Spectral triples for noncommutative solenoids and a Wiener's lemma

Carla Farsi, Therese Landry, Nadia S. Larsen, and Judith Packer

Abstract. In this paper, we construct odd finitely summable spectral triples based on length functions of bounded doubling on noncommutative solenoids. Our spectral triples induce a Leibniz Lip-norm on the state spaces of the noncommutative solenoids, giving them the structure of Leibniz quantum compact metric spaces. By applying methods of R. Floricel and A. Ghorbanpour, we also show that our odd spectral triples on noncommutative solenoids can be considered as inductive limits of spectral triples on rotation algebras. In the final section, we prove a noncommutative version of Wiener's lemma and show that our odd spectral triples can be defined to have an associated smooth dense subalgebra which is stable under the holomorphic functional calculus, thus answering a question of B. Long and W. Wu. The construction of the smooth subalgebra also extends to the case of nilpotent discrete groups.

1. Introduction

Our aim in this paper is to build and study noncommutative manifold structures (specifically spectral triples and their associated metric structures) on noncommutative solenoid C^*-algebras, as well as to analyze some of their associated smooth subalgebras. Our methods also apply to commutative solenoids as well, even though we will not emphasize the commutative case here.

In the 1980s, A. Connes initiated a program to adapt classical tools from topology and Riemannian geometry to the operator algebraic setting. For example, Connes showed in [10, 11] that the geodesic distance d_g on a closed spin Riemannian manifold M can be recovered from the C^*-algebra $C(M)$, the Hilbert space H of L^2-spinor fields, and the Dirac operator D, via, $\forall x, y \in M$,

$$d_g(x, y) = \sup \{|f(x) - f(y)| : f \in C(M), \|[D, f]\|_{\mathcal{B}(H)} \leq 1\}. \quad (1.1)$$

These ideas were generalized to the context of noncommutative C^*-algebras, which Connes formalized in the definition of a *spectral triple*. A spectral triple, in its most basic form, is given by a unital C^*-algebra A represented on a Hilbert space H, and a Dirac operator D behaving well with respect to taking the commutators with a dense $*$-subalgebra of A. With this structure, one can define a metric similar to equation (1.1)

using the state space $S(A)$ of A rather than only the pure states that metrizes the weak*-topology under appropriate circumstances. In this paper, all spectral triples will be odd, that is, ungraded.

M. Rieffel considered, more generally and in absence of a spectral triple, similar metric structures on unital C^*-algebras (and, in particular, on their state spaces) associated to seminorms; the resulting spaces were named *compact quantum metric spaces*; see [33]. Latrémolière (see e.g. [20]) considered a special case of Rieffel's spaces, and switched terminology to *quantum compact metric spaces*, terminology which we will follow in this paper. Rieffel also became interested in the problem of constructing spectral triples on reduced group C^*-algebras and twisted reduced group C^*-algebras associated to discrete groups by using the left regular representation together with a Dirac operator constructed from a proper length function, following pioneering work by Connes [10]. The class of groups where this approach could be used to derive not only spectral triples, but also examples of compact quantum metric spaces, was extended to include finitely generated nilpotent groups with a length function of *bounded doubling*; see [8, 35].

In three recent papers [25–27], Long and Wu followed up on the work of Christ and Rieffel, by constructing odd spectral triples for twisted group C^*-algebras $C^*(\Gamma, \sigma)$, in which Γ is not necessarily finitely generated, but it is endowed with a length function satisfying either the σ-*twisted rapid decay* or the *bounded θ-dilation* properties. (In this paper, we will call this latter condition *bounded **t**-dilation*). However, the question of whether or not one could construct length functions satisfying the desired conditions on discrete nilpotent groups that were not finitely generated remained an open one, even for the case of discrete abelian groups.

As far as this paper is concerned, the prime examples of twisted abelian group C^*-algebras in which the underlying abelian groups are not finitely generated are the so-called *noncommutative solenoids*. Noncommutative solenoids were introduced and studied by Latrémolière and Packer in [22–24]; they can be realized as the twisted group C^*-algebras $C^*(\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}], \sigma)$ for a certain multiplier σ (a multiplier is a 2-cocycle with values in $\mathbb{T}$). Here, as throughout the rest of the paper, p is a fixed prime number.

Latrémolière and Packer showed that these C^*-algebras have the structure of Leibniz quantum compact metric spaces, and are the limit in the direct sense and in the sense of the Gromov–Hausdorff propinquity of noncommutative rotation algebras. However, the question of constructing spectral triples on them was not considered in their papers.

Since Floricel and Ghorbanpour in [14] discuss the construction of inductive limits of odd spectral triples for unital C^*-algebras, it thus became a natural question to ask whether spectral triples as constructed by Connes [10] and Rieffel [35] (as well as by Long and Wu [25–27]) could be combined with the methods of Floricel and Ghorbanpour [14] to endow noncommutative solenoids with spectral triple structures (of inductive limit type). This is one of the main motivating questions that brought us to write this paper.

For the sake of completeness, we also note that Aiello, Guido, and Isola in [3] constructed semifinite spectral triples on periodic (but not general) rational noncommutative solenoids. Moreover, via Gabor analysis methods, Austad and Luef construct spectral

triples on noncommutative solenoids in [6]. However, neither of these approaches are entirely satisfactory for our purposes. The work in [3] gives semifinite and not finite spectral triples, and it does not apply when looking at aperiodic noncommutative solenoids, while the work of Austad and Luef does not provide specific examples of weights that satisfy their needed conditions to obtain any specific spectral triple on the noncommutative solenoid.

Therefore, in this paper one of our main aims is to combine the methods of Christ and Rieffel [8] and then Long and Wu [25–27] to construct a length function on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ with the bounded doubling property or bounded $\mathbf{t}$-dilation property. This allows us to construct the desired finitely summable spectral triples for arbitrary noncommutative solenoids. Our spectral triples define a quantum compact metric space structure on noncommutative solenoids as well. The spectral triples we construct are also inductive limits of spectral triples on rotation algebras in the sense of Floricel and Ghorbanpour [14].

In a more general setting, we also establish that for any countable discrete nilpotent (hence abelian, so that our results can be applied to $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$) group Γ, the $*$-subalgebra $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is stable under the holomorphic functional calculus in $\ell^1(\Gamma, \sigma)$, and that this latter $*$-algebra is stable under the holomorphic functional calculus in $C^*(\Gamma, \sigma)$. We do this by adapting the work of Jolissaint in [18] together with the work of Austad [5] involving the noncommutative Wiener lemma of Gröchenig and Leinert [15]. In doing so, we thus generalize some results of Chatterji (see the appendix of [30]), which requires σ-twisted rapid decay. This will show that when Γ is countable, discrete, and nilpotent, $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is stable under the holomorphic functional calculus in $C^*(\Gamma, \sigma)$. This also answers a question of Long and Wu from [27] in the solenoid case.

It remains to be seen whether the spectral triples presented here are limits of spectral triples on noncommutative rotation algebras in the sense of Latrémolière [19, 21].

The structure of our paper is as follows. In Section 2, we review some preliminaries by giving the required definitions. In Section 3, we construct the desired length function on

$$\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$$

and show that the Dirac operator arising from this length function gives rise to a spectral triple on the noncommutative solenoid. We also show that we can use the structure of a noncommutative solenoid as an inductive limit of noncommutative rotation algebras to view our spectral triple as one coming from an inductive system of spectral triples on noncommutative rotation algebras as defined by Floricel and Ghorbanpour [14]. In Section 4, for any countable discrete nilpotent group Γ and multiplier σ, we show that the dense $*$-subalgebra $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is stable in $\ell^1(\Gamma, \sigma)$ under the holomorphic functional calculus, and that in turn $\ell^1(\Gamma, \sigma)$ is stable under the holomorphic functional calculus in $C^*(\Gamma, \sigma)$ (noncommutative Wiener lemma). In the case where the length function has bounded $\mathbf{t}$-dilation for some $\mathbf{t} > 1$, it is possible to use $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ as the dense smooth subalgebra associated to our spectral triples, thus answering the question of Long and Wu posed in [27] in the solenoid case.

At a late stage in the preparation of this manuscript, the authors became aware of a paper by Antonini, Guido, Isola, and Rubin [4], in which they reconstructed the odd spectral triples of Long and Wu [27] for twisted group C^*-algebras, as well as an even version of them. We cite this paper for completeness.

2. Preliminaries

We start by reviewing the definition of spectral triples.

Definition 2.1 (Connes [10, 11]). A *spectral triple* (A, H, D) consists of a unital C^*-algebra A, a unital faithful representation π of A on a Hilbert space H, and a self-adjoint operator

$$D : \operatorname{dom}(D) \subseteq H \to H$$

such that

(ST1) the operator D has compact resolvent $R_\lambda(D) = (D - \lambda \operatorname{Id}_H)^{-1}$, $\lambda \in \mathbb{C} \setminus \operatorname{sp}(D)$,

(ST2) there exists a dense $*$-subalgebra $\mathcal{A}$ of A such that for every $a \in \mathcal{A}$ the commutator

$$\big[D, \pi(a)\big] := D\pi(a) - \pi(a)D$$

is densely defined and extends to a bounded operator on H. The subalgebra $\mathcal{A}$ is sometimes referred to as a *smooth subalgebra* of A with respect to D.

The operator D is called a (generalized) *Dirac operator*.

Spectral triples generalize differential structure, as seen from the pioneering example of Connes of a spectral triple associated to a spin manifold (in this case the spectral triple operator is the Dirac operator) [10]. The dense set in A described in condition (ST2) is therefore analogous to the dense set of C^∞ functions in the continuous functions $C(M)$ in the manifold case. The compact resolvent condition ensures that the eigenvalues of D exhibit properties that allow for the extraction of geometric information like measure and dimension from spectral data.

The following is another example due to Connes that is a special case of his result for more general group C^*-algebras. It was adapted to twisted group C^*-algebras for $\mathbb{Z}^d$ by Rieffel.

Example 2.2 ([10, 35]). To define a spectral triple for the group C^*-algebra $C^*(\mathbb{Z}^d)$, let $\| \cdot \|$ be any norm on $\mathbb{R}^d$ restricted to $\mathbb{Z}^d$. Let π be the regular representation of $C^*(\mathbb{Z}^d)$ on $\ell^2(\mathbb{Z}^d)$. Define the linear operator $D_{\|\cdot\|}$ on $\ell^2(\mathbb{Z}^d)$ by

$$D_{\|\cdot\|} f(v) = \|v\| \cdot f(v), \quad v \in \mathbb{Z}^d.$$

The results of [35, Theorem 0.1] show that $(C^*(\mathbb{Z}^d), \ell^2(\mathbb{Z}^d), D_{\|\cdot\|})$ is a spectral triple with smooth subalgebra $C_C(\mathbb{Z}^d) := \mathbb{C}\mathbb{Z}^d$. This example extends to twisted group C^*-algebras $C^*(\mathbb{Z}^d, \sigma)$ as well.

2.1. Inductive limit spectral triples

Christensen and Ivan built spectral triples for AF-algebras [9], as did Connes in the particular case of the commutative C^*-algebra of all continuous functions on a Cantor set [11]. Though slightly different, both constructions of spectral triples have C^*-algebras A, Hilbert spaces H, and Dirac operators D, which for some directed set $(J, \leqslant)$ can each, respectively, be written as inductive limits of C^*-algebras $\varinjlim_{j \in J} A_j$, Hilbert spaces $\varinjlim_{j \in J} H_j$, and self-adjoint operators $\varinjlim_{j \in J} D_j$. Furthermore, $\{(A_j, H_j, D_j)\}_{j \in J}$ is a family of spectral triples for which maps can be defined making this collection into an inductive system with inductive limit (A, H, D). Floricel and Ghorbanpour formalized a framework which includes these inductive systems of spectral triples, as we will review below.

The spectral triples for solenoids that we will construct in this paper will also be of this inductive limit kind. In the definition below, we slightly modify the definition of Floricel and Ghorbanpour to allow choices of different smooth subalgebras.

Definition 2.3 (Floricel–Ghorbanpour [14]). A *morphism between two spectral triples* (A_1, H_1, D_1) and (A_2, H_2, D_2) with respective smooth subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is a pair (ϕ, I) consisting of a unital $*$-homomorphism $\phi : A_1 \to A_2$ and a bounded linear operator $I : H_1 \to H_2$ satisfying the following conditions:

(1) $\phi(\mathcal{A}_1) \subseteq \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are as in (ST2),

(2) $I\pi_1(a) = \pi_2(\phi(a))I$, for every $a \in A_1$,

(3) $I(\mathrm{dom}(D_1)) \subseteq \mathrm{dom}(D_2)$ and $ID_1 = D_2 I$.

A morphism (ϕ, I) is said to be *isometric* if ϕ is injective and I is an isometry.

Definition 2.4 (Floricel–Ghorbanpour [14]). Let $(J, \leqslant)$ be a directed index set and let $\{(A_j, H_j, D_j)\}_{j \in J}$ be a family of spectral triples, with respective smooth subalgebras $\{\mathcal{A}_j\}_{j \in J}$. Suppose that for every $j, k \in J$, $j \leqslant k$, an isometric morphism $(\phi_{j,k}, I_{j,k})$ from (A_j, H_j, D_j) to (A_k, H_k, D_k) is given such that $\phi_{k,l}\phi_{j,k} = \phi_{j,l}$ and $I_{k,l}I_{j,k} = I_{j,l}$ for all $j, k, l \in J$, $j \leqslant k \leqslant l$. The resulting system $\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})\}_J$, with respective smooth subalgebras $\{\mathcal{A}_j\}_{j \in J}$, is called an *inductive system of spectral triples*.

Definition 2.5 (Floricel–Ghorbanpour [14]). Let $\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})\}_J$ be an inductive system of spectral triples, with respective smooth subalgebras $\{\mathcal{A}_j\}_{j \in J}$, with

$$A = \varinjlim_{j \in J} A_j, \quad H = \varinjlim_{j \in J} H_j,$$

and inductive limit isometric operators

$$I_j : H_j \to H,$$

and the $*$-monomorphisms $\phi_j : A_j \to A$. Let

$$\pi = \varinjlim_{j \in J} \pi_j.$$

be the inductive limit of the family of representations $\{\pi_j\}_{j \in J}$ associated with the family of spectral triples $\{(A_j, H_j, D_j)\}_{j \in J}$. Let

$$\mathcal{A} = \varinjlim_{j \in J} \mathcal{A}_j,$$

where the latter inductive limit above is taken in the sense of algebra, without completion. For every $\xi \in \bigcup_{j \in J} I_j(\mathrm{dom}(D_j))$ of the form $\xi = I_j \xi_j$, $\xi_j \in \mathrm{dom}(D_j)$, define

$$D\xi = I_j D_j \xi_j.$$

Then (A, H, D) is called the *inductive realization* of the inductive system $\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})\}_J$. It has corresponding subalgebra $\mathcal{A}$, which is norm dense in A. In [14], it is noted that the operator D is densely defined, symmetric, and essentially self-adjoint.

Not every inductive realization of spectral triples gives rise to a spectral triple, because it is necessary that the candidate for the Dirac operator D satisfies condition (ST1) for spectral triples, and that the candidate $\mathcal{A}$ for the smooth subalgebra of inductive limit algebra satisfies condition (ST2) for spectral triples. However, in two theorems, Floricel and Ghorbanpour give necessary and sufficient conditions for an inductive realization of spectral triples to give rise to a spectral triple, as follows.

Theorem 2.6 ([14, Theorem 3.1]). *Let $\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})\}_J$ be an inductive system of spectral triples, with respective smooth subalgebras $\{\mathcal{A}_j\}_{j \in J}$, and with inductive realization (A, H, D) and corresponding norm-dense subalgebra $\mathcal{A}$ as in Definition 2.5. The following conditions are equivalent:*

(i)   *D has compact resolvent;*

(ii)   *the sequence $\{I_j R_\lambda(D_j) I_j^*\}_{j \in \mathbb{N}}$ converges uniformly to $R_\lambda(D)$ for every $\lambda \in \mathbb{C} \backslash \mathbb{R}$;*

(iii) *the sequence $\{I_j R_\lambda(D_j) I_j^*\}_{j \in \mathbb{N}}$ converges uniformly to $R_\lambda(D)$ for some $\lambda \in \mathbb{C} \backslash \mathbb{R}$;*

(iv)   *the sequence $\{I_j f(D_j) I_j^*\}_{j \in \mathbb{N}}$ converges uniformly to $f(D)$ for every continuous function f on $\mathbb{R}$ vanishing at infinity.*

Notice in Theorem 2.6 that if the Dirac operators $\{D_j\}_{j \in \mathbb{N}}$ are diagonalizable, a necessary condition for any one of the equivalent conditions of this theorem to be satisfied is that for any $r > 0$, the closed disk around 0 of radius r in the complex plane should only contain finitely many eigenvalues of the $\{D_j\}$, where the eigenvalues are counted with multiplicity, or else it will be impossible for D to have compact resolvent. Here is the second main theorem and its fundamental corollary of Floricel and Ghorbanpour, which guarantees that condition (ST2) is satisfied.

