C_φ. Observe that in both cases, we use the *characterization* of compactness or membership to S_{2p}. ∎

Remark. In Theorem 4.2, we can only assume that D admits a Green function and use for R_D a universal covering map of D.

The most interesting case occurs when $\varphi(\mathbb{C}_0)$ is mapped into an angular sector contained in $\mathbb{C}_{1/2}$. This leads to Theorem 1.3 that we now prove.

Proof of Theorem 1.3. The Green function of the domain $\Omega = \{s \in \mathbb{C}_{1/2} : |\arg(s) - 1/2| < \pi/(2\alpha)\}$ is

$$g_\Omega(z, w) = \log \left| \frac{(z - 1/2)^\alpha + \overline{(w - 1/2)^\alpha}}{(z - 1/2)^\alpha - (w - 1/2)^\alpha} \right|.$$

By (4.2) and by Lemma 2.3 in [9], for $w \in \varphi(\mathbb{C}_0) \subset \Omega$,

$$M_\varphi(w) \ll \log \left| \frac{(w - 1/2)^\alpha + \overline{(\varphi(+\infty) - 1/2)^\alpha}}{(w - 1/2)^\alpha - (\varphi(+\infty) - 1/2)^\alpha} \right| \ll \operatorname{Re}\left(w - \frac{1}{2}\right)^\alpha \ll (\operatorname{Re}(w) - 1/2)^\alpha$$

provided $|w - \varphi(+\infty)| > \delta$ for some fixed $\delta > 0$. The proof of compactness follows from the characterization (1.1); the proof of the Schatten class part follows from Corollary 3.3.

Indeed, letting $\sigma_\infty = 2\operatorname{Re}\varphi(+\infty)$, let $T > 0$ be such that $|\operatorname{Im}(w)| \le T$ for all $w \in \varphi(\mathbb{C}_0) \cap (\mathbb{C}_{1/2} \setminus \mathbb{C}_{\sigma_\infty})$. Then

$$\begin{aligned}\int_{\mathbb{C}_{1/2}\setminus\mathbb{C}_{\sigma_\infty}} &\frac{(M_\varphi(w))^p}{(\operatorname{Re}(w)-1/2)^{p+2}}(1+|\operatorname{Im}(w)|)^{2(p-1)}\,dA(w)\\ &\ll \int_{1/2}^{\sigma_\infty}\int_{-T}^{T}\frac{(1+|\operatorname{Im}(w)|)^{2(p-1)}}{(\operatorname{Re}(w)-1/2)^{p+2-p\alpha}}\,dt\,d\sigma < +\infty, \quad w = \sigma + it,\end{aligned}$$

since $p > 1/(\alpha - 1)$. Moreover,

$$\begin{aligned}\int_{\mathbb{C}_{\sigma_\infty}} &\frac{(M_\varphi(w))^p}{(\operatorname{Re}(w)-1/2)^{p+2}}(1+|\operatorname{Im}(w)|)^{2(p-1)}\,dA(w)\\ &\ll \int_{\mathbb{C}_{\sigma_\infty}}\frac{dA(w)}{(1+|\operatorname{Im}(w)|)^2(\operatorname{Re}(w)-1/2)^2} < +\infty,\end{aligned}$$

by (2.8). ∎

Using properties of conformal maps, we can extend the compactness part of Theorem 1.3 to slightly more general domains. This is the analogue of the polygonal compactness theorem, [25], in our setting.

