Rev. Mat. Iberoam. 40 (2024), no. 5, 1917–1952
DOI 10.4171/RMI/1490

Transport-entropy and functional forms of Blaschke–Santaló inequalities

Matthieu Fradelizi, Nathaël Gozlan, Shay Sadovsky and Simon Zugmeyer

Abstract. We explore alternative functional or transport-entropy formulations of the Blaschke–Santaló inequality. In particular, we obtain new Blaschke–Santaló inequalities for s-concave functions. We also obtain new sharp symmetrized transport-entropy inequalities for a large class of spherically invariant probability measures, including the uniform measure on the unit Euclidean sphere and generalized Cauchy and Barenblatt distributions.

1. Introduction

The classical Blaschke–Santaló inequality [5, 28] states that if $K \subset \mathbb{R}^n$ is a convex body, then there exists $z \in \mathbb{R}^n$ such that

$$|K||(K-z)^\circ| \le |B_2^n|^2, \tag{1.1}$$

where the polar of a set $A \subset \mathbb{R}^n$ is defined by $A^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1, \forall x \in A\}$, and B_2^n denotes the Euclidean unit ball of $\mathbb{R}^n$. Equality holds in (1.1) if and only if K is an ellipsoid. Moreover, if one of the convex bodies K or K° has its barycenter at 0 (which is for instance the case for centrally symmetric convex bodies), then (1.1) holds with $z = 0$.

The inequality (1.1) admits a functional version, first proved by Ball [3] in the case of even functions, and then extended to arbitrary functions by Artstein-Avidan, Klartag and Milman [1]: for any function $\varphi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists $z \in \mathbb{R}^n$ such that

$$\int e^{-\varphi}\,dx \int e^{-(\varphi_z)^*}\,dx \le (2\pi)^n, \tag{1.2}$$

where $\varphi_z(x) = \varphi(x+z)$, $x \in \mathbb{R}^n$, and the Fenchel–Legendre transform of a function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y\rangle - \varphi(x)\}, \quad y \in \mathbb{R}^n.$$

Mathematics Subject Classification 2020: 52A40 (primary); 52A20, 60E15 (secondary).
Keywords: Blaschke–Santaló's inequality, optimal transport, functional form, Talagrand's transport inequality, transport-entropy inequality.

Lehec [17] gave another proof of inequality (1.2) and showed that, if $\int x e^{-\varphi(x)}\,dx = 0$, then (1.2) holds with $z = 0$. One sees that (1.2) gives back (1.1) by taking $\varphi = \|\cdot\|_K^2/2$.

Recently, a sharp form of Talagrand's transport-entropy inequality for the Gaussian standard measure γ on $\mathbb{R}^n$ has been deduced from (1.2) by Fathi [10]. More precisely, for all probability measures ν_1 and ν_2 on $\mathbb{R}^d$, with ν_2 centered, it holds

$$W_2^2(\nu_1,\nu_2) \le 2H(\nu_1\,|\,\gamma) + 2H(\nu_2\,|\,\gamma), \tag{1.3}$$

where W_2 denotes the usual quadratic Wasserstein distance (with respect to the usual Euclidean norm $|\cdot|$ on $\mathbb{R}^n$), defined by

$$W_2^2(\nu_1,\nu_2) = \inf \mathbb{E}[|X_1 - X_2|^2],$$

where the infimum runs over random vectors satisfying $X_1 \sim \nu_1$ and $X_2 \sim \nu_2$, and $H(\cdot\,|\,\mu)$ denotes the relative entropy functional with respect to some measure μ on $\mathbb{R}^n$, and is defined by

$$H(\nu\,|\,\mu) = \int \log \frac{d\nu}{d\mu}\,d\mu,$$

whenever ν is absolutely continuous with respect to μ, and $+\infty$ if this is not the case. Choosing $\nu_2 = \gamma$, inequality (1.3) immediately gives back the following classical inequality obtained by Talagrand in [29]: for all probability measures ν_1 on $\mathbb{R}^n$,

$$W_2^2(\nu_1,\gamma) \le 2H(\nu_1|\gamma). \tag{1.4}$$

Without centering assumptions on ν_2, the following inequality can be easily deduced from (1.4): for all probability measures ν_1 and ν_2 on $\mathbb{R}^n$,

$$W_2^2(\nu_1,\nu_2) \le 4H(\nu_1\,|\,\gamma) + 4H(\nu_2\,|\,\gamma). \tag{1.5}$$

Interestingly, inequalities (1.3), (1.4) and (1.5) are all sharp. We refer to [16] or [13] for applications of transport-entropy inequalities to the concentration of measure phenomenon.

The main objective of this paper is to extend the preceding results to other model probability spaces than the Gaussian space $(\mathbb{R}^n, |\cdot|, \gamma)$. For that purpose, we will rely on a more general functional version of the Blaschke–Santaló inequality that we shall now present. The functional inequality (1.2) is in fact a particular case of the following result, first proved by Ball [3] for even functions, then by the first named author and Meyer [11] for log-concave functions, and finally extended by Lehec [19] to arbitrary measurable functions: if $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is integrable, then there exists a point $z \in \mathbb{R}^n$ such that for any measurable function $g\colon \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$f(x+z)g(y) \le \rho(\langle x,y\rangle)^2, \quad \forall x,y \in \mathbb{R}^n \text{ such that } \langle x,y\rangle > 0,$$

it holds

$$\int f(x)\,dx \int g(y)\,dy \le \Big(\int \rho(|x|^2)\,dx\Big)^2, \tag{1.6}$$

where $\rho:\mathbb{R}_+ \to \mathbb{R}_+$ is some weight function such that $\int \rho(|x|^2)\,dx < +\infty$. As first proved by Ball [3], if f is even, then z can be chosen to be 0. Inequality (1.2) corresponds to the weight function $\rho_0(t) = e^{-t/2}$, $t \geq 0$.

In the spirit of Fathi's version of Talagrand's inequality (1.3), we show in Theorem 3.1 that the general functional version of the Blaschke–Santaló inequality (1.6) implies sharp transport-entropy inequalities for a class of spherically invariant probability measures that contain the standard Gaussian as a particular case. More precisely, we prove the following result in Theorem 3.1.

Theorem. *Suppose that $\rho:\mathbb{R}_+ \to (0,\infty)$ is a continuous nonincreasing function such that $\int \rho(|x|^2)\,dx < +\infty$ and $t \mapsto -\log \rho(e^t)$ is convex on $\mathbb{R}$. Then the probability measure*

$$\mu_\rho(dx) = \frac{\rho(|x|^2)}{\int \rho(|y|^2)\,dy}\,dx$$

satisfies the following inequality: for all $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n)$ with ν_1 and ν_2 symmetric,

$$\mathcal{T}_{\omega_\rho}(\nu_1,\nu_2) \leq H(\nu_1\,|\,\mu_\rho) + H(\nu_2\,|\,\mu_\rho), \tag{1.7}$$

where

$$\mathcal{T}_{\omega_\rho}(\nu_1,\nu_2) = \inf_{X_1\sim\nu_1, X_2\sim\nu_2} \mathbb{E}[\omega_\rho(X_1,X_2)]$$

is the optimal transport cost associated with the cost function ω_ρ defined, for $x, y \in \mathbb{R}^n$, by

$$\omega_\rho(x,y) = \begin{cases} \log\left(\dfrac{\rho(\langle x,y\rangle)^2}{\rho(|x|^2)\rho(|y|^2)}\right) & \text{if } \langle x,y\rangle \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

In the result above, and in the rest of the paper, a probability measure μ on $\mathbb{R}^n$ will be called symmetric if it is invariant under the map $\mathbb{R}^n \to \mathbb{R}^n : x \mapsto -x$.

The proof of this result relies on a classical duality argument due to Bobkov and Götze [6]. Since inequality (1.7) holds only for symmetric probability measures, it can be considered as some transport-entropy version of Ball's functional Blaschke–Santaló inequality for even functions. Linearizing inequality (1.7) around μ_ρ gives back a sharp Brascamp–Lieb type inequality recently used by Cordero-Erausquin and Rotem [8] in their study of the (B) conjecture and the Gardner–Zvavitch conjecture for rotationally invariant probability measures. More precisely, we get the following in Theorem 4.1.

Theorem. *Let $\rho:\mathbb{R}_+ \to \mathbb{R}_+$ be such that $t \mapsto v_\rho(t) = -\log \rho(e^t)$ is convex and increasing. Define the measure μ_ρ in the same way as in the previous theorem. Then, for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ even and such that $\int f\,d\mu_\rho = 0$,*

$$\int f^2\,d\mu_\rho \leq \frac{1}{2}\int \langle H_\rho^{-1}\nabla f, \nabla f\rangle\,d\mu_\rho, \tag{1.8}$$

where H_ρ is the positive matrix given by

$$\frac{1}{2}H_\rho(y) = \frac{1}{|y|^2}\left[\left(I_n - \frac{y\otimes y}{|y|^2}\right)v_\rho'(t) + \frac{y\otimes y}{|y|^2}\,v_\rho''(t)\right]$$

where, for simplicity, we used the notation $t = 2\log|y|$.

Since (1.8) admits equality cases, this shows in particular that inequality (1.7) is sharp.

In comparison to Fathi's inequality (1.3), it seems natural to ask if (1.7) can be extended to more general couples of probability measures, for instance for couples of the form (ν_1, ν_2) with ν_1 arbitrary and ν_2 centered with respect to μ_ρ. A closely related question is whether, for a given weight function ρ, the functional Blaschke–Santaló inequality (1.6) is true with $z = 0$ whenever f has its barycenter at 0, as proved by Lehec [17] in the particular case of the weight function ρ_0 defined above. As we will now explain, the answer to these questions depends on the weight function ρ. Consider the class of weight functions $(\rho_s)_{s\in\mathbb{R}}$, defined for $s \neq 0$ by

$$\rho_s(t) = (1 - st)_+^{1/(2s)}, \quad t \geq 0.$$

The associated probability measures are the following:

- For $s > 0$, we will denote

$$d\gamma_s(x) := \mu_{\rho_s}(dx) = \frac{1}{Z_s}\left[1 - s|x|^2\right]_+^{1/(2s)} dx, \tag{1.9}$$

  which is a particular case of the so-called *Barenblatt profiles*. Note that $\gamma_s \to \gamma$ as $s \to 0$ (in the sense of pointwise convergence of densities, for instance).
- For $\beta > n/2$, we will denote

$$d\mu_\beta(x) = \frac{1}{Z_\beta(1 + |x|^2)^\beta}\, dx,$$

  which is a *Cauchy-type distribution* and corresponds to (a dilation of) μ_{ρ_s} with $s = -1/(2\beta)$.

Let us first present our main contributions in the range $s > 0$. As we shall see in Theorem 3.2, the following is true.

Theorem. *Let $s > 0$. Consider the probability γ_s defined in (1.9). Then, for any ν_1 and ν_2 with compact support included in the open Euclidean ball B_s centered at the origin and of radius $1/\sqrt{s}$, and with ν_2 centered,*

$$\mathcal{T}_{k_s}(\nu_1, \nu_2) \leq H(\nu_1\,|\,\gamma_s) + H(\nu_2\,|\,\gamma_s), \tag{1.10}$$

where $k_s\colon B_s \times B_s \to \mathbb{R}$ is given by

$$k_s(x, y) = \frac{1}{s}\log\left(\frac{1 - s\langle x, y\rangle}{(1 - s|x|^2)^{1/2}(1 - s|y|^2)^{1/2}}\right), \quad x, y \in B_s.$$

This result is analogous to Fathi's result (1.3) in the Gaussian case, and gives back (1.3) by sending $s \to 0$. One can show that (1.10) (see Remark 3.3 for explanations) also implies the following version of the functional Blaschke–Santaló inequality: for all continuous $f\colon \mathbb{R}^n \to \mathbb{R}_+$ and $g\colon \mathbb{R}^n \to \mathbb{R}_+$ with supports in B_s and such that

$$\operatorname{bar}(f) := \frac{\int x f(x)\, dx}{\int f(y)\, dy} = 0$$

and

$$f(x)g(y) \leq \rho_s(\langle x, y\rangle)^2, \quad \forall x, y \in B_s, \tag{1.11}$$

it holds

$$\int f(x)\,dx \int g(y)\,dy \le \Big(\int \rho_s(|x|^2)\,dx\Big)^2.$$

This generalizes the Blaschke–Santaló inequality under a centering condition obtained by Lehec in [17] for the weight ρ_0 (which corresponds to the limit case $s \to 0$). As we will see with Theorem 2.9, one can go a step further:

Theorem. *If $f : \mathbb{R}^n \to \mathbb{R}_+$ is integrable and is such that $0 \in \operatorname{int}(\operatorname{Conv}(\operatorname{supp}(f)))$, then it holds*

$$\int f(x)\,dx \int \mathcal{L}_s f(y)\,dy \le \Big(\int \rho_s(|x|^2)\,dx\Big)^2 \big(1 - s\langle \operatorname{San}_s(\mathcal{L}_s(f)), \operatorname{bar}(f)\rangle\big)^{n+1+1/s},$$

where

$$\mathcal{L}_s f(y) = \inf_{x\in\mathbb{R}^n} \frac{(1 - s\langle x, y\rangle)_+^{1/s}}{f(x)}, \quad \text{for } s \neq 0,$$

the infimum being taken on $\{x \in \mathbb{R}^n : f(x) > 0\}$, and where $\operatorname{San}_s(g)$ denotes the s-Santaló point of g, whose definition is given in Lemma 2.8.

The proof of this theorem relies on the fact that the integral of $\mathcal{L}_s(f_z)$ with respect to Lebesgue measure, where $f_z(x) = f(z + x)$, $x \in \mathbb{R}^n$, can be expressed as the integral of $\mathcal{L}_s(f)$ under some weighted measure. The same type of arguments can also be used at the level of the Blaschke–Santaló inequality for sets. In particular, we show the following in Theorem 2.1.

Theorem. *If K is a compact set such that $|K| > 0$ and $0 \in \operatorname{int}(\operatorname{Conv}(K))$, then*

$$|K|\,|K^\circ| \le |B_2^n|^2 (1 - \langle \operatorname{San}(K^\circ), \operatorname{bar}(K)\rangle)^{n+1},$$

with equality if and only if K is a centered ellipsoid, where $\operatorname{San}(K^\circ)$ is defined in Section 2. In particular, if $\operatorname{bar}(K) := \frac{1}{|K|}\int_K x\,dx = 0$, then $|K|\,|K^\circ| \le |B_2^n|^2$.

The centered inequality above seems to be new, even for convex bodies, while the case where $\operatorname{bar}(K) = 0$ extends a result by Lutwak [20], also reproved differently by Lehec [17], who both obtained the same inequality but under the additional assumption that K is star-shaped.

Let us now turn to the range $s < 0$. Applying inequality (1.7) with the weight function $t \mapsto (1 + t)^{-\beta}$ and $\beta > n/2$, yields

$$\beta \mathcal{T}_\omega(\nu_1, \nu_2) \le H(\nu_1 \,|\, \mu_\beta) + H(\nu_2 \,|\, \mu_\beta), \tag{1.12}$$

where the optimal transport cost $\mathcal{T}_\omega$ is defined with respect to the cost function ω given, for $x, y \in \mathbb{R}^n$, by

$$\omega(x, y) = \begin{cases} -2\log\left(\dfrac{1 + \langle x, y\rangle}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}\right) & \text{if } \langle x, y\rangle > 0, \\ +\infty & \text{otherwise.}\end{cases}$$

The fact that the cost function ω can take the value $+\infty$ makes inequality (1.12) for Cauchy-type distributions more rigid than its counterpart (1.10) for Barenblatt-type distributions. Namely, it is not possible to extend (1.12) to couples of probability measures (ν_1, ν_2) with ν_1 arbitrary and ν_2 symmetric. See Remark 3.9 for more details. For the particular value $\beta = (n+1)/2$, it turns out that the canonical geometric framework for (1.12) is the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ equipped with the uniform probability measure, denoted by σ. In Theorem 3.7, we establish the following.