Theorem 2.7. *Let $\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})\}_J$ be an inductive system of spectral triples, with respective smooth subalgebras $\{\mathcal{A}_j\}_{j \in J}$, and let $\mathcal{A} = \lim_{j \to \infty} \mathcal{A}_j$ be the algebraic*

inductive limit and D the corresponding candidate for Dirac operator given in Definition 2.5. *Then the operator* $[D, \pi(\phi_j(a))]$ *is bounded if and only if the family of operators* $\{[D_k, \pi_k(\phi_{j,k}(a))]\}_{k \geqslant j}$ *is uniformly bounded.*

2.2. Noncommutative solenoids as twisted group C^*-algebras and inductive limits

Throughout this paper, all discrete groups we consider will be countable. We have mentioned earlier that group C^*-algebras of countable discrete groups equipped with length functions have been studied as noncommutative spaces by Christ, Rieffel, and Connes [8, 10, 32]. Connes used certain length functions on finitely generated groups to define metrics on the state space of the group C^*-algebras. Building on the work of Connes, Rieffel investigated when this same construction yields a metric that agrees with the weak*-topology of the state space. Rieffel also investigated this construction for twisted group C^*-algebras. In order to continue these studies, we first review the construction of twisted group C^*-algebras, and then concentrate on the specific examples given by noncommutative solenoids.

Let σ be any multiplier of the countable discrete group Γ, i.e., a 2-cocycle on Γ taking values in $\mathbb{T}$. For any $f_1, f_2 \in \ell^1(\Gamma)$, the twisted convolution $*_\sigma$ is given by

$$f_1 *_\sigma f_2 : \gamma \in \Gamma \mapsto \sum_{\gamma_1 \in \Gamma} f_1(\gamma_1) f_2\big((\gamma_1)^{-1}\gamma\big) \sigma\big(\gamma_1, (\gamma_1)^{-1}\gamma\big),$$

and the adjoint operation by

$$f_1^* : \gamma \in \Gamma \mapsto \overline{f_1(\gamma^{-1})\, \sigma(\gamma, \gamma^{-1})}.$$

Definition 2.8. Given a discrete group Γ and a multiplier σ on Γ, we define the *left-σ regular representation* λ_σ of the group algebra $\ell^1(\Gamma, \sigma)$ on $\ell^2(\Gamma)$ by, for all $f \in \ell^1(\Gamma, \sigma)$, $g \in \ell^2(\Gamma)$, $\gamma_1, \gamma \in \Gamma$,

$$\big(\lambda_\sigma(f)(g)\big)(\gamma) = \sum_{\gamma_1 \in \Gamma} \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1) g(\gamma_1^{-1}\gamma).$$

By restricting λ_σ to the delta functions $\{\delta_\gamma : \gamma \in \Gamma\}$, we obtain the left-$\sigma$ regular representation for the group Γ. The C^*-completion of $\ell^1(\Gamma, \sigma)$ in the norm given by the representation λ_σ is called *the reduced twisted group C^*-algebra associated to σ* and denoted by $C_r^*(\Gamma, \sigma)$. The *full twisted group algebra associated to σ* is defined to be the C^*-enveloping algebra of the Banach algebra $\ell^1(\Gamma, \sigma)$ given the standard ℓ^1-norm, and is denoted by $C^*(\Gamma, \sigma)$. It follows from the definition that $C^*(\Gamma, \sigma)$ is the universal C^*-algebra generated by all projective representations of Γ with multiplier σ, that is, by all families of unitaries $(W_\gamma)_{\gamma \in \Gamma}$ such that for any $\gamma_1, \gamma_2 \in \Gamma$, $W_{\gamma_1} W_{\gamma_2} = \sigma(\gamma_1, \gamma_2) W_{\gamma_1 \gamma_2}$. In the general case, $C_r^*(\Gamma, \sigma)$ is a quotient of $C^*(\Gamma, \sigma)$, but if Γ is amenable, we have $C^*(\Gamma, \sigma) \cong C_r^*(\Gamma, \sigma)$ [38]. When $\sigma = 1$ (untwisted case), $C^*(\Gamma) := C^*(\Gamma, 1)$ and $C_r^*(\Gamma) := C_r^*(\Gamma, 1)$. We will also denote by

$$C_C(\Gamma, \sigma)$$

the (dense) $*$-subalgebra of $\ell^1(\Gamma, \sigma)$ of complex-valued functions of finite support on Γ.

We will now describe some examples of multipliers we will use throughout the paper. First, fix an arbitrary prime p, let $\mathbb{Z}[\frac{1}{p}]$ denote the subgroup of rational numbers of the form $\mathbb{Z}[\frac{1}{p}] = \{\frac{a}{p^k} \in \mathbb{Q} : a \in \mathbb{Z}, \ k \in \mathbb{N}\}$ (endowed with the discrete topology).

Example 2.9. As examples of multipliers, we have:

(1) Fix $\theta \in [0, 1]$, and define σ_θ on $\mathbb{Z}^2$ by

$$\sigma_\theta : \mathbb{Z}^2 \times \mathbb{Z}^2 \ni \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \mapsto \exp\left(\pi i \theta (z_1 y_2 - z_2 y_1) \right).$$

(2) Let, for p a fixed prime,

$$\Omega_p := \{ \theta = (\theta_n) \in \Pi_{n=0}^{\infty} [0, 1)_n : \forall n \in \mathbb{N}, \ p\theta_n = \theta_{n-1} \bmod \mathbb{Z} \}.$$

Fix $\theta \in \Omega_p$, and define a multiplier σ_θ on $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ by

$$\sigma_\theta : \left(\mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \right)^2 \ni \left(\left(\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_2}} \right), \left(\frac{q_3}{p^{k_3}}, \frac{q_4}{p^{k_4}} \right) \right) \mapsto \exp(2\pi i \theta_{k_1+k_4} q_1 q_4).$$

We recall the following; the second definition is adapted from [23, Definition 3.1].

Definition 2.10. For $\theta \in [0, 1]$, define $C^*(\mathbb{Z}^2, \sigma_\theta)$ to be the *noncommutative two-torus of angle θ*; this C^*-algebra is also called the rotation algebra A_θ. That is, $C^*(\mathbb{Z}^2, \sigma_\theta) \cong A_\theta$ is the universal C^*-algebra generated by unitaries U_θ and V_θ such that

$$U_\theta V_\theta = e^{2\pi i \theta} V_\theta U_\theta. \tag{2.1}$$

For $\theta \in \Omega_p$, the twisted group C^*-algebra $C^*(\Gamma, \sigma_\theta)$ is called the *noncommutative θ-solenoid* and is denoted by $\mathcal{A}_\theta^{\mathcal{S}}$.

We now recall from [23, Theorem 3.6] how noncommutative solenoids can be viewed as inductive limits of noncommutative rotation algebras.

Theorem 2.11 ([23]). *Let $p \in \mathbb{N}$ be prime, and fix $\theta \in \Omega_p$. For all $n \in \mathbb{N}$, let φ_n be the unique $*$-morphism from $A_{\theta_{2n}}$ into $A_{\theta_{2n+2}}$ extending*

$$\varphi_n \begin{cases} U_{\theta_{2n}} \mapsto U_{\theta_{2n+2}}^p, \\ V_{\theta_{2n}} \mapsto V_{\theta_{2n+2}}^p. \end{cases}$$

Then the inductive limit C^*-algebra obtained from the above embeddings,

$$\varinjlim_{n \in \mathbb{N}} A_{\theta_{2n}} : A_{\theta_0} \xrightarrow{\varphi_0} A_{\theta_2} \xrightarrow{\varphi_1} A_{\theta_4} \xrightarrow{\varphi_2} \cdots A_{\theta_{2n}} \xrightarrow{\varphi_n} A_{\theta_{2n+2}} \cdots,$$

is $*$-isomorphic to $\mathcal{A}_\theta^{\mathcal{S}}$ (of Definition 2.10).

We briefly give the outline of the proof of Theorem 2.11. By the natural embeddings of subgroups $\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$ into $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$, the noncommutative solenoid

$$\mathcal{A}_\theta^S = C^*(\Gamma, \sigma_\theta)$$

can be viewed as an inductive limit of the C^*-algebras $\{C^*(\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}, (\sigma_\theta)_n)\}_{n \in \mathbb{N}}$, where here by $(\sigma_\theta)_n$, we mean the restriction of the multiplier σ_θ from Γ to its subgroup $\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$ for a fixed $n \in \mathbb{N}$. Then it is not hard to see that for every $n \in \mathbb{N}$, we have

$$C^*\left(\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}, (\sigma_\theta)_n\right) \cong A_{\theta_{2n}} \cong C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}),$$

because both of these C^*-algebras have two generating unitaries satisfying the fundamental commutation relation given in equation (2.1). With these identifications, it easily follows that $\mathcal{A}_\theta^S$ satisfies the universal properties that make it $*$-isomorphic to $\varinjlim_{n \in \mathbb{N}} A_{\theta_{2n}}$. We will make these identifications frequently in upcoming sections.

In [3], noncommutative solenoids of periodic type, such as the following example, were given the structure of noncommutative manifolds by Aiello, Guido, and Isola. This was done by constructing a semifinite spectral triple on them.

Example 2.12 (Aiello–Guido–Isola: NC solenoid, periodic rational case). Let $p = 2$. Consider $\theta = (\theta_n) \in \Omega_2$ given by

$$\theta = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \cdots\right).$$

By Theorem 2.11, $\mathcal{A}_\theta^S$ is the inductive limit of the single rotation algebra $A_{\frac{2}{3}}$.

In [23, Theorem 2.9], it is shown that, for $\theta \in \Omega_p$, $\mathcal{A}_\theta^S$ is simple if and only if, given any distinct $j, k \in \mathbb{N}$, we have $\theta_j \neq \theta_k$. In particular, the construction given by Aiello, Guido, and Isola in [3] cannot be applied to simple noncommutative solenoids like the one below.

Example 2.13. Let a and q be relatively prime integers and p a prime. Consider

$$\theta_0 = a/q, \ \theta_1 = a/(pq), \ \theta_2 = a/(p^2 q), \ldots, \ \theta_{2n} = a/(p^{2^n}q), \ldots.$$

The resulting noncommutative solenoid $\mathcal{A}_\theta^S$ is $*$-isomorphic to the inductive limit of non-isomorphic rotation algebras corresponding to rational rotations.

In Section 3, we will define spectral triples on all noncommutative solenoids.

3. Spectral triples for noncommutative solenoids, and their quantum compact metric space structure

We now come to one of the main results of the paper: the construction of spectral triples on noncommutative solenoids. (As a particular case, these results also hold for commutative solenoids.) Their associated metric structures in turn will give a new way of viewing noncommutative solenoids as quantum compact metric spaces.

3.1. Length functions on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ with bounded doubling

In this section, we define a length function on $\mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$. This will be used later in our construction of spectral triples on noncommutative solenoids.

We recall for the reader's convenience the definition of *length function* on a general discrete group.

Definition 3.1 (Connes [10], Christ–Rieffel [8], Long–Wu [27]). A *length function* on a discrete group Γ is a function $\mathbb{L} : \Gamma \to [0, \infty)$ such that

(1) $\mathbb{L}(\gamma) = 0$ if and only if $\gamma = e$, where e is the identity of Γ,

(2) $\mathbb{L}(\gamma) = \mathbb{L}(\gamma^{-1})$ for all $\gamma \in \Gamma$,

(3) $\mathbb{L}(\gamma_1 \gamma_2) \leqslant \mathbb{L}(\gamma_1) + \mathbb{L}(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Given a pair consisting of a discrete group Γ and a length function $\mathbb{L}$ on it, Connes and Rieffel associate to $(\Gamma, \mathbb{L})$ a "Dirac operator" by defining $D_{\mathbb{L}} : C_C(\Gamma) \to C_C(\Gamma)$ given by, for all $f \in C_C(\Gamma), \gamma \in \Gamma$,

$$D_{\mathbb{L}}(f)(\gamma) = \mathbb{L}(\gamma) f(\gamma).$$

As mentioned in the introduction, for many examples of $(\Gamma, \mathbb{L})$ (mostly with finitely generated Γ), given a multiplier σ on Γ, and corresponding left-σ regular representation λ_σ of $C^*(\Gamma, \sigma)$ on $\ell^2(\Gamma)$, $(C^*(\Gamma, \sigma), \ell^2(\Gamma), D_{\mathbb{L}})$ has been shown to be a spectral triple with smooth subalgebra $C_C(\Gamma, \sigma)$.

Definition 3.2. A length function $\mathbb{L}$ on a countable discrete group Γ (as in Definition 3.1) is, in addition, said to have the properties

(1) it is *proper* if $B_{\mathbb{L}}(R) := \{\gamma \in \Gamma : \mathbb{L}(\gamma) \leqslant R\}$ is a finite subset of Γ for each $0 \leqslant R < \infty$;

(2) it has (or is of) *bounded doubling* if $\mathbb{L}$ is proper and there exists a constant $C_{\mathbb{L}} < \infty$ such that $|B_{\mathbb{L}}(2R)| \leqslant C_{\mathbb{L}} |B_{\mathbb{L}}(R)|$ for each $R, 1 \leqslant R < \infty$.

We introduce for future reference the concept of bounded **t**-dilation for length functions on discrete groups, first introduced by Long and Wu [27].

Definition 3.3 (cf. [27, Definition 5.1]). Let $\mathbb{L}$ be a length function on the discrete group Γ. Then Γ is said to have the *property of bounded* **t**-*dilation* with respect to $\mathbb{L}$ for a fixed $\mathbf{t} > 1$ if $\mathbb{L}$ is proper and there exists $K_{\mathbb{L}} < \infty$ such that $|B_{\mathbb{L}}(\mathbf{t}R)| \leqslant K_{\mathbb{L}} |B_{\mathbb{L}}(R)|$ for each $R \geqslant 1$.

A simple argument shows that Γ has the property of bounded **t**-dilation with respect to $\mathbb{L}$ for some $\mathbf{t} > 1$ if and only if $\mathbb{L}$ has bounded doubling.

Suppose that $\mathbb{L}$ is a length function on a discrete group Γ. Let $D_{\mathbb{L}}$ denote the (usually unbounded) operator on $\ell^2(\Gamma)$ of pointwise multiplication by $\mathbb{L}$, σ a multiplier on Γ, and the left-σ regular representation of $\ell^1(\Gamma, \sigma)$ on $\ell^2(\Gamma)$. In [25, Proposition 3.3], Long and Wu show that, given any $\gamma \in \Gamma$,

$$\left\| [D_{\mathbb{L}}, \lambda_\sigma(\delta_\gamma)] \right\|_{\mathcal{B}(\ell^2(\Gamma))} = \mathbb{L}(\gamma).$$

In particular, when $f \in C_C(\Gamma, \sigma)$, $[D_{\mathbb{L}}, \lambda_\sigma(f)]$ extends to a bounded operator on $\ell^2(\Gamma)$; we note for later the following key inequality:

$$\big\|[D_{\mathbb{L}}, \lambda_\sigma(f)]\big\| \leqslant \sum_{\gamma \in \Gamma} \mathbb{L}(\gamma)|f(\gamma)|, \quad f \in C_C(\Gamma, \sigma). \tag{3.1}$$

As $C_C(\Gamma, \sigma)$ is a dense subalgebra of $C^*(\Gamma, \sigma)$, any proper length function in this setting can be used to build a candidate Dirac operator for a spectral triple for $C^*(\Gamma, \sigma)$ that satisfies conditions (ST1) and (ST2), but to guarantee that this spectral triple is finitely summable, one needs further conditions on $\mathbb{L}$. In this section, we will define length functions that exhibit the additional specifications allowing the related Dirac operators to give finitely summable spectral triples, as well as satisfy the conditions of Definition 2.1, and thus construct Dirac operators that also equip noncommutative solenoids with a quantum compact metric space structure.

To obtain a length function on $\mathbb{Z}[\frac{1}{p}]$ that will produce the Dirac operator with the desired properties, we will begin by recalling the embedding $\Lambda : \mathbb{Z}[\frac{1}{p}] \to \mathbb{R} \times \mathbb{Q}_p$ (where $\mathbb{Q}_p$ are the p-adic rationals) as a lattice. Also recall that the topology on $\mathbb{Q}_p$ is induced by the p-adic norm on $\mathbb{Q}$ defined by, if we denote the gcd by $(\cdot, \cdot)$,

$$\|r\|_p = \begin{cases} p^{-n} & \text{if } r \neq 0 \text{ and } r = \frac{ap^n}{b}, \ a, n \in \mathbb{Z}, \ b \in \mathbb{Z} \setminus \{0\}, \ (a, p) = (b, p) = (a, b) = 1, \\ 0 & \text{if } r = 0. \end{cases}$$

In particular, $\mathbb{Q}_p$ can be viewed as the completion of $\mathbb{Q}$ for the p-adic metric, which is the metric induced by this norm. Since addition and multiplication are uniformly continuous for this metric, $\mathbb{Q}_p$ can also be given the structure of a field. For further details on p-adic arithmetic and analysis, see the book by Robert [36]. Thus $\mathbb{Z}[\frac{1}{p}]$, being a subset of $\mathbb{Q}$, embeds naturally into $\mathbb{R}$ and also into the p-adic field $\mathbb{Q}_p$. To avoid confusion, we let $\iota : \mathbb{Z}[\frac{1}{p}] \to \mathbb{Q}_p$ denote the embedding of $\mathbb{Z}[\frac{1}{p}]$ into $\mathbb{Q}_p$, and if $r \in \mathbb{Z}[\frac{1}{p}] \subset \mathbb{R}$, we view it as being an element of $\mathbb{R}$. We then define $\Lambda : \mathbb{Z}[\frac{1}{p}] \to \mathbb{R} \times \mathbb{Q}_p$ by

$$\Lambda(r) = \big(r, -\iota(r)\big) \in \mathbb{R} \times \mathbb{Q}_p, \quad r \in \mathbb{Z}\left[\frac{1}{p}\right].$$

Following a suggestion of Kaminker and Spielberg, Latrémolière and Packer showed in [24] that Λ embeds $\mathbb{Z}[\frac{1}{p}]$ as a lattice in $\mathbb{R} \times \mathbb{Q}_p$ with, in fact, the Pontryagin dual

$$\widehat{\mathbb{Z}\left[\frac{1}{p}\right]} = \mathcal{S}_p$$

as its quotient. We note that Λ was used to construct different classes of finitely generated projective modules for the C^*-algebras $\mathcal{A}_\theta^{\mathcal{S}}$ in [24]. In [12], Enstad used the embedding Λ to give a new example of a non-trivial bundle motivating a Balian–Low theorem in a novel context. In [28], Lu used Λ and some of its generalizations to obtain some new sufficient conditions for two noncommutative solenoids to be strongly Morita equivalent

to one another. This gives yet another example where the diagonal embedding of $\mathbb{Z}[\frac{1}{p}]$ as a lattice in $\mathbb{R} \times \mathbb{Q}_p$ is the source of novel tools.