Theorem 4.3. *Let $\varphi \in \mathfrak{G}_0$ be such that for some $\delta > 0$, the set $\varphi(\mathbb{C}_0) \cap \{1/2 < \operatorname{Re} s \le 1/2 + \delta\}$ is contained in a finite union of angular sectors $\bigcup_{j=1}^{d}\{|\arg(z - 1/2 - i\tau_j)| < \alpha_j\}$, with $\tau_j \in \mathbb{R}$ and $\alpha_j \in (0, \pi/2)$. Then C_φ is compact on $\mathcal{H}^2$.*

Proof. Let us consider the Riemann map $f = R_D$, where

$$D = \mathbb{C}_{1/2+\delta} \cup \bigcup_{j=1}^{d}\left\{\left|\arg\left(z - \frac{1}{2} - i\tau_j\right)\right| < \alpha_j\right\},$$

and let $\{w_n\}_{n\ge1}$ be an arbitrary sequence such that $\operatorname{Re} w_n \to 1/2^+$. Since $M_\varphi(w_n) = 0$ if $w_n \notin D$, we can assume that the sequence $\{w_n\}$ converges to a corner boundary point $f(e^{i\theta_0}) \ne \infty$. Then, by (4.2) and Koebe's quarter theorem, see Corollary 1.4 in [20],

$$\begin{aligned}M_\varphi(w_n) &\ll g_D(w_n, f(0)) \ll 1 - |f^{-1}(w_n)|^2 \ll \operatorname{dist}(w_n, \partial D)\,|f'(f^{-1}(w_n))|^{-1}\\ &\le \left(\operatorname{Re} w_n - \frac{1}{2}\right)|f'(f^{-1}(w_n))|^{-1}.\end{aligned}$$

It is sufficient to prove that $|f'(f^{-1}(w_n))| \to \infty$ as $n \to \infty$. By the Kellogg–Warschawski theorem (see Theorem 3.9 in [20]) and the Carathéodory extension theorem, see Chapter 2 of [20],

$$|f'(f^{-1}(w_n))| \gg |f^{-1}(w_n) - e^{i\theta_0}|^{\alpha-1} \to \infty,$$

where $\alpha = \max\{a_j : 1 \le j \le d\}$. ∎

Remark. Our techniques apply also for symbols $\psi = c_0 s + \varphi \in \mathfrak{G}_{\ge1}$. Although, the range of such a symbol cannot meet the imaginary axis in an angular sector, or more generally, inside a domain D where $\operatorname{Im} w$ is bounded for $w \in D \bigcap(\mathbb{C}_0 \setminus \mathbb{C}_\varepsilon)$, $\varepsilon > 0$. If that were the case, then we would be able to find a point $s(\varepsilon) \in \mathbb{C}_0 \setminus \mathbb{C}_{\varepsilon/(4c_0)}$ such that $\operatorname{Re}\varphi(s(\varepsilon)) \le \varepsilon/4$. The Dirichlet series φ converges uniformly in $\mathbb{C}_{\operatorname{Re} s(\varepsilon)/2}$, see [7]. By almost periodicity, we can find an increasing unbounded sequence of positive numbers $\{T_n\}_{n\ge1}$ such that

$$\operatorname{Re}\varphi(s(\varepsilon) + iT_n)) \le \frac{\varepsilon}{2}$$

so that $\operatorname{Re}\psi(s(\varepsilon)+iT_n) \le 3\varepsilon/4$. We observe that $|\operatorname{Im}\psi(s(\varepsilon)+iT_n)| \to +\infty$. This contradicts our assumption.

4.3. On the boundedness on $\mathcal{H}^p$

We conclude this section with the proof of Theorem 1.4. We will use Hilbertian methods to prove that our assumption implies that C_φ is bounded as an operator from H to $\mathcal{H}^2$, where H is a Hilbert space of Dirichlet series containing $\mathcal{H}^p$. To do this, we need another class of Bergman spaces of Dirichlet series, the spaces $\mathcal{A}_a$, $\alpha \ge 1$. They are defined as

$$\mathcal{A}_\alpha = \Big\{ f(s) = \sum_{n\ge1} a_n n^{-s} : \|f\|^2_{\mathcal{A}_\alpha} = \sum_{n\ge1} \frac{|a_n|^2}{d_\alpha(n)} < \infty \Big\},$$

where by $d_\alpha(n)$ we denote the coefficients of the Dirichlet series $(\zeta(s))^\alpha$, $s\in\mathbb{C}_1$. In particular, $d_2(\cdot)$ is the divisor counting function. The space $\mathcal{A}_\alpha$ is a reproducing kernel Hilbert space, the reproducing kernel at a point $s_0 \in \mathbb{C}_{1/2}$ being the function $(\zeta(\overline{s_0}+\cdot))^\alpha$. The analogue of the embedding theorem for $\mathcal{A}_\alpha$ reads as follows.