Theorem. *Let $\alpha\colon \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}_+ \cup \{+\infty\}$ be the cost function defined, for $u, v \in \mathbb{S}^n$, by*

$$\alpha(u,v) = \begin{cases} \log\left(\frac{1}{\langle u,v\rangle}\right) & \text{if } \langle u,v\rangle > 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.13}$$

and denote by $\mathcal{T}_\alpha$ the corresponding transport cost on $\mathcal{P}(\mathbb{S}^n)$. Then, for all probability measures ν_1 and ν_2 on $\mathbb{S}^n$ which are invariant under the maps $\mathbb{S}^n \to \mathbb{S}^n : u \mapsto -u$ and $\mathbb{S}^n \to \mathbb{S}^n : u \mapsto (u_1, \dots, u_n, -u_{n+1})$, it holds

$$(n+1)\,\mathcal{T}_\alpha(\nu_1, \nu_2) \le H(\nu_1 \,|\, \sigma) + H(\nu_2 \,|\, \sigma). \tag{1.14}$$

This result is deduced from (1.12) using the fact that the standard Cauchy distribution $\mu_{(n+1)/2}$ is the image of σ_+, the uniform probability measure on the upper half sphere $\mathbb{S}^n_+$, under the so-called *gnomonic transformation*:

$$\mathbb{S}^n_+ \to \mathbb{R}^n : u \mapsto \left(\frac{u_1}{u_{n+1}}, \frac{u_2}{u_{n+1}}, \dots, \frac{u_n}{u_{n+1}}\right).$$

The cost function α defined above has been introduced by Oliker [25] (see also [4, 15]) in connection with the so-called Aleksandrov problem in convex geometry. Recently, Kolesnikov [15] proved the following inequality involving the transport cost $\mathcal{T}_\alpha$: for any symmetric probability measure ν on $\mathbb{S}^n$ (that is, invariant under the map $\mathbb{S}^n \to \mathbb{S}^n : u \mapsto -u$), it holds

$$(n+1)\,\mathcal{T}_\alpha(\nu, \sigma) \le H(\nu \,|\, \sigma). \tag{1.15}$$

Thus (1.14) already improves (1.15) for a special class of distributions. As it turns out, one can improve (1.15) further. We show in Theorem 3.7, by a direct proof using the Blaschke–Santaló inequality written in polar coordinates, together with the dual Kantorovich type formula for $\mathcal{T}_\alpha$, that (1.14) holds under the sole assumption that ν_1 and ν_2 are symmetric. We refer to the end of Section 3.3 for additional comments about the sharpness of this improvement of Kolesnikov inequality (1.15).

2. Blaschke–Santaló's inequality for compact sets and s-concave functions

In Subsection 2.1, we extend to arbitrary compact sets the result of Lutwak [20] and Lehec [17] stating that the Blaschke–Santaló inequality holds for star-shaped sets with barycenter at the origin. In Subsection 2.2, we generalize this to the Blaschke–Santaló inequality for s-concave functions, for $s \ge 0$. Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is s-concave, for $s > 0$, if f^s is concave on its support; for $s < 0$, if f^s is convex; and for $s = 0$, one says log-concave and it means that $\log(f)$ is concave. In fact, for sets as well as for functions, we prove an inequality valid also if the barycenter is not at the origin.

2.1. Blaschke–Santaló inequality for compact sets

For any set A in $\mathbb{R}^n$, we define its polar by $A^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1, \forall x \in A\}$. Then, one has $A^\circ = (\mathrm{Conv}(A, 0))^\circ$, thus the set A° is a closed convex set containing the origin and, from the bipolar theorem, one has that $(A^\circ)^\circ = \overline{\mathrm{Conv}(A, 0)}$. The classical Blaschke–Santaló [5,28] inequality asserts that, for any convex body K in $\mathbb{R}^n$, one has

$$\min_{z\in \mathrm{int}(K)} |K|\,|(K-z)^\circ| \le |B_2^n|^2,$$

with equality if and only if K is an ellipsoid. For any convex body K, we define its support function $h_K(y) = \sup_{x\in K}\langle x, y\rangle$, for $y \in \mathbb{R}^n$. If, moreover, K contains the origin, we define its radial function by $\rho_K(u) = \sup\{t : tu \in K\}$, for $u \in \mathbb{S}^{n-1}$, and one has $\rho_{K^\circ}(u) = h_K(u)^{-1}$, for all $u \in \mathbb{S}^{n-1}$. For any z in the interior of a convex body K and any $y \in \mathbb{R}^n$, one has

$$h_{K-z}(y) = \sup_{x\in K}\langle x - z, y\rangle = h_K(y) - \langle z, y\rangle.$$

Integrating in polar coordinates, we get

$$|(K-z)^\circ| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_{(K-z)^\circ}(u)^n\, d\sigma(u) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \frac{d\sigma(u)}{(h_K(u) - \langle z, u\rangle)^n}. \tag{2.1}$$

This formula shows that the map $z \mapsto |(K-z)^\circ|$ is strictly convex. Moreover, it is not difficult to see that $|(K-z)^\circ|$ tends to infinity when $z \to \partial K$. It follows that the minimum $\min_z |(K-z)^\circ|$ is reached at a unique point $\mathrm{San}(K)$, called the Santaló point of K, which is in the interior of K. It follows that the Blaschke–Santaló theorem may be reformulated as follows: for any convex body K such that $\mathrm{San}(K) = 0$, one has $|K||K^\circ| \le |B_2^n|^2$, with equality if and only if K is a centered ellipsoid. We say that a measurable set K with finite and positive volume is centered if its center of mass $\mathrm{bar}(K)$, defined by

$$\mathrm{bar}(K) = \int_K \frac{x\,dx}{|K|},$$

is at the origin. Since $\mathrm{San}(K)$ is the unique critical point of the function $z \mapsto |(K-z)^\circ|$, we get that $z = \mathrm{San}(K)$ if and only if $\nabla|(K-z)^\circ| = 0$. By differentiating (2.1) and integrating in spherical coordinates, we get

$$\begin{aligned}\nabla|(K-z)^\circ| &= \int_{\mathbb{S}^{n-1}} \frac{u\, d\sigma(u)}{(h_K(u) - \langle z, u\rangle)^{n+1}}\\ &= (n+1)\int_{(K-z)^\circ} x\,dx = (n+1)\,|(K-z)^\circ|\,\mathrm{bar}((K-z)^\circ).\end{aligned}$$

It follows that the Santaló point $\mathrm{San}(K)$ of the convex body K is also the unique point z such that $\mathrm{bar}((K-z)^\circ) = 0$. One deduces from this property that $\mathrm{San}((K-\mathrm{bar}(K))^\circ) = 0$ and that $\mathrm{San}(K) = 0$ if and only if $\mathrm{bar}(K^\circ) = 0$. Thus, the following third reformulation of the Blaschke–Santaló inequality holds: for any convex body K such that $\mathrm{bar}(K) = 0$, one has $|K||K^\circ| \le |B_2^n|^2$, with equality if and only if K is an ellipsoid. Lutwak noticed this in [20] and extended it to the case of compact star-shaped bodies. A compact set A is called star-shaped with respect to the origin if for any $a \in A$, the segment $\{ta : t \in [0,1]\}$ is contained in A. In his Theorem 3.15 in [20], Lutwak proved that if A is star-shaped with respect to the origin and has barycenter at the origin, then $|A||A^\circ| \le |B_2^n|^2$, with equality if and only if A is a centered ellipsoid. This result was also reproved by Lehec [17], who deduced it from a version of this theorem for log-concave functions. In the following theorem, we extend Lutwak's theorem to any compact set with a different proof.

Theorem 2.1. *Let K be a compact set such that $|K| > 0$ and $0 \in \mathrm{int}(\mathrm{Conv}(K))$. Then*

$$|K||K^\circ| \le |B_2^n|^2 (1 - \langle \mathrm{San}(K^\circ), \mathrm{bar}(K)\rangle)^{n+1}, \tag{2.2}$$

with equality if and only if K is a centered ellipsoid. In particular, if $\mathrm{bar}(K) = 0$, then $|K||K^\circ| \le |B_2^n|^2$, with equality if and only if K is a centered ellipsoid.

Remark 2.2. Formula (2.2) seems to be new even in the case of convex sets.

Remark 2.3. If K is convex, since $\mathrm{bar}(K) \in K$ and $\mathrm{San}(K^\circ) \in K^\circ$, one has the inequality $\langle \mathrm{San}(K^\circ), \mathrm{bar}(K)\rangle \le 1$, but it follows from the proof that actually $\langle \mathrm{San}(K^\circ), \mathrm{bar}(K)\rangle \le 0$, see Remark 2.6.

Remark 2.4. Another formulation of the Blaschke–Santaló inequality for compact sets follows directly from the case of convex sets, but with a less natural polarity point: given a compact set A, choosing $z = \mathrm{San}(\mathrm{Conv}(A))$ and applying the classical inequality to $\mathrm{Conv}(A)$, we get $(A - z)^\circ = (\mathrm{Conv}(A) - z)^\circ$, and we deduce that

$$|A||(A - z)^\circ| \le |\mathrm{Conv}(A)||(\mathrm{Conv}(A) - z)^\circ| \le |B_2^n|^2.$$

Before proving this theorem, we first give a lemma which is very classical in projective geometry.

Lemma 2.5. *For $z \neq 0$, we denote the open halfspace $H_z = \{y : 1 + \langle y, z\rangle > 0\}$, and we define the map $F_z \colon H_z \to \mathbb{R}^n$ by*

$$F_z(y) = \frac{y}{1 + \langle y, z\rangle}, \quad \textit{for any } y \in H_z.$$

Then

(i) *The map F_z is a bijection from H_z onto H_{-z} whose reciprocal is F_{-z}, and the Jacobian determinant of F_z is $J_z(y) := (1 + \langle y, z\rangle)^{-(n+1)}$.*

(ii) *For any compact set K in $\mathbb{R}^n$ such that $0, z \in \mathrm{int}(\mathrm{Conv}(K))$, we have $(K - z)^\circ = F_{-z}(K^\circ)$ and*

$$|(K - z)^\circ| = \int_{K^\circ} \frac{dx}{(1 - \langle z, x\rangle)^{n+1}}. \tag{2.3}$$

Notice that formula (2.3) is classical, and can be found for example in Lemma 3 of [24] by Meyer and Werner, who proved it by using (2.1) and a change of variable. We give here another proof which we shall extend to the functional case in the next section.

Proof. (i) From the definition of the map F_z, it is immediate that $F_z(H_z) \subset H_{-z}$ and that $F_{-z}(F_z(y)) = y$, for all $y \in H_z$. It follows that F_z is a bijection from H_z onto H_{-z} whose reciprocal is F_{-z}. The computation of the Jacobian matrix of F_z is direct and gives

$$\mathrm{Jac}(F_z)(y) = \frac{1}{1+\langle y,z\rangle}\Big(I_n - \frac{yz^{\mathsf{T}}}{1+\langle y,z\rangle}\Big).$$

Using the following Sylvester's identity, $\det(I_p + AB) = \det(I_q + BA)$, for any matrix $A \in M_{p,q}$ and $B \in M_{q,p}$, we conclude that the Jacobian determinant of F_z is

$$J_z(y) = \det(\mathrm{Jac}(F_z)(y)) = \frac{1}{(1+\langle y,z\rangle)^n}\Big(1 - \frac{\langle y,z\rangle}{1+\langle y,z\rangle}\Big) = \frac{1}{(1+\langle y,z\rangle)^{n+1}}.$$

(ii) One has

$$(K-z)^\circ = \{y : \langle y, x-z\rangle \le 1,\ \forall x \in K\} = \{y : \langle y,x\rangle \le 1 + \langle y,z\rangle,\ \forall x \in K\}.$$

Since $0 \in \mathrm{int}(\mathrm{Conv}(K))$, for any $y \in (K-z)^\circ$, one has $\langle y,-z\rangle < 1$, hence $1+\langle y,z\rangle > 0$, thus

$$(K-z)^\circ = \Big\{y : \Big\langle \frac{y}{1+\langle y,z\rangle}, x\Big\rangle \le 1,\ \forall x \in K\Big\} = \{y : F_z(y) \in K^\circ\} = F_{-z}(K^\circ).$$

The last equality follows from the fact that $K^\circ \subset H_{-z}$, which in turn follows from the hypothesis $z \in \mathrm{int}(\mathrm{Conv}(K))$. Formula (2.3) follows by using a change of variable and the formula for the Jacobian from (i). ■

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let K be a compact set such that $0 < |K| < +\infty$ and suppose that $0 \in \mathrm{int}(\mathrm{Conv}(K))$. Then K° is a convex body to which we apply the classical Blaschke–Santaló's inequality: for $z = \mathrm{San}(K^\circ)$, one has

$$|K^\circ||(K^\circ - z)^\circ| \le |B_2^n|^2,$$

with equality if and only if K° is an ellipsoid. Since $0 \in \mathrm{int}(K^\circ)$ and $z \in \mathrm{int}(K^\circ)$, we may apply formula (2.3) to K° and we get

$$|(K^\circ - z)^\circ| = \int_{K^{\circ\circ}} \frac{dx}{(1-\langle z,x\rangle)^{n+1}}.$$

Using now that $K \subset K^{\circ\circ}$ and applying Jensen's inequality to the function $\varphi(x) = (1-\langle z,x\rangle)^{-(n+1)}$, which is convex on K, we deduce that

$$|(K^\circ - z)^\circ| \ge \int_K \frac{dx}{(1-\langle z,x\rangle)^{n+1}} \ge \frac{|K|}{(1-\langle \mathrm{San}(K^\circ), \mathrm{bar}(K)\rangle)^{n+1}}. \tag{2.4}$$

This concludes the proof of the inequality. If there is equality in this inequality, then, from the equality case in Blaschke–Santaló's inequality, we deduce that K° is an ellipsoid. Moreover, from the equality case in Jensen's inequality, it follows that $\mathrm{San}(K^\circ) = 0$, thus $\mathrm{bar}(K) = 0$. Finally, one has $|K| = |K^{\circ\circ}|$, which implies that $|\mathrm{Conv}(K) \setminus K| = 0$. Since K is compact, it follows that $K = K^{\circ\circ}$. We thus conclude that K is a centered ellipsoid. ■

Remark 2.6. This is the proof of Remark 2.3: if K is convex then, using that in formula (2.4) one has $z = \mathrm{San}(K^\circ)$, it follows from the definition of the Santaló point that $|(K^\circ - z)^\circ| \le |K^{\circ\circ}| = |K|$. Thus, we conclude that $\langle \mathrm{San}(K^\circ), \mathrm{bar}(K)\rangle \le 0$. Notice that, applied to K°, this gives also $\langle \mathrm{San}(K), \mathrm{bar}(K^\circ)\rangle \le 0$.

2.2. Blaschke–Santaló inequality for the s-concave duality

The following general form of the functional Blaschke–Santaló inequality was proved by Ball [3] in the even case, by the first-named author and Meyer [11] in the log-concave case, and by Lehec [19] in the general case.

