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Approximate equivalence of representations of AH algebras into semifinite von Neumann factors

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Abstract. In this paper, we prove a non-commutative version of the Weyl–von Neumann theorem for representations of unital, separable AH algebras into countably decomposable, semifinite, properly infinite, von Neumann factors, where an AH algebra means an approximately homogeneous C^*-algebra. We also prove a result for approximate summands of representations of unital, separable AH algebras into finite von Neumann factors.

1. Introduction

1.1. Voiculescu's theorem and ideas in its proof

The classical Weyl–von Neumann theorem states that, for each bounded linear self-adjoint operator a on a separable Hilbert space $\mathcal{H}$, there is a diagonal self-adjoint operator d such that $a - d$ is of arbitrarily small Hilbert–Schmidt norm, which was proved in 1909 by Weyl in [39] with respect to compact perturbation and later was improved by von Neumann in [38] in 1935. This theorem provides important techniques in the perturbation theory for bounded linear operators on $\mathcal{H}$.

As a corollary of the main theorem in [37], Voiculescu proved that a normal operator is diagonalizable up to an arbitrarily small Hilbert–Schmidt perturbation. In [37], an important technique applied in the proof was Voiculescu's non-commutative Weyl–von Neumann theorem [36]. For convenience, we refer to this amazing theorem in [36] as *Voiculescu's theorem* in the current paper. Precisely, Voiculescu's theorem is cited as follows.

Voiculescu's Theorem ([36]). *Suppose that $\mathcal{A}$ is a separable unital C^*-algebra and $\mathcal{H}$ is a complex, separable Hilbert space. Let ϕ and ψ be unital $*$-representations of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. The following statements are equivalent:*

(1) $\phi \sim_a \psi$.

(2) $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}(\mathcal{H}))$.

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(3) $\ker\phi = \ker\psi$, $\phi^{-1}(\mathcal{K}(\mathcal{H})) = \psi^{-1}(\mathcal{K}(\mathcal{H}))$, *and the non-zero parts of the restrictions* $\phi|_{\phi^{-1}(\mathcal{K}(\mathcal{H}))}$ *and* $\psi|_{\psi^{-1}(\mathcal{K}(\mathcal{H}))}$ *are unitarily equivalent.*

In this theorem, by $\phi \sim_a \psi$, we mean the approximately unitary equivalence of ϕ and ψ; i.e., there exists a sequence of unitary operators $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathcal{H})$ such that

$$\lim_{n\to\infty} \|u_n^*\phi(a)u_n - \psi(a)\| = 0 \quad \forall\, a \in \mathcal{A},$$

where by $\|\cdot\|$ we denote the operator norm.

By $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}(\mathcal{H}))$, we mean the approximately unitary equivalence of ϕ and ψ relative to $\mathcal{K}(\mathcal{H})$; i.e., there exists a sequence of unitary operators $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathcal{H})$ such that, for each a in $\mathcal{A}$ and every $n \in \mathbb{N}$,

$$u_n^*\phi(a)u_n - \psi(a) \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad \lim_{n\to\infty} \|u_n^*\phi(a)u_n - \psi(a)\| = 0.$$

This theorem is important in both operator theory and operator algebras. Its other applications can be found in proving the eighth problem proposed by Halmos in [21], and in the proof of the well-known Brown–Douglas–Fillmore theory (see [7, Chapter IX]).

By introducing quasi-central approximate units of C^*-algebras, Arveson provided another beautiful proof of Voiculescu's theorem in [1]. Later, Hadwin [20] provided an algebraic characterization of approximate equivalence of representation. As another important application of [36], The authors of [6] characterized properly infinite injective von Neumann algebras and nuclear C^*-algebras by the uniqueness theorem.

While results related to the Weyl–von Neumann theorem being developed in $\mathcal{B}(\mathcal{H})$, several commutative versions of the Weyl–von Neumann theorem are proved in the setting of semifinite von Neumann algebras [19, 24, 25, 40]. Note that the class of semifinite von Neumann algebras is quite a large family. By the type decomposition theorem [23, Theorem 6.5.2], each semifinite von Neumann algebra can be written as a direct sum of von Neumann algebras with no direct summands of type III. In this sense, $\mathcal{B}(\mathcal{H})$ being a type I factor is a special semifinite von Neumann algebra. Thus, to consider the Weyl–von Neumann theorem in the setting of semifinite von Neumann algebras, it is natural to ask, for which class of C^*-subalgebras, Voiculescu's theorem is true.

One goal of the current paper is to set up the equivalence of (1) and (2) in Voiculescu's theorem for AH algebras in countably decomposable semifinite von Neumann algebras. Note that a type II_∞ factor is also a special semifinite von Neumann algebra, which is quite different from a type I_∞ factor. It turns out that the proof of "(1)⇔(2)" in a type I_∞ factor cannot be applied directly in a type II_∞ factor to obtain the counterpart. This means new techniques must be developed in semifinite von Neumann algebra. Before we explain the reason to choose AH algebras, we briefly recall the proof of Voiculescu's theorem in $\mathcal{B}(\mathcal{H})$.

Since (2)⇒(1) is trivial, it is sufficient to prove (1)⇒(2) to obtain the equivalence. Recall that the proof of (1)⇒(2) in $\mathcal{B}(\mathcal{H})$ involves two parts. Intuitively, to prepare for the two parts of the proof, a unital separable C^*-algebra $\mathcal{A}$ is cut into two C^*-algebras. One, denoted by $\mathcal{A}_0$, contains a $*$-subalgebra of compact operators with the identity operator $I_{\mathcal{A}_0}$ of $\mathcal{A}_0$ satisfying

$$I_{\mathcal{A}_0} = \bigvee_{a \in \mathcal{A}_0 \cap \mathcal{K}(\mathcal{H})} R(a),$$

where we denote by $R(a)$ the range projection of a. The other one, denoted by $\mathcal{A}_e$, contains no compact operators.

One part of the proof is to deal with

$$\phi \sim_a \psi \Rightarrow \phi|_{\mathcal{A}_0} \simeq \psi|_{\mathcal{A}_0}, \tag{1.1}$$

where the relation "$\phi|_{\mathcal{A}_0} \simeq \psi|_{\mathcal{A}_0}$" means that the two restrictions $\phi|_{\mathcal{A}_0}$ and $\psi|_{\mathcal{A}_0}$ are unitarily equivalent. The proof of this part depends on properties of $\mathcal{K}(\mathcal{H})$. It is worth noting that minimal projections in $\mathcal{B}(\mathcal{H})$ are compact, and they play an important role in the part of the proof.

The other part of the proof is much harder to establish the absorption theorem for $\mathcal{A}_e$. A combination of the two parts will complete the proof of (1)⇒(2).

It is more difficult to set up Voiculescu's theorem in type II_∞ factors. In [8], the authors mentioned that *in some certain type* II *factor, there exists a* C^**-algebra which is not nuclear such that* (1.1) *does not hold.*

Recently, in type II_∞ factors, we have proved the absorption theorem for separable nuclear C^*-algebras in [25, Theorem 5.3.1]. This corresponds to the part of the proof for $\mathcal{A}_e$ mentioned above. Thus, it is natural to ask whether or not the following *weak form* of (1.1) for $\mathcal{A}_0$

$$\phi \sim_a \psi \Rightarrow \phi|_{\mathcal{A}_0} \sim_{\mathcal{A}_0} \psi|_{\mathcal{A}_0} \quad \mathrm{mod}\ (\mathcal{K}(\mathcal{H})) \tag{1.2}$$

can be proved for unital separable nuclear C^*-algebras in type II_∞ factors.

1.2. The reason to choose AH algebras

In the aspect of C^*-algebras, we notice that a very rich results have been developed in Elliott's classification program during decades. It is proved that all simple separable stably finite C^*-algebras (with UCT) of finite nuclear dimension are ASH algebras (see [12, 17, 18, 35] for unital case, and [11, 14–16] for nonunital case). Thus, it seems reasonable to first consider unital ASH algebras to generalize (1.2) in type II_∞ factors. Besides, techniques about inductive limits (see Section 2.2 later) and locally homogeneous algebras are very helpful to our goal. After recalling several definitions, we explain that ASH algebras can be further replaced with AH algebras to generalize (1.2) in type II_∞ factors.

Definition 1.1. Let n be a positive integer. A C^*-algebra $\mathcal{A}$ is *n-homogeneous* if every irreducible representation of $\mathcal{A}$ is of dimension n. $\mathcal{A}$ is *n-subhomogeneous* if every irreducible representation of $\mathcal{A}$ has dimension less than or equal to n. $\mathcal{A}$ is said to be *homogeneous* (*resp.*, *subhomogeneous*) if it is n-homogeneous (resp., n-subhomogeneous) for some $n \in \mathbb{N}$. A subhomogeneous C^*-algebra is *locally homogeneous* if it is a (finite) direct sum of homogeneous C^*-algebra (see [2, Definition IV.1.4.1]).

A (not necessarily unital) C^*-algebra $\mathcal{A}$ is *approximately homogeneous*, or an AH algebra, if it is isomorphic to an inductive limit of locally homogeneous C^*-algebras. $\mathcal{A}$ is *approximately subhomogeneous*, or an ASH algebra, if it is isomorphic to an inductive limit of subhomogeneous C^*-algebras (see [2, Definition V.2.1.9]).

The standard example of an n-homogeneous C^*-algebra is $C_0(X, \mathbb{M}_n)$, where X is locally compact and $\mathbb{M}_n$ is the full matrix algebra of all $(n \times n)$-dimensional matrices over $\mathbb{C}$. Also, it is worth noting that AH and ASH algebras are usually required to be separable.

The reasons we choose AH algebras instead of ASH algebras to prove Voiculescu's theorem in type II_∞ factors are as follows:

(1) By [2, Proposition IV.1.4.6], a C^*-algebra $\mathcal{A}$ is *n-subhomogeneous* if and only if $\mathcal{A}^{**}$ is a (finite) direct sum of Type I_m von Neumann algebras for $m \leq n$. Since a locally homogeneous C^*-algebra is n-subhomogeneous for some certain $n \in \mathbb{N}$, its double dual is also a (finite) direct sum of Type I_m von Neumann algebras for $m \leq n$.

(2) We develop main techniques in a (finite) direct sum of Type I_m von Neumann algebras in the proofs. In this sense, this makes no difference between an AH algebra and an ASH algebra, since the weak-operator-closure of a locally homogeneous C^*-algebra and that of a subhomogeneous C^*-algebra are of the same class as mentioned in item (1).

Besides, there is one more thing about AH algebras to be mentioned here:

(3) The reader familiar with the classification program of C^*-algebras might doubt if a *slow-dimension-growth condition* on AH algebras is needed here. We would like to make it clear that we do not use the *slow-dimension-growth condition* on AH algebras in the current paper, since main techniques are developed in their weak-operator closures, i.e., finite direct sums of Type I_m von Neumann algebras. About the importance of AH algebras in the classification program of C^*-algebras, the reader is referred to [9, 10, 13, 26–28, 30].

Above all, the first goal in the current paper is to choose AH algebras to generalize (1.2) in type II_∞ factors.

1.3. Concepts in von Neumann algebras and main results

In the aspect of von Neumann algebras, we briefly recall several concepts. A *von Neumann algebra* is a $*$-algebra of bounded linear operators on a Hilbert space which is closed in the weak operator topology and contains the identity. A *factor* (or *von Neumann factor*) is a von Neumann algebra with trivial center. Factors are classified by Murray and von Neumann [29] into three types, i.e., type I, type II, and type III factors. A factor is called *semifinite* if it is of type I or II. This is equivalent to say that a factor equipped with a faithful, normal, semifinite, tracial weight is semifinite (see [23, Definition 7.5.1] for a weight on a C^*-algebra). A factor is called (*properly*) *infinite* if the identity is an infinite projection. The reader is referred to [2, 22, 23, 33, 34] for the theory of von Neumann algebras. Throughout this paper, let $\mathcal{H}$ be a complex, separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ be the set of all the bounded linear operators on $\mathcal{H}$. By definition, $\mathcal{B}(\mathcal{H})$ is a factor of type I. Since Voiculescu's theorem is proved for every separable C^*-algebra in type I_∞ factors, the proof of an analog of Voiculescu's theorem for AH algebras in type II_∞ factors implies that a generalized Voiculescu's theorem is true for AH algebras in properly infinite, semifinite factors.

Note that, in [20], Hadwin also proved the following result in type I factors.

Hadwin's Theorem ([20, Lemma 2.3]). *Suppose that $\mathcal{A}$ is a separable unital C^*-algebra, $\mathcal{H}_0$ and $\mathcal{H}_1$ are Hilbert spaces. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ and $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ be unital representations. The following are equivalent:*

(1) *There is a representation $\gamma : \mathcal{A} \to \mathcal{B}(\mathcal{H}_2)$ for some Hilbert space $\mathcal{H}_2$ such that*

$$\psi \oplus \gamma \sim_a \phi.$$

(2) *For every $A \in \mathcal{A}$,*

$$\operatorname{rank}(\psi(A)) \leq \operatorname{rank}(\phi(A)).$$

In [8], the authors extended some results of [20] to the case where $\mathcal{B}(\mathcal{H}_0)$ is replaced with a von Neumann algebra.

Inspired by the preceding interesting results, we focus on analogs of Voiculescu's theorem and Hadwin's theorem in the setting of semifinite factors.