Lemma 4.4 ([18]). *For every $f \in \mathcal{A}_\alpha$ and every interval $I \subset \mathbb{R}$, there exists a constant $C = C(|I|)$ such that*

$$\int_{1/2}^{1}\int_I |f'(\sigma+it)|^2 \Big(\sigma - \frac{1}{2}\Big)^\alpha \, dt\, d\sigma \le C\|f\|^2_{\mathcal{A}_\alpha}. \tag{4.3}$$

Proof of Theorem 1.4. Let us set $\alpha = 2k/p$. Working in a similar manner to Theorem 1.3, there exist $\varepsilon > 0$ and $C > 0$ such that $\operatorname{Re} w \in (1/2, 1/2+\varepsilon)$ implies

$$M_\varphi(w) \le C\Big(\operatorname{Re} w - \frac{1}{2}\Big)^\alpha.$$

Let $T > 0$ be such that $\varphi(\mathbb{C}_0) \cap (\mathbb{C}_{1/2} \setminus \mathbb{C}_{1/2+\varepsilon}) \subset [1/2, 1/2+\varepsilon] \times [-T, T]$. By (2.4), for $f \in \mathcal{H}^p \subset \mathcal{H}^{2k}$,

$$\begin{aligned}\|C_\varphi(f)\|^{2k}_{\mathcal{H}^{2k}} = \|C_\varphi(f^k)\|^2_{\mathcal{H}^2} &= |f^k(\varphi(+\infty))|^2 + \frac{2}{\pi}\int_{\mathbb{C}_{1/2}} |(f^k)'(w)|^2 M_\varphi(w)\, dA(w)\\ &\ll \int_{1/2}^{1/2+\varepsilon}\int_{-T}^{T} |(f^k)'(\sigma+it)|^2\Big(\sigma-\frac{1}{2}\Big)^\alpha dt\, d\sigma\\ &\quad + |f^k(\varphi(+\infty))|^2 + \frac{2}{\pi}\int_{\mathbb{C}_{1/2+\varepsilon}} |(f^k)'(w)|^2 M_\varphi(w)\, dA(w).\end{aligned}$$

Let us write $f^k = \sum_{j\geq 1} a_j j^{-s}$. By the Cauchy–Schwarz inequality, for all $w \in \mathbb{C}_{1/2+\varepsilon}$,

$$\begin{aligned}\big|(f^k)'(w)\big| &\leq \sum_{j\geq 2} \frac{|a_j|\log j}{j^{\operatorname{Re} w}} \leq \Big(\sum_{j\geq 2} \frac{|a_j|^2}{d_\alpha(j)}\Big)^{1/2}\Big(\sum_{j\geq 2} \frac{d_\alpha(j)\log^2 j}{j^{2\operatorname{Re} w}}\Big)^{1/2} \\ &\leq C(\varepsilon)\,|2^{-w}|\,\|f^k\|_{\mathcal{A}_\alpha}.\end{aligned}$$

Note that

$$(\zeta^\alpha(s))'' = \sum_{j\geq 2} \frac{d_\alpha(j)\log^2 j}{j^s}$$

converges absolutely for $\operatorname{Re} s > 1+\varepsilon$, $\varepsilon > 0$.

