

Deformation spaces, rescaled bundles, and the generalized Kirillov formula

Maxim Braverman and Ahmad Reza Haj Saeedi Sadegh

Abstract. In this paper, we construct a smooth vector bundle over the deformation to the normal cone $\text{DNC}(V, M)$ through a rescaling of a vector bundle $E \rightarrow V$, which generalizes the construction of the spinor rescaled bundle over the tangent groupoid by Nigel Higson and Zelin Yi. We also provide an equivariant version of their construction. As the main application, we recover the Kirillov character formula for the equivariant index of Dirac-type operators. As another application, we get an equivariant generalization of the description of the Witten and the Novikov deformations of the de Rham–Dirac operator using the deformation to the normal cone obtained recently by O. Mohsen.

1. Introduction

In the coadjoint orbit method of Kirillov, every character of a compact Lie group is given as an integral over some orbits of the coadjoint action of G on \mathfrak{g}^* . This is known as the *Kirillov character formula*. Berline and Vergne [3] (see also [2, Chapter 8]) extended this formula to a *delocalized* equivariant index theorem for a general equivariant Dirac-type operator on a closed manifold M . In this version of the index theorem, also known as the *generalized Kirillov formula*, the equivariant index is expressed as an integral of an equivariant differential form over the whole manifold. Applying the localization formula for the integral of an equivariant form to the Kirillov formula, one recovers Atiyah–Segal fixed-point formula for the index [1].

In [2, Chapter 8], a local version of the Kirillov formula is proven by adopting Bismut’s generalization of the Getzler rescaling technique (see [4]). In this paper, we give a coordinate-free version of the proof in [2] by using the tangent groupoid $\mathbb{T}M$. This gives an equivariant generalization of the method of Higson and Yi [20].

We also study the *deformation to the normal cone* $\text{DNC}(V, M)$ for an embedding $M \hookrightarrow V$ (see Section 4.1, or [10, 15, 18]) and generalize the rescaled bundle construction of [20] to this deformation space. In some sense, this generalization makes the exposition clearer and more conceptual since it makes it evident which structures are needed for the construction to work. We also describe an equivariant version of this construction. As

one of the applications, we provide an equivariant generalization of the description of the Witten deformation via the deformation to the normal cone, obtained in a recent paper by Mohsen [24], cf. Section 7.2. We also generalize this construction to the Novikov deformation [5, 6, 25].

We now give a slightly more detailed review of this construction.

To an embedding of smooth manifolds $M \hookrightarrow V$ one may associate the deformation to the normal cone, denoted $\text{DNC}(V, M)$, as a set is given by the disjoint union

$$\mathcal{N} \times \{0\} \bigsqcup_{t \neq 0} V \times \{t\},$$

where \mathcal{N} is normal bundle of the embedding. There is a smooth structure on $\text{DNC}(V, M)$ which can be given by an explicit atlas [10] or can be given using an algebraic point of view (see Section 4.1 or [18]).

One important example of the deformation to the normal cone is the *tangent groupoid*. Associated with the diagonal embedding $M \hookrightarrow M \times M$, the deformation space $\text{DNC}(M \times M, M)$, also denoted by $\mathbb{T}M$, is called the tangent groupoid of M . As a set, it is given by the disjoint union

$$TM \times \{0\} \bigsqcup_{t \neq 0} M \times M \times \{t\}.$$

In [12], Connes used the tangent groupoid whose K -theory carries the analytical index of elliptic operators. In [27], van Erp and Yuncken used the tangent groupoid to completely characterize the algebra of pseudodifferential operators. See also the preceding work by Debord and Skandalis [13] on the characterization of classical pseudodifferential operators using the action of \mathbb{R}_+^\times on the tangent groupoid.

In [20], Higson and Yi introduced a coordinate-free approach to Getzler’s rescaling method. This was achieved by introducing the *rescaled spinor bundle* over the tangent groupoid of a spin manifold. So, for a spin manifold M , with the spinor bundle $S \rightarrow M$, Higson and Yi [20] constructed a vector bundle $\mathbb{S} \rightarrow \mathbb{T}M$ over the tangent groupoid. This bundle is given in the following diagram:

$$\begin{array}{ccc} \pi^* \Lambda^* T^* M & & S \boxtimes S^* \\ \downarrow & & \downarrow \\ TM \times \{0\} & \bigsqcup_{t \neq 0} & M \times M \times \{t\}. \end{array}$$

Here, \boxtimes denotes the exterior tensor product of vector bundles.

In this paper, we generalized the construction of the rescaled bundle in [20] to deformation to the normal cones and gave some examples of the rescaled bundles. As an application, we give a proof of the Kirillov formula by introducing a family of rescaled bundles over the tangent groupoid of a manifold M that carries a G -equivariant Clifford module.

The paper is organized as follows: in Section 2, we give a quick overview of Cartan’s model of the equivariant cohomology and equivariant characteristic classes.

In Section 3, we state the (generalized) Kirillov formula and give an overview of the heat kernel proof of this formula. In particular, we introduce the Bismut Laplacian and explain its role in the proof.

In Section 4, we introduce the deformation to the normal cone $\text{DNC}(V, M)$ associated with an embedding of manifolds $M \hookrightarrow V$. We define the notion of scaling bundle (E, ∇) consisting of a vector bundle $E \rightarrow V$ and a connection along with filtration structures on the restriction $E|_M$ and on the endomorphism bundle $\text{End}(E)$ with some compatibility conditions. These data will give a vector bundle $\mathbb{E} \rightarrow \text{DNC}(V, M)$. In the construction of the deformation to the normal cone and the bundle over it, we follow an algebraic method introduced in [18] and [20].

In Section 5, we give an example of the rescaled bundle over the tangent groupoid $\mathbb{T}M$, where M carries a G -equivariant Clifford module structure. We give a family of rescaled bundles $\mathbb{E}_J \rightarrow \mathbb{T}M$ indexed by $J \in \mathbb{N}$.

In Section 6, we study the asymptotics of the heat kernel associated with the Bismut Laplacian and its relation to the rescaled bundles, and then we give a proof of Kirillov’s character formula for the equivariant index of the Dirac operator.

In Section 7, we gave two extra applications of the rescaled bundles in the study of the Witten and Novikov deformations and the equivariant index formula.

2. The equivariant cohomology

In this section, we recall the Cartan module for equivariant cohomology and the construction of the equivariant characteristic classes. In our exposition, we roughly follow [2, Section 7.1].

2.1. The polynomial algebra

Consider the space of polynomials $\mathbb{C}[\mathfrak{g}]$ that comes with the grading

$$\mathbb{C}[\mathfrak{g}] = \bigoplus_k \mathbb{C}_k[\mathfrak{g}]$$

with $\mathbb{C}^k[\mathfrak{g}]$ consists of homogeneous polynomials of degree k . We also consider the corresponding filtration

$$\mathbb{C} = \mathbb{C}^0[\mathfrak{g}] \subset \mathbb{C}^1[\mathfrak{g}] \subset \dots \subset \mathbb{C}[\mathfrak{g}].$$

Denote by $\mathbb{C}[\mathfrak{g}]_{(J)}$ the quotient of $\mathbb{C}[\mathfrak{g}]$ by the ideal of polynomials of order $J + 1$. Every element $a \in \mathbb{C}[\mathfrak{g}]_{(J)}$ has a unique polynomial representative of the form $\sum_{\alpha} a_{\alpha} X^{\alpha}$, and hence, the algebra $\mathbb{C}[\mathfrak{g}]_{(J)}$ inherits a grading (by the degree of the polynomials).

The polynomial algebra appears in the Kirillov formula context as

$$\mathbb{C}[\mathfrak{g}] \otimes \Lambda V,$$

where V is a finite-dimensional vector space. This tensor space has a grading given by

$$\deg(P \otimes a) = 2 \deg(P) + \deg_{\Lambda}(a), \tag{2.1}$$

where $P \in \mathbb{C}[\mathfrak{g}]$ and $a \in \Lambda V$. Similar grading will descend to tensor spaces

$$\mathbb{C}[\mathfrak{g}]_{(J)} \otimes \Lambda V.$$

2.2. G -equivariant differential forms

Assume G is a Lie group acting on the smooth manifold M . For $X \in \mathfrak{g}$, we denote by X^M or just X the vector field on M corresponding to the infinitesimal action on $C^\infty(M)$

$$Xf(m) := \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX)m).$$

Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of polynomial functions on \mathfrak{g} and consider the algebra

$$\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M),$$

which carries a left action of G : For $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ and $g \in G$, we have

$$(g \cdot \alpha)(X) = g \cdot (\alpha(\text{Ad}_{g^{-1}} X)).$$

This algebra has a grading for $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ and $g \in G$ given by the equivariant degree

$$o(\alpha) = o^{\text{ext}}(\alpha) + 2o^{\text{poly}}(\alpha),$$

where o^{ext} is the exterior degree of differential forms and o^{poly} is the polynomial degree.

We define the equivariant differential $d_{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ given by the formula

$$(d_{\mathfrak{g}}\alpha)(X) = d(\alpha(X)) - \iota(X^M)\alpha(X).$$

Note that the contraction $\iota(X^M)$ reduces the differential degree by 1 and increases the polynomial degree by 1, and hence, it increases the equivariant degree by 1. Hence, $d_{\mathfrak{g}}$ increases the equivariant degree by 1. Note that $d_{\mathfrak{g}}^2 \neq 0$, indeed, we have

$$(d_{\mathfrak{g}}^2\alpha)(X) = -\mathcal{L}(X)\alpha(X).$$

However, the square of the equivariant differential vanishes on the *Cartan subalgebra*

$$\mathcal{A}_G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))^G$$

of invariant elements or equivariant differential forms. Indeed, the following hold.

- For every $g \in G$,

$$(g \cdot \alpha)(X) = \alpha(\text{Ad}_g X),$$

or equivalently.

- For every $X \in \mathfrak{g}$,

$$\mathcal{L}(X)\alpha(X) = 0.$$

Since $(\mathcal{A}_G(M), d_{\mathfrak{g}})$ is a complex, we can define its cohomology, the *equivariant cohomology* of M , denoted by

$$H_G(M).$$

This is known as the *Cartan model of equivariant cohomology* [11]. When M is compact and oriented, we define the integration map

$$\int_M : \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M) \rightarrow \mathbb{C}[\mathfrak{g}]$$

by integrating only terms with the top differential degree part. This map restricts to

$$\int_M : \mathcal{A}_G(M) \rightarrow \mathbb{C}[\mathfrak{g}]^G,$$

where the image is the algebra of invariant polynomials. This map indeed descends to the cohomology level

$$\int_M : H_G(M) \rightarrow \mathbb{C}[\mathfrak{g}]^G.$$

2.3. The equivariant structure

Let M be compact oriented Riemannian manifold of M . Assume a Lie group G acts by positively oriented isometries on M . Assume $\mathcal{E} \rightarrow M$ is a G -equivariant vector bundle with a connection $\nabla^{\mathcal{E}}$ that commutes with the G -action. We define the moment $\mu^{\mathcal{E}} \in \mathcal{A}_G^2(M, \text{End}(\mathcal{E}))$ of the connection $\nabla^{\mathcal{E}}$ by the formula

$$\mu^{\mathcal{E}}(X) := \mathcal{L}^{\mathcal{E}}(X) - \nabla_X^{\mathcal{E}},$$

where $\mathcal{L}^{\mathcal{E}}(X)$ is the Lie derivative of $X \in \mathfrak{g}$. In particular, when $\mathcal{E} = TM$ with the Levi-Civita connection ∇ , we obtain the Riemannian moment

$$\mu^M \in \left(\Gamma(M, \mathfrak{so}(TM)) \otimes \mathfrak{g}^* \right)^G$$

so that, for $X \in \mathfrak{g}$ and $\xi \in \Gamma(TM)$, we have

$$\begin{aligned} \mu^M(X)\xi &= [X, \xi] - \nabla_X \xi \\ &= -\nabla_{\xi} X. \end{aligned}$$

Another example of a moment map is the Kosmann formula for the spinor bundle $\mathcal{E} = S$, when M is spin shown in the following proposition.