This paper is organized as follows. Since factors of type II contain no minimal projections, they are quite different from factors of type I. Thus, to prove the main theorems in the current paper, we need to prepare related notation, definitions, and technical lemmas in Section 2. In particular, we introduce the strongly-approximately-unitarily equivalent $*$-homomorphisms which was first defined in [25] to extend the concept of approximately-unitarily equivalence of $*$-homomorphisms relative to $\mathcal{K}(\mathcal{H})$ in the setting of $\mathcal{B}(\mathcal{H})$. In addition, we also cite properties of AH algebras from [2, 31] in Section 2.

In Section 3, we prove an extended Voiculescu's theorem for AH algebras in semifinite, (properly) infinite von Neumann factors.

Theorem 3.11. *Let $\mathcal{M}$ be a countably decomposable, properly infinite, semifinite factor with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a separable AH subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$.*

If ϕ and ψ are unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$, then the following statements are equivalent:

(i) $\phi \sim_a \psi$, *in $\mathcal{M}$;*

(ii) $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}(\mathcal{M}, \tau))$.

The reader is referred to Definition 2.5 in Section 2 for the notation $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}(\mathcal{M}, \tau))$. This notation is a special case of $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}_\Phi(\mathcal{M}, \tau))$ which was first introduced in [25, Definition 2.2.9], where by Φ we denote $\|\cdot\|$-dominating, unitarily invariant norms.

In Section 4, we prove the following theorem for AH algebras in II_1 factors. Recall that, for an operator x in a von Neumann algebra $\mathcal{M}$, we denote by $R(x)$ the range projection of x.

Theorem 4.7. *Let $\mathcal{A}$ be a unital separable AH subalgebra in a type II_1 factor $(\mathcal{N}, \tau)$ with separable predual. Let p be a projection in $(\mathcal{N}, \tau)$. Suppose that $\pi : \mathcal{A} \to \mathcal{N}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to p\mathcal{N}p$ is a unital $*$-homomorphism such that*

$$\tau(R(\rho(a))) \leq \tau(R(\pi(a))) \quad \forall\, a \in \mathcal{A}.$$

Then, there exists a unital $*$-homomorphism $\gamma : \mathcal{A} \to p^{\perp}\mathcal{N}p^{\perp}$ such that

$$\rho \oplus \gamma \sim_a \pi \quad \text{in } \mathcal{N}.$$

At the end of this section, we raise the following question, inspired by [25, Theorem 5.3.1] and Theorem 3.11 in the current paper.

Problem 1.2. For which class of C^*-subalgebras in a type II_∞ factor with separable predual the above Theorem 3.11 is true? Is it the class of separable amenable C^*-subalgebras?

2. Preliminary

2.1. Two-sided $\|\cdot\|$-norm closed ideals of semifinite von Neumann algebras

Recall that throughout the current paper we denote by $\|\cdot\|$ the operator norm. In $\mathcal{B}(\mathcal{H})$, the ideal $\mathcal{F}(\mathcal{H})$ of finite rank operators and the $\|\cdot\|$-norm closed two-sided ideal $\mathcal{K}(\mathcal{H})$ of compact operators are both important ingredients of Voiculescu's theorem. In the following, we define analogs of $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ in semifinite von Neumann algebras. These definitions will be frequently mentioned in this paper.

Definition 2.1. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ.

Define

$$\mathcal{PF}(\mathcal{M}, \tau) = \left\{p : p = p^* = p^2 \in \mathcal{M} \text{ and } \tau(p) < \infty\right\} \tag{2.1}$$

to be the set of finite trace projections in $(\mathcal{M}, \tau)$. In terms of $\mathcal{PF}(\mathcal{M}, \tau)$, define $\mathcal{F}(\mathcal{M}, \tau)$ to be the set in the form

$$\mathcal{F}(\mathcal{M}, \tau) = \left\{xpy : p \in \mathcal{PF}(\mathcal{M}, \tau) \text{ and } x, y \in \mathcal{M}\right\}.$$

Each element in $\mathcal{F}(\mathcal{M}, \tau)$ is said to be of $(\mathcal{M}, \tau)$-finite-rank. When no confusion can arise, elements in $\mathcal{F}(\mathcal{M}, \tau)$ are called finite-rank operators. If $\mathcal{M}$ is a factor of type I on a Hilbert space $\mathcal{H}$, then $\mathcal{F}(\mathcal{M}, \tau)$ coincides with $\mathcal{F}(\mathcal{H})$.

Define $\mathcal{K}(\mathcal{M}, \tau)$ to be the $\|\cdot\|$-norm closure of $\mathcal{F}(\mathcal{M}, \tau)$ in $\mathcal{M}$. Each element in $\mathcal{K}(\mathcal{M}, \tau)$ is said to be *compact in $\mathcal{M}$*.

Remark 2.2. Recall that, for an operator x in a von Neumann algebra $\mathcal{M}$, denote by $R(x)$ the range projection of x. From [23, Proposition 6.1.6], an operator a in $\mathcal{M}$ is of $(\mathcal{M}, \tau)$-finite-rank if and only if

$$\tau(R(a)) < \infty.$$

By [23, Theorem 6.8.3], $\mathcal{K}(\mathcal{M}, \tau)$ is a $\|\cdot\|$-norm closed two-sided ideal in $\mathcal{M}$. On the other hand, we denote by $\mathcal{K}(\mathcal{M})$ the $\|\cdot\|$-norm closed ideal generated by finite projections in $\mathcal{M}$. Notice that $\mathcal{K}(\mathcal{M})$ is usually called the Breuer ideal in $\mathcal{M}$ (see [3, 4] for references about the Breuer ideal).

In general, $\mathcal{K}(\mathcal{M}, \tau)$ is a subset of $\mathcal{K}(\mathcal{M})$. That is because a finite projection might not be a $(\mathcal{M}, \tau)$-finite-rank projection. However, if $\mathcal{M}$ is a countably decomposable, semifinite factor, then [23, Proposition 8.5.2] entails that

$$\mathcal{K}(\mathcal{M}, \tau) = \mathcal{K}(\mathcal{M})$$

for a faithful, normal, semifinite tracial weight τ.

To introduce *strongly-approximately-unitarily equivalence* of two unital $*$-homomorphisms of a separable C^*-algebra $\mathcal{A}$ into $\mathcal{M}$ (*relative to* $\mathcal{K}(\mathcal{M}, \tau)$), we need to develop the following notation and definitions. These were first introduced in [25] with $\|\cdot\|$-dominating, unitarily invariant norms Φ. The reader is referred to [25, Section 2] for more details. In the current paper, we only need the $\|\cdot\|$-norm instead of unitarily invariant norms.

Suppose that $\{e_{i,j}\}_{i,j=1}^{\infty}$ is a system of matrix units for $\mathcal{B}(l^2)$. For a countably decomposable, properly infinite von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite tracial weight τ, there exists a sequence $\{v_i\}_{i=1}^{\infty}$ of partial isometries in $\mathcal{M}$ such that

$$v_i v_i^* = I_{\mathcal{M}}, \quad \sum_{i=1}^{\infty} v_i^* v_i = I_{\mathcal{M}}, \quad v_j v_i^* = 0 \quad \text{when } i \neq j. \tag{2.2}$$

Definition 2.3. For all $x \in \mathcal{M}$ and all $\sum_{i,j=1}^{\infty} x_{i,j} \otimes e_{i,j} \in \mathcal{M} \otimes \mathcal{B}(l^2)$, define

$$\phi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(l^2) \quad \text{and} \quad \psi : \mathcal{M} \otimes \mathcal{B}(l^2) \to \mathcal{M}$$

by

$$\phi(x) = \sum_{i,j=1}^{\infty} (v_i x v_j^*) \otimes e_{i,j} \quad \text{and} \quad \psi\left(\sum_{i,j=1}^{\infty} x_{i,j} \otimes e_{i,j}\right) = \sum_{i,j=1}^{\infty} v_i^* x_{i,j} v_j,$$

where $\{v_i\}_{i=1}^{\infty}$ is a sequence of partial isometries in $\mathcal{M}$ as in (2.2) and $\{e_{i,j}\}_{i,j=1}^{\infty}$ is a system of matrix units for $\mathcal{B}(l^2)$ such that $\sum_{i=1}^{\infty} e_{i,i}$ equals the identity of $\mathcal{B}(l^2)$.

We further define a mapping $\tilde{\tau} : (\mathcal{M} \otimes \mathcal{B}(l^2))_+ \to [0, \infty]$ to be

$$\tilde{\tau}(y) = \tau(\psi(y)) \quad \forall\, y \in (\mathcal{M} \otimes \mathcal{B}(l^2))_+.$$

By [25, Lemma 2.2.2], both ϕ and ψ are normal $*$-homomorphisms satisfying

$$\psi \circ \phi = \mathrm{id}_{\mathcal{M}} \quad \text{and} \quad \phi \circ \psi = \mathrm{id}_{\mathcal{M} \otimes \mathcal{B}(l^2)}.$$

The following statements are proved in [25, Lemma 2.2.4]:

(i) $\tilde{\tau}$ is a faithful, normal, semifinite tracial weight of $\mathcal{M} \otimes \mathcal{B}(l^2)$.

(ii)

$$\tilde{\tau}\left(\sum_{i,j=1}^{\infty} x_{i,j} \otimes e_{i,j}\right) = \sum_{i=1}^{\infty} \tau(x_{i,i}) \quad \text{for all } \sum_{i,j=1}^{\infty} x_{i,j} \otimes e_{i,j} \in (\mathcal{M} \otimes \mathcal{B}(l^2))_+.$$

(iii)

$$\mathcal{PF}(\mathcal{M} \otimes \mathcal{B}(l^2), \tilde{\tau}) = \phi(\mathcal{PF}(\mathcal{M}, \tau)),$$
$$\mathcal{F}(\mathcal{M} \otimes \mathcal{B}(l^2), \tilde{\tau}) = \phi(\mathcal{F}(\mathcal{M}, \tau)),$$
$$\mathcal{K}(\mathcal{M} \otimes \mathcal{B}(l^2), \tilde{\tau}) = \phi(\mathcal{K}(\mathcal{M}, \tau)).$$

Remark 2.4. Note that $\tilde{\tau}$ is a natural extension of τ from $\mathcal{M}$ to $\mathcal{M} \otimes \mathcal{B}(l^2)$. If no confusion arises, $\tilde{\tau}$ will be also denoted by τ. By [25, Proposition 2.2.9], the ideal $\mathcal{K}(\mathcal{M} \otimes \mathcal{B}(l^2), \tilde{\tau})$ is independent of the choice of the system of matrix units $\{e_{i,j}\}_{i,j=1}^{\infty}$ of $\mathcal{B}(l^2)$ and the choice of the family $\{v_i\}_{i=1}^{\infty}$ of partial isometries in $\mathcal{M}$.

Now, we are ready to introduce the definition of approximate equivalence of $*$-homomorphisms of a separable C^*-algebra into $\mathcal{M}$ relative to $\mathcal{K}(\mathcal{M}, \tau)$.

Let $\mathcal{A}$ be a separable C^*-subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$. Suppose that ρ is a positive mapping from $\mathcal{A}$ into $\mathcal{M}$ such that $\rho(I_{\mathcal{A}})$ is a projection in $\mathcal{M}$. Then, for all $0 \le x \in \mathcal{A}$, we have

$$0 \le \rho(x) \le \|x\| \rho(I_{\mathcal{A}}).$$

Therefore, it follows that

$$\rho(x)\rho(I_{\mathcal{A}}) = \rho(I_{\mathcal{A}})\rho(x) = \rho(x)$$

for all positive $x \in \mathcal{A}$. In other words, $\psi(I_{\mathcal{A}})$ can be viewed as an identity of $\psi(\mathcal{A})$. Or,

$$\psi(\mathcal{A}) \subseteq \psi(I_{\mathcal{A}})\mathcal{M}\psi(I_{\mathcal{A}}).$$

The following definition is a special case of [25, Definition 2.3.1] when the norm is fixed to be the operator norm $\|\cdot\|$.

Definition 2.5. Let $\mathcal{A}$ be a separable C^*-subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$ and $\mathcal{B}$ a $*$-subalgebra of $\mathcal{A}$ such that $I_{\mathcal{A}} \in \mathcal{B}$. Suppose that $\{e_{i,j}\}_{i,j \ge 1}$ is a system of matrix units for $\mathcal{B}(l^2)$. Let $M, N \in \mathbb{N} \cup \{\infty\}$. Suppose that $\psi_1, \dots, \psi_M$ and $\phi_1, \dots, \phi_N$ are positive mappings from $\mathcal{A}$ into $\mathcal{M}$ such that $\psi_1(I_{\mathcal{A}}), \dots, \psi_M(I_{\mathcal{A}}), \phi_1(I_{\mathcal{A}}), \dots, \phi_N(I_{\mathcal{A}})$ are projections in $\mathcal{M}$.

