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Bi-Lipschitz arcs in metric spaces with controlled geometry

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Abstract. In this paper, we generalize a bi-Lipschitz extension result of David and Semmes from Euclidean spaces to complete metric measure spaces with controlled geometry (Ahlfors regularity and supporting a Poincaré inequality). In particular, we find sharp conditions on metric measure spaces X so that any bi-Lipschitz embedding of a subset of the real line into X extends to a bi-Lipschitz embedding of the whole line. Along the way, we prove that if the complement of an open subset Y of X has small Assouad dimension, then it is a uniform domain. Finally, we prove a quantitative approximation of continua in X by bi-Lipschitz curves.

1. Introduction

Given metric spaces (X, d_X) and (Y, d_Y), a map $f: X \to Y$ is said to be *an L-bi-Lipschitz embedding* (or simply *L-bi-Lipschitz*, or just *bi-Lipschitz*) if there is a constant $L \geq 1$ such that

$$L^{-1} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq L\, d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$. A *bi-Lipschitz arc* in a metric space X is the image of an interval in the real line $\mathbb{R}$ under a bi-Lipschitz map.

We will consider the following question: given a set $E \subset X$ which is the image of a subset of $\mathbb{R}$ under a bi-Lipschitz map, is E contained in a bi-Lipschitz arc? If E is any finite subset of $\mathbb{R}^n$, the answer is trivially "yes". For general sets $E \subset \mathbb{R}^n$, the question was answered in the positive when $n \geq 3$ by the following extension theorem of David and Semmes [6].

Theorem 1.1 (Proposition 17.1 in [6]). *Let $n \geq 3$ be an integer, let $A \subset \mathbb{R}$, and let the function $f: A \to \mathbb{R}^n$ be a bi-Lipschitz embedding. Then there exists a bi-Lipschitz extension $F: \mathbb{R} \to \mathbb{R}^n$.*

MacManus [24] extended the result of David and Semmes to the case $n = 2$, which is much more difficult since intersecting lines in $\mathbb{R}^3$ may be easily modified so that they no longer intersect, but this is not the case in $\mathbb{R}^2$. One may view these extension results as

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rougher versions of the classical Whitney extension theorem [35]; while the maps considered here are analytically weaker (as they are bi-Lipschitz rather than differentiable), they are metrically and topologically stronger.

Theorem 1.1 is a special case of a more general result in [6], where $A \subset \mathbb{R}^d$ and $n \geq 2d + 1$. The main motivation behind that result was to establish the equivalence of the boundedness of certain singular operators on $\mathbb{R}^n$ via quantitative rectifiability. More precisely, Theorem 1.1 was used in [6] to show that, when $n \geq 3$, every Ahlfors 1-regular set $A \subset \mathbb{R}^n$ (see (2.1) for the definition of Ahlfors regularity) which admits a corona decomposition (roughly speaking, A can be decomposed into a collection of subsets which are well-approximated by Lipschitz graphs and a collection of subsets which are not, and both of these collections have controlled measure) contains "big pieces" of bi-Lipschitz arcs, i.e., for any $\varepsilon > 0$, there exists an $M > 0$ such that, for any $x \in A$ and any $R > 0$, there is an M-bi-Lipschitz embedding $\rho\colon \mathbb{R} \to \mathbb{R}^n$ such that

$$|E \cap (B(x, R) \setminus \rho(\mathbb{R}))| \leq \varepsilon R.$$

Another application of Theorem 1.1 is in the problem of the *bi-Lipschitz rectifiability* of sets in Euclidean spaces. In other words, one hopes to classify those subsets of $\mathbb{R}^n$ that are contained in a bi-Lipschitz arc. While the classical characterization of the Lipschitz rectifiability of sets in Euclidean spaces has been completely resolved [17, 27], the problem of bi-Lipschitz rectifiability remains open mainly due to topological constraints. Theorem 1.1 can be used to show that, if a set $E \subset \mathbb{R}^n$ has Assouad dimension less than 1, then E is bi-Lipschitz rectifiable; see Corollary 3.5 in [1] for a different approach. See Section 2 for the definition of the Assouad dimension.

In this article, we generalize Theorem 1.1 to the setting in which Euclidean spaces $\mathbb{R}^n$ are replaced by a large class of metric measure spaces. There are two main difficulties in this generalization. Firstly, the target metric space X must contain many of rectifiable curves, and this notion of "many" must be understood quantitatively. A notable example (and, in fact, the initial motivation for this project) is the *Heisenberg group* $\mathbb{H}$, in which the classical Whitney extension theorem for curves has been well-studied recently; see [28, 30, 36, 37]. We will not define the Heisenberg group here, but only recall that it is a geodesic space homeomorphic to $\mathbb{R}^3$, and there exists a distribution $H\colon \mathbb{R}^3 \to \mathbf{Gr}(2, \mathbb{R}^3)$ such that if a curve $\gamma\colon [0, 1] \to \mathbb{H}$ is rectifiable, then it is differentiable almost everywhere and $\dot{\gamma}(t) \in H_{\gamma(t)}$ for almost every t. This fact implies that there must be many fewer rectifiable curves in $\mathbb{H}$ than in $\mathbb{R}^3$. Secondly, the proof in the Euclidean case relies on the existence of differentiable bump functions $\phi\colon \mathbb{R} \to \mathbb{R}^n$ with controlled derivatives, and we cannot hope to recover this idea in a general metric space.

The class of metric measure spaces to which the bi-Lipschitz extension result will be generalized will have two properties. The first is *Ahlfors regularity*: we say that a metric measure space (X, d, μ) is Ahlfors Q-regular (or simply Q-regular) if the measure of any ball of radius r is comparable to r^Q. The second property is the existence of a *Poincaré inequality*. Such an inequality roughly states that, if we use u_B to denote the average value of a function $u\colon X \to \mathbb{R}$ on a ball B, then the average of the variation $|u - u_B|$ is controlled by the average of a "weak derivative" of u on B. See Section 2 for all relevant definitions. It is known that Ahlfors regular spaces supporting a Poincaré inequality must contain quantitatively many rectifiable curves. Moreover, such spaces admit a notion of differentiation [5].

The following is the main result of this paper.

Theorem 1.2. *Let (X,d,μ) be a Q-regular, complete metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q-1$. If $A \subset \mathbb{R}$ and $f\colon A \to X$ is a bi-Lipschitz embedding, then f extends to a bi-Lipschitz embedding $F\colon I \to X$, where I is the smallest closed interval containing A.*

In Theorem 6.1, we prove a stronger quantitative version of this result in the sense that the bi-Lipschitz constant of F depends only the bi-Lipschitz constant of f and on the data of Ahlfors Q-regularity and the Poincaré inequality. Moreover, if X is unbounded, then we can choose $I = \mathbb{R}$.

A large variety of metric spaces satisfy the assumptions of Theorem 1.2, including orientable, n-regular, linearly locally contractible n-manifolds with $n \geq 3$ ([29]), Carnot groups ([16, 34]) (which include Euclidean spaces and the Heisenberg group), certain hyperbolic buildings [3], Laakso spaces ([20]), and certain Menger sponges ([8, 23]).

The assumptions of the theorem are sharp in that neither Ahlfors regularity nor the Poincaré inequality can be removed from the statement. For Ahlfors regularity, let $X = \mathbb{S}^2 \times \mathbb{R}$, with the length metric and the induced Hausdorff 3-measure. Then X is complete, has Ricci curvature bounded from below so it satisfies the 1-Poincaré inequality (see Chapter VI.5 of [4]), but is not Ahlfors regular. Define $f\colon \{2^n : n \in \mathbb{N}\} \to X$ by $f(2^n) = (p_0, (-2)^n)$, where $p_0 \in \mathbb{S}^2$. The map f is bi-Lipschitz, and if $F\colon \mathbb{R} \to X$ is any homeomorphic extension of f, then for any $n \in \mathbb{N}$, $F([2^n, 2^{n+1}])$ intersects with $(\mathbb{S}^2 \times \{0\})$, so F cannot be bi-Lipschitz.

Since the Poincaré inequality is an open ended condition [19], we may assume that $p < Q-1$ for the proof of the theorem. However, the bound $Q-1$ is sharp. To see this, let $n \geq 2$, let P_1 and P_2 be two n-dimensional planes in $\mathbb{R}^{2n-1}$ intersecting on a line ℓ, and let $p_0 \in \ell$. The metric space $X = (P_1 \cup P_2) \setminus B(p_0, 1)$ with the induced Euclidean metric and n-dimensional Lebesgue measure is complete, n-regular, and satisfies the p-Poincaré inequality for all $p > n-1$, see Theorem 6.15 in [10]. Let $f\colon (-\infty, -1] \cup \{-1/2, 1/2\} \cup [1, \infty) \to X$ be a map such that $f(-1/2) \in P_1 \setminus (\ell \cup B(p_0, 1))$, $f(1/2) \in P_2 \setminus (\ell \cup B(p_0, 1))$, and f maps $\mathbb{R} \setminus (-1, 1)$ isometrically onto $\ell \setminus B(p_0, 1)$. Then f is bi-Lipschitz but admits no homeomorphic (let alone bi-Lipschitz) extension $F\colon \mathbb{R} \to X$.

1.1. Related results

The first corollary of Theorem 1.2 gives a sufficient condition for bi-Lipschitz rectifiability in Ahlfors regular spaces satisfying a Poincaré inequality.

Corollary 1.3. *Let X be a complete Q-regular metric measure space which supports a p-Poincaré inequality for some $1 < p \leq Q-1$. If $E \subset X$ has Assouad dimension less than 1, then E is bi-Lipschitz rectifiable.*

The proof of the corollary follows the same ideas as in the Euclidean case. Since the Assouad dimension of E is less than 1, Lemma 15.2 in [7] implies that E must be uniformly disconnected, and hence it is bi-Lipschitz equivalent to an ultrametric space Z of Assouad dimension less than 1, see Proposition 15.7 in [7]. By Theorem 3.8 in [21], there exists a bi-Lipschitz embedding $g\colon E \to \mathbb{R}$, and, by Theorem 1.2, there exist a closed interval I and a bi-Lipschitz extension $f: I \to X$ of the map $g^{-1}: g(E) \to X$. Thus $E \subset f(I)$, so E is contained in a bi-Lipschitz arc.

The proof of Theorem 1.2 has two main ingredients. The first is the construction of short curves in $X \setminus f(A)$ that stay quantitatively far from $f(A)$. To build such curves, we will use the notion of the uniformity of a set. Given a set $U \subset X$, we say that U is *c-uniform* if, for every $x, y \in U$, there exists a path $\gamma: [0,1] \to U$ joining x to y such that

(1) the length of γ is at most $cd(x, y)$, and
(2) $\operatorname{dist}(\gamma(t), X \setminus U) \geq c^{-1} \operatorname{dist}(\gamma(t), \{x, y\})$ for all $t \in [0,1]$.

In other words, U is uniform if, for any $x, y \in U$, there exists a curve connecting them which is short compared to $d(x, y)$ and stays far from $X \setminus U$ quantitatively. If U satisfies only property (1) in this definition, then we say that U is *c-quasiconvex*.

It is an open problem to classify the closed sets $Y \subset X$ for which $X \setminus Y$ is quasiconvex or uniform. Hakobyan and Herron [9] showed that, if $Y \subset \mathbb{R}^n$ has Hausdorff $(n-1)$-measure $\mathcal{H}^{n-1}(Y) = 0$, then $\mathbb{R}^n \setminus Y$ is quasiconvex. Moreover, this assumption is sharp. Herron, Lukyanenko, and Tyson [14] proved the same result in the Heisenberg group $\mathbb{H}$ where, in this setting, it is assumed that $\mathcal{H}^3(Y) = 0$. The dimension 3 is natural as $\mathbb{H}$ is 4-regular, while $\mathbb{R}^n$ is n-regular. It is unknown if a similar result exists in all Carnot groups.

The question of whether $X \setminus Y$ is uniform has been studied in terms of uniform disconnectedness of Y, [25], and quasihyperbolicity of X and Y, [12, 13, 15]. Väisälä [33] showed that, if $\mathbb{R}^n \setminus Y$ is uniform, then the topological dimension of Y is at most $n-2$. The following proposition, which we prove in Section 3, works in the opposite direction: if X is Ahlfors regular and supports a Poincaré inequality, and if the Assouad dimension of Y is small, then $X \setminus Y$ is uniform.

Proposition 1.4. *Let (X, d, μ) be a complete Q-Ahlfors regular metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q$. If $Y \subset X$ is a closed set with Assouad dimension less than $Q - p$, then $X \setminus Y$ is a uniform domain.*

Note that if $Y \subset X$ and has Assouad dimension less than $Q - p$, then $\mathcal{H}^{Q-p}(Y) = 0$. The assumption on the Assouad dimension is sharp. For example, let $X = \mathbb{R}^n$, let P be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$, and let Y be a maximal 1-separated subset of P. Then it is easy to see that $\dim_A(Y) = n - 1$, $\mathcal{H}^{n-1}(Y) = 0$, and $\mathbb{R}^n \setminus Y$ is not uniform.

The second ingredient in the proof of Theorem 1.2 is a standard "straightening" argument for paths. In particular, Lytchak and Wenger (Lemma 4.2 in [22]) proved that, given any topological arc in a geodesic space, there exists a bi-Lipschitz arc with the same endpoints that is close to the original one; see also Lemma 4.2 in [26] for a similar result for topological circles. In Section 4, we prove a quantitative version of their result. Moreover, under the additional assumptions of Q-regularity and a Poincaré inequality, we show as a corollary of Theorem 1.2 that every continuum (i.e., every compact connected set) can be approximated by a bi-Lipschitz curve in the Hausdorff distance.