Theorem 2.7. *Let $f:\mathbb{R}^n \to \mathbb{R}_+$ be integrable. Then there exists $z\in\mathbb{R}^n$ such that whenever $g:\mathbb{R}^n\to\mathbb{R}_+$ is a measurable function satisfying $f(x+z)\,g(y) \le \rho(\langle x,y\rangle)^2$ for all $x, y \in \mathbb{R}^n$ such that $\langle x,y\rangle > 0$, for some weight function $\rho:\mathbb{R}_+\to\mathbb{R}_+$ such that $\int \rho(|x|^2)\,dx < +\infty$, it holds*

$$\int f(x)\,dx \int g(y)\,dy \le \Big(\int \rho(|x|^2)\,dx\Big)^2.$$

Moreover, the point z can be selected in the interior of the convex hull of the support of the measure with density f. In the case where f is even, then z can be chosen to be 0.

The fact that z can be chosen in the convex hull of the support of $\nu_f(dx) = f(x)\,dx$ follows from Lehec's construction of z as the center of a Yao–Yao partition for ν_f (see Theorem 9 in [19]) and from Proposition 5 of [18], which implies that the center of any such partition must belong to the convex hull of the support of ν_f. In the following, we shall denote $f_z = f(z+\cdot)$.

For $s\in\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}_+$ not identically zero, we define its s-concave dual function $\mathcal{L}_s g:\mathbb{R}^n\to\mathbb{R}_+$ in the following way: for every $y\in\mathbb{R}^n$,

$$\mathcal{L}_s g(y) = \inf_{x\in\mathbb{R}^n} \frac{(1-s\langle x,y\rangle)_+^{1/s}}{g(x)}, \quad \text{for } s\neq 0, \tag{2.5}$$

where the infimum is taken on $\{x\in\mathbb{R}^n : g(x)>0\}$. For $s=0$, we set

$$\mathcal{L}_0 g(y) = \inf_{x\in\mathbb{R}^n} \frac{e^{-\langle x,y\rangle}}{g(x)}.$$

Notice that the s-dual (even of a non-s-concave function) is s-concave, and that the 0-dual is very much related to the Legendre transform since for any function $\varphi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$, one has $\mathcal{L}_0(e^{-\varphi}) = e^{-\varphi^*}$, where φ^* is the Legendre transform of φ, defined by $\varphi^*(y) = \sup_x(\langle x,y\rangle - \varphi(x))$.

This class was previously studied by Artstein-Avidan and Milman in [2], where they proved that $\mathcal{L}_s$ is essentially the only order reversing transformation on s-concave functions. They also show that this duality is the usual polarity transform on the epigraphs of the functions for $s=1$.

Applied to the function $\rho_s(t) = (1-st)_+^{1/(2s)}$, for $s\neq 0$, and $\rho_0(t) = e^{-t/2}$, Theorem 2.7 implies that for any integrable function $f:\mathbb{R}^n\to\mathbb{R}_+$, there exists z such that for any $s > -1/n$,

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} \mathcal{L}_s(f_z)(y)\,dy \le \Big(\int_{\mathbb{R}^n} \rho_s(|x|^2)\,dx\Big)^2 =: c_s, \tag{2.6}$$

where a direct explicit computation gives that $c_0 = (2\pi)^n$ and

$$c_s = \begin{cases} \left(\frac{\pi}{s}\right)^n \left(\frac{\Gamma\left(1+\frac{1}{2s}\right)}{\Gamma\left(1+\frac{1}{2s}+\frac{n}{2}\right)}\right)^2 & \text{for } s > 0, \\ \left(\frac{\pi}{|s|}\right)^n \left(\frac{\Gamma\left(\frac{1}{2|s|}-\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2|s|}\right)}\right)^2 & \text{for } -1/n < s < 0. \end{cases}$$

Inequality (2.6) was established earlier in the cases where $1/s$ is an integer and $s = 0$ by Artstein-Avidan, Klartag, and Milman [1]. For $s < 0$, inequality (2.6) was proved by Rotem in [27]. In particular, for $s = 0$, this gives back the Blaschke–Santaló inequality for the Legendre transform established in [1], which states that for any function $\varphi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists $z \in \mathbb{R}^n$ such that

$$\int e^{-\varphi} \int e^{-(\varphi_z)^*} \le (2\pi)^n.$$

This theorem was reproved by Lehec [17], who also established that if the barycenter of $e^{-\varphi}$ (defined as $\mathrm{bar}(e^{-\varphi}) = \int x e^{-\varphi(x)}\, dx \,/ \int e^{-\varphi}$) is at the origin, then one may choose $z = 0$, that is,

$$\int e^{-\varphi} \int e^{-\varphi^*} \le (2\pi)^n.$$

We extend this theorem to the s-duality for any $s \ge 0$. First we define the barycenter of f to be $\mathrm{bar}(f) = \int x f(x)\, dx / \int f$. As in the case of sets, we first state a lemma that is the functional analogue of Lemma 2.5. Recall that

$$F_z(y) = \frac{y}{1 + \langle y, z\rangle}.$$

Lemma 2.8. *Let $s \ge 0$ and let $f\colon \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $f(0) > 0$.*

(1) *Then for every $z, y \in \mathbb{R}^n$, one has $\mathcal{L}_s(f_z)(y) = (1 + s\langle z, y\rangle)_+^{1/s} \mathcal{L}_s f(F_{sz}(y))$, for $s > 0$, and $\mathcal{L}_0(f_z)(y) = e^{\langle z,y\rangle} \mathcal{L}_0 f(y)$.*

(2) *Moreover, if $f(z) > 0$, then $\{x : \mathcal{L}_s f(x) > 0\} \subset H_{-sz} = \{x : 1 - s\langle z, x\rangle > 0\}$ and, for $s > 0$,*

$$\int \mathcal{L}_s(f_z) = \int \frac{\mathcal{L}_s f(x)}{(1 - s\langle z, x\rangle)^{n+1+1/s}}\, dx. \tag{2.7}$$

(3) *If f is bounded and $\mathcal{L}_s f$ is integrable, then the function $S(z) := \int \mathcal{L}_s(f_z)$ is strictly convex and admits a unique minimum at a point* $\mathrm{San}_s(f)$*, that we call the s-Santaló point of f, and which is in the interior of* $\mathrm{Conv}(\mathrm{supp}(f))$.

Proof. (1) For $s = 0$, the relation is clear. Let us assume that $s > 0$. From the definition, one has

$$\mathcal{L}_s(f_z)(y) = \inf_x \frac{(1 - s\langle x, y\rangle)_+^{1/s}}{f(x+z)} = \inf_x \frac{(1 + s\langle z, y\rangle - s\langle x, y\rangle)_+^{1/s}}{f(x)}.$$

Since the infimum runs on the set $\{x : f(x) > 0\}$, and since $f(0) > 0$, one deduces that

$$\mathcal{L}_s(f_z)(y) \leq \frac{(1 + s\langle z, y\rangle)_+^{1/s}}{f(0)}.$$

Hence $\mathcal{L}_s(f_z)(y) = 0$ if $1 + s\langle z, y\rangle \leq 0$. Moreover, for $y \in H_{sz}$, one has

$$\mathcal{L}_s(f_z)(y) = (1 + s\langle z, y\rangle)^{1/s}\, \mathcal{L}_s f\Big(\frac{y}{1 + s\langle z, y\rangle}\Big) = (1 + s\langle z, y\rangle)^{1/s}\, \mathcal{L}_s f(F_{sz}(y)).$$

(2) In the same way, from the definition of $\mathcal{L}_s$, if $f(z) > 0$ then, for all y,

$$\mathcal{L}_s(f)(y) \leq \frac{(1 - s\langle z, y\rangle)_+^{1/s}}{f(z)}.$$

Thus if $\mathcal{L}_s(f)(y) > 0$, then $1 - s\langle z, y\rangle > 0$, which means that $y \in H_{-sz}$. Thus, using the change of variable $y = F_{-sz}(x)$ for $y \in H_{sz}$, and the fact that $(1 + s\langle z, y\rangle)(1 - s\langle z, x\rangle) = 1$, we get

$$\begin{aligned}\int \mathcal{L}_s(f_z)(y)\, dy &= \int_{H_{sz}} (1 + s\langle z, y\rangle)^{1/s}\, \mathcal{L}_s f(F_{sz}(y))\, dy \\ &= \int_{H_{-sz}} \frac{\mathcal{L}_s f(x)}{(1 - s\langle z, x\rangle)^{n+1+1/s}}\, dx.\end{aligned}$$

(3) The convexity is a direct consequence of formula (2.7). The boundedness of f implies that $\mathcal{L}_s f(0) > 0$, and so 0 is in the interior of the support of $\mathcal{L}_s f$. The existence of a unique minimizer was recently proved by Ivanov and Werner in [14]. They assumed for their proof that f is s-concave, but using that $\mathcal{L}_s\mathcal{L}_s\mathcal{L}_s f_z = \mathcal{L}_s f_z$, we can actually assume that f is s-concave. Moreover, it is clear that $\mathrm{supp}(\mathcal{L}_s f_z) = (\mathrm{supp}(f_z))^\circ$, so if z is not in the interior of $\mathrm{Conv}(\mathrm{supp}(f))$, then 0 is not in the interior of $\mathrm{Conv}(\mathrm{supp}(f_z))$ and $\mathrm{supp}(\mathcal{L}_s f_z) = (\mathrm{supp}(f_z))^\circ$ is unbounded, which implies that $\int \mathcal{L}_s f_z = +\infty$. ■

Using the preceding lemma, we can now prove the following theorem.

Theorem 2.9. *Let $s \geq 0$ and let $f : \mathbb{R}^n \to \mathbb{R}_+$ be an integrable function such that $\int f > 0$ and $0 \in \mathrm{int}(\mathrm{Conv}(\mathrm{supp}(f)))$. Then, for $s > 0$,*

$$\int f \int \mathcal{L}_s f \leq c_s(1 - s\langle \mathrm{San}_s(\mathcal{L}_s(f)), \mathrm{bar}(f)\rangle)^{n+1+1/s}$$

and

$$\int f \int \mathcal{L}_0 f \leq (2\pi)^n\, e^{-\langle \mathrm{San}_0(\mathcal{L}_0(f)), \mathrm{bar}(f)\rangle}.$$

In particular, if $\mathrm{bar}(f) = 0$, then $\int f \int \mathcal{L}_s f \leq c_s$.

Proof of Theorem 2.9. The proof of this theorem is similar to that of Theorem 2.1. Fix a function f (without any concavity assumption) such that $0 \in \mathrm{int}(\mathrm{Conv}(\mathrm{supp}(f)))$ and $0 < \int_{\mathbb{R}^n} f < +\infty$. Then, from (2.6) applied to $\mathcal{L}_s f$, one has, for $z = \mathrm{San}_s(\mathcal{L}_s f)$,

$$\int_{\mathbb{R}^n} \mathcal{L}_s f(x)\, dx \int_{\mathbb{R}^n} \mathcal{L}_s((\mathcal{L}_s f)_z)(y)\, dy \leq c_s.$$

Since $\mathcal{L}_s f(z) > 0$, applying (2) of Lemma 2.8, we deduce that

$$\int_{\mathbb{R}^n} \mathcal{L}_s((\mathcal{L}_s f)_z)(y)\,dy = \int \frac{\mathcal{L}_s\mathcal{L}_s f(x)}{(1-s\langle z,x\rangle)^{n+1+1/s}}\,dx.$$

Using that $\mathcal{L}_s\mathcal{L}_s f(x) \geq f(x)$ and Jensen's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{L}_s((\mathcal{L}_s f)_z)(y)\,dy &\geq \int \frac{f(x)}{(1-s\langle z,x\rangle)^{n+1+1/s}}\,dx \\ &\geq \frac{\int f(x)\,dx}{(1-s\langle \mathrm{San}_s(\mathcal{L}_s(f)),\mathrm{bar}(f)\rangle)^{n+1+1/s}},\end{aligned}$$

which concludes the proof of the theorem. ■

3. Transport-entropy forms of Blaschke–Santaló inequality

Given a measurable cost function $c\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, bounded from below, the optimal transport cost between two probability measures $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n)$ is defined as follows:

$$\mathcal{T}_c(\nu_1,\nu_2) = \inf \int c(x,y)\,d\pi(x,y),$$

where the infimum runs over the set of all $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\pi(\mathbb{R}^n \times \cdot) = \nu_1(\cdot)$ and $\pi(\cdot \times \mathbb{R}^n) = \nu_2(\cdot)$, with $\mathcal{P}(\mathbb{R}^n)$ (respectively, $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$) denoting the set of all Borel probability measures on $\mathbb{R}^n$ (respectively, $\mathbb{R}^n \times \mathbb{R}^n$).

Relative entropy is another classical functional on $\mathcal{P}(\mathbb{R}^n)$ that we shall now recall. Whenever m is some measure on $\mathbb{R}^n$ (not necessarily of mass 1) and $d\nu = f dm \in \mathcal{P}(\mathbb{R}^n)$, the relative entropy of ν with respect to m is defined by

$$H(\nu|m) = \int f \log f\,dm,$$

as soon as the right-hand side makes sense (that is to say, $f\log^+ f$ or $f \log^- f$ is m-integrable). In particular, when m is a probability measure, $H(\nu|m)$ always makes sense in $\mathbb{R}_+ \cup \{+\infty\}$.

Comparing optimal transport costs to relative entropy is the purpose of the family of transport-entropy inequalities introduced by Marton [21–23] and Talagrand [29] in the nineties. We refer to the survey [13] for a presentation of this class of inequalities and their applications in the concentration of measure phenomenon. One of the most classical example of such an inequality is the so-called Talagrand's transport inequality for the standard Gaussian measure. It reads as follows:

$$W_2^2(\nu,\gamma) \leq 2H(\nu|\gamma), \quad \forall \nu \in \mathcal{P}(\mathbb{R}^n),$$

where γ is the standard Gaussian probability measure on $\mathbb{R}^n$, and $W_2^2(\nu,\gamma)$ is the squared Wasserstein distance, which is equal to $\mathcal{T}_c(\nu,\gamma)$ for $c(x,y) = |x-y|^2$, $x, y \in \mathbb{R}^n$. This inequality is optimal, with equality obtained when ν is a translation of γ. Using the triangle inequality for W_2, it is easily seen that the following variant involving two probability measures also holds:

$$W_2^2(\nu_1, \nu_2) \le 4H(\nu_1 \,|\, \gamma) + 4H(\nu_2 \,|\, \gamma), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n).$$

This inequality is still optimal, with equality achieved when ν_1 and ν_2 are two standard Gaussian with opposite means. Recently, a symmetrized version of this inequality was obtained by Fathi [10], namely

$$W_2^2(\nu_1, \nu_2) \le 2H(\nu_1 \,|\, \gamma) + 2H(\nu_2 \,|\, \gamma), \tag{3.1}$$

whenever ν_1 is centered and ν_2 is arbitrary. Fathi derived (3.1) from a functional version of Blaschke–Santaló's inequality.

The aim of this section is to further explore the relationships between transport-entropy inequalities and functional forms of the Blaschke–Santaló inequality given in Theorem 2.7. We will in particular derive from the latter some optimal transport-entropy inequalities for spherically invariant probability models that go beyond the Gaussian case.

3.1. General costs

Utilizing Theorem 2.7 gives us two different families of transport-entropy inequalities for a large class of spherically invariant probability measures.