Proposition 2.1 ([21]). *If M is spin, and $\mathcal{E} = S$ is the spinor bundle, then*

$$\mu^S(X) = -\frac{1}{4}c(d\theta_X).$$

Assume $\mathcal{E} \rightarrow M$ is a G -equivariant Clifford module whose Clifford action and Hermitian metric are compatible with the G -action. Locally, such vector bundle is given as $S \otimes E$, where S is the local spinor bundle and E is an equivariant auxiliary vector bundle with G -compatible connection ∇^E such that $\nabla^\mathcal{E} = \nabla^S \otimes 1 + 1 \otimes \nabla^E$. Then, we define the twisting moment map (locally) as the moment of the auxiliary bundle E or, equivalently, the difference

$$\mu^{\mathcal{E}/S} := \mu^\mathcal{E} - \mu^S.$$

So, we have

$$\mu^{\mathcal{E}/S} = \mu^\mathcal{E} + \frac{1}{4}c(d\theta_X),$$

and moreover, we have

$$\mu^{\mathcal{E}/S} \in \mathcal{A}_G^2(M, \text{End}_{\text{Cl}}(\mathcal{E}))$$

(see [2]). Here, End_{Cl} denotes the space of endomorphisms that commute with the Clifford action.

The equivariant curvature of M is defined by

$$R_g := R + \mu^M \in \mathcal{A}_G^2(M, \mathfrak{so}(TM)),$$

where R is the Riemannian curvature tensor. The equivariant \hat{A} -genus is the closed equivariant differential form

$$\hat{A}_g(X, M) := \det \left(\frac{R_g/2}{\sinh(R_g/2)} \right) \in \mathcal{A}_G(M).$$

Using this, we obtain the equivariant twisting curvature of \mathcal{E} given by

$$F_g^{\mathcal{E}/S}(X) = F^{\mathcal{E}/S} + \mu^{\mathcal{E}/S} \in \mathcal{A}_G^2(M, \text{End}_{\text{Cl}}(\mathcal{E})).$$

The equivariant relative Chern character of \mathcal{E} is then defined as

$$\text{Ch}_g(X, \mathcal{E}/S) = \text{Str}_{\mathcal{E}/S} \left(\exp(-F_g^{\mathcal{E}/S}(X)) \right) \in \mathcal{A}_G^2(M).$$

3. The equivariant index and the Kirillov formula

In this section, we formulate our main result—the Kirillov formula for the index of the equivariant Dirac-type operator. Then, we give an overview of the heat kernel proof of this formula [2, Chapter 8]. In particular, we define the Bismut Laplacian [4] and explain its role in the proof.

3.1. The settings

Let G be a compact Lie Group acting by positively oriented isometries on a compact oriented even-dimensional Riemannian manifold M^n . Consider a \mathbb{Z}_2 -graded Clifford module $\mathcal{E} \rightarrow M$, which carries an action of G that commutes with the grading, the Hermitian

product, and the Clifford action. We fix a Clifford a G -equivariant connection $\nabla^\mathcal{E}$ on \mathcal{E} and consider the Dirac-type operator

$$D : \Gamma(M, \mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(M, \mathcal{E} \otimes \Lambda^1 T^*M) \xrightarrow{c} \Gamma(M, \mathcal{E}).$$

Then, D commutes with the action of G . Any $g \in G$ takes $\ker(D^\pm)$ to themselves. The *equivariant index* $\text{ind}(g, D)$ of D is defined as the supertrace

$$\text{ind}(g, D) := \text{tr}(g, \ker(D^+)) - \text{tr}(g, \ker(D^-)).$$

Theorem 3.1 (Kirillov formula). *For $g = e^{-X}$, with $X \in \mathfrak{g}$ sufficiently small,*

$$\text{ind}(e^{-X}, D) = (2\pi i)^{-n/2} \int_M \hat{A}_\mathfrak{g}(X, M) \text{Ch}_\mathfrak{g}(X, \mathcal{E}/S).$$

In the rest of this section, we give a brief overview of the heat kernel proof of this theorem.

3.2. The modified McKean–Singer formula

The equivariant index of D can be related to the heat kernel of D via the McKean–Singer-type theorem

$$\text{ind}(g, D) = \text{str}(g e^{-tD^2}) \tag{3.1}$$

for $t > 0$. However, it turns out that the asymptotic equation of the heat kernel in the right-hand side of this formula is complicated, and the Getzler’s rescaling method does not go through well for it. The calculation becomes easier if we slightly modify the operator D . In fact, the following generalization of (3.1) is true: fix $X \in \mathfrak{g}$ and for $u \in \mathbb{C}$ define the deformed Dirac operator

$$D_u = D + uc(X).$$

Note that this is the Dirac operator associated with the connection

$$\nabla^{\mathcal{E}, -uX} := \nabla^\mathcal{E} + u\theta_X,$$

where $\theta_X \in \mathcal{A}^1(M)$ is the one-form dual to the vector field X^M

$$\theta_X(\xi) = (X^M, \xi),$$

in other words,

$$D_u : \Gamma(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}, -uX}} \Gamma(M, \mathcal{E} \otimes \Lambda^1 T^*M) \xrightarrow{c} \Gamma(M, \mathcal{E}).$$

Proposition 3.2. *For every $t > 0$ and $u \in \mathbb{C}$, we have*

$$\text{ind}(e^{-X}, D) = \text{str}(e^{-X} e^{-tD_u^2}).$$

Note that since D commutes with the infinitesimal action of \mathfrak{g} , we can write

$$e^{-X} e^{-tD_u^2} = e^{-\mathcal{L}(X)} e^{-tD_u^2} = e^{-tD_u^2 - \mathcal{L}(X)}.$$

We denote the generalized Laplacian operator that appears in the exponent of the right-hand side, for $t = 1$, by

$$H_u(X) = D_u^2 + \mathcal{L}^\mathcal{E}(X).$$

Therefore, the equivariant index corresponding to $e^{-X} \in G$ is given by

$$\text{ind}(e^{-X}, D) = \text{str}(e^{-H_u(X)}).$$

3.3. The Lichnerowicz formula

We now compute the operator H_u .

Lemma 3.3 ([2, Proposition 3.45]). *If $\theta \in \mathcal{A}^1(M)$ is dual to the vector field $X \in \mathfrak{X}(M)$, then*

$$[D, c(d\theta)] = -2\nabla_X^\mathcal{E} + c(d\theta) + d^*\theta.$$

Proposition 3.4. *We have a Lichnerowicz-type formula for $H_u(X)$:*

$$H_u(X) = (\nabla^{\mathcal{E}, uX})^* \nabla^{\mathcal{E}, uX} + \frac{1}{4}\kappa_M + c(F^{\mathcal{E}/S}) + \mu^\mathcal{E}(X) + uc(d\theta_X) + (1 - 4u)\nabla_X^\mathcal{E}.$$

With respect to a local orthonormal frame $\{e_i\}$,

$$(\nabla^{\mathcal{E}, uX})^* \nabla^{\mathcal{E}, uX} = -\sum_i (\nabla_{e_i}^{\mathcal{E}, uX})^2 - \nabla_{\sum_i e_i}^{\mathcal{E}, uX}.$$

Note that when $u = \frac{1}{4}$, using the equality

$$\mu^{\mathcal{E}/S} = \mu^\mathcal{E} + \frac{1}{4}c(d\theta_X),$$

we obtain

$$H_{1/4}(X) = (\nabla^{\mathcal{E}, \frac{1}{4}X})^* \nabla^{\mathcal{E}, \frac{1}{4}X} + \frac{1}{4}\kappa_M + c(F^{\mathcal{E}/S}) + \mu^{\mathcal{E}/S}(X).$$

Proof. Note that since the D_u is the Dirac operator associated to the connection $\nabla^{\mathcal{E}, -uX}$, and

$$\nabla^{\mathcal{E}, -uX} = \nabla^\mathcal{E} + u\theta_X$$

is a Clifford connection, then we have the Lichnerowicz formula

$$\begin{aligned} D_u^2 &= (\nabla^{\mathcal{E}, -uX})^* \nabla^{\mathcal{E}, -uX} + \frac{1}{4}\kappa_M + c(F^{\mathcal{E}/S, -u}) \\ &= (\nabla^{\mathcal{E}, -uX})^* \nabla^{\mathcal{E}, -uX} + \frac{1}{4}\kappa_M + c(F^{\mathcal{E}/S}) + uc(d\theta_X). \end{aligned}$$

We may write a similar formula for D_{-u}^2 , and by subtracting, we have

$$D_u^2 - D_{-u}^2 = (\nabla^{\mathcal{E}, -uX})^* \nabla^{\mathcal{E}, -uX} - (\nabla^{\mathcal{E}, uX})^* \nabla^{\mathcal{E}, uX} + 2uc(d\theta_X),$$

we also have

$$D_u^2 - D_{-u}^2 = 2u[D, c(\theta_X)] = -4u\nabla_X^\mathcal{E} + 2uc(d\theta_X) + 2ud^*\theta_X,$$

where we used Lemma 3.3 in the second equality. Note that $d^*\theta_X = \text{tr}(\mu^M(X)) = 0$. Combining these formulas, we find

$$\begin{aligned} H_u(X) &= D_u^2 + \mathcal{L}^\mathcal{E}(X) \\ &= (\nabla^{\mathcal{E}, uX})^* \nabla^{\mathcal{E}, uX} + \frac{1}{4}\kappa_M + c(F^{\mathcal{E}/S}) + uc(d\theta_X) - 4u\nabla_X^\mathcal{E} + \mathcal{L}^\mathcal{E}(X). \end{aligned}$$

Using the equality $\mathcal{L}^\mathcal{E}(X) = \nabla_X^\mathcal{E} + \mu^\mathcal{E}(X)$, we then achieve the result. ■

3.4. The Bismut–Laplacian

As we see, for $u = 1/4$, we obtain a simpler Lichnerowicz formula. The operator

$$H_{1/4} = D_{1/4}^2 + \mathcal{L}^\mathcal{E}(X) \tag{3.2}$$

is called the *Bismut Laplacian*. From now on, we drop the $1/4$ from the Bismut Laplacian notation and denote it by $H(X) := H_{1/4}(X)$. Following [2], we replace D^2 with $H_{1/4}$ in (3.1).

3.5. The heat kernel in normal coordinates

The proof of the Kirillov formula (Theorem 3.1), is done through rescaling of the smoothing kernel of the (a conjugate of the) heat operator $e^{-tH(X)}$ in normal coordinates. Roughly one needs to trivialize the bundle $\mathcal{E} \rightarrow M$ over a small neighborhood using the normal coordinates $\mathbf{x} \rightarrow \exp_m(\mathbf{x})$ through parallel transport along radial geodesics. Here, \mathbf{x} belongs to a small neighborhood of the origin in $T_m M$.

The Bismut Laplacian $H(X)$ is associated to the perturbed connection $\nabla^{\mathcal{E}, \frac{1}{4}X} = \nabla^\mathcal{E} - \frac{1}{4}\theta_X$. So, it is natural to use the parallel transport of this connection to define the trivialization of \mathcal{E} . However, it is more convenient to do a “two-step” trivialization: first, trivialize the bundle using the connection $\nabla^\mathcal{E}$ and compute the heat kernel $e^{-tH(X)}$ in this trivialization and then consider the conjugated kernel $\rho_{\frac{1}{4}X} e^{-tH(X)} \rho_{\frac{1}{4}X}^{-1}$, where $\rho_{\frac{1}{4}X}(\mathbf{x})$ is the parallel transport map on the trivial line bundle along the radial geodesics with respect to the connection $d - \frac{1}{4}\theta_X$ (see (5.4)). In this approach, in particular, the dependence of the result on X is more explicit.

4. Rescaled bundles

In this paper, we achieve the rescaling calculus through a coordinate-free method, i.e., through tangent groupoid of the manifold M . It is, however, more general and arguably

more natural to consider a more general deformation space $\text{DNC}(V, M)$. So, in this and in the next sections, we introduce a setup that gives vector bundles over the deformation space $\text{DNC}(V, M)$, generalizing the rescaled bundle of [20]. Specifying the case of the tangent groupoid, we recover the Getzler symbol calculus.