(a) Let $\mathcal{F} \subseteq \mathcal{A}$ be a finite subset and $\varepsilon > 0$. Then, we say that $\psi_1 \oplus \cdots \oplus \psi_M$ is $(\mathcal{F}, \varepsilon)$-strongly-approximately-unitarily-equivalent to $\phi_1 \oplus \cdots \oplus \phi_N$ over $\mathcal{B}$, denoted by

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathcal{B}}^{(\mathcal{F},\varepsilon)} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N \mod \mathcal{K}(\mathcal{M}, \tau)$$

if there exists a partial isometry v in $\mathcal{M} \otimes \mathcal{B}(l^2)$ such that

(i)

$$v^* v = \sum_{i=1}^{M} \psi_i(I_{\mathcal{A}}) \otimes e_{i,i} \quad \text{and} \quad v v^* = \sum_{i=1}^{N} \phi_i(I_{\mathcal{A}}) \otimes e_{i,i};$$

(ii)

$$\sum_{i=1}^{M} \psi_i(x) \otimes e_{i,i} - v^* \left(\sum_{i=1}^{N} \phi_i(x) \otimes e_{i,i} \right) v \in \mathcal{K}(\mathcal{M} \otimes \mathcal{B}(l^2), \tau) \quad \text{for all } x \in \mathcal{B};$$

(iii)

$$\left\| \sum_{i=1}^{M} \psi_i(x) \otimes e_{i,i} - v^* \left(\sum_{i=1}^{N} \phi_i(x) \otimes e_{i,i} \right) v \right\| < \varepsilon \quad \text{for all } x \in \mathcal{F}.$$

(b) We say that $\psi_1 \oplus \cdots \oplus \psi_M$ is strongly-approximately-unitarily-equivalent to $\phi_1 \oplus \cdots \oplus \phi_N$ over $\mathcal{B}$, denoted by

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathcal{B}} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N \mod \mathcal{K}(\mathcal{M}, \tau)$$

if, for any finite subset $\mathcal{F} \subseteq \mathcal{B}$ and $\varepsilon > 0$,

$$\psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_M \sim_{\mathcal{B}}^{(\mathcal{F},\varepsilon)} \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_N \mod \mathcal{K}(\mathcal{M}, \tau).$$

By virtue of the preceding definitions, assume that $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for a complex, separable, infinite dimensional, Hilbert space $\mathcal{H}$. Let ϕ and ψ be unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$. It follows that ϕ and ψ are strongly-approximately-unitarily equivalent over $\mathcal{A}$ if and only if ϕ and ψ are approximately-unitarily equivalent relative to $\mathcal{K}(\mathcal{H})$.

2.2. The inductive limit of C^*-algebras and properties of AH algebras

In the following, we recall the definitions of the inductive limit of C^*-algebras, AH algebras and certain useful properties.

Remark 2.6. By [31, Proposition 6.2.4], every inductive sequence of C^*-algebras

$$\mathcal{A}_1 \xrightarrow{\phi_1} \mathcal{A}_2 \xrightarrow{\phi_2} \mathcal{A}_3 \xrightarrow{\phi_3} \cdots$$

has an inductive limit $(\mathcal{A}, \{\varphi_n\}_{n\geq 1})$ which is also a C^*-algebra such that

(1) the diagram

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\phi_n} & \mathcal{A}_{n+1} \\ {\scriptstyle\varphi_n}\downarrow & \swarrow{\scriptstyle\varphi_{n+1}} & \\ \mathcal{A} & & \end{array} \tag{2.3}$$

commutes for each n in $\mathbb{N}$, where ϕ_n's and φ_n's are $*$-homomorphisms;

(2) the C^*-algebra $\mathcal{A}$ equals the norm-closure of the union of $\varphi_n(\mathcal{A}_n)$; i.e.,

$$\mathcal{A} = \overline{\bigcup_{n\geq 1} \varphi_n(\mathcal{A}_n)}^{\|\cdot\|}.$$

Note that the diagram in (2.3) implies that $\{\varphi_n(\mathcal{A}_n)\}_{n\geq 1}$ forms a monotone increasing sequence of C^*-algebras.

If this inductive limit C^*-algebra $\mathcal{A}$ is a subalgebra of $(\mathcal{M}, \tau)$ and $\mathcal{K}(\mathcal{M}, \tau)$ is as in (2.1), then [7, Lemma 3.4.1] entails that

$$\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau) = \overline{\bigcup_{n\geq 1} (\mathcal{A}_n \cap \mathcal{K}(\mathcal{M}, \tau))}^{\|\cdot\|}.$$

We recall AH and ASH algebras in Definition 1.1. By Sakai's Theorem (see [5, Theorem 1.4.1]), the double dual $\mathcal{A}^{**}$ of a C^*-algebra $\mathcal{A}$ is always viewed as the enveloping von Neumann algebra of $\mathcal{A}$. In addition, we cite a useful proposition in the following.

Proposition 2.7 ([2, Proposition III.5.2.10]). *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a bounded linear mapping between C^*-algebra, then by general considerations $\varphi^{**} : \mathcal{A}^{**} \to \mathcal{B}^{**}$ is a normal linear mapping of the same norm as φ. In addition, φ^{**} is a $*$-homomorphism if and only if φ is a $*$-homomorphism.*

As a quick application, if ϕ is a unital $*$-homomorphism of a unital locally homogeneous C^*-algebra $\mathcal{A}$ into another unital C^*-algebra $\mathcal{B}$, then $\phi(\mathcal{A})$ is also locally homogeneous.

For more details about inductive limit, the reader is referred to [34, Chapter XIV] and [31, Chapter 6]. It is convenient to assume that φ_n's are injective $*$-homomorphisms and $\{\mathcal{A}_n\}_{n\geq 1}$ is an increasing sequence of C^*-subalgebras of $\mathcal{A}$ whose union is $\|\cdot\|$-norm dense in $\mathcal{A}$.

3. Representations of AH algebras to semifinite, properly infinite factors

Let $(\mathcal{M}, \tau)$ be a countably decomposable von Neumann factor with a faithful, normal, semifinite, tracial weight τ. Recall that $\mathcal{F}(\mathcal{M}, \tau)$ is the set of all $(\mathcal{M}, \tau)$-finite-rank operators in $\mathcal{M}$ and $\mathcal{K}(\mathcal{M}, \tau)$ is the $\|\cdot\|$-norm closure of $\mathcal{F}(\mathcal{M}, \tau)$, where $\|\cdot\|$ denotes the operator norm. See Definition 2.1 in Section 2 for details.

Let $\mathcal{A}$ be a separable AH subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$. Let ϕ and ψ be unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$. The main goal of this section is to prove the equivalence of the following statements:

(1) $\phi \sim_a \psi$ in $\mathcal{M}$, i.e., ϕ and ψ are approximately unitarily equivalent in $\mathcal{M}$;

(2) $\phi \sim_{\mathcal{A}} \psi \mod (\mathcal{K}(\mathcal{M}, \tau))$ (see Definition 2.5).

In Section 1.1, we briefly recall the techniques in the proof of the equivalence of the above two statements in the setting of $\mathcal{B}(\mathcal{H})$. Based on [25, Theorem 5.3.1] in type II_∞ factors, the remaining difficulty to obtain "(1)⇒(2)" is to generalize relation (1.2).

In this section, we prove a generalization of (1.2) in Theorem 3.9. Then, we prove "(1)⇒(2)" in Theorem 3.11. In the following, we prepare several lemmas.

Notice that not each $*$-isomorphism between C^*-algebras can be extended to a WOT-WOT continuous $*$-isomorphism between the weak-operator closures of the C^*-algebras. In the following, we prove a sufficient condition for $*$-isomorphisms which can be extended to WOT-WOT continuous ones.

Lemma 3.1. *For $i = 1, 2$, let $\mathcal{M}_i$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ_i. Let $\mathcal{F}(\mathcal{M}_i, \tau_i)$ be the set of all $(\mathcal{M}_i, \tau_i)$-finite-rank operators in $(\mathcal{M}_i, \tau_i)$.*

Assume that $\mathcal{A}_i$ is a $*$-subalgebra of $\mathcal{F}(\mathcal{M}_i, \tau_i)$ such that $\mathcal{A}_i$ is weak**-dense in $\mathcal{M}_i$, for $i = 1, 2$.*

If $\rho : \mathcal{A}_1 \to \mathcal{A}_2$ is a $*$-isomorphism such that

$$\tau_2(\rho(x)) = \tau_1(x) \quad \forall\, x \in \mathcal{A}_1,$$

then ρ extends uniquely to a normal $*$-isomorphism $\rho' : \mathcal{M}_1 \to \mathcal{M}_2$ satisfying that

$$\tau_2(\rho'(x)) = \tau_1(x) \quad \textit{for each positive operator } x \in \mathcal{M}_1.$$

Proof. Since τ_i is a faithful, normal, semifinite tracial weight on $\mathcal{M}_i$ for $i = 1, 2$, then

$$(a, b) := \tau_i(b^* a) \tag{3.1}$$

defines a definite inner product on $\mathcal{A}_i$. Let $\mathcal{H}_i$ be the completion of $\mathcal{A}_i$ relative to the norm associated with the inner product defined in (3.1). From the fact that each $\mathcal{A}_i$ is weak*-dense in $\mathcal{M}_i$, it follows that $\mathcal{H}_i = L^2(\mathcal{M}_i, \tau_i)$, where $L^2(\mathcal{M}_i, \tau_i)$ is the completion of $\{x : x \in \mathcal{M}_i,\ \tau_i(x^*x) < \infty\}$ relative to the norm associated with the inner product in (3.1). By applying [23, Theorem 7.5.3], the faithful, normal tracial weight τ_i induces a faithful, normal representation π_i of $\mathcal{M}_i$ on $\mathcal{H}_i$ for $i = 1, 2$.

Let $\{a_\lambda\}_{\lambda \in \Lambda}$ be a bounded net in $\mathcal{A}_1$ such that a_λ converges to $a \in \mathcal{M}_1$ in the weak* topology. Note that, for each b in $\mathcal{A}_1$, the equality

$$(\pi_1(a_\lambda)b, b) = \tau_1(b^* a_\lambda b) = \tau_2(\rho(b^* a_\lambda b)) = (\pi_2(\rho(a_\lambda))\rho(b), \rho(b))$$

entails that $\pi_2(\rho(a_\lambda))$ converges to an operator x in $\pi_2(\mathcal{M}_2)$ in the weak operator topology. Note that π_2 is a normal $*$-isomorphism between von Neumann algebras $\mathcal{M}_2$ and $\pi_2(\mathcal{M}_2)$. It follows that $\rho(a_\lambda)$ converges to $\pi_2^{-1}(x)$ in the weak operator topology. For a, the weak* limit of a_λ, in $\mathcal{M}_1$, define $\rho'(a) := \pi_2^{-1}(x)$. It is easily verified that ρ' is well-defined. In this way, ρ extends uniquely to a normal $*$-isomorphism $\rho' : \mathcal{M}_1 \to \mathcal{M}_2$. Combining with the fact that each τ_i is normal, we can further conclude that

$$\tau_2(\rho'(x)) = \tau_1(x) \quad \text{for each positive operator } x \in \mathcal{M}_1.$$

This completes the proof. ∎

By Lemma 3.1, we can consider the normal-extensions of $*$-isomorphisms approximately equivalent to the identity mapping id.

Lemma 3.2. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{F}(\mathcal{M},\tau)$ and write $\mathcal{A}_+$ as the set of positive operators in $\mathcal{A}$.*

If ρ is a $*$-isomorphism of $\mathcal{A}$ into $\mathcal{M}$ such that ρ and the identity mapping id *of $\mathcal{A}$ are approximately equivalent in $\mathcal{M}$, written as*

$$\mathrm{id} \sim_a \rho \quad \textit{in } \mathcal{M},$$

then ρ extends uniquely to a normal $*$-isomorphism ρ' of the WOT-closure of $\mathcal{A}$ into $\mathcal{M}$ such that

$$\tau(\rho'(a)) = \tau(a) \quad \textit{for each positive operator } a \textit{ in the WOT-closure of } \mathcal{A}.$$

Proof. We will apply Lemma 3.1 to extend ρ uniquely to a von Neumann algebra isomorphism of the WOT-closure of $\mathcal{A}$ to the WOT-closure of $\rho(\mathcal{A})$ with the desired property. It is sufficient to prove the following equality:

$$\tau(a) = \tau(\rho(a)) \quad \forall\, a \in \mathcal{A}_+. \tag{3.2}$$

Recall that $\mathcal{PF}(\mathcal{M},\tau)$ is the set of projections p in $\mathcal{M}$ with $\tau(p) < \infty$. Let $a > 0$ be a $(\mathcal{M},\tau)$-finite-rank operator in $\mathcal{A}$. Note that

$$\tau(a) = \sup\big\{\tau(ap) : p \in \mathcal{PF}(\mathcal{M},\tau)\big\}.$$

We claim that $\tau(\rho(a)p) \le \tau(a)$ for each finite trace projection p in $\mathcal{M}$.

Since ρ and the identity mapping id are approximately equivalent in $\mathcal{M}$, there exists a sequence of unitary operators $\{u_k\}_{k\ge1}$ in $\mathcal{M}$ such that

$$\lim_{k\to\infty} \|\rho(a) - u_k^* a u_k\| = 0.$$

Let p be a finite trace projection in $\mathcal{M}$. Recall that $\|x\|_1 = \tau(|x|)$ for every x in $\mathcal{F}(\mathcal{M},\tau)$. In terms of the Holder inequality, we have

$$|\tau((\rho(a) - u_k^* a u_k)p)| \le \|\rho(a) - u_k^* a u_k\| \|p\|_1 \to 0 \quad \text{as } k \to \infty.$$

This implies that

$$|\tau(\rho(a)p)| = \lim_{k\to\infty} |\tau((u_k^* a u_k)p)| \le \|u_k^* a u_k\|_1 \|p\| = \tau(a).$$

This completes the proof of the claim.

