Interfaces Free Bound. 26 (2024), 489–541
DOI 10.4171/IFB/513

Multi-component conserved Allen–Cahn equations

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Abstract. We consider a multi-component version of the conserved Allen–Cahn equation proposed by J. Rubinstein and P. Sternberg in 1992 as an alternative model for phase separation. In our case, the free energy is characterized by a mixing entropy density which belongs to a large class of physically relevant entropies like, for example, the Boltzmann–Gibbs entropy. We establish the well-posedness of the Cauchy–Neumann problem with respect to a natural notion of (finite) energy solution which is more regular under appropriate assumptions and is strictly separated from pure phases if the initial datum is. We then prove that the energy solution becomes more regular and strictly separated instantaneously. Also, we show that any finite energy solution converges to a unique equilibrium. The validity of a dissipative inequality (identity for strong solutions) allows us to analyze the problem within the theory of infinite-dimensional dissipative dynamical systems. On account of the obtained results, we can associate to our problem a dissipative dynamical system and we can prove that it has a global attractor as well as an exponential attractor.

1. Introduction

Phase separation—that is, the creation of two (or more) distinct phases from a single homogeneous mixture—is an important phenomenon which characterizes many important processes. In particular, it has recently become a paradigm in cell biology (see, for instance, [5, 6] and references therein). A well-known mathematical model of phase separation for binary alloys was proposed by J. W. Cahn and J. E. Hilliard [3, 4]. This model leads to the so-called Cahn–Hilliard equation (see, for instance, [38] and references therein). More precisely, indicating by φ the concentration of one species, phase separation can be modeled as a competition between the Boltzmann–Gibbs mixing entropy

$$S(\varphi) = -\varphi \ln \varphi - (1-\varphi) \ln \varphi$$

and the demixing effects due to the reciprocal attraction of the molecules of the same species which can be described, for instance, as follows:

$$D(\varphi) = -\varphi(1-\varphi).$$

Mathematics Subject Classification 2020: 35K58 (primary); 35B36, 35B40, 35B41, 35B65, 35K40, 37L30 (secondary).
Keywords: conserved Allen–Cahn equations, well-posedness, global solutions, strict separation property, De Giorgi iteration, global attractors, exponential attractors, convergence to equilibrium.

Thus, the free energy density is given by the so-called Flory–Huggins potential (see, for instance, [2] and references therein)

$$W(\varphi) = -\Theta S(\varphi) + \Theta_c D(\varphi), \tag{1.1}$$

where $\Theta > 0$ is the absolute temperature of the mixture and $\Theta_0 > 0$ is its critical temperature (other constants are set equal to 1). If $\Theta < \Theta_c$, then W has a double well shape and phase separation takes place. Assuming that the mixture occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the previous considerations lead to the following free energy functional:

$$E(\varphi) := \int_\Omega W(\varphi)dx + \frac{\gamma}{2}\int_\Omega |\nabla\varphi|^2 dx,$$

where the penalization term allows the creation of diffuse interfaces between the two species and also allows a convenient mathematical treatment of the phenomenon (see [20]). Here $\gamma > 0$ is related to the thickness of the diffuse interface. The Cahn–Hilliard equation can be introduced as a conserved gradient flow generated by the gradient of the chemical potential μ defined by

$$\mu = \frac{\delta E}{\delta\varphi} = -\gamma\Delta\varphi + W'(\varphi),$$

that is, taking constant mobility equal to a constant $m > 0$,

$$\partial_t\varphi = m\Delta\mu.$$

This equation, subject to no-flux (or periodic) boundary conditions, entails the conservation of the total mass $\int_\Omega \varphi(t)dx$. An alternative model has been proposed by J. Rubinstein and P. Sternberg [42] by modifying another well-known equation proposed by S. M. Allen and J. W. Cahn [1] in order to ensure mass conservation. The equation has the form

$$\partial_t\varphi = \alpha(\overline{\mu} - \mu), \tag{1.2}$$

where $\alpha > 0$ and $\overline{f}$ is defined by

$$\overline{f} := |\Omega|_d^{-1}\int_\Omega f(x)dx,$$

for any integrable f. Here $|\Omega|_d$ stands for the d-dimensional Lebesgue measure of Ω. Equation (1.2) equipped with a homogeneous Neumann boundary condition preserves the total mass. In [42] a (formal) asymptotic analysis was performed with respect to a specific scaling in order to understand the motion of the separating interfaces (also see [8] for an important application). More rigorous results can be found in [17] where the authors show that, in a radially symmetric setting, the sharp interface problem of a suitable scaling of (1.2) is a nonlocal motion by mean curvature. Moreover, they also prove that both (1.2) and the Cahn–Hilliard equation can be seen as degenerate limits of the viscous Cahn–Hilliard equation introduced in [41].

The corresponding motion by mean curvature is also analyzed in [18] under more general assumptions on the evolving surface. In the quoted contributions, the mixing entropy is approximated, that is, the double well potential W is a fourth-order polynomial, also called the regular (or smooth) potential. However, on account of the nonlocal constraint, one cannot ensure that φ takes its values in the physical range $[0, 1]$ (see, however, [31] for an alternative model). Instead, if the mixing entropy is not approximated by a polynomial, then the image of φ is always contained in $[0, 1]$. Well-posedness issues in the case of a polynomial W are standard. However, if W is given by (1.1), then proving the existence of sufficiently regular global solutions is less trivial because S' is singular at the endpoints and cannot be controlled by S like a polynomial. In this case, it would be nice to show that φ stays uniformly away from 0 and 1, that is, if the strict separation property holds, then S' would be globally Lipschitz and the analysis would simplify a lot (see, for instance, [25] and references therein for the Cahn–Hilliard equation in two dimensions; see also [9] for the case of three dimensions). In the nonconserved case, the strict separation is trivial for regular potentials and a bit less straightforward for logarithmic-type potentials like (1.1) (see [32, Theorem 2.3]). Concerning (1.2), its instantaneous validity in dimension two has been proven (see [30]), while in dimension three the proof was given assuming that the initial datum is strictly separated (see [26]). Observe that the strict separation property combined with the uniqueness of a solution φ allows us to view the solution itself as the solution to a similar problem where S is replaced by a smooth approximation, defined on the whole real line, which coincides with S on the interval $[\delta, 1-\delta]$ and $\delta \in (0, 1)$ is such that $\varphi \in [\delta, 1-\delta]$. In other words, the validity of the strict separation can be interpreted as a rigorous justification of the entropy approximation with a polynomial.

In this paper we want to reconsider these issues and say more for a multi-component version of (1.2). In many applications, it is important to account for the presence of multiple interacting species (see, for instance, [11,12,15,16,29,33,34] and references therein; see also [22,40] and their references for the motion by mean curvature in the nonconserved case and [10] for the importance of the Flory–Huggins potential). Nevertheless, to our knowledge, a comprehensive theoretical analysis of multi-component conserved Allen–Cahn equations is missing. Nonetheless, it is worth recalling [23,43] and their references for nonconserved stationary problems with regular potential. Moreover, we recall that a rigorous solution to the so-called Keller–Rubinstein–Sternberg problem on the motion by curvature has recently been given in [21] (see also its references). On the contrary, multi-component Cahn–Hilliard equations were analyzed long ago in the pioneering paper [7] (see also [27] and its references for further results and recent developments). As we shall see, one of the advantages (and our main result) is the fact that any weak solution becomes instantaneously strong and strictly separated also in dimension three, while this property is known only in dimension two for the corresponding multi-component Cahn–Hilliard equation. This regularization allows us to investigate the longtime behavior of solutions in some details, that is, we prove the existence of a global and an exponential attractor. Also, we can show that any weak solution converges to a single stationary state. The present analysis can also be viewed as a first step towards the analysis of multi-component Navier–Stokes-Allen–Cahn systems (see, for instance, [14, 45, 46]; see also [30, 37] and references therein for binary fluids). We also believe that this contribution is a significant addition to [35, Section 9].

In the multi-component case, we denote by $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^N$ the vector-valued function of concentration species whose components must satisfy the constraint

$$\sum_{i=1}^{N} u_i \equiv 1. \tag{1.3}$$

The free energy density takes the form

$$\Psi(\mathbf{u}) = \sum_{i=1}^{N} \psi(u_i) - \frac{1}{2}\mathbf{u}^T \mathbf{A}\mathbf{u}, \tag{1.4}$$

where $\mathbf{A}$ is a constant symmetric $N \times N$ matrix with the largest eigenvalue $\lambda_{\mathbf{A}} > 0$. Concerning ψ, here we are mainly interested in the Boltzmann–Gibbs mixing entropy—namely,

$$\Psi^1(\mathbf{u}) := \theta \sum_{i=1}^{N} u_i \ln u_i = \sum_{i=1}^{N} \psi(u_i), \tag{1.5}$$

where $\theta > 0$ is the absolute temperature of the mixture. However, our framework also includes many other (physically relevant) entropy functions $\Psi^1 : [0, 1] \to \mathbb{R}_+$ (see papers [25, 27]). The free energy $\mathcal{E}$ is thus defined as

$$\mathcal{E}(\mathbf{u}) := W(\mathbf{u}) + \frac{\gamma}{2}\int_\Omega |\nabla \mathbf{u}|^2 dx,$$

where

$$W(\mathbf{u}) = \int_\Omega \Psi(\mathbf{u}) dx.$$

Setting

$$\mu_i^0 = \frac{\delta W}{\delta u_i} = \Psi_{,u_i}, \quad i = 1, \dots, N,$$

the vector $\boldsymbol{\mu}^0$ is the chemical potential without capillarity and

$$\boldsymbol{\mu} = -\gamma \Delta \mathbf{u} + \boldsymbol{\mu}^0$$

is the chemical potential.

Summing up, arguing as in [7] for the Cahn–Hilliard case, the goal of this work is to study the following initial and boundary value problem:

$$\begin{cases} \partial_t \mathbf{u} + \boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}) = \mathbf{0} & \text{in } \Omega \times (0, T), \\ \boldsymbol{w} = \mathbf{P}\boldsymbol{\mu} = -\gamma \Delta \mathbf{u} + \mathbf{P}\boldsymbol{\mu}^0 & \text{in } \Omega \times (0, T), \\ \nabla u_i \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T),\ i = 1, \dots, N, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

The (constant) mobility matrix $\boldsymbol{\alpha}$ is a symmetric, positive semidefinite $N \times N$ matrix such that its kernel is given by span$\{\boldsymbol{\zeta}\}$ (where $\zeta_i = 1$, for $i = 1, \ldots, N$). Here $\mathbf{P}$ is defined as follows (see also the next section):

$$(\mathbf{Pv})_l = v_l - \frac{1}{N}\sum_{m=1}^{N} v_m, \quad l = 1, \ldots, N. \tag{1.7}$$

Then, it is easy to check that, formally, a solution to the above problem with $\mathbf{P}\Delta\mathbf{u} = \Delta\mathbf{Pu}$ in place of $\Delta\mathbf{u}$ satisfies (1.3) if the initial datum does, using in $(1.6)_1$ the property

$$\sum_{l=1}^{N} (\mathbf{Pv})_l = 0$$

and the fact that (recalling that $\boldsymbol{\alpha}$ is also symmetric) $\sum_{i=1}^{N} \boldsymbol{\alpha}_{ij} = 0$ for any $j = 1, \ldots, N$. Therefore, $\mathbf{P}\Delta\mathbf{u} = \Delta\mathbf{u}$.

The plan of the paper goes as follows: In the next section we introduce the notation, the functional setup, and some basic assumptions on the mobility matrix $\boldsymbol{\alpha}$. Also, we discuss the basic assumptions on the potential (more general than (1.4)–(1.5)) and its regularization. The main results are stated in Section 3 and the last subsection contains the proof of the convergence to a single equilibrium. The proofs of the well-posedness and regularity results, including the strict separation property, can be found in Section 4. The existence of the global attractor and of an exponential attractor are proven in Sections 5 and 6, respectively.

2. The mathematical framework

The (real) Sobolev spaces are denoted as usual by $W^{k,p}(\Omega)$, where $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, with norm $\|\cdot\|_{W^{k,p}(\Omega)}$. The Hilbert space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k(\Omega)}$. Moreover, given a space X, we denote by $\mathbf{X}$ the space of vectors of three components, each one belonging to X. We then denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\|\cdot\|$ the induced norm. We indicate by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the canonical inner product and its induced norm in a generic (real) Hilbert space X, respectively. Further, we introduce the affine hyperplane

$$\Sigma := \left\{\mathbf{c}' \in \mathbb{R}^N : \sum_{i=1}^{N} c_i' = 1\right\},$$

the Gibbs simplex

$$\mathbf{G} := \left\{\mathbf{c}' \in \mathbb{R}^N : \sum_{i=1}^{N} c_i' = 1,\ c_i' \geq 0,\ i = 1, \ldots, N\right\},$$

and the tangent space to Σ

$$T\Sigma := \Big\{ \mathbf{d}' \in \mathbb{R}^N : \sum_{i=1}^{N} d_i' = 0 \Big\}.$$

We introduce the following notation:

$$\begin{aligned}
\mathbf{H}_0 &:= \Big\{ \boldsymbol{f} \in \mathbf{L}^2(\Omega) : \int_\Omega \boldsymbol{f}\, dx = \mathbf{0} \text{ and } \boldsymbol{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \Big\},\\
\widetilde{\boldsymbol{H}}_0 &:= \Big\{ \boldsymbol{f} \in \mathbf{L}^2(\Omega) : \boldsymbol{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \Big\},\\
\boldsymbol{V}_0 &:= \Big\{ \boldsymbol{f} \in \mathbf{H}^1(\Omega) : \int_\Omega \boldsymbol{f}\, dx = \mathbf{0} \text{ and } \boldsymbol{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \Big\},\\
\widetilde{\boldsymbol{V}}_0 &:= \Big\{ \boldsymbol{f} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{f}(x) \in T\Sigma \ \text{ for a.a. } x \in \Omega \Big\}.
\end{aligned}$$

Notice that the spaces above are still Hilbert spaces with the same inner products given in $\mathbf{L}^2(\Omega)$ for the first two, and $\mathbf{H}^1(\Omega)$, for the others. We also have (see [27]) the Hilbert triplets $\boldsymbol{V}_0 \hookrightarrow \boldsymbol{H}_0 \hookrightarrow \boldsymbol{V}_0'$ and $\widetilde{\boldsymbol{V}}_0 \hookrightarrow \widetilde{\boldsymbol{H}}_0 \hookrightarrow \widetilde{\boldsymbol{V}}_0'$.

Recalling (1.7), we now define rigorously the Euclidean projection $\mathbf{P}$ of $\mathbb{R}^N$ onto $T\Sigma$, which is, for $l = 1, \dots, N$,

$$(\mathbf{P}\mathbf{v})_l = \Big(\mathbf{v} - \Big(\frac{1}{N} \sum_{i=1}^{N} v_i \Big) \boldsymbol{\zeta} \Big)_l,$$

where $\boldsymbol{\zeta} := (1, 1, \dots, 1)$. Notice that the projector $\mathbf{P}$ is also an orthogonal $\mathbf{L}^2(\Omega)$-projector, being symmetric and idempotent. We now assume that $\boldsymbol{\alpha}$ is positive definite over $T\Sigma$. This will constitute the main assumption on the mobility matrix in this contribution, since it is enough to prove the existence of weak (and strong) solutions. Nevertheless, it is not enough to show the validity of a continuous dependence estimate. Thus, we need a second assumption (see assumption (**M1**)). More precisely, we assume that:

(**M0**) there exists $l_0 > 0$ such that

$$\boldsymbol{\alpha}\boldsymbol{\eta} \cdot \boldsymbol{\eta} \geq l_0 \boldsymbol{\eta} \cdot \boldsymbol{\eta}, \quad \forall\, \boldsymbol{\eta} \in T\Sigma; \tag{2.1}$$

(**M1**) $\boldsymbol{\alpha} \in \mathbb{R}^{N \times N}$ has the structure

$$\boldsymbol{\alpha} = \begin{bmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \vdots & \vdots \\ B & \dots & \dots & A \end{bmatrix}, \tag{2.2}$$

where $A > 0$ and $A + (N-1)B = 0$, so that $B = -\frac{A}{N-1} < 0$.

Remark 2.1. Note that assumption (**M1**) can be also rewritten as follows: there exists $\xi > 0$ such that

$$\boldsymbol{\alpha} = \xi \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & \dots & \dots & N-1 \end{bmatrix}. \tag{2.3}$$

A matrix of this kind is the natural extension to the case $N > 2$ of the admissible matrix $\boldsymbol{\alpha}$ when $N = 2$, which necessarily has the form in (2.3), as one can easily verify. Observe that when $\xi = 1$, the matrix $\boldsymbol{\alpha}$ is simply the representative matrix of the projector $\mathbf{P}$, that is, the identity operator over the space $T\Sigma$. We also point out that $\boldsymbol{\alpha}$ is positive semidefinite and satisfies (2.1), since it has a zero simple eigenvalue corresponding to the eigenspace $T\Sigma^{\perp}$, whereas on $T\Sigma$ we see by Lemma 4.1 below (with $\mathbf{C}$ equal to the $N \times N$ identity matrix) that $\boldsymbol{\alpha}$ is positive definite. In particular, one could show that the eigenvalues of $\boldsymbol{\alpha}$ are $\lambda_1 = 0$ (corresponding to the eigenvector $(1, 1, \dots, 1)$), and $\lambda_i = \xi N$, for $i = 2, \dots, N$, whose eigenspace is clearly $T\Sigma$.

Next, we define the set

$$\mathcal{K} := \Big\{ \boldsymbol{\eta} \in \mathbf{H}^1(\Omega) : \sum_{i=1}^{N} \eta_i = 1,\ \eta_i \ge 0,\ \forall i = 1, \dots, N \Big\}.$$

For the sake of simplicity we will adopt the compact notation $\mathbf{v} \ge k$, with $\mathbf{v} \in \mathbb{R}^N$ and $k \in \mathbb{R}$ to indicate the relations $v_i \ge k$, $i = 1, \dots, N$.

Recalling (1.5), we now set

$$(\boldsymbol{\phi}(\mathbf{u}))_i = \phi(u_i) := \psi'(u_i), \quad i = 1, \dots, N. \tag{2.4}$$

In order to include a large admissible class of entropy functionals in (1.4), we suppose that

$$\psi \in C[0, 1] \cap C^2(0, 1]$$

has the following properties:

(**E0**) $\psi''(s) \ge \zeta > 0$, for all $s \in (0, 1]$;

(**E1**) $\lim_{s \to 0^+} \psi'(s) = -\infty$;

(**E2**) $\lim_{s \to 0^+} (\psi'(s - 2s^2) - \psi'(2s^2)) = +\infty$.

