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Universal bounds on the entropy of toroidal attractors

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Abstract. A toroidal set is a compactum $K \subseteq \mathbb{R}^3$ which has a neighbourhood basis of solid tori. We study the topological entropy of toroidal attractors K, bounding it from below in terms of purely topological properties of K. In particular, we show that for a toroidal set K, either any smooth attracting dynamics on K has an entropy at least $\log 2$, or (up to continuation) K admits smooth attracting dynamics which are stationary (hence with a zero entropy).

1. Introduction

It is well known that attractors can be very complicated both topologically and dynamically. One wonders to what extent these two sorts of complexity are related, and in particular, whether "topological strangeness" of a compactum K alone may already force a certain degree of complexity on *any* possible attracting dynamics on K. With the usual interpretation that a strictly positive entropy h is indicative of complicated dynamics, we are therefore interested in bounds of the form

$$h(f|_K) \geq \log p(K) > 0 \tag{1.1}$$

where f is any $\mathcal{C}^\infty$ diffeomorphism having the compactum K as an attractor and, crucially, $p(K)$ depends *only* on K and *not* on f. In other words, the bound $\log p(K)$ is universal across all $\mathcal{C}^\infty$ attracting dynamics on a given set K.

We shall argue (Subsection 2.4) that, in a certain sense, the simplest class of compacta K for which universal bounds of the form (1.1) are possible is that of toroidal sets. These are compacta $K \subseteq \mathbb{R}^3$ which have a neighbourhood basis comprised of solid tori ([1]). These tori can wind inside each other and be knotted, and so K will usually be a very complicated continuum. Among toroidal sets, one finds every smooth or polygonal knot, many wild knots, the usual embeddings of n-adic solenoids and generalized solenoids (where n is replaced with a sequence $\{n_k\}$), knotted solenoids, every inverse limit of a self map of $\mathbb{S}^1$ (see Section 3 on pp. 675ff. of [19]), classical continua such as the Whitehead continuum, etc.

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To each toroidal set K (no dynamics yet), one can assign a collection of prime numbers $p_i \geq 2$ called its prime divisors ([2]). These capture purely topological properties of K; roughly speaking, they are related to the amount of "self-winding" of K. A toroidal set may have any number of prime divisors, possibly none, or possibly infinite.

In order to analyze dynamics on toroidal sets, we shall introduce the notion of the geometric degree of a local homeomorphism f which leaves a toroidal set K invariant, denoted by $d_{\mathcal{N}}(f; K)$. We shall concentrate almost exclusively on the case when K is an attractor for f. Then the geometric degree $d_{\mathcal{N}}(f; K)$ is a positive integer which, very roughly, counts the minimum number of "angular preimages" of a point. Unlike the ordinary (homological) degree, where preimages are counted as ± 1 depending on the local behaviour of the map, in the geometric degree each preimage contributes $+1$ to the count. This makes it more sensitive than the ordinary degree, while still retaining the usual properties of the latter. In particular, it is invariant under homotopies or, more precisely, under continuations in the sense of Conley: if $(f_\lambda)_{\lambda \in [0,1]}$ is a continuous family of local homeomorphisms, each having a toroidal attractor K_λ (also varying continuously with λ), the prime divisors of the K_λ and the geometric degrees $d_{\mathcal{N}}(f_\lambda; K_\lambda)$ are independent of λ.

The geometric degree provides a link between entropy and prime divisors. On the one hand, if K is an attractor for a local homeomorphism f, then the prime divisors of the integer $d_{\mathcal{N}}(f; K)$, in the ordinary sense of arithmetics, coincide with the prime divisors $\{p_i\}$ of the toroidal set K (Theorem 3.9) and so, in particular,

$$d_{\mathcal{N}}(f; K) \geq \prod_i p_i .$$

On the other hand, when f is $\mathcal{C}^\infty$, the geometric degree provides the lower bound $h(f|_K) \geq \log d_{\mathcal{N}}(f; K)$ for the entropy (Theorem 5.1). Therefore,

$$h(f|_K) \geq \log d_{\mathcal{N}}(f; K) \geq \log \prod_i p_i . \tag{1.2}$$

Since the prime divisors p_i depend only on K but not on the dynamics, this bound is of the desired form (1.1) with $p(K) = \prod_i p_i$. In particular, as soon as K has at least one prime divisor, we get $h(f|_K) \geq \log 2 > 0$. Notice that the bound $\log 2$ is universal not only across all $\mathcal{C}^\infty$ attracting dynamics on a given K, but in fact across all toroidal attractors with at least one prime divisor. In this sense it is sharp, since it is attained by the standard embedding of the dyadic solenoid in $\mathbb{R}^3$ with its usual dynamics.

If a toroidal attractor K has no prime divisors, then (1.2) reduces to the trivial bound $h(f|_K) \geq \log 1 = 0$ and, given the universal nature of the bound, the (somewhat wild) question arises of whether K actually supports some attracting dynamics with a zero entropy. Since the geometric degree and the prime divisors are invariant under continuation, we may also ask whether K can be continued to an attractor with a zero entropy. It turns out that the answer is in the affirmative: we shall prove that if K is a toroidal attractor with no prime divisors, then it can be continued to a smooth knot γ which is an attractor with stationary dynamics (Theorem 4.1). For this result, the use of the geometric degree instead of the ordinary one is crucial.

Summing up, up to continuation and for smooth dynamics, we obtain the following neat alternative.

Theorem 1.1. *Let K be a toroidal attractor for a $\mathcal{C}^\infty$ diffeomorphism of $\mathbb{R}^3$.*

(i) *If K has some prime divisor, then K and all of its $\mathcal{C}^\infty$ toroidal continuations have an entropy at least* $\log 2$.

(ii) *If K has no prime divisors, then K has a $\mathcal{C}^\infty$ continuation to a smooth knot which is an attractor with stationary dynamics* (*hence with a zero entropy*).

We finally discuss briefly the outline of the paper. In Section 2, we recall some definitions and in particular, that of the geometric index of a solid torus inside another. This was introduced by Schubert ([30]) to study knots, and is basic for our constructions. In Section 3, we define the prime divisors of a toroidal set K and the geometric degree $d_{\mathcal{N}}(f; K)$, and relate the two when K is an attractor. We also prove that they are invariant under continuation. In Section 4, we show that a toroidal attractor K with no prime divisors can be continued to a smooth knot. Section 5 relates the entropy of a toroidal attractor to the geometric degree and hence also to the prime divisors of the attractor. It is here where smoothness assumptions on the dynamics are needed for the first time in order to apply a bound of Yomdin on the entropy. Section 6 contains a comparison between the geometric and the ordinary degree as well as an open question related to the smoothness assumption on the dynamics.

We have included two appendices. Appendix A contains some technicalities about the definition of the geometric index. Appendix B is a brief outline of Yomdin's bound on entropy, tailored to the very particular case we use in this paper.

2. Some background

This section gathers some definitions and results that are needed for the rest of the paper. The most important are the definition and properties of the geometric index in Subsection 2.2.

2.1. Tame solid tori

A solid torus T is a topological space homeomorphic to $\mathbb{D}^2 \times \mathbb{S}^1$; any homeomorphism $h: \mathbb{D}^2 \times \mathbb{S}^1 \to T$ is called a framing of T. The image under h of $\mathbb{D}^2 \times \{*\}$ is called a meridional disk of T, and its boundary (which is a simple closed curve in ∂T), a meridian of T. The image under h of the curve $\{0\} \times \mathbb{S}^1$ is a called a core of T. One should bear in mind that meridional disks can lie in a very crooked way inside T; picturing them as being "radial" is misleading.

In dimensions three (and higher), there exist wild tori, and these are both inconvenient and unnecessary for our purposes. Henceforth, we will almost always confine ourselves to working with tame tori only. Recall that a compact subset $L \subseteq \mathbb{R}^3$ is tame if there exists a homeomorphism $g: \mathbb{R}^3 \to \mathbb{R}^3$ which sends L onto a polyhedron, and semilocally tame if there exist an open neighbourhood U of L and a homeomorphism $g: U \to g(U) \subseteq \mathbb{R}^3$ which sends L onto a polyhedron. Evidently, a tame set L is also semilocally tame, and a deep theorem of Moise states that the converse is also true (see Theorem 3 on p. 254 of [24]).

In the sequel, we will make use of the following facts:

(i) Any solid torus $T \subseteq \mathbb{R}^3$ can be perturbed an arbitrarily small quantity to make it tame. To prove this, let $h: \mathbb{D}^2 \times \mathbb{S}^1 \to T \subseteq \mathbb{R}^3$ be a framing of T. A classical approximation theorem ensures that for any $\varepsilon > 0$, there is another embedding $h': \mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ that is ε-close to h and is piecewise linear (see, for example, Theorem 2 on p. 251 of [24]). Then the image T' of h' is a polyhedral solid torus ε-close to T.

(ii) Suppose f is a homeomorphism of $\mathbb{R}^3$ or, more generally, a local homeomorphism of $\mathbb{R}^3$, i.e., a homeomorphism $f: U \to f(U) \subseteq \mathbb{R}^3$, where U is open in $\mathbb{R}^3$. If $T \subseteq U$ is a tame solid torus, then $f(T)$ is also tame. This follows because $f(T)$ is evidently semilocally tame and, as mentioned above, it is therefore tame.

2.2. The geometric index

We recall the notion of the geometric index of a solid torus inside another one. This was introduced by Schubert (who called it "order", see Section 9 in [30]), and will be essential in the paper. The original definition and results by Schubert are set up for polyhedral tori, but we need a purely topological version. This is not entirely straightforward because of the existence of wild objects in 3 dimensions, and sorting this out requires some nontrivial results from geometric topology. We have deferred the technical details to Appendix A, since the geometric content of the definition is very intuitive.

We first motivate the definition informally. Let T_0 be a solid torus containing a simple closed curve α in its interior. Let γ be a core of T_0, so that α is homologous to a multiple of γ; say $\alpha = m\gamma$, with $m \in \mathbb{Z}$. The integer m can be computed (up to a sign) by counting the number of times that α intersects any meridional disk D of T_0. This has to be performed algebraically: each point in $D \cap \alpha$ contributes ± 1 depending on the sense in which α crosses D at that point. This algebraic count is independent of the disk D used to perform it. We shall call it the homological winding number of α in T_0, and denote it by $m(\alpha \subseteq T_0)$.

The geometric index of α inside T_0 is defined in a similar manner, but counting the points of intersection in $D \cap \alpha$ geometrically; i.e., each contributes a $+1$, so the count is just the cardinality $|D \cap \alpha|$. This depends on the disk D, and one defines the geometric index $N(\alpha \subseteq T_0)$ by minimizing $|D \cap \alpha|$ over all meridional disks D. It is clear that $N(\alpha \subseteq T_0) \geq |m(\alpha \subseteq T_0)|$, and the inequality can be strict. For example, the Whitehead curve shown in the left panel of Figure 1 has $m = 0$ but geometric index 2.

We now proceed to the formal definitions. For our purposes, it is more convenient to define the geometric index of a solid torus (rather than a simple closed curve) inside another one, but the geometric motivation for the definition is the one just outlined.

Let $T_0 \subseteq \mathbb{R}^3$ be a solid torus, and let T_1 be another solid torus contained in the interior of T_0. We assume these to be tame. Let D be a meridional disk of T_0. We say that D is transverse to T_1 if there exist a neighbourhood N of D in T_0 and a homeomorphism $h: (N, N \cap T_1) \to (\mathbb{D}^2, D_1 \cup \cdots \cup D_r) \times [-1, 1]$ such that

(i) the D_i are pairwise disjoint closed disks,

(ii) h carries D onto $\mathbb{D}^2 \times \{0\}$.

This definition is intended to provide a local model for the intersection $D \cap T_1$. Observe that (ii) implies that $D \cap T_1$ consists of r disjoint closed disks (the preimages of the $D_i \times \{0\}$ under h). These are *not* assumed to be meridional disks of T_1, in contrast to the next definition.

Definition 2.1. The geometric index of T_1 in T_0 is the minimum r such that there exists a meridional disk D of T_0 which is transverse to T_1 and is such that $D \cap T_1$ consists of r meridional disks of T_1. We denote this number by $N(T_1 \subseteq T_0)$.

At this stage, it is not even clear that $N(T_1 \subseteq T_0)$ be well defined in general, since perhaps no meridional disk transverse to T_1 exists at all. However, for tame tori as we are considering, the geometric index is indeed well defined. Moreover, it has the following three fundamental properties:

(P1) If (T_0, T_1) and (T_0', T_1') are homeomorphic pairs, then $N(T_1 \subseteq T_0) = N(T_1' \subseteq T_0')$.

(P2) $N(T_1 \subseteq T_0) = 0$ if and only if there exists a tame ball B such that $T_1 \subseteq B \subseteq T_0$.

(P3) The geometric index is multiplicative: if $T_2 \subseteq T_1 \subseteq T_0$, then

$$N(T_2 \subseteq T_0) = N(T_2 \subseteq T_1) \cdot N(T_1 \subseteq T_0).$$

The first property is direct from the definition, since the notions of meridional disk and transversality are preserved by homeomorphisms. Proving the other two involves some woodworking which is best done when the tori are polyhedral. They are proved, under this assumption, by Schubert (see Hilfssatz 3 and Satz 3 on pp. 171 and 175 of [30]). We shall show in Appendix A how to translate those results to our topological setting. Here we discuss briefly (P2), because it is quite plausible and also provides a good illustration of why the geometric index is more powerful than the winding number m. If the geometric index $N(T_1 \subseteq T_0)$ is zero, there exists a meridional disk of T_0 which does not meet T_1. Then cutting T_0 along this meridional disk produces a ball B between T_1 and T_0. Conversely, should such a ball B exist, one can shrink it inwards by an ambient isotopy G_t of T_0 to a tiny size and find a meridional disk D disjoint from $G_1(B)$. Running the isotopy in the reverse produces a meridional disk disjoint from B and hence from T_1. Notice that (P2) is certainly not true for the winding number m, and again the Whitehead curve of Figure 1 is a classical example.

We finish by returning to the index $N(\alpha \subseteq T_0)$ of a simple closed curve α inside a solid torus T_0. This can be defined just as $N(T_1 \subseteq T_0)$ with the obvious adaptations: in the local model of transversality, one replaces the disks D_i with points p_i, and in Definition 2.1, one removes the condition that the D_i be meridional disks of T_0. The resulting definition agrees with the informal one given at the beginning of this section. It can be shown that $N(\gamma \subseteq T_0) = N(T_1 \subseteq T_0)$ whenever γ is a core of T_1. (In fact, in Schubert's original work the geometric index is defined first for curves, then for solid tori through this relation).

2.3. Dynamics

The following definitions are standard and can be found, for example, in [9] or [20].

(1) *Attractors.* Suppose f is a homeomorphism. A compact set N such that $f(N) \subseteq \operatorname{int} N$ is called a trapping region for f. The attractor defined by that trapping region is the set

$$K := \bigcap_{n \geq 0} f^n(N).$$

A set P is positively invariant if $f(P) \subseteq P$, and invariant if $f(P) = P$. An attractor K has a neighbourhood basis of compact positively invariant sets; namely, the $\{f^n(N)\}_{n\geq 0}$. The attractor K itself is invariant.