Since

$$\tau(\rho(a)) = \sup\big\{\tau(\rho(a)p) : p \in \mathcal{PF}(\mathcal{M},\tau)\big\},$$

it follows that

$$\tau(\rho(a)) \leq \tau(a).$$

Similarly, we have

$$\tau(a) \leq \tau(\rho(a)).$$

Thus, we have

$$\tau(a) = \tau(\rho(a)).$$

This completes the proof of equality (3.2). Thus, by virtue of Lemma 3.1, ρ can be extended uniquely to a von Neumann algebra isomorphism ρ' of the WOT-closure of $\mathcal{A}$ to the WOT-closure of $\rho(\mathcal{A})$ with the desired property. ∎

Remark 3.3. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. Let $\mathcal{A}$ be a (*separable*) AH subalgebra of $(\mathcal{M}, \tau)$. It is convenient to assume that there exists an increasing sequence $\{\mathcal{A}_n\}_{n\geq 1}$ of locally homogeneous C^*-algebras, as in Remark 2.6, such that

$$\mathcal{A} = \overline{\bigcup_{n\geq 1} \mathcal{A}_n}^{\|\cdot\|}. \tag{3.3}$$

By applying [7, Lemma 3.4.1], we have

$$\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau) = \overline{\bigcup_{n\geq 1} (\mathcal{A}_n \cap \mathcal{K}(\mathcal{M}, \tau))}^{\|\cdot\|}.$$

Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. For a separable, unital C^*-subalgebra $\mathcal{A}$ in $(\mathcal{M}, \tau)$, the reader is referred to [19, Lemma 3.1] and [19, Lemma 3.2] for several useful properties for operators in $\mathcal{K}(\mathcal{M}, \tau)$. By a routine continuous function calculus and [19, Lemma 3.1], there exists a sequence of $(\mathcal{M}, \tau)$-finite-rank operators $\|\cdot\|$-norm dense in $\mathcal{A}_+ \cap \mathcal{K}(\mathcal{M}, \tau)$.

Define a projection

$$p_{\mathcal{K}(\mathcal{A},\tau)} := \bigvee_{x\in\mathcal{A}\cap\mathcal{K}(\mathcal{M},\tau)} R(x). \tag{3.4}$$

In the following lemma, we prove that the projection $p_{\mathcal{K}(\mathcal{A},\tau)}$ can be constructed by a sequence of positive operators in $\mathcal{A}_+ \cap \mathcal{F}(\mathcal{M}, \tau)$. Furthermore, $p_{\mathcal{K}(\mathcal{A},\tau)}$ reduces $\mathcal{A}$. By definition, it is worth noting that the projection $p_{\mathcal{K}(\mathcal{A},\tau)}$ is unique in the sense that

$$\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau) \subseteq p_{\mathcal{K}(\mathcal{A},\tau)}\mathcal{A} \quad \text{and} \quad (I - p_{\mathcal{K}(\mathcal{A},\tau)})\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau) = \{0\}.$$

Lemma 3.4. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a separable C^*-subalgebra of $\mathcal{M}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of positive, $(\mathcal{M}, \tau)$-finite-rank operators in the unit ball of $\mathcal{A}_+ \cap \mathcal{F}(\mathcal{M}, \tau)$ such that $\{x_n\}_{n=1}^{\infty}$ is $\|\cdot\|$-dense in the unit ball of $\mathcal{A}_+ \cap \mathcal{K}(\mathcal{M}, \tau)$, where $\|\cdot\|$ is the operator norm. Then, the following statements are true:*

(1) $p_{\mathcal{K}(\mathcal{A},\tau)} = \bigvee_{n\geq 1} R(x_n)$, *where $p_{\mathcal{K}(\mathcal{A},\tau)}$ is defined as in (3.4);*

(2) $p_{\mathcal{K}(\mathcal{A},\tau)}x = xp_{\mathcal{K}(\mathcal{A},\tau)}\ \forall\, x \in \mathcal{A}$.

Proof. Assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau) \neq \mathbf{0}$ and the von Neumann algebra $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$.

Let p be the union of the range projections $R(x_n)$ of all these positive, finite-rank operators x_n's. For each element x in the unit ball of $\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau)$, let $x = |x^*|v$ be the polar decomposition of x (see [23, Theorem 6.1.2]). For every $\varepsilon > 0$, there is a positive, finite-rank operator x_n such that $\||x^*| - x_n\| \leq \varepsilon$. Thus, for each unit vector ξ in $\mathcal{H}$, it follows that

$$\|x\xi - x_n v\xi\| \leq \||x^*| - x_n\| \leq \varepsilon.$$

Since $px_n v\xi = x_n v\xi$, it follows, eventually, that $px = x$ holds for every x in $\mathcal{A} \cap \mathcal{K}(\mathcal{M}, \tau)$. Thus, we have that

$$p = p_{\mathcal{K}(\mathcal{A},\tau)}.$$

In the following, assume contrarily that there exists an operator a in $\mathcal{A}$ such that $pap^\perp \neq 0$. Then, there exists a positive, $(\mathcal{M}, \tau)$-finite-rank operator a_1 in the unit ball of $\mathcal{A}_+ \cap \mathcal{F}(\mathcal{M}, \tau)$ such that $R(a_1)ap^\perp \neq 0$.

Since the restriction of each bounded linear positive operator on the closure of its range is injective, the equality

$$\ker(a_1) = \ker(a_1^{1/2})$$

entails that

$$R(a_1^{1/2})ap^\perp = R(a_1)ap^\perp \neq 0 \quad \text{and} \quad a_1^{1/2}ap^\perp = a_1^{1/2}R(a_1^{1/2})ap^\perp \neq 0.$$

Thus, $p^\perp a^* a_1 ap^\perp \neq 0$ implies $pa^* a_1 a \neq a^* a_1 a$.

Note that the inequality $\tau(R(a^* a_1 a)) < \infty$ ensures that $a^* a_1 a$ is a positive, $(\mathcal{M}, \tau)$-finite-rank operator. Then, the fact that $pa^* a_1 a \neq a^* a_1 a$ contradicts the definition of p. It follows that p reduces $\mathcal{A}$. This completes the proof. ∎

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{A}$ a C^*-subalgebra of $\mathcal{M}$. Recall that by $W^*(\mathcal{A})$ we denote the WOT-closure of $\mathcal{A}$, which is also the von Neumann algebra generated by $\mathcal{A}$. Note that $W^*(\mathcal{A}) \subseteq \mathcal{M}$. But, in general, $W^*(\mathcal{A})$ does not contain $I_\mathcal{M}$ the identity of $\mathcal{M}$.

As mentioned in Remark 3.3, a separable AH algebra is an inductive limit of locally homogeneous C^*-algebras. The following three lemmas are developed with respect to locally homogeneous C^*-algebras in $(\mathcal{M}, \tau)$, which are prepared for Theorem 3.9.

Lemma 3.5. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a (separable) locally homogeneous C^*-subalgebra of $\mathcal{M}$.*

Let a be a non-zero positive $(\mathcal{M}, \tau)$-finite-rank operator in $\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau)$. Then, there exists a central projection e of $W^*(\mathcal{A})$, the WOT-closure of $\mathcal{A}$ in $\mathcal{M}$ such that

(1) *$e \in \mathcal{F}(\mathcal{M}, \tau)$;*

(2) *$R(a) \leq e$;*

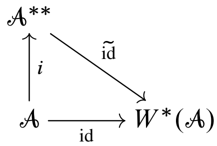
(3) *$W^*(\mathcal{A})e \subseteq W^*(\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau))$.*

Proof. By [2, Definition IV.1.4.1], a C^*-algebra $\mathcal{A}$ is locally homogeneous, if it is a finite direct sum of homogeneous C^*-algebras. For the sake of simplicity, we assume $\mathcal{A}$ to be n-homogeneous.

Claim 3.5.1. If $\mathcal{A}$ is n-homogeneous, then $W^*(\mathcal{A})$ is a type I_n von Neumann algebra.

Let $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ be the identity mapping of $\mathcal{A}$. Recall that the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ can be viewed as the enveloping von Neumann algebra of $\mathcal{A}$ by Sakai's Theorem ([5, Theorem 1.4.1]). Furthermore, by [2, Proposition IV.1.4.6], the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is a type I_n von Neumann algebra.

By applying [32, Proposition 1.21.13], the identity mapping id extends uniquely to a $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-WOT continuous $*$-homomorphism $\widetilde{\mathrm{id}}$ of $\mathcal{A}^{**}$ onto $W^*(\mathcal{A})$ as in the following commuting diagram:



It follows that $W^*(\mathcal{A})$ is a type I_n von Neumann algebra. This completes the proof of this claim.

End of the proof of Lemma 3.5. Without loss of generality, we assume that $\{e_{ij}\}_{i,j=1}^n$ is a system of matrix units for $W^*(\mathcal{A})$. Let

$$e := \bigvee_{i,j=1}^{n} R(e_{ij}a). \tag{3.5}$$

We can verify directly that $e_{ij}e = ee_{ij}e$ for all $1 \le i, j \le n$. Thus, e is a central projection of $W^*(\mathcal{A})$. Since $a \in \mathcal{F}(\mathcal{M}, \tau)$, the definition of e in (3.5) entails $e \in \mathcal{F}(\mathcal{M}, \tau)$. As $a = \sum_{i=1}^n e_{ii}a$, we have that $R(a) \le e$. From the fact that each $e_{ij}a$ belongs to $W^*(\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau))$, we further conclude that $W^*(\mathcal{A})e \subseteq W^*(\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau))$. ■

In the following lemma, we recall a well-known characterization for unitarily equivalent $*$-homomorphisms on $M_n(\mathbb{C})$. For completeness, we sketch its proof.

Lemma 3.6. *Let $\mathcal{M}$ be a von Neumann factor with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a C^*-algebra $*$-isomorphic to $M_n(\mathbb{C})$. Let ϕ and ψ be $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$ such that*

$$\tau(\phi(a)) = \tau(\psi(a)) \quad \forall\, a \in \mathcal{A}_+.$$

Then, there exists a partial isometry v in $\mathcal{M}$ such that

$$\phi(a) = v^*\psi(a)v \quad \forall\, a \in \mathcal{A}. \tag{3.6}$$

Moreover, if $\tau(\phi(a)) = \tau(\psi(a)) < \infty$, for each $a \in \mathcal{A}_+$, then there is a unitary operator u in $\mathcal{M}$ such that $\phi(a) = u^*\psi(a)u$, for every $a \in \mathcal{A}$.

Proof. Let $\{e_{ij}\}_{1\le i,j\le n}$ be a system of matrix units for $\mathcal{A}$ satisfying

(1) $e_{ij}^* = e_{ji}$ for each $i, j \in \mathbb{N}$;

(2) $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for each $i, j, k, l \in \mathbb{N}$;

(3) $\sum_{i=1}^n e_{ii} = I_{\mathcal{A}}$.

Then, $\{\phi(e_{ij})\}_{1\le i,j\le n}$ is a system of matrix units for $\phi(\mathcal{A})$. So is $\{\psi(e_{ij})\}_{1\le i,j\le n}$ for $\psi(\mathcal{A})$.

Note that

$$\tau(\phi(e_{ii})) = \tau(\psi(e_{ii})) \quad \forall\, 1 \le i \le n.$$

Since $\mathcal{M}$ is a factor and

$$\tau(\phi(e_{11})) = \tau(\psi(e_{11})),$$

there exists a partial isometry v_1 in $\mathcal{M}$ such that

$$\phi(e_{11}) = v_1^*v_1 \quad \text{and} \quad \psi(e_{11}) = v_1v_1^*.$$

Let v be defined as

$$v := \sum_{1\le i\le n} \phi(e_{i1})v_1^*\psi(e_{1i}).$$

Then, it is routine to verify that

(1) $v^*v = \psi(I_{\mathcal{A}})$ and $vv^* = \phi(I_{\mathcal{A}})$;

(2) $\phi(e_{ij})v = v\psi(e_{ij})$ for all $1 \le i, j \le n$.

This completes the proof of (3.6).

Furthermore, if $\tau(\phi(a)) = \tau(\psi(a)) < \infty$, for each $a \in \mathcal{A}_+$, then there exists a partial isometry w in $\mathcal{M}$ such that

$$w^*w = I - \psi(I_{\mathcal{A}}) \quad \text{and} \quad ww^* = I - \phi(I_{\mathcal{A}}).$$

Define $u = v + w$. It follows that u is a unitary operator in $\mathcal{M}$ as desired. ∎

The following technical lemma will be used in Theorem 3.9 to cut an operator into $(\mathcal{M}, \tau)$-finite-rank direct summands. Then, each summand is replaced with a finite direct sum of matrices up to an arbitrarily small perturbation.

Lemma 3.7. *Let $\mathcal{M}$ be a von Neumann factor with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a (separable) locally homogeneous C^*-subalgebra of $(\mathcal{M}, \tau)$. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ and e be a projection in $W^*(\mathcal{A})$ satisfying $ae = ea$ for each a in $\mathcal{F}$.*

For each $\varepsilon > 0$, there is a finite dimensional von Neumann subalgebra $\mathcal{B}$ of $W^*(\mathcal{A})$ with e being the identity such that for each a in $\mathcal{F}$, there exists an element b in $\mathcal{B}$ satisfying

$$\|ae - b\| < \varepsilon.$$

Proof. First, we prove a claim in type I_n von Neumann algebras. Then, we apply the claim to prove the lemma.

Claim 3.7.1. Suppose that $\mathcal{M}_n$ is a type I_n von Neumann algebra. For $x \in \mathcal{M}_n$ and $\varepsilon > 0$, there is a finite dimensional von Neumann subalgebra $\mathcal{B}_1$ of $\mathcal{M}_n$ and an operator $y \in \mathcal{B}_1$ such that $\|x - y\| < \varepsilon$.