Theorem 3.1. *Let $\rho\colon \mathbb{R}_+ \to (0, \infty)$ be a continuous nonincreasing function such that $\int \rho(|x|^2)\, dx < +\infty$, and $t \mapsto -\log \rho(e^t)$ is convex on $\mathbb{R}$. Let μ_ρ be the probability measure with density proportional to $\rho(|x|^2)$.*

(i) *For all $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n)$, we have*

$$\mathcal{T}_{\tilde\omega_\rho}(\nu_1, \nu_2) \le H(\nu_1 \,|\, \mu_\rho) + H(\nu_2 \,|\, \mu_\rho), \tag{3.2}$$

where the optimal transport cost $\mathcal{T}_{\tilde\omega_\rho}$ is defined with respect to the cost function $\tilde\omega_\rho$ given by

$$\tilde\omega_\rho(x, y) = \log\Big(\frac{\rho(|\langle x, y\rangle|)^2}{\rho(|x|^2)\,\rho(|y|^2)}\Big), \quad x, y \in \mathbb{R}^n.$$

(ii) *For all $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n)$ with ν_1 and ν_2 symmetric, we have*

$$\mathcal{T}_{\omega_\rho}(\nu_1, \nu_2) \le H(\nu_1 \,|\, \mu_\rho) + H(\nu_2 \,|\, \mu_\rho), \tag{3.3}$$

where the optimal transport cost $\mathcal{T}_{\omega_\rho}$ is defined with respect to the cost function ω_ρ given, for $x, y \in \mathbb{R}^n$, by

$$\omega_\rho(x, y) = \begin{cases} \log\Big(\dfrac{\rho(\langle x, y\rangle)^2}{\rho(|x|^2)\,\rho(|y|^2)}\Big) & \text{if } \langle x, y\rangle \ge 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Furthermore, there is equality in inequalities (3.2) and (3.3) when $\nu_1 = \nu_2 = \mu_\rho$.

Before turning to the proof of Theorem 3.1, let us make some comments. If (3.2) holds for all couples ν_1, ν_2 without restriction, note that the cost $\tilde{\omega}_\rho$ is not very standard. For instance, if $\rho_0(t) = e^{-t/2}$, for which $\mu_\rho = \gamma$ is the standard Gaussian, one gets $\tilde{\omega}_{\rho_0}(x, y) = \frac{1}{2}(|x| - |y|)^2$, $x, y \in \mathbb{R}$, instead of the usual quadratic cost $\frac{1}{2}|x - y|^2$. The cost ω_ρ seems better adapted to the geometry of the measure μ_ρ, but the corresponding transport-entropy inequality (3.3) requires symmetry assumptions on ν_1 and ν_2. Taking Fathi's result (3.1) into consideration, a natural question is to ask whether these symmetry assumptions can be relaxed or not. We will see in the next two sections that the answer to this question depends on the cost function ρ.

Proof. In this proof, we adapt the classical dualization argument by Bobkov and Götze [6] to our context. Let us first prove (i). Rewriting Theorem 2.7 (even case) with respect to the functions

$$F(x) = \log f(x) - \log \rho(|x|^2) \quad \text{and} \quad G(y) = \log g(y) - \log \rho(|y|^2),$$

we get the following: for all bounded measurable functions F, G such that F is even and

$$F \oplus G \leq \tilde{\omega}_\rho, \tag{3.4}$$

it holds

$$\int_{\mathbb{R}^n} e^F \, d\mu_\rho \int_{\mathbb{R}^n} e^G \, d\mu_\rho \leq 1, \tag{3.5}$$

where $F \oplus G(x, y) = F(x) + G(y)$, for $x, y \in \mathbb{R}^n$. We now introduce two probability measures ν_1 and ν_2. Then, taking the logarithm of inequality (3.5), we find that

$$\begin{aligned} H(\nu_1|m) + H(\nu_2|m) &\geq \int_{\mathbb{R}^n} F \, d\nu_1 - \log \int_{\mathbb{R}^n} e^F \, d\mu_\rho + \int_{\mathbb{R}^n} G \, d\nu_2 - \log \int_{\mathbb{R}^n} e^G \, d\mu_\rho \\ &\geq \int_{\mathbb{R}^n} F \, d\nu_1 + \int_{\mathbb{R}^n} G \, d\nu_2, \end{aligned} \tag{3.6}$$

where the first inequality comes from the duality formula for the relative entropy functional: if $\nu \in \mathcal{P}(\mathbb{R}^n)$ and $\log d\nu/dm \in L^1(\nu)$, then

$$H(\nu|m) = \sup_{f \in L^1(\nu)} \left\{ \int_{\mathbb{R}^n} f \, d\nu - \log \int_{\mathbb{R}^n} e^f \, dm \right\}.$$

Optimizing in (3.6) with respect to F and G, we thus find that

$$H(\nu_1|\mu_\rho) + H(\nu_2|\mu_\rho) \geq \sup_{(F,G) \in S} \left\{ \int_{\mathbb{R}^n} F \, d\nu_1 + \int_{\mathbb{R}^n} G \, d\nu_2 \right\}$$

where the supremum runs over the set S of couples of bounded measurable functions (F, G) with F even and satisfying (3.4).

Now, if (F, G) is a couple of bounded measurable functions satisfying (3.4) (with F not necessarily even), then by the symmetry of $\tilde{\omega}_\rho$, the even function given by $\tilde{F}(x) =$

$\max\{F(x), F(-x)\}$, $x \in \mathbb{R}^n$, is such that $(\tilde{F}, G) \in S$, and $\int_{\mathbb{R}^n} \tilde{F}\, d\nu_1 \geq \int_{\mathbb{R}^n} F\, d\nu_1$, and so we may remove the assumption on evenness of F and conclude that

$$\sup_{(F,G)\in S} \Big\{ \int_{\mathbb{R}^n} F\, d\nu_1 + \int_{\mathbb{R}^n} G\, d\nu_2 \Big\} = \sup_{(F,G): F\oplus G \leq \tilde{\omega}_\rho} \Big\{ \int_{\mathbb{R}^n} F\, d\nu_1 + \int_{\mathbb{R}^n} G\, d\nu_2 \Big\}$$
$$= \mathcal{T}_{\tilde{\omega}_\rho}(\nu_1, \nu_2),$$

where the second equality comes from the Kantorovich duality theorem (see, e.g., Theorem 5.10 in [30]), which applies since the cost function $\tilde{\omega}_\rho$ is lower semicontinuous (and even continuous) and bounded from below thanks to the log-concavity of $t \mapsto \rho(e^t)$ (it is, in fact, non-negative, a proof of which can be found in Lemma 4.3). This completes the proof of (i).

Let us now prove (ii). Reasoning exactly as before, one concludes that for any $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^n)$, it holds

$$H(\nu_1 \,|\, \mu_\rho) + H(\nu_2 \,|\, \mu_\rho) \geq \sup_{(F,G)\in \bar{S}} \Big\{ \int_{\mathbb{R}^n} F\, d\nu_1 + \int_{\mathbb{R}^n} G\, d\nu_2 \Big\},$$

where $\bar{S}$ is the set of couples of bounded measurable functions (F, G) with F even such that $F \oplus G \leq \omega_\rho$. Let (F, G) be a couple of bounded measurable functions (with F not necessarily even) such that $F \oplus G \leq \omega_\rho$. Since, for all $x, y \in \mathbb{R}^n$, $\omega_\rho(x, y) = \omega_\rho(-x, -y)$, defining $\bar{F}(x) = \frac{1}{2}(F(x) + F(-x))$ and $\bar{G}(y) = \frac{1}{2}(G(y) + G(-y))$, one gets that $(\bar{F}, \bar{G}) \in \bar{S}$. If ν_1 and ν_2 are further assumed to be symmetric, it holds $\int \bar{F}\, d\nu_1 = \int F\, d\nu_1$ and $\int \bar{G}\, d\nu_2 = \int G\, d\nu_2$. Thus, in this case,

$$\sup_{(F,G)\in \bar{S}} \Big\{ \int_{\mathbb{R}^n} F\, d\nu_1 + \int_{\mathbb{R}^n} G\, d\nu_2 \Big\} = \sup_{(F,G): F\oplus G \leq \omega_\rho} \Big\{ \int_{\mathbb{R}^n} F\, d\nu_1 + \int_{\mathbb{R}^n} G\, d\nu_2 \Big\}$$
$$= \mathcal{T}_{\omega_\rho}(\nu_1, \nu_2),$$

applying Kantorovich duality for the last equation, which completes the proof of (ii).

Finally, note that $\tilde{\omega}_\rho$ and ω_ρ are both non-negative and vanish on the diagonal, so that $\mathcal{T}_{\tilde{\omega}_\rho}(\mu_\rho, \mu_\rho) = \mathcal{T}_{\omega_\rho}(\mu_\rho, \mu_\rho) = 0$. There is thus equality in inequalities (3.2) and (3.3) when $\nu_1 = \nu_2 = \mu_\rho$. ∎

In the next subsections, we will study the consequences of Theorem 3.1 for two special costs, related respectively to Barenblatt-type and Cauchy-type distributions.

3.2. Barenblatt-type distributions

Let $s > 0$ and denote by $B_s = \{x \in \mathbb{R}^n : |x| < 1/\sqrt{s}\}$ the open Euclidean ball of center 0 and radius $1/\sqrt{s}$. Consider the probability measure

$$\gamma_s(dx) = \frac{1}{Z_s} (1 - s|x|^2)^{1/(2s)} \mathbf{1}_{B_s}(x)\, dx,$$

which is a particular case of the so-called Barenblatt profiles. Consider the cost function $k_s\colon B_s \times B_s \to \mathbb{R}$ defined by

$$k_s(x, y) = \frac{1}{s} \log\Big(\frac{1 - s\langle x, y\rangle}{(1 - s|x|^2)^{1/2}(1 - s|y|^2)^{1/2}} \Big), \quad x, y \in B_s.$$

For this particular cost, the conclusion of Theorem 3.1 can be improved, as shown in the following result.

Theorem 3.2. *For all $s > 0$, the probability measure γ_s satisfies the following transport-entropy inequality*:

$$\mathcal{T}_{k_s}(\nu_1, \nu_2) \leq H(\nu_1 \,|\, \gamma_s) + H(\nu_2 \,|\, \gamma_s),$$

for all probability measures ν_1 and ν_2, one of which is centered, and with supports $K_1, K_2 \subset B_s$.

This result is exactly analogous to Fathi's result (3.1) in the Gaussian case. Moreover, note that as $s \to 0$, it holds $\gamma_s \to \gamma$ (the standard Gaussian) and one recovers (3.1).

Proof of Theorem 3.2. Applying Theorem 2.7 to $\rho_s(t) = [1 - st]_+^{1/(2s)}$, $t \geq 0$, yields the following: for any $s > 0$ and $f : \mathbb{R}^n \to \mathbb{R}_+$ integrable, it holds

$$\int f(x)\, dx \inf_{z \in \operatorname{conv} S_f} \int \mathcal{L}_s(f_z)(y)\, dy \leq \Big(\int_{B_s} (1 - s|x|^2)^{1/(2s)}\, dx \Big)^2 = Z_s^2,$$

where S_f denotes the support of the measure $\nu_f(dx) = f(x)\, dx$, and the operator $\mathcal{L}_s$ is defined by (2.5). Let

$$b_s(x, y) = \frac{1}{s} \log[1 - s\langle x, y\rangle]_+, \quad x, y \in \mathbb{R}^n.$$

It is enough to prove that

$$\mathcal{T}_{b_s}(\nu_1, \nu_2) \leq H(\nu_1 \,|\, \mathrm{Leb}) + H(\nu_2 \,|\, \mathrm{Leb}) + 2 \log Z_s, \tag{3.7}$$

for all probability measures ν_1 and ν_2 with supports $K_1, K_2 \subset B_s$ and such that ν_1 is centered. Note that b_s is bounded and continuous on $K_1 \times K_2$. Therefore, applying Kantorovich duality theorem on $K_1 \times K_2$ yields the following identity:

$$\mathcal{T}_{b_s}(\nu_1, \nu_2) = \sup_{\varphi \in \mathcal{C}_b(K_2)} \Big\{ \int_{K_1} Q_s\varphi(x_1)\, d\nu_1(x_1) - \int_{K_2} \varphi(x_2)\, d\nu_2(x_2) \Big\}, \tag{3.8}$$

where $\mathcal{C}_b(K_2)$ denotes the set of bounded continuous functions on K_2 and

$$Q_s\varphi(x_1) = \inf_{x_2 \in K_2} \{\varphi(x_2) + b_s(x_1, x_2)\}, \quad x_1 \in \mathbb{R}^n.$$

Take $\varphi \in \mathcal{C}_b(K_2)$ and define $f : \mathbb{R}^n \to \mathbb{R}_+$ by $f(x_2) = e^{-\varphi(x_2)}$ if $x_2 \in K_2$, and 0 otherwise. Note the following relation:

$$e^{Q_s\varphi} = \mathcal{L}_s(f). \tag{3.9}$$

According to what precedes, it holds

$$\int f(x_2)\, dx_2 \inf_{z \in \operatorname{conv} K_2} \int \mathcal{L}_s(f_z)(x_1)\, dx_1 \leq Z_s^2.$$

Indeed, by its construction, the support of the measure $f(x)\,dx$ is K_2. Note that the following inequality holds, for any $z \in \mathbb{R}^n$:

$$\mathcal{L}_s(f_z)(y) \geq (1 + s\langle z, y\rangle)_+ \,\mathcal{L}_s f(F_{sz}(y)), \quad \forall y \in \mathbb{R}^n,$$

where, for any $a \in \mathbb{R}^n \setminus \{0\}$, the map $F_a(y) = \frac{y}{1+\langle z,a\rangle}$, $y \in H_a = \{y \in \mathbb{R}^n : 1 + \langle z,a\rangle > 0\}$ is a bijection from H_a onto H_{-a} (this is item (1) of Lemma 2.5; when $f(0) = 0$ there is equality, but this is not needed here). So it holds

$$\begin{aligned}\int \mathcal{L}_s(f_z)(x_1)\,dx_1 &\geq \int (1 + s\langle z, x_1\rangle)_+^{1/s} \,\mathcal{L}_s f(F_{sz}(x_1))\,dx_1 \\ &= \int_{H_{sz}} (1 + s\langle z, x_1\rangle)^{1/s} \,\mathcal{L}_s f(F_{sz}(x_1))\,dx_1 \\ &= \int_{H_{-sz}} \frac{1}{(1 - s\langle z, u\rangle)^{n+1+1/s}} \,\mathcal{L}_s f(u)\,du = \int e^{Q_s\varphi(u)}\,dm_z(u),\end{aligned}$$

where

$$dm_z(u) = \frac{1}{(1 - s\langle z, u\rangle)^{n+1+1/s}} \,\mathbf{1}_{H_{-sz}}(u)\,du.$$

Therefore,

$$-2\log Z_s \leq -\log \int_{K_2} e^{-\varphi(x_2)}\,dx_2 - \inf_{z \in \operatorname{conv} K_2} \log \int e^{Q_s\varphi(x_1)}\,dm_z(x_1),$$

and so,

$$\begin{aligned}&-2\log Z_s + \int Q_s\varphi\,d\nu_1 - \int \varphi\,d\nu_2 \\ &\leq \int -\varphi\,d\nu_2 - \log \int_{K_2} e^{-\varphi(x_2)}\,dx_2 + \int Q_s\varphi\,d\nu_1 - \inf_{z \in \operatorname{conv} K_2} \log \int e^{Q_s\varphi(x_1)}\,dm_z(x_1) \\ &\leq H(\nu_2|\mathrm{Leb}) + \sup_{z \in \operatorname{conv} K_2} H(\nu_1|m_z),\end{aligned}$$

where the last inequality follows from the bound

$$\int \psi\,d\nu - \log \int e^{-\psi}\,dm \leq H(\nu|m), \quad \forall \nu \ll m.$$

Note that if $z \in B_s$, then $B_s \subset H_{-sz}$ and so in particular $\nu_1 \ll m_z$.