Proposition 1.5. *Let (X, d, μ) be a complete Q-regular metric measure space supporting a p-Poincaré inequality for some $1 < p \leq Q - 1$, let $K \subset X$ be a continuum, and let $\varepsilon \in (0, 1)$. For any $x, y \in K$ with $d(x, y) \geq \varepsilon \operatorname{diam} K$, there exists a curve $\gamma: [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, and there exists a constant $L \geq 1$ depending only on ε, the* *constants of Q-regularity, and the data of the Poincaré inequality, such that*

$$\frac{1}{L}|s-t|\operatorname{diam} K \le d(\gamma(t),\gamma(s)) \le L|s-t|\operatorname{diam} K$$

for all $s,t \in [0,1]$, and the Hausdorff distance $\operatorname{dist}_H(K,\gamma([0,1])) \le \varepsilon \operatorname{diam} K$.

In particular, we have that every compact Ahlfors regular metric measure space supporting a Poincaré inequality contains "almost space-filling" bi-Lipschitz curves.

1.2. Outline of the proof of Theorem 1.2

We start with two simple reductions. First, since bi-Lipschitz maps extend on the completion of their domain, we may assume that A is a closed set. Second, it is well known that the Poincaré inequality, completeness, and Ahlfors regularity imply that X is quasiconvex (Theorem 17.1 in [5]). Every complete quasiconvex space is bi-Lipschitz equivalent to a geodesic metric space and since the properties of Ahlfors Q-regularity and the p-Poincaré inequality are preserved under bi-Lipschitz mappings (Lemma 8.3.18 in [11]), we may assume for the rest that X is geodesic.

For the proof of Theorem 1.2, similar to the proof of Theorem 1.1 and the Whitney extension theorem, we construct a *Whitney decomposition* $\{Q_i\}_{i\in\mathbb{N}}$ of $I\setminus A$, i.e., a collection of closed intervals in $I\setminus A$ with mutually disjoint interiors such that their union is $I\setminus A$ and the length of each interval is comparable to its distance from A.

In Section 5, we define two auxiliary embeddings. Specifically, in Section 5.1 we construct a bi-Lipschitz embedding π of E into X, where E is the set of endpoints of the Whitney intervals Q_i. The final map F will map elements of E very close to their image under π. In Section 5.2, we use the results of Sections 3 and 4 to define a second bi-Lipschitz embedding

$$g : A \cup \bigcup_{i\in\mathbb{N}} \hat{Q}_i \to X$$

of f. Here, $\hat{Q}_i$ denotes the middle third closed interval in Q_i. If we write $Q_i = [x,y]$, then the image $g(\hat{Q}_i)$ is a bi-Lipschitz curve that has endpoints very close to $\pi(x)$ and $\pi(y)$.

In Section 6, we describe a method to modify and extend the map g near the points $\pi(x)$ to build a curve on the entire interval I, and we verify that this curve is indeed bi-Lipschitz to complete the proof of Theorem 1.2.

2. Preliminaries

Given quantities $x,y \ge 0$ and constants $a_1,\dots,a_n > 0$, we write $x \lesssim_{a_1,\dots,a_n} y$ if there exists a constant C depending at most on $a_1,\dots,a_n$ such that $x \le Cy$. If C is universal, we write $x \lesssim y$. We write $x \simeq_{a_1,\dots,a_n} y$ if $x \lesssim_{a_1,\dots,a_n} y$ and $y \lesssim_{a_1,\dots,a_n} x$.

Given a metric space (X,d) and two points $x,y\in X$, we say that γ is a *path joining x with y* if there exists some continuous $\gamma\colon [0,1]\to X$ with $\gamma(0)=x$ and $\gamma(1)=y$.

Given a set $Y\subset X$ and $r>0$, we write $B(Y,r) := \{x\in X : \operatorname{dist}(x,Y) < r\}$.

2.1. Porosity and regularity

For a constant $C > 1$, a metric space X is called C*-doubling* if every ball of radius r can be covered by at most C balls of radii at most $r/2$. Given another constant $\alpha \geq 0$, X is called (C,α)*-homogeneous* if every ball of radius R can be covered by at most $C(R/r)^{\alpha}$ balls of radii at most r. We will occasionally refer to such a metric space as α*-homogeneous* when the constant C is not important. Clearly, a (C,α)-homogeneous space is $(C2^{\alpha})$-doubling. Conversely, given $C > 0$, there exist $C' > 0$ and $\alpha > 0$ such that a C-doubling space is (C',α)-homogeneous.

The *Assouad dimension* of a metric space X (denoted $\dim_A(X)$) is the infimum of all $\alpha \geq 0$ such that X is α-homogeneous.

A metric measure space (X, d, μ) is said to be Q*-Ahlfors regular* (or Q*-regular*) for $Q \geq 0$ if there exists $C \geq 1$ such that, for all $x \in X$ and all $r \in (0, \operatorname{diam} X)$,

$$C^{-1}r^{Q} \leq \mu(B(x,r)) \leq Cr^{Q}. \tag{2.1}$$

It is easy to see that if the space (X, d, μ) is Q-regular, then X is Q-homogeneous and $\dim_A(X) = Q$. If we want to emphasize the constant C in (2.1), then we say that (X, d, μ) is (C, Q)*-regular*.

Given $Y \subset X$, we say that Y is p*-porous* for some $p \geq 1$ if, for all $y \in Y$ and all $r \in (0, \operatorname{diam} X)$, there exists some $x \in B(y,r)$ such that $B(x, r/p) \subset B(y,r) \setminus Y$. In other words, Y contains relatively large "holes" near every point.

Lemma 2.1 (Lemma 3.12 in [2]). *Let $(X, d, \mathcal{H}^{Q})$ be (C, Q)-regular, where $\mathcal{H}^{Q}$ is the Q-dimensional Hausdorff measure. A set $Y \subset X$ is p-porous for some $p \geq 1$ if and only if $\dim_A(Y) \leq Q - \varepsilon$ for some $\varepsilon > 0$. Here, ε and p depend only on each other, Q, and C.*

2.2. Poincaré inequality

Given a locally Lipschitz function u defined on a metric space (X, d), we say that a function $g\colon X \to [0, \infty)$ is an *upper gradient* of u if

$$|u(x) - u(y)| \leq \int_{\gamma} g \, \mathrm{d}s$$

for all $x, y \in X$ and all paths γ in X joining x with y.

We say that a metric measure space (X, d, μ) supports a $(1, p)$*-Poincaré inequality* (or simply a p*-Poincaré inequality*) for some $1 \leq p < \infty$ if there exist $\lambda \geq 1$ and $C > 1$ with the following property: if $u\colon X \to \mathbb{R}$ is locally Lipschitz and $g\colon X \to [0, \infty)$ is an upper gradient of u, then, for all $x \in X$ and $r > 0$,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq C \operatorname{diam}(B(x, \lambda r)) \Big(\fint_{B(x,\lambda r)} g^{p} \, \mathrm{d}\mu \Big)^{1/p}, \tag{2.2}$$

where

$$\fint_A f \, \mathrm{d}\mu = \frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu \quad \text{and} \quad u_{B(x,r)} = \fint_{B(x,r)} u \, \mathrm{d}\mu.$$

It follows from Hölder's inequality that if $1 \leq p \leq q$ and (X, d, μ) satisfies a p-Poincaré inequality, then it satisfies a q-Poincaré inequality. Moreover, if the space is geodesic and doubling, then one can choose $\lambda = 1$; see for example Remark 9.1.19 in [11]. Henceforth, given a geodesic doubling space X that satisfies the p-Poincaré inequality, we will assume that $\lambda = 1$ in (2.2), and the constant C will be called the *data* of the Poincaré inequality.

For a detailed exposition on the Poincaré inequality on metric measure spaces, the reader is referred to [11].

2.3. Modulus of curve families

The basic tool in the proof of Theorem 1.2 and Proposition 1.4 is the notion of the modulus of curves. In a sense, the modulus is a measurement of "how many" rectifiable curves are contained in a curve family.

Given a family Γ of rectifiable curves in a metric measure space (X, d, μ), we say that a Borel function $\rho\colon X \to [0,\infty)$ is admissible for Γ if

$$\int_\gamma \rho \,\mathrm{d}s \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

For $p \geq 1$, we define the *p-modulus* of Γ by

$$\mathrm{Mod}_p(\Gamma) := \inf\Big\{\int_X \rho^p \,\mathrm{d}\mu : \rho \text{ is admissible for } \Gamma\Big\}.$$

It is well known that Mod_p is an outer measure on the space of all curve families in X.

The next lemma relates the modulus of curve families with the *locally Lipschitz capacity* between compact sets. Given two sets E and F in a metric space, we say that a curve γ joins E with F if there are points $x \in E$ and $y \in F$ such that γ joins x with y.

Lemma 2.2 (Theorem 1.1 in [18]). *Suppose that (X, d, μ) is a geodesic metric measure space equipped with a doubling measure μ and supporting a p-Poincaré inequality with $p > 1$, and suppose that Ω is a domain in X. Let E and F be disjoint, compact, non-empty subsets of Ω, and let Γ be the collection of curves in Ω that join E with F. Then the p-modulus of Γ is equal to the p-capacity of E and F:*

$$\mathrm{Mod}_p(\Gamma) = \mathrm{Cap}_p(E, F) := \inf \int_\Omega g^p \,\mathrm{d}\mu\,;$$

the infimum is taken over all Borel functions $g\colon \Omega \to [0,\infty)$ such that each g is an upper gradient of some locally Lipschitz function $u\colon \Omega \to \mathbb{R}$ satisfying $u|_E \geq 1$ and $u|_F \leq 0$.

3. Uniformity in metric measure spaces

The goal of this section is the proof of Proposition 1.4. The next lemmas are the crux of the proof.

Lemma 3.1. *Let (X, d, μ) be a (C_1, Q)-Ahlfors regular geodesic metric measure space supporting a p-Poincaré inequality with data C, for some $C, C_1 \geq 1$, and $1 < p < Q$. Let $x, y \in X$, let $r \in (0, d(x,y)/3)$, and let Γ be the collection of paths in $B(x, 2d(x,y))$ that connect $\overline{B}(x,r)$ with $\overline{B}(y,r)$. Then*

$$\mathrm{Mod}_p(\Gamma) \gtrsim_{p,C,C_1,Q} (d(x,y))^{Q-p}\Big(\frac{d(x,y)}{r}\Big)^{-Qp}.$$

Proof. Set $D = d(x, y)$. Let $u\colon B(x, 2D) \to \mathbb{R}$ be a locally Lipschitz function satisfying $u|_{B(x,r)} \geq 1$ and $u|_{B(y,r)} \leq 0$. Let also $g\colon B(x, 2D) \to [0, \infty)$ be an upper gradient of u. By the p-Poincaré inequality,

$$\begin{aligned}
&\int_{B(x,2D)} g^p \, \mathrm{d}\mu \\
&\quad \geq \frac{\mu(B(x,2D))}{C^p[\operatorname{diam}(B(x,2D))\,\mu(B(x,2D))]^p}\Big(\int_{B(x,2D)} |u - u_{B(x,2D)}| \, \mathrm{d}\mu\Big)^p \\
&\quad \gtrsim_{p,C,C_1,Q} D^{Q-p-Qp}\Big(\int_{B(x,r)\cup B(y,r)} |u - u_{B(x,2D)}| \, \mathrm{d}\mu\Big)^p \\
&\quad \geq D^{Q-p-Qp}\Big(\frac{1}{2}\min\{\mu(B(x,r)), \mu(B(y,r))\}\Big)^p \gtrsim_{p,C_1} D^{Q-p}\Big(\frac{D}{r}\Big)^{-Qp}.
\end{aligned}$$

Denote by Γ the collection of curves joining $\overline{B}(x, r)$ with $\overline{B}(y, r)$. By Lemma 2.2,

$$\operatorname{Mod}_p(\Gamma) \gtrsim_{p,C,C_1,Q} D^{Q-p}\left(\frac{D}{r}\right)^{-Qp}. \qquad \blacksquare$$

Lemma 3.2. *Let (X, d, μ) be a (C_1, Q)-Ahlfors regular metric measure space, let $R > 0$, let $\ell > 0$, and let Γ be the collection of paths in $B(x, R)$ that have length at least ℓR. Then,*

$$\operatorname{Mod}_p(\Gamma) \lesssim_{C_1} \ell^{-p} R^{Q-p}.$$

Proof. Note that the function $\rho = (\ell R)^{-1}\chi_{B(x,R)}$ is admissible for Γ. Therefore,

$$\operatorname{Mod}_p(\Gamma) \leq \int_X \rho^p \, \mathrm{d}\mu \leq C_1 \ell^{-p} R^{Q-p}. \qquad \blacksquare$$

Lemma 3.3. *Let (X, d, μ) be a (C_1, Q)-Ahlfors regular metric measure space, let $Y \subset X$ be a (C_2, α)-homogeneous set, let $R > 0$, let $\delta > 0$, and let Γ be the collection of paths in $B(x, R)$ with an endpoint outside of $B(Y, 2\delta R)$ and which intersect $B(Y, \delta R)$. Then*

$$\operatorname{Mod}_p(\Gamma) \lesssim_{Q,C_1,C_2} \delta^{Q-p-\alpha} R^{Q-p}.$$

Proof. Define the function

$$\rho := (\delta R)^{-1}\chi_{B(Y,2\delta R)\cap B(x,R)}$$

and note that ρ is admissible for Γ. Indeed, if $\gamma \in \Gamma$, then the total length of the part of γ that is inside $B(Y, 2\delta R)$ must be at least δR.