Let $\{p_{ij}\}_{1\le i,j\le n}$ be a system of matrix units for $\mathcal{M}_n$. Write $\mathcal{P}$ to be the von Neumann subalgebra generated by $\{p_{ij}\}_{1\le i,j\le n}$. Thus, $\mathcal{P}$ is $*$-isomorphic to $\mathbb{M}_n(\mathbb{C})$. Define $\mathcal{N} := \mathcal{P}' \cap \mathcal{M}_n$. Note that $\mathcal{N}$ is the center of $\mathcal{M}_n$. It follows that $\mathcal{M}_n = W^*(\mathcal{P} \cup \mathcal{N})$ and $\mathcal{M}_n$ is $*$-isomorphic to the von Neumann algebra tensor product $\mathcal{P} \otimes \mathcal{N}$.

For each x in $\mathcal{M}_n$, there exist $\{x_{ij}\}_{1\le i,j\le n} \subset \mathcal{N}$ such that $x = \sum_{1\le i,j\le n} x_{ij} p_{ij}$. Note that each x_{ij} is normal. For every $\varepsilon > 0$, there exist $\lambda_1^{ij}, \ldots, \lambda_{k_{ij}}^{ij}$ in $\mathbb{C}$ and $e_1^{ij}, \ldots, e_{k_{ij}}^{ij}$ in $\mathcal{N}$ satisfying

$$\sum_{1\le l\le k_{ij}} e_l^{ij} = I$$

such that

$$\Big\| x_{ij} p_{ij} - \sum_{1\le l\le k_{ij}} \lambda_l^{ij} e_l^{ij} p_{ij} \Big\| \le \varepsilon/n^2. \tag{3.7}$$

With respect to $\{e_l^{ij}\}_{1\le i,j\le n;\ 1\le l\le k_{ij}}$, there exists a finer central partition of I, denoted by $\{f_l\}_{1\le l\le m}$ (satisfying $\sum_{1\le l\le m} f_l = I$ and $f_l \in \mathcal{N}$ for all $1 \le l \le m$) such that

$$f_l e_t^{ij} = f_l \quad \text{or} \quad f_l e_t^{ij} = 0 \quad \forall\, 1 \le i,\ j \le n;\ 1 \le t \le k_{ij}.$$

For a fixed f_l, if $f_l e_t^{ij} = f_l$, then we rewrite λ_t^{ij} as λ_l^{ij}.

By using (3.7), we have

$$\Big\| x - \sum_{1\le l\le m} \sum_{1\le i,j\le n} \lambda_l^{ij} f_l p_{ij} \Big\| \le \varepsilon. \tag{3.8}$$

Note that the von Neumann algebra generated by $\{f_l p_{ij}\}_{1\le i,j\le n}$ is $\mathcal{P} f_l$, which is also $*$-isomorphic to $\mathbb{M}_n(\mathbb{C})$. Since $\sum_{1\le i,j\le n} \lambda_l^{ij} f_l p_{ij}$ is an element in $\mathcal{P} f_l$, we can identify it as an $n \times n$ matrix in $\mathbb{M}_n(\mathbb{C})$. Write

$$\mathcal{B}_1 = \mathcal{P} f_1 \oplus \cdots \oplus \mathcal{P} f_m \quad \text{and} \quad y = \sum_{1\le l\le m} \sum_{1\le i,j\le n} \lambda_l^{ij} f_l p_{ij}.$$

Therefore, we complete the proof of the claim by (3.8).

End of the proof of Lemma 3.7. By definition, $\mathcal{A}$ is a direct sum of finitely many homogeneous C^*-algebras. By Claim 3.5.1, $W^*(\mathcal{A})$ is a direct sum of finitely many type I_n von Neumann algebras, where these finitely many type I_n von Neumann algebras are different in general. Thus, the projection e can be written as $e = e_1 + \cdots + e_k$ such that $e_j W^*(\mathcal{A}) e_j$ is a type I_{n_j} von Neumann algebra for $j = 1, \ldots, k$. In fact, each e_j is the product of e and some certain central projection of $W^*(\mathcal{A})$.

Since $ae = ea$ for each a in $\mathcal{F}$, each ae_j belongs to $e_j W^*(\mathcal{A}) e_j$. Since $\mathcal{F}$ is a finite set, by applying Claim 3.7.1 repeatedly for finitely many times, in each $e_j W^*(\mathcal{A}) e_j$, there is a finite dimensional von Neumann subalgebra $\mathcal{B}_j$ and an element $b_j \in \mathcal{B}_j$ corresponding to each $a \in \mathcal{F}$ such that

$$\|ae_j - b_j\| < \varepsilon/k.$$

Write $\mathcal{B} = \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_k$ and $b = b_1 \oplus \cdots \oplus b_k$. Thus, we obtain $\|ae - b\| < \varepsilon$ for each $a \in \mathcal{F}$. This completes the proof. ∎

Definition 3.8. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite, tracial weight τ. Suppose that $\mathcal{A}$ is a separable C^*-subalgebra of $\mathcal{M}$.

By virtue of Lemma 3.4, the projection $p_{\mathcal{K}(\mathcal{A},\tau)}$ reduces $\mathcal{A}$. Define

$$\mathrm{id}_0(a) := ap_{\mathcal{K}(\mathcal{A},\tau)} \quad \text{and} \quad \mathrm{id}_e(a) := ap^{\perp}_{\mathcal{K}(\mathcal{A},\tau)} \quad \forall\, a \in \mathcal{A}.$$

Then, id_0 and id_e are well-defined $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{A}p_{\mathcal{K}(\mathcal{A},\tau)}$ and $\mathcal{A}p^{\perp}_{\mathcal{K}(\mathcal{A},\tau)}$, respectively.

Let ρ be a unital $*$-isomorphism of $\mathcal{A}$ into $\mathcal{M}$. Define

$$\rho_0(a) := \mathrm{id}_0(\rho(a)) \quad \text{and} \quad \rho_e(a) := \mathrm{id}_e(\rho(a)) \quad \forall\, a \in \mathcal{A}.$$

Then, ρ_0 and ρ_e are well-defined $*$-homomorphisms of $\mathcal{A}$ into $\rho(\mathcal{A})p_{\mathcal{K}(\rho(\mathcal{A}),\tau)}$ and $\rho(\mathcal{A})$ $p^{\perp}_{\mathcal{K}(\rho(\mathcal{A}),\tau)}$, respectively.

Theorem 3.9. *Let $\mathcal{M}$ be a countably decomposable, properly infinite, semifinite von Neumann factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is a separable AH subalgebra of $(\mathcal{M},\tau)$. Let* id *and ρ be unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$ such that* id *and ρ are approximately-unitarily equivalent.*

Then, id_0 *and ρ_0, as in Definition 3.8, are strongly-approximately-unitarily-equivalent over $\mathcal{A}$, as in Definition 2.5; i.e.,*

$$\mathrm{id}_0 \sim_{\mathcal{A}} \rho_0 \quad \mathrm{mod}\ (\mathcal{K}(\mathcal{M},\tau)). \tag{3.9}$$

Proof. Since $\mathcal{A}$ is AH, as in Remark 3.3, it is convenient to assume that there is a monotone increasing sequence of locally homogeneous C^*-subalgebras $\{\mathcal{A}_n\}_{n\geq 1}$ of $(\mathcal{M},\tau)$ such that

$$\mathcal{A} = \overline{\bigcup_{n\geq 1} \mathcal{A}_n}^{\|\cdot\|}.$$

We assume that $\mathcal{A} \cap \mathcal{K}(\mathcal{M},\tau) \neq \mathbf{0}$.

Let $\{x_n\}_{n\geq 1}$ be a sequence of positive, $(\mathcal{M},\tau)$-finite-rank operators in the unit ball of $\bigcup_{n\geq 1} \mathcal{A}_n$, which is $\|\cdot\|$-norm dense in the unit ball of $\mathcal{A}_+ \cap \mathcal{K}(\mathcal{M},\tau)$. Define two projections p and q as follows:

$$p := \bigvee_{n\geq 1} R(x_n) \quad \text{and} \quad q := \bigvee_{n\geq 1} R(\rho(x_n)). \tag{3.10}$$

Then, by virtue of Lemmas 3.2 and 3.4, we obtain that

$$p = p_{\mathcal{K}(\mathcal{A},\tau)} \quad \text{and} \quad q = p_{\mathcal{K}(\rho(\mathcal{A}),\tau)}.$$

Note that, by applying Lemma 3.2, the unital $*$-homomorphism ρ extends uniquely to a normal $*$-isomorphism ρ' of the WOT-closure of $\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau)$ into $\mathcal{M}$ such that

$$\tau(\rho'(a)) = \tau(a) \quad \forall\, a \in W^*\big(\mathcal{A} \cap \mathcal{F}(\mathcal{M}, \tau)\big)_+.$$

By the WOT-continuity of ρ' and (3.10), we obtain that

$$\rho'(p) = q. \tag{3.11}$$

Next, we cut p with respect to $\{x_n\}_{n\geq 1}$. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be a monotone increasing sequence of finite subsets of the unit ball of $\bigcup_{n\geq 1} \mathcal{A}_n$ such that $\bigcup_{n\geq 1} \mathcal{F}_n$ is $\|\cdot\|$-norm dense in the unit ball of $\mathcal{A}$.

In the following, we construct at most countably many, mutually orthogonal, $(\mathcal{M}, \tau)$-finite-rank projections in $W^*(\mathcal{A})$ with nice properties.

Choose $n_1 \geq 1$ such that $\mathcal{F}_1 \cup \{x_1\} \subset \mathcal{A}_{n_1}$. Since $\mathcal{A}_{n_1}$ is locally homogeneous, by virtue of Lemma 3.5, there is a central projection p_1 in $W^*(\mathcal{A}_{n_1})$ such that

(1) p_1 belongs to $\mathcal{F}(\mathcal{M}, \tau)$;

(2) $R(x_1) \leq p_1$ in $\mathcal{M}$;

(3) $W^*(\mathcal{A}_{n_1}) p_1 \subseteq W^*(\mathcal{A}_{n_1} \cap \mathcal{F}(\mathcal{M}, \tau))$.

Define $y_1 := x_1$. If $p_1 = p$, then we complete the construction. Otherwise, suppose $k \geq 2$, and we obtain $y_1, \ldots, y_k$ in $\{x_n\}_{n\geq 1}$ and $p_1, \ldots, p_k$ in $\mathcal{F}(\mathcal{M}, \tau)$ satisfying

(1) y_{i+1} is the first element after y_i in $\{x_n\}_{n\geq 1}$ such that $(p - p_i) y_{i+1} \neq 0$ for $1 \leq i \leq k-1$;

(2) $\mathcal{F}_{i+1} \cup \{y_1, \ldots, y_{i+1}\} \subset \mathcal{A}_{n_{i+1}}$ for all $1 \leq i \leq k-1$;

(3) $\mathcal{F}_{i+1} \cup \{y_1, \ldots, y_{i+1}, p_1, \ldots, p_i\} \subset W^*(\mathcal{A}_{n_{i+1}})$ and $n_i \leq n_{i+1}$ for all $1 \leq i \leq k-1$;

(4) p_{i+1} is a central projection in $W^*(\mathcal{A}_{n_{i+1}})$ such that

$$p_i \vee R(y_{i+1}) \leq p_{i+1}, \quad 1 \leq i \leq k-1.$$

If $p_k = p$, then we complete the construction. Otherwise, let y_{k+1} be the first element after y_k in $\{x_n\}_{n\geq 1}$ such that $(p - p_k) y_{k+1} \neq 0$. Choose $n_{k+1} \geq n_k$ such that

$$\mathcal{F}_{k+1} \cup \{y_1, \ldots, y_{k+1}\} \subset \mathcal{A}_{n_{k+1}}.$$

Note that the projections $p_1, \ldots, p_k$ are also in $W^*(\mathcal{A}_{n_{k+1}})$. In terms of Lemma 3.5, there is a central projection p_{k+1} of $W^*(\mathcal{A}_{n_{k+1}})$ such that

(1′) p_{k+1} belongs to $\mathcal{F}(\mathcal{M}, \tau)$;

(2′) $p_k \vee R(y_{k+1}) \leq p_{k+1}$ in $W^*(\mathcal{A}_{n_{k+1}})$;

(3′) $W^*(\mathcal{A}_{n_{k+1}}) p_{k+1} \subseteq W^*(\mathcal{A}_{n_{k+1}} \cap \mathcal{F}(\mathcal{M}, \tau))$.

Define

$$e_1 := p_1,$$
$$e_{k+1} := p_{k+1} - p_k$$

for each $k \geq 1$. Let $\mathcal{F}_0 := \mathcal{F}_1$. It follows that

$$ae_i = e_i a$$

for each a in $\mathcal{F}_j$, and $i = j+1, \ldots, k+1$, where $j \geq 0$.

Recursively, we obtain a sequence of at most countably many, mutually orthogonal, $(\mathcal{M}, \tau)$-finite-rank projections $\{e_i\}_{1\leq i\leq N}$ in $\mathcal{F}(\mathcal{M}, \tau)$ such that

$$\text{SOT-}\sum_{1\leq i\leq N} e_i = p,$$

where $N \in \mathbb{N} \cup \{\infty\}$.