As in [27], we also extend $\psi(s) = +\infty$, for any $s \in (-\infty, 0)$, and extend ψ for all $s \in [1, \infty)$ so that ψ is a C^2 function on $(0, +\infty)$ and assumption (**E0**) holds for any $s > 0$. In particular, we define

$$\psi(s) := As^3 + Bs^2 + Ds \quad \text{for all } s \ge 1,$$

with

$$\begin{cases} A = \psi(1) - \psi'(1) + \frac{1}{2}\psi''(1), \\ B = -3\psi(1) + 3\psi'(1) - \psi''(1), \\ D = 3\psi(1) - 2\psi'(1) + \frac{1}{2}\psi''(1). \end{cases}$$

We refer the reader to [25, Section 6.3] for some other important classes of mixing potentials that are singular at 0. Furthermore, following the general scheme developed in [24, Section 3.1], by assumptions (**E0**)–(**E1**), we can define an approximation of the potential ψ by means of a sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ of everywhere-defined nonnegative functions. More precisely, let

$$\psi_\varepsilon(s) = \frac{\varepsilon}{2}|\mathbb{T}_\varepsilon s|^2 + \psi(J_\varepsilon(s)), \quad s \in \mathbb{R},\ \varepsilon > 0, \tag{2.5}$$

where $J_\varepsilon = (I + \varepsilon\mathbb{A})^{-1} : \mathbb{R} \to (0, +\infty)$ is the resolvent operator and $\mathbb{T}_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon)$ is the Yosida approximation of $\mathbb{T}(s) := \psi'(s)$, for all $s \in \mathfrak{D}(\mathbb{T}) = (0, 1]$. According to the general theory of maximal monotone operators, as already developed in [27, Section 2], the following properties hold:

(i) ψ_ε is convex and $\psi_\varepsilon(s) \nearrow \psi(s)$, for all $s \in \mathbb{R}$, as ε goes to 0^+;

(ii) $\psi_\varepsilon'(s) = \mathbb{A}_\varepsilon(s)$ and $\phi_\varepsilon := \psi_\varepsilon'$ is globally Lipschitz with constant $\frac{1}{\varepsilon}$;

(iii) $|\psi_\varepsilon'(s)| \nearrow |\psi'(s)|$ for all $s \in (0, 1]$ and $|\psi_\varepsilon'(s)| \nearrow +\infty$, for all $s \in (-\infty, 0]$, as ε goes to 0^+;

(iv) for any $\varepsilon \in (0, 1]$, it holds that

$$\psi_\varepsilon''(s) \geq \frac{\zeta}{1+\zeta} \quad \text{for all } s \in \mathbb{R};$$

(v) for any compact subset $M \subset (0, 1]$, ψ_ε' converges uniformly to ψ' on M;

(vi) for any $\varepsilon_0 > 0$, there exists $\widetilde{K} = \widetilde{K}(\varepsilon_0) > 0$ such that

$$\sum_{i=1}^N \psi_\varepsilon(r_i) \geq \frac{1}{4\varepsilon_0}|\mathbf{r}|^2 - \widetilde{K}, \quad \forall \mathbf{r} \in \mathbb{R}^N,\ \forall\, 0 < \varepsilon < \varepsilon_0.$$

The final property directly follows from a simple adaptation of [24, Lemma 3.11], which entails that for any $\varepsilon_0 > 0$, there exists $C = C(\varepsilon_0) > 0$ such that $\psi_\varepsilon(s) \geq \frac{1}{4\varepsilon_0}s^2 - C$, for any $s \in \mathbb{R}$ and any $0 < \varepsilon < \varepsilon_0$ (see also [27, Section 2]). Let us now introduce

$$\Psi_\varepsilon(\mathbf{r}) := \sum_{i=1}^N \psi_\varepsilon(r_i) - \frac{1}{2}\mathbf{r}^T\mathbf{A}\mathbf{r} = \Psi_\varepsilon^1(\mathbf{r}) - \frac{1}{2}\mathbf{r}^T\mathbf{A}\mathbf{r},$$

where, as presented in the introduction, $\mathbf{A}$ is a symmetric $N \times N$ matrix with $\lambda_{\mathbf{A}} > 0$ as the largest eigenvalue. We thus have that for any $\varepsilon_0 > 0$ sufficiently small, there exist $K = K(\varepsilon_0) > 0$ and $C = C(\varepsilon_0) > 0$, with $C(\varepsilon_0) \nearrow +\infty$ as $\varepsilon_0 \to 0^+$, such that

$$\Psi_\varepsilon(\mathbf{r}) \geq C(\varepsilon_0)|\mathbf{r}|^2 - K, \quad \forall \mathbf{r} \in \mathbb{R}^N,\ \forall \varepsilon \in (0, \varepsilon_0).$$

In particular, this comes from the fact that $-\frac{1}{2}\mathbf{r}\cdot\mathbf{A}\mathbf{r} \geq -\frac{\lambda_{\mathbf{A}}}{2}|\mathbf{r}|^2$ and ε_0 has to be small enough so that, for example, $C(\varepsilon_0) = \frac{1}{4\varepsilon_0} - \frac{\lambda_{\mathbf{A}}}{2} > 0$.

Remark 2.2. We point out that, differently from the standard assumptions on ψ (see, e.g., [25, 30]), here we do not need the assumption

$$\phi'(s) = \psi''(s) \leq C\mathrm{e}^{C|\psi'(s)|^{\beta}} \qquad \text{for all } s \in (0,1],\ \beta \in [1,2),$$

since to deduce the validity of the instantaneous strict separation property we will make use only of assumptions (**E0**)–(**E2**). Clearly, the logarithmic potential in (1.4)–(1.5) satisfies assumptions (**E0**)–(**E2**) and is then included in our analysis. Indeed, assumption (**E2**) also certainly holds for the logarithmic potential, since $\psi'(s) = \theta(\ln(s)+1)$ and, thus, $\psi'(s-2s^2) - \psi'(2s^2) = \theta(\ln(s-2s^2) - \ln(2s^2)) = \theta\ln(\frac{1}{2s}-1) \to +\infty$ as $s \to 0^+$. Moreover, it seems that if we consider potentials exploding at infinity more slowly than the logarithm, then assumption (**E2**) is not satisfied. Indeed, if, for instance, we consider $\psi'(s) = -\ln(|\ln(s)|)$, then we get $\psi'(s-2s^2) - \psi'(2s^2) = -\ln(|\ln(s-2s^2)|) + \ln(|\ln(2s^2)|) \to \ln(2)$ as $s \to 0^+$.

3. Main results

This section is divided into several subsections according to the nature of the results.

3.1. Well-posedness and regularity

We first deal with well-posedness and regularity (see [27] for the multi-component Cahn–Hilliard system).

Theorem 3.1. *The following three scenarios hold:*

(1) *Assume* (**M0**) *and* (**E0**)–(**E1**), *and let* $\mathbf{u}_0 \in \mathcal{K}$. *Suppose that*

$$\delta_0 < \overline{\mathbf{u}}_0, \tag{3.1}$$

for some $0 < \delta_0 < \frac{1}{N}$. *Then, for any given* $T > 0$, *there exists a solution pair* $(\mathbf{u}, \boldsymbol{w})$ *defined on* $[0,T]$, *called a finite energy solution to* (1.6), *which has the following properties:*

$$\begin{aligned} \mathbf{u} &\in C([0,T];\boldsymbol{L}^2(\Omega)) \cap L^\infty(0,T;\mathbf{H}^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)),\\ \partial_t\mathbf{u} &\in L^2(0,T;\mathbf{L}^2(\Omega)),\\ \boldsymbol{w} &\in L^2(0,T;\mathbf{L}^2(\Omega)),\\ \phi(u_i) &\in L^2(0,T;L^2(\Omega)), \quad i = 1,\dots,N, \end{aligned}$$

and satisfies

$$\mathbf{u}(\cdot,t) \in \mathcal{K}, \quad \overline{\mathbf{u}}(\cdot,t) \equiv \overline{\mathbf{u}}_0 \qquad \textit{for a.a. } t \in (0,T), \tag{3.2}$$

$$0 < \mathbf{u}(x,t) < 1 \qquad \textit{for a.a. } (x,t) \in \Omega \times (0,T), \tag{3.3}$$

$$\partial_t \mathbf{u} + \boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}) = \mathbf{0} \qquad a.e.\ in\ \Omega \times (0,T), \tag{3.4}$$

$$\boldsymbol{w} = \mathbf{P}(-\mathbf{A}\mathbf{u} + \boldsymbol{\phi}(\mathbf{u})) - \gamma\Delta\mathbf{u} \qquad a.e.\ in\ \Omega \times (0,T), \tag{3.5}$$

$$\partial_{\boldsymbol{n}} \mathbf{u} = \mathbf{0} \qquad a.e.\ on\ \partial\Omega \times (0,T), \tag{3.6}$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad a.e.\ in\ \Omega. \tag{3.7}$$

Moreover, the following energy inequality holds:

$$\mathcal{E}(t) + \int_0^t (\boldsymbol{\alpha}(\boldsymbol{w}(s) - \overline{\boldsymbol{w}}(s)), \boldsymbol{w}(s) - \overline{\boldsymbol{w}}(s))ds \le \mathcal{E}(0), \quad \forall\, t \in [0,T]. \tag{3.8}$$

If, in addition, (**M1**) *holds and* $\overline{\mathbf{u}}_0^1 = \overline{\mathbf{u}}_0^2$*, then two solutions* $\mathbf{u}_1$, $\mathbf{u}_2$ *are such that*

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \le C\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|, \quad \forall\, t \in [0,T], \tag{3.9}$$

for some $C = C(T) > 0$ *and uniqueness follows.*

(2) *Assume* (**M0**) *and* (**E0**)–(**E1**) *and let* $\mathbf{u}_0 \in \mathcal{K} \cap \mathbf{H}^2(\Omega)$ *be such that* $\partial_{\boldsymbol{n}}\mathbf{u}_0 = \mathbf{0}$ *almost everywhere on* $\partial\Omega$*, and* $\phi(u_{0,i}) \in L^2(\Omega)$ *for any* $i = 1, \dots, N$*. Then, there is a finite energy solution pair* $(\mathbf{u}, \boldsymbol{w})$ *such that*

$$\mathbf{u} \in C([0,T]; \boldsymbol{H}^1(\Omega)) \cap L^\infty(0,T;\mathbf{H}^2(\Omega)),$$

$$\partial_t \mathbf{u} \in L^2(0,T;\mathbf{H}^1(\Omega)), \tag{3.10}$$

$$\boldsymbol{w} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)), \tag{3.11}$$

$$\phi(u_i) \in L^\infty(0,T;L^2(\Omega)), \quad i = 1, \dots, N.$$

Moreover, $\mathbf{u}$ *satisfies the energy identity*

$$\frac{d}{dt}\mathcal{E} + (\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}}) = 0 \quad for\ a.a.\ t \in [0,T]. \tag{3.12}$$

(3) *Let all the above assumptions hold along with* (**E2**) *and suppose that* $\mathbf{u}_0$ *is strictly separated, that is, there exists* $\delta_0 \in (0, \frac{1}{N})$ *such that* $\delta_0 < \mathbf{u}_0$ *everywhere in* $\overline{\Omega}$*. Then, the (unique) strong solution* $\mathbf{u}$ *is strictly separated as well, that is, there exists* $\delta = \delta(\tau, T) \in (0, \frac{1}{N}]$ *such that*

$$\delta \le \mathbf{u}(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [0,T]. \tag{3.13}$$

Remark 3.2. On account of (3.11), one could also prove that $\boldsymbol{\phi}(\mathbf{u}) \in L^2(0,T;\mathbf{L}^p(\Omega))$, where $\boldsymbol{\phi}$ is defined in (2.4), and $\mathbf{u} \in L^2(0,T;\boldsymbol{W}^{2,p}(\Omega))$ where $p = 6$ if $d = 3$, while $p \in [2,\infty)$ if $d = 2$, by slightly adapting part of the proof of [27, Theorem 3.1] (which is performed for the L^∞-in-time case). Again, the main issue is the presence of the projector $\mathbf{P}$ in the definition of $\boldsymbol{w}$ (cf. [13, Corollary 1] for the scalar case).

Remark 3.3. Notice that (3.1) implies that there exists $\rho > 0$ such that $\rho < \overline{u}_{0,i} < 1 - \rho$ for any $i = 1, \dots, N$. Indeed, we have, for any $i = 1, \dots, N$,

$$\delta_0 < \min_{j=1,\dots,N} \overline{u}_{0,j} \le \overline{u}_{0,i} = 1 - \sum_{j \ne i} \overline{u}_{0,j} \le 1 - (N-1) \min_{j=1,\dots,N} \overline{u}_{0,j} < 1 - (N-1)\delta_0,$$

and thus we can choose, for example, $\rho = \delta_0$, with $N \ge 2$.

Remark 3.4. Arguing as in [7, Proposition 2.1], we easily obtain $\sum_{i=1}^{N} u_i = 1$ and $\sum_{i=1}^{N} w_i = 0$. Moreover, by choosing $\boldsymbol{\eta} \equiv \boldsymbol{\eta}_i$, with $\boldsymbol{\eta}_i$ being the i-th unit vector, we get that the total mass of each component u_i is preserved, that is,

$$\overline{\mathbf{u}}(t) \equiv \overline{\mathbf{u}}_0, \quad \forall\, t \in [0, T].$$

Remark 3.5. From Theorem 3.1 part (2), we deduce that $\nabla u_i \cdot \mathbf{n} \in C_w([0, T]; H^{\frac{1}{2}}(\partial\Omega))$. Thus, $\nabla u_i \cdot \mathbf{n} = 0$ for any $t \in [0, T]$ almost everywhere on $\partial\Omega$, for $i = 1, \dots, N$. Furthermore, since we also have

$$\|\mathbf{u}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leq C(T),$$

with $\|\mathbf{u}(\cdot)\|_{\mathbf{H}^2(\Omega)}$ lower semicontinuous, we get

$$\|\mathbf{u}(t)\|_{\mathbf{H}^2(\Omega)} \leq C(T), \quad \forall\, t \in [0, T]. \tag{3.14}$$

Remark 3.6. Recalling Theorem 3.1 part (3), observe that (1.3) and (3.13) imply the existence of $\delta_1 := (N-1)\delta > 0$ such that $\mathbf{u} \leq 1 - \delta_1$ almost everywhere in $\Omega \times [0, T]$, that is, each component is strictly separated from the pure phases 0 and 1. Moreover, property (3.13) holds on $\overline{\Omega} \times [0, T]$, since from its proof (see Section 4.1) we deduce that, for any $t \in [0, T]$,

$$\mathbf{u}(t) \geq \delta \quad \text{a.e. in } \Omega.$$

Then, by Remark 3.5, we know that $\mathbf{u}(t) \in \mathbf{H}^2(\Omega) \hookrightarrow C(\overline{\Omega})$ for any $t \in [0, T]$, implying that

$$\mathbf{u}(x, t) \geq \delta, \quad \forall (x, t) \in \overline{\Omega} \times [0, T].$$

Remark 3.7. The quantity $\delta > 0$ in the separation property only depends on the initial data through the initial data energy $\mathcal{E}(0)$, $\overline{\mathbf{u}}_0$, δ_0, and $\|\mathbf{u}_0\|_{\mathbf{H}^2(\Omega)}$. The same goes for all the constants involved in the regularity estimates of part (2) of the Theorem 3.1, except δ_0.

Remark 3.8. As will be clear from the proof (see also Remark 4.4), in the case $N = 2$ assumption (**E2**) is not needed to prove (3.13). This agrees with the result obtained in [26] for binary mixtures.

On account of the dissipative nature of the system, we have the following uniform control of the energy $\mathcal{E}$:

Theorem 3.9. *Let the assumptions of Theorem 3.1 part (1) hold. Then, the energy of solution $\mathbf{u}$ satisfies the following inequality:*

$$\mathcal{E}(t) \leq C_1 e^{-\omega t} \mathcal{E}(0) + C_2, \quad \forall t \in [0, T], \tag{3.15}$$

where $C_1, C_2 > 0$ depend on Ω, $\boldsymbol{\alpha}$, Ψ, and $\overline{\mathbf{u}}_0$, while $\omega > 0$ is a universal constant.

We can prove that any weak solution given by Theorem 3.1 instantaneously regularizes. Thanks to this, we can show the instantaneous strict separation property in dimensions two and three. This means that, for any $\tau > 0$, there exists $0 < \delta = \delta(\tau) < \frac{1}{N}$ such that $\delta \le \mathbf{u}$ almost everywhere in $\Omega \times [\tau, +\infty)$. Again, notice that this implies the existence of $\delta_1 := (N-1)\delta > 0$ such that $\mathbf{u} \geq 1-\delta_1$ almost everywhere in $\Omega \times [\tau, +\infty)$, that is, each component is strictly separated from both the pure phases 0 and 1. More precisely, the following result holds:

Theorem 3.10. *Let the assumptions of Theorem 3.9 hold, together with* (**M1**) *and* (**E2**). *Then, the energy solution* $(\mathbf{u}, \boldsymbol{w})$ *is defined for all* $t \geq 0$ *and is such that, for any* $\tau > 0$,

$$\mathbf{u} \in C([\tau, \infty); \mathbf{H}^1(\Omega)) \cap L^\infty(\tau, \infty; \mathbf{H}^2(\Omega)), \tag{3.16}$$

$$\partial_t \mathbf{u} \in L^2(t, t+1; \mathbf{H}^1(\Omega)), \qquad \forall\, t \geq \tau, \tag{3.17}$$

$$\boldsymbol{w} \in L^\infty(\tau, \infty; \mathbf{L}^2(\Omega)), \qquad \forall\, t \geq \tau, \tag{3.18}$$

$$\phi(u_i) \in L^\infty(\tau, \infty; L^2(\Omega)), \qquad i = 1, \dots, N. \tag{3.19}$$

Moreover, $\mathbf{u}$ *and* $\boldsymbol{w}$ *are uniformly bounded in the above spaces by positive constants depending only on* Ω, $\boldsymbol{\alpha}$, Ψ, $\overline{\mathbf{u}}_0$, *and* $\mathcal{E}(0)$. *In particular, energy identity* (3.12) *holds for almost any* $t \geq \tau$. *Moreover, there exists* $0 < \delta = \delta(\tau) \leq \frac{1}{N}$ *such that*

$$\delta \leq \mathbf{u}(x,t), \quad \forall\, (x,t) \in \overline{\Omega} \times [\tau, +\infty), \tag{3.20}$$

that is, the instantaneous strict separation property holds.