If V is a (δR)-net of $Y \cap B(x, R)$, then

$$B(Y, 2\delta R) \cap B(x, R) \subset \bigcup_{v\in V} B(v, 3\delta R),$$

and, by the homogeneity of Y, it follows that $\operatorname{card}(V) \lesssim_{C_2} \delta^{-\alpha}$. Therefore

$$\operatorname{Mod}_p(\Gamma) \leq \int_X \rho^p \, \mathrm{d}\mu \lesssim_{Q,C_1,C_2} \delta^{Q-p-\alpha} R^{Q-p}. \qquad \blacksquare$$

Corollary 3.4. *Let (X, d, μ) be a (C_1, Q)-regular, geodesic metric measure space supporting a p-Poincaré inequality with $1 < p < Q$ and data C. Let $Y \subset X$ be a (C_2, α)-homogeneous set with $0 < \alpha < Q - p$. Given $x, y \in X \setminus Y$, there exists a path $\gamma\colon [0,1] \to X \setminus Y$ such that $\gamma(0) = x$, $\gamma(1) = y$,*

(1) $\gamma([0,1]) \subset B(x, 2d(x,y))$,

(2)
$$\mathrm{length}(\gamma) \lesssim_{p,C,C_1,Q} d(x,y) \max\left\{1, \left(\frac{d(x,y)}{\mathrm{dist}(\{x,y\},Y)}\right)^{Q}\right\},$$

(3) *for all z in the image of γ,*
$$\mathrm{dist}(z,Y) \gtrsim_{p,\alpha,Q,C,C_1,C_2} d(x,y) \min\left\{1, \left(\frac{\mathrm{dist}(\{x,y\},Y)}{d(x,y)}\right)^{\frac{Qp+Q-p-\alpha}{Q-p-\alpha}}\right\}.$$

Proof. Set $D := d(x,y)$ and
$$r := \frac{1}{4}\min\{D, \mathrm{dist}(\{x,y\},Y)\}.$$
Let Γ_1 be the collection of all curves in $B(x,2D)$ that join $\overline{B}(x,r)$ to $\overline{B}(y,r)$. Let Γ_ℓ be the collection of all curves in $B(x,2D)$ that have length at least $2D\ell$. Let Γ'_δ be the collection of all curves in $B(x,2D)$ that intersect a $(2D\delta)$-neighborhood of Y and have length at least $2D\delta$.

By Lemma 3.1, Lemma 3.2, and Lemma 3.3, there exist
$$\ell \simeq_{p,C,C_1,Q} \left(\frac{D}{r}\right)^{Q} \quad \text{and} \quad \delta \simeq_{p,\alpha,Q,C,C_1,C_2} \left(\frac{r}{D}\right)^{\frac{Qp}{Q-p-\alpha}}$$
such that
$$\mathrm{Mod}_p(\Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)) > 0.$$
It follows that $\Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)$ is non-empty. Fix now $\gamma \in \Gamma \setminus (\Gamma_l \cup \Gamma'_\delta)$ and concatenate γ with geodesic segments $[x, \gamma(0)]$ and $[\gamma(1), y]$. The resulting curve satisfies the conclusions of the corollary. ∎

Proof of Proposition 1.4. By Lemma 2.1, the regularity of X, and the homogeneity of Y, there exists $p_0 > 1$ such that Y is p_0-porous.

Fix now $x, y \in X \setminus Y$ and denote $r := d(x,y)$. There exists $z_0 \in B(x,r) \setminus (B(x, 2^{-1}r) \cup B(y, 2^{-1}r))$ such that
$$B(z_0, 2^{-1}r/p_0) \subset B(x,r) \setminus \big[(B(x,2^{-1}r) \cup B(y,2^{-1}r) \cup Y)\big]$$
by applying porosity of Y to a ball of radius $2^{-1}r$ contained in
$$B(x,r) \setminus \big[(B(x,2^{-1}r) \cup B(y,2^{-1}r))\big].$$
Moreover, for each $n \in \mathbb{N}$, there exist points $z_n \in B(x, 2^{-n}r) \setminus B(x, 2^{-n-1}r)$ and $z_{-n} \in B(y, 2^{-n}r) \setminus B(y, 2^{-n-1}r)$ such that
$$B(z_n, 2^{-n-1}r/p_0) \subset B(x, 2^{-n}r) \setminus (B(x, 2^{-n-1}r) \cup Y),$$
$$B(z_{-n}, 2^{-n-1}r/p_0) \subset B(y, 2^{-n}r) \setminus (B(y, 2^{-n-1}r) \cup Y),$$
again by applying the porosity of Y to balls in the annuli $B(x, 2^{-n}r) \setminus B(x, 2^{-n-1}r)$ and $B(y, 2^{-n}r) \setminus B(y, 2^{-n-1}r)$.

Applying Corollary 3.4, there exists $c > 1$ depending only on p_0, p, Q, C, and C_1 such that, for each $n \in \mathbb{Z}$, there exists a path $\gamma_n\colon [0,1] \to X \setminus Y$ with

(1) $\gamma_n(0) = z_n$, $\gamma_n(1) = z_{n+1}$,

(2) $\operatorname{length}(\gamma_n) \le c d(z_n, z_{n+1}) \le 2^{3-|n|} cr$, and

(3) for all $t \in [0,1]$, $\operatorname{dist}(\gamma_n(t), Y) \ge c^{-1} 2^{-2-|n|} r$.

Concatenating all the paths $\{\gamma_n\}_{n\in\mathbb{Z}}$ and adding the points x and y, we obtain a path $\gamma\colon [0,1] \to X \setminus Y$. Note that

$$\operatorname{length}(\gamma) = \sum_{n\in\mathbb{Z}} \operatorname{length}(\gamma_n) \le \sum_{n\in\mathbb{Z}} 2^{3-|n|} cr = 24cr = 24cd(x,y).$$

Let now $z \in \gamma([0,1])$. If z is either of x or y, then there is nothing to show. Otherwise, there exists $n \in \mathbb{Z}$ such that z is in the image of γ_n. Assume as we may that $n \ge 0$. Then

$$d(x,z) \le d(z_n, x) + d(z_n, z) \le (8c+1)\, 2^{-n} r \le 4c(8c+1) \operatorname{dist}(z, Y),$$

which completes the proof. ∎

4. Bi-Lipschitz approximation of curves

In this section, we show how paths in geodesic spaces can be approximated by bi-Lipschitz arcs with the same endpoints. The main goal will be the proof of Proposition 1.5.

The next lemma is important in the proof of Theorem 1.2, and is almost identical to Lemma 4.2 in [22]. The difference here is the quantitative control on the bi-Lipschitz constant L.

Lemma 4.1. *Given $C \ge 1$ and $\varepsilon > 0$, there exists $L = L(C, \varepsilon) \ge 1$ with the following property. Let (X, d) be a C-doubling geodesic metric space, and let $\sigma\colon [0,1] \to X$ be a curve with $\sigma(0) \ne \sigma(1)$. There exists a curve $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \sigma(0)$, $\gamma(1) = \sigma(1)$, for all $s, t \in [0,1]$,*

$$\frac{1}{L} |s-t| \operatorname{diam} \sigma([0,1]) \le d(\gamma(s), \gamma(t)) \le L |s-t| \operatorname{diam} \sigma([0,1]),$$

and

$$\operatorname{dist}(\gamma(t), \sigma([0,1])) \le \varepsilon \operatorname{diam} \sigma([0,1]).$$

The doubling property is not necessary to guarantee the existence of the bi-Lipschitz map γ; see Lemma 4.2 in [22]. It is, however, necessary to control the constant L. For example, let $X = \ell_2$, let $e_1, e_2, \dots$ be an orthonormal basis of ℓ_2, and let $n \in \mathbb{N}$. Define $\sigma\colon [0,1] \to \ell_2$ so that $\sigma(0) = e_0 := 0$, $\sigma(i/n) = e_i$ for $i \in \{1, \dots, n\}$, and $\sigma|_{[(i-1)/n, i/n]}$ is linear for each $i \in \{1, \dots, n\}$. Note that $\operatorname{diam} \sigma([0,1]) = \sqrt{2}$. It is easy to see that, if $\varepsilon < 1/6$, then for each $i \in \{1, \dots, n-1\}$, the set

$$B(\sigma([0,1]), \sqrt{2}\varepsilon) \setminus B(\sigma(i/n), 3\sqrt{2}\varepsilon)$$

is disconnected. Therefore, if γ is a path in ℓ_2 joining 0 with e_n and satisfying $\gamma([0,1]) \subset B(\sigma([0,1]), \sqrt{2}\varepsilon)$, then $\gamma([0,1])$ must intersect each ball $B(\sigma(i/n), 3\sqrt{2}\varepsilon)$ for $i = 1, \dots, n$. In particular, the length of γ is at least a fixed multiple of n, while $|\gamma(0) - \gamma(1)| = 1$. It follows that, if γ is L-bi-Lipschitz, then L must depend on n and not just on ε.

For the proof of Lemma 4.1, we require a simple lemma. Here and for the rest of this section, all geodesic curves are parameterized by arc-length.

Lemma 4.2. *Let X be a geodesic metric space, let $a \geq b > 0$, let $f: [0,a] \to X$ be L-bi-Lipschitz, let $p \in X$, and suppose $f(b)$ is the closest point in $f([0,a])$ to p, i.e.,*

$$c := \operatorname{dist}(f([0,a]), p) = d(f(b), p).$$

If $g: [b, b+c] \to X$ is the geodesic from $f(b)$ to p, then the concatenation of $f|_{[0,b]}$ and g is $(2L)$-bi-Lipschitz.

Proof. Let $h: [0, b+c] \to X$ be the concatenation of $f|_{[0,b]}$ and g. Clearly, $h|_{[0,b]}$ is L-bi-Lipschitz and $h|_{[b,b+c]}$ is 1-bi-Lipschitz. Fix now $s \in [0,b]$ and $t \in [b,c]$. Then

$$d(h(s), h(t)) \leq d(f(s), f(b)) + d(g(b), g(t)) \leq L(b-s) + t - b \leq L(t-s).$$

For the lower bound, we claim that $d(h(t), h(s)) \geq d(h(t), h(b))$. Indeed, if this were not the case, then

$$\begin{aligned} \operatorname{dist}(f([0,b]), p) &\leq d(h(s), h(t)) + d(h(t), p) < d(h(b), h(t)) + d(h(t), p) \\ &= d(f(b), p) = \operatorname{dist}(f([0,b]), p), \end{aligned}$$

which is impossible. Similarly, $d(h(s), h(t)) \geq d(h(s), h(b))$. Therefore,

$$\begin{aligned} d(h(s), h(t)) \geq \tfrac{1}{2} d(h(s), h(b)) + \tfrac{1}{2} d(h(t), h(b)) &\geq (2L)^{-1}|s-b| + \tfrac{1}{2}|b-t| \\ &\geq (2L)^{-1}|s-t|. \end{aligned}$$ ∎

We are now ready to show Lemma 4.1.

Proof of Lemma 4.1. Without loss of generality, assume that $\operatorname{diam}\sigma([0,1]) = 1$. Since X is doubling, it is (C', α)-homogeneous for some $C' > 0$ and $\alpha > 0$.

Fix $\varepsilon > 0$. If $d(\sigma(0), \sigma(1)) < 2\varepsilon$, then we can simply define γ to be the geodesic from $\sigma(0)$ to $\sigma(1)$ which is 1-bi-Lipschitz. Assume now that $d(\sigma(0), \sigma(1)) \geq 2\varepsilon$.

Let $Y \subset \sigma([0,1])$ be a maximal $(\varepsilon/4)$-separated set that contains $\sigma(0)$ and $\sigma(1)$. Since $\sigma([0,1])$ is connected, there exists a finite sequence of distinct points $x_0, \dots, x_n$ in Y such that $x_0 = \sigma(0)$, $x_n = \sigma(1)$, and $d(x_{i-1}, x_i) < \varepsilon/2$ for all $i \in \{1, \dots, n\}$. By the homogeneity of X, we have that $n \leq C'(\varepsilon/4)^{-\alpha}$.

We define a curve γ inductively. Let $\gamma_1: [0, s_1] \to X$ be a geodesic with $\gamma_1(0) = \sigma(0)$ and $\gamma_1(s_1) = x_1$. Clearly, γ_1 is 1-bi-Lipschitz, and for all $t \in [0, s_1]$,

$$\operatorname{dist}(\gamma_1(t), \sigma([0,1]) \leq \operatorname{length}(\gamma_1) \leq \varepsilon/2.$$

Suppose that for some $k \in \{1, \dots, n-1\}$ we have defined $s_k > 0$ and a 2^{k-1}-bi-Lipschitz curve $\gamma_k: [0, s_k] \to X$, parameterized by arc-length, such that $\gamma_k(0) = \sigma(0)$, $\gamma_k(s_k) = x_k$, and $\gamma_k([0, s_k]) \subset \overline{B}(\sigma([0,1]), \varepsilon/2)$. Let $r_k \in [0, s_k]$ be such that

$$c_k := \operatorname{dist}(\gamma_k([0, s_k]), x_{k+1}) = d(\gamma_k(r_k), x_{k+1}).$$

Define $s_{k+1} = r_k + c_k$ and let $\gamma_{k+1}\colon [0, s_{k+1}] \to X$ be the concatenation of $\gamma_k|_{[0,r_k]}$ with a geodesic joining $\gamma_k(r_k)$ to $x_{k+1} \in \sigma([0, 1])$. Note that

$$d(\gamma_k(r_k), x_{k+1}) \le d(x_k, x_{k+1}) < \varepsilon/2,$$

so for each $t \in [0, s_{k+1}]$, we have

$$\operatorname{dist}(\gamma_{k+1}(t), \sigma([0, 1])) < \varepsilon/2.$$

Moreover, by Lemma 4.2, the curve γ_{k+1} is 2^k-bi-Lipschitz.