By applying (3.11) and the preceding arguments, we have

$$\text{SOT-}\sum_{1\leq i\leq N} \rho'(e_i) = q.$$

And, for each projection e in $W^*(\mathcal{A}_{n_i})p_i$, we have

$$\tau(e) = \tau(\rho'(e)) < \infty.$$

Fix $\varepsilon > 0$. For each $i \geq 1$, in terms of Lemma 3.7, there exists a finite dimensional von Neumann algebra $\mathcal{B}_i$ containing e_i as its identity in $W^*(\mathcal{A}_{n_i})p_i$ such that, for each $a \in \mathcal{F}_{i-1}$, there is an operator $a_i \in \mathcal{B}_i$ satisfying

$$\|ae_i - a_i\| < \frac{\varepsilon}{2^{i+1}}.$$

Then, by virtue of Lemma 3.6, we obtain a partial isometry u_i in $\mathcal{M}$ for $1 \leq i \leq N$ such that

$$e_i = u_i^* u_i, \quad \rho'(e_i) = u_i u_i^*, \quad u_i b u_i^* = \rho'(b) \quad \forall\, b \in \mathcal{B}_i.$$

It follows that, for each $a \in \mathcal{F}_{i-1}$ and $i \geq 1$,

$$\|u_i(ae_i)u_i^* - \rho'(ae_i)\| \leq \|u_i(ae_i - a_i)u_i^*\| + \|\rho'(a_i - ae_i)\| < \frac{\varepsilon}{2^i}. \tag{3.12}$$

Define

$$u = \sum_{1\leq i\leq N} u_i$$

in $\mathcal{M}$. It follows that

$$u^*u = p \quad \text{and} \quad q = uu^*. \tag{3.13}$$

Note that, for each a in $\bigcup_{i\geq 1}\mathcal{F}_i$,

$$\mathrm{id}_0(a) = \text{SOT-}\sum_{1\leq i\leq N} ae_i,$$

$$\rho_0(a) = \text{SOT-}\sum_{1\leq i\leq N} \rho'(ae_i).$$

By (3.12) and (3.13), we have

(1) for every a in $\mathcal{F}_0$, we have $ae_i = e_i a$ for each e_i, and

$$\begin{aligned}\|u\,\mathrm{id}_0(a)u^* - \rho_0(a)\| &= \left\| \Big(\sum_{1\leq i\leq N} u_i\Big)\,\mathrm{id}_0(a)\Big(\sum_{1\leq i\leq N} u_i\Big)^* - \rho_0(a)\right\| \\ &= \left\| \sum_{1\leq i,k,l\leq N} u_l e_i a e_i u_k^* - \rho_0(a)\right\| \\ &= \left\| \sum_{1\leq i\leq N} \big(u_i e_i a e_i u_i^* - \rho'(ae_i)\big)\right\| \\ &\leq \sum_{1\leq i\leq N} \|u_i e_i a e_i u_i^* - \rho'(ae_i)\| < \varepsilon.\end{aligned}$$

(2) for every a in $\bigcup_{i\geq 1}\mathcal{F}_i$, there is an $i_0 \geq 1$ such that $a \in \mathcal{F}_{i_0}$ and $ae_i = e_i a$ for each $i > i_0$. Thus, a similar computation implies that

$$\|u\,\mathrm{id}_0(a)u^* - \rho_0(a)\| < \infty \quad \text{and} \quad u\,\mathrm{id}_0(a)u^* - \rho_0(a) \in \mathcal{K}(\mathcal{M},\tau).$$

For each $j \geq 1$, define

$$\{\mathcal{E}_i = \mathcal{F}_{i+j-1}\}_{i\geq 1} \quad \text{and} \quad \mathcal{E}_0 := \mathcal{E}_1.$$

We can iterate the preceding arguments to construct a partial isometry v_j in $\mathcal{M}$ with $\{\mathcal{E}_i\}_{i\geq 0}$ in place of $\{\mathcal{F}_i\}_{i\geq 0}$ such that

(1) for every a in $\bigcup_{i\geq 1}\mathcal{F}_i$ and $j \geq 1$,

$$\|v_j\,\mathrm{id}_0(a)v_j^* - \rho_0(a)\| < \infty \quad \text{and} \quad v_j\,\mathrm{id}_0(a)v_j^* - \rho_0(a) \in \mathcal{K}(\mathcal{M},\tau);$$

(2) for each a in $\mathcal{F}_j$,

$$\|v_j\,\mathrm{id}_0(a)v_j^* - \rho_0(a)\| < \frac{1}{2^j}.$$

Note that $\bigcup_{i\geq 1}\mathcal{F}_i$ is $\|\cdot\|$-norm dense in the unit ball of $\mathcal{A}$. Thus, for each a in $\mathcal{A}$, we obtain that

$$\lim_{j\to\infty} \|v_j^*\,\mathrm{id}_0(a)v_j - \rho_0(a)\| = 0.$$

This completes the proof of (3.9). ∎

We cite [25, Theorem 5.3.1] as another important tool. Note that, in the remainder, the symbol "$\sim_{\mathcal{A}}$" follows from Definition 2.5.

Theorem 3.10. *Let $\mathcal{M}$ be a countably decomposable, properly infinite, semifinite factor with a faithful, normal, semifinite, tracial weight τ. Let $\mathcal{K}(\mathcal{M},\tau)$ be the set of compact operators in $(\mathcal{M},\tau)$. Suppose that $\mathcal{A}$ is a separable nuclear C^*-subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$. If $\rho:\mathcal{A}\to\mathcal{M}$ is a $*$-homomorphism satisfying*

$$\rho(\mathcal{A}\cap\mathcal{K}(\mathcal{M},\tau))=\mathbf{0},$$

then

$$\mathrm{id}_{\mathcal{A}}\sim_{\mathcal{A}}\mathrm{id}_{\mathcal{A}}\oplus\rho \mod (\mathcal{K}(\mathcal{M},\tau)).$$

The following theorem is the main result in this section.

Theorem 3.11. *Let $\mathcal{M}$ be a countably decomposable, infinite, semifinite factor with a faithful normal semifinite tracial weight τ. Suppose that $\mathcal{A}$ is a separable AH subalgebra of $\mathcal{M}$ with an identity $I_{\mathcal{A}}$.*

If ϕ and ψ are unital $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{M}$, then the following statements are equivalent:

(i) $\phi\sim_a\psi$ *in $\mathcal{M}$;*

(ii) $\phi\sim_{\mathcal{A}}\psi \mod \mathcal{K}(\mathcal{M},\tau)$.

Proof. Note that the implication (ii)⇒(i) is easy by Definition 2.5. Thus, we only need to prove the implication (i)⇒(ii).

The assumption $\phi\sim_a\psi$ in $\mathcal{M}$ entails that ϕ and ψ have the same kernel. It follows that the mapping

$$\rho:\phi(\mathcal{A})\to\psi(\mathcal{A}),$$

defined by

$$\rho(b):=\psi(\phi^{-1}(b)) \quad \forall\, b\in\phi(\mathcal{A}),$$

is a well-defined $*$-isomorphism of $\phi(\mathcal{A})$ onto $\psi(\mathcal{A})$. Moreover, the following are equivalent:

(1) $\phi\sim_{\mathcal{A}}\psi \mod (\mathcal{K}(\mathcal{M},\tau))$;

(2) $\mathrm{id}_{\phi(\mathcal{A})}\sim_{\phi(\mathcal{A})}\rho \mod (\mathcal{K}(\mathcal{M},\tau))$.

In terms of Lemma 3.2, the restriction of ρ on $\phi(\mathcal{A})\cap\mathcal{F}(\mathcal{M},\tau)$ extends uniquely to a normal $*$-isomorphism of the WOT-closure of $\phi(\mathcal{A})\cap\mathcal{F}(\mathcal{M},\tau)$ into $\mathcal{M}$. Furthermore, Lemmas 3.2 and 3.4 guarantee that there exists a sequence $\{x_n\}_{n\ge1}$ of positive, $(\mathcal{M},\tau)$-finite-rank operators in the unit ball of $\phi(\mathcal{A})_+\cap\mathcal{F}(\mathcal{M},\tau)$ such that the projections

$$p:=\bigvee_{n\ge1}R(x_n),$$

$$q:=\bigvee_{n\ge1}R(\rho(x_n))$$

reduce $\phi(\mathcal{A})$ and $\psi(\mathcal{A})$, respectively. Moreover, we have that

$$px = x, \quad q\rho(x) = \rho(x) \quad \forall\, x \in \phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau).$$

Thus, the identity mapping id on $\phi(\mathcal{A})$ can be expressed in the form

$$\mathrm{id} = \mathrm{id}_0 \oplus \mathrm{id}_e, \tag{3.14}$$

where id_0 is the compression of $\mathrm{id}(\cdot)p$ on ran p, and id_e is the compression of $\mathrm{id}(\cdot)p^{\perp}$ on ran $p^{\perp}$. We also write that

$$\mathrm{id}_0(\phi(\mathcal{A})) = \phi_0(\mathcal{A}) \quad \text{and} \quad \mathrm{id}_e(\phi(\mathcal{A})) = \phi_e(\mathcal{A}).$$

It follows that $\mathrm{id}_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau)) = \mathbf{0}$.

Likewise, the $*$-isomorphism ρ of $\phi(\mathcal{A})$ can be expressed in the form

$$\rho = \rho_0 \oplus \rho_e, \tag{3.15}$$

where $\rho_0(A) = \rho(A)q|_{\operatorname{ran} q}$ and $\rho_e(A) = \rho(A)q^{\perp}|_{\operatorname{ran} q^{\perp}}$ for every a in $\phi(\mathcal{A})$. We also write that

$$\rho_0(\phi(\mathcal{A})) = \psi_0(\mathcal{A}) \quad \text{and} \quad \rho_e(\phi(\mathcal{A})) = \psi_e(\mathcal{A}).$$

It follows that

$$\rho_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau)) = \mathbf{0}.$$

By virtue of Theorem 3.9, there exists a partial isometry w in $\mathcal{M}$ such that

$$p = w^*w \quad \text{and} \quad q = ww^*.$$

It is worth noting that, in general, many operators in $\phi_0(\mathcal{A})$ do not belong to $\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau)$. This is a motivation to develop Theorem 3.9.

By virtue of (3.3), there exists a monotone increasing sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ of finite subsets of the unit ball of $\bigcup_{k\ge1} \mathcal{A}_k$ such that $\bigcup_{k\ge1} \mathcal{F}_k$ is $\|\cdot\|$-norm dense in the unit ball of $\mathcal{A}$. Likewise, the union $\bigcup_{k\ge1} \phi(\mathcal{F}_k)$ (resp., $\bigcup_{k\ge1} \psi(\mathcal{F}_k)$) is $\|\cdot\|$-norm dense in the unit ball of $\phi(\mathcal{A})$ (resp., $\psi(\mathcal{A})$). Similarly, $\bigcup_{k\ge1} \phi_0(\mathcal{F}_k)$ (resp., $\bigcup_{k\ge1} \psi_0(\mathcal{F}_k)$) is $\|\cdot\|$-norm dense in the unit ball of $\phi_0(\mathcal{A})$ (resp., $\psi_0(\mathcal{A})$).

By applying Theorem 3.9, for every $k \ge 1$, there exists a partial isometry v_k in $(\mathcal{M}, \tau)$ such that the inequality

$$\|v_k\phi_0(a)v_k^* - \psi_0(a)\| < \frac{1}{2^k}$$

holds for every a in $\mathcal{F}_k$.

Furthermore, for every a in $\mathcal{A}$, we have that $v_k\phi_0(a)v_k^* - \psi_0(a)$ belongs to the ideal $\mathcal{K}(\mathcal{M}, \tau)$. Therefore, there exists a sequence $\{v_k\}_{k\ge1}$ of partial isometries in $\mathcal{M}$ such that

(1) $\lim_{k\to\infty}\|v_k\phi_0(a)v_k^* - \psi_0(a)\| = 0$ for every a in $\mathcal{A}$;

(2) $v_k\phi_0(a)v_k^* - \psi_0(a)$ belongs to $\mathcal{K}(\mathcal{M}, \tau)$ for every a in $\mathcal{A}$ and $k \ge 1$.

Notice that

$$\mathrm{id}_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau)) = \rho_e(\phi(\mathcal{A}) \cap \mathcal{K}(\mathcal{M}, \tau)) = \mathbf{0}.$$

Thus, by applying Theorems 3.10 and 3.9 and the decompositions in (3.14) and (3.15), it follows that

$$\begin{aligned} \phi = (\mathrm{id}_0 \circ\phi) \oplus (\mathrm{id}_e \circ\phi) &\sim_{\mathcal{A}} (\mathrm{id}_0 \circ\phi) \oplus (\mathrm{id}_e \circ\phi) \oplus (\rho_e \circ \phi) && \mathrm{mod}\ (\mathcal{K}(\mathcal{M}, \tau)) \\ &= \phi_0 \oplus \phi_e \oplus \psi_e && \\ &\sim_{\mathcal{A}} \psi_0 \oplus \psi_e \oplus \phi_e && \mathrm{mod}\ (\mathcal{K}(\mathcal{M}, \tau)) \\ &= (\rho_0 \circ \phi) \oplus (\rho_e \circ \phi) \oplus (\mathrm{id}_e \circ\phi) && \\ &= (\rho \circ \phi) \oplus (\mathrm{id}_e \circ\phi) \sim_{\mathcal{A}} (\rho \circ \phi) = \psi && \mathrm{mod}\ (\mathcal{K}(\mathcal{M}, \tau)) \end{aligned}$$

This completes the proof. ■

4. Representations of AH algebras to type II_1 factors

In this section, we always assume that $(\mathcal{N}, \tau)$ is a type II_1 factor *with separable predual*, where τ is the faithful, normal, tracial state on $\mathcal{N}$. For two $*$-homomorphisms ρ and π of a unital C^*-algebra $\mathcal{A}$ into $\mathcal{N}$, if there is a unitary operator u in $\mathcal{N}$ such that the equality

$$u^* \rho(a) u = \pi(a)$$

holds for every a in $\mathcal{A}$, then ρ and π are *unitarily equivalent* (denoted by $\rho \simeq \pi$ in $\mathcal{N}$). Let $\mathcal{A}_+$ denote the set of positive elements of $\mathcal{A}$.