Remark 3.11. It is straightforward to see that $\mathbf{u}_\delta := (\mathbf{u}-\delta)^- \in C([\tau,\infty); \mathbf{L}^2(\Omega))$. In the proof of the strict separation property (see Section 4.3) we obtain

$$\delta \leq \mathbf{u}(x,t) \quad \text{for a.a. } (x,t) \in \Omega \times [\tau, +\infty),$$

which then implies $\|\mathbf{u}_\delta(t)\| \equiv 0$ for almost any $t \in [\tau, \infty)$, and thus it holds *for any* $t \in [\tau, \infty)$, by continuity. This means that we have

$$\delta \leq \mathbf{u}(t), \quad \forall\, t \in [\tau, +\infty), \text{ a.e. in } \Omega. \tag{3.21}$$

By (3.14) and its global nature ensured by Theorem 3.10, we have that $\mathbf{u}(t) \in \mathbf{H}^2(\Omega)$ for any $t \in [\tau, +\infty)$, entailing that (3.21) holds for any $(x,t) \in \overline{\Omega} \times [\tau, +\infty)$.

Remark 3.12. We point out that, as observed in Remark 3.8, assumption (**E2**) is not needed to prove (3.20) when $N = 2$ (i.e., for binary mixtures).

3.2. Existence of the regular global attractor

We now define a complete metric space which will be the phase space of the dissipative dynamical system (see, for instance, [44]) associated with (1.6). For a given $\mathbf{M} \in \Sigma$ such that $M_i \in (0,1)$, for any $i = 1, \dots, N$, we set

$$\mathcal{V}_{\mathbf{M}} := \left\{ \mathbf{u} \in \mathbf{H}^1(\Omega) :\ 0 \leq \mathbf{u}(x) \leq 1, \text{ for a.a. } x \in \Omega,\ \overline{\mathbf{u}} = \mathbf{M},\ \sum_{i=1}^N u_i = 1 \right\},$$

endowed with the $\mathbf{H}^1$-topology. In particular, we consider the one induced by the equivalent norm $\|\mathbf{u}\|_{\mathcal{V}_\mathbf{M}} = \|\nabla\mathbf{u}\| + |\overline{\mathbf{u}}|$. This is a complete metric space. Thus, under the same assumptions of Theorem 3.10, we can define a dynamical system $(\mathcal{V}_\mathbf{M}, S(t))$ where

$$S(t) : \mathcal{V}_\mathbf{M} \to \mathcal{V}_\mathbf{M}, \quad S(t)\mathbf{u}_0 = \mathbf{u}(t), \quad \forall\, t \geq 0.$$

Observe that $S(t)$ satisfies the following properties:

- $S(0) = Id_{\mathcal{V}_\mathbf{M}}$;
- $S(t+\tau) = S(t)S(\tau)$, for every $\mathbf{u}_0 \in \mathcal{V}_\mathbf{M}$;
- $t \mapsto S(t)\mathbf{u}_0 \in C((0,\infty); \mathcal{V}_\mathbf{M})$, for every $\mathbf{u}_0 \in \mathcal{V}_\mathbf{M}$;
- $\mathbf{u}_0 \mapsto S(t)\mathbf{u}_0 \in C(\mathcal{V}_\mathbf{M}; \mathcal{V}_\mathbf{M})$, for any $t \in [0, +\infty)$.

In particular, $t \mapsto S(t)\mathbf{u}_0 \in C((0,\infty); \mathcal{V}_\mathbf{M})$ comes from the instantaneous regularization so that for any $\tau > 0$, $\mathbf{u} \in C([\tau,\infty); \mathcal{V}_\mathbf{M})$, whereas the last property can be proved as follows: from (3.9) together with the $\mathbf{H}^2$-regularity (for any $t > 0$) and the interpolation estimate

$$\|\cdot\|_{\mathbf{H}^1(\Omega)} \leq C\|\cdot\|^{\frac{1}{2}}_{\mathbf{H}^2(\Omega)}\|\cdot\|^{\frac{1}{2}},$$

we deduce that $\mathbf{u}_0 \mapsto S(t)\mathbf{u}_0 \in C(\mathcal{V}_\mathbf{M}; \mathcal{V}_\mathbf{M})$, for any $t \in (0,\infty)$. This is indeed a consequence of (3.16), since $\mathbf{u} \in L^\infty(\tau,\infty; \mathbf{H}^2(\Omega))$ for any $\tau > 0$ entails that, given two initial data $\mathbf{u}_{0,1}, \mathbf{u}_{0,2} \in \mathcal{V}_\mathbf{M}$, for any $t > 0$,

$$\begin{aligned}\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|_{\mathbf{H}^1(\Omega)} &\leq C\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|^{\frac{1}{2}}_{\mathbf{H}^2(\Omega)}\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|^{\frac{1}{2}} \\ &\leq C(t)\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|^{\frac{1}{2}} \leq C(t)\|\mathbf{u}_{0,1} - \mathbf{u}_{0,2}\|^{\frac{1}{2}},\end{aligned}$$

where in the last step we also used (3.9). The case $t = 0$ is trivial.

Furthermore, we recall that the *global attractor* is the unique compact set $\mathcal{A} \subset \mathcal{V}_\mathbf{M}$ such that

- $\mathcal{A}$ is fully invariant, that is, $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$;
- $\mathcal{A}$ is attracting for the semigroup, that is,

$$\lim_{t\to+\infty} [\mathrm{dist}_{\mathcal{V}_\mathbf{M}}(S(t)B, \mathcal{A})] = 0$$

  for every bounded set $B \subset \mathcal{V}_\mathbf{M}$. Here $\mathrm{dist}_{\mathcal{V}_\mathbf{M}}$ stands for the Hausdorff semidistance.

The dissipative inequality given by (3.15) and the instantaneous regularization of the energy solution allow us to prove the next theorem.

Theorem 3.13. *Let the assumptions of Theorem 3.10 hold. Then, the dynamical system $(\mathcal{V}_\mathbf{M}, S(t))$ admits a (unique) connected global attractor $\mathcal{A} \subset \mathcal{V}_\mathbf{M}$ which is bounded in $\mathbf{H}^2(\Omega)$.*

Remark 3.14. The proof of this result is based on showing that the dynamical system $(\mathcal{V}_\mathbf{M}, S(t))$ admits a compact absorbing set $\mathcal{B}_0$ (see Section 5 below).

3.3. Existence of an exponential attractor

Thanks to the validity of the strict separation property in dimensions two and three, we can prove the existence of an exponential attractor in dimensions two and three. We first recall (see, e.g., [39]) that a compact set $\mathcal{M} \subset \mathcal{V}_{\mathbf{M}}$ is an exponential attractor for $(\mathcal{V}_{\mathbf{M}}, S(t))$ if

- $\mathcal{M}$ is positively invariant, that is, $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \geq 0$;
- $\mathcal{M}$ is exponentially attracting, that is, there exists $\overline{\omega} > 0$ such that

  $$\mathrm{dist}_{\mathcal{V}_{\mathbf{M}}}(S(t)\mathcal{B}, \mathcal{M}) \leq \mathcal{Q}(\|\mathcal{B}\|_{\mathcal{V}_{\mathbf{M}}})e^{-\overline{\omega}t}$$

  for every bounded $\mathcal{B} \subset \mathcal{V}_{\mathbf{M}}$, where $\mathcal{Q}(\cdot)$ denotes a generic increasing positive function;
- $\mathcal{M}$ has finite fractal dimension in $\mathcal{V}_{\mathbf{M}}$, where the fractal dimension is defined as

  $$\dim_{\mathcal{V}_{\mathbf{M}}}(\mathcal{M}) = \limsup_{\epsilon \to 0^+} \frac{\log N(\epsilon)}{-\log \epsilon},$$

  and $N(\epsilon)$ is the minimum number of ϵ-balls of $\mathcal{V}_{\mathbf{M}}$ necessary to cover $\mathcal{M}$.

Observe that the exponential attractor is not unique and that, by definition, $\mathcal{A} \subset \mathcal{M}$, so that from the existence result of an exponential attractor we deduce that the global attractor $\mathcal{A}$ is of finite fractal dimension. We thus have the following:

Theorem 3.15. *Let the assumptions of Theorem 3.10 hold. Moreover, assume that* $\psi \in C^3(0,1]$. *Then, the dynamical system* $(\mathcal{V}_{\mathbf{M}}, S(t))$ *possesses an exponential attractor* $\mathcal{M}$ *which is bounded in* $\mathbf{H}^2(\Omega)$. *Besides,* $\mathcal{A} \subset \mathcal{M}$ *has finite fractal dimension in* $\mathcal{V}_{\mathbf{M}}$.

3.4. Convergence to equilibrium

In this section we discuss the convergence of any weak solution to a single equilibrium. We have all the ingredients to state and prove the result.

We consider the phase space $\mathcal{V}_{\mathbf{M}}$ as in the previous section. Under the assumptions of Theorem 3.10, we define the ω-limit set $\omega(\mathbf{u}_0)$ of a given $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$

$$\omega(\mathbf{u}_0) = \left\{\mathbf{z} \in \mathbf{H}^{2r}(\Omega) \cap \mathcal{V}_{\mathbf{M}} : \exists\, t_n \nearrow +\infty \text{ s.t. } \mathbf{u}(t_n) \to \mathbf{z} \text{ in } \mathbf{H}^{2r}(\Omega)\right\},$$

where $r \in [\frac{1}{2}, 1)$. In particular, we fix $r \in (\frac{d}{4}, 1)$. We thus have the following:

Theorem 3.16. *Let the assumptions of Theorem 3.10 hold and suppose, in addition, that* ψ *is (real) analytic in* $(0,1)$. *Then, for any* $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$, *it holds that* $\omega(\mathbf{u}_0) = \{\mathbf{u}_\infty\}$, *where* $\mathbf{u}_\infty \in \mathcal{V}_{\mathbf{M}}$ *is a solution to*

$$\begin{cases} -\gamma\Delta\mathbf{u}_\infty + \mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}_\infty) = \boldsymbol{f} & \text{a.e. in } \Omega, \\ \partial_{\mathbf{n}}\mathbf{u}_\infty = 0 & \text{a.e. on } \partial\Omega, \\ \sum_{i=1}^N u_{\infty,i} = 1 & \text{in } \Omega, \end{cases}$$

with $\boldsymbol{f} = \mathbf{PA}\mathbf{u}_\infty + \overline{\mathbf{P}\Psi_{,\mathbf{u}}(\mathbf{u}_\infty)}$. *Moreover,* $\overline{\mathbf{u}}_\infty = \mathbf{M}$*; there exists* $\delta > 0$ *so that*

$$\delta < \mathbf{u}_\infty(x), \quad \forall\, x \in \overline{\Omega};$$

and the (unique) weak solution $\mathbf{u}(t)$ *is such that*

$$\mathbf{u}(t) = S(t)\mathbf{u}_0 \underset{t\to+\infty}{\to} \mathbf{u}_\infty \quad \textit{in } \mathbf{H}^{2r}(\Omega), \quad \forall\, r \in (0,1).$$

Proof. The proof of this theorem is exactly the same as the one of [27, Theorem 3.16]. Indeed, the only difference is in the energy estimate given by the application of Łojasiewicz–Simon inequality (see [27, Section 7.3]), in which we need to substitute $\nabla \boldsymbol{w}$ with $\boldsymbol{w} - \overline{\boldsymbol{w}}$ (basically, we do not need to apply Poincaré's inequality, but we keep $\|\boldsymbol{w} - \overline{\boldsymbol{w}}\|$ in the inequality for $\mathcal{E}'$). ■

Theorem 3.16 is still valid without assumption (**E2**). Indeed, in the proof we do not need the instantaneous strict separation property, for which that assumption is essential. It is also worth noticing that, without assuming (**E2**), by the same proof of [27, Theorem 3.13], we can show that the asymptotic strict separation property holds, that is, the next theorem holds.

Theorem 3.17. *Let the assumptions of Theorem 3.10 hold except for* (**E2**)*. Then, for any* $\mathbf{M} \in (0,1)$*,* $\mathbf{M} \in \Sigma$*, and for any initial datum* $\mathbf{u}_0 \in \mathcal{V}_{\mathbf{M}}$*, there exist* $\delta > 0$ *and* $t^* = t^*(\mathbf{u}_0)$ *such that the corresponding (unique) solution* $\mathbf{u}$ *satisfies*

$$\delta < \mathbf{u}(x,t), \quad \textit{for any } (x,t) \in \overline{\Omega} \times (t^*, +\infty).$$

4. Proofs of Section 3.1

Here we collect the proofs of Theorems 3.1, 3.9, and 3.10.

4.1. Proof of Theorem 3.1

This proof is divided into three parts. We first prove (3.9), which seems to require assumption (**M1**). The reason is related to the next lemma.

Lemma 4.1. *Let* (**M1**) *hold. Then, there exists* $\gamma_N > 0$ *such that, given any matrix* $\mathbf{C} = \mathrm{diag}(c_1, \dots, c_N)$*, with* $c_i \geq 0$ *for any* $i = 1, \dots, N$*,*

$$\boldsymbol{\zeta}^T (\mathbf{C}\boldsymbol{\alpha})\boldsymbol{\zeta} \geq \gamma_N \Big(\min_{i=1,\dots,N,\ c_i\boldsymbol{\alpha}_{ii}>0} c_i \boldsymbol{\alpha}_{ii}\Big)|\boldsymbol{\zeta}|^2 \geq 0, \tag{4.1}$$

for any $\boldsymbol{\zeta} \in T\Sigma$*. In particular, for any* $N \geq 2$*, considering the equivalent structure given by* (2.3)*, we have*

$$\gamma_N := \frac{N}{N-1}.$$

Remark 4.2. Notice that, since $\boldsymbol{\alpha}$ is positive semidefinite, $\boldsymbol{\alpha}_{ii} \geq 0$ for any $i = 1, \dots, N$.

Remark 4.3. What is needed to prove (3.9) is actually (4.1). Nevertheless, the matrix structure given by (2.2) is the only example case we know that implies (4.1).

Proof. Note that, for $\boldsymbol{\zeta} \in T\Sigma$, we have $\boldsymbol{\zeta}_N = -\sum_{i=1}^{N-1} \boldsymbol{\zeta}_i$. Thus, exploiting form (2.3) of the matrix $\boldsymbol{\alpha}$,

$$\boldsymbol{\zeta}^T \mathbf{C}\boldsymbol{\alpha}\boldsymbol{\zeta} = \xi\Big[\boldsymbol{\zeta}_1, \dots, -\sum_{i=1}^{N-1} \boldsymbol{\zeta}_i\Big] \begin{bmatrix} c_1(\boldsymbol{\zeta}_1(N-1) - \sum_{j\neq 1}^{N-1} \boldsymbol{\zeta}_j + \sum_{j=1}^{N-1} \boldsymbol{\zeta}_j) \\ \vdots \\ c_i(\boldsymbol{\zeta}_i(N-1) - \sum_{j\neq i}^{N-1} \boldsymbol{\zeta}_j + \sum_{j=1}^{N-1} \boldsymbol{\zeta}_j) \\ \vdots \\ c_N(-\sum_{j=1}^{N-1} \boldsymbol{\zeta}_j - (N-1)\sum_{j=1}^{N-1} \boldsymbol{\zeta}_j) \end{bmatrix}$$

$$= \xi\Big[\boldsymbol{\zeta}_1, \dots, -\sum_{i=1}^{N-1} \boldsymbol{\zeta}_i\Big] \begin{bmatrix} c_1 N \boldsymbol{\zeta}_1 \\ \vdots \\ c_i N \boldsymbol{\zeta}_i \\ \vdots \\ -c_N N \sum_{j=1}^{N-1} \boldsymbol{\zeta}_j \end{bmatrix}$$

$$= N\xi \sum_{i=1}^{N-1} c_i |\boldsymbol{\zeta}_i|^2 + \xi N c_N \Big|\sum_{j=1}^{N-1} \boldsymbol{\zeta}_j\Big|^2$$

$$\geq \xi N(\min_{i=1,\dots,N,\ c_i>0} c_i)|\boldsymbol{\zeta}|^2 \geq \gamma_N(\min_{i=1,\dots,N, c_i\boldsymbol{\alpha}_{ii}>0} c_i\boldsymbol{\alpha}_{ii})|\boldsymbol{\zeta}|^2$$

with $\gamma_N = \frac{N}{N-1}$. Thus, (4.1) holds. ∎

Continuous dependence estimate. We can now prove (3.9). Let us consider two solutions $\mathbf{u}^1$ and $\mathbf{u}^2$ and take the difference between the equations they solve. Taking $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$ as a test function in the resulting equation and recalling, by mass conservation, that $\overline{\mathbf{u}} \equiv \mathbf{0}$, we deduce (note that $\boldsymbol{\alpha}\mathbf{P}\Psi^1_{,\mathbf{u}}(\mathbf{u}^k) = \boldsymbol{\alpha}\Psi^1_{,\mathbf{u}}(\mathbf{u}^k)$ for $k = 1, 2$, since $\boldsymbol{\alpha}(\frac{1}{N}\sum_{i=1}^N \psi'(u_i^k)\boldsymbol{\zeta}) = (\frac{1}{N}\sum_{i=1}^N \psi'(u_i^k))\boldsymbol{\alpha}\boldsymbol{\zeta} = 0$, where $\boldsymbol{\zeta} = (1, \dots, 1)$) that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \gamma(\nabla\mathbf{u}, \boldsymbol{\alpha}\nabla\mathbf{u}) + \sum_{i,j=1}^N (\boldsymbol{\alpha}_{ij}(\psi'(u_j^1) - \psi'(u_j^2)), u_i) = (\boldsymbol{\alpha}\mathbf{A}\mathbf{u}, \mathbf{u}). \tag{4.2}$$