By the local embedding theorem (4.3), the boundedness of pointwise evaluation at $\varphi(+\infty)$, and the continuity of C_φ on $\mathcal{H}^2$, applied to 2^{-s}, we get

$$\|C_\varphi(f)\|_{\mathcal{H}^{2k}}^{2k} \ll \|f^k\|_{\mathcal{A}_\alpha}^2.$$

Now, the inclusion operator $i\colon \mathcal{H}^{p/k} \to \mathcal{A}_\alpha$ is contractive, [16]. Therefore,

$$\|C_\varphi(f)\|_{\mathcal{H}^p} \leq \|C_\varphi(f)\|_{\mathcal{H}^{2k}} \ll \|f^k\|_{\mathcal{A}_\alpha}^{1/k} \leq \|f^k\|_{\mathcal{H}^{p/k}}^{1/k} = \|f\|_{\mathcal{H}^p}.$$

Let us turn to compactness. Let $\{f_n\}_{n\geq 1}$ be a sequence of $\mathcal{H}^q$ converging weakly to 0. We set $g_n = f_n^k$ and observe that $\{g_n\}$ converges pointwise to 0 on $\mathbb{C}_{1/2}$ and that the Dirichlet coefficients $\widehat{g}_n(j)$ converge to 0 for each $j \geq 1$.

We work as above, but we now set $\alpha = 2k/q$ and consider $\delta \in (0,\varepsilon)$. Then

$$\begin{aligned}\|C_\varphi(f_n)\|_{\mathcal{H}^q}^{2k} &\leq \|C_\varphi(f_n)\|_{\mathcal{H}^{2k}}^{2k} = \|C_\varphi(g_n)\|_{\mathcal{H}^2}^2 \\ &\leq |g_n(\varphi(+\infty))|^2 + \delta^{1/p-1/q}\int_{1/2}^{1/2+\delta}\int_{-T}^{T} |g_n'(\sigma+it)|^2\Big(\sigma-\frac{1}{2}\Big)^\alpha\,dt\,d\sigma \\ &\quad + \frac{2}{\pi}\int_{\mathbb{C}_{1/2+\delta}} |g_n'(w)|^2\,M_\varphi(w)\,dA(w).\end{aligned}$$

The first term goes to zero as n tends to $+\infty$, and the second term is as small as we want for every n if we adjust δ small enough. Therefore it remains to show that, for a fixed $\delta > 0$, the last terms tends to 0 as n tends to $+\infty$. Now, for all $n \geq 1$ and all $w \in \mathbb{C}_{1/2+\delta}$,

$$\begin{aligned}|g_n'(w)| &\leq \sum_{j\geq 2}\frac{|\widehat{g}_n(j)|\log j}{j^{\operatorname{Re} w}} \leq \Big(\sum_{j\geq 2}\frac{d_\alpha(j)\log^2 j}{j^{2\operatorname{Re} w-\delta/2}}\Big)^{1/2}\Big(\sum_{j\geq 2}\frac{|\widehat{g}_n(j)|^2}{d_\alpha(j)j^{\delta/2}}\Big)^{1/2} \\ &\ll |2^{-w}|\Big(\sum_{j=2}^{N}\frac{|\widehat{g}_n(j)|^2}{d_\alpha(j)j^{\delta/2}} + \frac{1}{N^{\delta/2}}\,\|g_n\|_{\mathcal{A}_\alpha}^2\Big)^{1/2}.\end{aligned}$$

Since the sequence $\{g_n\}$ is bounded in $\mathcal{A}_\alpha$, for any $\eta > 0$, there exists $n_0 \in \mathbb{N}$ such that $|g_n'(w)| \leq \eta|2^{-w}|$. We now argue as above to conclude that $\{C_\varphi(f_n)\}$ tends to 0 in $\mathcal{H}^q$. ∎

Remark. We choose to work with symbols with range into angular sectors for the sake of simplicity. It will be interesting to know if our techniques can be applied to give other examples of geometric conditions related to the behavior of composition operators on Hardy spaces of Dirichlet series.