4.1. Deformation to the normal cone

Assume $i : M \hookrightarrow V$ is an embedding of smooth manifolds. The normal bundle of this embedding is the quotient bundle $\mathcal{N} = TV|_M/TM$. As a set the deformation space $\text{DNC}(V, M)$ is given by the disjoint union

$$\text{DNC}(V, M) := \mathcal{N} \times \{0\} \bigsqcup_{t \neq 0} V \times \{t\}.$$

To obtain a smooth structure, fix a Riemannian metric g on V and identify \mathcal{N} with TM^\perp the orthogonal complement of TM in $TV|_M$. Then, we have $\text{Exp}^g : \mathcal{W} \rightarrow U$, the Riemannian exponential map, from a neighborhood \mathcal{W} of 0-section of \mathcal{N} to a neighborhood U of M in V . The smooth structure on $\text{DNC}(V, M)$ away from the normal bundle is given by the product manifold $M \times \mathbb{R}^{\neq 0}$. To obtain the manifold structure in a neighborhood of \mathcal{N} we require the following map to be a diffeomorphism (cf. [14]):

$$\begin{aligned} \mathcal{W} \times \mathbb{R} &\rightarrow \text{DNC}(V, M), \\ (x, X, t) &\mapsto \begin{cases} (\text{Exp}_x^g(tX), t), & t \neq 0, \\ (x, X, 0), & t = 0. \end{cases} \end{aligned}$$

One may obtain local coordinates for this deformation space explicitly; see [10, 18].

For the construction of the rescaled bundle, we will use an algebraic approach to the deformation to the normal cone, following [18]. This is inspired by the algebraic geometric definition of the deformation to the normal cone as the prime spectrum of the *Rees algebra* (see, for example, [16, Chapter 5] and [19]).

Let $I_M \subset C^\infty(V)$ be the vanishing ideal of M . Consider the Rees algebra $\mathcal{A}(V, M) \subset C^\infty(V)[t, t^{-1}]$ given by

$$\mathcal{A}(V, M) = \bigoplus_{p=-\infty}^{\infty} I_M^p t^{-p}.$$

Therefore, the Rees algebra consists of the Laurent polynomials

$$\sum_p f_p t^{-p},$$

where f_p vanishes to p th order along M .

Definition 4.1. For a \mathbb{C} -algebra \mathcal{A} , the *character spectrum*, $\text{Spec}(\mathcal{A})$, is the space of all algebra homomorphisms

$$\phi : \mathcal{A} \rightarrow \mathbb{C}$$

with the weak topology of pointwise convergence.

It turns out the spectrum of the algebra $\mathcal{A}(V, M)$ consists of the following characters.

(1) For every $(v, \lambda) \in V \times \mathbb{R}^{\neq 0}$,

$$\begin{aligned} \varepsilon_{(v,\lambda)} : \mathcal{A}(V, M) &\rightarrow \mathbb{C}, \\ \sum_p f_p t^{-p} &\mapsto \sum_p f_p(v) \lambda^{-p}. \end{aligned}$$

(2) For $X_m \in \mathcal{N}_m = T_m V / T_m M$,

$$\begin{aligned} \varepsilon_{X_m} : \mathcal{A}(V, M) &\rightarrow \mathbb{C}, \\ \sum_p f_p t^{-p} &\mapsto \sum_p \frac{1}{p!} X^p f_p(m), \end{aligned} \tag{4.1}$$

where $X \in \mathfrak{X}(V)$ is a vector field that represents $X_m \in \mathcal{N}_m$ at $m \in M$.

Hence, there is a one-to-one correspondence:

$$\text{Spec}(\mathcal{A}(V, M)) \longleftrightarrow \mathcal{N} \times \{0\} \bigsqcup_{t \neq 0} V \times \{t\}.$$

This spectrum is indeed a manifold.

Theorem 4.2 ([18]). *The spectrum $\text{Spec}(\mathcal{A}(V, M))$ is a smooth manifold of dimension $\dim(V) + 1$ that has a canonical submersion to \mathbb{R} . This manifold is called the deformation to the normal cone and is denoted by $\text{DNC}(V, M)$. As a set, it is given by a disjoint union of fibers over \mathbb{R} as follows:*

$$\mathcal{N} \times \{0\} \bigsqcup_{t \neq 0} V \times \{t\}.$$

To gain insight into this theorem, it is useful to see algebra $\mathcal{A}(V, M)$ for $\text{DNC}(V, M)$ as a “homogeneous coordinate ring” for a variety. Note that the vanishing order along the submanifold M gives a filtration

$$C^\infty(V) = I_0 \supset I_1 \supset \dots,$$

where $I_p = I_M^p$ is the ideal of functions vanishing to order p along M . Denote by $\mathcal{A}_0(V, M)$ the quotient algebra $\mathcal{A}(V, M) / t \mathcal{A}(V, M)$ which is naturally isomorphic to the associated graded algebra

$$\begin{aligned} \mathcal{A}_0(V, M) &\xrightarrow{\cong} \bigoplus_{p=0} I_p / I_{p+1}, \\ \sum_p f_p t^{-p} &\mapsto \sum_{q \geq 0} \{f_p\}_p. \end{aligned}$$

We have the inclusion $C^\infty(M) \hookrightarrow \mathcal{A}_0(V, M)$ as the zero-degree part. This corresponds to the following fact.

Lemma 4.3. *The character spectrum of $\mathcal{A}_0(V, M)$ is the normal bundle $\mathcal{N} \rightarrow M$. The quotient $\mathcal{A}_{0,m}(V, M)$ of the algebra $\mathcal{A}_0(V, M)$ by the ideal of functions vanishing at $m \in M$ is isomorphic to the polynomial algebra over the normal space \mathcal{N}_m*

$$\sum_p f_p t^{-p} \mapsto \left[X_m \mapsto \sum_p \frac{1}{p!} X^p \cdot f_p|_m \right]$$

for $X_m \in \mathcal{N}_m$.

One important example of the deformation to the normal cone is the *tangent groupoid*.

Example 4.4. Let M be a smooth manifold. The deformation to the normal cone associated to the diagonal embedding $M \hookrightarrow M \times M$ is called the tangent groupoid and is denoted by

$$\mathbb{T}M := \text{DNC}(M \times M, M).$$

As a set, it is given by the disjoint union

$$M \times M \times \mathbb{R}^{\neq 0} \bigsqcup TM \times \{0\},$$

which is naturally fibered over \mathbb{R} .

The tangent groupoid is indeed a smooth groupoid, $\mathbb{T}M \rightrightarrows M \times \mathbb{R}$. The source and target maps are given so that for $t \neq 0$, they restrict to pair groupoid

$$M \times M \times \{t\} \rightrightarrows M \times \{t\},$$

and for $t = 0$, the source and target map are the same as the projection map of the vector bundle

$$TM \times \{0\} \rightarrow M \times \{0\}.$$

4.2. An Alternative formula for ε_{X_m}

We now describe Higson and Yi’s alternative presentation of the formula (4.1), (see [20]), which justifies many formulas in the upcoming subsections and sections. For every vector field $X \in \mathfrak{X}(V)$ we have the derivation map

$$\begin{aligned} tX : \mathcal{A}(V, M) &\rightarrow \mathcal{A}(V, M), \\ \sum_p f_p t^{-p} &\mapsto \sum_p X \cdot f_p t^{-p+1}, \end{aligned}$$

which descends to the quotient algebra

$$tX : \mathcal{A}_0(V, M) \rightarrow \mathcal{A}_0(V, M).$$

The action of tX on $\mathcal{A}_0(V, M)$ is locally nilpotent, in the sense that for each $f \in \mathcal{A}_0(V, M)$ there exists an integer $n = n(f)$ such that $(tX)^n f = 0$. Hence, we may define the exponential homomorphism

$$\exp(tX) : \mathcal{A}_0(V, M) \rightarrow \mathcal{A}_0(V, M)$$

as the *finite sum*

$$\exp(tX)f := \sum_j \frac{(tX)^j f}{j!}, \quad f \in \mathcal{A}_0(V, M).$$

Now, we may rewrite formula (4.1) as the composition

$$\varepsilon_{X_m} = \varepsilon_m \circ \exp(tX),$$

where

$$\varepsilon_m = \varepsilon_{0_m} : \mathcal{A}_0(V, M) \rightarrow \mathbb{C}, \quad \sum_p f_p t^{-p} \mapsto f_0(m)$$

is the evaluation map at $X_m = 0$.

4.3. The scaling order

Let $E \rightarrow V$ be a vector bundle with a connection ∇ .

Definition 4.5. We call (E, ∇) a scaling bundle if there is an algebra filtration on $\text{End}(E)$ and a filtration on the restriction bundle $F := E|_M$:

$$\text{End}(E)^0 \subset \text{End}(E)^1 \subset \dots \subset \text{End}(E)^{q'} = \text{End}(E)$$

and

$$M = F^{-1} \subset F^0 \subset F^1 \subset \dots \subset F^q = F$$

such that the following hold.

- The two filtration are compatible in the following sense:

$$A : F^\bullet \rightarrow F^{\bullet+k} \quad \text{for } A \in \text{End}(E)^k. \tag{4.2}$$

- The induced connection $i^*\nabla$ has filtration order zero:

$$i^*\nabla : \Gamma(F^j) \rightarrow \Gamma(F^j) \otimes \Omega^1(M), \quad j = 1, \dots, q.$$

- The curvature K of ∇ has filtration order at most 2:

$$K \in \Gamma(\text{End}(E)^2) \otimes \Omega^2(V). \tag{4.3}$$

- The induced connection $\nabla^{\text{End}(E)}$ has filtration order 0:

$$\nabla^{\text{End}(E)} : \Gamma(\text{End}(E)^j) \rightarrow \Gamma(\text{End}(E)^j) \otimes \Omega^1(V).$$

The space of sections $\Gamma(\text{End}(E))$ is filtered by the algebra filtration of $\text{End}(E)$. We use the notation $o^g(\phi)$ for the order of $\phi \in \Gamma(\text{End}(E))$ in this filtration. By definition, we have

$$o^g(\phi_1\phi_2) \leq o^g(\phi_1) + o^g(\phi_2). \tag{4.4}$$

The filtration F^\bullet induces a filtration on $\text{End}(E|_M)$, which might differ from the filtration induced from $\text{End}(E)$. For a section $\phi \in \Gamma(\text{End}(E|_M))$, we denote by $o^f(\phi)$ the order of this filtration order induced from F^\bullet . From (4.2) it then follows that

$$o^f(\phi) \leq o^g(\phi).$$

Note, however, that the order function o^f fails to satisfy the property (4.4), in general.

Definition 4.6. A differential operator D acting on $\Gamma(E)$ is of *Getzler order* at most p if locally it can be written as the sums of terms of the form

$$\phi \nabla_{X_1} \cdots \nabla_{X_l},$$

where $\phi \in \Gamma(\text{End}(E))$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$, where $o^g(\phi) + l \leq p$. We denote the Getzler order of D by $o^g(D)$.

We had to use the filtration order o^g , rather than o^f , on $\Gamma(\text{End}(E))$ in the definition of Getzler order to obtain the following lemma.

Lemma 4.7. For two differential operator D_1, D_2 acting on $\Gamma(E)$, we have

$$o^g(D_1 D_2) \leq o^g(D_1) + o^g(D_2). \quad \blacksquare$$

The space of sections $\Gamma(E)$ has natural filtration obtained by restricting to the submanifold M . We denote the corresponding filtration order by $o^f(\sigma)$ for $\sigma \in \Gamma(E)$ (if $\sigma = 0$ we set $o^f(\sigma) = -\infty$).

We need yet another filtration on $\Gamma(E)$, defined by the *scaling order*

$$o^{sc}(\sigma) = \min_D \{o^g(D) - o^f(D\sigma)\}.$$

Remark 4.8. The definition of the scaling order might seem unnatural at first. But it is modeled on the following formula for the vanishing order of a function.