Notice that $\overline{\boldsymbol{w}}_1 - \overline{\boldsymbol{w}}_2$ does not appear in (4.2), since we have

$$\begin{aligned}(\boldsymbol{\alpha}((\boldsymbol{w}_1 - \boldsymbol{w}_2) - (\overline{\boldsymbol{w}_1 - \boldsymbol{w}_2})), \mathbf{u}) &= (\boldsymbol{\alpha}((\boldsymbol{w}_1 - \boldsymbol{w}_2) - (\overline{\boldsymbol{w}_1 - \boldsymbol{w}_2})), \mathbf{u} - \overline{\mathbf{u}}) \\ &= (\boldsymbol{\alpha}(\boldsymbol{w}_1 - \boldsymbol{w}_2), \mathbf{u} - \overline{\mathbf{u}}) = (\boldsymbol{\alpha}(\boldsymbol{w}_1 - \boldsymbol{w}_2), \mathbf{u}),\end{aligned}$$

recalling in the last equality that $\overline{\mathbf{u}} \equiv \mathbf{0}$. Lemma 4.1 then entails

$$\gamma(\nabla\mathbf{u}, \boldsymbol{\alpha}\nabla\mathbf{u}) \geq 0.$$

Then, by the Cauchy–Schwarz inequality,

$$(\boldsymbol{\alpha}\mathbf{A}\mathbf{u}, \mathbf{u}) \leq C\|\mathbf{u}\|^2.$$

In conclusion, we have

$$\sum_{i,j=1}^{N} (\boldsymbol{\alpha}_{ij}(\psi'(u_j^1) - \psi'(u_j^2)), u_i) = \sum_{i,j=1}^{N} \int_\Omega \int_0^1 \boldsymbol{\alpha}_{ij}\psi''(su_j^1 + (1-s)u_j^2)u_j u_i \, ds dx$$
$$= \int_\Omega \mathbf{u}^T \boldsymbol{\alpha}\mathbf{C}\mathbf{u} dx,$$

where

$$\mathbf{C} = \mathrm{diag}(c_1, \ldots, c_N)$$
$$:= \mathrm{diag}\Big(\int_0^1 \psi''(su_1^1 + (1-s)u_1^2)ds, \ldots, \int_0^1 \psi''(su_N^1 + (1-s)u_N^2)ds\Big),$$

so that $c_i \geq 0$, for any $i = 1, \ldots, N$, by assumption (**E0**). Observing now that $\mathbf{u}(x,t) \in T\Sigma$ for almost any $(x,t) \in \Omega \times (0,T)$ and (by symmetry) $\mathbf{u}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{u} = \mathbf{u}^T\mathbf{C}\boldsymbol{\alpha}\mathbf{u}$, thanks to Lemma 4.1, for almost any $(x,t) \in \Omega \times (0,T)$, we have

$$\mathbf{u}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{u} \geq 0,$$

so that

$$\sum_{i,j=1}^{N} (\boldsymbol{\alpha}_{ij}(\psi'(u_j^1) - \psi'(u_j^2)), u_i) = \int_\Omega \mathbf{u}^T\boldsymbol{\alpha}\mathbf{C}\mathbf{u} dx \geq 0 \quad \text{a.e. in } (0,T).$$

Therefore, from (4.2), we infer

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|^2 \leq C\|\mathbf{u}(t)\|^2, \quad \text{for a.a. } t \in (0,T),$$

and the Gronwall lemma gives (3.9).

Existence of a solution. We consider approximation (2.5). In particular, for each $\varepsilon > 0$ sufficiently small, we set

$$\boldsymbol{\phi}_\varepsilon(\mathbf{y}) = \Psi^1_{\varepsilon,\mathbf{y}}(\mathbf{y}) = \{\psi'_\varepsilon(y_i)\}_{i=1,\ldots,N}, \quad \forall\, \mathbf{y} \in \mathbb{R}^N.$$

We then fix $0 < \varepsilon < \varepsilon_0$ and first define the Galerkin approximation of the problem. We consider the complete system of N-dimensional eigenfunctions $\{\mathbf{e}_i\}_i$ of the problem $-\Delta\mathbf{e}_i = \lambda_i\mathbf{e}_i$, with homogeneous Neumann boundary conditions $\partial_\mathbf{n}\mathbf{e}_i = 0$ on $\partial\Omega$ (λ_i is the eigenvalue corresponding to $\mathbf{e}_i$), subject to the constraints $\overline{\mathbf{e}}_i = 0$ and $\sum_{j=1}^N(\mathbf{e}_i)_j \equiv 0$. The family $\{\mathbf{e}_i\}_i$ can be tuned to form an orthogonal basis in $\boldsymbol{V}_0$, orthonormal in $\boldsymbol{H}_0$ (see also [27, Appendix 8.1]). We then set $\boldsymbol{m} := \overline{\mathbf{u}}_0$ and introduce the finite-dimensional spaces

$$\mathbf{V}_n := \mathrm{span}\{\mathbf{e}_i, i = 1, \ldots, n\}, \quad \forall\, n \geq 1,$$

and look for a function $\mathbf{u}_{n,\varepsilon} \in \mathbf{V}_n$ of the form

$$\mathbf{u}_{n,\varepsilon}(t) = \sum_{i=1}^{n} \widehat{\alpha}_i(t)\mathbf{e}_i \in \mathbf{V}_n,$$

and for $\boldsymbol{w}_{n,\varepsilon} \in \widetilde{\boldsymbol{V}}_0$ such that

$$\boldsymbol{w}_{n,\varepsilon}(t) - \overline{\boldsymbol{w}}_{n,\varepsilon}(t) = \sum_{i=1}^{n} \delta_i(t)\mathbf{e}_i \in \mathbf{V}_n,$$

solving the equations

$$(\partial_t \mathbf{u}_{n,\varepsilon}, \mathbf{v}) + (\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \mathbf{v}) = 0, \tag{4.3}$$

$$\begin{aligned}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}, \mathbf{v}) = \gamma(\nabla \mathbf{u}_{n,\varepsilon}, \nabla \mathbf{v}) + (\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}) - \mathbf{A}(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}))\\ - \overline{\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}) - \mathbf{A}(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}))}, \mathbf{v}),\end{aligned} \tag{4.4}$$

$$\overline{\boldsymbol{w}}_{n,\varepsilon} = \overline{\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}) - \mathbf{A}(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m}))}, \tag{4.5}$$

$$\mathbf{u}_{n,\varepsilon}(0) = \mathbf{u}_{n,0}, \tag{4.6}$$

for any $\mathbf{v} \in \mathbf{V}_n$ and for any $t \in [0, T]$ where $\mathbf{u}_{n,0}$ is the $\mathbf{L}^2(\Omega)$-projection on $\mathbf{V}_n$ of the vector $\mathbf{u}_0 - \boldsymbol{m} \in \boldsymbol{H}_0$.

Let us first notice that the quantity $\overline{\boldsymbol{w}}_{n,\varepsilon}$ must be specified, since any test function $\mathbf{v} \in \mathbf{V}_n$ has zero integral mean. Moreover, by construction,

$$\overline{\mathbf{u}}_{n,\varepsilon} \equiv \mathbf{0}, \quad \mathbf{P}\mathbf{u}_{n,\varepsilon} = \mathbf{u}_{n,\varepsilon}, \quad \mathbf{P}\boldsymbol{w}_{n,\varepsilon} = \boldsymbol{w}_{n,\varepsilon}.$$

In the rest of the paper, we will denote by C a generic positive constant independent of n. Any other dependence is explicitly pointed out if necessary.

Recalling that ψ_ε' is at least $C^1(\mathbb{R})$, we can locally solve the above Cauchy problem given by (4.3)–(4.4), (4.6) in the unknowns $\{\widehat{\alpha}_i\}_i$ and find a unique maximal solution $\widehat{\alpha}^{(n)} \in C^1([0, t_{n,\varepsilon}]; \mathbb{R}^n)$, from which we also obtain by comparison a unique $\delta^{(n)} \in C^1([0, t_{n,\varepsilon}]; \mathbb{R}^n)$. Then, by substitution in (4.5), we immediately obtain the complete quantity $\boldsymbol{w}_{n,\varepsilon}$. It is now standard to test (4.3) by $\mathbf{v} = \boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon} \in \mathbf{V}_n$ and obtain the energy identity

$$\frac{d}{dt}\mathcal{E}_{n,\varepsilon} + (\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}) = 0, \tag{4.7}$$

where

$$\mathcal{E}_{n,\varepsilon} := \frac{\gamma}{2}\|\nabla \mathbf{u}_{n,\varepsilon}\|^2 + \int_\Omega \Psi_\varepsilon(\mathbf{u}_{n,\varepsilon} + \boldsymbol{m})dx.$$

Let us observe that, since ψ_ε' is Lipschitz (see (2.5)), and recalling that $\Psi_\varepsilon(\mathbf{u}_0) \le \Psi(\mathbf{u}_0)$, we obtain

$$\begin{aligned}\int_\Omega \Psi_\varepsilon(\mathbf{u}_{n,\varepsilon}(0) + \boldsymbol{m})dx &= \int_\Omega (\Psi_\varepsilon(\mathbf{u}_{n,0} + \boldsymbol{m}) - \Psi_\varepsilon(\mathbf{u}_0))dx + \int_\Omega \Psi_\varepsilon(\mathbf{u}_0)dx\\ &\le C_\varepsilon\|\mathbf{u}_{n,0} + \boldsymbol{m} - \mathbf{u}_0\| + \int_\Omega \Psi(\mathbf{u}_0)dx.\end{aligned} \tag{4.8}$$

Therefore, since clearly $\|\mathbf{u}_{n,0}+\boldsymbol{m} - \mathbf{u}_0\| \to 0$ as $n \to \infty$, for any $\varepsilon > 0$, there exists an $\overline{n} = \overline{n}(\varepsilon)$ such that

$$\int_\Omega \Psi_\varepsilon(\mathbf{u}_{n,\varepsilon}(0) + \boldsymbol{m})dx \leq C, \quad \forall n > \overline{n}. \tag{4.9}$$

An application of Gronwall's lemma then gives, thanks to (4.9) and $\|\nabla \mathbf{u}_{n,0}\| \leq \|\nabla \mathbf{u}_0\|$,

$$\mathcal{E}_{n,\varepsilon}(t) + \int_0^t (\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon}(s) - \overline{\boldsymbol{w}}_{n,\varepsilon}(s)), \boldsymbol{w}_{n,\varepsilon}(s) - \overline{\boldsymbol{w}}_{n,\varepsilon}(s))ds \leq C, \quad \forall n > \overline{n}.$$

Now, recalling property (vi) of ψ_ε, it is immediate to see that for any $\varepsilon < \varepsilon_0$,

$$\int_\Omega \Psi_\varepsilon(\mathbf{u}_{n,\varepsilon}(t)+\boldsymbol{m})dx \geq -K$$

for some $K > 0$, so that we can conclude, for any $\varepsilon < \varepsilon_0$,

$$\|\mathbf{u}_{n,\varepsilon}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\Psi_\varepsilon(\mathbf{u}_{n,\varepsilon}+\boldsymbol{m})\|_{L^\infty(0,T;\mathbf{L}^1(\Omega))} + \|\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq C,$$
$$\forall n > \overline{n}(\varepsilon),$$

where we also exploited (2.1). Clearly, C does not depend on ε. From this we can easily deduce that local maximal time $t_{n,\varepsilon}$ is $+\infty$. Moreover, from these estimates we can clearly derive, by comparison, that

$$\|\partial_t \mathbf{u}_{n,\varepsilon}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq C, \quad \forall n > \overline{n}.$$

These estimates, together with the fact that ψ'_ε is Lipschitz, give from (4.5) that

$$\|\overline{\boldsymbol{w}}_{n,\varepsilon}\|_{L^\infty(0,T)} \leq C_\varepsilon, \quad \forall n > \overline{n}.$$

Here C_ε could depend on ε. The obtained bounds are enough to pass to the limit as $n \to \infty$ by standard compactness arguments. However, since we also need to prove the existence of strong solutions, we now assume $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$ such that $\partial_\mathbf{n} \mathbf{u}_0 = \mathbf{0}$ almost everywhere on $\partial\Omega$, together with $\phi(u_{0,i}) \in L^2(\Omega)$, for any $i = 1, \ldots, N$, and find a higher-order estimate, before passing to the limit. In particular, we test (4.3) with $\mathbf{v} = \partial_t(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}) \in \mathbf{V}_n$. Recalling that $\mathbf{P}$ is selfadjoint and $\overline{\partial_t \mathbf{u}_{n,\varepsilon}} \equiv \mathbf{0}$ by construction, we obtain

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}) + (\partial_t \mathbf{u}_{n,\varepsilon}, \partial_t \boldsymbol{w}_{n,\varepsilon}) = 0. \tag{4.10}$$

Using (4.4), since $\partial_t \mathbf{u}_{n,\varepsilon} \in \mathbf{V}_n$, we find

$$(\partial_t \mathbf{u}_{n,\varepsilon}, \partial_t \boldsymbol{w}_{n,\varepsilon}) = \sum_{i=1}^N \int_\Omega \phi'_\varepsilon(u_{n,\varepsilon,i} + \boldsymbol{m})|\partial_t \mathbf{u}_{n,\varepsilon}|^2 dx$$
$$- (\partial_t \mathbf{u}_{n,\varepsilon}, \mathbf{A}\partial_t \mathbf{u}_{n,\varepsilon}) + \gamma\|\nabla\partial_t \mathbf{u}_{n,\varepsilon}\|^2.$$

Since $\phi_\varepsilon' \geq 0$ by property (iv) of ψ_ε, we have only to treat the term related to the matrix $\mathbf{A}$. This is readily done by comparison with (4.3): indeed, since $\mathbf{v} = \partial_t \mathbf{u}_{n,\varepsilon} \in \mathbf{V}_n$, we get, by the Cauchy–Schwarz, Young, and Poincaré inequalities,

$$\begin{aligned}|(\partial_t \mathbf{u}_{n,\varepsilon}, \partial_t \mathbf{u}_{n,\varepsilon})| &\leq C\|\partial_t \mathbf{u}_{n,\varepsilon}\|^2 \leq C|(\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \partial_t \mathbf{u}_{n,\varepsilon})| \\ &\leq \frac{\gamma}{2}\|\nabla \partial_t \mathbf{u}_{n,\varepsilon}\|^2 + C\|\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}\|^2.\end{aligned}$$

Putting everything together in (4.10) and recalling (2.1), we end up with

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}) + \frac{\gamma}{2}\|\nabla \partial_t \mathbf{u}_{n,\varepsilon}\|^2 \\ &\quad \leq C(\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}), \boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}).\end{aligned} \tag{4.11}$$

Observe now that, from (4.4),

$$\begin{aligned}&\|(\boldsymbol{\alpha}(\boldsymbol{w}_{n,\varepsilon}(0) - \overline{\boldsymbol{w}}_{n,\varepsilon}(0)), \boldsymbol{w}_{n,\varepsilon}(0) - \overline{\boldsymbol{w}}_{n,\varepsilon}(0))\| \\ &\quad \leq C\Big(\|\Delta \mathbf{u}_{n,0}\|^2 + \|\mathbf{A}\mathbf{u}_{n,0}\|^2 + \sum_{i=1}^N \|\phi_\varepsilon(u_{n,0,i} + \boldsymbol{m})\|^2\Big).\end{aligned}$$

On the other hand, by the properties of the eigenfunctions, we have

$$\|\Delta \mathbf{u}_{n,0}\|^2 + \|\mathbf{A}\mathbf{u}_{n,0}\|^2 \leq C(|\boldsymbol{m}|^2 + \|\Delta \mathbf{u}_0\|^2 + \|\mathbf{u}_0\|^2) \leq C\|\mathbf{u}_0\|^2_{\mathbf{H}^2(\Omega)}.$$

Thus, recalling properties (ii)–(iii) of ψ_ε, we get

$$\begin{aligned}\sum_{i=1}^N \|\phi_\varepsilon(u_{n,0,i} + \boldsymbol{m})\|^2 &\leq 2\sum_{i=1}^N \|\phi_\varepsilon(u_{n,0,i} + \boldsymbol{m}) - \phi_\varepsilon(u_{0,i})\|^2 + 2\sum_{i=1}^N \|\phi_\varepsilon(u_{0,i})\|^2 \\ &\leq \frac{2}{\varepsilon^2}\|\mathbf{u}_{n,0} + \boldsymbol{m} - \mathbf{u}_0\|^2 + 2\sum_{i=1}^N \|\phi(u_{0,i})\|^2.\end{aligned}$$

Therefore, since $\|\mathbf{u}_{n,0} + \boldsymbol{m} - \mathbf{u}_0\| \to 0$ as $n \to \infty$, and by the stronger assumptions on the initial data, we deduce that for any $\varepsilon < \varepsilon_0$, there exists $\overline{\overline{n}} = \overline{\overline{n}}(\varepsilon) > 0$ such that

$$\sum_{i=1}^N \|\phi_\varepsilon(u_{n,0,i} + \boldsymbol{m})\|^2 \leq C + 2\sum_{i=1}^N \|\phi(u_{0,i})\|^2, \quad \forall n > \overline{\overline{n}}.$$

We can thus conclude that, for any $n > n_0(\varepsilon) = \max\{\overline{n}, \overline{\overline{n}}\}$, owing to Gronwall's lemma and (2.1), it holds that

$$\|\boldsymbol{w}_{n,\varepsilon} - \overline{\boldsymbol{w}}_{n,\varepsilon}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\partial_t \mathbf{u}_{n,\varepsilon}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C(T), \quad \forall n > n_0,$$

where $C(T)$ does not depend on ε. Furthermore, by comparison (choosing $\mathbf{v} = \partial_t \mathbf{u}_{n,\varepsilon}$ in (4.3)) it also holds that

$$\|\partial_t \mathbf{u}_{n,\varepsilon}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C(T), \quad \forall n > n_0.$$

We can now pass to the limit in n for both of the situations (according to the regularity of the initial data), to deduce, by standard compactness arguments, the following statement: for any $\varepsilon < \varepsilon_0$, there exists a pair $(\mathbf{u}_\varepsilon, \boldsymbol{w}_\varepsilon)$ defined on $[0, +\infty)$, with $\boldsymbol{w}_\varepsilon(t) \in \widetilde{\boldsymbol{V}}_0$ for almost any $t \geq 0$, such that (in the case of less regularity on $\mathbf{u}_0$) for each $T > 0$,

$$\begin{aligned} \mathbf{u}_\varepsilon &\in L^\infty(0, T; \mathbf{H}^1(\Omega)),\\ \partial_t \mathbf{u}_\varepsilon &\in L^2(0, T; \mathbf{L}^2(\Omega)),\\ \boldsymbol{w}_\varepsilon &\in L^2(0, T; \mathbf{L}^2(\Omega)). \end{aligned}$$

and

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\partial_t \mathbf{u}_\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{w}_\varepsilon - \overline{\boldsymbol{w}}_\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq C(T), \quad (4.12)$$

for some $C(T) > 0$ independent of ε, whereas there exists $C_\varepsilon > 0$ such that

$$\|\overline{\boldsymbol{w}}_\varepsilon\|_{L^\infty(0,T)} \leq C_\varepsilon.$$