5. Further discussion

5.1. Bergman spaces

We focused on the Hardy space $\mathcal{H}^2$, but we can extend our results to Bergman spaces of Dirichlet series $\mathcal{D}_{-a}$, $a \geq 0$. The class $\mathfrak{G}$ determines again the bounded composition operators on $\mathcal{D}_{-a}$, $a \geq 0$. For Dirichlet series symbols $\varphi \in \mathfrak{G}_0$, the compact composition operators C_φ have been characterized, [15], in terms of the weighted counting function

$$M_{\varphi,1+a}(w) = \lim_{\sigma \to 0^+} \lim_{T \to \infty} \frac{\pi}{T} \sum_{\substack{s \in \varphi^{-1}(\{w\}) \\ |\operatorname{Im} s| < T \\ \sigma < \operatorname{Re} s < \infty}} (\operatorname{Re} s)^{1+a}, \quad w \neq \varphi(+\infty),$$

and similarly with the Hardy space case, C_φ is compact on $\mathcal{D}_{-a}$, $a \geq 0$, if and only if

$$\lim_{\operatorname{Re} w \to 1/2^+} \frac{M_{\varphi,1+a}(w)}{(\operatorname{Re} w - 1/2)^{1+a}} = 0.$$

Theorem 5.1. *Let $\varphi \in \mathfrak{G}_0$ and let $p \geq 4$. A necessary condition for the composition operator C_φ to belong to the class S_p is the following*:

$$\int_{\mathbb{C}_{1/2}} \frac{(M_{\varphi,1+a}(w))^{p/2}}{(\operatorname{Re} w - 1/2)^{(a+1)p/2+2}} \, dA(w) < +\infty. \tag{5.1}$$

If we further assume that φ has bounded imaginary part, then C_φ belongs to the class S_p, $p \geq 2$, if and only if φ satisfies (5.1), and for $p > 0$, the condition remains necessary.

To prove Theorem 5.1, one can argue in a similar manner as with the Hardy space $\mathcal{H}^2$, using the analogue key ingredients, namely, the change of variables formula (Theorem 1.2 in [15]), the Littlewood-type inequality (Proposition 5.4 in [15]), the weak submean value property (Theorem 4.11 in [15]), and the behavior of reproducing kernels (2.10).

5.2. Carleson measures

E. Saksman and J.-F. Olsen [19] proved that if μ is a Carleson measure for $\mathcal{H}^2$, then it is a Carleson measure for $H^2(\mathbb{C}_{1/2})$. The converse is also true with the extra assumption that μ has compact support.

A direct consequence of the local embedding theorem is that a sufficient condition for a measure μ in $\{1/2 < \operatorname{Re} s < \sigma_\infty\}$ to be Carleson for $\mathcal{H}^2$ is $\{C(\mu_n, H^2(\mathbb{C}_{1/2}))\}_{n \in \mathbb{Z}} \in \ell^1$, where μ_n is the restriction of μ on the half-strip $\{s \in \mathbb{C}_{1/2} : n \leq \operatorname{Im} s < n+1\}$. Indeed,

$$\begin{aligned}\int_{\mathbb{C}_{1/2}} |f(w)|^2 \, d\mu(w) &\ll \sum_{n \in \mathbb{Z}} \int_{\mathbb{C}_{1/2}} \left| \frac{f(w)}{w - in} \right|^2 d\mu_n(w) \\ &\ll \sum_{n \in \mathbb{Z}} C(\mu_n) \left\| \frac{f(\cdot)}{\cdot - in} \right\|_{H^2(\mathbb{C}_{1/2})} \ll \sum_{n \in \mathbb{Z}} C(\mu_n) \|f\|^2_{\mathcal{H}^2} \ll \|f\|^2_{\mathcal{H}^2}.\end{aligned}$$

An example of such a measure is the restriction of $\frac{M_\varphi(w)}{\operatorname{Re} w - 1/2} \, dA(w)$ to $\{1/2 < \operatorname{Re} s < (1 + \operatorname{Re} \varphi(+\infty))/2\}$. The above condition is not necessary, as we will exemplify now.