For $f \in C^\infty(V)$, define $o^{val}(f)$ to be 0 if $f|_M$ is not the zero function; otherwise, define $o^{val}(f) = -\infty$. One easily sees that the vanishing order of f along the submanifold M , $o^{van}(f)$, is given by the equality

$$o^{van}(f) = \min_D \{o(D) - o^{val}(Df)\},$$

where $o(D)$ is the ordinary differential order and the minimum is taken over all differential operators on V .

It follows from the definition of the scaling order the following lemma.

Lemma 4.9. For a differential operator D acting on $\Gamma(E)$ with $o^g(D) \leq q$ and $\sigma \in \Gamma(E)$, we have

$$o^{sc}(D\sigma) \geq o^{sc}(\sigma) - q. \quad \blacksquare$$

4.4. The Taylor order

In this subsection, we obtain a formula for the scaling order in local coordinates.

Definition 4.10. Let M be an embedded submanifold of a manifold V . A vector field $\mathcal{R} \in \mathfrak{X}(V)$ is called Euler-like for the embedding $M \rightarrow V$, if for every function $f \in C^\infty(V)$ vanishing to p th order along M ,

$$\mathcal{R}f = pf + g,$$

where $g \in C^\infty(V)$ vanishes to order $p + 1$ along M .

The Euler-like vector fields can be used to obtain tubular neighborhood embeddings.

Theorem 4.11 ([9, 18]). *There is a bijection between the germs of Euler-like vector fields and germs of tubular neighborhood embeddings.*

The correspondence given by this theorem can be described as follows: given a tubular neighborhood embedding, we can choose local coordinates

$$(x^1, \dots, x^l, y^1, \dots, y^k) : \mathcal{N} \rightarrow \mathbb{R}^l \times \mathbb{R}^k \tag{4.5}$$

near a point of M , where x_i 's form local coordinates on M and y_j 's are the linear coordinates for the fibers of the normal bundle. Under the correspondence in Theorem 4.11, the associated Euler-like vector field in the trivializing neighborhood is given by

$$\mathcal{R} = \sum_j y^j \partial_{y^j}.$$

Using the trivialization (4.5), we call a vector field *horizontal* if it is a linear combination of coordinate vector fields ∂_{x^i} 's and *vertical* if it is a linear combination of coordinate vector fields ∂_{y^j} 's. In particular, the Euler-like vector field is a vertical vector field.

Definition 4.12. A section in $\sigma \in \Gamma(E)$ is called \mathcal{R} -synchronous if

$$\nabla_{\mathcal{R}}\sigma = 0$$

in a neighborhood of M .

Using this concept, we define the Taylor expansion of a section $\sigma \in \Gamma(E)$ in the trivializing neighborhood. The formal sum $\sum_I \sigma_I y^I$ is a Taylor expansion for σ if the following hold.

- $\sigma_I \in \Gamma(E)$ is an \mathcal{R} -synchronous section.
- For every $N \geq 0$, the difference

$$\sigma - \sum_{|I| < N} \sigma_I y^I$$

vanishes to N th order.

Definition 4.13. For a section $\sigma \in \Gamma(E)$ with support in the trivializing neighborhood, with Taylor expansion $\sum_I \sigma_I y^I$ we define the *Taylor order* by the formula

$$o^t(\sigma) := \min_{|I|} \{ |I| - o^f(\sigma_I) \}.$$

We will show that the Taylor order equals the scaling order. To prove this, we need the following lemma.

Lemma 4.14. *Let $\sigma \in \Gamma(E)$ be an \mathcal{R} -synchronous section.*

- *If Y is a vertical vector field, then we have the Taylor expansion*

$$\nabla_Y \sigma \simeq \sum_{|I|>0} \omega_I(\sigma) y^I,$$

where ω_I 's are endomorphisms of the vector bundle E of order at most 2.

- *If X is a horizontal vector field, then we have the following Taylor expansion:*

$$\nabla_X \sigma \simeq \sigma_0 + \sum_{|I|>0} \eta_I(\sigma) y^I,$$

where η_I 's are endomorphisms of the vector bundle E of order at most 2 and $o^f(\sigma_0) \leq o^f(\sigma)$.

Proof. For the first bullet point, without loss of generality, we may assume $Y = \partial_i$, where we have $[\mathcal{R}, Y] = -Y$. Consider the equation

$$\nabla_{\mathcal{R}} \nabla_Y \sigma - \nabla_Y \nabla_{\mathcal{R}} \sigma - \nabla_{[\mathcal{R}, Y]} \sigma = K(\mathcal{R}, Y) \sigma.$$

Since $\nabla_{\mathcal{R}} \sigma = 0$, we obtain

$$\nabla_{\mathcal{R}} \nabla_Y \sigma + \nabla_Y \sigma = K(\mathcal{R}, Y) \sigma. \tag{4.6}$$

Let $\sum_I \sigma_I y^I$ be the Taylor expansion for $\nabla_Y \sigma$, and let

$$\sum_{|I|>0} \omega_I y^I$$

be the Taylor expansion of the endomorphism $K(\mathcal{R}, Y)$, where ω_I 's are endomorphisms of E where $\nabla_{\mathcal{R}} \omega_I = 0$. Now, by writing the Taylor expansion of both sides of (4.6), we obtain

$$\sum_I (1 + |I|) \sigma_I y^I = \sum_{|I|>0} \omega_I(\sigma) y^I$$

from which the statement follows.

The argument for the second bullet point is quite similar, except for the inequality $o^f(\sigma_0) \leq o^f(\sigma)$, where it follows from the compatibility of the connection with filtration of the restricted bundle $E|_M$. ■

Corollary 4.15. *If $\sigma \in \Gamma(E)$ is an \mathcal{R} -synchronous section and D is a differential operator of Getzler order k . Then, for the Taylor expansion*

$$D\sigma \sim \sum_I y^I \sigma_I,$$

we have

$$o^f(\sigma_I) \leq o^f(\sigma) + |I| + k.$$

So, in particular,

$$o^f(D\sigma) \leq o^f(\sigma) + k.$$

Proof. If D is an endomorphism of the bundle $E \rightarrow V$, the statement is obvious. Consider a vector field X on V . We will prove the statement for $D = \nabla_X$, and by induction on the differential order, the full generality follows.

We can write $X = X^h + X^v$, where X^h and X^v are the horizontal and vertical components of X . Now, from Lemma 4.14, the corollary follows. ■

Recall that the Taylor order was defined in Definition 4.13.

Theorem 4.16. *For a section $\sigma \in \Gamma(E)$ with support in the trivializing neighborhood (4.5), we have*

$$o^{sc}(\sigma) = o^t(\sigma).$$

Proof. Choose $I = (\alpha_1, \dots, \alpha_k)$ with smallest $|I|$ such that

$$o^t(\sigma) = |\alpha| - o^f(\sigma_I),$$

and define the differential operator $D = \nabla_{\partial_{y^1}}^{\alpha_1} \dots \nabla_{\partial_{y^n}}^{\alpha_n}$. We claim that

$$o^f(D\sigma) = o^f(\sigma_\alpha).$$

For any $J \neq I$ with $|J| \geq |I|$, we have

$$D(y^J \sigma_J)|_M = 0.$$

If $|J| < |I|$, by the choice of I , we have

$$|J| - o^f(\sigma_J) > |I| - o^f(\sigma_I).$$

Also, by Corollary 4.15, we have

$$o^f(Dy^J \sigma_J) \leq o^f(\sigma_J) + |I| - |J|$$

and combining this with the previous inequality gives

$$o^f(Dy^J \sigma_J) < o^f(\sigma_I)$$

from which our claim follows. Therefore,

$$o^{sc}(\sigma) \leq o^t(\sigma).$$

Now, we prove the opposite inequality. From Corollary 4.15, we have

$$o^t(D\sigma) \geq o^t(\sigma) - o^f(D).$$

By definition, we have

$$o^t(D\sigma) \leq -o^f(D\sigma),$$

and consequently,

$$o^{sc}(\sigma) \geq o^t(\sigma). \quad \blacksquare$$

4.5. Rescaled module

We are ready to define a module over the algebra $\mathcal{A}(V, M)$ associated with a scaling bundle (E, ∇) , which will be later shown to be a module of sections of a certain vector bundle—the rescaled bundle—over the deformation space.

The space of Laurent polynomials $\Gamma(V, E)[t, t^{-1}]$ is a module over the algebra $\mathcal{A}(V, M)$, and hence, can be viewed as a sheaf over the deformation space $\text{DNC}(V, M)$. We are interested in the following submodule of $\Gamma(V, E)[t, t^{-1}]$.

Definition 4.17. The subspace $\mathcal{S}(E, \nabla) \subset \Gamma(V, E)[t, t^{-1}]$ of the space of Laurent polynomials, consisting of the sections of the form

$$s = \sum_p s_p t^{-p}, \quad \text{with } o^{sc}(s_p) \geq p,$$

is called the *rescaled module*.

The rescaled module is also a sheaf over $\text{DNC}(V, M)$. In order to show that $\mathcal{S}(E, \nabla)$ is isomorphic to a subspace of a space of section of a smooth bundle over $\text{DNC}(V, M)$, we need to study the restriction

$$\mathcal{S}_0(E, \nabla) := \mathcal{S}(E, \nabla) / t\mathcal{S}(E, \nabla)$$

of $\mathcal{S}(E, \nabla)$ to the zero fiber of $\text{DNC}(V, M)$. We denote by $\mathcal{S}_{0,m}(E, \nabla)$ the quotient of $\mathcal{S}_0(E, \nabla)$ by the ideal of functions vanishing at $m \in M$.

Let $\mathcal{P}(\mathcal{N}) \subset \Gamma(\mathcal{N}, \bigoplus_{p=1}^q F^p / F^{p-1})$ denote the space of sections whose restriction to each fiber \mathcal{N}_m of \mathcal{N} are vector-valued polynomial functions $\mathcal{N}_m \rightarrow \bigoplus_{p=1}^q F_m^p / F_m^{p-1}$.

Lemma 4.18. Let $I_p(E) \subset \Gamma(V, E)$ denote the submodule of sections with scaling order at least p . There is a canonical isomorphism

$$\mathcal{S}_0(E, \nabla) \xrightarrow{\cong} \bigoplus_p I_p(E) / I_{p+1}(E), \quad \sum_p s_p t^{-p} \mapsto \sum_{p \geq 0} \langle s_p \rangle_p.$$

Further, the map

$$\sum_p s_p t^{-p} \mapsto \left[X_m \mapsto \sum_p \frac{1}{p!} \langle \nabla_X^p s_p | m \rangle_p \right],$$

where $\langle \cdot \rangle_p$ denotes the corresponding class in the quotient space F_m^p / F_m^{p-1} , defines an isomorphism between $\mathcal{S}_{0,m}(E, \nabla)$ and the space of vector-valued polynomial functions $\mathcal{P}(\mathcal{N})$. ■

4.6. The construction of the rescaled bundle

We now give a rundown of how to obtain a smooth vector bundle for which a “big” subspace of the space of sections is isomorphic to $\mathcal{S}(E, \nabla)$.

As a set, this bundle is given by the union

$$\mathbb{E} = \left(\bigoplus_{p=1}^q F^p / F^{p-1} \times \{0\} \right) \bigsqcup_{t \neq 0} (E \times \{t\}).$$

To identify the elements of $\mathcal{S}(E, \nabla)$ with sections of \mathbb{E} , we define for each element $\mu \in \text{DNC}(V, M)$ an evaluation map $\varepsilon_\mu : \mathcal{S}(E, \nabla) \rightarrow \mathbb{E}$. Then, the fiber \mathbb{E}_μ of \mathbb{E} over μ is given by

$$\mathbb{E}_\mu = \begin{cases} E_v & \text{if } \mu = (v, t) \in V \times \mathbb{R}^{\neq 0}, \\ \bigoplus_{p=0}^q F_m^p / F_m^{p-1} & \text{if } \mu = (X_m, 0) \in \mathcal{N} \times \{0\}. \end{cases}$$

For (v, λ) , $v \in V$, $\lambda \neq 0$, the evaluation map is simply

$$\begin{aligned} \varepsilon_{(v,\lambda)} : \mathcal{S}(E, \nabla) &\rightarrow E_v, \\ \sum_p s_p t^{-p} &\mapsto \sum_p s_p(v) \lambda^{-p}. \end{aligned}$$

This map will be the evaluation map on the nonzero fiber element $(v, t) \in \text{DNC}(V, M)$.