If the stronger assumptions hold (see Theorem 3.1 part (2)), then there exists a constant $C > 0$, depending on the initial datum and on T, but independent of ε, such that

$$\begin{aligned} &\|\boldsymbol{w}_\varepsilon - \overline{\boldsymbol{w}}_\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega)))\cap L^2(0,T;\mathbf{H}^1(\Omega)))} + \|\partial_t \mathbf{u}_\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\\ &\qquad + \|\partial_t \mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C, \end{aligned} \quad (4.13)$$

where the $L^2(0, T; \mathbf{H}^1(\Omega))$ control on the chemical potential differences is obtained by comparison in (4.14) below. It is then standard to show that $(\mathbf{u}_\varepsilon, \boldsymbol{w}_\varepsilon)$ satisfies

$$\partial_t \mathbf{u}_\varepsilon + \boldsymbol{\alpha}(\boldsymbol{w}_\varepsilon - \overline{\boldsymbol{w}}_\varepsilon) = 0, \quad \text{a.e. in } \Omega \times (0, T), \quad (4.14)$$

$$(\boldsymbol{w}_\varepsilon, \boldsymbol{\eta}) = \gamma(\nabla \mathbf{u}_\varepsilon, \nabla \boldsymbol{\eta}) + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\eta}), \quad \forall\, \boldsymbol{\eta} \in \mathbf{H}^1(\Omega), \text{ a.e. in } (0, T), \quad (4.15)$$

$$\mathbf{u}_\varepsilon(0) = \mathbf{u}_0, \quad \text{a.e. in } \Omega.$$

Notice that, to be precise, we find that $\mathbf{u}_{n,\varepsilon}$ converges in suitable norms to a function $\tilde{\mathbf{u}}_\varepsilon(t) \in \boldsymbol{V}_0$ (for almost any $t \geq 0$) as $n \to \infty$. We then define $\mathbf{u}_\varepsilon := \tilde{\mathbf{u}}_\varepsilon + \boldsymbol{m}$ to obtain the results above. Then, by elliptic regularity, since $\boldsymbol{\phi}_\varepsilon$ is Lipschitz, from (4.15) we deduce its strong version—namely, $\mathbf{u}_\varepsilon \in L^2(0, T; \mathbf{H}^2(\Omega))$ and

$$\boldsymbol{w}_\varepsilon = -\gamma \Delta \mathbf{u}_\varepsilon + \mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \quad \text{a.e. in } \Omega \times (0, T), \quad (4.16)$$

$$\partial_{\mathbf{n}} \mathbf{u}_\varepsilon = 0, \quad \text{a.e. in } \partial\Omega \times (0, T). \quad (4.17)$$

By standard computations (see also [7] for similar results), we then have

- Conservation of mass:

$$\overline{\mathbf{u}}_\varepsilon(t) = \overline{\mathbf{u}}_0, \quad \forall\, t \geq 0.$$

- Conservation of total mass:

$$\sum_{i=1}^{N} u_{\varepsilon,i}(x,t) = 1, \qquad \text{for a.a. } x \in \Omega \text{ and for all } t \in [0,T]. \tag{4.18}$$

- Conservation of chemical potential differences

$$\sum_{i=1}^{N} w_{\varepsilon,i} = 0, \qquad \text{for a.a. } x \in \Omega \text{ and a.a. } t \in (0,T).$$

- From (4.7)–(4.8), by standard arguments, the following energy inequality holds:

$$\mathcal{E}_\varepsilon(t) + \int_0^t (\boldsymbol{\alpha}(\boldsymbol{w}_\varepsilon - \overline{\boldsymbol{w}}_\varepsilon), \boldsymbol{w}_\varepsilon - \overline{\boldsymbol{w}}_\varepsilon) \le \mathcal{E}_\varepsilon(0),$$

  for any $t \in [0,T]$, where

$$\mathcal{E}_\varepsilon := \frac{\gamma}{2}\|\nabla \mathbf{u}_\varepsilon\|^2 + \int_\Omega \Psi_\varepsilon(\mathbf{u}_\varepsilon)dx.$$

At this point, we can argue as in the proof of [27, Theorem 3.1] (which is based on [28]), in order to control $\overline{\boldsymbol{w}}_\varepsilon(t)$, which then allows us to control $\|\boldsymbol{w}_\varepsilon(t)\|$. Following the proof of [28, Lemma 3.3], we define

$$\boldsymbol{w}_{\varepsilon,0} := \boldsymbol{w}_\varepsilon - \boldsymbol{\lambda}_\varepsilon,$$

where, on account of the boundary conditions,

$$\boldsymbol{\lambda}_\varepsilon := \overline{\boldsymbol{w}}_\varepsilon = \overline{\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon))}.$$

Taking advantage of (4.15), we have,

$$\begin{aligned}(\boldsymbol{w}_{\varepsilon,0} + \boldsymbol{\lambda}_\varepsilon, \boldsymbol{\eta}) &= \gamma(\nabla \mathbf{u}_\varepsilon, \nabla \boldsymbol{\eta}) + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\eta}),\\ &\forall\, \boldsymbol{\eta} \in \mathbf{H}^1(\Omega), \quad \text{a.e. in } (0,T).\end{aligned} \tag{4.19}$$

Exploiting the convexity of Ψ_ε^1, for any $\mathbf{k} \in \mathbf{G}$ ($\mathbf{G}$ being the Gibbs simplex), because $\mathbf{k} - \mathbf{u}_\varepsilon \in T\Sigma$ almost everywhere in $\Omega \times (0,T)$, we find

$$\begin{aligned}C &\ge \int_\Omega \Psi_\varepsilon^1(\mathbf{k})dx \ge \int_\Omega \Psi_\varepsilon^1(\mathbf{u}_\varepsilon)dx + \int_\Omega \Psi_{\varepsilon,\mathbf{u}}^1(\mathbf{u}_\varepsilon)\cdot(\mathbf{k} - \mathbf{u}_\varepsilon)dx\\ &= \int_\Omega \Psi_\varepsilon^1(\mathbf{u}_\varepsilon)dx + \int_\Omega \mathbf{P}\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\cdot(\mathbf{k} - \mathbf{u}_\varepsilon)dx,\end{aligned} \tag{4.20}$$

where we used (see property (i) of ψ_ε)

$$\int_\Omega \Psi_\varepsilon^1(\mathbf{k})dx \le \int_\Omega \Psi^1(\mathbf{k})dx \le \max_{\mathbf{s}\in[0,1]} |\Psi^1(\mathbf{s})| = C.$$

Here and in what follows, $C > 0$ stands for a generic constant independent of ε. Recalling that $\Psi^1_{\varepsilon,\mathbf{u}}(\mathbf{u}_\varepsilon) = \{\phi_\varepsilon(u_{\varepsilon,i})\}_{i=1,\ldots,N}$ and choosing $\boldsymbol{\eta} = \mathbf{k} - \mathbf{u}_\varepsilon$ in (4.19), on account of (4.20), we deduce that

$$\begin{aligned} C &\geq \int_\Omega \Psi^1_\varepsilon(\mathbf{k})dx \\ &\geq \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon)dx + (\mathbf{P}(\mathbf{A}\mathbf{u}_\varepsilon), \mathbf{k} - \mathbf{u}_\varepsilon)dx \\ &\quad + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 + (\boldsymbol{w}_{\varepsilon,0}, \mathbf{k} - \mathbf{u}_\varepsilon) + (\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon), \end{aligned}$$

for almost all $t \in (0, T)$. On the other hand, we have ($\mathbf{k} \in \mathbf{G}$, and thus, $0 \leq \mathbf{k} \leq 1$)

$$\int_\Omega \sum_{i=1}^N k_i^2 dx \leq \int_\Omega \Big(\sum_{i=1}^N k_i\Big)^2 dx = |\Omega|_d.$$

Then, using Cauchy–Schwarz's and Young's inequalities and recalling property (vi) of ψ_ε, we obtain

$$\begin{aligned} (\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon) + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 - K &\leq (\boldsymbol{\lambda}_\varepsilon, \mathbf{k} - \mathbf{u}_\varepsilon) + \gamma\|\nabla \mathbf{u}_\varepsilon\|^2 + \int_\Omega \Psi^1_\varepsilon(\mathbf{u}_\varepsilon)dx \\ &\leq C - (\mathbf{P}(\mathbf{A}\mathbf{u}_\varepsilon), \mathbf{k} - \mathbf{u}_\varepsilon) - (\boldsymbol{w}_{\varepsilon,0}, \mathbf{k} - \mathbf{u}_\varepsilon) \\ &\leq C(1 + \|\mathbf{u}_\varepsilon\| + \|\mathbf{u}_\varepsilon\|^2 + \|\boldsymbol{w}_{\varepsilon,0}\|(1 + \|\mathbf{u}_\varepsilon\|)) \leq C(1 + \|\boldsymbol{w}_{\varepsilon,0}\|), \end{aligned} \tag{4.21}$$

where in the last estimate we have exploited (4.12). By the conservation of mass and Remark 3.3, we also deduce that for all $i = 1, \ldots, N$ and all $t \in [0, T]$,

$$0 < \delta_0 < \bar{\mathbf{u}}_{\varepsilon,i}(t) < 1 - (N-1)\delta_0 < 1 - \delta_0.$$

Therefore, for any fixed $k, l = 1, \ldots, N$, we choose

$$\mathbf{k} = \bar{\mathbf{u}}_\varepsilon + \delta_0 \operatorname{sign}(\lambda_{\varepsilon,k} - \lambda_{\varepsilon,l})(\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_l) \in \mathbf{G}$$

in (4.21), where

$$\boldsymbol{\zeta}_j := (0, \ldots, \underbrace{1}_{j}, \ldots, 0).$$

Thus, from (4.21) we get that

$$|(\lambda_{\varepsilon,k} - \lambda_{\varepsilon,l})(t)| \leq \frac{C}{\delta_0|\Omega|_d}(1 + \|\boldsymbol{w}_{\varepsilon,0}\|). \tag{4.22}$$

Integrating $|(\lambda_{\varepsilon,k} - \lambda_{\varepsilon,l})(t)|^2$ over $(0, T)$ and using the identity

$$\boldsymbol{\lambda}_\varepsilon = \frac{1}{N}\Big(\sum_{l=1}^N (\lambda_{\varepsilon,k} - \lambda_{\varepsilon,l})\Big)_{k=1,\ldots,N},$$

we find, owing to (4.12),

$$\int_0^T |\boldsymbol{\lambda}_\varepsilon(t)|^2 dt \le C.$$

This, using again (4.12), gives

$$\|\boldsymbol{w}_\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \le C. \tag{4.23}$$

As a consequence, we deduce from (4.22) that

$$|\boldsymbol{\lambda}_\varepsilon(t)|^2 \le C(1 + \|\boldsymbol{w}_\varepsilon(t) - \overline{\boldsymbol{w}}_\varepsilon(t)\|^2),$$

for almost any $t \in (0,T)$. Therefore, in the case of a more regular initial datum (see (4.13)), we have

$$\|\boldsymbol{w}_\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{w}_\varepsilon\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \le C. \tag{4.24}$$

We are now left with some estimates related to $\phi_\varepsilon(u_{\varepsilon,i})$. We follow again the proof of [27, Theorem 3.1]. Since ϕ_ε' is bounded for a fixed $\varepsilon \in (0,\varepsilon_0)$, we have that

$$\nabla\phi_\varepsilon(u_{\varepsilon,i}) = \phi_\varepsilon'(u_{\varepsilon,i})\nabla u_{\varepsilon,i} \in \mathbf{L}^2(\Omega),$$

for almost any $t \in (0,T)$. Thus, we can test (4.15) with $\boldsymbol{\eta} = \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon(t))$ to get

$$\begin{aligned}\sum_{i=1}^N (w_{\varepsilon,i}, \phi_\varepsilon(u_{\varepsilon,i})) &= \sum_{i=1}^N (\gamma(\nabla u_{\varepsilon,i}, \phi_\varepsilon'(u_{\varepsilon,i})\nabla u_{\varepsilon,i})) \\ &\quad + (\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon + \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)).\end{aligned} \tag{4.25}$$

Observe that

$$(\mathbf{P}(\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)), \boldsymbol{\phi}_\varepsilon(u_\varepsilon)) = \sum_{k=1}^N \int_\Omega (\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N}\sum_{l=1}^N \phi_\varepsilon(u_{\varepsilon,l}))\phi_\varepsilon(u_{\varepsilon,k}) dx,$$

and

$$\begin{aligned}&\sum_{k=1}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \frac{1}{N}\sum_{l=1}^N \phi_\varepsilon(u_{\varepsilon,l}))\phi_\varepsilon(u_{\varepsilon,k}) \\ &\quad = \frac{1}{N}\sum_{k,l=1}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))\phi_\varepsilon(u_{\varepsilon,k}) \\ &\quad = \frac{1}{N}\sum_{k<l}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))\phi_\varepsilon(u_{\varepsilon,k}) + \frac{1}{N}\sum_{k>l}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))\phi_\varepsilon(u_{\varepsilon,k}) \\ &\quad = \frac{1}{N}\sum_{k<l}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l})) \\ &\quad = \frac{1}{N}\sum_{k<l}^N (\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))^2.\end{aligned}$$

Thanks to (4.18), we have

$$u_{\varepsilon,m} := \min_{i=1,\dots,N} u_{\varepsilon,i} \le \frac{1}{N} \le \max_{i=1,\dots,N} u_{\varepsilon,i} =: u_{\varepsilon,M},$$

so that, ϕ_ε being monotone, we infer

$$\begin{aligned}
\frac{1}{N}\sum_{k<l}^{N}(\phi_\varepsilon(u_{\varepsilon,k}) - \phi_\varepsilon(u_{\varepsilon,l}))^2 &\ge \frac{1}{N}(\phi_\varepsilon(u_{\varepsilon,m}) - \phi_\varepsilon(u_{\varepsilon,M}))^2\\
&\ge \frac{1}{N}\max_{i=1,\dots,N}\Big(\phi_\varepsilon(u_{\varepsilon,i}) - \phi_\varepsilon\Big(\frac{1}{N}\Big)\Big)^2\\
&\ge \frac{1}{N}\max_{i=1,\dots,N}\Big(\frac{1}{2}\phi_\varepsilon(u_{\varepsilon,i})^2 - \phi_\varepsilon\Big(\frac{1}{N}\Big)^2\Big)\\
&\ge \frac{1}{2N}\max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2 - C,
\end{aligned}$$

owing to the inequality $(a-b)^2 \ge \frac{1}{2}a^2 - b^2$. Notice that C is independent of ε provided that we choose ε sufficiently small. Indeed, since we have the pointwise convergence $\phi_\varepsilon(\frac{1}{N}) \to \phi(\frac{1}{N})$ as $\varepsilon \to 0^+$, there exists $C > 0$, independent of ε, such that $|\phi_\varepsilon(\frac{1}{N})| \le C$ for any $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ sufficiently small. Then, we get

$$\sum_{i=1}^{N}(w_{\varepsilon,i}, \phi_\varepsilon(u_{\varepsilon,i})) \le \sum_{i=1}^{N}\|w_{\varepsilon,i}\|\|\phi_\varepsilon(u_{\varepsilon,i})\| \le C\|\boldsymbol{w}_\varepsilon\|^2 + \frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2 dx,$$

and (see (4.12))

$$\begin{aligned}
|(\mathbf{P}(-\mathbf{A}\mathbf{u}_\varepsilon, \boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon))| &\le C\|\mathbf{u}_\varepsilon\|^2 + \frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2 dx\\
&\le C + \frac{1}{8N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2 dx.
\end{aligned}$$

Therefore, on account of the above inequalities and recalling that $\phi_\varepsilon' \ge 0$, we deduce from (4.25) that

$$\frac{1}{4N}\int_\Omega \max_{i=1,\dots,N}\phi_\varepsilon(u_{\varepsilon,i})^2 dx \le C(1 + \|\boldsymbol{w}_\varepsilon\|^2), \tag{4.26}$$

which yields (see (4.23))

$$\|\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \le C(T). \tag{4.27}$$

From this result, together with (4.12) and (4.23), by elliptic regularity, we infer from (4.16)–(4.17) that

$$\|\mathbf{u}_\varepsilon\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \le C(T).$$

Moreover, from (4.26), assuming a more regular initial datum, we infer (see (4.24))

$$\|\boldsymbol{\phi}_\varepsilon(\mathbf{u}_\varepsilon)\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \le C(T),$$

as well as

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \le C(T).$$

We have obtained all the bounds we need to pass to the limit as $\varepsilon \to 0^+$. Being this step standard (see, e.g., [28]), we only present a sketch of the argument. By compactness we immediately deduce that, up to subsequences,

$$\begin{aligned}
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly* in } L^\infty(0,T;\mathbf{H}^1(\Omega)),\\
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^2(0,T;\mathbf{H}^2(\Omega)),\\
\partial_t\mathbf{u}_\varepsilon &\rightharpoonup \partial_t\mathbf{u} && \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
\mathbf{u}_\varepsilon &\to \mathbf{u} && \text{strongly in } L^2(0,T;\boldsymbol{L}^2(\Omega)),\\
\mathbf{u}_\varepsilon &\to \mathbf{u} && \text{a.e. in } \Omega\times(0,T),\\
\boldsymbol{w}_\varepsilon &\rightharpoonup \boldsymbol{w} && \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)).
\end{aligned}$$

Then, arguing as in [28, Section 6] and exploiting (4.27), we infer that

$$\begin{aligned}
\phi_\varepsilon(u_{\varepsilon,k}) &\to \phi(u_k) && \text{a.e. in } \Omega\times(0,T),\\
\phi_\varepsilon(u_{\varepsilon,k}) &\rightharpoonup \phi(u_k) && \text{weakly in } L^2(0,T;L^2(\Omega)),
\end{aligned}$$

for any $k = 1,\dots,N$. Thus, the pair $(\mathbf{u},\boldsymbol{w})$ satisfies (3.2)–(3.7). Energy inequality (3.12) is then retrieved by standard lower semicontinuity arguments. If the initial datum is more regular, then, up to subsequences, we also have the convergences

$$\begin{aligned}
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly* in } L^\infty(0,T;\mathbf{H}^2(\Omega)),\\
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^2(0,T;\mathbf{H}^2(\Omega)),\\
\partial_t\mathbf{u}_\varepsilon &\rightharpoonup \partial_t\mathbf{u} && \text{weakly* in } L^\infty(0,T;\mathbf{L}^2(\Omega) \text{ and weakly in } L^2(0,T;\mathbf{H}^1(\Omega)),\\
\boldsymbol{w}_\varepsilon &\rightharpoonup \boldsymbol{w} && \text{weakly* in } L^\infty(0,T;\mathbf{L}^2(\Omega)) \text{ and weakly in } L^2(0,T;\mathbf{H}^1(\Omega)),\\
\phi_\varepsilon(u_{\varepsilon,k}) &\rightharpoonup \phi(u_k) && \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \quad \forall k=1,\dots,N,
\end{aligned}$$

which ensure the regularity of Theorem 3.1 part (2). Energy identity (3.12) can be recovered, since $t \mapsto \|\nabla\mathbf{u}(t)\|^2$ is absolutely continuous in $[0,T]$ and because of $\Psi^1(\mathbf{u}) \in H^1(0,T;\boldsymbol{L}^1(\Omega))$ entailing that the function $t \mapsto \int_\Omega \Psi(\mathbf{u}(t))dx$ is absolutely continuous in $[0,T]$ as well. Indeed, $\|\partial_t\Psi^1(\mathbf{u})\|_{\boldsymbol{L}^1(\Omega)} \le \|\Psi^1_{,\boldsymbol{\varphi}}(\mathbf{u})\|\|\partial_t\mathbf{u}\| \le C$, from the regularity above. This concludes the proof of the existence part of Theorem 3.1.