We consider the sequence $\{s_n\}_{n\geq 1}$, where

$$s_n = \frac{1}{2} + \Big(\frac{1}{2}\Big)^n + i\Big(n + \frac{1}{2}\Big).$$

As we will prove in a moment, the measure $d\mu(w) = \sum_{n\geq 1}(\operatorname{Re} s_n - 1/2)\,\delta_{s_n}(w)$ is a Carleson measure for $\mathcal{H}^2$, where $\delta_{s_n}(w)$ is a Dirac mass at s_n . The restriction μ_n, $n \geq 1$, has the form

$$d\mu_n(w) = \Big(\operatorname{Re} s_n - \frac{1}{2}\Big)\,\delta_{s_n}(w).$$

Let Q_n, $n \geq 1$, be the square with centre at the point s_n and one side I_n on the line $\{\operatorname{Re} s = 1/2\}$. Then,

$$\mu_n(Q_n) = \frac{|I_n|}{2},$$

and thus $\{C(\mu_n)\}_{n\in\mathbb{Z}} \notin \ell^1$. It remains to prove that $d\mu(w) = \sum_{n\geq 1}(\operatorname{Re} s_n - 1/2)\,\delta_{s_n}(w)$ is a Carleson measure for $\mathcal{H}^2$. Actually, this is true for every sequence $\{s_n\}_{n\geq 1}$ in $\mathbb{C}_{1/2}$ such that

$$\operatorname{Re} s_{n+1} - \frac{1}{2} \leq a\Big(\operatorname{Re} s_n - \frac{1}{2}\Big), \quad n \in \mathbb{N}, \tag{5.2}$$

for some $a \in (0,1)$. We follow an argument of Section 4 in [23], see also [1]. It is sufficient to prove that the matrix

$$A = \Big[\frac{\zeta(s_i + \overline{s_j})}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}}\Big]_{i,j\geq 1}$$

defines a bounded operator on ℓ^2. We will prove that for every $j \in \mathbb{N}$,

$$\sum_{i\geq 1} \frac{|\zeta(s_i + \overline{s_j})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq C, \tag{5.3}$$

and the result will then follow from Schur's test, see [26], Section 3.3.

The Riemann zeta function has a simple pole at 1. Therefore, (5.2) yields the existence of $i_0 \geq 1$ and of $b \in (0,1)$ such that, for all $i \geq i_0$,

$$\frac{\zeta(2\operatorname{Re} s_i)}{\zeta(2\operatorname{Re} s_{i+1})} \leq b,$$

where $b \in (0,1)$. We only need to prove (5.3) for $j \geq i_0$. On the one hand,

$$\sum_{1\leq i\leq i_0} \frac{|\zeta(s_i + \overline{s_j})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq i_0 \frac{|\zeta(\operatorname{Re} s_{i_0} + \operatorname{Re} s_j)|}{\sqrt{\zeta(2\operatorname{Re} s_1)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \leq C.$$

On the other hand,

$$\begin{aligned}\sum_{i\geq i_0} \frac{|\zeta(s_i + \overline{s_j})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} &\leq \sum_{i\geq i_0} \frac{|\zeta(1/2 + \max\{\operatorname{Re} s_i, \operatorname{Re} s_j\})|}{\sqrt{\zeta(2\operatorname{Re} s_i)}\sqrt{\zeta(2\operatorname{Re} s_j)}} \\ \ll \sum_{i_0\leq i\leq j} \sqrt{\frac{\zeta(2\operatorname{Re} s_i)}{\zeta(2\operatorname{Re} s_j)}} + \sum_{i\geq j\geq i_0} \sqrt{\frac{\zeta(2\operatorname{Re} s_j)}{\zeta(2\operatorname{Re} s_i)}} &\ll \sum_{i_0\leq i\leq j} b^{(j-i)/2} + \sum_{i\geq j\geq i_0} b^{(i-j)/2} \leq C.\end{aligned}$$

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