To define the evaluation over the zero fiber, we need a bit more work. First, note that for a vector field $X \in \mathfrak{X}(V)$ we have a well-defined map

$$t \nabla_X : \mathcal{S}(E, \nabla) \rightarrow \mathcal{S}(E, \nabla).$$

This map descends to the quotient space $\mathcal{S}_0(E, \nabla)$ as a locally nilpotent map:

$$t \nabla_X : \mathcal{S}_0(E, \nabla) \rightarrow \mathcal{S}_0(E, \nabla),$$

cf. Section 4.2. Hence, we have a well-defined module homomorphism

$$\exp(t \nabla_X) : \mathcal{S}_0(E, \nabla) \rightarrow \mathcal{S}_0(E, \nabla).$$

For $m \in M$ define the map

$$\begin{aligned} \varepsilon_m : \mathcal{S}(E, \nabla) &\rightarrow \bigoplus_{p=1}^q F_m^p / F_m^{p-1}, \\ \sum_p s_p t^{-p} &\mapsto \sum_{p \leq 0} \langle s_p(m) \rangle_p, \end{aligned}$$

and finally, for $X_m \in \mathcal{N}$ define the map

$$\begin{aligned} \varepsilon_{X_m} : \mathcal{S}_0(E, \nabla) &\rightarrow \bigoplus_{p=1}^q F_m^p / F_m^{p-1}, \\ \varepsilon_{X_m} s &= \varepsilon_m(\exp(t \nabla_X) s), \end{aligned}$$

where X is an arbitrary extension of X_m . The map ε_{X_m} is the evaluation map over the zero fiber.

4.7. The smooth structure on \mathbb{E}

We now define a smooth structure on \mathbb{E} by constructing a locally free sheaf on $\text{DNC}(M, V)$ generated by $\mathcal{S}(E, \nabla)$. This sheaf is the sheaf of smooth functions on \mathbb{E} .

Definition 4.19. Define the sheaf \mathcal{E} on $\text{DNC}(V, M)$ that consists of maps

$$\begin{aligned} \text{DNC}(V, M) &\rightarrow \prod_{\mu \in \text{DNC}(V, M)} \mathbb{E}_\mu, \\ \mu &\mapsto \tau(\mu) \end{aligned}$$

such that for some $f_i \in C^\infty(\text{DNC}(V, M))$ and $s_i \in \mathcal{S}(E, \nabla)$

$$\tau(\mu) = \sum_{i=1}^N f_i(\mu) \varepsilon_\mu(s_i)$$

for every $\mu \in \text{DNC}(V, M)$.

Theorem 4.20. *The sheaf \mathcal{E} is a locally free sheaf of constant rank of modules over algebra $C^\infty(\text{DNC}(V, M))$. Its rank is equal to the rank of the vector bundle $E \rightarrow V$. The evaluation map of the previous subsections defines a smooth structure on \mathbb{E} and identifies \mathcal{E} with the space of smooth sections of \mathbb{E} . This gives \mathbb{E} the structure of a smooth bundle over $\text{DNC}(V, M)$. Fiber-wise \mathbb{E} is given as follows:*

$$\begin{array}{ccc} \pi^* \text{gr}(F) & & E \\ \downarrow & & \downarrow \\ \mathcal{N} \times \{0\} & \bigsqcup_{t \neq 0} & V \times \{t\}. \end{array}$$

Proof. In some neighborhood of every $\mu \in \text{DNC}(V, M)$, we show that the sheaf is locally free of constant rank. Fix a trivializing neighborhood (4.5) around $m \in M$ and consider the associated Euler-like vector field \mathcal{R} . Take a local frame $\{e_i\}_{i=1}^{\text{rank}(E)}$ for the restricted bundle $E|_M \rightarrow M$ such that each e_i gives a local homogeneous section of degree $q_i = o^f(e_i)$ of the associated graded bundle

$$\text{gr}(F) := \bigoplus_{p=0}^q F^p / F^{p-1}.$$

There are unique local \mathcal{R} -synchronous sections \tilde{e}_i of E so that $\tilde{e}_i|_M = e_i$. We show that the set $\{\tilde{e}_i t^{q_i}\}_{i=1}^{\text{rank}(E)}$ generates the sheaf locally. Clearly, in a neighborhood $\mu = (m, t) \in \text{DNC}(V, M)$, these sections generate the sheaf. For $\mu = X_m \in \mathcal{N}_m$, we have

$$\begin{aligned} \varepsilon_{X_m}(\tilde{e}_i t^{q_i}) &= \varepsilon_m \exp(t \nabla_X) \tilde{e}_i t^{q_i} \\ &= \varepsilon_m \sum_k \frac{t^{q_i+k}}{k!} \nabla_X^k \tilde{e}_i \\ &= \langle e_i \rangle_{q_i} + \text{higher - degree terms.} \end{aligned}$$

Hence, the sheaf is locally free of constant-rank equal $\text{rank}(E)$.

Now, it remains to show the same for $\mu = (v, t) \in \text{DNC}(V, M)$, with $v \notin M$. We may take a family of sections $\{s_i\}_i$ of E forming a frame near v and with supports away from M . In this case, the sections s_i have scaling order $-\infty$. So, the set $\{s_i\}_i$ generates the sheaf near (v, t) . ■

Definition 4.21. The smooth vector bundle $\mathbb{E} \rightarrow \text{DNC}(V, M)$ is called the *rescaled bundle*.

4.8. Example: the spinor rescaled bundle

The spinor rescaled bundle of Higson and Yi [20] was the main inspiration for this paper. We explain their setup briefly here.

Assume $V = M \times M$ and $M \hookrightarrow M \times M$ is the diagonal embedding. If $\mathcal{E} \rightarrow M$ is a Clifford module with a Clifford connection ∇ , then $E = \mathcal{E} \boxtimes \mathcal{E}^*$ carries the connection $\nabla^E = \nabla \boxtimes 1 + 1 \boxtimes \nabla$. Note that $E|_M$ is isomorphic to

$$\text{Cl}(TM) \otimes \text{End}_{\text{Cl}}(\mathcal{E}), \tag{4.7}$$

and it carries the Clifford filtration, which is compatible with the connection ∇^E . Thus, we are in the situation of Section 4.3, where the filtration is given by the Clifford filtration on the first factor of (4.7) (see Section 5 for more details on different filtrations in this case).

The curvature of this connection satisfies

$$K^E = c \circ q \circ \gamma(R) \boxtimes 1 + 1 \boxtimes c^* \circ q \circ \gamma(R) + F^{\mathcal{E}/S} \boxtimes 1 + 1 \boxtimes F^{\mathcal{E}^*/S}.$$

Here, R is the curvature of Levi–Civita connection, $\gamma : \mathfrak{so}(\mathcal{E}) \xrightarrow{\cong} \Lambda^2 \mathcal{E}$ is the canonical isomorphism [23, Section 2.2.10], $q : \Lambda^* TM \rightarrow \text{Cl}(TM)$ is the quantization [2, Proposition 3.5] map, $c : \text{Cl}(TM) \rightarrow \text{End}(\mathcal{E})$ is the Clifford action and $c^* : \text{Cl}(TM) \rightarrow \text{End}(\mathcal{E}^*)$ is the dual action. Thus, the first summand in the above formula has Clifford filtration of order 2, and, hence, when viewed as a differential operator, has Getzler filtration of order 2. Also, $F^{\mathcal{E}/S}$ and $F^{\mathcal{E}^*/S}$ are twisting curvatures of the Clifford modules \mathcal{E} and \mathcal{E}^* , respectively, cf. [2, Proposition 3.43]. Recall that the twisting curvatures commute with Clifford actions on \mathcal{E} and \mathcal{E}^* . So, as a differential operator of order 0, it has Getzler filtration of order 0. We conclude that K^E has Getzler filtration of order 2. Therefore, we obtain the rescaled bundle.

$$\mathbb{E} \rightarrow \text{DNC}(M \times M, M) = \mathbb{T}M.$$

This rescaled bundle recovers the Getzler symbol calculus [20]. This construction also recovers the local index formula, cf. [17, 22]

The next section is dedicated to a generalization of the spinor-rescaled bundle to the equivariant setting, from which we recover the Kirillov formula for the equivariant index.

5. Equivariant rescaled bundles

Let G be a compact Lie group acting on an oriented smooth manifold M by orientation-preserving isometries. Consider a $\mathbb{Z}/2$ -graded Clifford module $\mathcal{E} \rightarrow M$, which carries an even action of G . Let $\nabla^{\mathcal{E}}$ be a G -equivariant connection on \mathcal{E} .

5.1. The equivariant version of $\mathcal{E} \boxtimes \mathcal{E}^*$

As an equivariant version of the bundle $\mathcal{E} \boxtimes \mathcal{E}^*$ of Section 4.8, one would like to consider the bundle $\mathcal{E} \boxtimes \mathcal{E}^* \otimes \mathbb{C}[\mathfrak{g}]$. However, since this bundle has infinite dimensions, we prefer to work with its approximation:

$$E = \mathcal{E} \boxtimes \mathcal{E}^* \otimes \mathbb{C}[\mathfrak{g}]_{(J)},$$

where $\mathbb{C}[\mathfrak{g}]_{(J)}$ is defined in Section 2.1. This bundle is endowed with a natural connection

$$\nabla := \nabla^{\mathcal{E}} \boxtimes 1 \otimes 1 + 1 \boxtimes \nabla^{\mathcal{E}^*} \otimes 1 + 1 \boxtimes 1 \otimes d,$$

where d is the trivial connection on the trivial bundle $\mathbb{C}[\mathfrak{g}]_{(J)} \times M \rightarrow M$. The restricted bundle $F = E|_M$ is isomorphic to

$$F \simeq \text{Cl}(TM) \otimes \text{End}_{\text{Cl}}(\mathcal{E}) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}.$$

It carries the filtration $F^1 \subset F^2 \subset \dots \subset F^q$ defined by

$$F^p := \bigcup_{r \leq p/2} \text{Cl}^{p-2r}(TM) \otimes \text{End}_{\text{Cl}}(\mathcal{E}) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}^r.$$

The reason for this choice of filtration is that its associated graded space is isomorphic to

$$\text{gr}(F) \simeq \Lambda(T^*M) \otimes \text{End}_{\text{Cl}}(\mathcal{E}) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}$$

with the grading as in (2.1).

The bundle of endomorphisms

$$\text{End}(E) \simeq \text{End}(\mathcal{E}) \boxtimes \text{End}(\mathcal{E}^*) \otimes \text{End}(\mathbb{C}[\mathfrak{g}]_{(J)})$$

is filtered as follows:

$$\text{End}(E)^p = \bigcup_{r+s+2u=p} \text{End}(\mathcal{E})^r \boxtimes \text{End}(\mathcal{E}^*)^s \otimes \text{End}(\mathbb{C}[\mathfrak{g}]_{(J)})^u;$$

here we used Clifford filtrations on $\text{End}(\mathcal{E})$ and $\text{End}(\mathcal{E}^*)$, and the filtration on $\text{End}(\mathbb{C}[\mathfrak{g}]_{(J)})$ is induced from the filtration on $\mathbb{C}[\mathfrak{g}]_{(J)}$.

Clearly, (E, ∇) is a scaling bundle. Hence, by Theorem 4.20, we obtain a rescaled bundle

$$\mathbb{E}_J \rightarrow \mathbb{T}M.$$

5.2. Operators on the rescaled bundles

Let D be a differential operator of Getzler order p (cf. Definition 4.6) acting on $E \rightarrow M \times M$. Then, we have the induced map

$$t^p D : \mathcal{S}(E, \nabla) \rightarrow \mathcal{S}(E, \nabla)$$

on the rescaled modules, which gives a smooth differential operator on the rescaled bundle

$$D \curvearrowright \Gamma(\mathbb{E}_J).$$

Definition 5.1. We define the *Getzler symbol* of D , denoted by $\sigma^g(D)$, as the restriction of the operator D to the $t = 0$ fiber of the tangent groupoid

$$\sigma^g(D) \curvearrowright \Gamma(TM, \pi^* \Lambda(T^*M) \otimes \text{End}_{\text{Cl}}(\mathcal{E}) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}),$$

where $\pi : TM \rightarrow M$.