Strict separation property of strong solutions. We recall that (**M1**) is in force. Let us now introduce the following notation: we define $\mathcal{P}^s_\sigma$, with $s = 1,\dots,N-1$ and $\sigma \in \mathbb{N}$, as any possible subset of s (nonrepeated) indices from $1,\dots,N$. Note that σ indicates the choice of the subset, and $\sigma = 1,\dots,\binom{N}{s}$. In case $s = N-1$, we define the only index not belonging to $\mathcal{P}^{N-1}_\sigma$ by j_σ.

Step 1: Case $N-1$. Let us then start from $s = N-1$, having fixed σ. We consider the vector $\mathbf{e}_\sigma^{N-1}$ as, for $i = 1, \dots, N$,

$$(\mathbf{e}_\sigma^{N-1})_i = \begin{cases} 1 & \text{if } i \in \mathcal{P}_\sigma^{N-1}, \\ 0 & \text{if } i \notin \mathcal{P}_\sigma^{N-1}. \end{cases}$$

Then, we take $\boldsymbol{\eta} = \eta \mathbf{e}_\sigma^{N-1}$, for $\eta \in H^1(\Omega)$, in (3.4). This gives

$$\Big(\partial_t\Big(\sum_{i\in\mathcal{P}_\sigma^{N-1}} u_i\Big), \eta\Big) + \Big(\sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^N \boldsymbol{\alpha}_{ij}(w_j - \overline{w}_j), \eta\Big) = 0,$$

for almost any $t \in [0, T]$. We now fix $\delta > 0$ (to be chosen later on) and consider $u_{\sigma,\delta}^{N-1} = (\sum_{i\in\mathcal{P}_\sigma^{N-1}} u_i - \delta)^-$. Setting $\eta = -u_{\sigma,\delta}^{N-1}$ and integrating by parts, we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-1}\|^2 &- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^N \int_\Omega \alpha_{ij}\psi'(u_j) u_{\sigma,\delta}^{N-1} dx \\ &- \gamma \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^N \int_\Omega \alpha_{ij}\nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\ = - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j,k=1}^N \int_\Omega \alpha_{ij}\mathbf{A}_{jk} u_k u_{\sigma,\delta}^{N-1} dx &- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^N \int_\Omega \alpha_{ij}\overline{w}_j u_{\sigma,\delta}^{N-1} dx, \end{aligned} \tag{4.28}$$

where we used the property that, given any vector $\boldsymbol{\zeta} \in \mathbb{R}^N$, $\boldsymbol{\alpha}\mathbf{P}\boldsymbol{\zeta} = \boldsymbol{\alpha}\boldsymbol{\zeta}$. Now notice that, being $\alpha_{ii} = A > 0$ for any $i = 1, \dots, N$, we have

$$\begin{aligned} &- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^N \int_\Omega \alpha_{ij}\nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\ &= - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \int_\Omega (\alpha_{ii}\nabla u_i \cdot \nabla u_{\sigma,\delta}^{N-1}) dx - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j\neq i, j=1}^N \int_\Omega \alpha_{ij}\nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\ &= \int_\Omega (A\nabla u_{\sigma,\delta}^{N-1} \cdot \nabla u_{\sigma,\delta}^{N-1}) dx - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \Big(\sum_{j\neq i, j=1}^N \int_\Omega \alpha_{ij}\nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx\Big). \end{aligned}$$

Since $\alpha_{ij} = B < 0$ for any $i \neq j$ (clearly we have $A + (N-1)B = 0$), we see that the second summand becomes

$$\begin{aligned} &- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \Big(\sum_{j\neq i, j=1}^N \int_\Omega \alpha_{ij}\nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx\Big) \\ &= -B \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j\neq i, j\in\mathcal{P}_\sigma^{N-1}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx \end{aligned}$$

$$
\begin{aligned}
&- B \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j\notin\mathcal{P}_\sigma^{N-1}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\
&= -B(N-2) \sum_{j\in\mathcal{P}_\sigma^{N-1}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx - B(N-1) \int_\Omega \nabla u_{j_\sigma} \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\
&= B(N-2) \int_\Omega \nabla u_{\sigma,\delta}^{N-1} \cdot \nabla u_{\sigma,\delta}^{N-1} dx - B(N-1) \int_\Omega \nabla u_{j_\sigma} \cdot \nabla u_{\sigma,\delta}^{N-1} dx.
\end{aligned}
$$

Notice also that, recalling $u_{j_\sigma} = 1 - \sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j$ and that $A = -B(N-1)$, it holds

$$
\begin{aligned}
-B(N-1) \int_\Omega \nabla u_{j_\sigma} \cdot \nabla u_{\sigma,\delta}^{N-1} dx &= -B(N-1) \int_\Omega \nabla u_{\sigma,\delta}^{N-1} \cdot \nabla u_{\sigma,\delta}^{N-1} dx \\
&= A \int_\Omega \nabla u_{\sigma,\delta}^{N-1} \cdot \nabla u_{\sigma,\delta}^{N-1} dx,
\end{aligned}
$$

where we used the fact that, when $u_{\sigma,\delta}^{N-1} \le \delta$, it holds that

$$
\begin{aligned}
\nabla u_{j_\sigma} &= -\nabla \sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j = -\nabla\Big(\sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j - \delta \Big) \\
&= -\nabla\Big(\sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j - \delta \Big)^+ + \nabla\Big(\sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j - \delta \Big)^- = \nabla\Big(\sum_{j\in\mathcal{P}_\sigma^{N-1}} u_j - \delta \Big)^-.
\end{aligned}
$$

Therefore, in the end we get

$$
- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^{N} \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-1} dx = (2A + B(N-2)) \int_\Omega |\nabla u_{\sigma,\delta}^{N-1}|^2 dx \ge 0,
$$

recalling that $2A + B(N-2) = A - B \ge 0$.

Concerning the terms related to $\psi'(u_j)$, we can write, on account of (2.2) (which entails, in particular, $\sum_{i\in\mathcal{P}_\sigma^{N-1}} \alpha_{ij} = -\alpha_{j_\sigma j}$, for any $j = 1, \dots, N$)

$$
\begin{aligned}
&- \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j=1}^{N} \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-1} dx \\
&\quad = - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j\in\mathcal{P}_\sigma^{N-1}} \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-1} dx - \sum_{i\in\mathcal{P}_\sigma^{N-1}} \sum_{j\notin\mathcal{P}_\sigma^{N-1}} \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-1} dx \\
&\quad = \sum_{j\in\mathcal{P}_\sigma^{N-1}} \alpha_{j_\sigma j} \int_{E_{N-1}(t)} \psi'(u_j) u_{\sigma,\delta}^{N-1} dx + \alpha_{j_\sigma j_\sigma} \int_{E_{N-1}(t)} \psi'(u_{j_\sigma}) u_{\sigma,\delta}^{N-1} dx,
\end{aligned}
$$

where

$$
E_{N-1}(t) = \Big\{ x \in \Omega : \sum_{i\in\mathcal{P}_\sigma^{N-1}} u_i(x,t) \le \delta \Big\}.
$$

Observe that, in $E_{N-1}(t)$, it holds

$$1 \geq \sum_{j \notin \mathcal{P}_\sigma^{N-1}} u_j(t) = u_{j_\sigma}(t) = 1 - \sum_{i \in \mathcal{P}_\sigma^{N-1}} u_i(t) > 1 - \delta.$$

Thus, for $\delta \leq \frac{1}{2}$, we deduce

$$|\psi'(u_{j_\sigma})| \leq \max\left\{\left|\psi'\left(\frac{1}{2}\right)\right|, |\psi'(1)|\right\} \leq C, \tag{4.29}$$

since ψ' is monotonically increasing. Moreover, in $E_{N-1}(t)$ it also holds, being $\alpha_{j_\sigma j} = B = -|B|$ for $j \in \mathcal{P}_\sigma^{N-1}$ (recall that $B \leq 0$),

$$-|B|\psi'(u_j(t))u_{\sigma,\delta}^{N-1}(t) \geq -|B|\psi'(\delta)u_{\sigma,\delta}^{N-1}(t), \quad \forall j \in \mathcal{P}_\sigma^{N-1},$$

since we have $0 \leq u_i(t) \leq \delta$ for any $i \in \mathcal{P}_\sigma^{N-1}$ and

$$-\psi'(u_j) \geq -\psi'(\delta), \quad \forall j \in \mathcal{P}_\sigma^{N-1}.$$

Concerning the other terms in (4.28), we have, clearly, being $0 \leq u_k \leq 1$ for $k = 1, \ldots, N$, that

$$-\sum_{i \in \mathcal{P}_\sigma^{N-1}} \sum_{j,k=1}^{N} \int_\Omega \alpha_{ij} \mathbf{A}_{jk} u_k u_{\sigma,\delta}^{N-1} dx \leq C \int_\Omega u_{\sigma,\delta}^{N-1} dx,$$

and observing that (see (3.11)) $\overline{\boldsymbol{w}} \in L^\infty(0,T)$, we have, similarly,

$$-\sum_{i \in \mathcal{P}_\sigma^{N-1}} \sum_{j=1}^{N} \int_\Omega \alpha_{ij} \overline{w}_j u_{\sigma,\delta}^{N-1} dx \leq C(T) \int_\Omega u_{\sigma,\delta}^{N-1} dx.$$

Coming back to (4.28) and collecting all these results we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-1}\|^2 + \gamma(A-B)\int_\Omega |\nabla u_{\sigma,\delta}^{N-1}|^2 dx - (N-1)|B|\psi'(\delta)\int_\Omega u_{\sigma,\delta}^{N-1} dx \\
&\quad \leq C(T)\int_\Omega u_{\sigma,\delta}^{N-1} dx - \alpha_{j_\sigma j_\sigma}\int_{E_{N-1}(t)} \psi'(u_{j_\sigma}) u_{\sigma,\delta}^{N-1} dx \\
&\quad \leq C(T)\int_\Omega u_{\sigma,\delta}^{N-1} dx,
\end{aligned}$$

so that, assuming δ sufficiently small to satisfy (see assumption (**E1**))

$$-(N-1)|B|\psi'(\delta) - C(T) \geq 0,$$

we get, for almost any $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-1}\|^2 \leq 0.$$

Hence, having assumed the initial datum strictly separated, that is, there exists $0 < \delta_0 \leq \frac{1}{N}$ such that

$$u_{i,0} \geq \delta_0, \quad \forall i = 1, \dots, N, \tag{4.30}$$

we can choose $\delta \leq \delta_0$ in such a way that $u_{\sigma,\delta}^{N-1}(0) \equiv 0$ and Gronwall's lemma yields

$$\|u_{\sigma,\delta}^{N-1}(t)\| \equiv 0, \quad \forall\, t \in [0, T].$$

Notice now that the choice of the set $\mathcal{P}_\sigma^{N-1}$ is completely arbitrary, thus we infer that there exists δ_{N-1} such that $\delta_0 \geq \delta_{N-1} > 0$ and, for any possible $\mathcal{P}_\sigma^{N-1}$, with $\sigma = 1, \dots, N$,

$$\sum_{i \in \mathcal{P}_\sigma^{N-1}} u_i(t) \geq \delta > 0 \quad \text{in } \Omega, \quad \forall\, t \in [0, T], \ \forall\, \delta \in (0, \delta_{N-1}). \tag{4.31}$$

Remark 4.4. We point out that in the case $N = 2$, the proof is finished. This means that assumption **(E2)** is not necessary in this case, which is consistent with [26, Theorem 3.5].

Step 2: Case $N - 2$. If $N = 2$, we are done. Otherwise, we need to consider the sets $\mathcal{P}_\sigma^{N-2}$, $\sigma = 1, \dots, \frac{N(N-1)}{2}$. Let us fix σ and $0 < \delta \leq \delta_{N-1}$ (to be chosen later on). Then, we set

$$u_{\sigma,\delta}^{N-2} = \Big(\sum_{i \in \mathcal{P}_\sigma^{N-2}} u_i - 2\delta^2 \Big)^-$$

and define, for $i = 1, \dots, N$, the vector $\mathbf{e}_\sigma^{N-2}$ as

$$(\mathbf{e}_\sigma^{N-2})_i = \begin{cases} 1 & \text{if } i \in \mathcal{P}_\sigma^{N-2}, \\ 0 & \text{if } i \notin \mathcal{P}_\sigma^{N-2}. \end{cases}$$

We make a crucial observation: in the set

$$E_{N-2}(t) = \Big\{ x \in \Omega : \sum_{i \in \mathcal{P}_\sigma^{N-2}} u_i(x, t) \leq 2\delta^2 \Big\},$$

we infer from (4.31) that

$$u_j(t) \geq \delta - 2\delta^2, \quad \forall\, j \notin \mathcal{P}_\sigma^{N-2}. \tag{4.32}$$

Recall that $\delta < \frac{1}{2}$ and $0 < 2\delta^2 < \delta \leq \delta_{N-1} < 1$. Then, we take in (3.4), as in Step 1, the test function $\boldsymbol{\eta} = \eta \mathbf{e}_\sigma^{N-2}$ for $\eta \in H^1(\Omega)$, and we get

$$\Big(\partial_t \Big(\sum_{i \in \mathcal{P}_\sigma^{N-2}} u_i \Big), \eta \Big) + \Big(\sum_{i \in \mathcal{P}_\sigma^{N-2}} \sum_{j=1}^{N} \alpha_{ij} (w_j - \overline{w}_j), \eta \Big) = 0.$$

Choosing in the equation above $\eta = -u_{\sigma,\delta}^{N-2}$ and integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-2}\|^2 &- \sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx\\ &- \gamma\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ = -&\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j,k=1}^{N}\int_\Omega \alpha_{ij}\mathbf{A}_{jk}u_k u_{\sigma,\delta}^{N-2}dx - \sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\overline{w}_j u_{\sigma,\delta}^{N-2}dx. \quad (4.33)\end{aligned}$$

Recalling once more that $\alpha_{ii} = A > 0$ for any $i = 1,\dots,N$, and arguing exactly as in Step 1, we find

$$\begin{aligned}&-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ &= -\sum_{i\in\mathcal{P}_\sigma^{N-2}}\int_\Omega(\alpha_{ii}\nabla u_i\cdot\nabla u_{\sigma,\delta}^{N-2})dx - \sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\neq i,j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ &= \int_\Omega(A\nabla u_{\sigma,\delta}^{N-2}\cdot\nabla u_{\sigma,\delta}^{N-2})dx - \sum_{i\in\mathcal{P}_\sigma^{N-2}}\Big(\sum_{j\neq i,j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\Big).\end{aligned}$$

Since $\alpha_{ij} = B < 0$ for any $i \neq j$, the second summand becomes

$$\begin{aligned}&-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\Big(\sum_{j\neq i,j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\Big)\\ &= -B\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\neq i,j\in\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx - B\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ &= -B(N-3)\sum_{j\in\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx - B(N-2)\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ &= B(N-3)\int_\Omega \nabla u_{\sigma,\delta}^{N-2}\cdot\nabla u_{\sigma,\delta}^{N-2}dx - B(N-2)\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx.\end{aligned}$$

Recall now that $\sum_{j\notin\mathcal{P}_\sigma^{N-2}} u_j = 1 - \sum_{i\in\mathcal{P}_\sigma^{N-2}} u_i$ and $A = -B(N-1)$. Then, we have

$$\begin{aligned}-B(N-2)\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_\Omega \nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx &= -B(N-2)\int_\Omega \nabla u_{\sigma,\delta}^{N-2}\cdot\nabla u_{\sigma,\delta}^{N-2}dx\\ &= (A+B)\int_\Omega \nabla u_{\sigma,\delta}^{N-2}\cdot\nabla u_{\sigma,\delta}^{N-2}dx.\end{aligned}$$

This entails

$$-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\nabla u_j\cdot\nabla u_{\sigma,\delta}^{N-2}dx=(A+B(N-3)+A+B)\int_\Omega |\nabla u_{\sigma,\delta}^{N-2}|^2dx\geq 0,$$

since $A+B(N-3)+A+B=A-B\geq 0$.