By induction, we have defined a number $0 < s_n \le n(\varepsilon/2)$ and a 2^{n-1}-bi-Lipschitz curve $\gamma_n : [0, s_n] \to X$ such that $\gamma_n(0) = \sigma(0)$, $\gamma_n(s_n) = \sigma(1)$, and

$$\gamma_n([0, s_n]) \subset B(\sigma([0, 1]), \varepsilon).$$

The desired curve $\gamma\colon [0, 1] \to X$ is the reparameterization $\gamma(t) = \gamma_n(s_n t)$. ■

4.1. Proof of Proposition 1.5

The proof of Proposition 1.5 will rely on the quantitative version of Theorem 1.2; see Theorem 6.1.

We first review some elementary notions from graph theory. A (*combinatorial*) *graph* is a pair $G = (V, E)$ of a finite vertex set V and an edge set E, which contains elements of the form $\{v, v'\}$, where $v, v' \in V$ and $v \ne v'$. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$, $E' \subset E$, and $E' \subset V' \times V'$. A *simple path* joining $x, y \in V$ in G is a set $\gamma = \{v_0, \dots, , v_n\} \subset V$ of distinct points such that $v_0 = x$, $v_n = y$, and $\{v_{i-1}, v_i\} \in E$ for all $i \in \{1, \dots, n\}$. A graph G is *connected* if any two distinct vertices can be joined by a simple path in G.

Lemma 4.3. *Given a graph $G = (V, E)$ and two distinct $v, v' \in V$, there exists a finite sequence $(v_i)_{i=1}^N$ in V such that $\{v_1, \dots, v_n\} = V$, $v_1 = v$, $v_n = v'$, we have $\{v_i, v_{i+1}\} \in E$ for each $i \in \{1, \dots, n-1\}$, and for each $e \in E$, there exist at most two $i \in \{1, \dots, n-1\}$ such that $e = \{v_i, v_{i+1}\}$.*

Proof. We will use the fact that every connected graph admits a 2-to-1 Euler tour along its edges, that is, for each vertex z, there exists a finite sequence $(z_j)_{j=1}^m$ of vertices in G such that $z_1 = z_m = z$, $\{z_j, z_{j+1}\}$ is an edge for all j, and for each edge e there exists exactly two j such that $e = \{z_j, z_{j+1}\}$. See, e.g., the Euler tour technique introduced in [32].

Now let G, v and v' be as in the statement. Deleting some edges from E, we may assume that G is a (combinatorial) tree, that is, for any two distinct vertices, there exists a unique simple path in G that connects them. Let $\tilde{V} = \{v_1, \dots, v_k\}$ be the unique such path with $v_1 = v$ and $v_k = v'$. For each $i \in \{1, \dots, k\}$, let $G_i = (V_i, E_i)$ be the maximal subgraph of G with the property that any simple path connecting a vertex of G_i with a vertex of $\tilde{V}$ must contain v_i. Since G is connected, it follows that each G_i is connected. Moreover, since G is a tree, for any $i \ne j$ the graphs G_i and G_j are trees with mutually disjoint vertices (and hence edges).

The construction of the finite sequence $(v_i)_i$ is as follows. Firstly, do a 2-to-1 tour of G_1 starting and ending on v_1. Then proceed to v_2 and do a 2-to-1 tour of G_2 starting and ending on v_2. Continue in this way until reaching v_k, where we do a 2-to-1 tour of G_k starting and ending on v_k. ■

Proof of Proposition 1.5. If $\operatorname{diam} K = 0$, then there is nothing to prove. Assume now that $\operatorname{diam} K > 0$ and, rescaling, we may further assume that $\operatorname{diam} K = 1$.

Let Y be a maximal $(\varepsilon/4)$-separated subset of K that contains x and y. By the regularity of X, the cardinality of Y is at most $C'\varepsilon^{-Q}$ for some $C' > 0$ depending only on the constants of Q-regularity. Define a graph G with vertex set Y such that two points $z, z' \in G$ are connected by an edge if and only if $d(z, z') < \varepsilon/2$. Since K is connected, it follows that G is connected. By Lemma 4.3, there exists a tour $x = v_0, \ldots, v_n = y$ of the vertices Y such that each edge is visited at most twice.

For each $z \in Y$, denote by m_z the number of indices i such that $v_i = z$. There exists $C'' > 0$, depending only on the constants of Q-regularity, such that each vertex of G is contained in at most C'' edges. Therefore, for each $z \in Y$, $m_z \le C''$, and it follows that $n \le C''C'\varepsilon^{-Q}$. Moreover, there exists $c > 4$, depending only on the constants of Q-regularity, such that for each $z \in Y$ there exist points $v_{z,1}, \ldots, v_{z,m_z} \in B(z, \varepsilon/16)$ such that

$$d(v_{z,i}, v_{z,j}) \ge c^{-1}\varepsilon, \quad \text{for all } z \in Y \text{ and } i \ne j.$$

We may also assume that $v_{x,1} = x$ and $v_{y,m_y} = y$.

Given $i \in \{0, \ldots, n\}$, let $j(i)$ be the number of indices $l \in \{0, \ldots, i\}$ such that $v_l = v_i$. Define now $\tilde{v}_i = v_{v_i, j(i)}$. Note that the new sequence $\tilde{v}_0, \ldots, \tilde{v}_n$ satisfies

(1) $\tilde{v}_0 = x$, $\tilde{v}_n = y$,

(2) for each distinct $i, j \in \{0, \ldots, n\}$ we have $d(\tilde{v}_i, \tilde{v}_j) \ge c^{-1}\varepsilon$,

(3) for each $z \in K$ there exists $i \in \{0, \ldots, n\}$ such that $d(z, \tilde{v}_i) \le \varepsilon/2$,

(4) for each $i \in \{0, \ldots, n\}$, $\operatorname{dist}(\tilde{v}_i, K) \le \varepsilon/16$.

Define a map $f\colon \{i\varepsilon : i = 0, \ldots, n\} \to X$ by $f(i\varepsilon) = \tilde{v}_i$, and note that f is L'-bi-Lipschitz, with $L' = \max\{nc, 2/\varepsilon\}$. Indeed, for any distinct i and j, we have that $\varepsilon \le |i\varepsilon - j\varepsilon| \le n\varepsilon$ and $c^{-1}\varepsilon \le d(f(i\varepsilon), f(j\varepsilon)) \le 1 + 2c^{-1}\varepsilon < 2$.

By Theorem 6.1, there exists a constant L, depending on ε, the constants of Q-regularity and the data of the Poincaré inequality, and there exists a L-bi-Lipschitz arc $F\colon [0, n\varepsilon] \to X$ that extends f and

$$\operatorname{dist}_H(K, F([0, n\varepsilon])) \lesssim \varepsilon.$$

The arc $\gamma\colon [0, 1] \to X$ in question is obtained by reparameterizing F. ∎

5. Whitney intervals and a preliminary extension

Here and for the rest of this section, we assume that X is a complete geodesic (C_1, Q)-Ahlfors regular metric measure space supporting a p-Poincaré inequality with data C, where $p \in (1, Q-1)$ and $C_1, C > 1$. We also assume that $A \subset \mathbb{R}$ is a closed set, and that $f\colon A \to X$ is an L-bi-Lipschitz embedding.

Let I be the smallest closed interval with $A \subset I$ (possibly $\mathbb{R}$). We need a Whitney decomposition of $I \setminus A$ as in Whitney's classical proof of his extension theorem [35]. We may assume that A is not a closed interval itself, as then there is no extension to be made.

Lemma 5.1 (Theorem VI.1.1 and Proposition VI.1.1 in [31]). *There exists a collection of closed intervals* $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$ *such that*

(i) $\bigcup_{i=1}^{\infty} \mathcal{Q}_i = I \setminus A$,

(ii) *the intervals* $\{\mathcal{Q}_i\}$ *have disjoint interiors, and*

(iii) $\operatorname{diam} \mathcal{Q}_i \leq \operatorname{dist}(\mathcal{Q}_i, A) \leq 4 \operatorname{diam} \mathcal{Q}_i$ *for all* $i \in \mathbb{N}$.

Moreover, if the intervals $\mathcal{Q}_i$ *and* $\mathcal{Q}_j$ *share an endpoint, then*

$$\tfrac{1}{4} \operatorname{diam} \mathcal{Q}_j \leq \operatorname{diam} \mathcal{Q}_i \leq 4 \operatorname{diam} \mathcal{Q}_j. \tag{5.1}$$

Henceforth, the intervals $\{\mathcal{Q}_i\}_{i \in \mathbb{N}}$ will be called *Whitney intervals*.

5.1. Reference points

Let E denote the collection of endpoints of $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$. For each $x \in E$, fix a point $a_x \in A$ that is a closest point of A to x, that is, $|x - a_x| = \operatorname{dist}(x, A)$.

Proposition 5.2. *There exist* $\xi \in (0, 1)$ *and* $\tilde{L} > 1$*, depending only on* L*,* C_1*, and* Q*, and there exists an* $\tilde{L}$*-bi-Lipschitz map* $\pi\colon E \to X$ *such that, for all distinct* $x, y \in E$*,*

(1) $\frac{1}{4}|x - a_x| \leq d(\pi(x), f(a_x)) \leq 4|x - a_x|$,

(2) $\operatorname{dist}(\pi(x), f(A)) \geq \xi\,|x - a_x|$,

(3) $d(\pi(x), \pi(y)) \geq \xi(|x - a_x| + |y - a_y|)$.

Moreover, if $\mathcal{Q}_i = [x, y]$*, then*

$$d(\pi(x), \pi(y)) \leq d(f(a_x), f(a_y)) + 36 \operatorname{diam} \mathcal{Q}_i \leq 46\,L \operatorname{diam} \mathcal{Q}_i. \tag{5.2}$$

We start with a result that allows us to partition E into a finite number of subsets such that elements of the same subset are far apart quantitatively. Recall that, by Lemma 2.1, there exists $p_0 \geq 1$, depending only on L, C_1 and Q, such that $f(A)$ is p_0-porous.

Lemma 5.3. *There exists* $n \in \mathbb{N}$ *depending only on* L*,* C_1 *and* Q*, and there exists a partition of* E *into mutually disjoint sets* $E_1, \dots, E_n$ *such that, for any* $i \in \{1, \dots, n\}$ *and for any* $x, y \in E_i$*,*

(F1) *either* $|x - y| > (12L) \max\{|x - a_x|, |y - a_y|\}$,

(F2) *or* $\max\{|x - a_x|, |y - a_y|\} > (8p_0) \min\{|x - a_x|, |y - a_y|\}$.

Proof. Enumerate $E = \{x_1, x_2, \dots\}$, and for each $i \in \mathbb{N}$, define V_i be the set of all indices $j \in \mathbb{N}$ such that

$$|x_i - x_j| \leq (12L) \max\{|x_i - a_{x_i}|, |x_j - a_{x_j}|\}$$

and

$$(8p_0)^{-1}|x_i - a_{x_i}| \leq |x_j - a_{x_j}| \leq (8p_0)|x_i - a_{x_i}|.$$

Note that $i \in V_j$ if and only if $j \in V_i$.

We claim that there exists $n \in \mathbb{N}$, depending only on L, C_1 and Q, such that $\operatorname{card}(V_i) \leq n$ for each $i \in \mathbb{N}$. To this end, fix $i \in \mathbb{N}$ and note that for any $j, k \in V_i$ with $j \neq k$,

$$|x_j - x_k| \leq (24L) \max\{|x_j - a_{x_j}|, |x_k - a_{x_k}|, |x_i - a_{x_i}|\} \leq (192\, p_0 L)\,|x_i - a_{x_i}|. \tag{5.3}$$

Moreover, let $j, k \in V_i$ with $j \neq k$, let $\mathcal{Q}_{j_1}$ and $\mathcal{Q}_{j_2}$ be the two Whitney intervals which share the endpoint x_j, and let $\mathcal{Q}_{k_1}$ and $\mathcal{Q}_{k_2}$ be the two Whitney intervals which share the endpoint x_k. We have

$$\operatorname{dist}(\mathcal{Q}_{j_1}, A) \geq |x_j - a_{x_j}| - \operatorname{diam} \mathcal{Q}_{j_1} \geq |x_j - a_{x_j}| - \operatorname{dist}(\mathcal{Q}_{j_1}, A),$$

so $|x_j - a_{x_j}| \leq 2 \operatorname{dist}(\mathcal{Q}_{j_1}, A)$. By Lemma 5.1(iii),

$$\operatorname{diam} \mathcal{Q}_{j_1} \geq \tfrac{1}{4} \operatorname{dist}(\mathcal{Q}_{j_1}, A) \geq \tfrac{1}{8} |x_j - a_{x_j}|,$$

and a similar estimate holds for $\mathcal{Q}_{j_2}$, $\mathcal{Q}_{k_1}$, and $\mathcal{Q}_{k_2}$. Since $x_j \neq x_k$, one of the intervals for which they are endpoints lies between them. That is,

$$\begin{aligned}|x_j - x_k| &\geq \min\{\operatorname{diam} \mathcal{Q}_{j_1}, \operatorname{diam} \mathcal{Q}_{j_2}, \operatorname{diam} \mathcal{Q}_{k_1}, \operatorname{diam} \mathcal{Q}_{k_2}\} \\ &\geq \tfrac{1}{8} \min\{|x_j - a_{x_j}|, |x_k - a_{x_k}|\} \geq (64\, p_0)^{-1} |x_i - a_{x_i}|.\end{aligned}$$

Combining this with (5.3), we conclude that $\operatorname{card}(V_i) \leq 192 L (8 p_0)^2 =: n$.