Example 5.2. (1) For a vector field $\xi \in \mathcal{X}(M)$, consider the differential operator $D = \nabla_{\xi}^{\mathcal{E}}$ acting on $\Gamma(M \times M, E)$ by differentiating along the first component of $M \times M$. This operator has Getzler order 1.

(2) For a vector field $\xi \in \mathcal{X}(M)$, consider the operator $D = c(\xi)$ acting on $\Gamma(E)$ acting by left Clifford multiplication. This operator is of Getzler order 1.

(3) For a polynomial $p(X) \in \mathbb{C}[\mathfrak{g}]$ of degree k , the differential operator $D = p(X) \curvearrowright \Gamma(E)$ defined by multiplication by $p(x)$, has Getzler order $2k$.

5.3. A calculation of Getzler symbols of some operators

We now calculate the symbols of these operators. We need to fix some conventions for the curvatures involved. There is a Lie algebra isomorphism

$$\begin{aligned} \gamma : \mathfrak{so}(\mathcal{E}) &\xrightarrow{\cong} \Lambda^2 \mathcal{E}, \\ T &\mapsto \frac{1}{4} \sum_i T(e_i) \wedge e_i, \end{aligned}$$

where the formula is given with respect to an orthonormal basis $\{e_i\}$ (see [23, Section 2.2.10]). The curvature of the bundle \mathcal{E} satisfies

$$K^{\mathcal{E}} = c \circ q \circ \gamma(R) + F^{\mathcal{E}/S},$$

where $q : \Lambda^* TM \rightarrow \text{Cl}(TM)$ is the quantization (see [2, Proposition 3.5]) map and $c : \text{Cl}(TM) \rightarrow \text{End}(\mathcal{E})$ is the Clifford action. We will use the notation

$$\mathbb{K} = \gamma(R) + F^{\mathcal{E}/S} \in \mathcal{A}^2(M, \Lambda^2 \mathcal{E}),$$

where $F^{\mathcal{E}/S} := q^{-1} \circ c^{-1}(F^{\mathcal{E}/S})$.

Proposition 5.3. *For $\xi \in \mathfrak{X}(M)$ and $p(X) \in \mathbb{C}_J$ of order $k \leq J$, we have*

- the symbol $\sigma^g(\nabla_{\xi}^{\mathcal{E}})$ is given by

$$s \mapsto \left[\eta \mapsto \partial_{\xi} s(\eta) + \frac{1}{2} \mathbb{K}(\eta, \xi) \wedge s(\eta) \right],$$

- the symbol $\sigma^g(c(\xi))$ is given by

$$s \mapsto [\eta \mapsto \xi \wedge s(\eta)],$$

- the symbol $\sigma^g(p(X))$ is given by

$$s \mapsto [\eta \mapsto p(X) \cdot s(\eta)].$$

Here, these symbols are considered as operators acting on

$$\Gamma(T_m M, \wedge T_m^* M \otimes \text{End}_{\text{Cl}_1}(\mathcal{E}_m) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}) \quad \text{for } m \in M.$$

Proof. The first bullet point is quite verbatim as the proof of [20, Lemma 3.6.3]. The second bullet point has the same proof as [20, Lemma 3.6.2].

To see the third bullet point, note that the operators $\nabla_Y^{\mathcal{E}}$ and $p(X)$ commute; therefore,

$$\exp(t \nabla_Y^{\mathcal{E}}) t^{2k} p(X) = t^{2k} p(X) \exp(t \nabla_Y^{\mathcal{E}}),$$

and hence,

$$\varepsilon_{Y_m}(t^{2k} p(X)s) = t^{2k} p(X)\varepsilon_{(X_m, 0)}(s),$$

from which the statement follows. ■

5.4. A conjugate connection

By Proposition 3.4, up to lower-order terms, the Bismut Laplacian $H(X)$ equals

$$(\nabla^{\mathcal{E}, \frac{1}{4}X})^* \nabla^{\mathcal{E}, \frac{1}{4}X}.$$

Thus, it is related to the connection $\nabla^{\mathcal{E}, \frac{1}{4}X}$. But the rescaled bundle was defined using the connection

$$\nabla^{\mathcal{E}} = \nabla^{\mathcal{E}, \frac{1}{4}X} - \frac{1}{4}\theta_X.$$

In particular, the computations of Proposition 5.3 help to compute the symbol of operators expressed in terms of the connection $\nabla^{\mathcal{E}}$. Thus, it is convenient to express $H(X)$ in terms of this connection. In other words, we want to express it in a coordinate system obtained by local trivializing \mathcal{E} using the parallel transport of $\nabla^{\mathcal{E}}$ along geodesics (while naturally, it is expressed in trivialization obtained by parallel transport along $\nabla^{\mathcal{E}, \frac{1}{4}X}$). One way of doing it is by conjugating $H(X)$ with the bundle map $\rho : \mathcal{E} \rightarrow \mathcal{E}$ which intertwines those two trivializations. This bundle map is given by the parallel transport along geodesics of the trivial line bundle with respect to the connection $d - \frac{1}{4}\theta_X$. This parallel transform formula is given as $\rho = e^\alpha$, where $\alpha : M \times M \rightarrow \mathfrak{g}^*$ is a smooth map on $M \times M$ with values in \mathfrak{g}^* .

Specifically, consider a smooth function $\alpha : M \times M \rightarrow \mathfrak{g}^* \subset \mathbb{C}[\mathfrak{g}]$ defined on the neighborhood of the diagonal $\Delta M \hookrightarrow M \times M$ by the formula

$$\alpha_X(\exp_m(\xi), m) = -\frac{1}{4} \int_0^1 (\iota(\mathcal{R})\theta_X)(\exp(t\xi))t^{-1} dt,$$

where \mathcal{R} is the local Euler-like vector field on $M \times M$ defined by

$$\mathcal{R}_{(\exp_m(\xi), m)} = \frac{d}{dt} \Big|_{t=1} (\exp_m(t\xi), m). \tag{5.1}$$

Away from a neighborhood of the diagonal, we can extend the function by zero using a cut-off function, we have

$$\mathcal{R} \cdot \alpha_X(\exp_m(\xi), m) = -\frac{1}{4}\theta_X(\mathcal{R}_{(\exp_m(\xi), m)}).$$

Consider

$$\omega_X := -d_s \alpha_X - \frac{1}{4}\theta_X \in \Gamma(\Lambda^1 T_s^*(M \times M)) \otimes \mathfrak{g}^*,$$

which is a source-wise one-form and depends on X linearly, and d_s is the source-wise de Rham differential. We have the following equality:

- $\omega_X|_{\Delta M} = 0$,
- in a neighborhood of the diagonal

$$\iota(\mathcal{R})\omega_X = 0. \tag{5.2}$$

From (5.2), we obtain

$$\mathcal{L}(\mathcal{R})\omega_X = -\frac{1}{4}\iota(\mathcal{R})d\theta_X,$$

which means the first-order terms of Taylor expansion of ω are the same as $-\frac{1}{4}\iota(\mathcal{R})d\theta_X$.

Lemma 5.4. For $\xi, \eta \in \mathfrak{X}(M)$, we have

$$d\theta_X(\xi, \eta) = -2(\mu^M(X)\xi, \eta).$$

So, in particular, $-\frac{1}{4}\iota(\mathcal{R})d\theta_X$ is the one-form dual to the vector field $\frac{1}{2}\mu(X)\mathcal{R}$. ■

So, we have

$$\omega_X = \frac{1}{4} \sum_j (\mu(X)\mathcal{R}, \partial_j) dx^j + \mathcal{O}(2), \tag{5.3}$$

where $\mathcal{O}(2)$ is a linear in X .

Lemma 5.5. For $\eta \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^*) \otimes \mathbb{C}[\mathfrak{g}]_{(J)}$, we have

$$o^{\text{sc}}(\omega_X(\eta)\sigma) \geq o^{\text{sc}}(\sigma) - 1.$$

Proof. Since ω_X vanishes along the diagonal $\Delta M \hookrightarrow M \times M$, the scaling order increases by at least 1, and since it is linear in X , it increases the polynomial order by at most 1. So, the equivariant scaling (scaling order minus two times the polynomial order) order decreases by at most 1. ■

Therefore, $t\omega_X(\eta)$ acts on the rescaled bundle, and it gives corresponding operators on the quotient module and the rescaled bundle:

$$\omega_X(\eta) : \mathcal{S}_{J,0}(E, \nabla) \rightarrow \mathcal{S}_{J,0}(E, \nabla)$$

and

$$\omega_X(\eta) : \Gamma(\mathbb{T}M, \mathbb{E}_J) \rightarrow \Gamma(\mathbb{T}M, \mathbb{E}_J).$$

Lemma 5.6. As an operator on $\mathcal{S}_{J,0}(E)$, we have

$$\omega_X(\eta)\sigma = \frac{1}{4}(\mu(X)\mathcal{R}, \eta) \cdot \sigma.$$

Proof. This follows immediately from (5.3). ■

Consider the map

$$\rho : M \times M \rightarrow C^\infty(\mathfrak{g}) \tag{5.4}$$

and

$$\rho_X(m_1, m_2) = e^{\alpha_X(m_1, m_2)}.$$

Define the conjugate connection

$$\rho_X \nabla^{\mathcal{E}, X} \rho_X^{-1},$$

and denote it by

$$\widehat{\nabla}^X.$$

Note that

$$\widehat{\nabla}^X = \nabla^\varepsilon + \omega_X.$$

By Lemma 5.6, the scaled operator $t\widehat{\nabla}_\eta^X$ acts on the rescaled module and hence on the rescaled bundle. From Proposition 5.3, it follows the following proposition.

Proposition 5.7. *The Getzler’s symbol of $\widehat{\nabla}_\xi^X$ is given for $m \in M$ by*

$$\begin{aligned} \sigma^g(\widehat{\nabla}_\xi^X) &\curvearrowright \Gamma(T_m M, \Lambda T_m^* M \otimes \text{End}_{\mathbb{C}l}(\mathcal{E}_m) \otimes \mathbb{C}[g](J)), \\ s &\mapsto \left[\eta \mapsto \partial_\xi s(\eta) + \frac{1}{2} \mathbb{K}(\eta, \xi) \wedge s(\mu) + \frac{1}{4} (\mu(X)\eta, \xi) s(\xi) \right]. \quad \blacksquare \end{aligned}$$

Denote by $\widehat{H}(X)$ the conjugate operator $\rho_X H(X) \rho_X^{-1}$. From Propositions 3.4 and 5.7 it follows that the Getzler symbol of the operator $\widehat{H}(X)$, in a local orthonormal frame $\{e_i\}_i$, is given by the formula

$$\sigma^g(\widehat{H}(X)) = - \sum_i \left(\partial_i - \frac{1}{4} \sum_j (R_{ij} + \mu_{ij}^M(X)) x^j \right)^2 + F^{\varepsilon/S} + \mu^{\varepsilon/S}(X), \quad (5.5)$$

where $R_{ij} = (R\partial_i, \partial_j)$. We will use the notation $\widehat{\mathcal{H}}(X) := \sigma^g(\widehat{H}(X))$ for the symbol. So, in particular, the family of operators $t^2\widehat{H}(X)$ acting on the rescaled bundle $\Gamma(\mathbb{E}_J)$ over the t -fibers of the tangent groupoid along with the operator $\widehat{\mathcal{H}}(X)$ acting on the rescaled bundle over the 0-fibers give a smooth differential operator acting on $\Gamma(\mathbb{E}_J)$.