The terms $\psi'(u_j)$ can be handled as above. Indeed, observing that

$$\sum_{i\in\mathcal{P}_\sigma^{N-2}}\alpha_{ij}=-\sum_{l\notin\mathcal{P}_\sigma^{N-2}}\alpha_{lj}=-2B\geq 0,\quad \forall\, j\in\mathcal{P}_\sigma^{N-2},$$

we obtain

$$\begin{aligned}
&-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j=1}^{N}\int_\Omega \alpha_{ij}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx\\
&\quad=-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\in\mathcal{P}_\sigma^{N-2}}\int_\Omega \alpha_{ij}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx-\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_\Omega \alpha_{ij}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx\\
&\quad=\sum_{l\notin\mathcal{P}_\sigma^{N-2}}\sum_{j\in\mathcal{P}_\sigma^{N-2}}\alpha_{lj}\int_{E_{N-2}(t)}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx\\
&\qquad-B\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_{E_{N-2}(t)}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx.
\end{aligned}$$

Thanks to (4.32), we know that in $E_{N-2}(t)$, for δ sufficiently small, we have

$$|\psi'(u_j)|\leq -\psi'(\delta-2\delta^2),\quad \forall\, j\notin\mathcal{P}_\sigma^{N-2},$$

since ψ' is monotonically increasing. This entails that

$$\begin{aligned}
&B\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_{E_{N-2}(t)}\psi'(u_j)u_{\sigma,\delta}^{N-2}dx\\
&\qquad\leq -|B|\sum_{i\in\mathcal{P}_\sigma^{N-2}}\sum_{j\notin\mathcal{P}_\sigma^{N-2}}\int_{E_{N-2}(t)}\psi'(\delta-2\delta^2)u_{\sigma,\delta}^{N-2}dx\\
&\qquad\leq -2|B|(N-2)\int_\Omega \psi'(\delta-2\delta^2)u_{\sigma,\delta}^{N-2}dx.
\end{aligned}$$

Moreover, in $E_{N-2}(t)$ it also holds that, since $0\geq B=-|B|$ and $u_{\sigma,\delta}^{N-2}\geq 0$,

$$\begin{aligned}
\sum_{l\notin\mathcal{P}_\sigma^{N-2}}\sum_{j\in\mathcal{P}_\sigma^{N-2}}\alpha_{lj}\psi'(u_j)u_{\sigma,\delta}^{N-2}(t)&=-|B|\sum_{l\notin\mathcal{P}_\sigma^{N-2}}\sum_{j\in\mathcal{P}_\sigma^{N-2}}\psi'(u_j)u_{\sigma,\delta}^{N-2}(t)\\
&\geq -2(N-2)|B|\psi'(2\delta^2)u_{\sigma,\delta}^{N-2}(t),
\end{aligned}$$

since $0 \leq u_i(t) \leq 2\delta^2$ for any $i \in \mathcal{P}_\sigma^{N-2}$, and thus

$$-\psi'(u_j) \geq -\psi'(2\delta^2), \quad \forall j \in \mathcal{P}_\sigma^{N-2}.$$

Concerning the other terms in (4.28), we have (recall that $0 \leq u_k \leq 1$ for $k = 1, \dots, N$)

$$-\sum_{i \in \mathcal{P}_\sigma^{N-2}} \sum_{j,k=1}^{N} \int_\Omega \alpha_{ij} \mathbf{A}_{jk} u_k u_{\sigma,\delta}^{N-2} dx \leq C \int_\Omega u_{\sigma,\delta}^{N-2} dx,$$

and, arguing similarly (see (3.11)), we find

$$-\sum_{i \in \mathcal{P}_\sigma^{N-2}} \sum_{j=1}^{N} \int_\Omega \alpha_{ij} \overline{w}_j u_{\sigma,\delta}^{N-2} dx \leq C(T) \int_\Omega u_{\sigma,\delta}^{N-2} dx.$$

Combining (4.33) with the obtained estimates, we end up with

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-2}\|^2 + \gamma(A-B)\int_\Omega |\nabla u_{\sigma,\delta}^{N-1}|^2 dx - 2|B|(N-2)\psi'(2\delta^2)\int_\Omega u_{\sigma,\delta}^{N-1} dx \\ \leq C(T)\int_\Omega u_{\sigma,\delta}^{N-1} dx - 2|B|(N-2)\int_{E_{N-2}(t)} \psi'(\delta - 2\delta^2) u_{\sigma,\delta}^{N-1} dx,\end{aligned}$$

that is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-2}\|^2 + \gamma(A-B)\int_\Omega |\nabla u_{\sigma,\delta}^{N-1}|^2 dx \\ + [2|B|(N-2)(-\psi'(2\delta^2) + \psi'(\delta - 2\delta^2)) - C(T)]\int_\Omega u_{\sigma,\delta}^{N-1} dx \leq 0.\end{aligned}$$

Therefore, on account of assumption (**E2**), for $0 < \delta \leq \delta_{N-1} \leq \delta_0 \leq \frac{1}{N}$ and δ sufficiently small, we can ensure that

$$2|B|(N-2)(-\psi'(\delta^2) + \psi'(\delta - \delta^2)) - C(T) \geq 0.$$

Recalling that $A - B \geq 0$, we deduce, for almost any $t \in [0, T]$,

$$\frac{1}{2}\frac{d}{dt}\|u_{\sigma,\delta}^{N-2}\|^2 \leq 0.$$

Then, thanks to (4.30) and to the choice $\delta \leq \delta_{N-1}$ (entailing also $2\delta^2 \leq \delta_{N-1}$), we get $u_{\sigma,\delta}^{N-2}(0) \equiv 0$. Therefore, by Gronwall's lemma, we get

$$\|u_{\sigma,\delta}^{N-2}(t)\| \equiv 0, \quad \forall t \in [0, T].$$

Again the choice of the set $\mathcal{P}_\sigma^{N-2}$ is completely arbitrary, meaning that there exists a $0 < \delta_{N-2} \leq \delta_{N-1}$ such that, for any possible $\mathcal{P}_\sigma^{N-2}$, with $\sigma = 1, \dots, \frac{N(N-1)}{2}$,

$$\sum_{i \in \mathcal{P}_\sigma^{N-2}} u_i(t) \geq \delta > 0 \quad \text{in } \Omega, \quad \forall t \in [0, T], \ \forall 0 < \delta \leq \delta_{N-2}. \tag{4.34}$$

Step 3: Iterative procedure and conclusion. If $N = 3$, we are done. Otherwise, we consider the sets $\mathcal{P}_\sigma^{N-3}$, $\sigma = 1, \dots, \binom{N}{N-3}$. Let us fix σ and $\delta \le \delta_{N-2}$ (to be chosen later on), introduce as before

$$u_{\sigma,\delta}^{N-3} = \Big(\sum_{i \in \mathcal{P}_\sigma^{N-3}} u_i - 2\delta^2 \Big)^-,$$

and define the vector $\mathbf{e}_\sigma^{N-3}$ as

$$(\mathbf{e}_\sigma^{N-3})_i = \begin{cases} 1 & \text{if } i \in \mathcal{P}_\sigma^{N-3}, \\ 0 & \text{if } i \notin \mathcal{P}_\sigma^{N-3}, \end{cases}$$

for $i = 1, \dots, N$. The essential observation is again the following: in the set

$$E_{N-3}(t) = \Big\{ x \in \Omega : \sum_{i \in \mathcal{P}_\sigma^{N-3}} u_i(x,t) \le 2\delta^2 \Big\}$$

from (4.34), since $0 < 2\delta^2 < \delta \le \delta_{N-2} \le \frac{1}{N}$, we deduce that

$$u_j(t) \ge \delta - 2\delta^2, \quad \forall\, j \notin \mathcal{P}_\sigma^{N-3}.$$

This implies that in $E_{N-3}(t)$, for $\delta > 0$ sufficiently small, we have

$$|\psi'(u_j(t))| \le -\psi'(\delta - 2\delta^2), \quad \forall\, j \notin \mathcal{P}_\sigma^{N-3},$$

and

$$-\psi'(u_i(t)) \ge -\psi'(2\delta^2), \quad \forall\, i \in \mathcal{P}_\sigma^{N-3}.$$

We can now argue as in Step 2 and conclude that there exists a $\delta_{N-3} \in (0, \delta_{N-2}]$ such that, for any possible $\mathcal{P}_\sigma^{N-3}$, with $\sigma = 1, \dots, \binom{N}{N-3}$,

$$\sum_{i \in \mathcal{P}_\sigma^{N-3}} u_i(t) \ge \delta > 0 \quad \text{in } \Omega, \quad \forall\, t \in [0,T], \ \forall\, \delta \in (0, \delta_{N-3}].$$

Applying these arguments iteratively, we reach a generic step m and we find $\delta_{N-m} \in (0, \delta_{N-m+1}]$ such that, for any $\mathcal{P}_\sigma^{N-m}$ with $\sigma = 1, \dots, \binom{N}{N-m}$, we have

$$\sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i(t) \ge \delta > 0 \quad \text{in } \Omega, \quad \forall\, t \in [0,T], \ \forall\, \delta \in (0, \delta_{N-m}].$$

Therefore, we can continue the procedure until $N - m = 1$, which entails in the end that there exists a $0 < \delta \le \delta_0 \le \frac{1}{N}$ such that, for any $i = 1, \dots, N$,

$$u_i(t) \ge \delta > 0 \quad \text{in } \Omega, \quad \forall\, t \in [0,T],$$

that is, the strict separation property holds. This concludes the proof of Theorem 3.1.

4.2. Proof of Theorem 3.9

Let us take $\boldsymbol{\eta} = \mathbf{u}(t) - \overline{\mathbf{u}}(t)$ in equation (3.5). This gives

$$(\mathbf{P}\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) + \gamma\|\nabla\mathbf{u}\|^2 = (\boldsymbol{w} - \overline{\boldsymbol{w}}, \mathbf{u} - \overline{\mathbf{u}}) + (\mathbf{A}\mathbf{u}, \mathbf{u} - \overline{\mathbf{u}}). \tag{4.35}$$

Moreover, by the convexity of Ψ^1 (recall that $\mathbf{u} - \overline{\mathbf{u}} \in T\Sigma$), we have

$$(\mathbf{P}\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) = (\boldsymbol{\phi}(\mathbf{u}), \mathbf{u} - \overline{\mathbf{u}}) \geq \int_\Omega \Psi^1(\mathbf{u})dx - \int_\Omega \Psi^1(\overline{\mathbf{u}})dx,$$

but, since $\overline{\mathbf{u}} \equiv \overline{\mathbf{u}}_0$, it holds that

$$|\Psi^1(\overline{\mathbf{u}})| \leq C,$$

where $C > 0$ depends on $\overline{\mathbf{u}}_0$. Applying standard inequalities, from (4.35) we infer that

$$\int_\Omega \Psi^1(\mathbf{u})dx + \gamma\|\nabla\mathbf{u}\|^2 \leq C + C\|\nabla\mathbf{u}\|\|\boldsymbol{w} - \overline{\boldsymbol{w}}\| + (\mathbf{A}\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{A}\mathbf{u}),$$

and using (2.1), we get

$$\begin{aligned}
&\int_\Omega \Psi^1(\mathbf{u})dx - \frac{1}{2}(\mathbf{A}\mathbf{u}, \mathbf{u}) + \frac{\gamma}{4}\|\nabla\mathbf{u}\|^2 \\
&\quad \leq C(\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}}) + \frac{1}{2}(\mathbf{A}\mathbf{u}, \mathbf{u}) - (\overline{\mathbf{u}}, \mathbf{A}\mathbf{u}) \\
&\quad \leq C(1 + (\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}})) + C\|\mathbf{u}\|^2 \\
&\quad \leq C(1 + (\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}})) + \frac{1}{2}\int_\Omega \Psi(\mathbf{u})dx,
\end{aligned}$$

where in the last step we applied property (vi) of the potential ψ_ε (recall that these estimates must be obtained in an approximating scheme, so for ε sufficiently small, see above). Therefore, we obtain

$$\frac{\gamma}{4}\|\nabla\mathbf{u}\|^2 + \frac{1}{2}\int_\Omega \Psi(\mathbf{u})dx \leq C(1 + (\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}})). \tag{4.36}$$

Combining (3.8) with (4.36) (multiplied by the sufficiently small $\epsilon > 0$), we end up with

$$\frac{d}{dt}\mathcal{E}(t) + \frac{\epsilon}{2}\mathcal{E}(t) \leq \frac{d}{dt}\mathcal{E}(t) + \frac{\epsilon}{2}\mathcal{E}(t) + (1 - \epsilon C)(\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}}) \leq C,$$

and the result follows from Gronwall's lemma.

4.3. Proof of Theorem 3.10

Proof. Instantaneous regularization of weak solutions. Thanks to part (1) of Theorem 3.1, for any $\tau > 0$, we can find $\widetilde{\tau} \leq \tau$ such that $\mathbf{u}(\widetilde{\tau}) \in \mathbf{H}^2(\Omega)$ and $\partial_\mathbf{n}\mathbf{u}(\widetilde{\tau}) = 0$ on $\partial\Omega$ such that the solution starting from $\widetilde{\tau}$ is more regular. Having assumed (**M1**), this solution coincides with the weak one (generated from $\mathbf{u}_0$) and it can be easily extended to $[\widetilde{\tau}, +\infty)$, by uniqueness, whence its instantaneous regularization and the validity of properties (3.16)–(3.19). Concerning the global bounds, for the sake of brevity, here we simply show the formal estimates. A rigorous argument can be formulated within an approximation scheme like the previous one. First, we observe that (3.8) entails

$$\|\mathbf{u}\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} + \|\boldsymbol{w} - \overline{\boldsymbol{w}}\|_{L^2(t,t+1;\mathbf{L}^2(\Omega))} \le C, \quad \forall\, t \ge 0. \tag{4.37}$$

Notice that the constant $C > 0$ only depends on the initial energy $\mathcal{E}(0)$. Then, arguing as in (4.11), we obtain

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}}) + \frac{\gamma}{2}\|\nabla\partial_t\mathbf{u}\|^2 \le C(\boldsymbol{\alpha}(\boldsymbol{w} - \overline{\boldsymbol{w}}), \boldsymbol{w} - \overline{\boldsymbol{w}}).$$

Due to (4.37), we can apply the uniform Gronwall's lemma (see, e.g., [44] by choosing, e.g., $r = \frac{\tau}{2}$) to deduce, for any given $\tau > 0$,

$$\|\boldsymbol{w} - \overline{\boldsymbol{w}}\|_{L^\infty(\tau,\infty;\mathbf{L}^2(\Omega))} + \gamma\|\nabla\partial_t\mathbf{u}\|_{L^2(t,t+1;\mathbf{L}^2(\Omega))} \le C, \quad \forall\, t \ge \tau.$$

From now on, we can argue as in the proof of Theorem 3.1 to get

$$\|\boldsymbol{w}\|_{L^\infty(\tau,\infty;\mathbf{L}^2(\Omega))} \le C, \quad \forall\, t \ge \tau,$$

where $C > 0$, now and in the rest of the paper, stands for a generic constant depending on Ω, $\boldsymbol{\alpha}$, Ψ, $\overline{\mathbf{u}}_0$, and $\mathcal{E}(0)$. This allows us to deduce

$$\|\boldsymbol{\phi}(\mathbf{u})\|_{L^\infty(\tau,\infty;\boldsymbol{L}^2(\Omega))} + \|\mathbf{u}_\varepsilon\|_{L^\infty(\tau,\infty;\mathbf{H}^2(\Omega))} \le C, \quad \forall\, t \ge \tau. \tag{4.38}$$

Also, by comparison in (3.7), we find

$$\|\boldsymbol{w}\|_{L^2(t,t+1;\mathbf{H}^1(\Omega))} \le C, \quad \forall\, t \ge \tau.$$

The proof is finished. ∎

Instantaneous strict separation. We are in the case $\|\mathbf{u}_0\|_{\mathbf{L}^\infty(\Omega)} \le 1$, that is, $\mathbf{u}_0$ is not necessarily strictly separated like in Section 4.1. Therefore, we need to adapt the proof we performed in Section 4.1. In order to do that, we perform a De Giorgi-type iterative scheme at each step.

The basic tool is the next lemma.

Lemma 4.5. *Let $\{y_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathbb{R}^+$ satisfy the recursive inequalities*

$$y_{n+1} \le Cb^n y_n^{1+\varepsilon}, \quad \forall n \ge 0,$$

for some $C > 0$, $b > 1$, and $\varepsilon > 0$. If

$$y_0 \le \theta := C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}},$$

then

$$y_n \le \theta b^{-\frac{n}{\varepsilon}}, \quad \forall n \ge 0,$$

and consequently, $y_n \to 0$ for $n \to \infty$.