If the forward orbit of a point x enters N, then it converges to K asymptotically, this meaning that for every neighbourhood V of K, there exists n_0 such that $f^n(x) \in V$ for $n \geq n_0$. The set of points with this property is called the basin of attraction of K, and is denoted by $\mathcal{A}(K)$. It is an open, invariant neighbourhood of K. One can easily check that in fact not only points in $\mathcal{A}(K)$ are attracted by K, but also compact sets as well. Explicitly: for every neighbourhood V of K and every compact set $C \subseteq \mathcal{A}(K)$, there exists n_0 such that $f^n(C) \subseteq V$ for all $n \geq n_0$.

An attractor can also be defined intrinsically (i.e., without reference to a trapping region) as a compact invariant set which attracts nearby points and has a neighbourhood basis of positively invariant sets. In turn, the latter condition can be replaced by requiring that K be stable in the sense of Lyapunov. Thus the type of attractors we are considering are sometimes called stable attractors.

In this paper, we will consider dynamics generated by a local homeomorphism of $\mathbb{R}^3$. By this we mean a map f defined on an open subset $U \subseteq \mathbb{R}^3$ and which is a homeomorphism onto its image $f(U) \subseteq \mathbb{R}^3$. By the invariance of domain theorem, it suffices to require that f be continuous and injective. The definition of an attractor given above generalizes immediately to this situation.

(2) *Continuations.* We now recall the notion of a continuation introduced by Conley [5], tayloring the definition to our particularly simple case of attracting dynamics.

Suppose we have a parametrized family of local homeomorphisms f_λ all defined on some open set $U \subseteq \mathbb{R}^3$. Here λ ranges in some set of parameters which for definiteness we shall take to be the interval $[0,1] \subseteq \mathbb{R}$. Formally, we consider a continuous map $f\colon [0,1] \times U \to \mathbb{R}^3$ such that every partial mapping $f_\lambda\colon U \to \mathbb{R}^3$ defined by $f_\lambda(x) := f(\lambda, x)$ is injective (hence a local homeomorphism).

Now let $[0,1] \ni \lambda \mapsto K_\lambda$ be a map, where each K_λ is an attractor for f_λ. We want to provide a reasonable definition of "continuity" of this map. Fix some $\lambda_0 \in [0,1]$ and let N be a trapping region for K_{λ_0}, so that $f_{\lambda_0}(N) \subseteq \operatorname{int} N$. Evidently, the latter inclusion still holds for λ close enough to λ_0, and so N contains an attractor for f_λ. We say that $\lambda \mapsto K_\lambda$ is continuous at λ_0 if there is a neighbourhood $I \subseteq [0,1]$ of λ_0 such that for every $\lambda \in I$ the attractor K_λ is precisely the one determined by the trapping region N under the dynamics f_λ; that is,

$$K_\lambda := \bigcap_{n\geq 0} f_\lambda^n(N).$$

It is easy to check that this condition is independent of N, although I will usually depend on it. Of course, we say that $\lambda \mapsto K_\lambda$ is continuous if it is continuous at every $\lambda_0 \in [0,1]$. This *ad hoc* definition should be reasonable on intuitive grounds and will suffice for our purposes, but in fact one can set up a topology in the collection of attractors of the family f_λ so that this map is continuous in the ordinary sense of Topology.

A continuous mapping as just described is called a continuation, or a continuation from $K_{\lambda=0}$ to $K_{\lambda=1}$. Each intermediate K_λ is also called a continuation of $K_{\lambda=0}$.

(3) *Entropy.* We also recall the definition of topological entropy. For this purpose, we consider a compact metric space, (X, d), and a continuous map $g\colon X \to X$.

For each $n \geq 0$, we define the metric

$$\mathrm{d}_n(x, y) := \max_{0 \leq i \leq n} \{\mathrm{d}(g^i(x), g^i(y))\}$$

and, for a given $x \in X$ and the open ball $B(x, \varepsilon) := \{y \in X \mid \mathrm{d}(x, y) < \varepsilon\}$, we consider the (n, ε)-dynamical ball

$$B(x, n, \varepsilon) = \bigcap_{i=0}^{n} g^{-i}(B(x, \varepsilon)).$$

The continuity of g ensures that $B(x, n, \varepsilon)$ is open. Since X is compact, for any $\varepsilon > 0$ there is a minimum number of (n, ε)-dynamical balls needed to cover X. We denote this number by $S(n, \varepsilon)$. It can be interpreted as the minimum amount of initial conditions which, at a scale ε and up to time n, are representative of all possible trajectories of the system.

Now, the topological entropy of g is defined as the following double limit:

$$h(g) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon).$$

Although the definition of h involves the distance d, it is actually invariant among distances which define the same topology. This can be proved explicitly or by means of a purely topological definition of the entropy that does not involve distances (as it was defined originally).

2.4. Why toroidal sets?

It was mentioned in the introduction that toroidal sets are the simplest for which universal bounds of the form (1.1) are possible. The following example justifies this. For part (iv), recall that a compactum $K \subseteq \mathbb{R}^n$ is called cellular if it has a neighbourhood basis comprised of cells; that is, of sets homeomorphic to the standard closed n-ball in $\mathbb{R}^n$.

Example 2.2. Let $K \subseteq \mathbb{R}^n$. If

(i) K is an attractor for a flow, or

(ii) K is a global attractor, or

(iii) K is an attractor in dimension $n \leq 2$, or

(iv) K is a cellular set,

then K can be realized as an attractor with stationary dynamics and so with a zero entropy.

Proof. (i) One just needs to stop the flow on K. This is straightforward to do when the flow is $\mathcal{C}^1$ (by multiplying its vectorfield by a nonnegative function which vanishes precisely on K), but also true when the flow is merely continuous; see the modification of Beck's theorem in Theorem 1.3.3 on p. 22 of [22].

(ii) and (iv) Every global attractor is cellular and, conversely, every cellular set can be realized as a (global) attractor of a flow which is stationary on the set (these are results of Garay [10]).

(iii) An attractor in $\mathbb{R}^n$ has a finitely generated Čech cohomology (see Theorem 1 on p. 2827 of [28]), and then, in dimensions $n \leq 2$, a result of Günther and Segal (see Corollary 3 on p. 326 of [15]) reduces the situation to that of flows. ∎

Remark 2.3. Regarding smoothness of the dynamics, in case (i) the slowed down flow can evidently be made as smooth as the original flow, and in the remaining cases in dimension $n \neq 4$, the dynamics can be made $\mathcal{C}^\infty$. This follows from results by Grayson, Norton and Pugh ([13] and [26]).

Notice that the example shows that several well-known strange attractors (for instance, the Lorenz or Hénon attractors) can also be realized as attractors with stationary dynamics. Thus, "dynamical strangeness" is not always necessary for "topological strangeness".

Bearing in mind our goal of obtaining positive bounds on the entropy which are universal across all attracting dynamics on a given compactum K, the preceding example justifies that we focus on discrete dynamical systems and on dimension n at least 3. Also, we need to go beyond cellular sets. If one regards cells as handlebodies of genus zero, a natural next step in complexity consists in considering compacta $K \subseteq \mathbb{R}^3$ that have a neighbourhood basis comprised of handlebodies of genus one; i.e., solid tori. These are precisely toroidal sets.

2.5. Toroidal sets and toroidal attractors

A compactum $K \subseteq \mathbb{R}^3$ is toroidal if it is not cellular and has a neighbourhood basis comprised of (not necessarily tame) solid tori $\{T_k\}$. One can always choose the $\{T_k\}$ to satisfy the following conditions:

(i) Each T_k is a tame solid torus.

(ii) Each T_{k+1} is contained in the interior of T_k.

(iii) The geometric indices $N(T_{k+1} \subseteq T_k)$ are all nonzero.

To show that these bases exist, first start with any neighbourhood basis $\{T_k\}$ of K satisfying (ii). This exists by the definition of a toroidal set. Then one perturbs each T_k to a tame torus T'_k by a perturbation of size $\varepsilon_k \to 0$ chosen inductively to ensure that the tame T'_k still contains T_{k+1} in its interior and is contained in the interior of T'_{k-1}. The new tame tori $\{T'_k\}$ form a neighbourhood basis of K which satisfy (i) and (ii). Finally, condition (iii) can be achieved by discarding finitely many of the T'_k. Indeed, for each k such that $N(T'_{k+1} \subseteq T'_k) = 0$, we have (by property (P2) of the geometric index) a ball $T'_{k+1} \subseteq B_k \subseteq T'_k$. Thus if $N(T'_{k+1} \subseteq T'_k) = 0$ for infinitely many k, then we have $\{B_k\}$ a neighbourhood basis for K comprised of balls, so K would be cellular. This contradicts the definition of a toroidal set. Hence $N(T'_{k+1} \subseteq T'_k) = 0$ for finitely many k, and so, by discarding these, we may achieve the three conditions enumerated above. From now on, whenever we speak of "a basis" of a toroidal set, we will always mean a neighbourhood basis that satisfies the conditions enumerated above.

When a toroidal set K is an attractor for a local homeomorphism f, there is a natural way of constructing bases $\{T_k\}$. Since K is toroidal and its basin of attraction is an open neighbourhood of K, it contains some solid torus which is a neighbourhood of K. This compact set is attracted by K, and so there exists n_0 such that $f^n(T) \subseteq \operatorname{int} T$ for every $n \geq n_0$. Thus $\{T_k\} := \{T, f^{n_0}(T), f^{2n_0}(T), \ldots\}$ is a neighbourhood basis of K comprised of nested solid tori. If T is taken to be tame (this can always be done by perturbing it to a polyhedral torus), then all the iterates $f^{kn_0}(T)$ are semilocally tame, and hence tame. Thus $\{T_k\}$ satisfies properties (i) to (iii) listed above. It has the additional crucial property that each pair $(f^{(k+1)n_0}(T), f^{kn_0}(T))$ is homeomorphic to $(T, f^{n_0}(T))$, and so the geometric indices $N(T_{k+1} \subseteq T_k)$ are all equal. We will call a basis constructed in this manner a dynamically generated basis.

Remark 2.4. A toroidal attractor has a neighbourhood basis of solid tori (because it is toroidal) and a neighbourhood basis of positively invariant sets (because it is an attractor), but there is in principle no guarantee that it has a basis of neighbourhoods which satisfy simultaneously both conditions; i.e., which are positively invariant solid tori. We do not know if this is generally true.

We conclude by observing that not every toroidal set can be realized as an attractor, and characterizing topologically which can is an open problem. For example, (i) any smooth knot in $\mathbb{R}^3$ can be realized as a (toroidal) attractor with stationary dynamics, but (ii) there are toroidal knots which are smooth everywhere except at a single point and cannot be realized as attractors whatsoever; however, (iii) there are toroidal knots which are nowhere smooth (in fact, they are wild everywhere in the sense of geometric topology) and they can again be realized as attractors with stationary dynamics. Thus a rather natural geometric gradation of complexity (smooth everywhere, smooth but at a single point, smooth nowhere) does not have a consistent dynamical counterpart.

3. Prime divisors and the geometric degree

In this section, we associate to each toroidal set a collection of prime numbers (possibly empty or infinite) called its prime divisors. They capture purely topological information about the self-winding of K. We also associate to each local homeomorphism between two toroidal sets a rational number called its geometric degree. This is then particularized to the case of a toroidal attractor of a local homeomorphism. Some motivation for the definitions to come can be found in Section 6.

3.1. Prime divisors

Let K be a toroidal set, and let $\{T_k\}$ be a basis for K as described in Subsection 2.5.

Definition 3.1. A prime divisor of K is a prime number $p \geq 2$ which divides $N(T_{k+1} \subseteq T_k)$ for infinitely many k.

By multiplicativity of the geometric index and the primality of p, this is equivalent to requiring that the number $N(T_k \subseteq T_1)$ contains arbitrarily large powers of p as $k \to +\infty$. A yet equivalent condition is that for any fixed k_1, the numbers $N(T_k \subseteq T_{k_1})$ should contain arbitrarily large powers of p as $k \to +\infty$.

Proposition 3.2. *Being a prime divisor is independent of the basis* $\{T_k\}$.

Proof. Let $\{T_k\}$ and $\{T'_\ell\}$ be two bases for K, and suppose that p satisfies Definition 3.1 for $\{T_k\}$. We show that it also satisfies the definition for $\{T'_\ell\}$. Pick k_1 so that $T_{k_1} \subseteq T'_1$. Now, for a given power p^n of p, let $k \geq k_1$ be big enough so that $p^n | N(T_k \subseteq T_{k_1})$. Finally, pick ℓ so that $T'_\ell \subseteq T_k$. Then by multiplicativity of the geometric index applied to $T'_1 \supseteq T_{k_1} \supseteq T_k \supseteq T'_\ell$, we have $N(T_k \subseteq T_{k_1}) | N(T'_\ell \subseteq T'_1)$. Thus $p^n | N(T'_\ell \subseteq T'_1)$, and so $N(T'_\ell \subseteq T'_1)$ contains arbitrarily large powers of p as $\ell \to +\infty$, as was to be shown. ∎

The prime divisors of a toroidal set were first defined in [2] in a less elementary fashion. The definition given here is equivalent.

Example 3.3. (1) Suppose $K \subseteq \mathbb{R}^3$ is a smooth knot. Picking a closed tubular neighbourhood T of K and a diffeomorphism $(T, K) \cong (\mathbb{D}^2 \times \mathbb{S}^1, 0 \times \mathbb{S}^1)$, it is straightforward to construct a neighbourhood basis of K which consists of nested, solid tori such that the geometric index of each consecutive pair is 1. Thus any smooth knot is a toroidal set with no prime divisors. The same holds for a polygonal knot taking pl regular neighbourhoods instead of tubular neighbourhoods.

(2) Suppose K is the intersection of a nested family of solid tori $\{T_i\}$ such that each T_{i+1} winds monotonically (i.e., without doubling back) $n_i \geq 1$ times inside T_i. The monotonicity condition ensures that there exists a meridional disk of T_i which intersects T_{i+1} along n_i disks (so $N(T_{i+1} \subseteq T_i) \leq n_i$), and at each of these intersections, T_{i+1} crosses the meridional disk in the same sense (so $m(T_{i+1} \subseteq T_i) = n_i$). Thus $N(T_{i+1} \subseteq T_i) = n_i$, and so the prime divisors of K are exactly those prime numbers that divide infinitely many of the n_i. In particular, when $n_i = n$ (as in the standard embedding of an n-adic solenoid), the prime divisors of K are the prime divisors of n without multiplicity.

A more interesting example is provided by Whitehead continua.