It is worth noting that, in the setting of finite factors, Voiculescu's theorem is automatically true. The reason is that, for a finite factor $(\mathcal{M}, \tau)$, $\mathcal{M} = \mathcal{K}(\mathcal{M}, \tau)$. Thus, for a separable C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$ and two unital $*$-homomorphisms ϕ and ψ of $\mathcal{A}$ into $\mathcal{M}$, the following relation is naturally true:

$$\phi \sim_a \psi \Longleftrightarrow \phi \sim_{\mathcal{A}} \psi \quad \mathrm{mod}\ (\mathcal{K}(\mathcal{M}, \tau)).$$

Furthermore, suppose that ϕ is *tracially weaker* than ψ which means

$$\tau\big(R(\phi(a))\big) \leq \tau\big(R(\psi(a))\big) \quad \forall\, a \in \mathcal{A}. \tag{4.1}$$

It is interesting to ask can ψ be approximately decomposed with respect to ϕ which means

$$\phi \oplus \gamma \sim_a \psi, \tag{4.2}$$

where γ is another $*$-homomorphism of $\mathcal{A}$ into $\mathcal{M}$.

In Theorem 4.7 of the current paper, we prove that (4.2) is true for $*$-homomorphisms satisfying (4.1) of AH algebras into a type II_1 factor $(\mathcal{N}, \tau)$.

The following Lemmas 4.1 and 4.2 are prepared for Lemma 4.3, where we extend π and ρ such that the inequality (4.1) holds for each projection in $\mathcal{A}^{**}$ the double dual of $\mathcal{A}$.

Lemma 4.1. *Let $C(X)$ be a unital, separable, abelian C^*-algebra with X a compact metric space. Suppose that p is a projection in a type II_1 factor $(\mathcal{N}, \tau)$.*

If $\pi : C(X) \to \mathcal{N}$ is a unital $*$-homomorphism and $\rho : C(X) \to p\mathcal{N}p$ is a unital $*$-homomorphism such that

$$\tau\big(R(\rho(f))\big) \le \tau\big(R(\pi(f))\big) \quad \forall\, f \in C(X), \tag{4.3}$$

then, for every positive function h in $C(X)$,

$$\tau(\rho(h)) \le \tau(\pi(h)). \tag{4.4}$$

Proof. By applying [7, Theorem II.2.5], there are regular Borel measures μ_ρ and μ_π on X such that ρ (resp., π) extends to a weak*-WOT continuous $*$-isomorphism $\tilde{\rho}$ (resp., $\tilde{\pi}$) of $L^\infty(\mu_\rho)$ (resp., $L^\infty(\mu_\pi)$) onto $\rho(C(X))''$ (resp., $\pi(C(X))''$).

Let Δ be a Borel subset of X and χ_Δ be the characteristic function on Δ. Note that, for each regular Borel measure μ on X, every μ-measurable set is a disjoint union of a Borel set and a set of μ-measure 0. Thus, we only need to concentrate on χ_Δ for every Borel subset Δ of X instead of considering measurable subsets.

If Δ is a non-empty open subset of X, then there exists a positive function f in $C(X)_+$ such that $f(\lambda) \ne 0$ for $\lambda \in \Delta$ and $f(\lambda) = 0$ for $\lambda \in X \backslash \Delta$. The weak*-WOT continuity of $\tilde{\rho}$ entails that

$$R(\rho(f)) = \text{WOT-}\lim_{n\to\infty} \rho(f)^{\frac{1}{n}} = \text{WOT-}\lim_{n\to\infty} \tilde{\rho}(f^{\frac{1}{n}}) = \tilde{\rho}(\chi_\Delta).$$

Thus, the hypothesis in (4.3) implies that the inequality

$$\tau(\tilde{\rho}(\chi_\Delta)) \le \tau(\tilde{\pi}(\chi_\Delta)) \tag{4.5}$$

holds for each open subset Δ of X.

If Δ is a Borel subset of X, then all the open subsets O_α's of X with $\Delta \subseteq O_\alpha$, form a net with respect to each regular Borel measure μ; i.e.,

$$\mu(\Delta) = \lim_\alpha \mu(O_\alpha).$$

Let χ_α be the characteristic function on O_α. It follows that χ_α converges to χ_Δ in the weak* topology. Thus, the inequality in (4.5) holds for every Borel subset Δ of X.

Given $\varepsilon > 0$ and a positive function h in $C(X)$, there exist positive numbers $\lambda_1, \dots, \lambda_m$ and a Borel partition $\Delta_1, \dots, \Delta_m$ of X such that

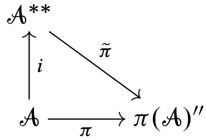
$$\Big\| h - \sum_{k=1}^{m} \lambda_k \chi_{\Delta_m} \Big\| \le \varepsilon.$$

It follows the inequality in (4.4). This completes the proof. ∎

A lemma from [33] is prepared as follows.

Lemma 4.2 ([33, Lemma 2.2]). *Let $\mathcal{A}$ be a C^*-algebra and $\{\pi, \mathcal{H}\}$ be a representation of $\mathcal{A}$. Then, there is a unique linear mapping $\tilde{\pi}$ of the second conjugate space $\mathcal{A}^{**}$ of $\mathcal{A}$ onto $\pi(\mathcal{A})''$ such that*

(1) *the diagram*



is commutative, where i is the canonical imbedding of $\mathcal{A}$ into $\mathcal{A}^{**}$.

(2) *the mapping $\tilde{\pi}$ is continuous with respect to the $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-topology and the weak operator topology of $\pi(\mathcal{A})''$.*

By virtue of Lemmas 4.1 and 4.2, we are ready for the following lemma.

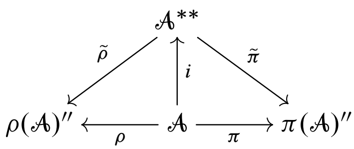
Lemma 4.3. *Let $\mathcal{A}$ be a unital, separable C^*-algebra and let $(\mathcal{N}, \tau)$ be a type II_1 factor. Let p be a projection in $(\mathcal{N}, \tau)$. Suppose that $\pi : \mathcal{A} \to \mathcal{N}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to p\mathcal{N}p$ is a unital $*$-homomorphism such that*

$$\tau\big(R(\rho(a))\big) \le \tau\big(R(\pi(a))\big) \quad \forall\, a \in \mathcal{A}.$$

Let $\tilde{\pi} : \mathcal{A}^{**} \to \pi(\mathcal{A})''$ and $\tilde{\rho} : \mathcal{A}^{**} \to \rho(\mathcal{A})''$ be the weak**-WOT continuous $*$-homomorphisms extended by π and ρ, respectively. Then, for every projection e in $\mathcal{A}^{**}$,*

$$\tau(\tilde{\rho}(e)) \le \tau(\tilde{\pi}(e)).$$

Proof. As an application of Lemma 4.2, we have the following commutative diagram:



where i is the canonical imbedding of $\mathcal{A}$ into $\mathcal{A}^{**}$.

Given a projection e in $\mathcal{A}^{**}$, by virtue of [22, Theorem 1.6.5] and Kaplansky's Density Theorem ([22, Corollary 5.3.6]), there exists a sequence $\{a_n\}_{n\ge 1}$ of positive operators in the unit ball of $\mathcal{A}$ such that $i(a_n)$ is SOT-convergent to e. Then, [23, Lemma 7.1.14] entails that $\rho(a_n) = \tilde{\rho} \circ i(a_n)$ is SOT-convergent to $\tilde{\rho}(e)$ in $\rho(\mathcal{A})''$. Likewise, $\pi(a_n) = \tilde{\pi} \circ i(a_n)$ is SOT-convergent to $\tilde{\pi}(e)$ in $\pi(\mathcal{A})''$.

In terms of Lemma 4.1, we have that

$$\tau(\rho(a_n)) \le \tau(\pi(a_n)) \quad \forall\, n \ge 1.$$

Since τ is a normal mapping, it follows that the inequality

$$\tau(\tilde{\rho}(e)) \le \tau(\tilde{\pi}(e))$$

holds for every projection e in $\mathcal{A}^{**}$. This completes the proof. ∎

Lemma 4.3 and the following Lemma 4.4 are prepared for Lemma 4.6. Note that the following Lemma 4.4 is a special case of Lemma 3.6. For convenience, we list it here.

Lemma 4.4. *Let $(\mathcal{N}, \tau)$ be a type* II_1 *factor with tracial state τ. Suppose that $\mathcal{A}$ is a unital C^*-algebra, $*$-isomorphic to $M_n(\mathbb{C})$, with an identity $I_{\mathcal{A}}$. Let ϕ and ψ be $*$-homomorphisms of $\mathcal{A}$ into $\mathcal{N}$ such that*

$$\tau(\phi(a)) = \tau(\psi(a)) \quad \forall\, a \in \mathcal{A}.$$

Then, there exists a partial isometry v in $\mathcal{N}$ such that

$$\phi(a) = v^*\psi(a)v \quad \forall\, a \in \mathcal{A}.$$

Recall that a locally homogeneous C^*-algebra is a (finite) direct sum of homogeneous C^*-algebra. A C^*-algebra is homogeneous if it is n-homogeneous for some n. A C^*-algebra $\mathcal{A}$ is n-homogeneous if every irreducible representation of $\mathcal{A}$ is of dimension n. The reader is referred to [2, Definition IV.1.4.1] for the definition.

Remark 4.5. It follows from [2, Proposition IV.1.4.6] that if a C^*-algebra is n-homogeneous, then its double dual is a type I_n von Neumann algebra. In the following lemmas, we will frequently mention type I_n von Neumann algebras. Thus, the following facts about type I_n von Neumann algebras are useful.

Let $\mathcal{A}$ be a type I_n von Neumann algebra on a Hilbert space $\mathcal{H}$. Then, there exists a system of matrix units $\{e_{ij}\}_{1\le i,j\le n}$ for $\mathcal{A}$. Let $\mathcal{R}_n$ be the von Neumann algebra generated by $\{e_{ij}\}_{1\le i,j\le n}$. Then, $\mathcal{R}_n$ is $*$-isomorphic to $\mathcal{M}_n(\mathbb{C})$. Define $\mathcal{P} = \mathcal{R}_n' \cap \mathcal{A}$. Since $\mathcal{A}$ is a type I_n von Neumann algebra, it follows that $\mathcal{P}$ is abelian and

$$\mathcal{A} = (\mathcal{P} \cup \mathcal{R}_n)''.$$

Moreover, $\mathcal{A}$ is $*$-isomorphic to the von Neumann tensor product $\mathcal{P} \otimes \mathcal{M}_n(\mathbb{C})$.

The following observation is useful in the sequel. For each element a in $\mathcal{A}$ and $\varepsilon > 0$, there are projections $p_1, \ldots, p_m$ in $\mathcal{P}$ with

$$1_{\mathcal{P}} = \sum_{1\le i\le m} p_i$$

and matrices $a_1, \ldots, a_m$ in $\mathcal{R}_n$ such that

$$\left\| a - \sum_{1\le i\le m} p_i a_i \right\| < \varepsilon. \tag{4.6}$$

For Theorem 4.7, we prepare the following lemma.

Lemma 4.6. *Let $\mathcal{A}$ be a unital, separable, locally homogeneous C^*-algebra and let $(\mathcal{N}, \tau)$ be a type* II_1 *factor. Let p be a projection in $(\mathcal{N}, \tau)$. Suppose that $\pi : \mathcal{A} \to \mathcal{N}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to p\mathcal{N}p$ is a unital $*$-homomorphism such that*

$$\tau(R(\rho(a))) \le \tau(R(\pi(a))) \quad \forall\, a \in \mathcal{A}.$$

Let $\tilde{\pi}$ (resp., $\tilde{\rho}$) be the weak**-WOT continuous $*$-homomorphism of $\mathcal{A}^{**}$ onto $\pi(\mathcal{A})''$ (resp., $\rho(\mathcal{A})''$) extended by the $*$-homomorphism π (resp., ρ).*

For a finite subset $\mathcal{F}$ of $\mathcal{A}^{**}$ and $\varepsilon > 0$, there exists a finite dimensional von Neumann subalgebra $\mathcal{B}$ in $\mathcal{A}^{**}$ such that

(1) *for each a in $\mathcal{F}$, there is an element b in $\mathcal{B}$ satisfying*

$$\|a - b\| < \varepsilon;$$

(2) *there is a unital $*$-homomorphism $\gamma : \mathcal{B} \to p^{\perp}\mathcal{N}p^{\perp}$ satisfying*

$$\tilde{\rho}|_{\mathcal{B}} \oplus \gamma \simeq \tilde{\pi}|_{\mathcal{B}} \quad \text{in } \mathcal{N}.$$

Moreover, if $\gamma' : \mathcal{B} \to p^{\perp}\mathcal{N}p^{\perp}$ is another unital $*$-homomorphism satisfying

$$\tilde{\rho}|_{\mathcal{B}} \oplus \gamma' \simeq \tilde{\pi}|_{\mathcal{B}} \quad \text{in } \mathcal{N},$$

then $\gamma' \simeq \gamma$ in $p^{\perp}\mathcal{N}p^{\perp}$.