Define now a map $\mathbf{c} \colon \mathbb{N} \to \{1, \ldots, n\}$ such that $\mathbf{c}(1) = 1$, and for each $i \geq 2$,

$$\mathbf{c}(i) := \min\{\ell \in \mathbb{N} : \ell \neq \mathbf{c}(k) \text{ for all } k \in V_i \cap \{1, \ldots, i-1\}\}.$$

It is clear that if $i \in V_j$ and $i \neq j$, then $\mathbf{c}(i) \neq \mathbf{c}(j)$. For each $i \in \{1, \ldots, n\}$, define $E_i := \{x_j : \mathbf{c}(j) = i\}$. Given $x_j, x_k \in E_i$, $\mathbf{c}(i) = \mathbf{c}(j)$, so $j \notin V_k$ (equivalently, $k \notin V_j$). Properties (F1) and (F2) follow. ∎

We now turn to the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $n \in \mathbb{N}$ and $E_1, \ldots, E_n$ be the integer and the partition, respectively, from Lemma 5.3. For each $k \in \{1, \ldots, n\}$, define $E^{(k)} = E_1 \cup \cdots \cup E_k$.

Let $i \in \{1, \ldots, n\}$, $x \in E_i$, and $x' \in \partial B(f(a_x), |x - a_x|)$. By the porosity of $f(A)$, there exists a point $\tilde{x} \in X$ such that

$$B(\tilde{x}, (2p_0)^{-1} |x - a_x|) \subset B\big(x', \tfrac{1}{2} |x - a_x|\big) \setminus f(A).$$

Then

$$\tfrac{1}{2} |x - a_x| \leq d(\tilde{x}, f(a_x)) \leq \tfrac{3}{2} |x - a_x| \quad \text{and} \tag{5.4}$$

$$\operatorname{dist}(\tilde{x}, f(A)) \geq (2p_0)^{-1} |x - a_x|. \tag{5.5}$$

For any $i \in \{1, \ldots, n\}$, and for any $x, y \in E_i$, we will show that

$$d(\tilde{x}, \tilde{y}) \geq (8p_0)^{-1} (|x - a_x| + |y - a_y|). \tag{5.6}$$

Fix such i, x, and y, and assume without loss of generality that $|x - a_x| \geq |y - a_y|$. If (F1) holds, then by (5.4),

$$\begin{aligned}d(\tilde{x}, \tilde{y}) &\geq d(f(a_x), f(a_y)) - d(f(a_x), \tilde{x}) - d(f(a_y), \tilde{y}) \\ &\geq L^{-1} |a_x - a_y| - \tfrac{3}{2} (|x - a_x| + |y - a_y|) \geq L^{-1} |x - y| - 6 |x - a_x| \\ &> 6 |x - a_x| \geq 3 (|x - a_x| + |y - a_y|).\end{aligned}$$

Assume now that (F2) holds and (F1) fails. By (5.4) and (5.5),

$$\begin{aligned} d(\tilde{x},\tilde{y}) &\geq d(\tilde{x}, f(a_y)) - d(f(a_y),\tilde{y}) \geq (2p_0)^{-1}|x-a_x| - \tfrac{3}{2}|y-a_y| \\ &\geq (8p_0)^{-1}(|x-a_x|+|y-a_y|). \end{aligned}$$

We define the map π on each $E^{(k)}$ in an inductive manner. Define $\pi\colon E_1 \to X$ by $\pi(x) = \tilde{x}$. Properties (1)–(3) of the proposition for E_1 follow from (5.4), (5.5), and (5.6) with $\xi_1 = (8p_0)^{-1}$.

Assume now that for some $k \in \{1,\dots,n-1\}$ we have defined a constant $\xi_k \in (0,1)$ and a function $\pi\colon E^{(k)} \to X$ such that, for all distinct $x,y \in E^{(k)}$,

$$\tfrac{1}{4}|x-a_x| \leq d(\pi(x), f(a_x)) \leq 4|x-a_x|, \tag{5.7}$$

$$\operatorname{dist}(\pi(x), f(A)) \geq (4p_0)^{-1}|x-a_x|, \quad \text{and} \tag{5.8}$$

$$d(\pi(x),\pi(y)) \geq \xi_k(|x-a_x|+|y-a_y|). \tag{5.9}$$

Fix $x \in E_{k+1}$, and assume that there exist $y_1,\dots,y_N \in E^{(k)}$ such that

$$d(\tilde{x},\pi(y_j)) < (8p_0)^{-1}(|x-a_x|+|y_j-a_{y_j}|) \tag{5.10}$$

for each $j \in \{1,\dots,N\}$. First, for each such j, by (5.7), (5.5), and (5.10),

$$\begin{aligned} |y_j-a_{y_j}| &\geq \tfrac{1}{4} d(\pi(y_j), f(a_{y_j})) \geq \tfrac{1}{4}\big(d(\tilde{x}, f(a_{y_j})) - d(\tilde{x},\pi(y_j))\big) \\ &\geq \tfrac{1}{4}\big((2p_0)^{-1}|x-a_x| - (8p_0)^{-1}(|x-a_x|+|y_j-a_{y_j}|)\big). \end{aligned}$$

This gives

$$|y_j-a_{y_j}| \geq (12p_0)^{-1}|x-a_x|. \tag{5.11}$$

Next, for any $j,\ell \in \{1,\dots,N\}$, (5.9) and (5.11) yield

$$d(\pi(y_j),\pi(y_\ell)) \geq (12p_0)^{-1}\xi_k|x-a_x|, \tag{5.12}$$

so $\{\pi(y_1),\dots,\pi(y_N)\}$ is a $((12p_0)^{-1}\xi_k|x-a_x|)$-separated set. By (5.4), (5.8), and (5.10), for any $j \in \{1,\dots,N\}$,

$$\begin{aligned} |x-a_x| &\geq \tfrac{2}{3} d(\tilde{x}, f(a_x)) \geq \tfrac{2}{3}\big(d(\pi(y_j), f(a_x)) - d(\tilde{x},\pi(y_j))\big) \\ &\geq (6p_0)^{-1}|y_j-a_{y_j}| - (12p_0)^{-1}(|x-a_x|+|y_j-a_{y_j}|). \end{aligned}$$

Therefore,

$$|y_j-a_{y_j}| \leq 24p_0|x-a_x|. \tag{5.13}$$

By Ahlfors regularity, (5.10), (5.13), and (5.12), we obtain $N \lesssim_{L,C_1,Q} \xi_k^{-Q}$.

By Lemma 2.1, $\{\pi(y_1),\dots,\pi(y_N)\}$ is p_k-porous for some $p_k \geq 1$ depending only on C_1, Q, L, and k. Hence, there exists a point $\pi(x) \in B(\tilde{x},(32p_0)^{-1}|x-a_x|)$ such that

$$B(\pi(x),(32p_0p_k)^{-1}|x-a_x|) \subset X \setminus \{\pi(y_1),\dots,\pi(y_N)\}. \tag{5.14}$$

To complete the inductive step, we show that π defined on $E^{(k+1)}$ satisfies properties (1)–(3) of the proposition for some appropriate $\xi_{k+1} \in (0,1)$.

For the first property, fix $x \in E_{k+1}$. By (5.4),

$$d(\pi(x), f(a_x)) \le d(\pi(x), \tilde{x}) + d(f(a_x), \tilde{x}) \le 2|x - a_x| \quad \text{and}$$
$$d(\pi(x), f(a_x)) \ge d(f(a_x), \tilde{x}) - d(\pi(x), \tilde{x}) \ge \tfrac{1}{4}|x - a_x|.$$

For the second property, fix $x \in E_{k+1}$. By (5.5),

$$\operatorname{dist}(\pi(x), f(A)) \ge \operatorname{dist}(\tilde{x}, f(A)) - d(\pi(x), \tilde{x}) \ge (4p_0)^{-1}|x - a_x|.$$

For the third property, fix distinct $x, y \in E^{(k+1)}$ and assume that $x \in E_{k+1}$. If $y \in E_{k+1}$, then by (5.6),

$$d(\pi(x), \pi(y)) \ge d(\tilde{x}, \tilde{y}) - d(\pi(x), \tilde{x}) - d(\pi(y), \tilde{y}) \ge (16p_0)^{-1}(|x - a_x| + |y - a_y|).$$

Assume now that $y \in E^{(k)}$. If

$$d(\tilde{x}, \tilde{y}) \ge (8p_0)^{-1}(|x - a_x| + |y - a_y|),$$

then we work as in the preceding case. If

$$d(\tilde{x}, \tilde{y}) < (8p_0)^{-1}(|x - a_x| + |y - a_y|),$$

then by (5.13) and (5.14),

$$d(\pi(x), \pi(y)) \ge (32p_0 p_k)^{-1}|x - a_x| \ge (2^9 3 p_0^2 p_k)^{-1}(|x - a_x| + |y - a_y|).$$

After n steps, we have defined $\xi := \xi_n$ and the map $\pi\colon E \to X$ that satisfies properties (1)–(3) in the statement of the lemma.

To show that π is bi-Lipschitz, fix distinct $x, y \in E$ and assume without loss of generality that $|x - a_x| \ge |y - a_y|$. By Lemma 5.1 (iii), $|x - a_x| \le 4|x - y|$. By property (3),

$$\begin{aligned} d(\pi(x), \pi(y)) &\le d(\pi(x), f(a_x)) + d(f(a_x), f(a_y)) + d(\pi(y), f(a_y)) \\ &\le 4|x - a_x| + L|a_x - a_y| + 4|y - a_y| \le 41L|x - y|. \end{aligned}$$

For the lower bound, suppose first that $|a_x - a_y| \ge 16L|x - a_x|$. Then we have that $|x - y| \le 2|a_x - a_y|$, and by property (1),

$$d(\pi(x), \pi(y)) \ge d(f(a_x), f(a_y)) - d(\pi(x), f(a_x)) - d(\pi(y), f(a_y)) \ge (2L)^{-1}|a_x - a_y|.$$

Suppose now that $|a_x - a_y| \le 16L|x - a_x|$. Then, $|x - y| \le (2 + 16L)|x - a_x|$ and by property (3), $d(\pi(x), \pi(y)) \ge \xi|x - a_x|$.

For (5.2), fix a Whitney interval $\mathcal{Q}_i = [x, y]$ and assume, without loss of generality, that $|x - a_x| \le |y - a_y|$. There are two cases to consider. Assume first that $a_x = a_y$. By property (1),

$$d(\pi(x), \pi(y)) \le 4|x - a_x| + 4|y - a_x| \le 8|x - a_x| + 4 \operatorname{diam} \mathcal{Q}_i \le 36 \operatorname{diam} \mathcal{Q}_i.$$

Assume now that $a_x \ne a_y$. Then

$$|y - a_y| \le |y - a_x| \le |x - y| + |x - a_x| \le 5 \operatorname{diam} \mathcal{Q}_i,$$

which yields that $|a_x - a_y| \le 11 \operatorname{diam} \mathcal{Q}_i$. By property (1),

$$\begin{aligned} d(\pi(x), \pi(y)) &\le 4|x - a_x| + d(f(a_x), f(a_y)) + 4|y - a_y| \\ &\le d(f(a_x), f(a_y)) + 40 \operatorname{diam} \mathcal{Q}_i \\ &\le L|a_x - a_y| + 40 \operatorname{diam} \mathcal{Q}_i \le 51L \operatorname{diam} \mathcal{Q}_i. \end{aligned}$$

■

5.2. The middle third of each Whitney interval

The goal of this subsection is to extend f to the union of the middle-thirds of all Whitney intervals $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$ in a bi-Lipschitz way. From here on, for each Whitney interval $\mathcal{Q}_i$, we denote by $\hat{\mathcal{Q}}_i$ the middle third interval of $\mathcal{Q}_i$. Recall the constants $\xi \in (0,1)$ and $\tilde{L}$ from Proposition 5.2, depending only on L, C_1, and Q.

Proposition 5.4. *There exists a constant $\hat{L} \geq 1$ depending only on p, C, C_1, L, and Q, and there exists an $\hat{L}$-bi-Lipschitz extension of f,*

$$g : A \cup \bigcup_{i\in\mathbb{N}} \hat{\mathcal{Q}}_i \to X,$$

such that for each $i \in \mathbb{N}$, if $\mathcal{Q}_i = [w,z]$ and $\hat{\mathcal{Q}}_i = [\hat{w},\hat{z}]$, then

(1) $d(g(\hat{w}),\pi(w)) \leq (2^8\tilde{L})^{-1}\xi \operatorname{diam}\mathcal{Q}_i$,

(2) $d(g(\hat{z}),\pi(z)) \leq (2^8\tilde{L})^{-1}\xi \operatorname{diam}\mathcal{Q}_i$, *and*

(3) $g(\hat{\mathcal{Q}}_i) \subset B(\pi(w),4R_i) \cap B(\pi(z),4R_i)$, *where $R_i = d(\pi(w),\pi(z))$.*

Recall that, since f is bi-Lipschitz, the set $f(A)$ is 1-homogeneous in X.

Lemma 5.5. *There exist constants $\beta_0, \ell_0, \delta_0 > 0$, depending only on p, C, C_1, L, and Q, with the following property. Let $\mathcal{Q}_i = [w,z]$ be a Whitney interval and let Γ_i be the collection of curves $\gamma\colon [0,1] \to X$ such that*

(1) $\gamma([0,1]) \subset B(\pi(w),3R_i) \cap B(\pi(z),3R_i)$, *where $R_i = d(\pi(w),\pi(z))$,*

(2) $\max\{d(\gamma(0),\pi(w)), d(\gamma(1),\pi(z))\} < (2^8\tilde{L})^{-1}\xi \operatorname{diam}\mathcal{Q}_i$,

(3) $\operatorname{length}(\gamma) \leq \ell_0 \operatorname{diam}\mathcal{Q}_i$,

(4) $\operatorname{dist}(\gamma(t), f(A)) \geq \delta_0 \operatorname{diam}\mathcal{Q}_i$ *for all $t \in [0,1]$.*

Then,

$$\operatorname{Mod}_p(\Gamma_i) \geq \beta_0 (\operatorname{diam}\mathcal{Q}_i)^{Q-p}.$$

Proof. Since $B(\pi(w),2R_i) \subset B(\pi(w),3R_i) \cap B(\pi(z),3R_i)$, we may apply Lemma 3.1, Proposition 5.2(3), and (5.2) to conclude that the family $\Gamma_i^{(1)}$ of curves

$$\gamma : [0,1] \to B(\pi(w),3R_i) \cap B(\pi(z),3R_i)$$

such that $\gamma(0)$ lies in the closed ball $\overline{B}(\pi(w),(2^8\tilde{L})^{-1}\xi \operatorname{diam}\mathcal{Q}_i)$ and $\gamma(1)$ lies in the closed ball $\overline{B}(\pi(z),(2^8\tilde{L})^{-1}\xi \operatorname{diam}\mathcal{Q}_i)$ has p-modulus

$$\operatorname{Mod}_p(\Gamma_i^{(1)}) \geq \alpha (\operatorname{diam}\mathcal{Q}_i)^{Q-p},$$

where $\alpha > 0$ is some constant depending only on p, C, C_1, Q, and L.