Example 3.4. Start with an unknotted solid torus T_1 and place a thinner one T_2 in its interior along the black curve depicted in the left panel of Figure 1. Then place an even thinner torus T_3 inside T_2 following the same pattern (by this we mean that (T_2, T_3) is homeomorphic to (T_1, T_2)), and so on. This produces a nested sequence of solid tori $\{T_k\}$ whose intersection K is a toroidal set called a Whitehead continuum, an example of which is shown in the right panel of Figure 1. This prescription only determines the isotopy class of the core of T_{k+1} inside T_k, so K is far from being uniquely determined.

To compute the prime divisors of K, notice that $N(T_{k+1} \subseteq T_k) = N(T_2 \subseteq T_1)$ by construction (and invariance of the geometric index under homeomorphisms). Thus the prime divisors of K are exactly the prime numbers that divide $N(T_2 \subseteq T_1)$. It is clear from the drawing that $N(T_2 \subseteq T_1) \leq 2$, since there are obvious meridional disks of T_1 which intersect T_2 in exactly two points. In fact, one has $N(T_2 \subseteq T_1) = 2$; i.e., no meridional disk of T_1 transverse to T_2 intersects it in less than two disks. This seems intuitively reasonable, but is not completely trivial to prove. A quick argument goes as follows. Since the geometric index $N(T_2 \subseteq T_1)$ and the homological winding number $m(T_2 \subseteq T_1)$ are both given by counting intersections with a meridional disk, one with a sign and the other without, they must differ in an even number. But $m(T_2 \subseteq T_1)$ is clearly zero, so $N(T_2 \subseteq T_1)$ must be even and hence either 0 or 2. The core of T_2 is linked with a meridian of T (in the sense of knot theory), and therefore it cannot be contained in a ball in T_1, so $N(T_2 \subseteq T_1)$

must be nonzero, showing that $N(T_2 \subseteq T_1) = 2$ indeed. Thus K has exactly the prime divisor $p = 2$.

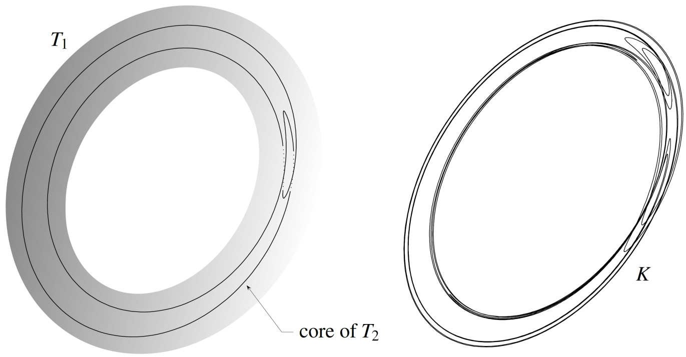

Figure 1. Whitehead's continuum.

3.2. The geometric degree

Let K and K' be two toroidal sets, and let f be a continuous map such that $f(K) = K'$ and f is injective on some open neighbourhood U of K. By invariance of domain, this implies that f is a homeomorphism onto the open set $U' = f(U)$.

Let $\{T_k\}$ and $\{T'_\ell\}$ be bases for K and K', respectively. Notice that for large enough k, the tori T_k are contained in the domain of f, and in fact $\{f(T_k)\}$ is also a basis of K': tameness of $f(T_k)$ is ensured by the discussion at the end of Subsection 2.1, while the invariance of the geometric index under homeomorphisms ensures that $N(f(T_{k+1}) \subseteq f(T_k)) = N(T_{k+1} \subseteq T_k) \neq 0$.

Let k be big enough so that T_k is contained in the domain of f and $f(T_k) \subseteq T'_1$. We define a rational number by

$$d := \frac{N(f(T_k) \subseteq T'_1)}{N(T_k \subseteq T_1)} \in \mathbb{Q}. \tag{3.1}$$

The denominator is nonzero because $N(T_k \subseteq T_1)$ is the product of the geometric indices of the consecutive pairs $T_2 \subseteq T_1$, $T_3 \subseteq T_2$, etc., and these are nonzero by definition of a basis.

Proposition 3.5. *The quantity d does not depend on k.*

Proof. We provisionally denote d by d_k to reflect its dependence on k. Suppose that the condition $f(T_k) \subseteq T'_1$ holds. Then it also holds with $k + 1$ instead of k, and to prove the proposition, it suffices to show that $d_k = d_{k+1}$. We have

$$d_{k+1} = \frac{N(f(T_{k+1}) \subseteq T'_1)}{N(T_{k+1} \subseteq T_1)}.$$

Inserting $f(T_k)$ in the numerator and T_k in the denominator using multiplicativity gives

$$d_{k+1} = \frac{N(f(T_{k+1}) \subseteq T_1')}{N(T_{k+1} \subseteq T_1)} = \frac{N(f(T_{k+1}) \subseteq f(T_k))N(f(T_k) \subseteq T_1')}{N(T_{k+1} \subseteq T_k)N(T_k \subseteq T_1)}.$$

By the invariance of the geometric index under homeomorphisms,

$$N(f(T_{k+1}) \subseteq f(T_k)) = N(T_{k+1} \subseteq T_k),$$

and so a cancellation occurs in the expression above, yielding $d_{k+1} = d_k$. ■

We call d the geometric degree of f with respect to the bases $\{T_k\}$ and $\{T_\ell'\}$, and denote it by $d_{\mathcal{N}}(f; \{T_k\}, \{T_\ell'\})$. This degree is nonzero. This follows readily from Proposition 3.6 below, but can also seen directly. If $d_{\mathcal{N}}(f; \{T_k\}, \{T_\ell'\}) = 0$, then $N(f(T_k) \subseteq T_1') = 0$ for every k. For every ℓ, we may choose k large enough so that $f(T_k) \subseteq T_\ell'$, and then $0 = N(f(T_k) \subseteq T_1') = N(f(T_k) \subseteq T_\ell')N(T_\ell' \subseteq T_1')$ implies $N(f(T_k) \subseteq T_\ell') = 0$, since the second factor is nonzero by definition of a basis. But this implies that there is a ball B_k between $f(T_k)$ and T_ℓ', and this being true for every ℓ would imply that K' is cellular, a contradiction.

As with the usual degree, the geometric degree is multiplicative under composition. Suppose we have three toroidal sets K, K' and K'' with bases $\{T_k\}$, $\{T_\ell'\}$ and $\{T_m''\}$, and local homeomorphisms f and g that carry K to K' and K' to K'', respectively.

Proposition 3.6. *The geometric degree is multiplicative*:

$$d_{\mathcal{N}}(g \circ f; \{T_k\}, \{T_m''\}) = d_{\mathcal{N}}(f; \{T_k\}, \{T_\ell'\})\, d_{\mathcal{N}}(g; \{T_\ell'\}, \{T_m''\}).$$

Proof. Choose ℓ so that $g(T_\ell') \subseteq T_1''$, and then k so that $f(T_k) \subseteq T_\ell'$. Then $(g \circ f)(T_k) \subseteq g(T_\ell')$, so again using the multiplicativity and invariance under homeomorphisms of the geometric index, we have

$$\begin{aligned} N((g \circ f)(T_k) \subseteq T_1'') &= N((g \circ f)(T_k) \subseteq g(T_\ell'))\, N(g(T_\ell') \subseteq T_1'') \\ &= N(f(T_k) \subseteq T_\ell')\, N(g(T_\ell') \subseteq T_1'') \\ &= \frac{N(f(T_k) \subseteq T_1')}{N(T_\ell' \subseteq T_1')}\, N(g(T_\ell') \subseteq T_1'') \\ &= N(f(T_k) \subseteq T_1')\, d_{\mathcal{N}}(g; \{T_\ell'\}, \{T_m''\}). \end{aligned}$$

Dividing through by $N(T_k \subseteq T_1)$ gives

$$\frac{N((g \circ f)(T_k) \subseteq T_1'')}{N(T_k \subseteq T_1)} = d_{\mathcal{N}}(f; \{T_k\}, \{T_\ell'\})\, d_{\mathcal{N}}(g; \{T_\ell'\}, \{T_m''\}).$$

The left-hand side is, by definition, $d_{\mathcal{N}}(g \circ f; \{T_k\}, \{T_m''\})$. This proves the desired equality. ■

Now we particularize to the case when $K' = K$; i.e., f is a local homeomorphism which leaves a toroidal set K invariant. Then there is a natural choice of bases for the geometric degree of f; namely, the same basis $\{T_k\}$ of K for its role both as a source and target space.

Proposition 3.7. *Let f be a local homeomorphism which leaves a toroidal set K invariant. Then $d_{\mathcal{N}}(f;\{T_k\},\{T_k\})$ is independent of the basis $\{T_k\}$.*

Proof. Let $\{T_k\}$ and $\{T'_\ell\}$ be two bases for K. Observe that by multiplicativity of the geometric degree,

$$d_{\mathcal{N}}(\mathrm{Id};\{T'_\ell\},\{T_k\})\, d_{\mathcal{N}}(\mathrm{Id};\{T_k\},\{T'_\ell\}) = d_{\mathcal{N}}(\mathrm{Id};\{T'_\ell\},\{T'_\ell\}) = 1,$$

where the last equality is trivial from the definition. Writing $f = \mathrm{Id} \circ f \circ \mathrm{Id}$, again by multiplicativity,

$$d_{\mathcal{N}}(f;\{T'_\ell\},\{T'_\ell\}) = d_{\mathcal{N}}(\mathrm{Id};\{T'_\ell\},\{T_k\})\, d_{\mathcal{N}}(f;\{T_k\},\{T_k\})\, d_{\mathcal{N}}(\mathrm{Id};\{T_k\},\{T'_\ell\}),$$

and the first and third terms on the right-hand side cancel each other by the previous paragraph. ■

The previous proposition justifies the following definition.

Definition 3.8. If f is a local homeomorphism which leaves a toroidal set K invariant and $\{T_k\}$ is any basis for K, we call $d_{\mathcal{N}}(f;\{T_k\},\{T_k\}) \in \mathbb{Q}$ the geometric degree of f (on K), and denote it by $d_{\mathcal{N}}(f;K)$.

3.3. The case of attractors

We now discuss the case when K is an attractor for f (and not merely an invariant set as in Proposition 3.7). The following theorem is the main result in this section. It relates a purely topological property of K (its prime divisors) to the geometric degree of any attracting dynamics on it.

Theorem 3.9. *Let K be a toroidal attractor for a local homeomorphism f. Then*:

(i) *K has only finitely many prime divisors p_i (possibly none).*

(ii) *$d_{\mathcal{N}}(f;K)$ is an integer whose prime divisors in the ordinary sense of arithmetic are exactly those of K (perhaps with multiplicities).*

As a consequence, $d_{\mathcal{N}}(f;K) \geq \prod_i p_i$, and $d_{\mathcal{N}}(f;K) = 1$ if and only if K has no prime divisors.

Proof. Let $T \subseteq \mathcal{A}(K)$ be a solid torus neighbourhood of K. Pick r big enough so that $f^r(T) \subseteq T$ and set $n := N(f^r(T) \subseteq T)$. We are free to perform our computations in any convenient basis, and we choose the basis $\{T_k\}$ generated from T by using the dynamics; namely, $\{T \supseteq f^r(T) \supseteq f^{2r}(T) \supseteq \cdots\}$. Thus, by definition, $T_{k+1} = f^r(T_k)$, starting with $T_1 := T$.

(i) By construction, each pair $T_{k+1} \subseteq T_k$ is homeomorphic to $f^r(T) \subseteq T$, and so $N(T_{k+1} \subseteq T_k) = N(f^r(T) \subseteq T) = n$ for every k. Thus a prime p divides infinitely many of the $N(T_{k+1} \subseteq T_k)$ if and only if it divides n; in other words, the prime divisors of K are precisely those of n. In particular, if $n = 1$, then K has no prime divisors.

(ii) We first compute the geometric degree of f^r instead of f. By definition,

$$d_{\mathcal{N}}(f^r;K) = \frac{N(f^r(T_k) \subseteq T_1)}{N(T_k \subseteq T_1)} = \frac{N(T_{k+1} \subseteq T_1)}{N(T_k \subseteq T_1)}$$

and, since $T_{k+1} \subseteq T_k$ by construction of the basis $\{T_k\}$, we may interpolate a T_k in the numerator and cancel with the denominator to get

$$d_{\mathcal{N}}(f^r; K) = N(T_{k+1} \subseteq T_k) = n.$$

By multiplicativity of the geometric degree, we must have

$$n = d_{\mathcal{N}}(f^r; K) = (d_{\mathcal{N}}(f; K))^r.$$

An elementary argument shows that if an integer (n, which is an integer because it is a geometric index) has an rth root in $\mathbb{Q}$ (namely $d_{\mathcal{N}}(f; K)$), then in fact the root is an integer, so we deduce $d_{\mathcal{N}}(f; K) \in \mathbb{Z}$. Again using the equality $n = (d_{\mathcal{N}}(f; K))^r$, we see that the prime factors of n must be exactly those of $d_{\mathcal{N}}(f; K)$. ■

For later use, we record here the following fact, established in the proof of Theorem 3.9.

Remark 3.10. If $T \subseteq \mathcal{A}(K)$ is any torus neighbourhood of K and r is such that $f^r(T) \subseteq T$, then $d_{\mathcal{N}}(f; K)^r = N(f^r(T) \subseteq T)$ and the prime divisors of K are the prime divisors of $N(f^r(T) \subseteq T)$. When T is positively invariant, this is trivial from the definitions, but as mentioned in Remark 2.4, we do not know if a toroidal attractor always has a positively invariant neighbourhood which is a solid torus.

Example 3.11. (1) A variation on the construction of Example 3.4 produces Whitehead continua which are automatically attractors. First fix a pair (T_1, T_2) with the usual pattern. Now observe that there is a homeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$ that carries T_1 onto T_2. This must be the case since both are unknotted solid tori, but it is also easy to picture such an f. One starts with T_1, stretches it along some direction to obtain a tube shaped like an ellipse, then twists it one whole turn along its long axis, and finally folds it over. This makes the tube resemble the pattern shown in Figure 1. It is then just a matter of shrinking the diameter of the tube and moving it around to make it fit inside T_1. This whole procedure defines an isotopy of $\mathbb{R}^3$ whose final stage is the required homeomorphism f. Then setting $T_{k+1} := f^k(T_1)$, one obtains a Whitehead continuum $K := \bigcap_{k \geq 1} T_k$ which is by construction an attractor for f. We emphasize that the choice of f is far from unique and can very well lead to different Whitehead continua or even to the same continuum but with essentially different dynamics. For example Garity, Jubran and Schori ([11, 12, 18]) show how to construct two homeomorphisms f and f' of $\mathbb{R}^3$ which both fit the framework just described, and even have the same attractor K but have the property that $f|_K$ is not transitive whereas $f'|_K$ is transitive (and in fact, chaotic in the sense of Devaney).