As a means of obtaining a proper length function on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ that is of bounded doubling, we will first consider an initial length function defined on the abelian group $\mathbb{Z}[\frac{1}{p}]$.

Lemma 3.4. *Fix a prime p. Consider the function $\mathbb{L}_p : \mathbb{Z}[\frac{1}{p}] \to [0, \infty)$ given by*

$$\mathbb{L}_p(r) = |r| + \|r\|_p.$$

Then $\mathbb{L}_p$ is a length function on $\mathbb{Z}[\frac{1}{p}]$ that is both proper and unbounded.

Proof. The basic requirements for a length function on a group will first be checked. Suppose that $r, r' \in \mathbb{Z}[\frac{1}{p}]$. Since $0 \leqslant |r| \leqslant \mathbb{L}_p(r)$, $\mathbb{L}_p(r) = 0$ if and only if $|r| = 0$. As $|\cdot|$ is a norm on $\mathbb{Z}[\frac{1}{p}]$, $\mathbb{L}_p$ vanishes only on the identity of $\mathbb{Z}[\frac{1}{p}]$. It is easily seen that $\mathbb{L}_p(r) = \mathbb{L}_p(-r), \ \forall r \in \mathbb{Z}[\frac{1}{p}]$. Moreover,

$$\mathbb{L}_p(r + r') = |r + r'| + \|r + r'\|_p \leqslant |r| + |r'| + \|r\|_p + \|r'\|_p = \mathbb{L}_p(r) + \mathbb{L}_p(r'),$$

thereby completing the verification.

To show that $\mathbb{L}_p$ is also proper, fix a choice of $R > 0$. We will demonstrate that $B_{\mathbb{L}_p}(R)$ is a finite set by making some very crude estimates on $|B_{\mathbb{L}_p}(p^d)|$, where d is the least nonnegative integer such that $R < p^d$. To do so we will first prove that, for all integers $d \geqslant 1$:

$$\left\{ \frac{m}{p^{d-1}} : m \in \mathbb{Z}, \ -p^{2(d-1)} \leqslant m \leqslant p^{2(d-1)} \right\}$$

$$\subseteq B_{\mathbb{L}_p}(p^d) \subseteq \left\{ \frac{m}{p^{d-1}} : m \in \mathbb{Z}, \ -p^{2d} \leqslant m \leqslant p^{2d} \right\}, \tag{3.2}$$

which via a counting argument readily implies that

$$2p^{2(d-1)} + 1 \leqslant \left| B_{\mathbb{L}_p}(p^d) \right| \leqslant 2p^{2d} + 1. \tag{3.3}$$

To prove equation (3.2), let $x = \frac{m}{p^{2(d-1)}}$, with $|m| \leqslant p^{2(d-1)}$. Then

$$\mathbb{L}_p\left(\frac{m}{p^{d-1}} \right) \leqslant p^{d-1} + p^{d-1} = 2p^{d-1} \leqslant p^d,$$

which shows the set inclusion on the left-hand side.

To show the set inclusion on the right-hand side, we will actually use complements and show that if $x \in \mathbb{Z}[\frac{1}{p}]$ is not in $\{ \frac{m}{p^{d-1}} : m \in \mathbb{Z}, \ -p^{2d} \leqslant m \leqslant p^{2d} \}$, then it is not in $B_{\mathbb{L}_p}(p^d)$.

Fix any $x \notin \{ \frac{m}{p^{d-1}} : m \in \mathbb{Z}, \ -p^{2d} \leqslant m \leqslant p^{2d} \}$. We first assume that we have $x = \frac{m}{p^{d-1}}$ with $|m| > p^{2d}$ and $\gcd(m, p) = 1$. In this case, we have

$$\mathbb{L}_p\left(\frac{m}{p^{d-1}} \right) = \frac{|m|}{p^{d-1}} + p^{d-1} > \frac{p^{2d}}{p^{d-1}} + p^{d-1} > \frac{p^{2d}}{p^d} = p^d.$$

For the other case, suppose that $x = \frac{m}{p^j}$ with $\gcd(m, p) = 1$ and $j \geqslant d$, then

$$\mathbb{L}_p\left(\frac{m}{p^j}\right) > p^j \geqslant p^d,$$

so that $x \notin B_{\mathbb{L}_p}(p^d)$. This completes the proof of equation (3.2).

Now notice that, if d is the least nonnegative integer such that $R < p^d$,

$$B_{\mathbb{L}_p}(R) \subseteq B_{\mathbb{L}_p}(p^d).$$

An application of equation (3.3) ends the proof that $\mathbb{L}_p$ is a proper length function on $\mathbb{Z}[\frac{1}{p}]$.

Finally, because $\mathbb{L}_p(r) > |r|, \forall r \in \mathbb{Z}[\frac{1}{p}]$, and the absolute value function is unbounded on $\mathbb{Z}[\frac{1}{p}]$, we see that $\mathbb{L}_p$ is unbounded. $\blacksquare$

Remark 3.5. When a length function $\mathbb{L}$ on a discrete group is also of bounded doubling, Christ and Rieffel showed that $D_{\mathbb{L}}$ induces a quantum compact metric space structure on the group C^*-algebra [8]. These results were extended by Long and Wu to the setting of group C^*-algebras twisted by a 2-cocycle [27]. Furthermore, Christ and Rieffel demonstrated that such length functions also exhibit a property called *polynomial growth* [8]. Connes proved that for a finitely generated discrete group Γ with length function $\mathbb{L}$, if there exists a positive constant s such that $(\mathrm{Id} + [D_{\mathbb{L}}]^2)^{\frac{-s}{2}}$ is a trace-class operator, then $\mathbb{L}$ will have polynomial growth in which case $D_{\mathbb{L}}$ is also the Dirac operator for a spectral triple. Under these hypotheses, the associated spectral triple is said to be *finitely summable*. Conversely, if Γ is a finitely generated discrete group with length function $\mathbb{L}$ such that $\mathbb{L}$ has polynomial growth, then Connes proved that the associated spectral triple is finitely summable [10]. In our case, the groups we consider are not finitely generated. However, as a consequence of the following lemma, our Dirac operators arise from length functions on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ that can also be shown to be of bounded doubling, and therefore by [8, Proposition 1.2], the length functions that we construct will have polynomial growth. Moreover, we will show in Theorem 3.9 that bounded doubling of the length function, even in the non-finitely generated case, implies that the associated spectral triple is finitely summable.

Lemma 3.6. *For a fixed prime* p, $\mathbb{Z}[\frac{1}{p}]$ *has the property of bounded* p-*dilation with respect to* $\mathbb{L}_p$. *In particular,* $\mathbb{L}_p$ *is of bounded doubling with* $C_{\mathbb{L}_p} = 4p^8$.

Proof. As a consequence of the proof of Lemma 3.4, particularly equation (3.3), we have

$$\begin{aligned}
\left|B_{\mathbb{L}_p}(p^{d+1})\right| &\leqslant \left|B_{\mathbb{L}_p}(p^{d+1})\right| + \left|B_{\mathbb{L}_p}(p^d)\right| \leqslant (2p^{2(d+1)} + 1) + (2p^{2d} + 1) \\
&< 2p^{2(d+1)} + 2p^{2d} + 2p^4 = p^4(2p^{2(d-1)}) + p^2(2p^{2(d-1)}) + 2p^4 \\
&< 2p^4(2p^{2(d-1)} + 1) \leqslant 2p^4\left(\left|B_{\mathbb{L}_p}(p^d)\right|\right). \tag{3.4}
\end{aligned}$$

When d is zero, note that $\mathbb{L}_p(0) = 1$ implies that $|B_{\mathbb{L}_p}(1)| \geqslant 1$, which yields

$$\left|B_{\mathbb{L}_p}(p)\right| \leqslant 2p^2 + 1 < 2p^4 \leqslant 2p^4\left|B_{\mathbb{L}_p}(1)\right|.$$

In particular, given $R \geqslant 1$, there exists $d \in \mathbb{N}$ such that $p^d \leqslant R < p^{d+1}$, and so

$$B_{\mathbb{L}_p}(p^d) \subseteq B_{\mathbb{L}_p}(R) \subseteq B_{\mathbb{L}_p}(pR) \subseteq B_{\mathbb{L}_p}(p^{d+2}),$$

which yields, together with equation (3.4),

$$\left|B_{\mathbb{L}_p}(pR)\right| \leqslant \left|B_{\mathbb{L}_p}(p^{d+2})\right| \leqslant 4p^8\left|B_{\mathbb{L}_p}(p^d)\right| \leqslant 4p^8\left|B_{\mathbb{L}_p}(R)\right|. \tag{3.5}$$

Since the same reasoning holds for each $R \geqslant 1$, $\mathbb{Z}[\frac{1}{p}]$ has the property of bounded p-dilation with respect to $\mathbb{L}_p$. The conclusion that $\mathbb{L}_p$ is also of bounded doubling follows by our remark made after Definition 3.3. ∎

Latrémolière and Packer used the map Λ to build an embedding of $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ as a lattice in $[\mathbb{R} \times \mathbb{Q}_p] \times [\mathbb{R} \times \mathbb{Q}_p]$ [24]. As a consequence, $\mathbb{L}_p$ can be used to build a length function on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ that is of bounded doubling. More generally, the following proposition holds.

Proposition 3.7. *Let Γ_1 and Γ_2 be countable discrete groups and $\Gamma = \Gamma_1 \times \Gamma_2$. Given proper length functions $\mathbb{L}^{(i)}$ on Γ_i for $i = \{1, 2\}$, define $\mathbb{L}^\Sigma : \Gamma \to [0, \infty)$ by, for all $x_1 \in \Gamma_1$, $x_2 \in \Gamma_2$,*

$$\mathbb{L}^\Sigma((x_1, x_2)) := \mathbb{L}^{(1)}(x_1) + \mathbb{L}^{(2)}(x_2). \tag{3.6}$$

Then $\mathbb{L}^\Sigma$ is a proper length function on Γ. If $\mathbb{L}^{(i)}$ has the bounded doubling property with $C_{\mathbb{L}^{(i)}}$ for $i \in \{1, 2\}$, then $\mathbb{L}^\Sigma$ is of bounded doubling with $C_{\mathbb{L}^\Sigma} = C^4$, where $C = \max\{C_{\mathbb{L}^{(1)}}, C_{\mathbb{L}^{(2)}}\}$.

Proof. A length function on a Cartesian product of two groups can be built from a sum of length functions on each group. Since length functions take only non-negative values, $\mathbb{L}^\Sigma$ vanishes if and only if $\mathbb{L}^{(1)}$ and $\mathbb{L}^{(2)}$ both vanish. Because $\mathbb{L}^{(1)}$ and $\mathbb{L}^{(2)}$ each evaluate to zero only on e_{Γ_1} and e_{Γ_2}, respectively, $\mathbb{L}^\Sigma$ likewise takes null values only on e_Γ. Next suppose that $(x_1, x_2), (x_1', x_2') \in \Gamma$. The remaining length function properties of $\mathbb{L}^{(1)}$ and $\mathbb{L}^{(2)}$ imply that

$$\begin{aligned} \mathbb{L}^\Sigma((x_1, x_2)^{-1}) = \mathbb{L}^\Sigma((x_1^{-1}, x_2^{-1})) &= \mathbb{L}^{(1)}(x_1^{-1}) + \mathbb{L}^{(2)}(x_2^{-1}) \\ &= \mathbb{L}^{(1)}(x_1) + \mathbb{L}^{(2)}(x_2) = \mathbb{L}^\Sigma((x_1, x_2)), \end{aligned}$$

as well as

$$\begin{aligned} \mathbb{L}^\Sigma((x_1, x_2)(x_1', x_2')) = \mathbb{L}^\Sigma((x_1 x_1', x_2 x_2')) &= \mathbb{L}^{(1)}(x_1 x_1') + \mathbb{L}^{(2)}(x_2 x_2') \\ &\leqslant \mathbb{L}^{(1)}(x_1) + \mathbb{L}^{(1)}(x_1') + \mathbb{L}^{(2)}(x_2) + \mathbb{L}^{(2)}(x_2') \\ &= \mathbb{L}^\Sigma((x_1, x_2)) + \mathbb{L}^\Sigma((x_1', x_2')). \end{aligned}$$

Thus $\mathbb{L}^\Sigma$ is also a length function on Γ.

Fix a $0 < R < \infty$. If $\mathbb{L}^{\Sigma}((x_1, x_2)) \leqslant R$, then this same upper bound holds both for $\mathbb{L}^{(1)}(x_1)$ and for $\mathbb{L}^{(2)}(x_2)$. For each $i \in \{1, 2\}$, define $B_{\mathbb{L}^{(i)}}(R)$ and $B_{\mathbb{L}^{\Sigma}}(R)$ as in part (1) of Definition 3.2. Since $\mathbb{L}^{(1)}$ and $\mathbb{L}^{(2)}$ are also each proper length functions, $B_{\mathbb{L}^{(1)}}(R)$ and $B_{\mathbb{L}^{(2)}}(R)$ are finite sets, as is $B_{\mathbb{L}^{\Sigma}}(R) \subseteq B_{\mathbb{L}^{(1)}}(R) \times B_{\mathbb{L}^{(2)}}(R)$. Because the same reasoning holds for each $R < \infty$, $\mathbb{L}^{\Sigma}$ is a proper length function on Γ. Next, assume that $\mathbb{L}_i$ satisfies the bounded doubling property with constants $C_{\mathbb{L}_i}$ for $i \in \{1, 2\}$, and choose an $R \geqslant 1$. Repeated application of the bounded doubling condition yields

$$\left| B_{\mathbb{L}^{(1)}}(2R) \right| \cdot \left| B_{\mathbb{L}^{(2)}}(2R) \right| \leqslant C^4 \left| B_{\mathbb{L}^{(1)}}(R/2) \right| \cdot \left| B_{\mathbb{L}^{(2)}}(R/2) \right|.$$

Note that $B_{\mathbb{L}^{\Sigma}}(2R) \subseteq B_{\mathbb{L}^{(1)}}(2R) \times B_{\mathbb{L}^{(2)}}(2R)$ and $B_{\mathbb{L}^{(1)}}(R/2) \times B_{\mathbb{L}^{(2)}}(R/2) \subseteq B_{\mathbb{L}^{\Sigma}}(R)$, hence $|B_{\mathbb{L}^{\Sigma}}(2R)|$ is bounded above by $C^4 |B_{\mathbb{L}^{\Sigma}}(R)|$. As the choice of $R \geqslant 1$ was arbitrary, $\mathbb{L}^{\Sigma}$ is of bounded doubling with $C_{\mathbb{L}^{\Sigma}} = C^4$. ∎

Corollary 3.8. *Fix a prime p, and let $\mathbb{L}_p^{\Sigma} : \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}] \to [0, \infty)$ be given by*

$$\mathbb{L}_p^{\Sigma}(\gamma_1, \gamma_2) := \mathbb{L}_p(\gamma_1) + \mathbb{L}_p(\gamma_2).$$

Then $\mathbb{L}_p^{\Sigma}$ is a length function on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ that is of bounded doubling with $C_{\mathbb{L}_p^{\Sigma}} = (4p^8)^4$.

3.2. Spectral triples on noncommutative solenoids from length functions with bounded doubling

Throughout this section, fix a prime p and let $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$. Denote $\ell^2(\Gamma)$ by H. To build our Dirac operators, we recall from Definition 2.10 that the noncommutative solenoid $\mathcal{A}_{\theta}^{\mathcal{S}}$ is defined to be the twisted group C^*-algebra $C^*(\Gamma, \sigma_\theta)$. Moreover, $C^*(\Gamma, \sigma_\theta)$ has a faithful representation as bounded operators on the Hilbert space H via the left-σ regular representation λ_{σ_θ}. Let $\mathcal{D}_p$ be the unbounded operator $D_{\mathbb{L}_p^{\Sigma}}$ defined on a dense subspace of H. Since $\mathbb{L}_p^{\Sigma}$ is proper, $\mathcal{D}_p$ can also be shown to satisfy conditions (ST1) and (ST2) of Definition 2.1 for a Dirac operator. Moreover, we will show that the spectral triple coming from $\mathcal{D}_p$ is finitely summable. That is, we have the following theorem.

Theorem 3.9. *Fix a prime p. Let $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ and let $H = \ell^2(\Gamma)$. For every $\theta \in \Omega_p$, $(\mathcal{A}_{\theta}^{\mathcal{S}}, H, \mathcal{D}_p)$ with representation λ_σ is a spectral triple for the noncommutative solenoid $\mathcal{A}_{\theta}^{\mathcal{S}}$ with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$. Furthermore, this spectral triple is finitely summable.*

Proof. The following is a slight modification of the proof of Long and Wu's Proposition 3.3 in [27].