By Lemma 3.2, there exists $\ell_0 > 0$, depending only on p, C, C_1, Q, and L, such that the subfamily

$$\Gamma_i^{(2)} := \left\{\gamma \in \Gamma_i^{(1)} : \operatorname{length}(\gamma) \leq \ell_0 \operatorname{diam}\mathcal{Q}_i\right\}$$

satisfies

$$\operatorname{Mod}_p(\Gamma_i^{(2)}) \geq \tfrac{1}{2}\alpha (\operatorname{diam}\mathcal{Q}_i)^{Q-p}.$$

By Lemma 3.3, there exists $\delta_0 > 0$, depending only on Q, p, C, C_1, and L, such that the subfamily

$$\Gamma_i := \{\gamma \in \Gamma_i^{(2)} : \operatorname{dist}(\gamma(t), f(A)) \geq \delta_0 \operatorname{diam} \mathcal{Q}_i \text{ for each } t \in [0,1]\}$$

satisfies

$$\operatorname{Mod}_p(\Gamma_i) \geq \tfrac{1}{4}\alpha(\operatorname{diam} \mathcal{Q}_i)^{Q-p}. \qquad \blacksquare$$

We now need a filtration of the Whitney decomposition, in the vein of the following result of David and Semmes. The proof of the lemma is almost identical to that of Lemma 5.3, and is left to the reader.

Lemma 5.6 (Proposition 17.4 in [6]). *There exists an integer N depending only on L, C_1, and Q, and there exists a partition of $\mathbb{N}$ into sets $\{\mathfrak{I}_1, \dots, \mathfrak{I}_N\}$ such that for any $k \in \{1, \dots, N\}$ and for any $i, j \in \mathfrak{I}_k$,*

(i) *either* $\operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) > 800L^2 \max\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\}$,

(ii) *or* $\max\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\} > 800\, L\, \delta_0^{-1} \min\{\operatorname{diam} \mathcal{Q}_i, \operatorname{diam} \mathcal{Q}_j\}$.

We are now ready to prove Proposition 5.4.

Proof of Proposition 5.4. The construction is in an inductive fashion. Let N be the integer and let $\mathfrak{I}_1, \dots, \mathfrak{I}_N$ be the sets of indices from Lemma 5.6. Denote $A_0 := A$, and for each $k \in \{1, \dots, N\}$, denote

$$A_k := A_0 \cup \bigcup_{j=1}^{k} \bigcup_{i \in \mathfrak{I}_j} \hat{\mathcal{Q}}_i.$$

For each $k \in \{0, \dots, N\}$, we find some $L_k \geq 1$, depending only on p, C, C_1, L, Q, and k, and we find an L_k-bi-Lipschitz embedding $f_k : A_k \to X$ such that for all $k \in \{1, \dots, N\}$, $f_k|_{A_{k-1}} = f_{k-1}$ and such that, if $i \in \mathfrak{I}_k$, $\mathcal{Q}_i = [w, z]$, and $\hat{\mathcal{Q}}_i = [\hat{w}, \hat{z}]$, then

(a) $d(f_k(\hat{w}), \pi(w)) \leq (2^8 \tilde{L})^{-1} \xi \operatorname{diam} \mathcal{Q}_i$,

(b) $d(f_k(\hat{z}), \pi(z)) \leq (2^8 \tilde{L})^{-1} \xi \operatorname{diam} \mathcal{Q}_i$,

(c) $f_k(\hat{\mathcal{Q}}_i) \subset B(\pi(w), 4R_i) \cap B(\pi(z), 4R_i)$, where $R_i = d(\pi(w), \pi(z))$.

The map g of Proposition 5.4 will then be the map f_N.

For $k = 0$, set $L_0 = L$ and $f_0 = f$. Properties (a)–(c) are vacuous.

Assume now that, for some $k \in \{0, \dots, N-1\}$, there exist a constant L_k and an L_k-bi-Lipschitz map $f_k \colon A_k \to X$ satisfying (a)–(c).

Fix $i \in \mathfrak{I}_{k+1}$ and write $\mathcal{Q}_i = [w, z]$ and $\hat{\mathcal{Q}}_i = [\hat{w}, \hat{z}]$. Recall the family of curves Γ_i from Lemma 5.5. By Lemma 3.3, there exists $\delta_{k+1} \in (0, \xi)$, depending only on Q, p, C, C_1, L, and k (in particular, on the homogeneity constant of $f(A_k)$), such that the subfamily

$$\Gamma'_{k,i} := \{\gamma \in \Gamma_i : \operatorname{dist}(\gamma(t), f_k(A_k)) \geq \delta_{k+1} \operatorname{diam} \mathcal{Q}_i \text{ for each } t \in [0,1]\}$$

satisfies

$$\operatorname{Mod}_p(\Gamma'_{k,i}) \geq \tfrac{1}{2}\beta_0(\operatorname{diam} \mathcal{Q}_i)^{Q-p} > 0.$$

In particular, $\Gamma'_{k,i}$ is non-empty, so we can pick a curve $\sigma_i \in \Gamma'_{k,i}$. Applying Lemma 4.1 to σ_i with a suitable reparameterization, we find a constant L'_{k+1}, depending only on Q, p, C, C_1, L, and k, and we find an L'_{k+1}-bi-Lipschitz curve $\gamma_i \colon \hat{\mathcal{Q}}_i \to X$ such that $\gamma_i(\hat{w}) = \sigma_i(0)$, $\gamma_i(\hat{z}) = \sigma_i(1)$, and the inductive hypothesis (c) for f_k gives

$$\begin{aligned} \gamma_i(\hat{\mathcal{Q}}_i) &\subset B(\sigma_i([0,1]), \tfrac{1}{2}\delta_{k+1} \operatorname{diam} \mathcal{Q}_i) \\ &\subset B(\pi(w), 3R_i + \tfrac{1}{2}\delta_{k+1} \operatorname{diam} \mathcal{Q}_i) \cap B(\pi(z), 3R_i + \tfrac{1}{2}\delta_{k+1} \operatorname{diam} \mathcal{Q}_i) \\ &\subset B(\pi(w), 4R_i) \cap B(\pi(z), 4R_i). \end{aligned} \tag{5.15}$$

In particular, we have that $\operatorname{dist}(\gamma_i(\hat{\mathcal{Q}}_i), f_k(A_k)) \geq \frac{1}{2}\delta_{k+1} \operatorname{diam} \mathcal{Q}_i$.

Define now $f_{k+1} \colon A_{k+1} \to X$ by setting $f_{k+1}|A_k = f_k$ and $f_{k+1}|\hat{\mathcal{Q}}_i = \gamma_i$ for each $i \in \mathcal{I}_{k+1}$. By (5.2), we have for all $i \in \mathcal{I}_{k+1}$,

$$\operatorname{diam} f_{k+1}(\hat{\mathcal{Q}}_i) \leq 9R_i \leq 414L \operatorname{diam}(\mathcal{Q}_i). \tag{5.16}$$

Clearly, $f_{k+1}|A_k = f_k$. Properties (a)–(c) are clear from the design of f_{k+1} and Lemma 5.5. To complete the inductive step, we claim that f_{k+1} is L_{k+1}-bi-Lipschitz for some $L_{k+1} \geq 1$ depending only on Q, p, C, C_1, L, and k. Fix $x, y \in A_{k+1}$.

Firstly, if $x, y \in A_k$, then the claim follows by the fact that $f_{k+1}|A_k = f_k$ and the inductive hypothesis that f_k is L_k-bi-Lipschitz.

Secondly, assume that $x \in \hat{\mathcal{Q}}_i$ for some $i \in \mathcal{I}_{k+1}$ and $y \in A$. Let w be the endpoint of $\mathcal{Q}_i$ closest to A, let $\hat{w}$ be the endpoint of $\hat{\mathcal{Q}}_i$ between x and w, and note that $|w - x| \leq |x - a_w| \leq |x - y|$. By (5.16), Proposition 5.2(1), the fact $\operatorname{diam} \mathcal{Q}_i \leq |w - a_w|$, and properties (a) and (b) for f_{k+1},

$$\begin{aligned} &d(f_{k+1}(x), f_{k+1}(y)) \\ &\quad \leq d(f_{k+1}(x), f_{k+1}(\hat{w})) + d(f_{k+1}(\hat{w}), \pi(w)) + d(\pi(w), f(a_w)) + d(f(a_w), f(y)) \\ &\quad \leq (414L + 5)|w - a_w| + L|a_w - y| \leq (414L + 5)|x - a_w| + L|a_w - y| \\ &\quad \leq (416L + 5)|x - y|. \end{aligned}$$

For the lower bound, we have by Lemma 5.5(4) and the design of γ_i,

$$d(f_{k+1}(x), f_{k+1}(y)) \geq \operatorname{dist}(f_{k+1}(x), f(A)) \geq \tfrac{1}{2}\delta_0 \operatorname{diam} \mathcal{Q}_i \geq \tfrac{1}{8}\delta_0 |w - a_w|,$$

and, by (5.16), property (c) for f_{k+1}, and Proposition 5.2(2),

$$\begin{aligned} d(f_{k+1}(x), f(a_w)) &\leq d(f_{k+1}(x), f_{k+1}(\hat{w})) + d(f_{k+1}(\hat{w}), \pi(w)) + d(f(a_w), \pi(w)) \\ &\leq 414L \operatorname{diam} \mathcal{Q}_i + (2^8 \tilde{L})^{-1} \xi \operatorname{diam} \mathcal{Q}_i + 4|w - a_w| \leq 419L|w - a_w|. \end{aligned}$$

Therefore, since $|x - a_w| \leq 2|w - a_w|$,

$$\begin{aligned} |x - y| &\leq |x - a_w| + |a_w - y| \\ &\leq 2|w - a_w| + L[d(f(a_w), f_{k+1}(x)) + d(f_{k+1}(x), f(y))] \\ &\leq 419L^2|w - a_w| + L\,d(f_{k+1}(x), f(y)) \\ &\leq 3352L^2 \delta_0^{-1} d(f_{k+1}(x), f_{k+1}(y)). \end{aligned}$$

Thirdly, assume that $x \in \hat{\mathcal{Q}}_i$ and $y \in \hat{\mathcal{Q}}_j$, for some $i, j \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{k+1}$. Assume that $\operatorname{diam} \mathcal{Q}_i \geq \operatorname{diam} \mathcal{Q}_j$. For the upper bound, note that

$$|x - y| \geq \operatorname{dist}(\hat{\mathcal{Q}}_i, \hat{\mathcal{Q}}_j) \geq \operatorname{diam} \hat{\mathcal{Q}}_i + \operatorname{diam} \hat{\mathcal{Q}}_j = \tfrac{1}{3}(\operatorname{diam} \mathcal{Q}_i + \operatorname{diam} \mathcal{Q}_j). \tag{5.17}$$

Let a_i be the closest point of A to $\mathcal{Q}_i$, let a_j be the closest point of A to $\mathcal{Q}_j$, let e_i be the endpoint of $\mathcal{Q}_i$ that lies between x and a_i, and let e_j be the endpoint of $\mathcal{Q}_j$ that lies between y and a_j. By Proposition 5.2 (1), (5.15), (5.2), Lemma 5.1 (iii), and (5.17),

$$\begin{aligned} d(f_{k+1}(x), f_{k+1}(y)) &\leq d(f_{k+1}(x), \pi(e_i)) + d(\pi(e_i), f(a_i)) + d(f(a_i), f(a_j)) \\ &\quad + d(f(a_j), \pi(e_j)) + d(\pi(e_j), f_{k+1}(y)) \\ &\leq (16 + 184\,L)(\operatorname{diam} \mathcal{Q}_i + \operatorname{diam} \mathcal{Q}_j) + L\,|a_i - a_j| \\ &\leq (16 + 189\,L)(\operatorname{diam} \mathcal{Q}_i + \operatorname{diam} \mathcal{Q}_j) + L\,|x - y| \leq 616\,L\,|x - y| \end{aligned}$$

since $|x - a_i| \leq \operatorname{diam} \mathcal{Q}_i + |e_i - a_i| \leq 5 \operatorname{diam} \mathcal{Q}_i$ and, similarly, $|y - a_j| \leq 5 \operatorname{diam} \mathcal{Q}_j$.

For the lower bound, there are two cases to consider.

Case 1: $\operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) > 800 L^2 \operatorname{diam} \mathcal{Q}_i$.

By Proposition 5.2 (1), (5.15), and Lemma 5.1 (iii),

$$\begin{aligned} d(f_{k+1}(x), f_{k+1}(y)) &\geq d(f(a_i), f(a_j)) - d(f(a_i), \pi(e_i)) - d(\pi(e_i), f_{k+1}(x)) \\ &\quad - d(f(a_j), \pi(e_j)) - d(\pi(e_j), f_{k+1}(y)) \\ &\geq L^{-1}|a_i - a_j| - (184\,L + 16)(\operatorname{diam} \mathcal{Q}_i + \operatorname{diam} \mathcal{Q}_j) \\ &\geq L^{-1}|x - y| - L^{-1}(|x - a_i| + |a_j - y|) - 400\,L \operatorname{diam} \mathcal{Q}_i \\ &\geq L^{-1}|x - y| - (10\,L^{-1} + 400\,L) \operatorname{diam} \mathcal{Q}_i \\ &> L^{-1}|x - y| - 410\,L(800\,L^2)^{-1} \operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) \geq (3L)^{-1}|x - y|. \end{aligned}$$

Case 2: $\operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) \leq 800 L^2 \operatorname{diam} \mathcal{Q}_i$.