(2) Whenever a local homeomorphism f has a Whitehead continuum K as an attractor, its geometric degree must be $d_{\mathcal{N}}(f; K) = 2, 4, 8, \ldots$ This is a consequence of Theorem 3.9 and of the fact that K has 2 as its only prime divisor. All possible powers of 2 can appear: for the specific construction of part (1), one has that $d_{\mathcal{N}}(f; K) = 2$ (because T_1 is positively invariant and Remark 3.10(1) applies with $r = 1$), and then the homeomorphisms $f^2, f^3, \ldots$ have the same continuum K as an attractor, and thus have geometric degrees $4, 8, \ldots$

3.4. Invariance under continuation

Recall the setup for continuations: $U \subseteq \mathbb{R}^3$ is some open set and $f_\lambda : U \to \mathbb{R}^3$ is a family of continuous, injective maps (hence homeomorphisms onto their image) which depends continuously on a parameter $\lambda \in [0,1]$. We assume that $\lambda \mapsto K_\lambda$ is a continuation of attractors for the f_λ. We begin by analyzing the local situation.

Proposition 3.12. *Let $\lambda \mapsto K_\lambda$ be a continuation of attractors and suppose that K_{λ_0} is toroidal. Then for λ close enough to λ_0, the attractor K_λ is also toroidal. Moreover, its prime divisors and the geometric degree of f_λ at K_λ are the same as those at λ_0.*

Proof. Let N be a trapping region for f_{λ_0}. Since K_{λ_0} is toroidal, there exists a (tame) solid torus $T \subseteq N$ which is a neighbourhood of K_{λ_0}. This T is contained in the region of attraction of K_{λ_0}, so there exists an iterate $r \geq 1$ such that $f_{\lambda_0}^r(T) \subseteq \operatorname{int} T$. Finally, pick another trapping region N' for K_{λ_0} contained in T. There is an interval of parameters $I \subseteq [0,1]$ which is a neighbourhood of λ_0 and such that for every $\lambda \in I$ we still have the conditions (i) $f_\lambda(N) \subseteq \operatorname{int} N$, $f_\lambda(N') \subseteq \operatorname{int} N'$ (that is, N and N' are still trapping regions), (ii) the continuation K_λ derived from N and N' is the same, and (iii) $f_\lambda^r(T) \subseteq \operatorname{int} T$.

Since $N \supseteq T \supseteq N'$, evidently

$$\underbrace{\bigcap_{k\geq 0} f_\lambda^k(N)}_{=K_\lambda} \supseteq \bigcap_{k\geq 0} f_\lambda^k(T) \supseteq \underbrace{\bigcap_{k\geq 0} f_\lambda^k(N')}_{=K_\lambda},$$

and so

$$K_\lambda = \bigcap_{k\geq 0} f_\lambda^k(T).$$

The same computation holds with $(f_\lambda^r)^k$ instead of f_λ^k. Thus we see that K_λ is a toroidal set; in fact, it has the dynamical basis $\{(f_\lambda^r)^k(T)\}$. If we now show that $N(f_\lambda^r(T) \subseteq T)$ is independent of λ the theorem will follow from Remark 3.10(1).

Let

$$C := f_{\lambda_0}^r(T).$$

Pick any $\lambda \in I$ and define an isotopy of C inside T by $h_t : C \to T$ with

$$h_t := f_{\lambda_0+t(\lambda-\lambda_0)}^r \circ f_{\lambda_0}^{-r} \quad \text{and} \quad t \in [0,1].$$

This isotopy actually takes place in the interior of T, since $h_t(C) = f_{\lambda_0+t(\lambda-\lambda_0)}^r(T) \subseteq \operatorname{int} T$ because $\lambda_0 + t(\lambda - \lambda_0) \in I$ for every $t \in [0,1]$ and condition (iii) holds on I. Trivially, h_t can be extended to an open neighbourhood of C in $\operatorname{int} T$; the same definition given above provides the extension. Now a deep result of Edwards and Kirby (see Corollary 1.2 on p. 63 of [7]) ensures that the isotopy h_t extends to an ambient isotopy, that is, there exists an isotopy $H_t : T \to T$ such that $h_t = H_t|_C$. Then H_1 is a homeomorphism of T which sends $C = f_{\lambda_0}^r(T)$ onto $h_1(C) = f_\lambda^r(T)$, and by the invariance of the geometric index under homeomorphisms, we have $N(f_{\lambda_0}^r(T) \subseteq T) = N(f_\lambda^r(T) \subseteq T)$ as desired. ∎

The previous proposition shows in particular that the set $\{\lambda \in [0,1] : K_\lambda \text{ is toroidal}\}$ is open. In general, it need not be closed, so to obtain a global continuation theorem one needs to place some extra assumption:

Theorem 3.13. *Let $\lambda \mapsto K_\lambda$ be a continuation through toroidal sets (i.e., each K_λ is toroidal). Then all the K_λ have the same prime divisors and all the f_λ have the same geometric degree at K_λ.*

Proof. This is completely straightforward. Since each K_λ is toroidal by assumption, Proposition 3.12 shows that the prime divisors of K_λ and the degree of f_λ at K_λ depend on λ in a locally constant manner. Since $[0, 1]$ is connected, they must be constant. ∎

4. Toroidal attractors with no prime divisors

Suppose that $\gamma \subseteq \mathbb{R}^3$ is a smooth knot. We saw earlier (Example 3.3(1)) that γ is a toroidal set with no prime divisors, and that it can be easily realized as an attractor for homeomorphism (even a $\mathcal{C}^\infty$ diffeomorphism) of $\mathbb{R}^3$. Thus, smooth knots provide a particularly simple example of toroidal attractors with no prime divisors. In this section we show that, up to continuation, this is the only model for such attractors.

Theorem 4.1. *Let K be a toroidal attractor for a homeomorphism f of all $\mathbb{R}^3$. Then K has no prime divisors if and only if it can be continued through toroidal attractors to a smooth knot γ. Moreover, the dynamics on γ can be taken to be stationary.*

If f is $\mathcal{C}^\infty$, then the continuation can also be made $\mathcal{C}^\infty$.

Remark 4.2. (1) Notice that the dynamical system f is assumed to be a homeomorphism of all of $\mathbb{R}^3$, and not only a local homeomorphism as in the previous sections. Without this assumption, it is easy to see that the theorem is false.

(2) The use of the geometric index (through the prime divisors of K) instead of the homological winding number is crucial. To see this, consider Example 6.2 in Section 6. The toroidal attractor K defined there has no homological prime divisors, but it cannot be continued to a smooth knot since it has $p = 3$ as a geometric prime divisor.

Implication ($\Leftarrow$) of Theorem 4.1 is a direct consequence of Example 3.3(1) and the invariance of prime divisors under continuation (Theorem 3.13). The remaining of this section is devoted to a proof of the converse implication. We will prove it in the $\mathcal{C}^\infty$ case. The (slightly simpler) argument in the topological case follows essentially the same steps, and we will just make a couple of comments where appropriate.

Recall that a diffeotopy of $\mathbb{R}^3$ is a $\mathcal{C}^\infty$ map $G\colon [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that each partial map G_t is a diffeomorphism of $\mathbb{R}^3$ and $G_0 = \mathrm{Id}$. A diffeotopy is supported on a set U if each G_t is the identity outside U. This implies in particular that $G_t(U) = U$ for each t. Diffeotopies can be concatenated; i.e., given two diffeotopies $G^{(1)}$ and $G^{(2)}$, one can first perform $G^{(1)}_{2t}$ for $0 \le t \le 1/2$ and then apply $G^{(2)}$ to the end result of the first diffeotopy, namely $G^{(2)}_{2t-1} \circ G^{(1)}_1$ for $1/2 \le t \le 1$. This is generally not differentiable at $t = 1/2$, but can be easily smoothed out (see [17], p. 111).

We will use diffeotopies to produce $\mathcal{C}^\infty$ continuations as follows. Suppose that $g = g_0$ is a diffeomorphism of $\mathbb{R}^3$ which has an attractor K with a trapping region N. If G_t is a diffeotopy supported on N, then $[0,1] \ni \lambda \mapsto G_\lambda \circ g$ produces a continuation of g_0, and the condition that G be supported on N implies that $g_\lambda(N) = G_\lambda \circ g(N) \subseteq G_\lambda(\operatorname{int} N) = \operatorname{int} N$ for every λ. Thus N is a trapping region throughout the whole continuation, and in particular, its maximal invariant subset K_λ is a continuation of K.

The proof of ($\Rightarrow$) of Theorem 4.1 requires two auxiliary lemmas. We state them, then prove the theorem, and finally prove the lemmas. Recall that two solid tori $T_1 \subseteq \operatorname{int} T_0$ are concentric if $\overline{T_0 \setminus T_1}$ is homeomorphic (or equivalently diffeomorphic, if the tori are smooth) to $\mathbb{T}^2 \times [0,1]$. The first auxiliary lemma states that given two concentric solid tori in $\mathbb{R}^3$, we can drag the smaller one via an ambient deformation until it matches the bigger one.

Lemma 4.3. *Let $T_1 \subseteq \operatorname{int} T_0 \subseteq \mathbb{R}^3$ be two concentric smooth solid tori. There exists a diffeotopy G of all $\mathbb{R}^3$ such that*

(i) *$G_t(T_1) \subseteq T_0$ for all $t \in [0,1]$,*

(ii) *$G_1(T_1) = T_0$.*

Moreover, G can be taken to be supported on any prescribed neighbourhood U of $\overline{T_0 \setminus T_1}$.

The second lemma essentially states that if a diffeomorphism of $\mathbb{R}^3$ leaves a solid torus T invariant and is homologically the identity on its boundary, then it can be deformed to be the identity on T.

Lemma 4.4. *Let $T \subseteq \mathbb{R}^3$ be a differentiable solid torus. Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism such that $g(T) = T$ and $g|_{\partial T}$ induces the identity in $H_1(\partial T; \mathbb{Z})$. Then there exists a diffeotopy G of $\mathbb{R}^3$ such that*

(i) *$G_t(T) = T$ for every $t \in [0,1]$,*

(ii) *$G_1 \circ g|_T = \mathrm{Id}_T$.*

Moreover, G can be taken to be supported on any prescribed neighbourhood U of T.

Proof of ($\Rightarrow$) in Theorem 4.1. We assume that K is a toroidal attractor for some diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ and that K has no prime divisors. We are going to show that some power of f can be continued to a smooth curve with trivial dynamics.

Let T be a smooth solid torus in $\mathcal{A}(K)$ which is a neighbourhood of K. Choose a power r such that $f^r(T) \subseteq \operatorname{int} T$. By Remark 3.10, the prime divisors of K are the prime divisors of $N(f^r(T) \subseteq T)$, and so the latter must be 1. Moreover, since f is defined on all of $\mathbb{R}^3$, the tori $f^r(T)$ and T are equivalently knotted. A result of Edwards (see Theorem 3 on p. 4 of [6]) then implies that $f^r(T)$ and T are concentric.

Set $g_0 := f^{2r}$, where the role of the extra factor 2 in the exponent will be clear shortly. This diffeomorphism will be the starting point of our continuation of the dynamics. Notice that $g_0(T) \subseteq \operatorname{int} T$. Let T' be a solid differentiable torus which is a thickening of T thin enough so that $g_0(T') \subseteq \operatorname{int} T$ still.

Let G be the diffeotopy given by Lemma 4.3 applied to $T_0 = T$, $T_1 = g_0(T)$, and $U = T'$. Then $g_\lambda := G_\lambda \circ g_0$ defines a continuation of g_0 such that $g_1(T) = T$ and T' is a trapping region for each g_λ because G is supported on T'.

Claim. *The map $g_1|_{\partial T}$ induces the identity in $H_1(\partial T; \mathbb{Z})$.*

Proof of Claim. The homology group $H_1(\partial T;\mathbb{Z})$ is generated by the homology classes m and ℓ of a meridian and a longitude of ∂T. The element m is uniquely determined up to sign by the condition that when included in T, it becomes zero. The element ℓ depends on the framing of T, but one can fix it up to sign by requiring it to be nullhomologous in $\overline{\mathbb{R}^3 \setminus T}$ (this is called a preferred longitude in knot theory). Now, since g_1 is defined not only on ∂T but in all of $\mathbb{R}^3$ and leaves both T and $\overline{\mathbb{R}^3 \setminus T}$ invariant, $(g_1|_{\partial T})_*(m)$ and $(g_1|_{\partial T})_*(\ell)$ are again nullhomologous when included in T and $\overline{\mathbb{R}^3 \setminus T}$ respectively, so

$$(g_1|_{\partial T})_*(m) = \pm m \quad \text{and} \quad (g_1|_{\partial T})_*(\ell) = \pm \ell \quad \text{in } H_1(\partial T;\mathbb{Z}).$$

Observe that both g_0 and G_1 are orientation preserving; the first because it is an even power of a homeomorphism, and the second because it is isotopic to the identity. Thus g_1 is also orientation preserving, and so are $g_1|_T$, and consequently $g_1|_{\partial T}$. This implies that the determinant of the endomorphism $(g_1|_{\partial T})_*$ of $H_1(\partial T;\mathbb{Z})$ must be positive. The computation from the previous paragraph shows that, in the basis $\{m,\ell\}$, the matrix of $(g_1|_{\partial T})_*$ is diagonal with ± 1 entries; so we conclude that these must either both be positive or both be negative. Thus to prove that $(g_1|_{\partial T})_* = \mathrm{Id}$, we only need to show that the sign in $(g_1|_{\partial T})_*(\ell) = \pm\ell$ is actually a $+$.

Consider the map $f^r|_T : T \to T$. It can be regarded as the composition of f^r and the inclusion $f^r(T) \subseteq T$, both of which induce isomorphisms in homology: the first because it is a homeomorphism, the second because $f^r(T) \subseteq T$ are concentric. Hence $(f^r|_T)_*$ is an isomorphism of $H_1(T;\mathbb{Z}) \cong \mathbb{Z}$, i.e., multiplication by ± 1. Since by definition $g_0 = f^{2r}$, we have that $(g_0|_T)_*$ is the square of $(f^r|_T)_*$, so it follows that $(g_0|_T)_* = \mathrm{Id}$ in $H_1(T;\mathbb{Z})$. The map $G_t \circ g_0|_T$ is a homotopy between $g_0|_T$ and $g_1|_T$, and this homotopy takes place in T by condition (i) of Lemma 4.3, so $(g_1|_T)_* = (g_0|_T)_* = \mathrm{Id}$ in $H_1(T;\mathbb{Z})$.

We can now show that the sign in $(g_1|_{\partial T})_*(\ell) = \pm\ell$ is in fact a $+$. Regarding this as an equality in $H_1(T;\mathbb{Z})$ via the inclusion $\partial T \subseteq T$, we have $(g_1|_T)_*(\ell) = \pm\ell$, and since $(g_1|_T)_* = \mathrm{Id}$, we get $\ell = \pm\ell$. Since $\ell \neq -\ell$ in $H_1(T;\mathbb{Z}) \cong \mathbb{Z}$, we conclude that the sign on the right-hand side must be a $+$. ∎

Now we apply Lemma 4.4 to g_1 and $U = T'$ to obtain a second diffeotopy G_t. Letting λ run from 1 to 2 and setting $g_\lambda := G_{\lambda-1} \circ g_1$, we obtain a further continuation of g_1 to some g_2 such that $g_2|_T$ is the identity and again T' is a trapping region for each g_λ.