Proof. We first assume that $\mathcal{A}$ is n-homogeneous. It follows from Remark 4.5 that the double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ can be expressed as $\mathcal{A}^{**} = (\mathcal{P} \cup \mathcal{R}_n)''$ on some Hilbert space $\mathcal{H}$, where $\mathcal{R}_n$ is $*$-isomorphic to $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{P} = \mathcal{R}_n' \cap \mathcal{A}^{**}$ is an abelian von Neumann subalgebra. Let $\mathcal{F} = \{a_1, \ldots, a_k\}$ be a finite subset of $\mathcal{A}$. Since $\mathcal{A}$ is isometrically imbedded into $\mathcal{A}^{**}$, we can also view a_i as an element in $\mathcal{A}^{**}$ for each $1 \le i \le k$.

For each $\varepsilon > 0$, by (4.6) in Remark 4.5, there are finitely many projections $p_1, \ldots, p_m$ in $\mathcal{P}$ with $I_{\mathcal{P}} = \sum_{1\le j\le m} p_j$ and there are matrices $a_{i1}, \ldots, a_{im}$ in $\mathcal{R}_n$ for $1 \le i \le k$ such that

$$\left\| a_i - \sum_{1\le j\le m} p_j a_{ij} \right\| < \varepsilon.$$

Note that each p_j is in the center of $\mathcal{A}^{**}$ and $I_{\mathcal{P}}$ is the identity of $\mathcal{A}^{**}$. Define a finite dimensional von Neumann subalgebra $\mathcal{B}$ in $\mathcal{A}^{**}$ as follows:

$$\mathcal{B} := \sum_{j=1}^{m} p_j \mathcal{R}_n.$$

Thus, by Lemma 4.3, there is a finite dimensional von Neumann subalgebra $\mathcal{M}$ of $p^{\perp}\mathcal{N}p^{\perp}$ in the form

$$\mathcal{M} := \sum_{j=1}^{m} \mathcal{M}_j$$

such that

(1) for each $1 \le j \le m$, $\mathcal{M}_j$ is $*$-isomorphic to $M_n(\mathbb{C})$;

(2) the identity q_j of $\mathcal{M}_j$ satisfies

$$p^{\perp} = \sum_{1\le j\le m} q_j \quad \text{and} \quad \tau(q_j) = \tau(\tilde{\pi}(p_j)) - \tau(\tilde{\rho}(p_j)) \quad \text{for } 1 \le j \le m.$$

Since each $\mathcal{M}_j$ is $*$-isomorphic to $M_n(\mathbb{C})$, we obtain that $\mathcal{B}$ is $*$-isomorphic to $\mathcal{M}$. In terms of Lemma 4.4, we can define a unital $*$-isomorphism γ of $\mathcal{B}$ into $\mathcal{M}$ satisfying $\gamma(p_j) = q_j$ for each $1 \leq j \leq m$, and

$$\tilde{\rho}|_{\mathcal{B}} \oplus \gamma \simeq \tilde{\pi}|_{\mathcal{B}} \quad \text{in } \mathcal{N}.$$

Moreover, γ is unique up to unitary equivalence.

If $\mathcal{A}$ is a unital, separable, locally homogeneous C^*-subalgebra of $\mathcal{N}$, then $\mathcal{A}$ can be expressed as $\mathcal{A} = \sum_{k=1}^{m} \mathcal{A}_k$ such that

(1) for each $1 \leq k \leq m$, $\mathcal{A}_k$ is n_k-homogeneous for some $n_k \in \mathbb{N}$;

(2) the identities $I_{\mathcal{A}_k}$ of $\mathcal{A}_k$ are mutually orthogonal.

By composing the preceding arguments for each $\mathcal{A}_k$, we can complete the proof. ■

The following theorem is the main result of this section. For a unital, separable C^*-subalgebra $\mathcal{A}$ of $\mathcal{N}$ and two unital $*$-homomorphisms ϕ and ψ of $\mathcal{A}$ into $\mathcal{N}$, recall that $\phi \sim_a \psi$ in $\mathcal{N}$ means the approximately unitary equivalence of ϕ and ψ in $\mathcal{N}$; i.e., there exists a sequence of unitary operators $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{N}$ such that

$$\lim_{n\to\infty} \|u_n^* \phi(a) u_n - \pi(a)\| = 0 \quad \forall\, a \in \mathcal{A}.$$

Theorem 4.7. *Let $\mathcal{A}$ be a unital, separable AH algebra and let $(\mathcal{N}, \tau)$ be a type II_1 factor. Let p be a projection in $(\mathcal{N}, \tau)$. Suppose that $\pi : \mathcal{A} \to \mathcal{N}$ is a unital $*$-homomorphism and $\rho : \mathcal{A} \to p\mathcal{N}p$ is a unital $*$-homomorphism such that*

$$\tau(R(\rho(a))) \leq \tau(R(\pi(a))) \quad \forall\, a \in \mathcal{A}.$$

Then, there exists a unital $*$-homomorphism $\gamma : \mathcal{A} \to p^{\perp}\mathcal{N}p^{\perp}$ such that

$$\rho \oplus \gamma \sim_a \pi \quad \textit{in } \mathcal{N}.$$

Proof. As in Remark 2.6, we can assume that

$$\mathcal{A} = \overline{\bigcup_{n\geq 1} \mathcal{A}_n}^{\|\cdot\|},$$

where $\{\mathcal{A}_n\}_{n\geq 1}$ is a monotone increasing sequence of unital, locally homogeneous C^*-algebras.

Since $\mathcal{A}$ is separable, let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be a monotone increasing sequence of finite subsets in $\bigcup_{n\geq 1} \mathcal{A}_n$ such that $\bigcup_{i\geq 1} \mathcal{F}_i$ is $\|\cdot\|$-dense in $\mathcal{A}$, where $\|\cdot\|$ is the operator norm. By dropping to a subsequence, we can assume that $\mathcal{F}_i \subset \mathcal{A}_i$ for each i in $\mathbb{N}$. We further require that $1_{\mathcal{A}} \in \mathcal{F}_1$.

In terms of [2, Proposition III.5.2.10], for each $n \in \mathbb{N}$, we have that

$$\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \subseteq \mathcal{A} \implies \mathcal{A}_n^{**} \subseteq \mathcal{A}_{n+1}^{**} \subseteq \mathcal{A}^{**}.$$

In the following, we identify $\mathcal{A}$ as a unital, separable C^*-subalgebra of $\mathcal{A}^{**}$. Note that each $\mathcal{A}_i^{**}$ is a direct sum of finitely many type I_m von Neumann algebra for m less than a certain n.

As an application of Lemma 4.6, there is a finite dimensional von Neumann subalgebra $\mathcal{B}_1$ of $\mathcal{A}_1^{**}$ corresponding to $\mathcal{F}_1$ such that, for each a in $\mathcal{F}_1$, there is an a_1 in $\mathcal{B}_1$ satisfying

$$\|a - a_1\| < \frac{1}{2^2}.$$

Applying Lemma 4.6 once more for $\mathcal{F}_2$ and the system of matrix units of $\mathcal{B}_1$, there exists a finite dimensional von Neumann subalgebra $\mathcal{B}_2$ of $\mathcal{A}_2^{**}$ such that

(1) $\mathcal{B}_2 \supseteq \mathcal{B}_1$;

(2) for each a in $\mathcal{F}_2$, there is an a_2 in $\mathcal{B}_2$ satisfying

$$\|a - a_2\| < \frac{1}{2^3}.$$

By induction, there is a finite dimensional von Neumann subalgebra $\mathcal{B}_i$ of $\mathcal{A}_i^{**}$ corresponding to each $\mathcal{F}_i$ such that

(1) $\mathcal{B}_i \supseteq \mathcal{B}_{i-1}$ for each $i \geq 2$;

(2) for each a in $\mathcal{F}_i$, there is an a_i in $\mathcal{B}_i$ satisfying

$$\|a - a_i\| < \frac{1}{2^{i+1}}. \tag{4.7}$$

Moreover, there is a unital $*$-homomorphism $\phi_i : \mathcal{B}_i \to p^\perp \mathcal{N} p^\perp$ such that

$$\tilde{\rho}|_{\mathcal{B}_i} \oplus \phi_i \simeq \tilde{\pi}|_{\mathcal{B}_i} \quad \text{in } \mathcal{N}. \tag{4.8}$$

Note that $\mathcal{B}_{i+1} \supseteq \mathcal{B}_i$ implies that

$$\tilde{\rho}|_{\mathcal{B}_i} \oplus \phi_{i+1}|_{\mathcal{B}_i} \simeq \tilde{\rho}|_{\mathcal{B}_i} \oplus \phi_i \simeq \tilde{\pi}|_{\mathcal{B}_i} \quad \text{in } \mathcal{N}.$$

Thus, we have

$$\phi_{i+1}|_{\mathcal{B}_i} \simeq \phi_i \quad \text{in } p^\perp \mathcal{N} p^\perp.$$

Define $\gamma_1 := \phi_1$. Let u_2 be the unitary operator in $p^\perp \mathcal{N} p^\perp$ such that $u_2^*(\phi_2|_{\mathcal{B}_1})u_2 = \gamma_1$ and define

$$\gamma_2(\cdot) := u_2^* \phi_2(\cdot) u_2.$$

Likewise, let u_{i+1} be the unitary operator in $p^\perp \mathcal{N} p^\perp$ such that $u_{i+1}^*(\phi_{i+1}|_{\mathcal{B}_i})u_{i+1} = \gamma_i$ and define

$$\gamma_{i+1}(\cdot) := u_{i+1}^* \phi_{i+1}(\cdot) u_{i+1}. \tag{4.9}$$

With respect to the choice of the family $\{\mathcal{B}_i\}_{i=1}^\infty$ of finite dimensional von Neumann algebras, we construct a sequence of $*$-homomorphisms $\{\gamma_i\}_{i \geq 1}$ such that the equality

$$\gamma_{i+k}(b) = \gamma_i(b) \tag{4.10}$$

holds for every b in $\mathcal{B}_i$ and each $i \geq 1$, $k \geq 1$.

By virtue of (4.7), for every a in $\mathcal{F}_i$, there is a sequence $\{a_k : a_k \in \mathcal{B}_k\}_{k\geq 1}$ such that

$$a_k = 0 \quad \text{for } k < i \quad \text{and} \quad \|a - a_k\| < \frac{1}{2^{k+1}} \quad \text{for } k \geq i. \tag{4.11}$$

It follows that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence in the operator norm topology. Moreover, we assert that $\{\gamma_k(a_k)\}_{k=1}^{\infty}$ is a Cauchy sequence in the operator norm topology. Notice that, for $k \geq i$ and each $p \geq 1$, (4.10) and (4.11) imply that

$$\|\gamma_{k+p}(a_{k+p}) - \gamma_k(a_k)\| = \|\gamma_{k+p}(a_{k+p}) - \gamma_{k+p}(a_k)\| \leq \|a_{k+p} - a_k\| < \frac{1}{2^k}.$$

This guarantees that $\{\gamma_k(a_k)\}_{k\geq 1}$ is also a Cauchy sequence in the operator norm topology.

Note that, for each fixed a in $\mathcal{F}_i$, if there is another sequence $\{a'_k : a'_k \in \mathcal{B}_k\}$ satisfying

$$a'_k = 0 \quad \text{for } k < i \quad \text{and} \quad \|a - a'_k\| < \frac{1}{2^{k+1}} \quad \text{for } k \geq i,$$

then both $\{a'_k\}_{k=1}^{\infty}$ and $\{\gamma_k(a'_k)\}_{k=1}^{\infty}$ are Cauchy sequences in the operator norm topology. Furthermore, the limit $\lim_{k\to\infty}\|a_k - a'_k\| = 0$ entails that the equality

$$\lim_{k\to\infty} \gamma_k(a'_k) = \lim_{k\to\infty} \gamma_k(a_k)$$

holds in the operator norm topology. Since $\|\gamma_k\| \leq 1$ for each $k \in \mathbb{N}$, the mapping

$$\gamma : \bigcup_{i\geq 1} \mathcal{F}_i \to p^{\perp}\mathcal{N}p^{\perp}, \quad \gamma(a) := \lim_{k\to\infty} \gamma_k(a_k)$$

extends to a well-defined unital $*$-homomorphism of $\mathcal{A}$ into $p^{\perp}\mathcal{N}p^{\perp}$.

Note that, for each $i \geq 1$, (4.8) and (4.9) entail that there is a unitary operator v_i in $\mathcal{N}$ such that $v_i^*(\rho_i \oplus \gamma_i)v_i = \pi_i$, where ρ_i (resp., π_i) is the restriction of $\tilde{\rho}$ (resp., $\tilde{\pi}$) on $\mathcal{B}_i$. Thus, for each $a \in \mathcal{F}_i$, it follows that

$$\begin{aligned}&\|\pi(a) - v_i^*(\rho(a) \oplus \gamma(a))v_i\| \\ &\quad \leq \|\pi(a) - \pi_i(a_i)\| + \|\rho_i(a_i) \oplus \gamma_i(a_i) - \rho(a) \oplus \gamma(a)\| < \frac{1}{2^i}.\end{aligned}$$

Since $\bigcup_{i\geq 1} \mathcal{F}_i$ is $\|\cdot\|$-dense in $\mathcal{A}$, we obtain that

$$\lim_{i\to\infty} \|\pi(a) - v_i^*(\rho(a) \oplus \gamma(a))v_i\| = 0 \quad \forall\, a \in \mathcal{A}.$$

This completes the proof. ∎

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