For every $\gamma \in \Gamma$, δ_γ is an eigenvector of $\mathcal{D}_p$ with eigenvalue $\mathbb{L}_p^{\Sigma}(\gamma)$. Hence each such δ_γ is also an eigenvector of $(\mathcal{D}_p - it\mathrm{Id}_H)^{-1}, t \in \mathbb{R} \setminus \{0\}$, with eigenvalue $(\mathbb{L}_p^{\Sigma}(\gamma) - it)^{-1}$. To show that $(\mathcal{D}_p - it\mathrm{Id}_H)^{-1}$ is compact for a fixed $t \in \mathbb{R} \setminus \{0\}$, choose a $\varepsilon > 0$. Then there exists $R > 0$ such that for $z \in \mathbb{R}, z > R$ implies that $|z - it|^{-1} < \varepsilon$. Given such an R, define $B_{\mathbb{L}_p^{\Sigma}}(R)$ as in part (1) of Definition 3.2. Since $\mathbb{L}_p^{\Sigma}$ is proper, $B_{\mathbb{L}_p^{\Sigma}}(R)$ is a finite set.

Let $\{\gamma_j\}_{1 \leqslant j \leqslant m}$ be an enumeration of the elements in this set. For each $1 \leqslant j \leqslant m$, let P_j signify the orthogonal projection taking H onto the one-dimensional subspace spanned by δ_{γ_j}. Then

$$\left\|(\mathcal{D}_p - it\mathrm{Id}_H)^{-1} - \Sigma_{j=1}^m \big(\mathbb{L}_p^\Sigma(\gamma_j) - it\big)^{-1} P_j\right\|_{\mathcal{B}(H)}$$
$$= \sup\left\{\big|\mathbb{L}_p^\Sigma(\gamma) - it\big|^{-1} : \gamma \notin B_{\mathbb{L}_p^\Sigma}(R)\right\} < \varepsilon.$$

As the choice of ε was arbitrary, $(\mathcal{D}_p - it\mathrm{Id}_H)^{-1}$ can be approximated in norm by finite rank operators. Hence $\mathcal{D}_p$ satisfies condition (ST1). Since $\mathbb{L}_p^\Sigma$ is an unbounded, proper length function as a consequence of Lemma 3.4 and Proposition 3.7, the associated Dirac operator $\mathcal{D}_p$ and subalgebra $C_C(\Gamma, \sigma_\theta)$ also satisfy condition (ST2), by [27, Proposition 3.3].

We finally prove that the spectral triple $(\mathcal{A}_\theta^s, H, \mathcal{D}_p)$ is finitely summable; i.e., we prove that there exists $s > 0$ such that for all $t > s$, $(\mathrm{Id}_H + [\mathcal{D}_p]^2)^{\frac{-t}{2}}$ is a trace class operator. For this, we use the bounded doubling condition proved for $\mathcal{D}_p$ in Corollary 3.8. Recall that there exists a constant $C_{\mathbb{L}_p^\Sigma} > 1$ such that for every $R \geqslant 1$, $|B_{\mathbb{L}_p^\Sigma}(2R)| \leqslant C_{\mathbb{L}_p^\Sigma}|B_{\mathbb{L}_p^\Sigma}(R)|$. Taking $R = 1$, by induction, we obtain for every $n \in \mathbb{N}$ that

$$\big|B_{\mathbb{L}_p^\Sigma}(2^n)\big| \leqslant C_{\mathbb{L}_p^\Sigma}\big|B_{\mathbb{L}_p^\Sigma}(2^{n-1})\big| \quad \text{and} \quad \big|B_{\mathbb{L}_p^\Sigma}(2^n)\big| \leqslant (C_{\mathbb{L}_p^\Sigma})^n\big|B_{\mathbb{L}_p^\Sigma}(1)\big|.$$

Fix $t > 0$. The eigenfunctions for $(\mathrm{Id}_H + [\mathcal{D}_p]^2)^{\frac{-t}{2}}$ are exactly the delta functions $\{\delta_\gamma : \gamma \in \Gamma\}$ with corresponding eigenvalues equal to $\{\frac{1}{(1+(\mathbb{L}_p^\Sigma(\gamma))^2)^{t/2}} : \gamma \in \Gamma\}$. Let δ_{γ_0} be a fixed eigenfunction for $\mathcal{D}_p$. To obtain an upper bound on the modulus of the eigenvalue corresponding to δ_{γ_0}, we need to determine where γ_0 sits in the partition

$$\Gamma = B_{\mathbb{L}_p^\Sigma}(1) \sqcup \bigsqcup_{n=1}^{\infty} \big[B_{\mathbb{L}_p^\Sigma}(2^n) \backslash B_{\mathbb{L}_p^\Sigma}(2^{n-1})\big].$$

To check that $(\mathrm{Id}_H + [\mathcal{D}_p]^2)^{\frac{-t}{2}}$ is trace class, we just need to check that the sum of moduli of the eigenvalues of this operator, counted with multiplicity, converge. We note that within the ball $B_{\mathbb{L}_p^\Sigma}(1)$, $(\mathrm{Id}_H + [\mathcal{D}_p]^2)^{\frac{-t}{2}}$ has $|B_{\mathbb{L}_p^\Sigma}(1)|$ eigenfunctions having eigenvalues with modulus less than or equal to $\frac{1}{(1+(0)^2)^{t/2}} = 1$. Then for $n \geqslant 1$, each annulus $[B_{\mathbb{L}_p^\Sigma}(2^n) \backslash B_{\mathbb{L}_p^\Sigma}(2^{n-1})]$ has cardinality less than or equal to

$$C_{\mathbb{L}_p^\Sigma}\big|B_{\mathbb{L}_p^\Sigma}(2^{n-1})\big| - \big|B_{\mathbb{L}_p^\Sigma}(2^{n-1})\big| \leqslant (C_{\mathbb{L}_p^\Sigma} - 1)(C_{\mathbb{L}_p^\Sigma})^{n-1}\big|B_{\mathbb{L}_p^\Sigma}(1)\big|.$$

For each eigenfunction δ_{γ_0} corresponding to γ_0 lying in this nth annulus, the corresponding eigenvalues have modulus less than $(\frac{1}{(1+2^{2(n-1)})})^{t/2}$. These estimates show that the trace of $(\mathrm{Id}_H + [\mathcal{D}_p]^2)^{\frac{-t}{2}}$ is bounded from above by

$$\big|B_{\mathbb{L}_p^\Sigma}(1)\big| \cdot 1 + \sum_{n=1}^{\infty}(C_{\mathbb{L}_p^\Sigma} - 1)(C_{\mathbb{L}_p^\Sigma})^{n-1}\big|B_{\mathbb{L}_p^\Sigma}(1)\big| \cdot \frac{1}{(1 + 2^{2(n-1)})^{t/2}}$$
$$\leqslant \max\big\{1, (C_{\mathbb{L}_p^\Sigma} - 1)\big\} \cdot \big|B_{\mathbb{L}_p^\Sigma}(1)\big| \cdot \left[1 + \sum_{n=1}^{\infty} \frac{(C_{\mathbb{L}_p^\Sigma})^{n-1}}{(1 + 2^{2(n-1)})^{t/2}}\right]$$

$$\leqslant \max\left\{1, (C_{\mathbb{L}_p^\Sigma} - 1)\right\} \cdot \left|B_{\mathbb{L}_p^\Sigma}(1)\right| \cdot \left[1 + \sum_{n=1}^{\infty} \frac{(C_{\mathbb{L}_p^\Sigma})^{n-1}}{(2^t)^{n-1}}\right]$$

$$= \max\left\{1, (C_{\mathbb{L}_p^\Sigma} - 1)\right\} \cdot \left|B_{\mathbb{L}_p^\Sigma}(1)\right| \cdot \left[1 + \sum_{n=1}^{\infty} \left[\frac{C_{\mathbb{L}_p^\Sigma}}{2^t}\right]^{n-1}\right],$$

and this last series converges as long as $2^t > C_{\mathbb{L}_p^\Sigma}$; that is, it converges for all $t > \log_2(C_{\mathbb{L}_p^\Sigma})$. Therefore, the spectral triple $(\mathcal{A}_\theta^{\mathcal{S}}, H, \mathcal{D}_p)$ is finitely summable. ∎

3.3. Noncommutative solenoids as quantum compact metric spaces

In the context of spectral triples and other more general settings, Rieffel developed a theory (later expanded by Latrémolière) that can be termed *noncommutative metric geometry*; see Rieffel [33, 34]. The basis for this perspective is built on a noncommutative analogue of a compact metric space. We review some of this work below. Whereas Rieffel gave the definition below for a seminorm defined on an order-unit space taking finite values, Latrémolière restricted to the unital C^*-algebra setting, where the seminorm in question takes on finite values on a dense subalgebra of a unital C^*-algebra. Because of the extra properties required, he decided to switch the order of "compact" and "quantum". We will give below an adaptation of the slightly more specialized definition given by Latrémolière; see [20]. We refer to [20, Remark 2.7] for a precise comparison of Rieffel and Latrémolière's definitions.

Definition 3.10. A *quantum compact metric space* (A, L) is an ordered pair, where A is a unital C^*-algebra and L is a seminorm defined on a dense $*$-subalgebra $\mathrm{dom}(L)$ of the self-adjoint elements $\mathrm{sa}(A)$ such that

  (1) $\{a \in \mathrm{sa}(A) : L(a) = 0\} = \mathbb{R}1_A$,

  (2) the Monge–Kantorovich metric mk_L, defined on the state space $\mathcal{S}(A)$ of A by setting for all $\varphi, \psi \in \mathcal{S}(A)$,

$$\mathrm{mk}_L(\varphi, \psi) = \sup\left\{\left|\varphi(a) - \psi(a)\right| : a \in \mathrm{dom}(L), \ L(a) \leqslant 1\right\},$$

metrizes the weak* topology restricted to the state space $\mathcal{S}(A)$ of A.

Moreover, seminorms L on A that satisfy conditions (1) and (2) are called *Lip-norms*.

Example 3.11 (Rieffel [34]). Let (X, d) be a compact metric space. Set L_d equal to the Lipschitz seminorm on $C(X)$ associated to d; that is, for every f in $C(X)$,

$$L_d(f) = \sup\left\{\frac{\left|f(x) - f(y)\right|}{d(x, y)} : x, y \in X, \ x \neq y\right\}.$$

Then $(C(X), L_d)$ is a quantum compact metric space. It is well known that the metric d can be recovered from the restriction of mk_{L_d} to the pure states.

Remark 3.12. In [31, Proposition 1.3], it is shown that for a unital C^*-algebra A with a trace τ, a Lipschitz seminorm L (that is, a seminorm that has null-space which is exactly the scalar multiples of the identity) for A will be a Lip-norm on A if and only if $\{a \in \mathrm{dom}(L) : L(a) \leqslant 1, \ \tau(a) = 0\}$ is totally bounded for the topology induced by $\| \cdot \|_A$ on $\mathrm{dom}(L)$.

There is an additional property that can be assigned to Lipschitz seminorms, which is the *Leibniz property*: a Lipschitz seminorm L on A is said to satisfy the *Leibniz property* if, for all $a, b \in \mathrm{dom}(L)$,

$$L(ab) \leqslant \|a\|_A \cdot L(b) + \|b\|_A \cdot L(a).$$

In his studies and generalizations of compact quantum metric spaces, Latrémolière in [20] introduces an additional Leibniz condition for Lip-norms of Jordan–Lie type; that is, for all $a, b \in \mathrm{dom}(L)$,

$$\frac{ab + ba}{2}, \frac{ab - ba}{2i} \in \mathrm{dom}(L),$$

and L satisfies the *Leibniz inequality of Jordan–Lie type*:

$$\max\left\{ L\left(\frac{ab + ba}{2} \right), L\left(\frac{ab - ba}{2i} \right) \right\} \leqslant L(a) \|b\|_A + \|a\|_A L(b).$$

Therefore, we make the following definition.

Definition 3.13 ([20, Definition 2.19], [22, Definition 1.4]). Let A be a unital C^*-algebra, and let L be a Lip-norm on a dense subspace $\mathrm{dom}(L)$ of A. We say that the ordered pair (A, L) is a *Leibniz quantum compact metric space* if in addition to conditions (1) and (2) of Definition 3.10, the following conditions are satisfied:

(3) the seminorm L satisfies the Leibniz inequality of Jordan–Lie type;

(4) the seminorm L is lower semicontinuous with respect to the norm $\| \cdot \|_A$ topology restricted to $\mathrm{dom}(A)$.

In [20, Proposition 2.17], Latrémolière proves that if (A, L) is a quantum compact metric space in the sense of Definition 3.10, with L satisfying the Leibniz property, then (A, L) will satisfy the Leibniz inequality of Jordan–Lie type. By Long and Wu's Proposition 6.3 in [27], it is the case that the Lip-norms we discuss coming from length functions will satisfy the Leibniz property, and therefore they satisfy the Leibniz inequality of Jordan–Lie type. We will use this fact in proving Theorem 3.14.

Noncommutative metric geometry allows the theory of metric geometry to be applied to noncommutative C^*-algebras via their state spaces. As with classical compact metric spaces, the set of quantum compact metric spaces can be equipped with a metric. In analogy with its classical counterpart, Latrémolière's Gromov–Hausdorff propinquity on Leibniz quantum compact metric spaces, which builds on the earlier work of Rieffel, has many of the same properties as in the compact metric space setting [20].

Inductive limit C^*-algebras admit natural candidates for quantum metric approximations. Aguilar and Latrémolière have identified some conditions under which inductive limit convergence can also be described by metric convergence [1,2], and Landry, Lapidus, and Latrémolière in [19] studied metric approximation of spectral triples on fractals of Sierpinski gasket type. Latrémolière and Packer also considered the question of metric convergence in the setting of noncommutative solenoids [22]. When a compact group acts ergodically on a unital C^*-algebra, any length function on the group induces a quantum compact metric space structure on the C^*-algebra [33]. Working in the more general context of inductive limits of C^*-algebras on which projective limits of compact metrizable groups act ergodically, Latrémolière and Packer showed that noncommutative solenoids are limits, for the Gromov–Hausdorff propinquity, of rotation algebras. Their result relied on the definition of certain length functions to construct Lip-norms on an inductive sequence of rotation algebras and their inductive limit. These particular Lip-norms did not come from Dirac operators, as in the earlier work of Connes [10]. Since we want to study noncommutative solenoids as both quantum compact metric spaces as well as noncommutative Riemannian manifolds, we now describe how our results of Sections 3.1 and 3.2 give noncommutative solenoids the structure of quantum compact metric spaces.

In [10], Connes studied spectral triples on reduced group C^*-algebras with Dirac operators built in the same way as $\mathcal{D}_p$ from length functions on discrete groups. Given a unital faithful representation $\pi : A \to \mathcal{B}(H)$, let L_D be the seminorm on the associated smooth subalgebra $\mathcal{A}$ given by

$$L_D(a) = \big\|[D, \pi(a)]\big\|_{\mathcal{B}(H)}. \tag{3.7}$$

As in the unital commutative case, Connes showed that a seminorm defined as in equation (3.7) from such a Dirac operator gives rise to a metric on the state space of the reduced group C^*-algebra [8, 10]. This metric is precisely the Monge–Kantorovich metric given in part (2) of Definition 3.10. When D is the Dirac operator constructed by Connes from a proper length function $\mathbb{L}$ on a discrete group Γ, Christ and Rieffel showed that mk_{L_D} also metrizes the weak*-topology on $\mathcal{S}(C_r^*(\Gamma))$ if $\mathbb{L}$ is in addition assumed to be of bounded doubling. In a more general setting, Rieffel, as well as Ozawa in collaboration with Rieffel, identified alternate characterizations of Lip-norms [31, 33]. In [27], Long and Wu used one of these equivalent sets of conditions to extend Christ and Rieffel's work to discrete groups equipped with a 2-cocycle. Since noncommutative solenoids are defined to be twisted discrete group C^*-algebras, our Dirac operators can also be shown to give rise to Leibniz Lip-norms on noncommutative solenoids.

Theorem 3.14. *Fix a prime p. For every $\theta \in \Omega_p$, let $(\mathcal{A}_\theta^{\mathcal{S}}, H, \mathcal{D}_p)$ be the spectral triple from Theorem* 3.9. *Then $(\mathcal{A}_\theta^{\mathcal{S}}, L_{\mathcal{D}_p})$ is a Leibniz quantum compact metric space in the sense of Definition* 3.13.

Proof. Because of Corollary 3.8, the conclusion follows immediately from [27, Propositions 6.4 and 6.9], which use the property of bounded doubling for the length function

to derive that the seminorm $L_{\mathcal{D}_p}$ has the Leibniz property and is lower semicontinuous, respectively. ∎

Remark 3.15. Once again we note the fact that, equipped with the weak*-topology, the state space of a unital C^*-algebra is compact and second countable, so that any two metrics which induce this same topology on the compact space must be uniformly equivalent. The Leibniz Lip-norm on the state space of the noncommutative solenoid obtained by Latrémolière and Packer in [22] induces the weak*-topology, as does the Leibniz Lip-norm induced by the seminorm coming from the spectral triple, by virtue of the bounded doubling condition being satisfied by the length function $\mathbb{L}_p^\Sigma$. Therefore, the metrics on the state space given by these two seminorms must be uniformly equivalent.