Since T' is a trapping region for g_λ for every $\lambda \in [0,2]$ and its maximal invariant subset for g_0 is precisely K, the map $\lambda \mapsto K_\lambda = \bigcap_{k\geq 0} g_\lambda^k(T')$ defines a continuation of K through toroidal attractors. The attractor K_2 is contained in T' and contains T, since the latter is compact and invariant under g_2. The last part of the proof consists in perturbing the dynamics further by gradually adding a "radial" component on T' towards a core γ of T. We first describe this idea in the abstract.

Consider the nested triple of solid tori $\mathbb{D}^2 \times \mathbb{S}^1 \subseteq (2\mathbb{D}^2) \times \mathbb{S}^1 \subseteq (3\mathbb{D}^2) \times \mathbb{S}^1$, where $r\mathbb{D}^2$ denotes the closed unit disk of radius r. Let ρ be a diffeotopy of the interval $[0,3]$ which is supported on $[0,5/2]$ and such that $\rho_t|_{[0,2]}(r) = (1-t/2)r$. We use this to define a diffeotopy f_t of the solid torus $(3\mathbb{D}^2) \times \mathbb{S}^1$ by $f_t(x,s) := (\rho_t(\|x\|)x/\|x\|, s)$. The action of f_t on any meridional disk $(2\mathbb{D}^2) \times \{s\}$ is just given by $x \mapsto (1-t/2)x$, so for $t \in (0,1]$, it is a radial contraction towards the origin. Hence f_t sends $(2\mathbb{D}^2) \times \mathbb{S}^1$ into its interior and has the core $\{0\} \times \mathbb{S}^1$ as an attractor with stationary dynamics. The same is true of the restriction $f_t|_{\mathbb{D}^2 \times \mathbb{S}^1}$. At the final stage $t = 1$, we have $f_1((2\mathbb{D}^2) \times \mathbb{S}^1) = \mathbb{D}^2 \times \mathbb{S}^1$.

Now we copy this abstract construction to our setting. Recall that T' was obtained by thickening T very slightly. Let T'' be obtained from T' in the same way, and consider a diffeomorphism $h\colon (T'', T', T) \to (3\mathbb{D}^2, 2\mathbb{D}^2, \mathbb{D}^2) \times \mathbb{S}^1$. Then we set G to be the diffeotopy of T'' obtained by copying f_t through h; i.e., $G_t(x) := h^{-1} \circ f_t \circ h(x)$. By construction, each G_t is the identity on a neighbourhood of $\partial T''$, and so we can extend G to a diffeotopy of all $\mathbb{R}^3$. This has the following properties:

(i) $G_t(T') \subseteq T'$ for each $t \in [0, 1]$ and $G_1(T') \subseteq T$,

(ii) $G_1(T) \subseteq T$ and the restriction $G_1|_T$ is conjugate (via h) to a radial contraction of $\mathbb{D}^2 \times \mathbb{S}^1$ onto its core $\gamma := h^{-1}(\{0\} \times \mathbb{S}^1)$.

Letting λ run from 2 to 3, consider the continuation $g_\lambda := G_{\lambda-2} \circ g_2$ of g_2. We have that $g_\lambda(T') \subseteq \operatorname{int} T'$ by (i) and the corresponding property for g_2, so again the map $\lambda \mapsto \bigcap_{k\geq 0} g_\lambda^k(T')$ is a continuation of K_2 through toroidal attractors to the attractor K_3 of g_3. We claim that $K_3 = \gamma$. To check this, first notice that for $x \in T$ we have $g_3(x) = G_1 \circ g_2(x) = G_1(x)$, which belongs to $G_1(T) \subseteq T$ again, so it follows inductively that $g_3^k|_T = G_1^k|_T = (G_1|_T)^k$ for every $k \geq 0$. Thus, by (ii), the dynamics of g_3 on T is conjugate (via h) to the dynamics of a radial contraction on $\mathbb{D}^2 \times \mathbb{S}^1$. The latter clearly has $\{0\} \times \mathbb{S}^1$ as an attractor, and so γ is an attractor for $g_3|_T$. Since T' is positively invariant under g_2 and $G_1(T') \subseteq T$, we have $g_3(T') \subseteq T$, and so γ is an attractor also for $g_3|_{T'}$, so in particular, $K_3 = \gamma$. Moreover, $g_3|_\gamma = G_1|_\gamma = \mathrm{Id}|_\gamma$.

The proof is complete. The full continuation from K to K_3 is given by the (smoothed out) concatenation of the G's obtained in the successive steps of the proof. ■

Now we prove the auxiliary Lemmas 4.3 and 4.4. The first is an easy exercise in differential topology, and we will only sketch the proof.

Proof of Lemma 4.3. Let $C_1 \subseteq U$ be a closed collar of ∂T_1 inside T_1, let C be the set $\overline{T_0 \setminus T_1}$, and let $C_0 \subseteq U$ be a closed collar of ∂T_0 in $\overline{\mathbb{R}^3 \setminus T_0}$. (In the topological category, one needs to require that T_0 be tame to ensure that this last collar exists). Each of these sets is diffeomorphic to a 2-torus S times an interval: the first and third are collars; the second is a product because of the hypothesis that T_0 and T_1 are concentric. It is then a standard exercise to construct an ambient diffeotopy that is the identity outside $C_1 \cup C \cup C_0$ and stretches C_1 so much that it becomes $C_1 \cup C$, while shrinking $C \cup C_1$ appropriately so that they fit in C_0. (Pick a diffeomorphism $b\colon C_1 \cup C \cup C_0 \to \mathbb{T}^2 \times [-1, 2]$ such that $b(C_1) = \mathbb{T}^2 \times [-1, 0]$, $b(C) = \mathbb{T}^2 \times [0, 1]$ and $b(C_0) = \mathbb{T}^2 \times [1, 2]$, so that ∂T_1 and ∂T_0 correspond to $\mathbb{T}^2 \times 0$ and $\mathbb{T}^2 \times 1$, respectively. Let $\rho\colon [-1, 2] \to [-1, 2]$ be a diffeotopy which is stationary near -1 and 2 and whose final stage sends 0 to 1 and 1 to $3/2$. Define a diffeotopy a_t of $\mathbb{T}^2 \times [-1, 2]$ by $a_t(x, s) := (x, \rho_t(s))$, and use this to construct a diffeotopy G_t of $\mathbb{R}^3$ given by $G_t(x) = b^{-1} \circ a_t \circ b(x)$ if $x \in C_1 \cup C \cup C_0$, and the identity outside). ■

For the second lemma, we need the following result: any diffeomorphism $g\colon T \to T$ of a solid torus T such that $g|_{\partial T} = \mathrm{Id}_{\partial T}$ is diffeotopic to the identity Id_T; i.e., there exists a diffeotopy G of T such that $G_1 \circ g = \mathrm{Id}_T$. This is relatively easy to prove in the topological case, but in the differentiable category the proof is much more involved, and in fact the result is equivalent to a conjecture of Smale which was settled by Hatcher in [16] (for the formulation used here, see statement (9) on p. 606 of [16]).

Proof of Lemma 4.4. Since diffeomorphisms of the 2-torus are classified up to diffeotopy by the homomorphism they induce in homology (see Theorem 2.5 on p. 55 of [8]), it follows from the hypothesis that there exists a diffeotopy $G^{(1)}$ defined on ∂T such that $G^{(1)}_1 \circ g|_{\partial T} = \mathrm{Id}_{\partial T}$. That diffeotopy can be extended, by using a collar of ∂T in T, to a diffeotopy of all of T which we still denote by $G^{(1)}$. Applying the result recalled before the proof to $G^{(1)}_1 \circ g$, there is a diffeotopy $G^{(2)}$ of T such that $G^{(2)}_1 \circ G^{(1)}_1 \circ g = \mathrm{Id}_T$. Then the concatenation of $G^{(1)}$ and $G^{(2)}$ gives a diffeotopy G of T which carries $g|_T$ onto the identity. Using a sufficiently thin collar of ∂T in $\overline{\mathbb{R}^3 \setminus T}$, one can extend the diffeotopy to all of $\mathbb{R}^3$ having it be the identity outside any prescribed neighbourhood of T. (The proof of the lemma in the topological case requires that one assumes T to be tame, so that ∂T indeed has a collar in $\overline{\mathbb{R}^3 \setminus T}$). ■

5. The entropy of toroidal attractors

The goal of this section is to prove that the geometric degree provides a lower bound on the entropy of a toroidal attractor. It is here where for the first time we need the dynamics to be smooth.

Theorem 5.1. *Let K be a toroidal attractor for a $\mathcal{C}^\infty$ local diffeomorphism f. Then $h(f|_K) \geq \log d_{\mathcal{N}}(f;K)$.*

Coupling the above with the inequality $d_{\mathcal{N}}(f;K) \geq \prod_i p_i$ from Theorem 3.9 yields

$$h(f|_K) \geq \log d_{\mathcal{N}}(f;K) \geq \log \prod_i p_i,$$

which is bound (1.2) from the introduction. The rest of the argument leading to Theorem 1.1 was already detailed there.

The proof of Theorem 5.1 depends, in turn, on the following result concerning the entropy of dynamics on a solid torus. For definiteness, we denote by $V \subset \mathbb{R}^3$ the solid torus obtained by rotating around the z-axis the disk of radius $1/2$ and center $(0, 3/2, 0)$ contained in the yz-plane.

Theorem 5.2. *Let $f : V \to V$ a $\mathcal{C}^\infty$ embedding. Then the entropy of f is bounded below by $h(f) \geq \log N(f(V) \subseteq V)$.*

Proof of Theorem 5.1 *from Theorem* 5.2. Let $T \subseteq \mathcal{A}(K)$ be a smooth solid torus neighbourhood of K, and let r be big enough so that $f^r(T) \subseteq T$. Consider the restriction $f^r|_T : T \to T$, which is a dynamical system on T that still has K as its (global in T) attractor. This implies that K contains the non-wandering set of $f^r|_T$, and therefore, by a result of Bowen (see [4] or [27]), it concentrates all the entropy: $h(f^r|_K) = h(f^r|_T)$. Now observe that Theorem 5.2 is valid not only for dynamics on V but on any smooth solid torus. This is a direct consequence of the invariance of both entropy and the geometric index under conjugation. Thus, for f^r on the smooth solid torus T, we may write

$$h(f^r|_T) \geq \log N(f^r(T) \subseteq T) = r \log d_{\mathcal{N}}(f;K),$$

where in the last step we have used Remark 3.10 (1). Finally, a standard property of entropy ensures that $h(f^r|_K) = rh(f|_K)$, and putting all this together yields $h(f|_K) \geq \log d_{\mathcal{N}}(f;K)$. ■

It remains to prove Theorem 5.2. The argument proceeds by showing that the length of a curve in a solid torus is bounded below by its geometric index (Lemma 5.3), and then applying an inequality of Yomdin ([31]) which relates volume growth rate and topological entropy for smooth dynamics. We only need a very particular case of the inequality, which we describe now.

Given a $\mathcal{C}^\infty$ path in V, $\sigma\colon [0,1] \to V$, its length is given by the usual formula

$$\ell(\sigma) := \int_{[0,1]} \|\sigma'(t)\|\, dt.$$

Suppose $f\colon V \to V$ is a $\mathcal{C}^\infty$ map (not necessarily injective). We consider the iterates of σ under the dynamics generated by f; i.e., the paths $f^n \circ \sigma$, and the exponential growth rate of their lengths

$$\overline{\lim_{n\to\infty}}\ \frac{1}{n} \log \ell(f^n \circ \sigma).$$

The inequality that we need is the following:

$$\overline{\lim_{n\to\infty}}\ \frac{1}{n} \log \ell(f^n \circ \sigma) \leq h(f). \tag{5.1}$$

We have included a proof of (5.1) in Appendix B. This is done both for completeness and because, while in the general case the proof is very involved, in our particular setting it becomes fairly accessible while still retaining the essential ideas.

An extremely crude intuition of why (5.1) might be reasonable is as follows. Suppose $[0,1]$ is partitioned into intervals I_i so that each portion $\sigma(I_i)$ of the curve is contained in an (n,ε)-dynamical ball. This implies that the endpoints of $f^k \circ \sigma|_{I_i}$ lie at a distance less than ε for $k = 0,\dots,n$ and, if the curve $f^k \circ \sigma|_{I_i}$ does not oscillate too much, its length will then be of order ε. Thus the total length $\ell(f^k \circ \sigma)$ will be of the order of ε times the number of intervals I_i, which in turn is related to the minimal number of dynamical balls $S(n,\varepsilon)$ needed to cover V, and therefore to the entropy $h(f)$. This heuristic idea breaks down if the curve $f^k \circ \sigma$ oscillates a lot. If it does so confined within a small ball, the oscillations will have a large contribution to the length but not to the entropy. If it oscillates entering and exiting a dynamical ball a large number of times, then the number of intervals I_i might grossly overestimate $S(n,\varepsilon)$. The assumption on the smoothness of f provides control over this phenomenon.

In order to apply (5.1), we need to bound from below the length of paths in the solid torus V. This is the content of the following lemma.

Lemma 5.3. *For every regular $\mathcal{C}^\infty$ parametrization $\gamma\colon [0,1] \to V$ of a simple closed curve, its length $\ell(\gamma)$ is bounded as follows*:

$$\ell(\gamma) \geq 2\pi N(\gamma \subseteq V).$$

Proof. We shall regard γ as a diffeomorphic embedding $\gamma\colon \mathbb{S}^1 \to V$, and denote its image also by γ.

Let $S \subseteq V$ be the shortest longitude of the solid torus V; namely, the circumference in the $\{z = 0\}$ plane, of radius 1 and centered at the origin. Fix some orientation on S. We define a mapping $h\colon \mathbb{S}^1 \to S$ which captures just the angular information in γ as follows: h is the composition of (i) the parameterization γ, followed by (ii) the orthogonal projection of $\mathbb{R}^3 \setminus z$-axis onto the punctured plane $\{z = 0\} \setminus \{(0, 0, 0)\}$, followed finally by (iii) the radial retraction of the latter onto S. Each of these maps is differentiable, so h is differentiable as well.

Fix $0 < \varepsilon < 2\pi$. By Sard's theorem, the set of critical values of h can be covered by a family of open arcs c_i whose lengths add up to less than ε. The c_i can be taken to be mutually disjoint and finite in number. It is possible that h has no critical values: this happens precisely when γ winds monotonically inside V, without doubling back. To avoid having to discuss that somewhat trivial case separately, we then take the family of arcs $\{c_i\}$ to consist of a single arc of length less than ε.

Write $c_1, \ldots, c_n$ for the arcs in the covering with the convention that indices are taken cyclically (modulo n) as we move along S in the positive orientation, and denote by e_i^1 and e_i^2 the endpoints of c_i. Let D_i^1 and D_i^2 be the radial meridional disks of V that go through the points e_i^1 and e_i^2 (that is, D_i^j is the intersection of V with the plane that contains the z-axis and the point e_i^j). Figure 2 illustrates the definitions showing V viewed from the top.