In this case, we have

$$|x - y| \leq \operatorname{diam} \mathcal{Q}_i + \operatorname{dist}(\mathcal{Q}_i, \mathcal{Q}_j) + \operatorname{diam} \mathcal{Q}_j \leq 802\,L^2 \operatorname{diam} \mathcal{Q}_i.$$

Case 2 splits now into two subcases.

Case 2.1: $i \in \mathcal{J}_{k+1}$ and $j \in \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_k$. According to the line following (5.15),

$$\begin{aligned} d(f_{k+1}(x), f_{k+1}(y)) \geq d(f_{k+1}(x), f_k(A_k)) &\geq \tfrac{1}{2}\delta_{k+1} \operatorname{diam} \mathcal{Q}_i \\ &\geq \delta_{k+1}(1604\,L^2)^{-1}|x - y|. \end{aligned}$$

Case 2.2: $i, j \in \mathcal{J}_{k+1}$. By Lemma 5.6, we have that $\operatorname{diam} \mathcal{Q}_i > 800\,L\delta_0^{-1} \operatorname{diam} \mathcal{Q}_j$. By Lemma 5.5 (4), the design of γ_i, Proposition 5.2 (1), and (5.15),

$$\begin{aligned} &d(f_{k+1}(x), f_{k+1}(y)) \geq \operatorname{dist}(f_{k+1}(\hat{\mathcal{Q}}_i), f_{k+1}(y)) \\ &\quad \geq \operatorname{dist}(f_{k+1}(\hat{\mathcal{Q}}_i), f(a_j)) - d(\pi(e_j), f(a_j)) - d(\pi(e_j), f_{k+1}(y)) \\ &\quad \geq \tfrac{1}{2}\delta_0 \operatorname{diam} \mathcal{Q}_i - (16 + 184\,L) \operatorname{diam} \mathcal{Q}_j \geq \tfrac{1}{4}\delta_0 \operatorname{diam} \mathcal{Q}_i \geq \tfrac{1}{4}\delta_0(802 L^2)^{-1}|x - y|. \quad \blacksquare \end{aligned}$$

6. Proof of Theorem 1.2

In this section, we will give the proof of the following quantitative version of Theorem 1.2.

Theorem 6.1. *Given $C, C_1 > 0$, $Q > 2$, $p \in (1, Q-1)$, and $L \geq 1$, there exists $L' \geq 1$ with the following property.*

Let (X, d, μ) be a complete geodesic (C_1, Q)-Ahlfors regular metric measure space supporting a p-Poincaré inequality with data C. Let $A \subset \mathbb{R}$ be a closed set, let I be the smallest closed interval of $\mathbb{R}$ containing A, and let $f: A \to X$ be an L-bi-Lipschitz embedding. Then there exists an L'-bi-Lipschitz extension $F: I \to X$ of f.

Moreover, if (x, y) is a component of $I \setminus A$, then

$$\operatorname{diam} F([x, y]) \leq 75 \max\{|x-y|, d(f(x), f(y))\}. \tag{6.1}$$

The remainder of this section is devoted to the proof of this theorem. Let $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$ be the Whitney decomposition of $I \setminus A$ from Lemma 5.1, and let

$$\hat{A} := A \cup \bigcup_{i\in\mathbb{N}} \hat{\mathcal{Q}}_i.$$

Recall that $\hat{\mathcal{Q}}_i$ denotes the middle third of the Whitney interval $\mathcal{Q}_i$ and that E denotes the set of endpoints of Whitney intervals $\{\mathcal{Q}_i\}_{i\in\mathbb{N}}$.

There is a map $\pi: E \to X$ satisfying the properties of Proposition 5.2, there exists a constant $\hat{L} \geq 1$ depending only on C, C_1, Q, p, and L, and there exists an $\hat{L}$-bi-Lipschitz extension of f,

$$g : \hat{A} \to X,$$

satisfying the properties outlined in Proposition 5.4. In particular, if (x, y) is a component of $I \setminus A$, if $\mathcal{Q}_i \subset (x, y)$, and if x is the closest point of A to $\mathcal{Q}_i$, then by (5.2) and (5.15),

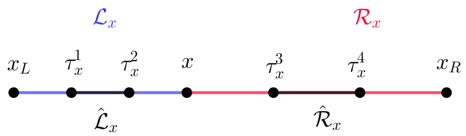
$$\max_{z\in\hat{\mathcal{Q}}_i} d(f(x), g(z)) \leq 2d(f(x), f(y)) + 73 \operatorname{diam} \mathcal{Q}_i. \tag{6.2}$$

We introduce several pieces of notation. Given $x \in E$, we denote by $\mathcal{L}_x$ (respectively, $\mathcal{R}_x$) the Whitney interval for which x is the right (respectively, left) endpoint. As above, $\hat{\mathcal{L}}_x$ and $\hat{\mathcal{R}}_x$ are the middle thirds of intervals $\mathcal{L}_x$ and $\mathcal{R}_x$. By (5.1), for any $x \in E$ we have

$$\tfrac{1}{4} \operatorname{diam} \mathcal{L}_x \leq \operatorname{diam} \mathcal{R}_x \leq 4 \operatorname{diam} \mathcal{L}_x.$$

Further, for any $x \in E$ we write

$$\mathcal{L}_x = [x_L, x], \quad \hat{\mathcal{L}}_x = [\tau_x^1, \tau_x^2], \quad \mathcal{R}_x = [x, x_R] \quad \text{and} \quad \hat{\mathcal{R}}_x = [\tau_x^3, \tau_x^4].$$



Since g is $\hat{L}$-bi-Lipschitz, there exists $C_2 > 0$ depending only on C, C_1, Q, p, and L such that the set $g(\hat{A})$ (and each of its subsets) is $(C_2, 1)$-homogeneous.

6.1. Local modifications around points in E

We divide E into two sets, E' and E'', such that for any two points in E', there exists a point in E'' between them and vice-versa. That is, for any $x \in E'$ we have $x_L, x_R \in E''$, and for any $x \in E''$ we have $x_L, x_R \in E'$.

We perform local modifications around points in E starting with points in E'.

6.1.1. Local modifications around points in E'. Fix a point $x \in E'$. By the $(C_2, 1)$-homogeneity of $g(\hat{A} \setminus (\hat{\mathcal{L}}_x \cup \hat{\mathcal{R}}_x))$, by Corollary 3.4 and Proposition 5.4 (1), (2), there exists a constant $C' \geq 1$, depending only on C, C_1, Q, p, L, and there exists a curve $\sigma_x\colon [0,1] \to X$ such that

(1) $\sigma_x(0) = g(\tau_x^2)$, $\sigma_x(1) = g(\tau_x^3)$;

(2) $\sigma_x([0,1]) \subset B(g(\tau_x^2), 2d(g(\tau_x^2), g(\tau_x^3)))$, so for each $t \in [0,1]$,

$$\begin{aligned} d(\sigma_x(t), \pi(x)) &\leq 2d(g(\tau_x^2), g(\tau_x^3)) + d(g(\tau_2^x), \pi(x)) \\ &\leq 5(2^8\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}; \end{aligned}$$

(3) $\operatorname{length}(\sigma_x) \leq C' \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}$;

(4) $\operatorname{dist}(\sigma_x([0,1]), g(\hat{A} \setminus (\hat{\mathcal{L}}_x \cup \hat{\mathcal{R}}_x))) \geq (C')^{-1} \min\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}$.

By Lemma 4.1, there exists $L^* > 1$, depending only on C, C_1, Q, p, and L, and there exists an L^*-bi-Lipschitz map

$$\gamma_x : [\tau_x^2, \tau_x^3] \to B\big(\pi(x), 6(2^8\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}\big)$$

such that $\gamma_x(\tau_x^2) = \sigma_x(0) = g(\tau_x^2)$, $\gamma_x(\tau_x^3) = \sigma_x(1) = g(\tau_x^3)$, and for all $t \in [\tau_x^2, \tau_x^3]$,

$$\operatorname{dist}(\gamma_x(t), \sigma_x([0,1])) \leq (2^{11}C'\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}.$$

In particular,

$$\operatorname{dist}\big(\gamma_x([\tau_x^2, \tau_x^3]), g(\hat{A} \setminus (\mathcal{L}_x \cup \mathcal{R}_x))\big) \geq (2C')^{-1} \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\}. \tag{6.3}$$

Set $\varepsilon = (2^{50}\tilde{L}\hat{L}L^*C')^{-2}\xi$. Define

$$\begin{aligned} t_x^1 &= \min\{t \in [\tau_x^1, \tau_x^2] : \operatorname{dist}(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \varepsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\}, \\ t_x^2 &= \max\{t \in [\tau_x^2, \tau_x^3] : d(g(t_x^1), \gamma_x(t)) = \varepsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x)\}. \end{aligned}$$

By (5.1), Proposition 5.2 (3), and Proposition 5.4 (1), (2),

$$d(g(\tau_x^1), g(\tau_x^2)) \geq d(\pi(x_L), \pi(x)) - d(\pi(x_L), g(\tau_x^1)) - d(\pi(x), g(\tau_x^2)) \geq \tfrac{1}{2}\xi \operatorname{diam}\mathcal{L}_x,$$

so we have that

$$\begin{aligned} &d(g(t_x^1), g(\tau_x^1)) \\ &\quad\geq d(g(\tau_x^1), g(\tau_x^2)) - \max_{t \in [\tau_x^2, \tau_x^3]} d(\gamma_x(t), g(\tau_x^2)) - \operatorname{dist}(g(t_x^1), \gamma_x([\tau_x^2, \tau_x^3])) \\ &\quad\geq \tfrac{1}{2}\xi \operatorname{diam}\mathcal{L}_x - \max_{t \in [0,1]} d(\sigma_x(t), g(\tau_x^2)) \\ &\qquad - (2^{11}C'\tilde{L})^{-1}\xi \max\{\operatorname{diam}\mathcal{L}_x, \operatorname{diam}\mathcal{R}_x\} - \varepsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\quad\geq \big(\tfrac{1}{2}\xi - 2^{-5}\xi - 2^{-9}\xi - 5\varepsilon\big) \operatorname{diam}\mathcal{L}_x \geq \tfrac{1}{4}\xi \operatorname{diam}\mathcal{L}_x \end{aligned} \tag{6.4}$$

and

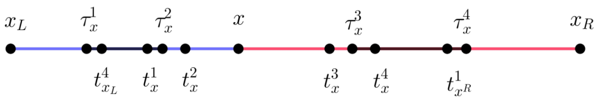
$$d(g(t_x^1), g(\tau_x^2)) \geq \operatorname{dist}(g(t_x^1), \gamma_x([\tau_x^2, \tau_x^3])) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \tag{6.5}$$

Moreover,

$$\begin{aligned} d(\gamma_x(t_x^2), \gamma_x(\tau_x^3)) &\geq \operatorname{dist}(g(\tau_x^3), g([\tau_x^1, \tau_x^2])) - \operatorname{dist}(\gamma_x(t_x^2), g([\tau_x^1, \tau_x^2])) \\ &\geq \tfrac{1}{3}\, \hat{L}^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) - \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \\ &\geq \tfrac{1}{4}\, \hat{L}^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \end{aligned}$$

Define now

$$\begin{aligned} t_x^4 &= \max\left\{t \in [\tau_x^3, \tau_x^4] : d(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\right\} \\ t_x^3 &= \min\left\{t \in [t_x^2, \tau_x^3] : d(\gamma_x(t), g(t_x^4)) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\right\}. \end{aligned}$$



As in (6.4), we have that

$$d(g(t_x^4), g(\tau_x^4)) \geq \tfrac{1}{4}\, \xi \operatorname{diam} \mathcal{R}_x \tag{6.6}$$

and

$$d(g(t_x^4), g(\tau_x^3)) \geq \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x).$$

Moreover, if $t \in [\tau_x^2, \tau_x^3]$ satisfies $d(\gamma_x(t), g(t_x^4)) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)$, then

$$\begin{aligned} d(\gamma_x(t_x^2), \gamma_x(t)) &\geq \operatorname{dist}(g([\tau_x^1, \tau_x^2]), g([\tau_x^3, \tau_x^4])) - 2\varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \\ &\geq (\tfrac{1}{3}\, \hat{L}^{-1} - 2\varepsilon)(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \geq \tfrac{1}{4}\, \hat{L}^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \end{aligned}$$

Therefore, t_x^3 is well defined and

$$t_x^3 - t_x^2 \geq (4\hat{L}L^*)^{-1}(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x). \tag{6.7}$$

6.1.2. Local modifications around points in E''. Fix $x \in E''$. We proceed to define γ_x and points $t_x^1, \dots, t_x^4$ as in Section 6.1.1. The only difference is that we take into account the modifications done for points in $x_L, x_R \in E'$. In particular, we define

$$\begin{aligned} t_x^1 &= \min\{t \in [t_{x_L}^4, \tau_x^2] : \operatorname{dist}(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}, \\ t_x^2 &= \max\{t \in [\tau_x^2, \tau_x^3] : d(g(t_x^1), \gamma_x(t)) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}, \\ t_x^4 &= \max\{t \in [\tau_x^3, t_{x_R}^1] : d(g(t), \gamma_x([\tau_x^2, \tau_x^3])) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}, \\ t_x^3 &= \min\{t \in [t_x^2, \tau_x^3] : d(\gamma_x(t), g(t_x^4)) = \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x)\}. \end{aligned}$$

Equations (6.4), (6.6), (6.7) are still valid for $x \in E''$ as well.