The operator (5.5) is a *generalized harmonic oscillator operator* (cf. [2, Definition 4.11]) and the Mehler kernel associated to the heat operator $e^{-\tau\widehat{\mathcal{H}}(X)}$, $\tau > 0$, is given by the formula

$$\begin{aligned} K_\tau^{\widehat{\mathcal{H}}}(X, \xi) &= (4\pi\tau)^{-n/2} \det \left(\frac{\tau(R + \mu^M(X))/2}{\sinh \tau(R + \mu^M(X))/2} \right) \exp(-\tau(F^{\varepsilon/S} + \mu^{\varepsilon/S}(X))) \\ &\quad \times \exp \left(-\frac{1}{4\tau} \langle \xi | \tau(R + \mu^M(X))/2 \coth \tau(R + \mu^M(X))/2 | \xi \rangle \right), \end{aligned}$$

which can be written in terms of equivariant curvature terms as follows:

$$\begin{aligned} (4\pi\tau)^{-n/2} \det \left(\frac{\tau R_g(X)/2}{\sinh \tau R_g(X)/2} \right) \\ \times \exp \left(-\frac{1}{4\tau} \langle \xi | \tau R_g/2 \coth \tau R_g/2 | \xi \rangle \right) \exp(-F_g^{\varepsilon/S}(X)). \quad (5.6) \end{aligned}$$

6. The heat kernel asymptotics and proof of the Kirillov formula

We denote by $K_\tau(x, y, X) \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^*)$ the heat kernel of $e^{-\tau\widehat{H}(X)}$ where $\widehat{H}(X)$ is the conjugate operator $\rho_X H(X) \rho_X^{-1}$ and $H(X) = H_{\frac{1}{4}}(X)$ is the Bismut Laplacian (3.2).

Lemma 6.1 ([26, Theorem 7.15]). *Fix $X \in \mathfrak{g}$. For $J \geq 0$, there exists an asymptotic expansion*

$$K_\tau(x, y, X) \sim (4\pi t)^{-n/2} \exp\left(-\frac{d^2(x, y)}{4t}\right) \sum_{j \geq 0} \tau^j \Phi_{j,J}(x, y, X),$$

where $\Phi_{j,J}(x, y, X) \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^*)$. In a neighborhood of the diagonal, we have the recurrence relations (cf. [26, equation (7.17)])

$$(\nabla_{\mathcal{R}} + j + \nabla_{\mathcal{R}} \log g^{1/4})\Phi_{j,J} = -\widehat{H}(X)\Phi_{j-1,J}, \tag{6.1}$$

where \mathcal{R} is the Euler-like vector field (5.1), $\Phi_{0,J}(x, y, X) : \mathcal{E}_y \rightarrow \mathcal{E}_x$ is the parallel transport along the geodesic between x and y with respect to the connection $\nabla^{\mathcal{E}}$. Furthermore,

$$g = \det(g_{ij})$$

where $g_{ij}(x, y)$ are functions defined in a neighborhood of the diagonal on $M \times M$ whose restriction to the source fibers $M \times \{y\}$ are the components of the metric in the normal coordinate at y . ■

Remark 6.2. From the recurrence relations (6.1), it follows that the coefficients $\Phi_{j,J}(x, y, X)$ are polynomials in variable X . Hence, we have

$$\Phi_{j,J}(x, y, X) \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^* \otimes \mathbb{C}[\mathfrak{g}]_{(J)}).$$

For every $N \geq 0$, we will denote by

$$K_\tau^N := (4\pi\tau)^{-n/2} e^{-\frac{d^2(x,y)}{4\tau}} \sum_{j \geq 0}^N \tau^j \Phi_{j,J}(x, y, X)$$

the approximate heat kernel. Consider the rescaled heat kernel

$$\sigma_\tau^N(x, y, X) := t^n \psi(d(x, y)) K_{t^2\tau}^N,$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}^{\geq 0}$ is the cut-off functions such that $\psi|_{[-r/2, r/2]} = 1$ and ψ vanishes outside $[-r, r]$ where r is the injectivity radius of M .

Proposition 6.3. *For every $j \geq 0$, we have*

$$o^{\text{sc}}(\Phi_{j,J}) \geq -2j.$$

Proof. We use induction on j . Note that for $j = 0$ the case is clear by Remark 6.2 and Theorem 4.16, since $\Phi_{0,J}$ is the parallel transport map and hence synchronous. We show for $j > 0$

$$o^{\text{sc}}(\Phi_{j,J}) = o^{\text{sc}}((\nabla_{\mathcal{R}} + j + \nabla_{\mathcal{R}} \log g^{1/4})\Phi_{j,J}), \tag{6.2}$$

from which we then deduce

$$o^{\text{sc}}(\Phi_{j,J}) \geq o^{\text{sc}}(-\widehat{H}(X)\Phi_{j-1,J}) \geq -2(j-1) - 2 = -2j.$$

So, we just need to prove (6.2). Using Theorem 4.16, we only need to show the equality in a trivializing neighborhood associated with the Euler-like vector field \mathcal{R} . So if the Taylor expansion of $\Phi_{i,J}$ is given by

$$\Phi_{j,J} \sim \sum_I y^I \sigma_I,$$

then we have the Taylor expansion

$$(\nabla_{\mathcal{R}} + j)\Phi_{j,J} \sim \sum_I (|I| + j)y^I \sigma_I,$$

which has the same scaling order (i.e., Taylor order) as $\Phi_{j,J}$. The Euler-like vector field \mathcal{R} vanishes along the diagonal; hence so does $\nabla_{\mathcal{R}} \log g^{1/4}$. Thus, this term has no effect on the scaling order, and therefore, the equality (6.2) follows. ■

Note that for $\tau > 0$ we may consider $\psi(d(x, y))e^{-\frac{d^2(x,y)}{4\tau t^2}}$ as a smooth function on $\mathbb{T}M$. By Proposition 6.3, we have

$$\sum_{j \geq 0}^N \tau^j t^{2j} \Phi_{j,J} \in \mathcal{S}_J(E, \nabla),$$

and hence, $\sigma_t^N(x, y, X) \in \Gamma(\mathbb{T}M, \mathbb{E}_J)$.

The following lemma is a generalization of [22, Lemma 2.8]. Consider a section $s \in \Gamma(M \times M \times \mathbb{R}, \mathcal{E} \boxtimes \mathcal{E}^*)$ and the natural projection $\pi : \mathbb{T}M \rightarrow M \times M \times \mathbb{R}$.

Lemma 6.4. *For $N \geq n$, the section $t^{N+1}s$, after composing with $\pi : \mathbb{T}M \rightarrow M \times M \times \mathbb{R}$, may be considered as a smooth section of the rescaled bundle $\mathbb{E}_J \rightarrow \mathbb{T}M$ that vanishes to order $2N - n + 1$ on the zero fiber of the tangent groupoid.*

Proof. Since the function, $t : \mathbb{T}M \rightarrow \mathbb{R}$ is the canonical projection. This projection, as a function, vanishes on the zero fiber. Since we may write

$$t^{2N+1}s = t^{2N-n} \cdot t^{n+1}s,$$

it is enough to show that $t^n s$ gives a smooth section of the rescaled bundle that vanishes along the zero fiber. Note that we may find sections $\sigma_1, \dots, \sigma_p \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^*)$ and functions $f_1, \dots, f_p \in C^\infty(M \times M \times \mathbb{R}, \mathcal{E} \boxtimes \mathcal{E}^*)$ such that

$$s = \sum_{j=1}^p f_j(x, y, t)\sigma_j(x, y).$$

Since $t^{n+1}\sigma_j$ is an element of the rescaled module that gives zero values on the zero fiber of the tangent groupoid and $f_i \circ \pi \in C^\infty(\mathbb{T}M)$, we obtain a smooth section of the rescaled bundle of the form

$$\sum_{j=j}^p f_i \circ \pi \cdot t^{n+1}\sigma_j$$

that vanishes along the zero fiber of $\mathbb{T}M$. ■

From Lemma 6.1, we have

$$K_{\tau t^2} - \sigma_\tau^N = \mathcal{O}(t^{2N+1}),$$

where $\mathcal{O}(t^{2N+1}) \in \Gamma(M \times M \times \mathbb{R}, \mathcal{E} \boxtimes \mathcal{E}^*)$ and if $N > n/2$, by Lemma 6.4, it gives a smooth section of the rescaled bundle vanishing over the zero fiber of the tangent groupoid. In particular, we have the following theorem.

Theorem 6.5. *The rescaled heat kernel $K_\tau := K_{\tau t^2}$ gives a smooth section of the rescaled bundle $\mathbb{E}_J \rightarrow \mathbb{T}M$, which over the zero fiber of the tangent groupoid agrees with values of sections σ_τ^N for $N > n/2$.* ■

The section $K_\tau \in \Gamma(\mathbb{T}M, \mathbb{E}_J)$ satisfies the heat equation

$$(\partial_\tau + \widehat{H}(X))K_\tau = 0,$$

where $\widehat{H}(X)$ is the smooth extension of the operator $t^2\widehat{H}(X)$ to zero fiber of the tangent groupoid. Note that on zero fiber of the tangent groupoid, $\widehat{H}(X)$ acts as the harmonic oscillator operator $\widehat{\mathcal{H}}(X)$ given in (5.5). Therefore, we have

$$(\partial_\tau + \widehat{\mathcal{H}}(X))(K_\tau|_{t=0}) = 0.$$

The uniqueness property of the Mehler kernel then proves

$$K|_{t=0} = K_\tau^{\widehat{\mathcal{H}}}, \tag{6.3}$$

where $K_\tau^{\widehat{\mathcal{H}}}$ is as in (5.6).

6.1. Supertraces on the rescaled bundle

Let $M \times \mathbb{R} \hookrightarrow \mathbb{T}M$ denote the embedding of the unit space of the tangent groupoid.

Lemma 6.6. *The following map is well defined:*

$$\begin{aligned} \varepsilon : \Gamma(\mathbb{T}M, \mathbb{E}_J) &\rightarrow C^\infty(M \times \mathbb{R}, \mathbb{C}[\mathfrak{g}]_{(J)}), \\ \sigma &\mapsto \begin{cases} (m, t) \mapsto t^{-n} \operatorname{str}(\sigma(m, m, t)), & t \neq 0, \\ (m, 0) \mapsto \operatorname{str}(\sigma(0_m, 0)), & t = 0. \end{cases} \end{aligned}$$

Here, $C^\infty(M \times \mathbb{R}, \mathbb{C}[\mathfrak{g}]_{(J)})$ denotes the $\mathbb{C}[\mathfrak{g}]_{(J)}$ -valued smooth functions on $M \times \mathbb{R}$. The supertrace $\operatorname{str}(\sigma(m, m, t))$ is induced by the supertrace on $\mathcal{E} \otimes \mathcal{E}^*$ and the supertrace $\operatorname{str}(\sigma(0_m, 0))$ is given by the Berezin integral (cf. [2, page 40]).

Proof. Since every $\sigma \in \Gamma(\mathbb{T}M, \mathbb{S})$ can be written as a sum

$$\sigma = \sum_k f_k \sigma_k,$$

where $f_k \in C^\infty(\mathbb{T}M)$ and $\sigma_k \in \mathcal{S}_J(E, \nabla)$, it is enough to prove the lemma for $\sigma \in \mathcal{S}_J(E, \nabla)$. So, assume $\sigma = \sum_k s_k t^{-k}$, where $s_k \in \Gamma(M \times M, \mathcal{E} \boxtimes \mathcal{E}^* \otimes \mathbb{C}[\mathfrak{g}]_{(J)})$ is of scaling order k or more. We claim $\text{str}(s_k(m, m)) = 0$ for $k > -n$. To see why, note that by definition

$$o^{\text{sc}}(s_k) \leq -o^f(s_k).$$

So, when $k > -n$,

$$o^f(s_k) < n,$$

and therefore, $\text{str}(s_k(m, m)) = 0$. Also, for $k < -n$, $t^{-n} \text{str}(s_k)(x, x) t^{-k}$ extends smoothly to zero at $t = 0$, since $k - n > 0$. For $k = -n$, the function $t^{-n} \text{str}(s_{-n}(m, m)) t^n$ is constant in t . Therefore,

$$t^{-n} \text{str}(\sigma(m, m, t))$$

extends smoothly to

$$\text{str}(s_{-n}(m, m))$$

at $t = 0$ which equals $\text{str}(\sigma(0_m, 0))$. ■

Using Lemma 6.6, we define a family of functionals on $\Gamma(\mathbb{T}M, \mathbb{E}_J)$

$$\text{Str}_t : \Gamma(\mathbb{T}M, \mathbb{E}_J) \rightarrow \mathbb{C}$$

given as

$$\text{Str}_t(\sigma) = \int_M \mathfrak{s}(\sigma)(m, t) dm, \tag{6.4}$$

and obviously, $\text{Str}_t(\sigma) \rightarrow \text{Str}_0(\sigma)$.