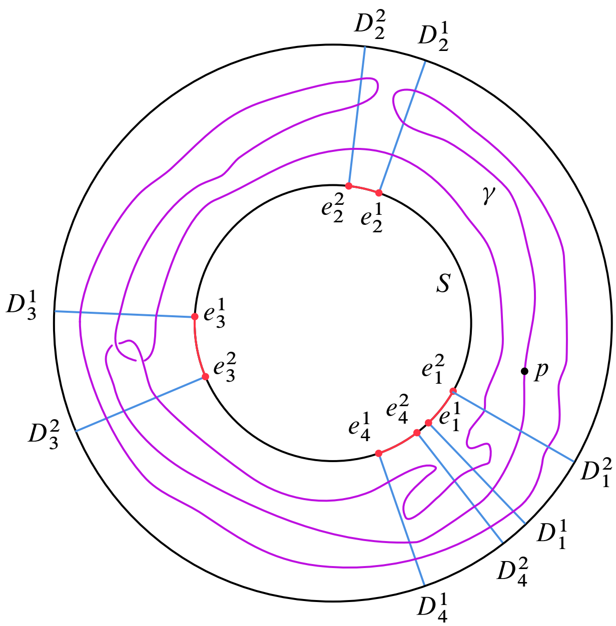

Figure 2. Setup for Lemma 5.3.

Fix any one of the indices i, and denote by V_i the closed sector of V comprised between the disks D_i^2 and D_{i+1}^1. Suppose $t \in \mathbb{S}^1$ is such that $p := \gamma(t)$ belongs to the interior of the sector V_i. Then:

Claim. *The point p lies in an arc of γ which is entirely contained in the sector V_i and joins the disks D_i^2 and D_{i+1}^1.*

Proof of Claim. Define $J \subseteq \mathbb{S}^1$ to be the closure of the connected component of $\mathbb{S}^1 \setminus \gamma^{-1}(\bigcup_{l,m} D_l^m)$ which contains t. Notice that $\gamma^{-1}(\bigcup_{l,m} D_l^m)$ is closed in $\mathbb{S}^1$, so it is disjoint from int J and contains ∂J. Hence $\gamma(J)$ is an arc in V whose interior is disjoint from $\bigcup_{l,m} D_l^m$, and whose endpoints are contained in $\bigcup_{l,m} D_l^m$. Since the arc is connected and contains p, it must be completely contained in the sector V_i and its endpoints must lie in $D_i^2 \cup D_{i+1}^1$. Thus $h(J)$ is an arc in S whose endpoints are contained in $\{e_i^2, e_{i+1}^1\}$ and which is disjoint from all the c_j. The latter implies that $h|_J$ has no critical points, and so, in particular, it cannot map both endpoints of J onto the same point e_i^2 or e_{i+1}^1 (otherwise, it would reach some local extremum on J). Hence the endpoints of $\gamma(J)$ indeed lie one in D_i^2 and the other in D_{i+1}^1. ■

Now pick a point $p_0 \in D_i^2 \cap \gamma$. The following holds.

Claim. *The point p_0 is an endpoint of an arc of γ which is entirely contained in the sector V_i and whose other endpoint is contained in D_{i+1}^1.*

Proof of Claim. Write $p_0 = \gamma(t_0)$ for some $t_0 \in \mathbb{S}^1$. Since γ is transverse to the disk D_i^2, there exists a small arc $I \subseteq \mathbb{S}^1$ centered at t_0 such that $\gamma(I)$ intersects D_i^2 precisely at p_0 and both components of $\gamma(I \setminus \{t_0\})$ lie on different sides of D_i^2. Since the latter is the common boundary of V_i and V_{i-1}, one of the components I_0 of $I \setminus \{t_0\}$ satisfies that $\gamma(I_0)$ is contained in the interior of V_i. It only remains to pick any $t \in I_0$ and apply the previous claim to $p := \gamma(t)$. ■

The two claims above imply that the number of arcs of γ in the sector V_i is precisely $|D_i^2 \cap \gamma|$, and each of them has a length at least d_i^{i+1}, where d_i^{i+1} is the angle between D_i^2 and D_{i+1}^1. Since $|D_i^2 \cap \gamma| \geq N(\gamma \subseteq V)$ by the definition of the geometric index, the total contribution of these arcs to the length of γ is bounded below by $d_i^{i+1} N(\gamma \subseteq V)$. Hence

$$\ell(\gamma) \geq N(\gamma \subseteq V) \sum_i d_i^{i+1} \geq N(\gamma \subseteq V)(2\pi - \varepsilon),$$

where the last inequality makes use of the fact that $\sum_i d_i^{i+1} \geq 2\pi - \varepsilon$ by the choice of the covering $\{c_i\}$. Since this is true for every $\varepsilon > 0$, the lemma follows. ■

The proof of Theorem 5.2 is now straightforward. Recall that $f : V \to V$ is a $\mathcal{C}^\infty$ embedding. Let σ be a smooth core of V (for example its centerline). By the preceding lemma applied to $\gamma = f^n \circ \sigma$, we have

$$\ell(f^n \circ \sigma) \geq 2\pi N(f^n \circ \sigma \subseteq V) = 2\pi N(f^n(V) \subseteq V),$$

where the last equality owes to the definition of the geometric index of a curve and the fact that $f^n \circ \sigma$ is a core of $f^n(V)$.

Observe that for every $n \geq 1$, we have

$$f^n(V) \subseteq f^{n-1}(V) \subseteq \cdots \subseteq f(V) \subseteq V,$$

and so, by multiplicativity and invariance under homeomorphisms of the geometric index,

$$N(f^n(V) \subseteq V) = \prod_{k=0}^{n-1} N(f^{k+1}(V) \subseteq f^k(V)) = \prod_{k=0}^{n-1} N(f(V) \subseteq V) = N(f(V) \subseteq V)^n.$$

Therefore,

$$\ell(f^n \circ \sigma) \geq 2\pi N(f(V) \subseteq V)^n,$$

and now by inequality (5.1),

$$h(f) \geq \overline{\lim_{n\to\infty}} \frac{1}{n} \log \ell(f^n \circ \sigma) \geq \log N(f(V) \subseteq V),$$

as was to be shown.

6. Concluding remarks

6.1. Comparing the homological and geometric degrees

The formal similarities between the homological winding number m and the geometric index N imply that one can obtain "homological" counterparts to the definitions and results above by replacing N with m throughout. Thus one can define homological prime divisors q_i of a toroidal set K, a homological degree $d(f; K)$, etc. These turn out to be describable in terms of Čech cohomology. One needs to assume that $\check{H}^1(K; \mathbb{Z}) \neq 0$ for the definitions to make sense. Then:

(D1) The homological prime divisors of K correspond to those prime numbers q which divide every element in $\check{H}^1(K; \mathbb{Z})$ in the sense that for every $x \in \check{H}^1(K; \mathbb{Z})$ there exists $y \in \check{H}^1(K; \mathbb{Z})$ such that $x = y + \overset{(q)}{\cdots} + y$.

(D2) Suppose f is a local homeomorphism which leaves a toroidal set K invariant. Then the induced homomorphism $f^*: \check{H}^1(K; \mathbb{Z}) \to \check{H}^1(K; \mathbb{Z})$ turns out to be multiplication by the homological degree $d(f; K)$.

A straightforward adaptation of the proof of Theorem 3.9 yields:

(D3a) When K is an attractor for f, the homological degree $d(f; K)$ is an integer whose prime divisors are exactly the prime divisors q_i of K. Moreover,

(D3b) if $T \subseteq \mathcal{A}(K)$ is a solid torus neighbourhood of K and r is such that $f^r(T) \subseteq T$, then the q_i are precisely the prime divisors of $|m(f^r(T) \subseteq T)|$ and the homological degree $d(f; K)$ satisfies $|d(f; K)|^r = |m(f^r(T) \subseteq T)|$.

Since $|m| \leq N$ in general, (D3b) and Remark 3.10 yield the inequality $|d(f; K)| \leq d_{\mathcal{N}}(f; K)$.

For a $\mathcal{C}^0$ embedding $f: V \to V$, the induced map $f_*: H_1(V) \to H_1(V)$ is just multiplication by $m(f(V) \subseteq V)$, and so a classical theorem of Manning ([21]) implies that $h(f) \geq |m(f(V) \subseteq V)|$. This is the homological counterpart to Theorem 5.2, where notably the smoothness assumption is not needed. Replacing N with m in the derivation of Theorem 5.1 from Theorem 5.2 shows that:

(D4) If K is a toroidal attractor for a local homeomorphism and $\check{H}^1(K;\mathbb{Z}) \neq 0$, then $h(f|_K) \geq \log |d(f;K)|$.

Combining (D3a) and (D4), one has $d(f;K) \geq \prod_i q_i$, and as a consequence the universal bound

$$h(f|_K) \geq \log |d(f;K)| \geq \log \prod_i q_i, \tag{6.1}$$

where f is now a local homeomorphism having a toroidal set K as an attractor. This is structurally similar to (1.2), and holds without any smoothness assumptions.

Example 6.1. Let $K \subseteq \mathbb{R}^3$ be the standard embedding of an n-adic solenoid in $\mathbb{R}^3$ (see, for example, Section 17.1 in [20]). This particular embedding is toroidal and its prime divisors, both geometric and homological, are just the prime divisors $p_i = q_i$ of the integer n (counted just once). Thus if f is any local homeomorphism having K as an attractor, then $h(f|_K) \geq \log \prod_i q_i$. There exists a homeomorphism of $\mathbb{R}^3$ (in fact, a $\mathcal{C}^\infty$ diffeomorphism) which realizes the embedded solenoid K as an attractor with its standard dynamics. These have an entropy $\log \prod_i q_i$, and so in this case the universal bound $\log \prod_i q_i$ for any attracting dynamics on K is in fact sharp.

In spite of the previous example, the homological approach has several shortcomings. It is not applicable when $\check{H}^1(K;\mathbb{Z}) = 0$, so for instance it leaves out examples as simple as the Whitehead continua described in Example 3.4. Also, Theorem 4.1 fails if one replaces (geometric) prime divisors with homological prime divisors. The consequence of this is that the bound $h(f|_K) \geq \log \prod_i q_i$ is not sharp even up to continuation, and so the homological degree is not powerful enough to prove Theorem 1.1. The following example provides an illustration of these phenomena.

Example 6.2. Consider a variation of Example 3.11 (1) where T_1 and $T_2 = f(T_1)$ are still unknotted and the core of $f(T_1)$ looks like the pattern in the left panel of Figure 1, but with an extra full winding inside T_1. The resulting toroidal set K can still be realized as an attractor for a $\mathcal{C}^\infty$ diffeomorphism f of $\mathbb{R}^3$. Now $m(T_2 \subseteq T_1) = 1$ and $N(T_2 \subseteq T_1) = 3$. Therefore,

- K has no homological prime divisors and $d(f;K) = 1$,
- K has $p = 3$ as a (geometric) prime divisor, and $d_{\mathcal{N}}(f;K) = 3$.

This illustrates that the inequality $|d(f;K)| \leq d_{\mathcal{N}}(f;K)$ can be strict. Since f is $\mathcal{C}^\infty$, according to Theorem 5.1 we have $h(f|_K) \geq \log d_{\mathcal{N}}(f;K) = \log 3$, whereas the homological bound only says $h(f|_K) \geq \log d(f;K) = \log 1 = 0$. Notice that K cannot be continued to a smooth knot through toroidal attractors (because it has 3 as a prime divisor), but the homological bound on the entropy vanishes.

6.2. An open question

It is natural to ask whether the bound $h(f|_K) \geq \log d_{\mathcal{N}}(f; K)$ is valid when f is just a (local) homeomorphism, without any smoothness assumptions. The derivation of Theorem 5.1 from Theorem 5.2 works equally well in the $\mathcal{C}^0$ case, so this boils down to the following:

Question. *Let $f: V \to V$ be continuous and injective. Is it true that*

$$h(f) \geq \log N(f(V) \subseteq V)?$$

A heuristic argument which might suggest an affirmative answer goes as follows. Let again σ be the centerline of V. After some technical work, it can be arranged that the angular behaviour of $f^n \circ \sigma$ be piecewise monotone, and then the number of monotonicity intervals is bounded below by $N(f^n \circ \sigma \subseteq V) = N(f(V) \subseteq V)^n$. A classical result of Misiurewicz and Szlenk (see Theorem 1 on p. 48 of [23]) concerning the entropy of self-maps of $\mathbb{S}^1$ then suggests that this angular behaviour alone should contribute $\log N(f(V) \subseteq V)$ to the entropy of f.

If the answer to the question posed above is affirmative then, since $d_{\mathcal{N}}(f; K) \geq \prod_i p_i$ holds generally, one would have the universal bound $h(f|_K) \geq \log \prod_i p_i$ for any local homeomorphism f having K as an attractor. In turn, this leads to the following theorem, which improves Theorem 1.1 from the introduction in that it gives a conclusion about K itself, with no continuations involved, and makes no smoothness assumptions:

Theorem. *Let K be a toroidal attractor for a homeomorphism f of $\mathbb{R}^3$. Then either K admits stationary attracting dynamics, or every attracting dynamics on K has an entropy at least* $\log 2$.

Proof. If K has a prime divisor, then $h(f|_K) \geq \log 2$. If K has no prime divisors, as in the proof of Theorem 4.1, this implies that K has a basis of concentric solid tori. Using these, one can construct a $\mathcal{C}^0$ flow which has K as an attractor (see Theorem B on p. 71 of [2]) and is stationary on it. In particular, the time-one map of such a flow provides a homeomorphism of $\mathbb{R}^3$ which has K as an attractor and is stationary on it. ∎

A. Appendix: the topological definition of the geometric index

In this appendix, we relate our topological definition of the geometric index to the standard piecewise linear (pl) one, and show how the properties (P2) and (P3) given after Definition 2.1 can be derived from their pl counterparts. Since we will be working simultaneously with the topological and the piecewise linear geometric indices, we shall distinguish them notationally with a subscript thus: N_{top} and N_{pl}.

We recall the pl definition. Let $T \subseteq \mathbb{R}^3$ be a polyhedral solid torus and $\gamma \subseteq \operatorname{int} T$ a polygonal simple closed curve. We consider all possible polyhedral meridional disks D of T (which may be very contorted) and count the number of points of intersection in $D \cap \gamma$. The geometric index of γ inside T is defined as

$$N_{\text{pl}}(\gamma \subseteq T) := \min\{|D \cap \gamma| : D \text{ is a polyhedral meridional disk of } T\}.$$

It should be intuitively clear that this number is finite: since the objects involved are polyhedral, a slight perturbation of any meridional disk D will always make it transverse to γ, and then $D \cap \gamma$ will consist of finitely many points.