Furthermore, suppose that $x < y$ are consecutive points in E; that is, $x = y_L$ (or equivalently, $y = x_R$). Then,

$$
\begin{aligned}
(6.8)\quad & d(g(t_x^4), g(t_y^1)) \\
&\geq d(\pi(x), \pi(y)) - d(\pi(x), g(t_x^4)) - d(\pi(y), g(t_y^1)) \\
&\geq \xi \operatorname{diam} \mathcal{L}_y - \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) - 6(2^8 \tilde{L})^{-1} \xi (\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \\
&\quad - \varepsilon(\operatorname{diam} \mathcal{L}_y + \operatorname{diam} \mathcal{R}_y) - 6(2^8 \tilde{L})^{-1} \xi (\operatorname{diam} \mathcal{L}_y + \operatorname{diam} \mathcal{R}_y) \\
&\geq (\xi - 10\varepsilon - 60(2^8 \tilde{L})^{-1} \xi) \operatorname{diam} \mathcal{L}_y \geq \tfrac{1}{2} \xi \operatorname{diam} \mathcal{L}_y.
\end{aligned}
$$

6.2. Definition of the extension F and proof of Theorem 6.1

Set

$$\tilde{A} = \hat{A} \setminus \bigcup_{x \in E} [t_x^1, t_x^4].$$

Define the map $F: I \to X$ so that

(1) $F|\tilde{A} = g|\tilde{A}$,
(2) for each $x \in E$, $F|[t_x^2, t_x^3] = \gamma_x|[t_x^2, t_x^3]$,
(3) for each $x \in E$, $F|[t_x^1, t_x^2]$ is the geodesic from $g(t_x^1)$ to $\gamma_x(t_x^2)$ of constant speed,
(4) for each $x \in E$, $F|[t_x^3, t_x^4]$ is the geodesic from $\gamma_x(t_x^3)$ to $g(t_x^4)$ of constant speed.

Clearly, F is an extension of f. In view of (6.2), the following proposition completes the proof of Theorem 6.1.

Proposition 6.1. *The map F is an L'-bi-Lipschitz embedding for some $L' \geq 1$ depending only on C, C_1, Q, p, and L.*

Proof. Fix $s, t \in I$ with $s < t$. We may assume that one of s or t is in $[t_x^1, t_x^4]$ for some $x \in E$, since otherwise $F = g$, which is $\hat{L}$-bi-Lipschitz. Assume without loss of generality that $t \in [t_x^1, t_x^4]$ for some $x \in E$. The proof is a case study.

Case 1. Assume that $s \in [t_x^1, t_x^4]$. There are a few subcases to consider.

Case 1.1. Assume that $s, t \in [t_x^1, t_x^2]$ or $s, t \in [t_x^3, t_x^4]$. Without loss of generality, assume the former. In this case, $F(s)$ and $F(t)$ lie on a geodesic of unit speed joining $g(t_x^1)$ and $\gamma_x(t_x^2)$, and by (6.5),

$$\hat{L}^{-1} \varepsilon(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \leq |t_x^1 - \tau_x^2| \leq |t_x^1 - t_x^2| \leq \operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x,$$

so

$$\frac{d(F(s), F(t))}{|s - t|} = \frac{d(g(t_x^1), \gamma_x(t_x^2))}{|t_x^1 - t_x^2|} \in [\varepsilon, \hat{L}].$$

Case 1.2. Assume that $s, t \in [t_x^2, t_x^3]$. Here $F|[t_x^2, t_x^3] = \gamma_x|[t_x^2, t_x^3]$, and γ_x is L^*-bi-Lipschitz.

Case 1.3. Assume that $s \in [t_x^1, t_x^2]$ and $t \in [t_x^2, t_x^3]$ or $s \in [t_x^2, t_x^3]$ and $t \in [t_x^3, t_x^4]$. Without loss of generality, we assume the former. Then $F(s)$ lies on a geodesic of unit speed joining $g(t_x^1)$ and $\gamma_x(t_x^2)$, and $F(t) = \gamma_x(t)$. Since $\gamma_x(t_x^2)$ is a closest point of $\gamma_x([t_x^2, t_x^3])$

to $g(t_x^1)$, Lemma 4.2 implies that the gluing $F([t_x^1, t_x^3]) = g([t_x^1, t_x^2]) \cup \gamma_x([t_x^2, t_x^3])$ is bi-Lipschitz with a constant depending only on that of γ_x, which itself depends only on C, C_1, Q, p, and L.

Case 1.4. Assume that $s \in [t_x^1, t_x^2]$ and $t \in [t_x^3, t_x^4]$. By (6.7),

$$(4\hat{L}L^*)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \le |t_x^2 - t_x^3| \le |s-t| \le \operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x.$$

On one hand, using the fact that $F(s)$ and $F(t_x^2)$, and $F(t_x^3)$ and $F(t)$ lie on unit speed geodesics joining $g(t_x^1)$ to $\gamma_x(t_x^2)$ and $\gamma_x(t_x^3)$ to $g(t_x^4)$ respectively, we get

$$\begin{aligned} d(F(s),F(t)) &\le d(F(s),F(t_x^2)) + d(\gamma_x(t_x^2),\gamma_x(t_x^3)) + d(F(t_x^3),F(t)) \\ &\le d(g(t_x^1),\gamma_x(t_x^2)) + d(\gamma_x(t_x^2),\gamma_x(t_x^3)) + d(\gamma_x(t_x^3),g(t_x^4)) \\ &\le (2\varepsilon + 12\,(2^8\tilde{L})^{-1}\xi)\,(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

On the other hand, arguing similarly gives

$$\begin{aligned} d(F(s),F(t)) &\ge d(\gamma_x(t_x^2),\gamma_x(t_x^3)) - d(F(s),F(t_x^2)) - d(F(t),F(t_x^3)) \\ &\ge (L^*)^{-1}\,|t_x^2 - t_x^3| - 2\varepsilon\,(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\ge ((4\hat{L}(L^*)^2)^{-1} - 2\varepsilon)\,(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\ge (8\hat{L}(L^*)^2)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Case 2. Assume that $s \in [t_y^1, t_y^4]$ for some $y \in E$ with $y < x$. First, using (6.8),

$$(10\,\hat{L})^{-1}\,\xi(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \le |t_y^4 - t_x^1| \le |s-t|.$$

As with Case 1.4,

$$\begin{aligned} d(F(s),F(t_y^4)) &\le d(F(t_y^1),F(t_y^2)) + d(\gamma_y(t_y^2),\gamma_y(t_y^3)) + d(F(t_y^3),F(t_y^4)) \\ &\le (2\varepsilon + 12\,(2^8\tilde{L})^{-1}\xi)\,(\operatorname{diam}\mathcal{L}_y + \operatorname{diam}\mathcal{R}_y), \end{aligned}$$

and similarly,

$$d(F(t_x^1),F(t)) \le (2\varepsilon + 12\,(2^8\tilde{L})^{-1}\xi)\,(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x).$$

Thus

$$\begin{aligned} &d(F(s),F(t)) \le d(F(s),F(t_y^4)) + d(g(t_y^4),g(t_x^1)) + d(F(t_x^1),F(t)) \\ &\quad \le 5(2\varepsilon+1)(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) + \hat{L}\,|t_y^4 - t_x^1| \le 51\,\hat{L}\,\xi^{-1}(2\varepsilon+1)|s-t|. \end{aligned}$$

For the lower bound, if $y = x_L$, then $|s-t| \le 9|x-y|$, and (6.8) gives

$$\begin{aligned} d(F(s),F(t)) &\ge d(g(t_y^4),g(t_x^1)) - d(F(s),F(t_y^4)) - d(F(t_x^1),F(t)) \\ &\ge \tfrac{1}{2}\,\xi\operatorname{diam}\mathcal{L}_x - 10\,(2\varepsilon + 12\,(2^8\tilde{L})^{-1}\xi)\operatorname{diam}\mathcal{L}_x \\ &\ge \tfrac{1}{50}\,\xi\operatorname{diam}\mathcal{L}_x \ge \tfrac{1}{450}\,|s-t|. \end{aligned}$$

If instead $y < x_L$, then

$$\begin{aligned} d(F(s), F(t)) &\geq d(\pi(x), \pi(y)) - d(\pi(y), F(s)) - d(\pi(x), F(t)) \\ &\geq \tilde{L}^{-1}|x-y| - 5(12(2^8\tilde{L})^{-1}\xi + 2\varepsilon)(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \\ &\geq (\tilde{L}^{-1} - 10(12(2^8\tilde{L})^{-1}\xi + 2\varepsilon))|x-y| \geq (18\tilde{L})^{-1}|s-t|. \end{aligned}$$

Case 3. Assume that $s \in \tilde{A}$. Then, $y \in [t^4_{y_L}, t^1_y]$ for some $y \in E$. There are two subcases to consider.

Case 3.1. Assume that $y = x$. There are further subcases here.

Case 3.1.1. Assume first that $t \in [t^1_x, t^2_x]$. As in Case 1.3, $g(t^1_x)$ is a closest point of $g([t^4_{x_L}, t^1_x])$ to $\gamma_x(t^2_x)$, so Lemma 4.2 tells us that $F([t^4_{x_L}, t^1_x])$ is bi-Lipschitz with a constant depending only on C, C_1, Q, p, and L.

Case 3.1.2. Assume now that $t \in [t^2_x, t^3_x]$. By (6.5),

$$\varepsilon\hat{L}^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \leq |t^1_x - \tau^2_x| \leq |s-t| \leq \operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x,$$

so our desired bounds come from

$$\begin{aligned} d(F(s), F(t)) &\geq \operatorname{dist}(g([\tau^1_x, t^1_x]), \gamma_x([t^2_x, t^3_x])) = \varepsilon(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \quad \text{and} \\ d(F(s), F(t)) &\leq \operatorname{diam} g(\hat{\mathcal{L}}_x) + \operatorname{diam}\gamma_x([\tau_2, \tau_3]) \\ &\leq (\hat{L} + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Case 3.1.3. Finally, assume that $t \in [t^3_x, t^4_x]$. By (6.7),

$$(4\hat{L}L^*)^{-1}(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \leq |t^2_x - t^3_x| \leq |t-s| \leq \operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x.$$

Now, on one hand,

$$\begin{aligned} d(F(s), F(t)) &\leq \operatorname{diam} g(\hat{\mathcal{L}}_x) + \operatorname{diam}\gamma_x([\tau^2_x, \tau^3_x]) + \operatorname{diam} g(\hat{\mathcal{R}}_x) \\ &\leq (\hat{L} + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(F(s), F(t)) &\geq \operatorname{dist}(g(\hat{\mathcal{L}}_x), g(t^4_x)) - \operatorname{diam} F([t^3_x, t^4_x]) \\ &\geq ((3\hat{L})^{-1} - \varepsilon)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x). \end{aligned}$$

Case 3.2. Assume that $y < x$. Then

$$3^{-1}(\operatorname{diam}\mathcal{R}_y + \operatorname{diam}\mathcal{L}_x) \leq |\tau^2_y - \tau^1_x| \leq |s-t|.$$

As in Case 2, we have

$$\begin{aligned} d(F(s), F(t)) &\leq d(g(s), g(t^1_y)) + d(g(t^1_y), g(t^1_x)) + d(F(t^1_x), F(t)) \\ &\leq \hat{L}|s - t^1_y| + \hat{L}|t^1_y - t^1_x| + (2\varepsilon + 12(2^8\tilde{L})^{-1}\xi)(\operatorname{diam}\mathcal{L}_x + \operatorname{diam}\mathcal{R}_x) \\ &\leq 3(\hat{L} + L^* + 2\varepsilon + 1)|s-t|. \end{aligned}$$

For the lower bound, set $M := (2\varepsilon + 6(2^8 \tilde{L})^{-1}\xi)$. If $|s-t| \leq M \operatorname{diam} \mathcal{L}_x$, then the desired bound is a result of the following application of (6.3):

$$\begin{aligned} d(F(s), F(t)) &\geq \operatorname{dist}\big(F([t_x^1, t_x^4]), g(\hat{A} \setminus (\mathcal{L}_x \cup \mathcal{R}_x))\big) \\ &\geq ((2C')^{-1} - 4\varepsilon) \max\{\operatorname{diam} \mathcal{L}_x, \operatorname{diam} \mathcal{R}_x\} \geq (16C')^{-1} \operatorname{diam} \mathcal{L}_x. \end{aligned}$$

If $|s-t| > M \operatorname{diam} \mathcal{L}_x$, then

$$\begin{aligned} d(F(s), F(t)) &\geq d(g(s), g(t_x^1)) - d(F(t_x^1), F(t)) \geq \hat{L}^{-1}|s - t_x^1| - \operatorname{diam} F([t_x^1, t_x^4]) \\ &\geq \tfrac{1}{16}\hat{L}^{-1}|s-t| - (2\varepsilon + 6(2^8\tilde{L})^{-1}\xi)(\operatorname{diam} \mathcal{L}_x + \operatorname{diam} \mathcal{R}_x) \\ &\geq \tfrac{1}{16}\hat{L}^{-1}|s-t| - 5(2\varepsilon + 6(2^8\tilde{L})^{-1}\xi)\operatorname{diam} \mathcal{L}_x \geq \tfrac{1}{32}\hat{L}^{-1}|s-t|. \quad \blacksquare \end{aligned}$$

6.3. The unbounded case

Assuming that X is unbounded, one can replace I in Theorem 6.1 by $\mathbb{R}$. The difference here is that we consider a Whitney decomposition of $\mathbb{R} \setminus A$. The unboundedness of X guarantees the existence of function $\pi: E \to X$ as in Proposition 5.2. The rest of the proof is verbatim.

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