3.4. Restriction of spectral triples to noncommutative tori

Recall from Theorem 2.11 that noncommutative solenoids can be identified with inductive limits of rotation algebras. We will next build spectral triples for rotation algebras via suitably defined restrictions of the spectral triples on noncommutative solenoids constructed in Theorem 3.9. For each $n \in \mathbb{N}$, denote the subgroup $\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$ of $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}] = \Gamma$ by Γ_n. Recall from Section 2 that for each prime p, the group Γ can be realized as an inductive limit of the groups $\{\Gamma_n\}_{n\in\mathbb{N}}$, where the group monomorphisms $\alpha_n : \Gamma_n \to \Gamma_{n+1}$ are given by inclusion maps. We will show that the length function on Γ_n that is the restriction of that given for Γ is still of bounded doubling.

Lemma 3.16. *For fixed prime p and $n \in \mathbb{N}$, let $\mathbb{L}_{p,n}$ denote the restriction of $\mathbb{L}_p$ to $\frac{1}{p^n}\mathbb{Z}$. Then $\frac{1}{p^n}\mathbb{Z}$ has the property of bounded p-dilation with respect to $\mathbb{L}_{p,n}$. In particular, $\mathbb{L}_{p,n}$ is of bounded doubling with $C_{\mathbb{L}_{p,n}} = 4p^8$.*

Proof. In order to qualify as a length function on $\frac{1}{p^n}\mathbb{Z}$, $\mathbb{L}_{p,n}$ must be well defined on the identity of the group, the inverses of its elements, and the closure of those elements under the group operation. Since $\frac{1}{p^n}\mathbb{Z} \leqslant \mathbb{Z}[\frac{1}{p}]$, $\mathbb{L}_p$ restricts to a length function in this setting. Define $B_{\mathbb{L}_{p,n}}(R)$ as in part (1) of Definition 3.2. Since $B_{\mathbb{L}_{p,n}}(R) \subseteq B_{\mathbb{L}_p}(R)$ for each $0 < R < \infty$, $\mathbb{L}_{p,n}$ is also a proper length function on $\frac{1}{p^n}\mathbb{Z}$.

The argument that $\mathbb{L}_{p,n}$ has the property of bounded p-dilation follows the same strategy as in Lemma 3.6. Fix $R \geqslant 1$. Choose $d \in \mathbb{N}$ so that equation (3.5) holds for balls defined with $\mathbb{L}_{p,n}$. Consider first the case when $1 \leqslant d \leqslant n$. Since $B_{\mathbb{L}_{p,n}}(pR) \subseteq B_{\mathbb{L}_p}(pR)$, the inequalities described in equation (3.5) can be applied to obtain the desired bounded doubling constant if $B_{\mathbb{L}_p}(R)$ can be shown to be a subset of $B_{\mathbb{L}_{p,n}}(R)$. Note that every nonzero element in $B_{\mathbb{L}_p}(R)$ can be written as $\frac{a}{p^k}$, where a is an integer coprime with p and $k \in \mathbb{N}$. Since $\mathbb{L}_p(\frac{a}{p^k}) > p^k$, $\frac{a}{p^k} \in B_{\mathbb{L}_p}(R)$ implies that $k < d$. Because $d \leqslant n$, $\mathbb{L}_p(\frac{a}{p^k})$ coincides with $\mathbb{L}_{p,n}(\frac{a}{p^k})$. As a result,

$$\left|B_{\mathbb{L}_{p,n}}(pR)\right| \leqslant \left|B_{\mathbb{L}_p}(pR)\right| \leqslant 4p^8\left|B_{\mathbb{L}_p}(R)\right| \leqslant 4p^8\left|B_{\mathbb{L}_{p,n}}(R)\right|.$$

For the case when $d > n$, note that $a \in \mathbb{Z}$ and $|a| \geqslant p^{d+n}$ implies that

$$\mathbb{L}_{p,n}\left(\frac{a}{p^n}\right) > \left|\frac{a}{p^n}\right| \geqslant p^d.$$

Moreover, $a \in \mathbb{Z}$ and $|a| \leqslant p^{d+n-1}$, as well as $n < d$, gives

$$\mathbb{L}_{p,n}\left(\frac{a}{p^n}\right) = \left|\frac{a}{p^n}\right| + \left\|\frac{a}{p^n}\right\|_p \leqslant p^{d-1} + p^n \leqslant 2p^{d-1} \leqslant p^d.$$

Therefore,

$$\left\{\frac{a}{p^n} : a \in \mathbb{Z}, \; -p^{d+n-1} \leqslant a \leqslant p^{d+n-1}\right\}$$

$$\subseteq B_{\mathbb{L}_{p,n}}(p^d) \subseteq \left\{\frac{a}{p^n} : a \in \mathbb{Z}, \; -p^{d+n} \leqslant a \leqslant p^{d+n}\right\},$$

which yields, in analogy with equation (3.3),

$$2p^{d+n-1} + 1 \leqslant \left|B_{\mathbb{L}_{p,n}}(p^d)\right| \leqslant 2p^{d+n} + 1,$$

Consequently, an argument similar to that given in Lemma 3.6 yields

$$\left|B_{\mathbb{L}_{p,n}}(p^{d+1})\right| < 2p^4\left(\left|B_{\mathbb{L}_{p,n}}(p^d)\right|\right).$$

In particular, we see that the inequalities described in equation (3.5) still hold with $\mathbb{L}_p$ replaced by $\mathbb{L}_{p,n}$. Since the same reasoning holds for each $R \geqslant 1$, $\frac{1}{p^n}\mathbb{Z}$ has the property of bounded p-dilation with respect to $\mathbb{L}_{p,n}$. As in the setting of $\mathbb{Z}[\frac{1}{p}]$, the conclusion that $\mathbb{L}_p$ is also of bounded doubling when restricted to $\frac{1}{p^n}\mathbb{Z}$ follows from the fact that $|B_{\mathbb{L}_{p,n}}(2R)| \leqslant |B_{\mathbb{L}_{p,n}}(pR)|$ for $R \geqslant 1$. ∎

Proposition 3.7, together with Lemma 3.16, now yields the following corollary.

Corollary 3.17. *Fix a prime p and $n \in \mathbb{N}$. Let $\mathbb{L}_{p,n}^\Sigma : \Gamma_n = \frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z} \to [0, \infty)$ be given by*

$$\mathbb{L}_{p,n}^\Sigma(\gamma_1, \gamma_2) = \mathbb{L}_{p,n}(\gamma_1) + \mathbb{L}_{p,n}(\gamma_2).$$

The function $\mathbb{L}_{p,n}^\Sigma$ is a proper length function on Γ_n that is of bounded doubling with $C_{\mathbb{L}_{p,n}^\Sigma} = (4p^8)^4$.

The proof of Corollary 3.17 can also be adapted to a more general setting.

Corollary 3.18. *Let Γ_1 and Γ_2 each be countable discrete groups and $\Gamma = \Gamma_1 \times \Gamma_2$. Define $\mathbb{L}^\Sigma$ as in equation (3.6). For each $i \in \{1, 2\}$, suppose that G_i is a subgroup of Γ_i. Denote the restriction of $\mathbb{L}^{(i)}$ to G_i by $\mathbb{L}_{G_i}^{(i)}$. If each $\mathbb{L}_{G_i}^{(i)}$ has the bounded doubling property with $C_{\mathbb{L}_{G_i}^{(i)}}$, then the restriction of $\mathbb{L}^\Sigma$ to $G_1 \times G_2$ is of bounded doubling with constant given by $[\max\{C_{\mathbb{L}_{G_1}^{(1)}}, C_{\mathbb{L}_{G_2}^{(2)}}\}]^4$.*

As a consequence of Corollary 3.17, our Dirac operators for noncommutative solenoids can be used to define Dirac operators for rotation algebras. Given $C^*(\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}], \sigma_\theta)$, recall from Section 2.2 that $(\sigma_\theta)_n$ is the restriction of the multiplier σ_θ to the subgroup $\Gamma_n = \frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$. For the representation of $C^*(\Gamma_n, (\sigma_\theta)_n)$ as bounded operators on the

Hilbert space $H_n := \ell^2(\Gamma_n)$, set $\lambda_{(\sigma_\theta)_n}$ equal to the left $(\sigma_\theta)_n$-regular representation of $C^*(\Gamma_n, (\sigma_\theta)_n)$ on H_n. For every $n \in \mathbb{N}$, there exists a $*$-isomorphism υ_n which takes $C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}})$ to $C^*(\Gamma_n, (\sigma_\theta)_n)$. Then $\pi_{\theta_{2n}} : C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}) \to \mathcal{B}(H_n)$ defined by

$$\pi_{\theta_{2n}} = \lambda_{(\sigma_\theta)_n} \circ \upsilon_n$$

gives a representation of $C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}})$ that is unital and faithful. Also using υ_n to denote the abstract isomorphism between the groups $\mathbb{Z}^2$ and Γ_n, we let $D_{p,n}$ be the operator $D_{\mathbb{L}_{p,n}^\Sigma \circ \upsilon_n}$. We note that for every $n \in \mathbb{N}$, the length function $\mathbb{L}_{p,n}^\Sigma \circ \upsilon_n$ on $\mathbb{Z}^2$ is of bounded doubling, since $\mathbb{L}_{p,n}^\Sigma$ is of bounded doubling on Γ_n by Corollary 3.17. Since unbounded length functions on groups that are of bounded doubling are also proper, the same kinds of arguments as used in the proof of Theorem 3.9 can be applied to demonstrate that $D_{p,n}$ is a Dirac operator.

Proposition 3.19. *Fix a prime p and $n \in \mathbb{N}$. For every $\theta \in \Omega_p$ and $n \in \mathbb{N}$, $(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}),$ $\ell^2(\Gamma_n), D_{p,n})$ with representation $\pi_{\theta_{2n}}$ is a spectral triple for the noncommutative torus $C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}})$, with associated smooth subalgebra $C_C(\mathbb{Z}^2, \sigma_{\theta_{2n}})$.*

We next consider the topology induced by $\mathrm{mk}_{L_{D_{p,n}}}$ on $\mathcal{S}(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}))$, where $L_{D_{p,n}}$ is the seminorm on $C_C(\mathbb{Z}^2, \sigma_{\theta_{2n}})$ induced by the Dirac operator $D_{p,n}$. When equipped with the weak*-topology, the state space of a unital C^*-algebra is compact and second countable. The same argument given in the proof of Theorem 3.14 and Corollary 3.17 will still hold here, and the discussion of Remark 3.15 applies to show that length functions on $\mathbb{Z}^2$ that are of bounded doubling will give rise to uniformly equivalent metrics on $\mathcal{S}(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}))$, since they both determine the same compact weak-$*$ topology on $\mathcal{S}(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}))$. Therefore, we obtain the following proposition.

Proposition 3.20. *Fix a prime p and $n \in \mathbb{N}$. For every $\theta \in \Omega_p$, $(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}), L_{D_{p,n}})$ is a Leibniz quantum compact metric space.*

3.5. Spectral triples on noncommutative solenoids produced as limits of spectral triples on noncommutative tori

We now use the set-up of the previous section to build a countable inductive system of spectral triples for rotation algebras, in the sense of Floricel and Ghorbanpour (see Definition 2.4), which will satisfy the hypotheses of Theorems 2.6 and 2.7. Fix a prime p. Given $\theta \in \Omega_p$, recall the construction of $\{C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}})\}_{n \in \mathbb{N}}$, where the multiplier σ_θ on $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ is chosen to construct the noncommutative solenoid $\mathcal{A}_\theta^{\mathcal{S}}$.

Again, throughout this subsection, denote $\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ by Γ. For each $n \in \mathbb{N}$, denote the subgroup $\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$ of Γ by Γ_n. Since Γ_n embeds into Γ_{n+1} via inclusion maps and $(\sigma_\theta)_n$ is the restriction of σ_θ from Γ to Γ_n, there exist, for every $j, k \in \mathbb{N}$, $j \leqslant k$, unital $*$-monomorphisms

$$\beta_{j,k} : C^*(\Gamma_j, (\sigma_\theta)_j) \to C^*(\Gamma_k, (\sigma_\theta)_k).$$

Let $\phi_{j,k} : C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}) \to C^*(\mathbb{Z}^2, \sigma_{\theta_{2k}})$ be the map given by the $*$-monomorphism

$$\phi_{j,k} = \upsilon_k^{-1} \circ \beta_{j,k} \circ \upsilon_j$$

and let
$$I_{j,k} : H_j \to H_k$$

be the isometry given by the inclusion map, where, as in previous sections, υ_j is the $*$-isomorphism between $C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$ and $C^*(\Gamma_j, (\sigma_\theta)_j)$, and we use the notation $H_j = \ell^2(\Gamma_j)$, $j \in \mathbb{N}$. Since $C_C(\mathbb{Z}^2, \sigma_{2j})$ is equal to $\upsilon_j^{-1}(C_C(\Gamma_j, (\sigma_\theta)_j))$ and $\beta_{j,k}(C_C(\Gamma_j, (\sigma_\theta)_j)) \subseteq C_C(\Gamma_k, (\sigma_\theta)_k)$,
$$\phi_{j,k}\big(C_C(\mathbb{Z}^2, \sigma_{\theta_{2j}})\big) \subseteq C_C(\mathbb{Z}^2, \sigma_{\theta_{2k}}).$$

Moreover, for every $f \in C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$,
$$I_{j,k} \circ \pi_{\theta_{2j}}(f) = \pi_{\theta_{2k}}(\phi_{j,k}(f)) \circ I_{j,k}.$$

By definition, $\mathbb{L}_{p,n}^\Sigma$ agrees with $\mathbb{L}_p^\Sigma$ restricted to Γ_n for each $n \in \mathbb{N}$, so that our choice of $I_{j,k}$ yields $I_{j,k}(\mathrm{dom}(D_{\mathbb{L}_{p,j}^\Sigma})) \subseteq \mathrm{dom}(D_{\mathbb{L}_{p,k}^\Sigma})$. In particular,
$$I_{j,k} D_{p,j} = D_{p,k} I_{j,k}.$$

Note that by construction, $\phi_{k,l}\phi_{j,k} = \phi_{j,l}$ and $I_{k,l}I_{j,k} = I_{j,l}$ for all $j, k, l \in \mathbb{N}, j \leqslant k \leqslant l$. Therefore,
$$\big\{\big(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), H_j\big), (\phi_{j,k}, I_{j,k})\big\}_{j \in \mathbb{N}}$$
$$\text{with associated smooth subalgebras } \big\{C_C(\mathbb{Z}^2, \sigma_{\theta_{2j}})\big\}_{j \in \mathbb{N}}$$

is a countable inductive system of spectral triples (see Definitions 2.3, 2.4, and 2.5).

We now are in a position to deduce that the spectral triples for noncommutative solenoids constructed in Theorem 3.9 can each be obtained as the inductive realization of a countable inductive system of finite-dimensional spectral triples. For the inductive system of Hilbert spaces, take $\{(H_j = \ell^2(\Gamma_j), I_{j,k})\}_{j \in \mathbb{N}}$. Then
$$\ell^2(\Gamma) = \varinjlim_{j \in J} H_j.$$

By the way that the morphisms $\phi_{j,k}$ are defined, the inductive limit of the inductive system of C^*-algebras
$$\big\{\big(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \phi_{j,k}\big)\big\}_{j \in \mathbb{N}}$$

is $*$-isomorphic to that of $\{(C^*(\Gamma_j, (\sigma_\theta)_j), \beta_{j,k})\}_{j \in \mathbb{N}}$. The inductive limit of the second inductive system is $C^*(\Gamma, \sigma_\theta) = \mathcal{A}_\theta^S$, with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$. Consequently,
$$\varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$$

is also $*$-isomorphic to $\mathcal{A}_\theta^S$ with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$. We denote this $*$-isomorphism by
$$\rho : \varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}) \to \mathcal{A}_\theta^S.$$

We calculate that at the level of representations on $\varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$,

$$\pi_\theta = \lambda_{\sigma_\theta} \circ \rho = \varinjlim_{j \in J} \lambda_{\sigma_{\theta_{2j}}} \circ \upsilon_j = \varinjlim_{j \in J} \pi_{\theta_{2j}}.$$

Thus we obtain a representation of

$$\varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}) \cong \mathcal{A}_\theta^{\mathcal{S}}$$

that is an inductive limit of the family of representations $\{\pi_{\theta_{2j}}\}_{j \in \mathbb{N}}$ associated with the family of spectral triples $\{(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), H_j, D_{p,j})\}_{j \in \mathbb{N}}$. By its very definition, the $*$-isomorphism $\rho : \varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}) \to \mathcal{A}_\theta^{\mathcal{S}}$ intertwines the C^*-algebra representation pairs $(\varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \pi_\theta)$ and $(C^*(\Gamma, \sigma_\theta), \lambda_{\sigma_\theta})$. For each fixed $j \in \mathbb{N}$, set $I_j : H_j \to H = \ell^2(\Gamma)$ equal to

$$\varinjlim_{k \in J} I_{j,k}$$

and $\phi_j : C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}) \to \mathcal{A}_\theta^{\mathcal{S}}$ equal to

$$\varinjlim_{k \in J} \beta_{j,k} \circ \upsilon_j.$$

The combination of these choices yields π_θ as the unique representation of

$$\varinjlim_{j \in \mathbb{N}} C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$$

on $\ell^2(\Gamma)$ such that for every $f \in C^*(\mathbb{Z}^2, \theta_{2j})$,

$$\pi_\theta(\phi_j(f)) \circ I_j = I_j \circ \pi_{\theta_{2j}}(f).$$

To see that

$$\mathcal{D}_p = \varinjlim_{j \in J} D_{p,j},$$

observe that $\mathcal{D}_p$ agrees with $D_{p,j}$ on $\ell^2(\Gamma_j)$. Hence $(\mathcal{A}_\theta^{\mathcal{S}}, \ell^2(\Gamma), \mathcal{D}_p)$ is the inductive realization of our inductive system of spectral triples for rotation algebras.