Now suppose $T_1 \subseteq T_0$ is a pair of polyhedral solid tori, with T_1 contained in the interior of T_0. Let γ_1 be a polyhedral core of T_1. Then one defines the geometric index of T_1 inside T_0 as $N_{\mathrm{pl}}(T_1 \subseteq T_0) := N_{\mathrm{pl}}(\gamma_1 \subseteq T_0)$. This definition is correct because cores are unique up to isotopy. An equivalent definition, which does not make use of cores and is more directly related to our definition of N_{top}, is the following. Consider a meridional disk D of T_0. By perturbing it slightly if necessary, we may achieve that it intersects ∂T_1 along disjoint simple closed curves γ_i which bound meridional disks E_i of T_1. Then $N_{\mathrm{pl}}(T_1 \subseteq T_0)$ is the smallest possible number of these curves. The equivalence of this definition and the previous one is a consequence of Hilfssatz 5 on p. 174 of [30].

For a polyhedral pair of tori $T_1 \subseteq T_0$ we now have two definitions of a geometric index. The following proposition shows that both are equivalent.

Proposition A.1. *Suppose $T_1 \subseteq T_0$ is a pair of polyhedral solid tori. Then*

$$N_{\mathrm{top}}(T_1 \subseteq T_0) = N_{\mathrm{pl}}(T_1 \subseteq T_0).$$

Proof. Let p and t be the polyhedral and topological geometric indices of T_1 inside T_0, respectively. As mentioned above, there is a pl meridional disk of T_0 which intersects T_1 in precisely p meridional disks. This can clearly be made transverse to T_1 by slightly perturbing its vertices if necessary; hence $t \leq p$.

To prove the converse, we shall show that any topological meridional disk D of T_0 which is transverse to T_1 in the sense of Definition 2.1 can be used to construct a polyhedral disk which is still transverse to T_1 and intersects it in the same number of meridional disks, thus showing that $p \leq t$.

Since T_1 is polyhedral, the set $M := T_0 \setminus \operatorname{int} T_1$ is a compact 3-manifold whose boundary is the disjoint union of ∂T_0 and ∂T_1. Consider $D^* := D \cap M$, which is a disk with holes. The boundaries of these holes are simple closed curves $\gamma_1, \ldots, \gamma_k$ in ∂T_1 which are meridional curves of T_1.

By the topological transversality condition in Definition 2.1, the set D^* is semilocally tame and hence tame (Theorem 9 on p. 157 of [3]). Thus there exists a homeomorphism h of M onto itself which moves points less than any prescribed ε and sends D^* to a polyhedral disk with holes. Due to the invariance of the boundary of a manifold under homeomorphisms, h sends ∂T_0 and ∂T_1 to themselves. By choosing ε sufficiently small, each $h(\gamma_i)$ is homotopic to γ_i in ∂T_1; in particular, each is still a meridional curve (nonnullhomotopic in ∂T_1 but nullhomotopic in T_1). Thus each $h(\gamma_i)$ bounds a polyhedral meridional disk E_i of T_1; these can clearly be taken to be disjoint by removing their possible intersections. Then $h(D^*) \cup E_1 \cup \cdots \cup E_k$ is a polyhedral meridional disk of T_0 which intersects T_1 in k disks. ∎

The analogues of properties (P1) to (P3) of the geometric index were established, in the pl context, by Schubert. The above proposition then implies that they also hold in the topological context. We shall argue this for the multiplicativity property, for example.

Consider three nested solid tori $T_2 \subseteq T_1 \subseteq T_0$, where as usual we take the tori to be tame. Pick any homeomorphism h_0 from T_0 onto any polyhedral solid torus P_0. This sends $h_0(T_1)$ onto a topological solid torus inside T_0 which is semilocally tame (it being the image under a local homeomorphism of a tame object). Thus it is tame; in fact, there is a homeomorphism $h_1\colon P_0 \to P_0$ which is the identity outside a neighbourhood of $h_0(T_1)$ and sends the latter onto a polyhedral solid torus P_1. Thus the homeomorphism $h_1 \circ h_0$ sends both T_0 and T_1 onto polyhedral tori. Applying the same argument once more, we obtain a homeomorphism h which sends the triple (T_0, T_1, T_2) onto a triple of polyhedral solid tori (P_0, P_1, P_2). By the invariance of N_{top} under homeomorphisms and the proposition above,

$$N_{\mathrm{top}}(T_j \subseteq T_i) = N_{\mathrm{top}}(P_j \subseteq P_i) = N_{\mathrm{pl}}(P_j \subseteq P_i).$$

Then the pl multiplicativity property

$$N_{\mathrm{pl}}(P_2 \subseteq P_0) = N_{\mathrm{pl}}(P_2 \subseteq P_1) \cdot N_{\mathrm{pl}}(P_1 \subseteq P_0)$$

implies directly its topological counterpart

$$N_{\mathrm{top}}(T_2 \subseteq T_0) = N_{\mathrm{top}}(T_2 \subseteq T_1) \cdot N_{\mathrm{top}}(T_1 \subseteq T_0).$$

B. Appendix B: Yomdin's inequality

In this appendix, we prove Yomdin's inequality (5.1) for our very specific case of a map $f\colon V \to V$ and exponential growth rates of lengths of curves. The mathematics here are not new; rather, the purpose of this exercise is to use this relatively simple case to illustrate the main ideas involved in the general Yomdin inequality. We have followed the papers by Yomdin [31] and Gromov [14]. Although we will eventually take f to be $\mathcal{C}^\infty$, for the proof we work with a fixed degree of smoothness k.

Define $M(f)$ to be the maximum of $\|df\|_\infty$ over V, and $R(f)$ as the limit

$$R(f) := \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in V} \|df^n\|_\infty,$$

or in other words, the exponential growth rate of $M(f^n)$. Notice that

$$M(f^{n+m}) \le M(f^n) M(f^m),$$

so the sequence $\log M(f^n)$ is subadditive. Thus $R(f)$ exists and satisfies

$$R(f) \le M(f) < +\infty.$$

For the sake of brevity, we shall denote by EGR a_n the exponential growth rate of any sequence $a_n \ge 0$; i.e., $\overline{\lim}_{n\to\infty} \frac{1}{n} \log a_n$. Now, the topological entropy of f is bounded as follows:

Theorem B.1. *Let $f\colon V \to V$ be of class $\mathcal{C}^k$. For any $\mathcal{C}^k$ curve $\sigma\colon [0,1] \to V$, the following inequality holds*:

$$\text{(B.1)} \qquad \mathrm{EGR}\, \ell(f^n \circ \sigma) \le h(f) + \frac{1}{k} R(f).$$

Notice that $R(f)$ is finite and does not depend on k. Hence, when f and σ are of class $\mathscr{C}^\infty$, then k can be taken to be arbitrarily large and so the $R(f)/k$ term can be removed, leading to inequality (5.1).

Remark B.2. Inequality (B.1) actually holds for arbitrary compact manifolds and simplices of arbitrary dimension instead of paths, replacing length with the appropriate volume. This is the celebrated theorem of Yomdin. Combining this with an inequality of Newhouse ([25]) shows that when f is $\mathscr{C}^\infty$, its entropy $h(f)$ is in fact equal to the supremum of the left-hand side of (5.1) over all smooth simplices of arbitrary dimensions.

Examination of (B.1) under time rescaling (i.e., replacing f with f^q for some fixed $q \geq 1$) leads to the observation that it is enough to prove the following a priori slightly weaker bound:

$$\text{EGR}\, \ell(f^n \circ \sigma) \leq h(f) + A(k) + \frac{1}{k} M(f), \tag{B.2}$$

where $A(k)$ is some number which only depends on k but not on f or σ. This implies inequality (B.1) as follows. Fix the path $\sigma\colon [0,1] \to V$ and pick some sequence $n_k \to +\infty$ such that $1/n_k \log \ell(f^{n_k} \circ \sigma)$ converges to the exponential growth rate $\text{EGR}\, \ell(f^n \circ \sigma)$. Fix some $q \geq 1$ and write

$$n_k = m_k q + r_k, \quad \text{with } 0 \leq r_k < q.$$

After passing to a subsequence, we may take all r_k to be equal to some fixed r. Trivially,

$$\frac{1}{n_k} \log \ell(f^{n_k} \circ \sigma) = \frac{m_k}{n_k} \frac{1}{m_k} \log \ell((f^q)^{m_k} \circ (f^r \circ \sigma)),$$

and taking limits as $k \to +\infty$ yields

$$\text{EGR}\, \ell(f^n \circ \sigma) \leq \frac{1}{q} \text{EGR}\, \ell((f^q)^m \circ (f^r \circ \sigma)),$$

where the exponential growth rate on the right-hand side is taken as $m \to +\infty$. Now the weaker inequality (B.2) applied to the map f^q and the path $f^r \circ \sigma$ on the right-hand side yields

$$\text{EGR}\, \ell(f^n \circ \sigma) \leq \frac{1}{q} \Big(h(f^q) + A(k) + \frac{1}{k} \log M(f^q) \Big).$$

A standard property of entropy ensures that $h(f^q) = q h(f)$. Plugging this above and letting $q \to +\infty$ removes the constant term $A(k)$ and leads to (5.1) by the definition of $R(f)$.

B.1. Basic strategy of the proof

Ultimately, the entropy is related to the minimum number of dynamical balls needed to cover V. One possible way of estimating this from below is the following.

Fix some curve $\sigma\colon [0,1] \to V$ and any open covering $\mathscr{B}$. For each member B of the covering, we want to analyze the intersection of (the image of) σ with B, which will generally be a union of curves, and how the length of these grows as we iterate them with f. Thus we consider the set $\sigma^{-1}(B) \subseteq [0,1]$, which is a disjoint union of compact intervals (its connected components), and the length of $f^n \circ \sigma|_{\sigma^{-1}(B)}$ for each $B \in \mathcal{B}$. Obviously, as B ranges over the covering $\mathcal{B}$, the sets $\sigma^{-1}(B)$ provide a covering of $[0,1]$, and so $\ell(f^n \circ \sigma) \leq \sum_{B\in\mathcal{B}} \ell(f^n \circ \sigma|_{\sigma^{-1}(B)})$. In particular, when $\mathcal{B}$ is a subcover of $\mathcal{B}(n,\varepsilon)$ with the minimal number of elements, namely $S(n,\varepsilon)$, we have

$$\ell(f^n \circ \sigma) \leq S(n,\varepsilon) \sup_{B\in\mathcal{B}(n,\varepsilon)} \ell(f^n \circ \sigma|_{\sigma^{-1}(B)})$$

because the number of summands is $S(n,\varepsilon)$ and every summand can be overestimated by the supremum of the lengths over the whole cover $\mathcal{B}(n,\varepsilon)$ instead of only the minimal subcover. Taking exponential growth rates on both sides and then letting $\varepsilon \to 0$ leads to

$$\text{(B.3)} \qquad \operatorname{EGR} \ell(f^n \circ \sigma) \leq h(f) + \lim_{\varepsilon\to 0} \operatorname{EGR} \sup_{B\in\mathcal{B}(n,\varepsilon)} \ell(f^n \circ \sigma|_{\sigma^{-1}(B)}).$$

Comparing this with (B.2), we see that one needs to show that the second summand is bounded above by an expression of the form $A(k) + M(f)/k$. The key to doing this is an analysis of the set $\sigma^{-1}(B)$. This is provided by Lemma B.3 below. We introduce a preliminary definition and then state the lemma.

We shall say that a curve $\tau\colon [0,1] \to V$ is normalized if all the derivatives of τ up to order k have a norm ≤ 1. If τ is defined only on a subinterval $J \subseteq [0,1]$, we reparameterize it by letting $\psi\colon [0,1] \to J$ be the unique affine bijection which preserves orientation, and say that τ is normalized if $\tau \circ \psi$ is normalized. Clearly, the length of a normalized curve is bounded above by 1.

Lemma B.3. *Let $f\colon V \to V$ be a $\mathcal{C}^k$ map. There exist numbers $\mu(k)$ and $\varepsilon_0(f,k)$ with the following property. For any $\mathcal{C}^k$ curve $\sigma\colon [0,1] \to V$ and any (n,ε)-dynamical ball $B \subseteq V$ with $\varepsilon < \varepsilon_0$, the set $\sigma^{-1}(B)$ can be covered by at most $c(\sigma,\varepsilon)(\mu(k)M(f)^{1/k})^n$ intervals J_i such that, for each of these, the curve $f^n \circ \sigma|_{J_i}$ is normalized.*

The notation $c(\sigma,\varepsilon)$ means that c depends only on σ and ε, but not on f or B. It follows immediately from the lemma that for $\varepsilon < \varepsilon_0(f,k)$,

$$\ell(f^n \circ \sigma|_{\sigma^{-1}(B)}) \leq \sum_i \ell(f^n \circ \sigma|_{J_i}) \leq c(\sigma,\varepsilon)\,(\mu(k)M(f)^{1/k})^n,$$

since each piece $f^n \circ \sigma|_{J_i}$ is normalized. The right-hand side of the above inequality is independent of B, and so the supremum of these lengths over $B \in \mathcal{B}(n,\varepsilon)$ has the same upper bound. Therefore, extracting the exponential growth rate, one has

$$\operatorname{EGR} \sup_{B\in\mathcal{B}(n,\varepsilon)} \ell(f^n \circ \sigma|_{\sigma^{-1}(B)}) \leq \log\mu(k) + \frac{1}{k}\log M(f)$$

whenever $\varepsilon < \varepsilon_0(f,k)$. In particular, the inequality holds in the limit $\varepsilon \to 0$, so coupled with (B.3) we get

$$\operatorname{EGR} \ell(f^n \circ \sigma) \leq h(f) + \log\mu(k) + \frac{1}{k}\log M(f),$$

which is valid for any $\sigma\colon [0,1] \to V$. With $A(k) = \log\mu(k)$, this has the desired form (B.2), and therefore proves (B.1) via time rescaling as explained above.

B.2. The inductive step for Lemma B.3

Suppose $t \in [0,1]$ belongs to $\sigma^{-1}(B)$, where B is an (n,ε)-dynamical ball. By definition, there is some x such that $\sigma(t) \in B(x,\varepsilon)$, $f \circ \sigma(t) \in B(f(x),\varepsilon)$, etc., up to $f^n \circ \sigma(t) \in B(f^n(x),\varepsilon)$. Thus finding the set $\sigma^{-1}(B)$ can be approached iteratively as follows. For each r, let $S_r \subseteq [0,1]$ be the set of parameters t such that $\sigma(t) \in B(x,\varepsilon)$, $f \circ \sigma(t) \in B(f(x),\varepsilon) \ldots$, up to iterate r. Clearly, the S_r form a decreasing sequence of compact sets and $\sigma^{-1}(B) = S_n$. Suppose we have already found some S_r. To find S_{r+1}, one may proceed as follows. Each S_r is a disjoint union of closed intervals (its connected components; maybe some degenerate). If we write $S_r = \bigcup J$ for that decomposition, one can describe S_{r+1} by restricting attention to each $(f^r \circ \sigma)|_J$ at a time and finding what parameter values $t \in J$ satisfy the additional condition $f^{r+1} \circ \sigma(t) \in B(f^{r+1}(x),\varepsilon)$. In other words, letting τ be the curve $f^r \circ \sigma|_J$, which is contained in the ball $B(z,\varepsilon)$ with $z = f^r(x)$, we need to find the set of parameters $t \in J$ such that $f \circ \tau(t) \in B(f(z),\varepsilon)$. This set might be quite complicated (for instance, it might have infinitely many connected components), and it will be technically convenient to overestimate it slightly; that is, we will actually show that it is contained in a union of closed intervals which we shall be able to bound in number.