Finally, for all $z \in B_s$, it holds

$$\begin{aligned}H(\nu_1|m_z) &= \int_{B_s} \log \frac{d\nu_1}{dm_z}\,d\nu_1 = H(\nu_1|\mathrm{Leb}) - \int_{B_s} \log \frac{dm_z}{dx}\,d\nu_1 \\ &= H(\nu_1|\mathrm{Leb}) + \left(n + 1 + \frac{1}{s}\right) \int_{B_s} \log\left(1 - s\langle z, x_1\rangle\right)\,d\nu_1(x_1) \\ &\leq H(\nu_1|\mathrm{Leb}) + \left(n + 1 + \frac{1}{s}\right) \log\left(1 - s\left\langle z, \int x_1\,d\nu_1(x_1)\right\rangle\right) = H(\nu_1|\mathrm{Leb}),\end{aligned}$$

using the concavity of the logarithm and the fact that ν_1 is centered. This completes the proof. ■

Remark 3.3. Suppose that $f:\mathbb{R}^n\to\mathbb{R}^+$ is a continuous function such that $\int xf(x)\,dx=0$ and $f=0$ outside B_s. Denote by $K_2=\{x\in B_s : f(x)\neq 0\}$ and let $\varphi=-\log f\in\mathcal{C}_b(K_2)$. Then, using (3.7) and (3.8), one gets

$$\int Q_s\varphi\,d\nu_1 - H(\nu_1\,|\,\mathrm{Leb}) + \int -\varphi\,d\nu_2 - H(\nu_2\,|\,\mathrm{Leb}) \le 2\log Z_s,$$

for all ν_1,ν_2 with compact support in B_s, and ν_2 centered. Taking

$$d\nu_1(x) = \frac{e^{Q_s\varphi}(x)}{\int e^{Q_s\varphi(y)}\,dy}\,dx = \frac{\mathcal{L}_s f(x)}{\int \mathcal{L}_s f(y)\,dy}\,dx$$

and

$$d\nu_2(x) = \frac{e^{-\varphi}(x)}{\int e^{-\varphi(y)}\,dy}\,dx = \frac{f(x)}{\int f(y)\,dy}\,dx$$

(thanks to (3.9)) and noting that ν_2 is centered, one gets

$$\int f \int \mathcal{L}_s f \le \Big(\int \rho_s(|x|^2)\,dx\Big)^2,$$

which essentially gives back the conclusion of Theorem 2.9 in the centered case.

3.3. Cauchy-type distributions

In this section, we consider the cost function

$$\rho_\beta(t) = \frac{1}{(1+t)^\beta},\qquad t\ge 0,$$

for which $x\mapsto\rho_\beta(|x|^2)$ is integrable whenever $\beta>n/2$. For $\beta>n/2$, we consider the following Cauchy-type distribution:

$$d\mu_\beta(x) = \frac{1}{Z_\beta(1+|x|^2)^\beta}\,dx,\quad \text{with}\quad Z_\beta = \pi^{n/2}\frac{\Gamma(\beta-n/2)}{\Gamma(\beta)}.$$

The following result follows immediately from item (ii) of Theorem 3.1.

Corollary 3.4. *For any $\beta>n/2$, the Cauchy-type probability measure μ_β satisfies the following transport-entropy inequality: for all $\nu_1,\nu_2\in\mathcal{P}(\mathbb{R}^n)$ with ν_1 and ν_2 symmetric, we have*

$$\beta\mathcal{T}_\omega(\nu_1,\nu_2) \le H(\nu_1\,|\,\mu_\beta) + H(\nu_2\,|\,\mu_\beta), \tag{3.10}$$

where the optimal transport cost $\mathcal{T}_\omega$ is defined with respect to the cost function ω given, for $x,y\in\mathbb{R}^n$, by

$$\omega(x,y) = \begin{cases} -2\log\Big(\dfrac{1+\langle x,y\rangle}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\Big) & \textit{if } \langle x,y\rangle>0,\\ +\infty & \textit{otherwise.}\end{cases} \tag{3.11}$$

Note that a similar transport-entropy inequality holds with respect to the cost function

$$\tilde{\omega}(x,y) = -2\log\Big(\frac{1+|\langle x,y\rangle|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\Big), \quad x,y\in\mathbb{R}^n,$$

without symmetry restrictions on ν_1,ν_2.

Proof. The function $t\mapsto\log(1+e^t)$ being convex on $\mathbb{R}$, the conclusion immediately follows from Theorem 3.1 (item (ii)). ∎

It turns out that sharp transport-entropy inequalities for a family of probability measures on the Euclidean unit sphere can be derived from Corollary 3.4. To state this result, we need to introduce additional notation. Let

$$\mathbb{S}^n = \Big\{u=(u_1,\dots,u_{n+1}) : \sum_{i=1}^{n+1} u_i^2 = 1\Big\} \quad\text{and}\quad \mathbb{S}^n_+ = \mathbb{S}^n\cap\{u\in\mathbb{R}^{n+1} : u_{n+1}\geq 0\}$$

be, respectively, the n-dimensional Euclidean unit sphere and the upper half unit sphere of $\mathbb{R}^{n+1}$. Denote by σ the uniform probability measure on $\mathbb{S}^n$ and by $\sigma_+(\cdot)=2\sigma(\mathbb{S}^n_+\cap\cdot)$ the normalized restriction of σ to $\mathbb{S}^n_+$ (the dimension n is omitted in the notation of σ and σ_+). For any $\beta>n/2$, let $\sigma_{\beta,+}\in\mathcal{P}(\mathbb{S}^n_+)$ (respectively, $\sigma_\beta\in\mathcal{P}(\mathbb{S}^n)$) be the probability measure with a density proportional to

$$u\mapsto|u_{n+1}|^{2\beta-(n+1)}$$

with respect to σ_+ (respectively, to σ). Note that σ and σ_+ correspond to the parameter $\beta=(n+1)/2$.

The set of Borel probability measures on $\mathbb{S}^n$ (respectively, $\mathbb{S}^n_+$) will be denoted by $\mathcal{P}(\mathbb{S}^n)$ (respectively, $\mathcal{P}(\mathbb{S}^n_+)$). A probability measure $\mu\in\mathcal{P}(\mathbb{S}^n)$ will be called symmetric if it is invariant under the map $\mathbb{S}^n\to\mathbb{S}^n : u\mapsto -u$. The set of all symmetric probability measures on $\mathbb{S}^n$ will be denoted by $\mathcal{P}_s(\mathbb{S}^n)$.

Finally, let $\alpha:\mathbb{S}^n\times\mathbb{S}^n\to\mathbb{R}_+\cup\{+\infty\}$ be the cost function defined, for $u,v\in\mathbb{S}^n$, by

$$\alpha(u,v) = \begin{cases}\log\Big(\dfrac{1}{\langle u,v\rangle}\Big) & \text{if } \langle u,v\rangle>0,\\ +\infty & \text{otherwise,}\end{cases}$$

and let $\mathcal{T}_\alpha$ be the associated optimal transport cost on $\mathcal{P}(\mathbb{S}^n)$. This cost function has been introduced by Oliker [25] (see also [4] and [15]) in connection with the so-called Aleksandrov problem in convex geometry.

Recall the definition of the geodesic distance $d_{\mathbb{S}^n}$ on $\mathbb{S}^n$:

$$d_{\mathbb{S}^n}(u,v) = \arccos(\langle u,v\rangle), \quad u,v\in\mathbb{S}^n.$$

The cost α can thus also be expressed, for $u,v\in\mathbb{S}^n$, as

$$\alpha(u,v) = \begin{cases}-\log\cos(d_{\mathbb{S}^n}(u,v)) & \text{if } d_{\mathbb{S}^n}(u,v)<\pi/2,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.12}$$

Remark 3.5. Characterizing couples (μ, ν) for which the transport cost $\mathcal{T}_\alpha(\mu, \nu)$ is finite is a delicate question, discussed in particular in [4]. Note that, according to Lemma 3.3 of [15] and Remark 4.9 of [4], if μ, ν are symmetric probability measures such that μ has a positive density with respect to σ and ν is such that $\nu(\mathbb{S}^n \cap L) = 0$ for any hyperplane L passing through the origin, then $\mathcal{T}_\alpha(\mu, \nu) < +\infty$.

Corollary 3.6. *Let $\beta > n/2$.*

(i) *For any $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{S}^n_+)$ which are invariant under the map*

$$\mathbb{S}^n_+ \to \mathbb{S}^n_+ : u \mapsto (-u_1, \dots, -u_n, u_{n+1}),$$

it holds

$$2\beta \mathcal{T}_\alpha(\nu_1, \nu_2) \le H(\nu_1 | \sigma_{\beta,+}) + H(\nu_2 | \sigma_{\beta,+}).$$

(ii) *For any $\nu_1, \nu_2 \in \mathcal{P}_s(\mathbb{S}^n)$ which are also invariant under the map*

$$\mathbb{S}^n \to \mathbb{S}^n : u \mapsto (u_1, \dots, u_n, -u_{n+1}),$$

it holds

$$2\beta \mathcal{T}_\alpha(\nu_1, \nu_2) \le H(\nu_1 | \sigma_\beta) + H(\nu_2 | \sigma_\beta).$$

Proof. Let us prove (i), following the proof of Theorem 19 in [12]. Denote by $\mu = \mu_{(n+1)/2}$ the multivariate Cauchy distribution with density $Z^{-1}(1 + |x|^2)^{-(n+1)/2}$. Consider the map

$$T : \mathbb{R}^n \to \mathbb{S}^n_{++}, \quad x \mapsto \frac{1}{(1 + |x|^2)^{1/2}} (x, 1),$$

denoting by $\mathbb{S}^n_{++} = \mathbb{S}^n \cap \{u \in \mathbb{R}^{n+1} : u_{n+1} > 0\}$. This transformation is bijective, with inverse

$$T^{-1} : \mathbb{S}^n_{++} \to \mathbb{R}^n, \quad u \mapsto \frac{1}{u_{n+1}} (u_1, \dots, u_n),$$

which is sometimes called *gnomonic projection*. It is easy to check that T^{-1} pushes forward σ_+ onto μ, or equivalently, that T pushes forward μ onto σ_+. For any $\beta > n/2$, the probability measure μ_β has density

$$g_\beta(x) = \frac{C_\beta}{(1 + |x|^2)^{\beta-(n+1)/2}}, \quad x \in \mathbb{R}^n,$$

with respect to μ. Therefore, the probability measure $T_\# \mu_\beta$ has density $g_\beta(T^{-1})$ with respect to $T_\# \mu = \sigma_+$. A simple calculation shows that

$$g_\beta(T^{-1}(u)) = C_\beta\, u_{n+1}^{2\beta-(n+)}, \quad u \in \mathbb{S}^n_+,$$

and so

$$\sigma_{\beta,+} = T_\# \mu_\beta.$$

Note the following relation between the cost functions ω (of Corollary 3.4) and α:

$$\alpha(u, v) \le \frac{1}{2} \omega(T^{-1}(u), T^{-1}(v)), \quad \forall u, v \in \mathbb{S}^n_{++}.$$

Let $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{S}^n_+)$ be measures invariant under the map $u \mapsto (-u_1, \dots, -u_n, u_{n+1})$. If $H(\nu_1 | \sigma_{\beta,+}) = +\infty$ or $H(\nu_2 | \sigma_{\beta,+}) = +\infty$, there is nothing to prove. Let assume that $H(\nu_1 | \sigma_{\beta,+}) < +\infty$ and $H(\nu_2 | \sigma_{\beta,+}) < +\infty$. In particular, ν_1 and ν_2 do not give mass to $\mathbb{S}^n \cap \{u \in \mathbb{R}^{n+1} : u_{n+1} = 0\}$, and can thus be seen as elements of $\mathcal{P}(\mathbb{S}^n_{++})$. Define $\nu_1' := T_\#^{-1}\nu_1$ and $\nu_2' := T_\#^{-1}\nu_2$, which are symmetric and so, according to Corollary 3.4 applied to μ_β, it holds

$$\beta \mathcal{T}_\omega(\nu_1', \nu_2') \leq H(\nu_1' | \mu_\beta) + H(\nu_2' | \mu_\beta).$$

If π' is a coupling between ν_1' and ν_2' and π is the push forward of π' under the map $(x, y) \mapsto (T(x), T(y))$, it holds

$$\begin{aligned}\frac{1}{2}\iint \omega(x, y)\, d\pi'(x, y) &= \frac{1}{2}\iint \omega(T^{-1}(u), T^{-1}(v))\, d\pi(u, v)\\ &\geq \iint \alpha(u, v)\, d\pi(u, v) \geq \mathcal{T}_\alpha(\nu_1, \nu_2),\end{aligned}$$

since π has ν_1 and ν_2 as marginals. Therefore, $\mathcal{T}_\alpha(\nu_1, \nu_2) \leq \frac{1}{2}\mathcal{T}_\omega(\nu_1', \nu_2')$. Finally, a simple calculation shows that

$$H(\nu_i' | \mu_\beta) = H(T_\#^{-1}\nu_i \,|\, T_\#^{-1}\sigma_{\beta,+}) = H(\nu_i | \sigma_{\beta,+}),$$

which completes the proof of (i).

Let us now prove (ii). Let $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{S}^n)$ be invariant under the maps $u \mapsto -u$ and $u \mapsto (u_1, \dots, u_n, -u_{n+1})$, with densities f_1 and f_2 with respect to σ_β. For $i = 1, 2$, it holds $\nu_i(\mathbb{S}^n_+) = 1/2$. Define $d\nu_{i,+}(u) = 2 f_i \mathbf{1}_{\mathbb{S}^n_+}(u)\, d\sigma_\beta(u) = f_i(u)\, d\sigma_{\beta,+}(u)$. Then it holds

$$H(\nu_i | \sigma_\beta) = \int f_i \log f_i\, d\sigma_\beta = 2\int_{\mathbb{S}^n_+} f_i \log f_i\, d\sigma_\beta = \int f_i \log f_i\, d\sigma_{\beta,+} = H(\nu_{i,+} | \sigma_{\beta,+}).$$

On the other hand, if (U, V) is a coupling between $\nu_{1,+}$ and $\nu_{2,+}$ and ε is such that $\mathbb{P}(\varepsilon = \pm 1) = 1/2$ and is independent of (U, V), then $X = (U_1, \dots, U_n, \varepsilon U_{n+1})$, $Y = (V_1, \dots, V_n, \varepsilon V_{n+1})$ is a coupling between ν_1 and ν_2, and it holds that $\mathbb{E}[\alpha(X, Y)] = \mathbb{E}[\alpha(U, V)]$. From this follows that $\mathcal{T}_\alpha(\nu_1, \nu_2) \leq \mathcal{T}_\alpha(\nu_{1,+}, \nu_{2,+})$. Thus (ii) immediately follows from (i), which completes the proof. ∎

For the probability measure σ (corresponding to $\beta = (n+1)/2$), the conclusion of Corollary 3.6 can be improved, as the following result shows.