Theorem 6.7 (McKean–Singer). *For $t \neq 0$ and $\tau > 0$, we have*

$$\text{Str}_t(\mathbf{K}_\tau)$$

is independent of t, τ and by Proposition 3.2 equals the equivariant index

$$\text{ind}(e^{-X}, \mathcal{D}).$$

Since the supertraces Str_t (6.4) are defined for every t , Theorem 6.7 also applies for $t = 0$. So, by evaluating at $\tau = 1$ and $t = 0$ and using (6.3), we obtain

Theorem 6.8 (Kirillov formula). *The equivariant index is given by the integral of equivariant differential forms as follows:*

$$\text{ind}(e^{-X}, \mathcal{D}) = (2\pi i)^{-n/2} \int_M \widehat{\mathbf{A}}_{\mathfrak{g}}(X, M) \text{Ch}_{\mathfrak{g}}(X, \mathcal{E}/S).$$

7. The other examples of the rescaled bundles

We conclude with several more examples and applications of our construction of the rescaled bundle in different geometric situations.

7.1. The case with no filtration

Consider an embedding $M \hookrightarrow V$. For every vector bundle $E \rightarrow V$ with trivial filtrations on $E|_M$ and $\text{End}(E)$ and for every connection ∇ on $E \rightarrow V$, the pair (E, ∇) is a scaling bundle. Hence, we obtain a rescaled bundle

$$\mathbb{E} \rightarrow \text{DNC}(V, M).$$

Since, in this case, the scaling order coincides with the vanishing order, this bundle is independent of the choice of the connection ∇ . Indeed, the rescaled bundle \mathbb{E} is isomorphic to the pullback bundle $\pi_V^* E$, where $\pi_V : \text{DNC}(V, M) \rightarrow V$ is the canonical projection.

One important case of this situation is the description of the Witten deformation via the deformation to normal cone construction, recently obtained by Omar Mohsen [24]. In this case,

$$E = \Lambda^* T^* V \rightarrow V.$$

In the next subsection, we discuss an equivariant generalization of Mohsen’s construction.

7.2. The equivariant Witten and Novikov deformations

Let G be a compact Lie group action on a closed manifold M . Then, $\Lambda^* T^* M \rightarrow M$ is a G -equivariant bundle induced with the Levi–Civita connection ∇^{LC} . As in Section 5.1, we consider the equivariant version of this bundle:

$$E := \Lambda^* T^* M \otimes \mathbb{C}[\mathfrak{g}]_{(J)}$$

and endow it with the connection

$$\nabla := \nabla^{LC} \otimes 1 + 1 \otimes d.$$

Let $f : M \rightarrow \mathbb{R}$ be a G -equivariant Morse–Bott function, with the critical submanifold $Z \hookrightarrow M$. We consider the filtrations on $E|_Z$ and on E defined by the trivial filtrations on $\Lambda^* T^* M|_Z$ and $\Lambda^* T^* M$ and the filtration by $2 \deg P$ on $\mathbb{C}[\mathfrak{g}]_{(J)}$. (Note that this is different from the filtration (2.1), where we also consider a non-trivial filtration on $\Lambda^* T^* M$.) This gives a scaling bundle (E, ∇) , and thus, one obtains a rescaled bundle $\mathbb{E} \rightarrow \text{DNC}(M, Z)$. The arguments of [24] extend naturally to the equivariant case and show that the “rescaled Witten deformation” of the de Rham–Dirac operator

$$t(e^{\frac{1}{t^2} f} d e^{-\frac{1}{t^2} f} + e^{-\frac{1}{t^2} f} d^* e^{\frac{1}{t^2} f}) = t \left(d + d^* + \frac{1}{t^2} c(df) \right)$$

acts on the rescaled module $\mathcal{S}(E, \nabla)$ and hence extends to a smooth G -equivariant differential operator acting on the section of the rescaled bundle $\Gamma(\mathbb{E})$. One can also generalize it to the Novikov deformation, defined by a closed differential form ω :

$$t \left(d + d^* + \frac{1}{t^2} c(\omega) \right),$$

cf. [5, 8, 25] for the non-equivariant case and [6, 7] for the equivariant case.

7.3. The fixed-point formula for the equivariant index

Another example of the rescaled bundle arises in the context of the equivariant index formula, which will appear in a joint paper of the second author with Yiannis Loizides, Jesus Sanchez, and Shiqi Liu.

Consider a compact Lie group G acting isometrically on M . For an element $g \in G$, denoted by M^g the fixed submanifold under the action of g . Associated to the diagonal embedding $M^g \hookrightarrow M \times M$, the deformation space $\text{DNC}(M \times M, M^g)$ is called the *relative tangent groupoid* and denoted by $\mathbb{T}_g M$.

Assume $\mathcal{E} \rightarrow M$ is a G -equivariant Clifford module that carries a G -invariant connection ∇ . The bundle

$$E = \mathcal{E} \boxtimes \mathcal{E}^* \rightarrow M \times M$$

carries the induced connection ∇^E . Using a similar argument as in Example 4.8, one obtains the scaling bundle (E, ∇^E) and the rescaled bundle

$$\mathbb{E}_g \rightarrow \mathbb{T}_g M.$$

Using this vector bundle, one recovers the fixed-point index formula for equivariant Dirac operators.

References

- [1] M. F. Atiyah and G. B. Segal, [The index of elliptic operators. II](#). *Ann. of Math. (2)* **87** (1968), 531–545 Zbl [0164.24201](#) MR [0236951](#)
- [2] N. Berline, E. Getzler, and M. Vergne, [Heat kernels and Dirac operators](#). Grundlehren Math. Wiss. 298, Springer, Berlin, 1992 Zbl [0744.58001](#) MR [1215720](#)
- [3] N. Berline and M. Vergne, [The equivariant index and Kirillov's character formula](#). *Amer. J. Math.* **107** (1985), no. 5, 1159–1190 Zbl [0604.58046](#) MR [0805808](#)
- [4] J.-M. Bismut, [Index theorem and equivariant cohomology on the loop space](#). *Comm. Math. Phys.* **98** (1985), no. 2, 213–237 Zbl [0591.58027](#) MR [0786574](#)
- [5] M. Braverman and M. Farber, [The Novikov–Bott inequalities](#). *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), no. 7, 897–902 Zbl [0846.58016](#) MR [1355849](#)
- [6] M. Braverman and M. Farber, [Novikov inequalities with symmetry](#). *C. R. Acad. Sci. Paris Sér. I Math.* **323** (1996), no. 7, 793–798 Zbl [0863.57028](#) MR [1416178](#)

- [7] M. Braverman and M. Farber, [Equivariant Novikov inequalities](#). *K-Theory* **12** (1997), no. 4, 293–318 Zbl [0889.57040](#) MR [1485432](#)
- [8] M. Braverman and M. Farber, [Novikov type inequalities for differential forms with non-isolated zeros](#). *Math. Proc. Cambridge Philos. Soc.* **122** (1997), no. 2, 357–375 Zbl [0894.58012](#) MR [1458239](#)
- [9] H. Bursztyn, H. Lima, and E. Meinrenken, [Splitting theorems for Poisson and related structures](#). *J. Reine Angew. Math.* **754** (2019), 281–312 Zbl [1425.53103](#) MR [4000576](#)
- [10] P. Carrillo Rouse, [A Schwartz type algebra for the tangent groupoid](#). In *K-theory and non-commutative geometry*, pp. 181–199, EMS Ser. Congr. Rep., EMS Press, Zürich, 2008 Zbl [1165.58003](#) MR [2513337](#)
- [11] H. Cartan, Notions d’algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pp. 15–27, Georges Thone, Liège, 1951 MR [0042426](#)
- [12] A. Connes, *Noncommutative geometry*. Academic Press, San Diego, CA, 1994 Zbl [0818.46076](#) MR [1303779](#)
- [13] C. Debord and G. Skandalis, [Adiabatic groupoid, crossed product by \$\mathbb{R}_+^*\$ and pseudodifferential calculus](#). *Adv. Math.* **257** (2014), 66–91 Zbl [1300.58007](#) MR [3187645](#)
- [14] C. Debord and G. Skandalis, Blowup constructions for lie groupoids and a boutet de monvel type calculus. 2017, arXiv:[1705.09588](#)
- [15] C. Debord and G. Skandalis, [Lie groupoids, pseudodifferential calculus, and index theory](#). In *Advances in noncommutative geometry—on the occasion of Alain Connes’ 70th birthday*, pp. 245–289, Springer, Cham, 2019 Zbl [1442.58006](#) MR [4300554](#)
- [16] W. Fulton, *Intersection theory*. *Ergeb. Math. Grenzgeb.* (3) **2**, Springer, Berlin, 1984 Zbl [0541.14005](#) MR [0732620](#)
- [17] A. R. Haj Saeedi Sadegh, *The local index theorem and the tangent groupoid*. ProQuest LLC, Ann Arbor, MI, 2021. PhD thesis, Pennsylvania State University MR [4419931](#)
- [18] A. R. Haj Saeedi Sadegh and N. Higson, [Euler-like vector fields, deformation spaces and manifolds with filtered structure](#). *Doc. Math.* **23** (2018), 293–325 Zbl [1393.57018](#) MR [3846057](#)
- [19] N. Higson, The tangent groupoid and the index theorem. In *Quanta of maths*, pp. 241–256, Clay Math. Proc. 11, American Mathematical Society, Providence, RI, 2010 Zbl [1230.58016](#) MR [2732053](#)
- [20] N. Higson and Z. Yi, [Spinors and the tangent groupoid](#). *Doc. Math.* **24** (2019), 1677–1720 Zbl [1429.58018](#) MR [4033830](#)
- [21] Y. Kosmann, *Dérivées de Lie des spineurs*. *Ann. Mat. Pura Appl.* (4) **91** (1972), 317–395 Zbl [0231.53065](#) MR [0312413](#)
- [22] M. Ludewig and Z. Yi, [A short proof of the localization formula for the loop space Chern character of spin manifolds](#). *J. Noncommut. Geom.* **17** (2023), no. 1, 189–210 Zbl [1516.53046](#) MR [4565431](#)
- [23] E. Meinrenken, *Clifford algebras and Lie theory*. *Ergeb. Math. Grenzgeb.* (3) **58**, Springer, Heidelberg, 2013 Zbl [1267.15021](#) MR [3052646](#)
- [24] O. Mohsen, [Witten deformation using Lie groupoids](#). *Adv. Math.* **398** (2022), article no. 108142 Zbl [1486.58010](#) MR [4375625](#)
- [25] A. V. Pazhitnov, An analytic proof of the real part of Novikov’s inequalities. *Soviet Math. Dokl.* **35** (1987), no. 2, 456–457
- [26] J. Roe, *Elliptic operators, topology and asymptotic methods*. 2nd edn., Pitman Res. Notes Math. Ser. 395, Longman, Harlow, 1998 Zbl [0919.58060](#) MR [1670907](#)

- [27] E. van Erp and R. Yuncken, [A groupoid approach to pseudodifferential calculi](#). *J. Reine Angew. Math.* **756** (2019), 151–182 Zbl [1433.58025](#) MR [4026451](#)

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Maxim Braverman

Department of Mathematics, Northeastern University, 115 Richards Hall, Boston, MA 02115, USA; m.braverman@northeastern.edu

Ahmad Reza Haj Saeedi Sadegh

Department of Mathematics, Northeastern University, 115 Richards Hall, Boston, MA 02115, USA; a.hajsaeedisadegh@northeastern.edu