Since noncommutative solenoids are inductive limit C^*-algebras, our spectral triples for noncommutative solenoids can be viewed as "inductive limit spectral triples" on inductive limit C^*-algebras. Because all of the test cases examined by Floricel and Ghorbanpour involve countable inductive systems of finite-dimensional spectral triples whose inductive realizations are spectral triples for AF-algebras, our spectral triples supply important new examples where Floricel and Ghorbanpour's results can be applied. In particular, our spectral triples for noncommutative solenoids, when viewed as inductive realizations of countable inductive systems of spectral triples for rotation algebras, satisfy the requirements for spectral triple structure given in [14, Theorems 3.1 and 3.2], as we show in the following theorem.

Theorem 3.21. *Fix a prime p. Let $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ and for each $n \in \mathbb{N}$, set $\Gamma_n = \frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$. For every $\theta \in \Omega_p$, the triple $(\mathcal{A}_\theta^S, \ell^2(\Gamma), \mathcal{D}_p)$ with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$ can be written as the inductive realization of*

$$\{(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \ell^2(\Gamma_j), D_{p,j}), (\phi_{j,k}, I_{j,k})\}_{j \in \mathbb{N}},$$
$$\text{with smooth subalgebras } \{C_C(\mathbb{Z}^2, \sigma_{\theta_{2j}})\}_{j \in \mathbb{N}}.$$

Furthermore, the above inductive system of spectral triples satisfies the hypotheses of Floricel and Ghorbanpour's theorems (Theorems 2.6 and 2.7), so that the inductive realization $(\mathcal{A}_\theta^S, \ell^2(\Gamma), \mathcal{D}_p)$ of the inductive system is itself a spectral triple, with associated smooth subalgebra $C_C(\Gamma, \sigma_\theta)$.

Proof. To show that $\mathcal{D}_p$, as the inductive limit of the family of operators $\{D_{p,j}\}_{j \in \mathbb{N}}$, has compact resolvent, fix $t \in \mathbb{R} \setminus \{0\}$. By [14, Theorem 3.1], it suffices to show that the sequence of operators $\{I_j(D_{p,j} - it)^{-1}I_j^*\}_{j \in \mathbb{N}}$ converges in norm to $(\mathcal{D}_p - it)^{-1}$. Choose a $\varepsilon > 0$. Then there exists $R > 0$ such that for $z \in \mathbb{R}$, $z > R$ implies that $|z - it|^{-1} < \varepsilon$. Given such an R, define $B_{\mathbb{L}_p^\Sigma}(R)$ as in part (1) of Definition 3.2, where we recall that $\mathbb{L}_p^\Sigma$ was defined in Corollary 3.17. As in the proof of Theorem 3.9, the fact that $\mathbb{L}_p^\Sigma$ is proper implies that $B_{\mathbb{L}_p^\Sigma}(R)$ is a finite set. Let $\{\gamma_k\}_{1 \leq k \leq m}$ be an enumeration of the elements in this set. For each $1 \leq k \leq m$, find $n_k \in \mathbb{N}$ and $a_k \in \mathbb{Z}$ such that $\gamma_k = \frac{a_k}{p^{n_k}}$ with $\gcd(a_k, p) = 1$. Set $N = \max\{n_1, \dots, n_m\}$ and define $B_{\mathbb{L}_p^\Sigma}(R)$ to be the ball of radius R in the length metric in Γ as in part (1) of Definition 3.2. Note that

$$B_{\mathbb{L}_p^\Sigma}(R) \subset \Gamma_N.$$

For each $1 \leq k \leq m$, now let P_k denote the orthogonal projection taking $\ell^2(\Gamma_N)$ onto the one-dimensional subspace spanned by δ_{γ_k}. Then $I_N P_k I_N^*$ describes the orthogonal projection taking $\ell^2(\Gamma)$ onto the one-dimensional subspace spanned by δ_{γ_k} viewed as an element of $\ell^2(\Gamma)$. Since each δ_{γ_k} is an eigenvector of $(\mathcal{D}_p - it)^{-1}$ with eigenvalue $(\mathbb{L}_p^\Sigma(\gamma_k) - it)^{-1}$, $j > N$ implies, by the triangle inequality, that

$$\left\|(\mathcal{D}_p - it)^{-1} - I_j(D_{p,j} - it)^{-1}I_j^*\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$
$$\leq \left\|(\mathcal{D}_p - it)^{-1} - I_N\left(\sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} (\mathbb{L}_p^\Sigma(\gamma_k) - it)^{-1} P_k\right)I_N^*\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$
$$+ \left\|I_N\left(\sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} (\mathbb{L}_p^\Sigma(\gamma_k) - it)^{-1} P_k\right)I_N^* - I_j(D_{p,j} - it)^{-1}I_j^*\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$
$$= \left\|(\mathcal{D}_p - it)^{-1} - \sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} (\mathbb{L}_p^\Sigma(\gamma_k) - it)^{-1} I_N P_k I_N^*\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$
$$+ \left\|I_j\left(I_{N,j}\left(\sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} (\mathbb{L}_p^\Sigma(\gamma_k) - it)^{-1} P_k\right)I_{N,j}^* - (D_{p,j} - it)^{-1}\right)I_j^*\right\|_{\mathcal{B}(\ell^2(\Gamma))}$$

$$= \left\| (\mathcal{D}_p - it)^{-1} - \sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} \left(\mathbb{L}_p^\Sigma(\gamma_k) - it \right)^{-1} I_N P_k I_N^* \right\|_{\mathcal{B}(\ell^2(\Gamma))}$$

$$+ \left\| \sum_{\gamma_k \in B_{\mathbb{L}_p^\Sigma}(R)} \left(\mathbb{L}_p^\Sigma(\gamma_k) - it \right)^{-1} I_{N,j} P_k I_{N,j}^* - (D_{p,j} - it)^{-1} \right\|_{\mathcal{B}(\ell^2(\Gamma))}$$

$$= \sup \left\{ \left| \mathbb{L}_p^\Sigma(\gamma) - it \right|^{-1} : \gamma \notin B_{\mathbb{L}_p^\Sigma}(R) \right\} + \sup \left\{ \left| \mathbb{L}_p^\Sigma(\gamma) - it \right|^{-1} : \gamma \notin B_{\mathbb{L}_p^\Sigma}(R) \right\}$$

$$< 2\varepsilon,$$

where the bound on the second term of the last equality holds because each δ_{γ_k} is also an eigenvector of $D_{p,j}$ with eigenvalue $(\mathbb{L}_p^\Sigma(\gamma) - it)^{-1}$ if $j > N$. As the choice of ε was arbitrary, the sequence $\{I_j(D_{p,j} - it)^{-1} I_j^*\}_{j \in \mathbb{N}}$ converges uniformly to $(\mathcal{D}_p - it)^{-1}$. Hence the inductive realization of $\{(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \ell^2(\Gamma_j), D_{p,j}), (\phi_{j,k}, I_{j,k})\}_{j \in \mathbb{N}}$ satisfies condition (ST1) in Definition 2.1 for a Dirac operator.

To see that $\mathcal{D}_p$, as the inductive limit of the family of operators $\{D_{p,j}\}_{j \in \mathbb{N}}$, satisfies the remaining denseness condition given in (ST2) of Definition 2.1, fix $J \in \mathbb{N}$ and a choice of $g \in C_C(\mathbb{Z}^2, \theta_{2J})$. By Theorem 2.7, it suffices to show that the family of commutators $\{[D_{p,k}, \pi_{\theta_{2k}}(\phi_{J,k}(g))]\}_{k \geqslant J}$ is uniformly bounded. The key norm inequality for commutators in equation (3.1) shows that, for all $g \in C_C(\mathbb{Z}^2, \theta_{2J})$,

$$\left\| [D_{p,k}, \pi_{\theta_{2k}}(\phi_{J,k}(g))] \right\|_{\mathcal{B}(H_k)} = \left\| [D_{p,k}, \lambda_{\sigma_{\theta_{2k}}} \circ \upsilon_k(\phi_{J,k}(g))] \right\|_{\mathcal{B}(H_k)}$$

$$\leqslant \sum_{\gamma \in \Gamma_k} \mathbb{L}_{p,k}^\Sigma(\gamma) \left| \upsilon_k(\phi_{J,k}(g))(\gamma) \right|.$$

Since $\mathbb{L}_{p,k}^\Sigma$ agrees with $\mathbb{L}_{p,J}^\Sigma$ on $\Gamma_J \subseteq \Gamma_k$ for each $k \geqslant J$, and since both agree with the restriction of $\mathbb{L}_p^\Sigma$ to the respective subgroups, we obtain

$$\left\| [D_{p,k}, \pi_{\theta_{2k}}(\phi_{J,k}(g))] \right\|_{\mathcal{B}(H_k)} \leqslant \sum_{\gamma \in \Gamma} \mathbb{L}_p^\Sigma(\gamma) \left| \phi_J(g)(\gamma) \right|, \quad \forall k \geqslant J$$

which, in combination with the observation that $g \in C_C(\mathbb{Z}^2, \theta_{2J})$ implies that

$$\phi_J(g) \in C_C(\Gamma_J, (\sigma_\theta)_J) \subset C_C(\Gamma, \sigma_\theta),$$

proves the desired uniform boundedness condition.   ∎

Remark 3.22. If we had tried to take inductive limits of Dirac operators arising from the standard length function on $\mathbb{Z}^2$ and $\Gamma_j = \frac{1}{p^j} \mathbb{Z}^2$ as described in Example 2.2, we would not have obtained this result, because some of the standard lengths of non-zero elements in Γ_j go to 0 as $j \to +\infty$. This would have caused major problems if we tried to verify condition (ST1). Fortunately, our adjusted lengths on Γ_j are bounded from below by 1 on non-zero elements.

4. A dense subalgebra of twisted Abelian group C^*-algebras with spectral invariance

We will now examine dense subdomains associated to our spectral triples which arise from our Dirac operators in greater detail. Notable properties to be discussed include identification, for fixed prime p and $\theta \in \Omega_p$, of a dense $*$-subalgebra of $\mathcal{A}_\theta^s$ having the property of spectral invariance in the sense of Schweitzer [37], on which $L_{\mathcal{D}_p}$ is finite. For similar $*$-subalgebras in the σ-twisted rapid decay case, see Chatterji's appendix to [30], and see also [13] for some related results. The existence of such a $*$-subalgebra was conjectured in a question posed by Long and Wu at the conclusion of [27]. The existence of this additional condition on $L_{\mathcal{D}_p}$ was also described as "useful" by Christ and Rieffel in the discussion after [8, Proposition 1.6]. To obtain such results, our efforts will rely heavily on methods from both noncommutative geometry and noncommutative harmonic analysis.

Our length function on the discrete abelian group $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$ will be used to define a Fréchet $*$-algebra that is a dense $*$-subalgebra having the property of spectral invariance in the twisted group C^*-algebra $C^*(\Gamma, \sigma_\theta)$. Indeed, some of our results will apply to twisted group C^*-algebras associated to arbitrary countable discrete nilpotent groups. As discussed by Christ and Rieffel in [8], subalgebras equipped with these properties also capture the K-theory of the C^*-algebra. With this aim in mind, we recall the following definition.

Definition 4.1 (Cf. [18, Definition 2.1]). Let Γ be a countable discrete group and σ a multiplier on Γ. Given a length function $\mathbb{L}$ on Γ, set $H_\mathbb{L}^{1,\infty}(\Gamma, \sigma)$ equal to the *space of complex-valued functions f* on Γ such that for each $s \geqslant 0$, the function $f(1 + \mathbb{L})^s$ belongs to $\ell^1(\Gamma, \sigma)$. For $f \in H_\mathbb{L}^{1,\infty}(\Gamma, \sigma)$ and $s \geqslant 0$, define the norm

$$\|\cdot\|_{1,s,\mathbb{L}} : H_\mathbb{L}^{1,\infty}(\Gamma, \sigma) \to [0, \infty)$$

by

$$\|f\|_{1,s,\mathbb{L}} = \sum_{\gamma \in \Gamma} |f(\gamma)| (1 + \mathbb{L}(\gamma))^s.$$

Remark 4.2. Notice that if we set, for fixed $q \in \mathbb{N}$ and for $\phi \in \ell^1(\Gamma, \sigma)$,

$$\mu_{q,\sigma}(\phi) = \sup_{N \in \mathbb{N}} \{N^q \|(1 - \chi_N)\phi\|_{\ell^1(\Gamma, \sigma)}\},$$

then the same proof as in [18, Lemma 2.2] shows that $\phi \in H_\mathbb{L}^{1,\infty}(\Gamma, \sigma)$ if and only if $\mu_{q,\sigma}(\phi) < \infty, \forall q \in \mathbb{N}$.

Remark 4.3. Our definition is a modification of that given in [18], which is specified for discrete group C^*-algebras rather than twisted discrete group C^*-algebras. Although the multiplier σ does not play a role in the definition of the norm $\|\cdot\|_{1,s,\mathbb{L}}$, each such norm for $s \geqslant 0$ will be shown in the proof of Proposition 4.7 to exhibit submultiplicativity with respect to the convolution product twisted by σ.

Remark 4.4. A Fréchet algebra which arises from a twisted group C^*-algebra and whose topology can be induced by a collection of seminorms which are each submultiplicative with respect to the twisted convolution product will be called a *twisted Fréchet algebra*. The collection of seminorms used in our definition of $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ happens to also be a collection of norms.

We will demonstrate that $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma) \subset C^*(\Gamma, \sigma)$ is stable under the holomorphic functional calculus (see Theorem 4.12). We first introduce the notion of a spectral invariant subalgebra, as introduced by Schweitzer in [37].

Definition 4.5 ([37]). Let A be a unital $*$-subalgebra of a unital Banach algebra B. We say that A is a *spectral invariant (SI) $*$-subalgebra* of B if the invertible elements of A are precisely those elements in A that are invertible in B. Note that A is SI in B if and only if for every $a \in A$, the spectrum of a in A is the same as the spectrum of a in B.

Remark 4.6. As a consequence of Schweitzer's work [37, Lemma 1.2], a unital Fréchet $*$-subalgebra A of a unital C^*-algebra B *is SI if and only if it is stable under the holomorphic functional calculus*, that is, if a function f is holomorphic in a neighborhood of the spectrum of $a \in A$, the element $f(a)$ of B lies in A. This is important particularly when considering K-theoretic invariants of B, because under these hypotheses, the K-theory of A will be the same as the K-theory of B.

Working in the more general setting of countable nilpotent groups, we will first apply ideas developed by Jolissaint in [18], as well as techniques used by Ji in [17], to show that $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is a twisted Fréchet $*$-subalgebra that is dense and spectral invariant in $\ell^1(\Gamma, \sigma)$. To deduce that $\ell^1(\Gamma, \sigma)$ is SI in $C^*(\Gamma, \sigma)$, we will rely on results obtained by Austad in [5] building on the work of Gröchenig and Leinert in [15], Ludwig in [29], and Hulanicki in [16]. We will then be able to equip our spectral triples for noncommutative solenoids with $H_{\mathbb{L}_p^{\Sigma}}^{1,\infty}(\Gamma, \sigma_\theta)$,

$$\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right],$$

as a choice of dense smooth subalgebra (see Corollary 4.14).

4.1. Spectral invariance of dense twisted Fréchet $*$-subalgebras of $\ell^1(\Gamma, \sigma)$

For twisted group C^*-algebras, the multiplier plays a role in adjoint and convolution product computations. Since multipliers on groups take complex values of unit modulus, the twisted Fréchet $*$-subalgebra $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma) \subset \ell^1(\Gamma, \sigma)$ can be shown to exhibit additional structure with respect to these operations. For generalities on Fréchet algebras in our context, see e.g. [17].

Proposition 4.7. *Let Γ be a countable discrete group and σ a multiplier on Γ. Suppose that $\mathbb{L}$ is a length function on Γ. Then $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is a (twisted) $*$-subalgebra of $\ell^1(\Gamma, \sigma)$. Furthermore, $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is also a* twisted Fréchet $*$-algebra *with respect to the topology defined by the increasing sequence of norms $\{\|\cdot\|_{1,s,\mathbb{L}}\}_{s \geqslant 0}$.*

Proof. To show that $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is invariant under the $*$-operation in $\ell^1(\Gamma,\sigma)$, suppose that f is in $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. Then

$$\|f^*\|_{1,s,\mathbb{L}} = \sum_{\gamma\in\Gamma} \left|\overline{f(\gamma^{-1})\sigma(\gamma,\gamma^{-1})}\right|\left(1+\mathbb{L}(\gamma)\right)^s$$

$$= \sum_{\gamma\in\Gamma} \left|f(\gamma)\right|\left(1+\mathbb{L}(\gamma)\right)^s = \|f\|_{1,s,\mathbb{L}}.$$

Since the choice of f was arbitrary, $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma) \subset \ell^1(\Gamma,\sigma)$ is a $*$-subalgebra.