Theorem 3.7. *For all symmetric probability measures ν_1 and ν_2 on $\mathbb{S}^n$, it holds*

$$(n+1)\mathcal{T}_\alpha(\nu_1, \nu_2) \leq H(\nu_1 | \sigma) + H(\nu_2 | \sigma). \tag{3.13}$$

The preceding result is an improvement of a result by Kolesnikov [15], who obtained the following transport-entropy inequality on $\mathbb{S}^n$:

$$(n+1)\mathcal{T}_\alpha(\nu, \sigma) \leq H(\nu | \sigma), \tag{3.14}$$

for all symmetric probability $\nu \in \mathcal{P}(\mathbb{S}^n)$. The proof by Kolesnikov is based on the Monge–Ampère equation relating the density of ν to the optimal transport map T transporting σ on μ. The determinant of the Jacobian matrix of T is controlled with the help of the classical Blaschke–Santaló inequality for convex bodies (see the proof of Theorem 7.3 in [15]). Kolesnikov also establishes links between minimizers of the functional

$$\nu_1 \mapsto H(\nu_1 | \sigma) - (n+1)\mathcal{T}_\alpha(\nu_1, \nu_2),$$

with ν_1 and ν_2 symmetric, and the log-Minkowski problem; we refer to [15] for further explanations and references. Remark 3.11 below gathers further comments on (3.13) and (3.14).

Before turning to the proof of (3.13), let us comment on the role of the symmetry assumption. It turns out that for any constant $C > 0$, the inequality

$$C\mathcal{T}_\alpha(\nu, \sigma) \leq H(\nu | \sigma),$$

cannot be true for all $\nu \in \mathcal{P}(\mathbb{S}^n)$. This follows immediately from the following lemma.

Lemma 3.8. *There exists $\nu \in \mathcal{P}(\mathbb{S}^n)$ such that $\mathcal{T}_\alpha(\nu, \sigma) = +\infty$ and $H(\nu | \sigma) < +\infty$.*

In particular, contrary to Fathi's inequality (3.1) for the standard Gaussian measure, the inequality (3.13) is not true if only one of the probability measures ν_1, ν_2 is assumed to be symmetric.

Proof of Lemma 3.8. Let $A \subset \mathbb{S}^n$ be some spherical cap, and define

$$d\nu = \frac{\mathbf{1}_A}{\sigma(A)}\, d\sigma.$$

Then $H(\nu | \sigma) = -\log \sigma(A) < +\infty$. On the other hand, if (X, Y) is a coupling between σ and ν, then denoting by

$$A_{\pi/2} = \{y \in \mathbb{S}^n : \exists x \in A \text{ such that } d_{\mathbb{S}^n}(x, y) < \pi/2\},$$

it holds

$$\mathbb{P}(d(X, Y) < \pi/2) \leq \mathbb{P}(Y \in A_{\pi/2}) = \sigma(A_{\pi/2}).$$

If A is small enough, then $\sigma(A_{\pi/2}) < 1$ and so $\mathbb{P}(d(X, Y) \geq \pi/2) > 0$. Therefore, by definition of α, $\mathbb{E}[\alpha(X, Y)] = +\infty$. The coupling being arbitrary, one concludes that $\mathcal{T}_\alpha(\nu, \sigma) = +\infty$. ∎

Remark 3.9. The preceding construction can be easily adapted to show that, for any $\beta > n/2$, (3.10) can be false if only one of the measures ν_1, ν_2 is assumed to be symmetric.

Our proof of Theorem 3.7 is based on the following Kantorovich type duality for the cost function α. To state this result, let us introduce additional notation. Recall that if $C \subset \mathbb{R}^{n+1}$ is a convex body, the support function of C is the function denoted by h_C defined by

$$h_C(y) = \sup_{x \in C} \langle x, y \rangle, \quad \forall y \in \mathbb{R}^{n+1},$$

and when C contains 0 in its interior, the radial function of C is the function denoted by ρ_C defined by

$$\rho_C(x) = \sup\{r \geq 0 : rx \in C\}, \quad \forall x \in \mathbb{R}^{n+1}.$$

Lemma 3.10. *For all probability measures ν_1 and ν_2 on $\mathbb{S}^n$, it holds*

$$\mathcal{T}_\alpha(\nu_1, \nu_2) = \sup_C \int -\ln h_C \, d\nu_1 + \int \ln \rho_C \, d\nu_2,$$

where the supremum runs over the set of all convex bodies C containing 0 in their interiors. Moreover, when ν_1 and ν_2 are symmetric, the supremum can be restricted to centrally symmetric convex bodies C.

This duality relation was first established by Oliker in [25] in his transport approach to Alexandrov's problem on the Gauss curvature prescription of Euclidean convex sets (see also [4] in particular for the question of dual attainment). For the sake of completeness, we briefly sketch the proof of Lemma 3.10.

Proof. For any probability measures ν_1 and ν_2 on $\mathbb{S}^n$, Kantorovich duality (see Theorem 5.10 (i) in [30]) yields to

$$\mathcal{T}_\alpha(\nu_1, \nu_2) = \sup_{\phi,\psi} \int \phi \, d\nu_1 + \int \psi \, d\nu_2, \tag{3.15}$$

where the supremum runs over the set of couples (ϕ, ψ) of bounded continuous functions on $\mathbb{S}^n$ such that

$$\phi(x) + \psi(y) \le \alpha(x, y), \quad \forall x, y \in \mathbb{S}^n. \tag{3.16}$$

Whenever ν_1 and ν_2 are symmetric, and (ϕ, ψ) satisfies (3.16), then defining

$$\bar{\phi}(x) = \frac{1}{2}(\phi(x) + \phi(-x)) \quad \text{and} \quad \bar{\psi}(y) = \frac{1}{2}(\psi(y) + \psi(-y)), \quad x, y \in \mathbb{S}^n,$$

the couple $(\bar{\phi}, \bar{\psi})$ satisfies (3.16) (because $\alpha(-x, -y) = \alpha(x, y)$) and is such that

$$\int \bar{\phi} \, d\nu_1 + \int \bar{\psi} \, d\nu_2 = \int \phi \, d\nu_1 + \int \psi \, d\nu_2.$$

Therefore, in this symmetric case, the supremum in (3.15) can be further restricted to couples of even functions (ϕ, ψ). Let us now consider the α-transform f^α of a function $f : \mathbb{S}^n \to \mathbb{R}$ defined by

$$f^\alpha(y) = \inf_{x \in \mathbb{S}^n} \{\alpha(x, y) - f(x)\}, \quad y \in \mathbb{S}^n.$$

It is not difficult to check that whenever f is bounded on $\mathbb{S}^n$, then f^α is bounded and continuous on $\mathbb{S}^n$, and that if f is even, then f^α is also even. Using a well-known double conjugation argument (see Theorem 5.10 (i) in [30] for details), one sees that the duality formula (3.15) can be further restricted to couples (ϕ, ψ) of α-conjugate functions, that is to say such that $\phi^\alpha = \psi$ and $\psi^\alpha = \phi$. Moreover, in the case where ν_1 and ν_2 are symmetric, (3.15) can be restricted to couples (ϕ, ψ) of even α-conjugate functions. With the change of functions $h = e^{-\phi}$ and $\rho = e^{\psi}$, we see that (ϕ, ψ) is a couple of continuous (even) α-conjugate functions if and only if (h, ρ) is a couple of continuous (even) positive functions such that

$$h(x) = \sup_{y \in \mathbb{S}^n} \rho(y)\langle x, y\rangle, \quad \forall x \in \mathbb{S}^n \quad \text{and} \quad \frac{1}{\rho(y)} = \sup_{x \in \mathbb{S}^n} \frac{\langle x, y\rangle}{h(x)}, \quad \forall y \in \mathbb{S}^n.$$

It is well-known that to any such couple (h,ρ) uniquely corresponds a convex body C containing 0 in its interior such that $h = h_C$ and $\rho = \rho_C$; we refer to Theorem 2 in [25] for details. In the case where h and ρ are both even, then C is centrally symmetric, which completes the proof. ■

Proof of Theorem 3.7. Let C be a centrally symmetric convex body in $\mathbb{R}^{n+1}$. According to the classical Blaschke–Santaló inequality, it holds

$$|C||C^\circ| \leq |B_2^{n+1}|^2.$$

Calculating the volume of C in polar coordinates yields to

$$|C| = (n+1)|B_2^{n+1}| \int_{\mathbb{S}^n} \Big(\int_{\mathbb{R}^+} \mathbf{1}_C(ru)\, r^n \, dr \Big) \, d\sigma(u) = |B_2^{n+1}| \int_{\mathbb{S}^n} \rho_C(u)^{n+1} \, d\sigma(u),$$

where ρ_C denotes the radial function of C. Similarly,

$$|C^\circ| = |B_2^{n+1}| \int_{\mathbb{S}^n} \rho_{C^\circ}(u)^{n+1} \, d\sigma(u) = |B_2^{n+1}| \int_{\mathbb{S}^n} \frac{1}{h_C(u)^{n+1}} \, d\sigma(u),$$

using the well-known (and easy to check) relation $\rho_{C^\circ} = 1/h_C$, where h_C is the support function of C. So, for every symmetric convex C body in $\mathbb{R}^{n+1}$, it holds

$$\int_{\mathbb{S}^n} \rho_C(u)^{n+1} \, d\sigma(u) \int_{\mathbb{S}^n} \frac{1}{h_C(u)^{n+1}} \, d\sigma(u) \leq 1. \tag{3.17}$$

On the other hand, if ν_1 and ν_2 are two symmetric probability measures on $\mathbb{S}^n$, Lemma 3.10 yields

$$(n+1)\,\mathcal{T}_\alpha(\nu_1,\nu_2) = \sup_C \int -\ln(h_C^{n+1}) \, d\nu_1 + \int \ln(\rho_C^{n+1}) \, d\nu_2,$$

where the supremum runs over the set of all centrally symmetric convex bodies C containing 0 in their interiors. Reasoning exactly as in the proof of Theorem 3.1, one sees that (3.17) implies (and is in fact equivalent to)

$$(n+1)\,\mathcal{T}_\alpha(\nu_1,\nu_2) \leq H(\nu_1|\sigma) + H(\nu_2|\sigma),$$

for all ν_1, ν_2 symmetric. This completes the proof. ■

In order to discuss inequalities (3.13) and (3.14), let us recall that the uniform probability measure σ on $\mathbb{S}^n$ satisfies the following Poincaré inequality: for any smooth function $f : \mathbb{S}^n \to \mathbb{R}$,

$$\lambda_1(\mathbb{S}^n)\,\mathrm{Var}_\sigma(f) \leq \int |\nabla_{\mathbb{S}^n} f|^2 \, d\sigma, \tag{3.18}$$

with the sharp constant $\lambda_1(\mathbb{S}^n) = n$ (corresponding to the spectral gap of the Laplace operator on $\mathbb{S}^n$). Equality in (3.18) is reached for every linear form. Under symmetry assumptions, the constant in (3.18) can be improved. More precisely, for all smooth functions $f: \mathbb{S}^n \to \mathbb{R}$ such that $f(-u) = f(u)$, for all $u \in \mathbb{S}^n$, it holds

$$\lambda_2(\mathbb{S}^n)\,\mathrm{Var}_\sigma(f) \leq \int |\nabla_{\mathbb{S}^n} f|^2\, d\sigma, \tag{3.19}$$

where $\lambda_2(\mathbb{S}^n) = 2(n+1)$ is the second non-zero eigenvalue of the Laplace operator on $\mathbb{S}^n$. Moreover, equality in (3.19) is reached whenever f is the restriction to $\mathbb{S}^n$ of a homogeneous polynomial of degree 2. For the sake of completeness, we recall the classical argument leading to (3.19).

Proof of (3.19). For all $d = 0, 1, 2 \ldots$, denote by $H_d \subset L^2(\sigma)$ the space of degree d homogeneous harmonic polynomials (restricted to $\mathbb{S}^n$). It is well known that

$$L^2(\sigma) = \bigoplus_{d=0}^{+\infty} H_d,$$

and that for all $f \in H_d$, it holds

$$\Delta_{\mathbb{S}^n} f = -d(d+n-1) f.$$

If $f: \mathbb{S}^n \to \mathbb{R}$ is a smooth *even* function, then it can be written as $f = \sum_{k=0}^{+\infty} f_{2k}$, with $f_{2k} \in H_{2k}$, for all $k \geq 0$. Therefore, by integration by parts,

$$\begin{aligned}\int |\nabla_{\mathbb{S}^n} f|^2\, d\sigma &= -\int f.\Delta_{\mathbb{S}^n} f\, d\sigma = \sum_{k=0}^{+\infty} 2k(2k+n-1)\int f_k^2\, d\sigma \\ &\geq 2(n+1)\sum_{k=1}^{+\infty}\int f_k^2\, d\sigma = 2(n+1)\,\mathrm{Var}_\sigma(f),\end{aligned}$$

which proves (3.19). Whenever $f \in H_0 \oplus H_2$, equality clearly holds. This is, in particular, the case if f is the restriction to the sphere of a degree 2 homogeneous polynomial. Indeed, suppose that $f = P_{|\mathbb{S}^n}$, where $P: \mathbb{R}^{n+1} \to \mathbb{R}$ is some degree 2 homogeneous polynomial. Then there is some constant c such that $\Delta_{\mathbb{R}^{n+1}} P = c$. The polynomial Q defined by $Q(x) = P(x) - \frac{c}{2(n+1)}|x|^2$, $x \in \mathbb{R}^{n+1}$, is homogeneous of degree 2 and harmonic. Moreover, it holds $f = Q_{|\mathbb{S}^n} + \frac{c}{2(n+1)}$ and so $f \in H_0 \oplus H_2$. ∎

Recall the expression (3.12), which will be used in the following remark on the optimality of (3.13).

Remark 3.11. (a) First, let us relate Kolesnikov's inequality (3.14) to existing transport-entropy inequalities on $\mathbb{S}^n$. A simple calculation shows that $-\log\cos u \geq u^2/2$ for all $u \in [0, \pi/2]$. Therefore, (3.14) implies that for all symmetric probability measures ν on $\mathbb{S}^n$, it holds

$$\frac{n+1}{2} W_2^2(\nu, \sigma) \leq H(\nu|\sigma), \tag{3.20}$$

with W_2 being the usual Wasserstein distance on $\mathbb{S}^n$ (with respect to the geodesic distance $d_{\mathbb{S}^n}$). The inequality (3.20) is an improvement of the following classical transport-entropy inequality:

$$\frac{n}{2}\, W_2^2(\nu,\sigma) \le H(\nu\,|\,\sigma), \tag{3.21}$$

that holds for all $\nu \in \mathcal{P}(\mathbb{S}^n)$. Inequality (3.21) can for instance be deduced from the log-Sobolev inequality on $\mathbb{S}^n$ that holds with the optimal constant $2/n$ using the Otto–Villani theorem [26]. The constant $n/2$ in (3.21) is optimal. Indeed, according to a well-known general linearization argument of [26], (3.21) implies the sharp Poincaré inequality (3.18). Using the fact that the function $u \mapsto -\log\cos\sqrt{u}$ is convex and increasing on $[0, (\pi/2)^2]$, it follows from Jensen's inequality that (3.14) implies the following transport-entropy inequality:

$$-(n+1)\log\cos W_2(\nu,\sigma) \le H(\nu\,|\,\sigma), \tag{3.22}$$

for all symmetric $\nu \in \mathcal{P}(\mathbb{S}^n)$. Inequality (3.22) improves the conclusion of Corollary 3.29 in [9] in the case of symmetric probability measures on $\mathbb{S}^n$. See Remark 7.4 of [15] for other transport-entropy inequalities derived from (3.14).