The proof of Lemma B.3 thus boils down to a one-step estimate. Much as we did with the time rescaling earlier on, by taking advantage of a degree of freedom related to the metric size of the problem, we can reduce the one-step estimate to the convenient case when the derivatives of f of orders $2,\ldots,k$ are bounded above by 1 and the radius of the ball B is $\varepsilon = 1$. We therefore state the one-step estimate for a solid torus V_λ which is just the solid torus V scaled up by a factor $\lambda \geq 1$:

Lemma B.4. *Let $f_\lambda\colon V_\lambda \to V_\lambda$ be a $\mathcal{C}^k$ map with $\|d^s f\|_\infty \leq 1$ for every $s = 2,\ldots,k$. Let $\sigma_\lambda\colon [0,1] \to V_\lambda$ be a $\mathcal{C}^k$ normalized curve, and let $B \subseteq V_\lambda$ be a closed ball of radius* 1 *and an arbitrary center. Finally, let $S = \{t \in [0,1] : f_\lambda \circ \sigma_\lambda(t) \in B\}$. Then there exists a family of no more than $\mu(k)M(f_\lambda)^{1/k}$ intervals J_i such that*

(i) *the set S is covered by the J_i,*

(ii) *for each of the J_i, the curve $f_\lambda \circ \sigma_\lambda|_{J_i}$ is normalized.*

Applying this one-step case to each of the curves $f_\lambda \circ \sigma_\lambda|_{J_i}$ (after the standard affine reparametrization) and repeating this inductively as described earlier, we obtain that for an $(n,1)$-dynamical ball, the set $\sigma^{-1}(B)$ is covered by at most $(\mu(k)M(f_\lambda)^{1/k})^n$ intervals $J_{i_1 i_2 \ldots i_n}$ such that for each of them, the curve $f_\lambda^n \circ \sigma_\lambda|_{J_{i_1 i_2 \ldots i_n}}$ is normalized.

Proof of Lemma B.3 *from Lemma* B.4. First we scale up all the elements in our problem by a factor $\lambda \geq 1$ to be fixed later. The solid torus V becomes the solid torus $V_\lambda := \lambda \cdot V$, and the map f is replaced with its conjugate via the rescaling; i.e., $f_\lambda\colon V_\lambda \to V_\lambda$ given by $f_\lambda(x) := \lambda \cdot f(x/\lambda)$. Notice that this rescaling does not change the first derivatives of f (so $M(f) = M(f_\lambda)$), but divides its sth derivatives, $s \geq 1$, by a factor λ^{s-1}. Hence, by choosing λ large enough, we can achieve that all the derivatives of f_λ of orders $2,3,\ldots,k$ be bounded above by 1 on V_λ. The cutoff value λ_0 of λ at which this happens depends only on the derivatives of f on V. Let $\varepsilon_0(f,k) := 1/\lambda_0$.

Now consider an (n,ε)-dynamical ball $B \subseteq V$, with $\varepsilon < \varepsilon_0$. It is straightforward to see that its scaled up version $\lambda \cdot B$ is just an $(n,\lambda\varepsilon)$-dynamical ball for f_λ. Choose $\lambda := 1/\varepsilon$

as the scaling factor, so that λB is actually an $(n, 1)$-dynamical ball. Since $\varepsilon < \varepsilon_0$, we have $\lambda > \lambda_0$, and so the derivatives of f_λ of order $2 \leq s \leq k$ are all bounded by 1. This sets us under the assumptions of Lemma B.4 save for the normalization of the curve σ. This is the origin of the $c(\sigma, \varepsilon)$ factor in the statement of the lemma, as follows. Split $[0, 1]$ into N intervals I_j of equal length $1/N$. It is then trivial to check that the standard reparameterization $\sigma_\lambda \circ \psi_j$ satisfies

$$\|d^s \sigma_\lambda \circ \psi_j\|_\infty = \lambda \|d^s \sigma\|_\infty / N^s.$$

Set

$$c(\sigma, \varepsilon) := \max_{1 \leq s \leq k} (\lambda \|d^s \sigma\|_\infty)^{1/s} = \max_{1 \leq s \leq k} (\varepsilon^{-1} \|d^s \sigma\|_\infty)^{1/s}.$$

Clearly, by choosing $N \approx c(\sigma, \varepsilon)$, by which we mean $c(\sigma, \varepsilon)$ rounded up to the closest integer, we have that $\sigma_\lambda \circ \psi_j$ is normalized for all j. Then an inductive application of Lemma B.4 as described before to each piece $\sigma_\lambda \circ \psi_j$ in turn shows that each $S_j := \{t \in [0, 1] : \sigma_\lambda \circ \psi_j(t) \in \lambda \cdot B\}$ can be covered by at most $(\mu(k) M(f_\lambda)^{1/k})^n = (\mu(k) M(f)^{1/k})^n$ intervals $J_{j i_1 i_2 \ldots i_n}$. Scaling back down to V, one has $S_j = \{t \in [0, 1] : \sigma \circ \psi_j(t) \in B\}$, and so $\sigma^{-1}(B) = \bigcup_j S_j \subseteq \bigcup_{j i_1 \ldots i_n} J_{j i_1 \ldots i_n}$ is then covered by at most $c(\sigma, \varepsilon)(\mu(k) M(f)^{1/k})^n$ intervals. ∎

B.3. The one-step estimate

Here we finally prove Lemma B.4. The basic strategy of the proof consists in replacing $f \circ \sigma$ with local Taylor approximations Q of degree k, and solving the problem for these.

The polynomial case. In all the following statements, $Q \colon [0, 1] \to \mathbb{R}^3$ is a polynomial curve of degree k which will eventually be a Taylor approximation to $f \circ \sigma$. Any number expressed in the form $\alpha(k)$ is meant to depend only on k, and not on Q, λ, etc.

(1) Let $B' \subseteq V_\lambda$ be a ball of radius $3/2$, and set

$$S' = \{t \in [0, 1] : Q(t) \in B'\}.$$

Then S' can be covered by at most $\alpha_1(k)$ intervals J_i with the property that $Q(J_i)$ is contained in a ball of radius 2.

Proof. The ball B' is the intersection of an Euclidean ball with the solid torus V_λ. These can be described by polynomial inequations of degrees 2 and 4, respectively, and so S' is described by two polynomial inequalities of degrees $2k$ and $4k$. The set S' is therefore a union of intervals (possibly degenerated) whose endpoints are roots of those polynomials or 0 or 1. Thus S' consists of at most $\alpha_1(k) := 6k + 2$ connected components (if all of them are singletons; if all of them are nondegenerate intervals, there can be at most $3k + 1$ of these). If a component of S' is a singleton, we just enlarge it ever so slightly that its image under Q is contained in the ball of radius 2 concentric with B'.

(2) Assume that the image of Q is contained in a ball of radius 2. Then $[0, 1]$ can be partitioned into $\alpha_2(k)$ intervals J_i such that the reparameterized curves $Q \circ \psi_i$ satisfy the $1/2$-normalization condition $\|d^s(Q \circ \psi_i)\|_\infty \leq 1/2$ for $s = 1, \ldots, k$.

Proof. A result of Markov (see for example [29]) implies that there exists a constant $c(k)$ such that for every polynomial $Q\colon[0,1]\to\mathbb{R}^3$ of degree k, one has $\|d^sQ\|_\infty \le c(k)\|Q\|_\infty$ for $s=1,\dots,k$. Under our assumptions, this is bounded above by $2c(k)$. Thus splitting $[0,1]$ into $N\approx 4c(k)$ intervals J_i of equal length, and reparameterizing affinely via ψ_i, each curve $Q_i := Q\circ\psi_i$ will satisfy

$$\|d^s(Q\circ\psi_i)\|_\infty = N^{-s}\|d^sQ\|_\infty \le N^{-1}\|d^sQ\|_\infty \le N^{-1}2c(k) \le \frac{1}{2} \quad \text{for } s=1,\dots,k.$$

Combining (1) and (2), we get the following.

Lemma B.5. *Let $Q\colon[0,1]\to\mathbb{R}^3$ be a polynomial curve of degree k. Let B' be a ball of radius $3/2$ in V_λ. Then the set*

$$S' = \{t\in[0,1] : Q(t)\in B'\}$$

can be covered by at most $\alpha(k)$ intervals J_i such that $Q|_{J_i}$ is $1/2$-normalized for each i.

Proof. By (1), the set S' can be covered by $\alpha_1(k)$ intervals J_i. For each of these, we consider the reparameterized curve $Q\circ\psi_i$. Its image is contained in a ball of radius 2, as stated in (1), and so applying (2) to $Q\circ\psi_i$, the interval $[0,1]$ can be split into no more than $\alpha_2(k)$ intervals such that $(Q\circ\psi_i)$ restricted to each of them is $1/2$-normalized. ∎

Taylor approximations. The basic strategy of the proof of Lemma B.4 will consist in partitioning the interval $[0,1]$ into subintervals, and replacing $f\circ\sigma$ with its Taylor polynomial of degree k on each of those subintervals. We now make some comments about these approximations. Recall that we have a $\mathcal{C}^k$ map $f_\lambda\colon V_\lambda\to V_\lambda$ with $\|d^sf_\lambda\|_\infty\le 1$ for $s=2,\dots,k$ and also a $\mathcal{C}^k$ normalized curve $\sigma_\lambda\colon[0,1]\to V_\lambda$. For notational ease, we will henceforth drop the subindex λ from all objects except for V_λ itself.

(1) The derivatives of order s of $f\circ\sigma$ can be expressed via the chain rule and, using the fact that f has derivatives ≤ 1 for orders $s\ge 2$ and σ has derivatives ≤ 1 for all orders, we see that for every $s=1,\dots,k$, one has $\|d^s(f\circ\sigma)\|\le A(k)+B(k)M(f)$ for some constants $A(k)$ and $B(k)$ that depend only on k. Assuming that $M(f)\ge 1$, this can be written as $\|d^s(f\circ\sigma)\|_\infty\le c_1(k)M(f)$ for some constant $c_1(k)=A(k)+B(k)$ which depends only on k. When $M(f)<1$, we can just write it as $\|d^s(f\circ\sigma)\|_\infty\le c_1(k)$, but this case will be somewhat trivial in what follows.

(2) Let $J\subseteq[0,1]$ be a closed interval of length δ (which one thinks of as being short). Let t_0 be the midpoint of J, and denote by P the Taylor polynomial of degree k for $f\circ\sigma$ at t_0. Starting with the very crude trivial bound

$$\|(f\circ\sigma)^{(k)}(t)-(f\circ\sigma)^{(k)}(t_0)\|\le 2\|(f\circ\sigma)^{(k)}\|_\infty$$

and integrating it, one obtains

$$\|(f\circ\sigma)^{(s)}|_J - P^{(s)}|_J\|_\infty \le \frac{2\|(f\circ\sigma)^{(k)}\|_\infty}{(k-s)!}\,\delta^{k-s}\le 2c_1(k)M(f)\,\delta^{k-s}$$

for $s=0,1,\dots,k$.

Suppose that $\psi\colon [0,1] \to J$ is the standard affine reparameterization, so that $\psi' = \delta$. Clearly,

$$(f \circ \sigma \circ \psi)^{(s)} = \delta^s (f \circ \sigma)^{(s)} \circ \psi,$$

and so,

$$\|(f \circ \sigma \circ \psi)^{(s)} - (P \circ \psi)^{(s)}\|_\infty = \delta^s \|(f \circ \sigma)^{(s)}|_J - P_J^{(s)}\|_\infty \le 2c_1(k) M(f) \delta^k. \tag{B.4}$$

Proof of Lemma B.4. We are finally ready to prove the one-step estimate. Let $B \subseteq V_\lambda$ be a ball of radius 1, and denote by B' the closed ball with the same center as B but radius $3/2$. As in the statement of the lemma, set $S = \{t \in [0,1] : f \circ \sigma(t) \in B\}$.

Partition $[0,1]$ as the union of $N \approx (4c_1(k)M(f))^{1/k}$ closed intervals J_i of equal length $\delta = 1/N$, so that $2c_1(k)M(f)\delta^k \le 1/2$. Denote by P_i the Taylor approximation to $(f \circ \sigma)|_{J_i}$ as described above, and by $\psi_i\colon [0,1] \to J_i$ the standard affine reparametrization. Let us focus on one of the intervals J_i. By (B.4), we have that

$$\|(f \circ \sigma \circ \psi_i) - (P_i \circ \psi_i)\|_\infty \le \frac{1}{2}.$$

This ensures that if $t \in S \cap J_i$, then $P_i(t)$ belongs to the ball B'. Therefore,

$$S \cap J_i \subseteq \{t \in J_i : P_i(t) \in B'\} = \psi_i^{-1}(\{t \in [0,1] : P_i \circ \psi_i(t) \in B'\}).$$

By Lemma B.5 applied to $Q_i = P_i \circ \psi_i$, the set $\{t \in [0,1] : P_i \circ \psi_i(t) \in B'\}$ can be covered by at most $\alpha(k)$ intervals J'_{ij} such that $(P_i \circ \psi_i)|_{J'_{ij}}$ is 1/2-normalized. Taking the preimage of these via ψ_i produces a family of at most $\alpha(k)$ intervals J_{ij} which cover $S \cap J_i$. Therefore S itself is covered by no more than $\alpha(k)N \approx \mu(k)M(f)^{1/k}$ intervals J_{ij}, where $\mu(k)$ simply absorbs all factors other than $M(f)$.

To conclude the proof, it only remains to show that each $f \circ \sigma|_{J_{ij}}$ is normalized. Let $\psi_{ij}\colon [0,1] \to J_{ij}$ and $\psi'_{ij}\colon [0,1] \to J'_{ij}$ be the standard affine reparameterizations. Observe that the definition of J_{ij} as $\psi_i^{-1}(J'_{ij})$ ensures that $\psi_{ij} = \psi_i \circ \psi'_{ij}$. The condition that $(P_i \circ \psi_i)|_{J'_{ij}}$ be 1/2-normalized thus reads $\|d^s(P_i \circ \psi_{ij})\| \le 1/2$ for $s = 1, \dots, k$. We also have

$$\|d^s(f \circ \sigma \circ \psi_{ij}) - d^s(P_i \circ \psi_{ij})\| \le \|d^s(f \circ \sigma \circ \psi_i) - d^s(P_i \circ \psi_i)\| \le \frac{1}{2},$$

where the first inequality follows because ψ'_{ij} has a derivative less than 1, and the second from (B.4) and the choice of N. Therefore

$$\|d^s(f \circ \sigma \circ \psi_{ij})\| \le 1,$$

as required. ∎

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