To determine bounds on the behavior of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ under the σ-twisted convolution product in $\ell^1(\Gamma,\sigma)$, a bound on the length function will first be obtained. For every $\gamma,\eta\in\Gamma$, the fact that $\mathbb{L}$ is a length function on Γ yields, by taking $\gamma=\eta\eta^{-1}\gamma$,

$$1+\mathbb{L}(\gamma) \leqslant 1+\mathbb{L}(\eta)+\mathbb{L}(\eta^{-1}\gamma)$$

$$\leqslant 1+\mathbb{L}(\eta)+\mathbb{L}(\eta^{-1}\gamma)+L(\eta)L(\eta^{-1}\gamma)$$

$$= \left(1+\mathbb{L}(\eta)\right)\left(1+\mathbb{L}(\eta^{-1}\gamma)\right). \tag{4.1}$$

Thus, for every $f,f'\in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$, we have by equation (4.1)

$$\left\|(f *_\sigma f')\right\|_{1,s,\mathbb{L}} = \sum_{\gamma\in\Gamma} \left|(f *_\sigma f')(\gamma)\right|\left(1+\mathbb{L}(\gamma)\right)^s$$

$$\leqslant \sum_{\gamma\in\Gamma}\sum_{\eta\in\Gamma} \left|f(\eta)f'(\eta^{-1}\gamma)\sigma(\eta,\eta^{-1}\gamma)\right|\left(1+\mathbb{L}(\gamma)\right)^s$$

$$\leqslant \sum_{\gamma\in\Gamma}\sum_{\eta\in\Gamma} \left|f(\eta)f'(\eta^{-1}\gamma)\right|\left(1+\mathbb{L}(\eta)\right)^s\left(1+\mathbb{L}(\eta^{-1}\gamma)\right)^s$$

$$= \sum_{\eta\in\Gamma} \left|f(\eta)\right|\left(1+\mathbb{L}(\eta)\right)^s \sum_{\gamma\in\Gamma} \left|f'(\eta^{-1}\gamma)\right|\left(1+\mathbb{L}(\eta^{-1}\gamma)\right)^s$$

$$= \|f\|_{1,s,\mathbb{L}}\|f'\|_{1,s,\mathbb{L}} < \infty.$$

Thus, for each $s\geqslant 0$, $\|\cdot\|_{1,s,\mathbb{L}}$ has been shown to be submultiplicative with respect to the twisted convolution product. Hence $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is also a twisted Fréchet $*$-subalgebra of $\ell^1(\Gamma,\sigma)$. ∎

Because $C_C(\Gamma,\sigma) \subset H_{\mathbb{L}}^{1,\infty}(\Gamma)$, $H_{\mathbb{L}}^{1,\infty}(\Gamma)$ is dense in $\ell^1(\Gamma,\sigma)$. We now begin to lay the groundwork towards extending a result of Jolissaint in [18] to these twisted Fréchet $*$-subalgebras.

Proposition 4.8. *Let Γ be a countable discrete group and σ a multiplier on Γ. Suppose that $\mathbb{L}$ is a length function on Γ. Then the twisted Fréchet $*$-subalgebra $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is SI and dense in $\ell^1(\Gamma,\sigma)$.*

Proof. In this proof, we follow the proof of Jolissaint's Proposition 2.3 [18]. To check Definition 4.5, we first consider the case in which $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma) \ni h = (1-f)$, with

$\|f\|_{\ell^1(\Gamma,\sigma)} < \frac{1}{2}$. Hence h is first of all invertible in $\ell^1(\Gamma,\sigma)$, with inverse $h^{-1} = \sum_{n=0}^\infty f^n$, where exponentiation signifies repeated application of the twisted convolution product. We will show below that $h^{-1} \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$, as needed for spectral invariance. Hence our goal is to obtain a bound on $\mu_{q,\sigma}(\sum_{n=1}^\infty f^n)$, which will imply that $\sum_{n=0}^\infty f^n \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$; see Remark 4.2.

Given $N \in \mathbb{N}$, define $B_{\mathbb{L}}(N)$ as in part (1) of Definition 3.2. Let χ_N denote the characteristic function supported on $B_{\mathbb{L}}(N)$. Fix $q \in \mathbb{N}$. (Note that we changed Jolissaint's notation from χ^N to χ_N for the characteristic function of $B_{\mathbb{L}}(N)$.) Now fix $N \in \mathbb{N}$, $N \geqslant 1$. For $\mathbb{N} \ni n \leqslant \sqrt{N}$, we have

$$
\begin{aligned}
(1-\chi_N)f^n &= (1-\chi_N)\big[\big((1-\chi_{\sqrt{N}})f\big) *_\sigma f^{n-1}\big] \\
&\quad + (1-\chi_N)\big[(\chi_{\sqrt{N}}f) *_\sigma \big((1-\chi_{\sqrt{N}})f\big) *_\sigma f^{n-2}\big] + \cdots \\
&\quad + (1-\chi_N)\big[(\chi_{\sqrt{N}}f)^k *_\sigma \big((1-\chi_{\sqrt{N}})f\big) *_\sigma f^{n-k-1}\big] + \cdots \\
&\quad + (1-\chi_N)\big[(\chi_{\sqrt{N}}f)^{n-1} *_\sigma \big((1-\chi_{\sqrt{N}})f\big)\big] + (1-\chi_N)(\chi_{\sqrt{N}}f)^n. \quad (4.2)
\end{aligned}
$$

This equality can be confirmed using the associative structure of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ in combination with a telescoping/cancellation argument involving the expansion of f^n.

Now note that the last term is zero because $(1-\chi_N)$ is supported outside the ball of radius N. Consequently, we only have the first n terms left. Using these n terms, the triangle inequality, and the submultiplicativity of the norm, we obtain, for $n \geqslant 1$,

$$
\big\|(1-\chi_N)f^n\big\|_{\ell^1(\Gamma,\sigma)} \leqslant \big\|n\big((1-\chi_{\sqrt{N}})f\big)\big\|_{\ell^1(\Gamma,\sigma)}.
$$

We also remark that if $\|f\| := \|f\|_{\ell^1(\Gamma,\sigma)} < 1/2$, then, for $P \in \mathbb{N}$,

$$
\sum_{n=P}^\infty \|f\|^n < \|f\|^P \frac{1}{1-1/2} = 2\|f\|^P < \frac{2\|f\|}{2^{P-1}} = \frac{4\|f\|}{2^P}. \quad (4.3)
$$

Therefore, equations (4.2)–(4.3) imply that, if we denote by "$[\sqrt{N}]$" the "greatest integer less than or equal to $\sqrt{N}$",

$$
\begin{aligned}
&\bigg\|(1-\chi_N)\sum_{n=1}^\infty f^n\bigg\|_{\ell^1(\Gamma,\sigma)} \\
&\leqslant \sum_{n=1}^{[\sqrt{N}]} \big\|(1-\chi_N)f^n\big\|_{\ell^1(\Gamma,\sigma)} + \sum_{n=[\sqrt{N}]+1}^\infty \big\|(1-\chi_N)f^n\big\|_{\ell^1(\Gamma,\sigma)} \\
&\leqslant \sum_{n=1}^{[\sqrt{N}]} n\big\|\big((1-\chi_{\sqrt{N}})f\big)\big\|_{\ell^1(\Gamma,\sigma)} + \sum_{n=[\sqrt{N}]+1}^\infty \|f\|^n \\
&\leqslant [\sqrt{N}]^2 \big\|(1-\chi_{\sqrt{N}})f\big\|_{\ell^1(\Gamma,\sigma)} + \frac{4\|f\|}{2^{[\sqrt{N}]+1}} \\
&\leqslant [\sqrt{N}]^2 \big\|(1-\chi_{\sqrt{N}})f\big\|_{\ell^1(\Gamma,\sigma)} + \frac{4\|f\|}{2^{\sqrt{N}}}. \quad (4.4)
\end{aligned}
$$

So equation (4.4) becomes

$$\left\|(1-\chi_N)\sum_{n=1}^{\infty}f^n\right\|_{\ell^1(\Gamma,\sigma)} \leq [\sqrt{N}]^2\left\|(1-\chi_{\sqrt{N}})f\right\|_{\ell^1(\Gamma,\sigma)} + \frac{4\|f\|_{\ell^1(\Gamma,\sigma)}}{2\sqrt{N}}.$$

Then, multiplying both sides of the above inequality by $[\sqrt{N}]^{2q}$, for $q \in \mathbb{N}$, we obtain

$$[\sqrt{N}]^{2q}\left\|(1-\chi_N)\sum_{n=1}^{\infty}f^n\right\|_{\ell^1(\Gamma,\sigma)}$$

$$\leq [\sqrt{N}]^{2q+2}\left\|(1-\chi_{\sqrt{N}})f\right\| + [\sqrt{N}]^{2q}\frac{4\|f\|_{\ell^1(\Gamma,\sigma)}}{2\sqrt{N}}$$

$$\leq [\sqrt{N}]^{2q+2}\left\|(1-\chi_{[\sqrt{N}]})f\right\|_{\ell^1(\Gamma,\sigma)} + N^q\frac{4\|f\|}{2\sqrt{N}}.$$

Now recall that $f \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ implies that $\mu_{2q+2,\sigma}(f)$ is finite, and that, $\forall N \in \mathbb{N}$,

$$[\sqrt{N}]^{2q+2}\left\|(1-\chi_{[\sqrt{N}]})f\right\|_{\ell^1(\Gamma,\sigma)} \leq \mu_{2q+2,\sigma}(f) < \infty.$$

Also $\lim_{N\to\infty}\frac{N^q}{2\sqrt{N}} = 0$, so that the set of nonnegative numbers $\{N^q\frac{4\|f\|}{2\sqrt{N}}\}_{N=1}^{\infty}$ is bounded above. Thus

$$\mu_q\left(\sum_{n=1}^{\infty}f^n\right) < \infty, \quad \forall q \in \mathbb{N},$$

so that $\sum_{n=1}^{\infty}f^n$ is an element of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. Therefore, $\sum_{n=0}^{\infty}f^n = 1 + \sum_{n=1}^{\infty}f^n$ is an element of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ as well. Hence $h = (1-f)$ is invertible in $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ when $\|1-h\|_{\ell^1(\Gamma,\sigma)} < \frac{1}{2}$.

We now consider the case of arbitrary $h \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$, where h is invertible in the larger Banach $*$-algebra $\ell^1(\Gamma,\sigma)$, but *a priori* it is unknown whether or not $h^{-1} \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. Recall that $1_{\ell^1(\Gamma,\sigma)} = 1_{H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)} = 1 \cdot \delta_{0_\Gamma}$, and $1_{H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)} = h *_\sigma h^{-1}$. Since $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is dense in $\ell^1(\Gamma,\sigma)$, an adaptation of arguments used by Ji in the proof of [17, Theorem 1.2] yields that if we take $h' \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ such that $\|h^{-1} - h'\|_{\ell^1(\Gamma,\sigma)} < \frac{1}{2\cdot\|h\|_{\ell^1(\Gamma,\sigma)}}$, then

$$\left\|1-(h *_\sigma h')\right\|_{\ell^1(\Gamma,\sigma)} = \left\|(h *_\sigma h^{-1}) - (h *_\sigma h')\right\|_{\ell^1(\Gamma,\sigma)}$$

$$\leq \|h\|_{\ell^1(h*_\sigma)} \cdot \|h^{-1} - h'\|_{\ell^1(\Gamma,\sigma)}$$

$$< \frac{1}{2}.$$

We now note that $(h *_\sigma h')$ and $1 - (h *_\sigma h')$ are both elements of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. By our proof given above, $(h *_\sigma h')$ is invertible in $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$, with $(h *_\sigma h')^{-1}$ being an element of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. Because $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is an associative algebra, the element $h^{-1} = h' *_\sigma (h *_\sigma h')^{-1}$ is an element of $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$, as well as being an element of $\ell^1(\Gamma,\sigma)$.

It follows that, if $h \in H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is invertible in $\ell^1(\Gamma,\sigma)$, then it is also invertible in $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$. Therefore, $H_{\mathbb{L}}^{1,\infty}(\Gamma,\sigma)$ is a subalgebra that is SI in $\ell^1(\Gamma,\sigma)$. ∎

4.2. A noncommutative Wiener lemma for twisted discrete nilpotent group C^*-algebras

Because of the equivalence between spectral invariance and being closed under the holomorphic functional calculus for unital subalgebras of twisted convolution algebras, a statement about the spectral invariance of $\ell^1(\Gamma, \sigma) \subset C^*(\Gamma, \sigma)$ can be viewed as a noncommutative Wiener lemma. Many such results [5, 15, 29] rely on showing that the spectrum of positive elements in the unital subalgebra is positive. We will demonstrate that $\ell^1(\Gamma, \sigma)$ belongs to a class of algebras which exhibit this property.

Definition 4.9 (E.g. Ludwig [29]). A Banach $*$-algebra A is called *symmetric* if for all $a \in A$, $\mathrm{spec}_A(a^*a) \subseteq [0, \infty)$.

As noted by Gröchenig and Leinert in their work on noncommutative Wiener lemmas [15], locally compact nilpotent groups have symmetric L^1–algebras. They also cite and build upon the following result due to Ludwig.

Theorem 4.10 ([15, 29]). *If G is a locally compact nilpotent group, then $L^1(G)$ is symmetric. Consequently, $\mathrm{spec}_{L^1(G)}(f) = \mathrm{spec}_{B(L^2(G))}(\lambda(f))$ for $f = f^* \in L^1(G)$, and λ the regular representation of $L^1(G)$ as bounded operators on $L^2(G)$.*

Our twisted discrete group C^*-algebras are in fact quotients of nilpotent group C^*-algebras, as follows. Let Γ be a countable discrete nilpotent group and σ a multiplier on Γ. Set $\mathbb{H}_\sigma = \Gamma \times \mathbb{T}$. Equip $\mathbb{H}_\sigma$ with multiplication by defining

$$(\gamma, z) \cdot_\sigma (\gamma', z') = \big(\gamma + \gamma', zz' \, \sigma(\gamma, \gamma')\big).$$

Given the product topology, $\mathbb{H}_\sigma$ becomes a locally compact second countable group. Moreover, $\mathbb{H}_\sigma$ is nilpotent. For non-trivial σ and abelian groups Γ, $\mathbb{H}_\sigma$ is also in fact a two-step nilpotent group. The Haar measure dm on $\mathbb{H}_\sigma$ can be described explicitly by

$$\int_{\mathbb{H}_\sigma} F(\gamma, z) \, dm = \sum_{\gamma \in \Gamma} \int_{\mathbb{T}} F(\gamma, z) \, dz.$$

As in the more specific setting of $\Gamma = \mathbb{Z}^d \times \mathbb{Z}^d$ considered by Gröchenig and Leinert in [15], we define convolution $*_\sigma$ of integrable functions on $\mathbb{H}_\sigma$ with respect to this measure. We also follow their terminology in describing $\mathbb{H}_\sigma$ as a *Heisenberg-type group*. These groups are also called Mackey groups in [5], where they use the notation Γ_σ for them. By constructing a unitary group representation of $\mathbb{H}_\sigma$ on $\ell^2(\Gamma)$, it is not hard to see that the twisted group C^*-algebra $C^*(\Gamma, \sigma)$ is a quotient of the group C^*-algebra $C^*(\mathbb{H}_\sigma)$. Note that nilpotent groups are amenable, so that the reduced and full group C^*-algebras of $\mathbb{H}_\sigma$ are the same. In [5], Austad extended Gröchenig and Leinert's arguments to identify conditions for spectral invariance in the more general context of twisted locally compact group C^*-algebras. In both [5,15], a condition on the spectrum of self-adjoint elements in $L^1(G)$ for G locally compact with polynomial growth, first obtained by Hulanicki in [16] and

generalized by Barnes in [7] (see [16, the theorem on p. 137] and [7, Theorem 4]), played an important role in obtaining these results. We will now apply Austad's result, together with Ludwig's theorem, to deduce that there is a noncommutative Wiener lemma for twisted discrete nilpotent group C^*-algebras.

Lemma 4.11 (Cf. [5, Theorem 3.1]). *Let Γ be a countable discrete nilpotent group and σ a multiplier on Γ. Restrict the left-σ regular representation $\lambda_\sigma : C^*(\Gamma, \sigma) \to B(\ell^2(\Gamma))$ to $\ell^1(\Gamma, \sigma)$ and let λ_σ also denote this restriction. If $f \in \ell^1(\Gamma, \sigma)$, then*

$$\operatorname{spec}_{\ell^1(\Gamma,\sigma)}(f) = \operatorname{spec}_{B(\ell^2(\Gamma))}\big(\lambda_\sigma(f)\big). \tag{4.5}$$

In particular, f is invertible in the Banach $*$-algebra $\ell^1(\Gamma, \sigma)$ if and only if $\lambda_\sigma(f)$ is invertible in $B(\ell^2(\Gamma))$.

Proof. Second countable locally compact nilpotent groups are also amenable locally compact groups. For a twisted group algebra arising as a quotient of a nilpotent (thus amenable) locally compact group and admitting a faithful $*$-representation, Austad's proof of [5, Theorem 3.1 (i)] gives that the maximal C^*-norm is the unique C^*-norm. In fact, this unique C^*-norm in the case of $\ell^1(\Gamma, \sigma)$ is completely determined by using $\|\cdot\|_{B(L^2(\mathbb{H}_\sigma, m))}$ and decomposing the left regular representation of $C^*(\mathbb{H}_\sigma)$ (see [5, Lemmas 3.4 and 3.5]). Since $\mathbb{H}_\sigma$ is nilpotent, Ludwig's theorem (Theorem 4.10) implies that $L^1(\mathbb{H}_\sigma, m)$ is also symmetric. This shows that the hypotheses of [5, Theorem 3.1] are satisfied, and that [5, Theorem 3.1 (ii)] gives equation (4.5). ∎

The noncommutative Wiener lemma above allows us to equip a twisted discrete nilpotent group C^*-algebra with a smooth $*$-subalgebra that is spectral invariant, so has the same K-theory as the original C^*-algebra. We have thus achieved the promised generalization of Jolissaint's proposition [18] to twisted discrete nilpotent group C^*-algebras.