(b) Now let us discuss the sharpness of inequality (3.13). Reasoning as above, we see that (3.13) implies the following variant of (3.20):

$$\frac{n+1}{2}\, W_2^2(\nu_1,\nu_2) \le H(\nu_1\,|\,\sigma) + H(\nu_2\,|\,\sigma) \tag{3.23}$$

for all symmetric probability measure ν_1 and ν_2 on $\mathbb{S}^n$. Adapting the linearization argument of [26] (see below for a sketch of proof), one can see that (3.23) implies the Poincaré inequality (3.19) for smooth even functions $f:\mathbb{S}^n \to \mathbb{R}$. In comparison, for the same class of functions f, (3.20) only yields to Poincaré's inequality with the sub-optimal constant $\lambda = n+1$, so that (3.13) is a strict improvement of (3.14). As explained above, the constant $2(n+1)$ is sharp, with equality obtained for instance for $f(u) = u_1^2$, $u \in \mathbb{S}^n$.

For the sake of completeness, let us recall how to deduce the Poincaré inequality (3.19) from (3.23).

Proof of (3.23) $\Rightarrow$ (3.19). Let $f:\mathbb{S}^n \to \mathbb{R}$ be a smooth and even function. Without loss of generality, one can also assume that $\int f\, d\sigma = 0$. Bounding the second-order derivatives, one sees there is some constant $C > 0$ such that

$$f(v) \le f(u) + |\nabla_{\mathbb{S}^n} f|(u)\, d_{\mathbb{S}^n}(u,v) + C d_{\mathbb{S}^n}^2(u,v), \qquad \forall u, v \in \mathbb{S}^n.$$

For all $t > 0$, consider $\nu_{1,t} = (1-tf)\sigma$ and $\nu_{2,t} = (1+tf)\sigma$. For all t small enough, $\nu_{1,t}$ and $\nu_{2,t}$ are symmetric probability measures on $\mathbb{S}^n$. If π is an coupling between $\nu_{1,t}$ and $\nu_{2,t}$ for W_2, it holds

$$\begin{aligned}
\int f^2\, d\sigma = \int f\, d\Big(\frac{\nu_{2,t}-\nu_{1,t}}{2t}\Big) &= \frac{1}{2t}\int f(v) - f(u)\, d\pi(u,v)\\
&\le \frac{1}{2t}\int |\nabla_{\mathbb{S}^n} f|(u)\, d_{\mathbb{S}^n}(u,v) + C d_{\mathbb{S}^n}^2(u,v)\, d\pi(u,v)\\
&\le \frac{1}{2t}\Big(\int |\nabla_{\mathbb{S}^n} f|^2\, d\sigma\Big)^{1/2} W_2(\nu_{1,t},\nu_{2,t}) + \frac{C}{2t}\, W_2^2(\nu_{1,t},\nu_{2,t}).
\end{aligned}$$

According to (3.23), it holds

$$\frac{1}{t^2} W_2^2(\nu_{1,t}, \nu_{2,t}) \leq \frac{2}{n+1} \Big(\frac{H(\nu_{1,t} \,|\, \mu)}{t^2} + \frac{H(\nu_{2,t} \,|\, \mu)}{t^2} \Big),$$

and a simple calculation shows that

$$\frac{H(\nu_{i,t} \,|\, \mu)}{t^2} \to \frac{1}{2} \int f^2 \, d\sigma.$$

Therefore,

$$\limsup_{t \to 0} \frac{1}{t^2} W_2^2(\nu_{1,t}, \nu_{2,t}) \leq \frac{2}{n+1} \int f^2 \, d\sigma.$$

Passing to the limit above yields

$$\int f^2 \, d\sigma \leq \frac{1}{2} \Big(\int |\nabla_{\mathbb{S}^n} f|^2 \, d\sigma \Big)^{1/2} \Big(\frac{2}{n+1} \int f^2 \, d\sigma \Big)^{1/2},$$

which amounts to (3.19). ∎

In the following, we derive some simple consequences of inequality (3.13) in terms of measure concentration for symmetric sets of the sphere. Whenever $A, B \subset \mathbb{S}^n$, we will set

$$d_{\mathbb{S}^n}(A, B) = \inf_{x \in A, y \in B} d_{\mathbb{S}^n}(x, y)$$

to denote the distance between A and B.

Corollary 3.12. *Suppose $A, B \subset \mathbb{S}^n$ are two symmetric subsets of $\mathbb{S}^n$. Then $d_{\mathbb{S}^n}(A, B) \leq \pi/2$, and it holds*

$$\sigma(A)\sigma(B) \leq \cos^{n+1}(d_{\mathbb{S}^n}(A, B)). \tag{3.24}$$

Proof. The fact that $d_{\mathbb{S}^n}(A, B) \leq \pi/2$ is obvious. Inequality (3.24) is then immediately derived from the transport entropy inequality (3.13) using a general argument by Marton which is detailed in, e.g., Theorem 10 in [12]. ∎

Remark 3.13. Inequality (3.24) is not always true for general sets A and B such that $d_{\mathbb{S}^n}(A, B) \leq \pi/2$. Indeed, if A and B are two (small enough) spherical caps such that $d_{\mathbb{S}^n}(A, B) = \pi/2$, then inequality (3.24) would imply that $\sigma(A)\sigma(B) = 0$, which is obviously false.

In particular, if A is some symmetric set of $\mathbb{S}^n$ such that $\sigma(A) \geq 1/2$ and $B = \mathbb{S}^n \setminus A_r$, where $0 < r \leq \pi/2$, and $A_r = \{y \in \mathbb{S}^n : d_{\mathbb{S}^n}(y, A) < r\}$ is the r-enlargement of A, it holds

$$\sigma(\mathbb{S}^n \setminus A_r) \leq 2 \cos^{n+1}(r), \quad \forall\, 0 \leq r \leq \pi/2. \tag{3.25}$$

In comparison, for a general set $A \subset \mathbb{S}^n$ such that $\sigma(A) \geq 1/2$, the classical Talagrand inequality (3.21) yields to

$$\sigma(\mathbb{S}^n \setminus A_r) \leq 2 e^{-nr^2/4}, \quad \forall\, 0 \leq r \leq \pi/2, \tag{3.26}$$

and, if A is supposed symmetric, inequality (3.23) gives

$$\sigma(\mathbb{S}^n \setminus A_r) \leq 2e^{-(n+1)r^2/2}, \quad \forall\, 0 \leq r \leq \pi/2. \tag{3.27}$$

Since $\cos(r) \leq e^{-r^2/2}$ for $r \leq 0 \leq \pi/2$, the bound (3.25) is clearly better than bounds (3.26) and (3.27). On the other hand, the classical isoperimetric inequality on $\mathbb{S}^n$ implies that if a general set $A \subset \mathbb{S}^n$ is such that $\sigma(A) \geq 1/2$, then

$$\sigma(\mathbb{S}^n \setminus A_r) \leq \psi_n(r) := \frac{1}{2s_n} \int_r^{\pi/2} \cos^{n-1}(u)\, du, \quad \forall r \geq 0, \tag{3.28}$$

with

$$s_n = \int_0^{\pi/2} \cos^{n-1}(u)\, du$$

(see, e.g., [16]), with equality if A is a spherical cap of measure $1/2$. It is not difficult to see that

$$\frac{\cos^n(r)}{n} \leq \int_r^{\pi/2} \cos^{n-1}(u)\, du \leq \frac{1}{\sin(r)} \frac{\cos^n(r)}{n}, \quad \forall\, 0 < r \leq \pi/2,$$

and $s_n \sim \sqrt{\pi/(2n)}$, so that for any $0 < a < b < \pi/2$,

$$c \frac{\cos^{n+1}(r)}{\sqrt{n}} \leq \psi_n(r) \leq \frac{c'}{\sin(a)\cos(b)} \frac{\cos^{n+1}(r)}{\sqrt{n}}, \quad \forall r \in [a, b],$$

where c and c' are constants independent of a, b and n. Thus for $r \in [a, b]$, the bound (3.25) is off only by a factor of order $1/\sqrt{n}$ from the optimal bound (3.28).

4. Linearization of transport-entropy inequalities

In this section, we show that, by linearizing the transport-entropy inequality (3.3), one recovers the following sharp Brascamp–Lieb type inequality due to Cordero-Erausquin and Rotem [8]. Notice that the same inequality can be also obtained by linearizing the functional Blaschke–Santaló inequality (1.6).

Theorem 4.1. *Assume that $t \mapsto v_\rho(t) = -\log \rho(e^t)$ is convex and increasing. Then, for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ even and such that $\int f\, d\mu_\rho = 0$,*

$$\int f^2\, d\mu_\rho \leq \frac{1}{2} \int \langle H_\rho^{-1} \nabla f, \nabla f \rangle\, d\mu_\rho, \tag{4.1}$$

where the positive matrix H_ρ is given by

$$\frac{1}{2} H_\rho(y) = \frac{1}{|y|^2} \Big[\Big(I_n - \frac{y \otimes y}{|y|^2} \Big) v_\rho'(s) + \frac{y \otimes y}{|y|^2} v_\rho''(s) \Big],$$

where we set $s = 2\log|y|$.

Remark 4.2. This result is exactly the one obtained in Theorem 3 of [8] for the probability μ_ρ. Namely, using the same notation as in [8], if $v_\rho(s) = w(e^{s/2})$, we find

$$2H_\rho(y) = \frac{w'(|y|)}{|y|}\Big(2I_n - \frac{y\otimes y}{|y|^2}\Big) + \frac{y\otimes y}{|y|^2}\, w''(|y|),$$

which is easily seen to be the same matrix as the one appearing in Theorem 3 of [8]. As observed in [8], the Poincaré inequality (4.1) admits non-trivial equality cases, and is therefore sharp. Note however that Theorem 3 in [8] is much stronger than Theorem 4.1 above since it shows that the weighted Poincaré inequality (4.1) is satisfied not only by the model probability measure μ_ρ, but also by any log-concave perturbation of μ_ρ. This raises the question to know if (3.3) is also true for log-concave perturbations of μ_ρ.

Our proof, adapted from [7], relies on a well-known linearization technique involving the following Hopf–Lax operator:

$$RF(y) = \inf_{x\in\mathbb{R}^n}\{F(x) + \omega_\rho(x,y)\}, \quad y\in\mathbb{R}^n, \tag{4.2}$$

where we recall that the cost function ω_ρ is defined by

$$\omega_\rho(x,y) = \begin{cases} \log\Big(\dfrac{\rho(\langle x,y\rangle)^2}{\rho(|x|^2)\rho(|y|^2)}\Big) & \text{if } \langle x,y\rangle > 0,\\ +\infty & \text{otherwise.}\end{cases} \tag{4.3}$$

The following result collects some properties of the cost function ω_ρ, and in particular relates the matrix H_ρ to the behavior of ω_ρ near the diagonal.

Lemma 4.3. *Assume that $\rho\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is nonincreasing, and that $t\mapsto\rho(e^t)$ is log-concave. The cost function ω_ρ defined in (4.3) then satisfies the following:*

(1) $\omega_\rho \geq 0$.

If $t\mapsto\rho(e^t)$ is furthermore assumed to be strictly *log-concave, then*

(2) *if ρ is of class $\mathcal{C}^3$, then for every $y\neq 0$, there exists a symmetric definite positive matrix H_ρ such that*

$$\omega_\rho(y+h,y) = \frac{1}{2}\langle H_\rho h, h\rangle + o(|h|^2) \quad \textit{when } h\to 0;$$

(3) *for every compact subset K and $\rho > 0$, there exists a constant $\eta > 0$ such that, for all $x\in K$ and $y\in\mathbb{R}^n$,*

$$|x-y| > \delta \implies \omega_\rho(x,y)\geq \eta.$$

Remark 4.4. The log-concavity of $t\mapsto\rho(e^t)$ is, in fact, equivalent to the nonnegativity of ω_ρ if ρ is assumed nonincreasing.

Proof. First, note that by monotonicity, for any $x,y\in\mathbb{R}^n$,

$$\omega_\rho(x,y)\geq \log\Big(\frac{\rho(|x|\,|y|)^2}{\rho(|x|^2)\,\rho(|y|^2)}\Big).$$

To prove point (1), it suffices to show that, for any $s, t > 0$,

$$\log\Big(\frac{\rho(e^{s/2}\,e^{t/2})^2}{\rho(e^s)\,\rho(e^t)}\Big) \geq 0.$$

Rewriting this inequality in terms of $v_\rho(t) = -\log(\rho(e^t))$, we find that it is equivalent to

$$v_\rho\Big(\frac{s+t}{2}\Big) \leq \frac{1}{2}\,v_\rho(s) + \frac{1}{2}\,v_\rho(t),$$

which in turn is equivalent to the convexity of v_ρ.

Item (2) is a direct consequence of the computation of the second derivative of $\varphi(h) = \omega_\rho(y, y+h)$ or, in terms of the function v_ρ,

$$\varphi(h) = -2v_\rho(\log(\langle y, y+h\rangle)) + v_\rho(\log(|y|^2)) + v_\rho(\log(|y+h|^2)).$$

We find that

$$\nabla\varphi(0) = 0 \quad \text{and} \quad \nabla^2\varphi(0) = \frac{2}{|y|^2}\Big[\Big(I_n - \frac{y\otimes y}{|y|^2}\Big)v_\rho'(s) + \frac{y\otimes y}{|y|^2}\,v_\rho''(s)\Big] =: H_\rho,$$

where we wrote $|y|^2 = e^s$ for brevity. Strict convexity implies monotonicity of v_ρ, so both matrices appearing in the Hessian are nonnegative. Moreover, the second matrix is positive on the line spanned by y, and the first matrix is positive on its orthogonal, thus their sum must be positive. For future reference, we may rewrite H_ρ in terms of ρ rather than v_ρ:

$$\frac{1}{2}\,H_\rho = -\frac{\rho'(s)}{\rho(s)}\,I_n + \Big(\frac{\rho'^2(s)}{\rho^2(s)} - \frac{\rho''(s)}{\rho(s)}\Big)(y\otimes y).$$

A Taylor expansion yields the formula of item (2).

The last point is an immediate (but useful enough to be stated) consequence of the strict convexity of v_ρ. Notice that $\omega_\rho(x, y) > 0$ whenever $x \neq y$. This is true because the monotonicity and the convexity of ρ are strict. The stated result is then simply the consequence of continuity, if x and y are taken in some compact sets. However, we want a uniform estimate when y is any point in $\mathbb{R}^n$, which is a bit more than we can say with just continuity. Fix $R > 0$. So far, we proved that the property is true for all x, y such that $|x| < R$ and $|y| < 2R$. If $|y| \geq 2R$, then

$$\omega_\rho(x, y) \geq \log\Big(\frac{\rho(|x|\,|y|)^2}{\rho(|x|^2)\rho(|y|^2)}\Big) = -2v_\rho\Big(\frac{s+t}{2}\Big) + v_\rho(s) + v_\rho(t)$$

if we once again write $|x|^2 = e^s$ and $|y|^2 = e^t$. Since v_ρ is convex, v_ρ' is nondecreasing, and we find that

$$\omega_\rho(x, y) \geq -2v_\rho\Big(\frac{\log(R^2) + \log(4R^2)}{2}\Big) + v_\rho(\log(R^2)) + v_\rho(\log(4R^2)) > 0.$$

Combining this estimate at infinity with the local one we had due to continuity, we may conclude. ∎

The next result establishes some Hamilton–Jacobi type (in)equation for $R(\varepsilon f)$ as ε goes to 0.