Theorem 4.12 (Cf. [18, Proposition 2.3]). *Let Γ be a countable discrete nilpotent group and σ a multiplier on Γ. Suppose that $\mathbb{L}$ is a length function on Γ. Then the twisted Fréchet $*$-subalgebra $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is dense and has the property of spectral invariance in $C^*(\Gamma, \sigma)$. Therefore, $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is stable in $C^*(\Gamma, \sigma)$ under the holomorphic functional calculus.*

Proof. Proposition 4.8 and Lemma 4.11 together give that f is invertible in $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ if and only if $\lambda_\sigma(f)$ is invertible in $\mathcal{B}(\ell^2(\Gamma))$. Hence $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ has the property of spectral invariance in $C^*(\Gamma, \sigma)$; i.e., f is invertible in $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ if and only if f is invertible in $C^*(\Gamma, \sigma)$. Since $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ contains $C_C(\Gamma, \sigma)$, it follows that $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is dense in $C^*(\Gamma, \sigma)$. By Proposition 4.7, $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is a twisted Fréchet $*$-subalgebra of $C^*(\Gamma, \sigma)$ (see Remark 4.4). Consequently, by Remark 4.6, $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ is stable in $C^*(\Gamma, \sigma)$ under the holomorphic functional calculus. ∎

Remark 4.13. In contrast to our spectral triple constructions for noncommutative solenoids, where we needed additional properties on the length functions to ensure that we had

Leibniz quantum compact metric spaces, our dense smooth subalgebras do not require the length function to have bounded doubling to obtain spectral invariance for $H_{\mathbb{L}}^{1,\infty}(\Gamma, \sigma)$ in $C^*(\Gamma, \sigma)$. However, our results do rely on the requirement that the group Γ be countable, discrete, and nilpotent.

We now use the above results to modify our spectral triples for noncommutative solenoids by choosing the dense smooth subalgebras we just constructed. Fix a prime p and a choice of $\theta \in \Omega_p$. Let $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$. In Section 3, we showed that $(\mathcal{A}_\theta^S, \ell^2(\Gamma), \mathcal{D}_p)$ with representation λ_{σ_θ} is a spectral triple for the noncommutative solenoid $\mathcal{A}_\theta^S$. Recall that for every $\gamma \in \Gamma$,

$$L_{\mathcal{D}_p}(\delta_\gamma) = \left\| [\mathcal{D}_p, \lambda_{\sigma_\theta}(\delta_\gamma)] \right\|_{\mathcal{B}(\ell^2(\Gamma))} = \mathbb{L}_p^\Sigma(\gamma).$$

Thus for every $f \in H_{\mathbb{L}_p^\Sigma}^{1,\infty}(\Gamma, \sigma)$,

$$
\begin{aligned}
L_{\mathcal{D}_p}(f) &= \left\| [\mathcal{D}_p, \lambda_{\sigma_\theta}(f)] \right\|_{\mathcal{B}(\ell^2(\Gamma))} \\
&= \left\| \left[\mathcal{D}_p, \sum_{\gamma \in \Gamma} f(\gamma) \lambda_{\sigma_\theta}(\delta_\gamma) \right] \right\|_{\mathcal{B}(\ell^2(\Gamma))} \\
&= \left\| \sum_{\gamma \in \Gamma} f(\gamma) [\mathcal{D}_p, \lambda_{\sigma_\theta}(\delta_\gamma)] \right\|_{\mathcal{B}(\ell^2(\Gamma))} \\
&\leqslant \sum_{\gamma \in \Gamma} |f(\gamma)| \left\| [\mathcal{D}_p, \lambda_{\sigma_\theta}(\delta_\gamma)] \right\|_{\mathcal{B}(\ell^2(\Gamma))} \\
&= \sum_{\gamma \in \Gamma} |f(\gamma)| \mathbb{L}_p^\Sigma(\gamma) \\
&\leqslant \sum_{\gamma \in \Gamma} |f(\gamma)| \left(1 + \mathbb{L}_p^\Sigma(\gamma) \right) \\
&= \|f\|_{1,1,\mathbb{L}_p^\Sigma} < \infty.
\end{aligned}
$$

Induction can then be used to verify that for $k \in \mathbb{N}$,

$$\Big\| \underbrace{[\mathcal{D}_p, [\mathcal{D}_p, \ldots, [\mathcal{D}_p, \lambda_{\sigma_\theta}(f)], \ldots]]}_{k} \Big\|_{\mathcal{B}(\ell^2(\Gamma))} \leqslant \|f\|_{1,k,\mathbb{L}_p^\Sigma} < \infty.$$

Taking the point of view that the iterated commutants above can be seen as a noncommutative form of higher-order differentiation, the above calculations validate our choice of $H_{\mathbb{L}_p^\Sigma}^{1,\infty}(\Gamma, \sigma)$ as a smooth subalgebra in $C^*(\Gamma, \sigma)$. With the selection of $H_{\mathbb{L}_p^\Sigma}^{1,\infty}(\Gamma, \sigma)$ as dense smooth subalgebra, our spectral triples for noncommutative solenoids can support further study of these inductive limit C^*-algebras as noncommutative manifolds. We now are able to adjust the results of Theorem 3.9 using a dense subalgebra that has the property of spectral invariance.

Corollary 4.14. *Fix a prime p. Let $\Gamma = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}[\frac{1}{p}]$. For every $\theta \in \Omega_p$, $(\mathcal{A}_\theta^S, \ell^2(\Gamma), \mathcal{D}_p)$ with representation λ_{σ_θ} is a spectral triple for the noncommutative solenoid*

$$C^*(\Gamma, \sigma_\theta) = \mathcal{A}_\theta^S,$$

with associated smooth subalgebra $H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta)$. Furthermore, the twisted Fréchet $*$-subalgebra $H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta)$ is a proper dense subalgebra of $C^*(\Gamma, \sigma_\theta) = \mathcal{A}^{\mathcal{S}}_\theta$ that is stable under the holomorphic functional calculus.

Proof. Theorem 4.12 gives that $H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta)$ is a dense twisted Fréchet $*$-subalgebra that is stable in $C^*(\Gamma, \sigma_\theta)$ under the holomorphic functional calculus. Since

$$\big[\mathcal{D}_p, \lambda_{\sigma_\theta}(f)\big] \in \mathcal{B}\big(\ell^2(\Gamma)\big) \quad \text{for every } f \in H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta),$$

$H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta)$ is also a proper dense subalgebra of $C^*(\Gamma, \sigma_\theta) = \mathcal{A}^{\mathcal{S}}_\theta$. ∎

We also revisit and enhance the countable inductive system of finite-dimensional spectral triples considered in Theorem 3.21. For each $j \in \mathbb{N}$, $(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \ell^2(\Gamma_j), D_{p,j})$ is a spectral triple for the twisted discrete abelian group C^*-algebra $C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$ coming from the length function $\mathbb{L}^\Sigma_{p,j}$. Theorem 4.12 gives that

$$H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}}\big(\Gamma_j, (\sigma_\theta)_j\big)$$

is a proper dense twisted Fréchet $*$-subalgebra that is stable in $C^*(\Gamma_j, (\sigma_\theta)_j)$ under the holomorphic functional calculus. Recall that the representation π_{θ_j} is defined using a $*$-isomorphism υ_j which takes $C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$ to $C^*(\Gamma_j, (\sigma_\theta)_j)$. Our inductive system of C^*-algebras

$$\big\{\big(C^*\big(\Gamma_j, (\sigma_\theta)_j\big), \beta_{j,k}\big)\big\}_{j\in\mathbb{N}}$$

therefore supports the inductive system of twisted Fréchet $*$-algebras

$$\big\{\big(H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}}\big(\Gamma_j, (\sigma_\theta)_j\big), \beta_{j,k}\big)\big\}_{j\in\mathbb{N}}.$$

Furthermore, each $H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}}(\Gamma_j, (\sigma_\theta)_j)$ is $*$-isomorphic to a proper dense subalgebra of the corresponding C^*-algebra in our inductive system of rotation algebras

$$\big\{\big(C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \phi_{j,k}\big)\big\}_{j\in\mathbb{N}}.$$

In particular, each $\upsilon_j^{-1}(H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}}(\Gamma_j, (\sigma_\theta)_j))$ can be explicitly identified with the twisted Fréchet $*$-algebra

$$H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}\circ\upsilon_j}(\mathbb{Z}^2, \sigma_{\theta_{2j}}),$$

and thus is stable under the holomorphic functional calculus in $C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}})$. Hence our inductive system of twisted Fréchet $*$-algebras encodes the respective K-theories of our inductive system of rotation algebras. Thus $H^{1,\infty}_{\mathbb{L}^\Sigma_p}(\Gamma, \sigma_\theta)$ can be expressed as the inductive limit of our inductive system of twisted Fréchet $*$-algebras

$$\big\{H^{1,\infty}_{\mathbb{L}^\Sigma_{p,j}\circ\upsilon_j}(\mathbb{Z}^2, \sigma_{\theta_{2j}})\big\}_{j\in\mathbb{N}}.$$

Since noncommutative solenoids can be identified with inductive limits of rotation algebras, we hope that equipping our inductive system of spectral triples for rotation algebras

with the corresponding inductive system of twisted Fréchet ∗-algebras as the underlying smooth subalgebras in the limiting case will enrich the development, study, and understanding of the noncommutative geometry of noncommutative solenoids.

References

[1] K. Aguilar, Inductive limits of C*-algebras and compact quantum metric spaces. *J. Aust. Math. Soc.* **111** (2021), no. 3, 289–312 Zbl 1491.46072 MR 4337940

[2] K. Aguilar and F. Latrémolière, Quantum ultrametrics on AF algebras and the Gromov-Hausdorff propinquity. *Studia Math.* **231** (2015), 149–193 Zbl 1348.46071 MR 3465284

[3] V. Aiello, D. Guido, and T. Isola, Spectral triples for noncommutative solenoidal spaces from self-coverings. *J. Math. Anal. Appl.* **448** (2017), 1378–1412 Zbl 1369.46063 MR 3582288

[4] P. Antonini, D. Guido, T. Isola, and A. Rubin, A note on twisted crossed products and spectral triples. *J. Geom. Phys.* **180** (2022), article no. 104640 Zbl 1514.46047 MR 4464553

[5] A. Austad, Spectral invariance of ∗-representations of twisted convolution algebras with applications in Gabor analysis. *J. Fourier Anal. Appl.* **27** (2021), no. 3, article no. 56 Zbl 1480.46063 MR 4269451

[6] A. Austad and F. Luef, Modulation spaces as a smooth structure in noncommutative geometry. *Banach J. Math. Anal.* **15** (2021), no. 2, article no. 35 Zbl 1470.46098 MR 4216371

[7] B. A. Barnes, When is the spectrum of a convolution operator on L^p independent of p? *Proc. Edinburgh Math. Soc. (2)* **33** (1990), no. 2, 327–332 Zbl 0715.47004 MR 1057760

[8] M. Christ and M. A. Rieffel, Nilpotent group C*-algebras as compact quantum metric spaces. *Canad. Math. Bull.* **60** (2017), no. 1, 77–94 Zbl 1372.46054 MR 3612100

[9] E. Christensen and C. Ivan, Spectral triples for AF C*-algebras and metrics on the Cantor set. *J. Operator Theory* **56** (2006), no. 1, 17–46 Zbl 1111.46052 MR 2261610

[10] A. Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems* **9** (1989), no. 2, 207–220 Zbl 0718.46051 MR 1007407

[11] A. Connes, *Noncommutative geometry.* Academic Press, San Diego, CA, 1994 Zbl 0818.46076 MR 1303779

[12] U. Enstad, The Balian-Low theorem for locally compact abelian groups and vector bundles. *J. Math. Pures Appl. (9)* **139** (2020), 143–176 Zbl 1508.42034 MR 4108410

[13] G. Fendler, K. Gröchenig, and M. Leinert, Convolution-dominated operators on discrete groups. *Integral Equations Operator Theory* **61** (2008), no. 4, 493–509 Zbl 1194.47030 MR 2434338

[14] R. Floricel and A. Ghorbanpour, On inductive limit spectral triples. *Proc. Amer. Math. Soc.* **147** (2019), no. 8, 3611–3619 Zbl 1429.58015 MR 3981138

[15] K. Gröchenig and M. Leinert, Wiener's lemma for twisted convolution and Gabor frames. *J. Amer. Math. Soc.* **17** (2004), no. 1, 1–18 Zbl 1037.22012 MR 2015328

[16] A. Hulanicki, On the spectrum of convolution operators on groups with polynomial growth. *Invent. Math.* **17** (1972), 135–142 Zbl 0264.43007 MR 0323951

[17] R. Ji, Smooth dense subalgebras of reduced group C^*-algebras, Schwartz cohomology of groups, and cyclic cohomology. *J. Funct. Anal.* **107** (1992), no. 1, 1–33 Zbl 0787.46043 MR 1165864

[18] P. Jolissaint, K-theory of reduced C^*-algebras and rapidly decreasing functions on groups. *K-Theory* **2** (1989), no. 6, 723–735 Zbl 0692.46062 MR 1010979

[19] T.-M. Landry, M. L. Lapidus, and F. Latrémolière, Metric approximations of spectral triples on the Sierpiński gasket and other fractal curves. *Adv. Math.* **385** (2021), article no. 107771 Zbl 1478.46072 MR 4256136

[20] F. Latrémolière, The quantum Gromov-Hausdorff propinquity. *Trans. Amer. Math. Soc.* **368** (2016), no. 1, 365–411 Zbl 1334.46056 MR 3413867

[21] F. Latrémolière, The Gromov-Hausdorff propinquity for metric spectral triples. *Adv. Math.* **404** (2022), article no. 108393 Zbl 1500.46055 MR 4411527

[22] F. Latrémolière and J. Packer, Noncommutative solenoids and the Gromov-Hausdorff propinquity. *Proc. Amer. Math. Soc.* **145** (2017), no. 5, 2043–2057 Zbl 1368.46067 MR 3611319

[23] F. Latrémolière and J. Packer, Noncommutative solenoids. *New York J. Math.* **24A** (2018), 155–191 Zbl 1472.46075 MR 3904875

[24] F. Latrémolière and J. A. Packer, Noncommutative solenoids and their projective modules. In *Commutative and noncommutative harmonic analysis and applications*, pp. 35–53, Contemp. Math. 603, American Mathematical Society, Providence, RI, 2013 Zbl 1322.46046 MR 3204025

[25] B. Long and W. Wu, Isometry groups of twisted reduced group C^*-algebras. *Internat. J. Math.* **28** (2017), no. 14, 1750101, 28 Zbl 1458.46061 MR 3741728

[26] B. Long and W. Wu, Twisted group C^*-algebras as compact quantum metric spaces. *Results Math.* **71** (2017), no. 3-4, 911–931 Zbl 1384.46049 MR 3648451

[27] B. Long and W. Wu, Twisted bounded-dilation group C^*-algebras as C^*-metric algebras. *Sci. China Math.* **64** (2021), no. 3, 547–572 Zbl 1475.46064 MR 4215999

[28] S. Lu, Constructing equivalence bimodules between noncommutative solenoids: a two-pronged approach. *J. Math. Anal. Appl.* **507** (2022), no. 2, article no. 125794 Zbl 1490.46069 MR 4335792

[29] J. Ludwig, A class of symmetric and a class of Wiener group algebras. *J. Functional Analysis* **31** (1979), no. 2, 187–194 Zbl 0402.22003 MR 0525950

[30] V. Mathai, Heat kernels and the range of the trace on completions of twisted group algebras. In *The ubiquitous heat kernel*, pp. 321–345, Contemp. Math. 398, American Mathematical Society, Providence, RI, 2006 Zbl 1122.46053 MR 2218025

[31] N. Ozawa and M. A. Rieffel, Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces. *Canad. J. Math.* **57** (2005), no. 5, 1056–1079 Zbl 1101.46047 MR 2164594

[32] M. A. Rieffel, C^*-algebras associated with irrational rotations. *Pacific J. Math.* **93** (1981), no. 2, 415–429 Zbl 0499.46039 MR 0623572

[33] M. A. Rieffel, Metrics on states from actions of compact groups. *Doc. Math.* **3** (1998), 215–229 Zbl 0993.46043 MR 1647515

[34] M. A. Rieffel, Metrics on state spaces. *Doc. Math.* **4** (1999), 559–600 Zbl 0945.46052 MR 1727499

[35] M. A. Rieffel, Group C^*-algebras as compact quantum metric spaces. *Doc. Math.* **7** (2002), 605–651 Zbl 1031.46082 MR 2015055

[36] A. M. Robert, *A course in p-adic analysis*. Grad. Texts in Math. 198, Springer, New York, 2000 Zbl 0947.11035 MR 1760253

[37] L. B. Schweitzer, A short proof that $M_n(A)$ is local if A is local and Fréchet. *Internat. J. Math.* **3** (1992), no. 4, 581–589 Zbl 0804.46054 MR 1168361

[38] G. Zeller-Meier, Produits croisés d'une C^*-algèbre par un groupe d'automorphismes. *J. Math. Pures Appl. (9)* **47** (1968), 101–239 Zbl 0165.48403 MR 0241994

Carla Farsi
Department of Mathematics, University of Colorado Boulder, Boulder, CO 80309-0395, USA;
carla.farsi@colorado.edu

Therese Landry
Department of Mathematics, University of California, Santa Barbara, Santa Barbara,
CA 93106-3080, USA; tlandry@ucsb.edu

Nadia S. Larsen
Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway;
nadiasl@math.uio.no

Judith Packer
Department of Mathematics, University of Colorado Boulder, Boulder, CO 80309-0395, USA;
packer@colorado.edu