Lemma 4.5. *Let $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, and assume that ρ is strictly decreasing, and that $t \mapsto \rho(e^t)$ is log-concave. Then*

$$R(\varepsilon f) \geq \varepsilon f - \frac{1}{2}\varepsilon^2 \langle H_\rho^{-1}\nabla f, \nabla f\rangle + o(\varepsilon^2), \quad \text{when } \varepsilon \to 0, \tag{4.4}$$

with

$$\frac{1}{2} H_\rho = -\frac{\rho'(s)}{\rho(s)} I_n + \left(\frac{\rho'^2(s)}{\rho^2(s)}\right) - \frac{\rho''(s)}{\rho(s)}(y \otimes y), \quad s = 2\log|y|. \tag{4.5}$$

Proof. As is usual when linearizing such semigroups, the key is to localize the infimum. Namely, recalling (4.2),

$$R(\varepsilon f)(y) = \inf_{x \in \mathbb{R}^n} \{\varepsilon f(x) + \omega_\rho(x, y)\},$$

if x_ε is a minimizer of this expression, we want to prove that $|x_\varepsilon - y|$ goes to 0 uniformly in y as ε goes to 0. Of course, we must also prove that such a x_ε exists.

We would like the result to be independent from the variable y. To that end, notice that since f has compact support, we may restrict the study to y in a compact subset of $\mathbb{R}^n$. Indeed, notice that, in general, $R(\varepsilon f)(y) \leq \varepsilon f(y)$. Assume more specifically now that $y \in \operatorname{supp}(f)^c$. In that case, $R(\varepsilon f)(y) \leq 0$. Since $\omega_\rho \geq 0$, the infimum in the Hopf–Lax semigroup can only be reached for $x = y$, or for $x \in \operatorname{supp}(f)$. In other words, whenever $y \in \operatorname{supp}(f)^c$,

$$R(\varepsilon f)(y) = \inf_{x \in \mathbb{R}^n} \{\varepsilon f(x) + \omega_\rho(x, y)\} = \min\left(0, \inf_{x \in \operatorname{supp} f} \{\varepsilon f(x) + \omega_\rho(x, y)\}\right).$$

Furthermore, according to point (3) of Lemma 4.3, there exists $\nu > 0$ such that $x \in \operatorname{supp}(f)$ and $|x - y| > 1$ implies that $\omega_\rho(x, y) > \eta$. As such, if $\varepsilon < \eta/\|f\|_\infty$, $d(y, \operatorname{supp}(f)) > \delta$ implies that $R(\varepsilon f)(y) = 0$.

We now restrict our study to some ball B that contains $\operatorname{supp}(f) + B(0, 1)$. Assume that $y \in B$. To make the calculations a little bit clearer, we rewrite (4.2) as

$$R(\varepsilon f)(y) = \inf_{h \in \mathbb{R}^n} \{\varepsilon f(y + h) + \omega_\rho(y + h, y)\}.$$

The immediate estimate $R(\varepsilon f) \leq \varepsilon \|f\|_\infty$ means that, in order to find the infimum, we may restrict h to be in the set

$$\left\{h \in \mathbb{R}^n, \varepsilon f(y + h) + \omega_\rho(y + h, y) \leq \varepsilon \|f\|_\infty\right\} \subset \left\{h \in \mathbb{R}^n, \omega_\rho(y + h, y) \leq 2\varepsilon \|f\|_\infty\right\}.$$

Now, recall that for any $y \in B$,

$$\omega_\rho(y + h, y) = \frac{1}{2}\langle H_\rho h, h\rangle + o(|h|^2),$$

where H_ρ is a continuous (positive definite) function of y, and the remainder term is uniform in y. This implies that there exist $r, \delta > 0$ such that $|h| < r$ implies

$$\omega_\rho(y+h, y) \geq \delta |h|^2.$$

Owing to point (3) of Lemma 4.3, there also exists $\eta' > 0$ such that if $|h| > r$, then

$$\omega_\rho(y+h, y) \geq \eta'.$$

If $2\varepsilon \|f\|_\infty < \eta'$, then $\omega_\rho(y+h, y) \leq 2\varepsilon \|f\|_\infty$ implies that $|h| < r$, and thus

$$R(\varepsilon f)(y) = \inf_{|h|<r} \{\varepsilon f(y+h) + \omega_\rho(y+h, y)\}.$$

The fact that $B(0, r)$ is compact implies the existence of a minimizer h_ε such that

$$R(\varepsilon f)(y) = \varepsilon f(y+h_\varepsilon) + \omega_\rho(y+h_\varepsilon, y).$$

Since $\omega_\rho(y+h, y) \geq \eta|h|^2$, we can already state that $|h_\varepsilon| \leq C\sqrt{\varepsilon}$ for some constant C independent from y, but we can do better. The function f is Lipschitz for some constant $L > 0$. Then,

$$\varepsilon f(y) - \varepsilon L|h_\varepsilon| + \delta|h_\varepsilon|^2 \leq R(\varepsilon f)(y) \leq \varepsilon f(y),$$

and thus $|h_\varepsilon| \leq C'\varepsilon$ for $C' = L/\delta > 0$, which we emphasize is independent from y.

Now that the minimizer h_ε is localized, the rest follows naturally:

$$\begin{aligned} R(\varepsilon f)(y) &= \varepsilon f(y+h_\varepsilon) + \omega_\rho(y+h_\varepsilon, y) \\ &= \varepsilon f(y) + \varepsilon\langle \nabla f(y), h_\varepsilon\rangle + \frac{1}{2}\langle H_\rho h_\varepsilon, h_\varepsilon\rangle + o(\varepsilon^2) \\ &\geq \varepsilon f(y) - \frac{1}{2}\varepsilon^2 \langle H_\rho^{-1}\nabla f(y), \nabla f(y)\rangle + o(\varepsilon^2), \end{aligned}$$

since $\langle H_\rho z, z\rangle \geq 0$, where $z = h_\varepsilon + \varepsilon H_\rho^{-1}\nabla f(y)$. ∎

We are now in a position to prove Theorem 4.1. Let us underline that in order to retrieve the sharp constant in the final inequality, one needs to consider a two sided linearization involving $\mathcal{T}_{\omega_\rho}((1-\varepsilon f)\mu_\rho, (1+\varepsilon f)\mu_\rho)$, rather than $\mathcal{T}_{\omega_\rho}((1+\varepsilon f)\mu_\rho, \mu_\rho)$.

Proof of Theorem 4.1. Choose $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that its integral against μ_ρ is 0, and consider, for $\varepsilon > 0$,

$$\nu_1 = (1+\varepsilon f)\mu_\rho \quad \text{and} \quad \nu_2 = (1-\varepsilon f)\mu_\rho.$$

Linearizing the entropy is straightforward: since

$$(1+\varepsilon)\ln(1+\varepsilon) = \varepsilon^2/2 + o(\varepsilon^2),$$

the right-hand side of inequality (3.3) is equal to

$$H((1+\varepsilon f)\mu_\rho \,|\, \mu_\rho) + H((1-\varepsilon f)\mu_\rho \,|\, \mu_\rho) = \varepsilon^2 \int f^2\, d\mu_\rho + o(\varepsilon^2).$$

For the left-hand side, note that since $R(\varepsilon f)(y) - \varepsilon f(x) \le \omega_\rho(x,y)$,

$$\begin{aligned}\mathcal{T}_{\omega_\rho}(\nu_1,\nu_2) \ge \int R(\varepsilon f)\,d\nu_1 - \int \varepsilon f\,d\nu_2 = \int R(\varepsilon f)(1+\varepsilon f)\,d\mu_\rho \\ - \int \varepsilon f(1-\varepsilon f)\,d\mu_\rho.\end{aligned}$$

Lemma 4.5 applies, and assuming that ε is sufficiently small, the remainder term is uniform and zero outside of a compact. We may integrate it to find

$$\begin{aligned}\mathcal{T}_{\omega_\rho}(\nu_1,\nu_2) &\ge \int \Big(\varepsilon f - \frac{\varepsilon^2}{2}\langle H_\rho^{-1}\nabla f, \nabla f\rangle\Big)(1+\varepsilon f)\,d\mu_\rho - \int \varepsilon f(1-\varepsilon f)\,d\mu_\rho + o(\varepsilon^2) \\ &= \varepsilon^2\Big(-\frac{1}{2}\int \langle H_\rho^{-1}\nabla f, \nabla f\rangle\,d\mu_\rho + 2\int f^2\,d\mu_\rho + o(1)\Big).\end{aligned}$$

Combine these two observations to find that, after dividing by ε^2, letting it go to 0 leads to the claimed Poincaré inequality. ∎

Acknowledgements. We would like to thank the referees for their insightful remarks that helped us to improve the quality of this work.

Funding. The first and second named author are supported by a grant of the Simone and Cino Del Duca Foundation. The third named author was partially supported by the Chateaubriand training research fellowship program, and is also grateful to the Azrieli Foundation for the award of an Azrieli fellowship. The fourth named author has benefited from a post doctoral position funded by the Simone and Cino Del Duca Foundation. This research has been conducted within the FP2M federation (CNRS FR 2036).

References

[1] Artstein-Avidan, S., Klartag, B. and Milman, V.: The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika* **51** (2004), no. 1-2, 33–48. Zbl 1121.52021 MR 2220210

[2] Artstein-Avidan, S. and Milman, V.: The concept of duality for measure projections of convex bodies. *J. Funct. Anal.* **254** (2008), no. 10, 2648–2666. Zbl 1145.26003 MR 2406688

[3] Ball, K.: *Isometric problems in l_p and sections of convex sets*. PhD Thesis, Cambridge, 1986.

[4] Bertrand, J.: Prescription of Gauss curvature using optimal mass transport. *Geom. Dedicata* **183** (2016), 81–99. Zbl 1350.49069 MR 3523119

[5] Blaschke, W.: Vorlesungen über Differentialgeometrie II: Affine Differentialgeometrie. Grundlehren Math. Wiss. 8, Springer, Berlin, 1923. Zbl 49.0499.01 MR 1548296

[6] Bobkov, S. G. and Götze, F.: Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.* **163** (1999), no. 1, 1–28. Zbl 0924.46027 MR 1682772

[7] Cordero-Erausquin, D.: Transport inequalities for log-concave measures, quantitative forms, and applications. *Canad. J. Math.* **69** (2017), no. 3, 481–501. Zbl 1388.60057 MR 3679684

[8] Cordero-Erausquin, D. and Rotem, L.: Improved log-concavity for rotationally invariant measures of symmetric convex sets. *Ann. Probab.* **51** (2023), no. 3, 987–1003. Zbl 07690053 MR 4583060

[9] Erbar, M., Kuwada, K. and Sturm, K.-T.: On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. *Invent. Math.* **201** (2015), no. 3, 993–1071. Zbl 1329.53059 MR 3385639

[10] Fathi, M.: A sharp symmetrized form of Talagrand's transport-entropy inequality for the Gaussian measure. *Electron. Commun. Probab.* **23** (2018), article no. 81, 9 pp. Zbl 1400.35008 MR 3873788

[11] Fradelizi, M. and Meyer, M.: Some functional forms of Blaschke–Santaló inequality. *Math. Z.* **256** (2007), no. 2, 379–395. Zbl 1128.52007 MR 2289879

[12] Gozlan, N.: Characterization of Talagrand's like transportation-cost inequalities on the real line. *J. Funct. Anal.* **250** (2007), no. 2, 400–425. Zbl 1135.46022 MR 2352486

[13] Gozlan, N. and Léonard, C.: Transport inequalities. A survey. *Markov Process. Related Fields* **16** (2010), no. 4, 635–736. Zbl 1229.26029 MR 2895086

[14] Ivanov, G. and Werner, E. M.: Geometric representation of classes of concave functions and duality. *J. Geom. Anal.* **34** (2024), no. 8, article no. 260, 25 pp. Zbl 07880804 MR 4758440

[15] Kolesnikov, A. V.: Mass transportation functionals on the sphere with applications to the logarithmic Minkowski problem. *Mosc. Math. J.* **20** (2020), no. 1, 67–91. Zbl 1460.49037 MR 4060313

[16] Ledoux, M.: *The concentration of measure phenomenon*. Math. Surveys Monogr. 89, American Mathematical Society, Providence, RI, 2001. Zbl 0995.60002 MR 1849347

[17] Lehec, J.: A direct proof of the functional Santaló inequality. *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 1-2, 55–58. Zbl 1159.52010 MR 2536749

[18] Lehec, J.: On the Yao–Yao partition theorem. *Arch. Math. (Basel)* **92** (2009), no. 4, 366–376. Zbl 1185.68784 MR 2501292

[19] Lehec, J.: Partitions and functional Santaló inequalities. *Arch. Math. (Basel)* **92** (2009), no. 1, 89–94. Zbl 1168.39010 MR 2471991

[20] Lutwak, E.: Extended affine surface area. *Adv. Math.* **85** (1991), no. 1, 39–68. Zbl 0727.53016 MR 1087796

[21] Marton, K.: A simple proof of the blowing-up lemma. *IEEE Trans. Inform. Theory* **32** (1986), no. 3, 445–446. Zbl 0594.94003 MR 0838213

[22] Marton, K.: Bounding $\bar{d}$-distance by informational divergence: a method to prove measure concentration. *Ann. Probab.* **24** (1996), no. 2, 857–866. Zbl 0865.60017 MR 1404531

[23] Marton, K.: A measure concentration inequality for contracting Markov chains. *Geom. Funct. Anal.* **6** (1996), no. 3, 556–571. Zbl 0856.60072 MR 1392329

[24] Meyer, M. and Werner, E.: The Santaló-regions of a convex body. *Trans. Amer. Math. Soc.* **350** (1998), no. 11, 4569–4591. Zbl 0917.52004 MR 1466952

[25] Oliker, V.: Embedding $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ with given integral Gauss curvature and optimal mass transport on $\mathbb{S}^n$. *Adv. Math.* **213** (2007), no. 2, 600–620. Zbl 1233.49024 MR 2332603

[26] Otto, F. and Villani, C.: Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173** (2000), no. 2, 361–400. Zbl 0985.58019 MR 1760620

[27] Rotem, L.: A sharp Blaschke–Santaló inequality for α-concave functions. *Geom. Dedicata* **172** (2014), 217–228. Zbl 1320.52010 MR 3253780

[28] Santaló, L. A.: An affine invariant for convex bodies of n-dimensional space. *Portugal. Math.* **8** (1949), 155–161. Zbl 0038.35702 MR 0039293

[29] Talagrand, M.: Transportation cost for Gaussian and other product measures. *Geom. Funct. Anal.* **6** (1996), no. 3, 587–600. Zbl 0859.46030 MR 1392331

[30] Villani, C.: *Optimal transport. Old and new*. Grundlehren Math. Wiss. 338, Springer, Berlin, 2009. Zbl 1156.53003 MR 2459454

Received July 15, 2023; revised March 8, 2024.

Matthieu Fradelizi
Université Gustave Eiffel, Université Paris Est Creteil, CNRS, LAMA UMR8050
5 Bd. Descartes, 77447 Marne-la-Vallée, France;
matthieu.fradelizi@univ-eiffel.fr

Nathaël Gozlan
MAP5 (UMR 8145), Université Paris Cité
45 rue des Saints-Pères, 75006 Paris, France;
nathael.gozlan@u-paris.fr

Shay Sadovsky
School of Mathematical Sciences, Tel Aviv University
69978 Tel Aviv, Israel;
shaysadovsky@mail.tau.ac.il

Simon Zugmeyer
Département de mathématiques, ENS de Lyon, UMPA (UMR 5669)
46 Allée d'Italie, 69364 Lyon Cedex 07, France;
simon.zugmeyer@ens-lyon.fr