Lemma 4.5 can be found, for example, in [19, Ch. I, Lemma 4.1] (see also [36, Ch.2, Lemma 5.6]) and can be easily proven by induction (see, e.g., [9, Lemma 3.8]). Since the iterative argument in which we sum up some components of **u** (in decreasing number at each step) is exactly the same as in the case treated in Section 4.1, we directly assume to be at Step $m > 1$ and show the differences with respect to estimate (4.28) (Step 1 is even easier, as we have seen in Section 4.1 thanks to relation (4.29), thus it can be easily adapted following the analysis of the other steps). We assume to know, for an arbitrary $\tau > 0$, that there exists $0 < \delta_{N-m+1} \le \frac{1}{N}$ such that, for any σ,

$$\sum_{i \in \mathcal{P}_\sigma^{N-m+1}} u_i \ge \delta, \quad \text{in } \overline{\Omega} \times \left[\frac{\tau}{2} + \frac{m\tau}{2N}, +\infty\right), \quad \forall \delta \le \delta_{N-m+1}, \tag{4.39}$$

with the same notation as in Section 4.1. Notice that the upper bound $\delta_{N-m+1} \le \frac{1}{N}$ is set, since clearly in the end the necessary condition for the separation will be that $\delta \le \frac{1}{N}$. We aim at showing that (4.39) also holds at Step m. We now consider the set of indices $\mathcal{P}_\sigma^{N-m}$ for a certain σ. Then, for $i = 1, \dots, N$, we set

$$(\mathbf{e}_\sigma^{N-m})_i = \begin{cases} 1 & \text{if } i \in \mathcal{P}_\sigma^{N-m}, \\ 0 & \text{if } i \notin \mathcal{P}_\sigma^{N-m}. \end{cases}$$

We can now perform De Giorgi's scheme. Let us set δ sufficiently small such that $\delta \le \delta_{N-m+1}$ and fix $\widetilde{\tau}$ such that

$$2\widetilde{\tau} + \frac{\tau}{2} + \frac{m\tau}{2N} = \frac{\tau}{2} + \frac{(m+1)\tau}{2N}, \tag{4.40}$$

that is, $\widetilde{\tau} = \frac{\tau}{4N}$. Choose now $T > 0$ such that $T - 3\widetilde{\tau} = \frac{\tau}{2} + \frac{m\tau}{2N} \ge \frac{\tau}{2}$, that is, $T = \frac{\tau}{2} + \frac{3+2m}{4N}\tau$. Notice that condition (4.40) implies

$$T - \widetilde{\tau} = \frac{\tau}{2} + \frac{(m+1)\tau}{2N}. \tag{4.41}$$

Let us define the sequence

$$k_n = \delta^2 + \frac{\delta^2}{2^n}, \quad \forall n \ge 0,$$

where

$$\delta^2 < k_{n+1} < k_n < 2\delta^2, \quad \forall n \ge 1, \qquad k_n \to \delta^2 \quad \text{as } n \to \infty$$

and the sequence of times

$$\begin{cases} t_{-1} = T - 3\widetilde{\tau}, \\ t_n = t_{n-1} + \frac{\widetilde{\tau}}{2^n}, \quad n \ge 0, \end{cases}$$

which satisfies

$$t_{-1} < t_n < t_{n+1} < T - \widetilde{\tau}, \quad \forall n \ge 0.$$

Then, introduce a cutoff function $\eta_n \in C^1(\mathbb{R})$ by setting

$$\eta_n(t) := \begin{cases} 0, & t \le t_{n-1}, \\ 1, & t \ge t_n \end{cases} \quad \text{and} \quad |\eta_n'(t)| \le \frac{2^{n+1}}{\widetilde{\tau}},$$

on account of the above definition of $\{t_n\}_n$, and set

$$u_{\sigma,\delta}^{N-m,n}(x,t) := \Big(\sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i - k_n \Big)^{-}.$$

Also, for any $n \ge 0$, let us introduce the interval $I_n = [t_{n-1}, T]$ and the set

$$\mathcal{A}_n(t) := \Big\{ x \in \Omega : \sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i(x,t) - k_n \le 0 \Big\}, \quad \forall t \in I_n,$$

so that on $\mathcal{A}_n(t)$ it holds that (see (4.39)), since $0 < 2\delta^2 < \delta \le \delta_{N-m+1} \le \frac{1}{N}$,

$$u_j(t) \ge \delta - 2\delta^2, \quad \forall j \notin \mathcal{P}_\sigma^{N-m}.$$

This means that, on $\mathcal{A}_n(t)$ and for $\delta > 0$ sufficiently small, we have

$$|\psi'(u_j(t))| \le -\psi'(\delta - 2\delta^2), \quad \forall j \notin \mathcal{P}_\sigma^{N-m}, \tag{4.42}$$

and

$$-\psi'(u_i(t)) \ge -\psi'(2\delta^2), \quad \forall i \in \mathcal{P}_\sigma^{N-m}. \tag{4.43}$$

Observe now that

$$\begin{aligned} I_{n+1} &\subseteq I_n, && \forall n \ge 0, \\ \mathcal{A}_{n+1}(t) &\subseteq \mathcal{A}_n(t), && \forall n \ge 0,\ \forall t \in I_{n+1}, \end{aligned}$$

and set

$$y_n = \int_{I_n} \int_{\mathcal{A}_n(s)} 1 dx ds, \quad \forall n \ge 0.$$

For any $n \ge 0$, we take the test function $\boldsymbol{\eta} = -\mathbf{e}_\sigma^{N-m} u_{\sigma,\delta}^{N-m,n} \eta_n^2$ in (3.4) and integrate over $[t_{n-1}, t]$, $t_n \le t \le T$. After an integration by parts, we get

$$\frac{1}{2}\eta_n^2(t)\|u_{\sigma,\delta}^{N-m,n}(t)\|^2 \underbrace{- \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j=1}^N \int_{t_{n-1}}^t \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds}_{I_1}$$

$$\underbrace{-\gamma \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j=1}^N \int_{t_{n-1}}^t \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds}_{I_2}$$

$$= \underbrace{- \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j,k=1}^{N} \int_{t_{n-1}}^{t} \int_\Omega \alpha_{ij} \mathbf{A}_{jk} u_k u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds}_{I_3}$$

$$\underbrace{- \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j=1}^{N} \int_{t_{n-1}}^{t} \int_\Omega \alpha_{ij} \overline{w}_j u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds}_{I_4}$$

$$+ \underbrace{\int_{t_{n-1}}^{t} \partial_t \eta_n \eta_n \| u_{\sigma,\delta}^{N-m,n} \|^2 ds,}_{I_5}$$

where we used

$$\frac{1}{2} \eta_n^2(t) \| u_{\sigma,\delta}^{N-m,n}(t) \|^2 = \int_{t_{n-1}}^{t} < \partial_t u_{\sigma,\delta}^{N-m,n}, u_{\sigma,\delta}^{N-m,n} \eta_n^2 ds$$
$$- \int_{t_{n-1}}^{t} \partial_t \eta_n \eta_n \| u_{\sigma,\delta}^{N-m,n} \|^2 ds.$$

As in Section 4.1, recalling that $\alpha_{ii} = A > 0$ for any $i = 1, \ldots, N$, we obtain

$$I_2 = -\gamma \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j=1}^{N} \int_{t_{n-1}}^{t} \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds$$
$$= -\gamma \sum_{i \in \mathcal{P}_\sigma^{N-m}} \int_{t_{n-1}}^{t} \int_\Omega (\alpha_{ii} \nabla u_i \cdot \nabla u_{\sigma,\delta}^{N-m,n}) \eta_n^2 dx ds$$
$$- \gamma \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j \neq i, j=1}^{N} \int_{t_{n-1}}^{t} \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds$$
$$= \gamma \int_{t_{n-1}}^{t} \int_\Omega (A \nabla u_{\sigma,\delta}^{N-m,n} \cdot \nabla u_{\sigma,\delta}^{N-m,n}) \eta_n^2 dx ds$$
$$- \gamma \sum_{i \in \mathcal{P}_\sigma^{N-m}} \Big(\sum_{j \neq i, j=1}^{N} \int_{t_{n-1}}^{t} \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} \eta_n^2 dx ds \Big).$$

Since $\alpha_{ij} = B < 0$ for any $i \neq j$, the second summand becomes

$$- \sum_{i \in \mathcal{P}_\sigma^{N-m}} \Big(\sum_{j \neq i, j=1}^{N} \int_\Omega \alpha_{ij} \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx \Big)$$
$$= -B \sum_{i \in \mathcal{P}_\sigma^{N-m}} \sum_{j \neq i, j \in \mathcal{P}_\sigma^{N-m}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx$$

$$
\begin{aligned}
&- B \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\notin\mathcal{P}_\sigma^{N-2}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m} dx \\
&= B(N-m-1) \sum_{j\in\mathcal{P}_\sigma^{N-m}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx \\
&\quad - B(N-m) \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx.
\end{aligned}
$$

On the other hand, observe that $\sum_{j\notin\mathcal{P}_\sigma^{N-m}} u_j = 1 - \sum_{i\in\mathcal{P}_\sigma^{N-m}} u_i$. Hence, we get

$$
-B(N-m) \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_\Omega \nabla u_j \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx = -B(N-m) \int_\Omega \nabla u_{\sigma,\delta}^{N-m,n} \cdot \nabla u_{\sigma,\delta}^{N-m,n} dx.
$$

This entails (recall that $A = -B(N-1)$)

$$
\begin{aligned}
I_2 &= \gamma(A + B(N-m-1) - B(N-m)) \int_{t_{n-1}}^t \int_\Omega |\nabla u_{\sigma,\delta}^{N-2}|^2 \eta_n^2 dx ds \\
&= \gamma N|B| \int_{t_{n-1}}^t \int_\Omega |\nabla u_{\sigma,\delta}^{N-m,n}|^2 \eta_n^2 dx ds,
\end{aligned}
$$

since $A + B(N-m-1) - B(N-m) = A - B = N|B| > 0$.

Concerning I_1, recall that for any $j \in \mathcal{P}_\sigma^{N-m}$, $\sum_{i\in\mathcal{P}_\sigma^{N-m}} \alpha_{ij} = -\sum_{l\notin\mathcal{P}_\sigma^{N-m}} \alpha_{lj}$, we can write

$$
\begin{aligned}
&- \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j=1}^N \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx \\
&= - \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\in\mathcal{P}_\sigma^{N-m}} \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx \\
&\quad - \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_\Omega \alpha_{ij} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx \\
&= \sum_{l\notin\mathcal{P}_\sigma^{N-m}} \sum_{j\in\mathcal{P}_\sigma^{N-m}} \alpha_{lj} \int_{\mathcal{A}_n(t)} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx \\
&\quad - B \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_{\mathcal{A}_n(t)} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx.
\end{aligned}
$$

Thus, by (4.42), we deduce

$$
\begin{aligned}
&B \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_{\mathcal{A}_n(t)} \psi'(u_j) u_{\sigma,\delta}^{N-m,n} dx \\
&\quad \le -|B| \sum_{i\in\mathcal{P}_\sigma^{N-m}} \sum_{j\notin\mathcal{P}_\sigma^{N-m}} \int_{\mathcal{A}_n(t)} \psi'(\delta - 2\delta^2) u_{\sigma,\delta}^{N-m,n} dx
\end{aligned}
$$

$$\le -m|B|(N-m)\int_\Omega \psi'(\delta-2\delta^2)u_{\sigma,\delta}^{N-m,n}dx.$$

Moreover, since $0 \ge B = -|B|$ and thanks to (4.43), in $\mathcal{A}_n(t)$ it also holds that

$$\begin{aligned}\sum_{l\notin\mathcal{P}_\sigma^{N-m}}\sum_{j\in\mathcal{P}_\sigma^{N-m}}\alpha_{lj}\psi'(u_j)u_{\sigma,\delta}^{N-m,n} &= -|B|\sum_{l\notin\mathcal{P}_\sigma^{N-m}}\sum_{j\in\mathcal{P}_\sigma^{N-m}}\psi'(u_j)u_{\sigma,\delta}^{N-m,n}\\ &\ge -m(N-m)|B|\psi'(2\delta^2)u_{\sigma,\delta}^{N-m,n}.\end{aligned}$$

About the other terms in (4.28), recalling that $0 \le u_k \le 1$ for $k = 1,\dots,N$, we clearly have

$$I_3 = -\sum_{i\in\mathcal{P}_\sigma^{N-m}}\sum_{j,k=1}^{N}\int_{t_{n-1}}^{t}\int_\Omega \alpha_{ij}\mathbf{A}_{jk}u_k u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds \le C\int_{t_{n-1}}^{t}\int_\Omega u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds,$$

and by (3.18) on $(\frac{\tau}{2},+\infty)$ we have, similarly,

$$I_4 = -\sum_{i\in\mathcal{P}_\sigma^{N-m}}\sum_{j=1}^{N}\int_{t_{n-1}}^{t}\int_\Omega \alpha_{ij}\overline{w}_j u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds \le C\int_{t_{n-1}}^{t}\int_\Omega u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds.$$

We are left with I_5, which is not present when $\mathbf{u}_0$ is strictly separated. Note that, since $\sum_{i=1}^N u_i = 1$ and $0 \le u_i \le 1$ almost everywhere in $\Omega\times[0,+\infty)$, for any $i = 1,\dots,N$, we have

$$0 \le \sum_{i\in\mathcal{P}_\sigma^{N-m}} u_i \quad \text{a.e. in } \Omega\times[0,+\infty),$$

and thus

$$0 \le u_{\sigma,\delta}^{N-m,n} \le 2\delta^2 \quad \text{a.e. in } \Omega\times[0,+\infty).$$

Then, thanks to the above inequality, we infer

$$\begin{aligned}I_5 &= \int_{t_{n-1}}^{t}\|u_{\sigma,\delta}^{N-m,n}(s)\|^2\eta_n\partial_t\eta_n ds = \int_{t_{n-1}}^{t}\int_\Omega (u_{\sigma,\delta}^{N-m,n}(s))^2\eta_n\partial_t\eta_n dxds\\ &= \int_{t_{n-1}}^{t}\int_{\mathcal{A}_n(s)}(u_{\sigma,\delta}^{N-m,n}(s))^2\eta_n\partial_t\eta_n dxds \le \int_{t_{n-1}}^{t}\int_{\mathcal{A}_n(s)}4\delta^4\frac{2^{n+1}}{\widetilde{\tau}}dxds\\ &\le \frac{2^{n+3}\delta^4}{\widetilde{\tau}}y_n.\end{aligned}$$

Therefore, collecting all the above results, we end up with

$$\begin{aligned}&\frac{1}{2}\eta_n^2(t)\|u_{\sigma,\delta}^{N-m,n}(t)\|^2 - m(N-m)|B|\psi'(2\delta^2)\int_{t_{n-1}}^{t}\int_\Omega u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds\\ &\quad+\gamma N|B|\int_{t_{n-1}}^{t}\int_\Omega|\nabla u_{\sigma,\delta}^{N-m,n}|^2\eta_n^2dxds\\ &\le (C-m|B|(N-m)\psi'(\delta-2\delta^2))\int_{t_{n-1}}^{t}\int_\Omega u_{\sigma,\delta}^{N-m,n}\eta_n^2dxds + \frac{2^{n+3}\delta^4}{\widetilde{\tau}}y_n,\end{aligned}$$

for any $t \in [t_n, T]$, that is,

$$\frac{1}{2}\eta_n^2(t)\|u_{\sigma,\delta}^{N-m,n}(t)\|^2 + [m(N-m)|B|(-\psi'(2\delta^2)+\psi'(\delta-2\delta^2))-C]\int_{t_{n-1}}^t\int_\Omega u_{\sigma,\delta}^{N-m,n}\eta_n^2 dxds + \gamma N|B|\int_{t_{n-1}}^t\int_\Omega |\nabla u_{\sigma,\delta}^{N-m,n}|^2\eta_n^2 dxds \le \frac{2^{n+3}\delta^4}{\widetilde{\tau}}y_n.$$

We now recall assumption (**E2**) and see that

$$\psi'(\delta-2\delta^2)-\psi'(2\delta^2)\to+\infty \quad \text{as } \delta\to 0^+.$$

Therefore, for $0<\delta\le\delta_{N-m+1}$ sufficiently small, we have

$$m|B|(N-m)(-\psi'(2\delta^2)+\psi'(\delta-2\delta^2))-C\ge 0.$$

This entails

$$\max_{t\in I_{n+1}}\|u_{\sigma,\delta}^{N-m,n}(t)\|^2\le X_n, \quad 2\gamma N|B|\int_{I_{n+1}}\|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 ds\le X_n, \tag{4.44}$$

where

$$X_n := \frac{2^{n+4}\delta^4}{\widetilde{\tau}}y_n.$$

On the other hand, for any $t\in I_{n+1}$ and for almost any $x\in A_{n+1}(t)$, we get

$$u_{\sigma,\delta}^{N-m,n}(x,t) = \delta^2+\frac{\delta^2}{2^n}-\sum_{i\in\mathcal{P}_\sigma^{N-m}}u_i(x,t) = \underbrace{-\sum_{i\in\mathcal{P}_\sigma^{N-m}}u_i(x,t)+\left[\delta^2+\frac{\delta^2}{2^{n+1}}\right]}_{u_{\sigma,\delta}^{N-m,n+1}(x,t)\ge 0}+\delta^2\left[\frac{1}{2^n}-\frac{1}{2^{n+1}}\right]\ge\frac{\delta^2}{2^{n+1}},$$

which implies

$$\int_{I_{n+1}}\int_\Omega |u_{\sigma,\delta}^{N-m,n}|^3 dxds \ge \int_{I_{n+1}}\int_{\mathcal{A}_{n+1}(s)}|u_{\sigma,\delta}^{N-m,n}|^3 dxds \ge \left(\frac{\delta^2}{2^{n+1}}\right)^3\int_{I_{n+1}}\int_{\mathcal{A}_{n+1}(s)}dxds = \left(\frac{\delta^2}{2^{n+1}}\right)^3 y_{n+1}.$$

Then, for $d=2,3$, we find

$$\left(\frac{\delta^2}{2^{n+1}}\right)^3 y_{n+1}\le\int_{I_{n+1}}\int_\Omega |u_{\sigma,\delta}^{N-m,n}|^3 dxds$$

$$
\begin{aligned}
&= \int_{I_{n+1}} \int_{\mathcal{A}_n(s)} |u_{\sigma,\delta}^{N-m,n}|^3 dxds \\
&\le \Big(\int_{I_{n+1}} \int_{\Omega} |u_{\sigma,\delta}^{N-m,n}|^{\frac{2d+4}{d}} dxds\Big)^{\frac{3d}{2d+4}} \Big(\int_{I_{n+1}} \int_{\mathcal{A}_n(s)} dxds\Big)^{\frac{4-d}{2d+4}}.
\end{aligned}
\tag{4.45}
$$

Thanks to the Sobolev–Gagliardo–Nirenberg inequality and Poincaré's inequality, we have

$$
\|u\|_{L^{\frac{2d+4}{d}}(\Omega)} \le \widetilde{C}(\|u-\overline{u}\|^{\frac{2}{d+2}} \|\nabla u\|^{\frac{d}{d+2}} + |\overline{u}|), \quad \forall\, u \in H^1(\Omega),
$$

so we get

$$
\begin{aligned}
&\int_{I_{n+1}} \int_{\Omega} |u_{\sigma,\delta}^{N-m,n}|^{\frac{2d+4}{d}} dxds \\
&\quad \le \widetilde{C} \int_{I_{n+1}} (\|u_{\sigma,\delta}^{N-m,n}\|^{\frac{4d+8}{d(d+2)}} \|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 + |\overline{u_{\sigma,\delta}^{N-m,n}}|^{\frac{2d+4}{d}}) ds \\
&\quad \le \widehat{C} \int_{I_{n+1}} (\|u_{\sigma,\delta}^{N-m,n}\|^{\frac{4d+8}{d(d+2)}} \|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 + \|u_{\sigma,\delta}^{N-m,n}\|^{\frac{2d+4}{d}}) ds.
\end{aligned}
$$

On the other hand, by (4.44), we obtain

$$
\begin{aligned}
&\widehat{C} \int_{I_{n+1}} \|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 \|u_{\sigma,\delta}^{N-m,n}\|^{\frac{4d+8}{d(d+2)}} ds \\
&\quad \le \widehat{C} \max_{t \in I_{n+1}} \|u_{\sigma,\delta}^{N-m,n}(t)\|^{\frac{4d+8}{d(d+2)}} \int_{I_{n+1}} \|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 ds \\
&\quad \le \frac{\widehat{C}}{2\gamma N|B|} X_n^{\frac{2d+4}{d(d+2)}} 2\gamma N|B| \int_{I_{n+1}} \|\nabla u_{\sigma,\delta}^{N-m,n}\|^2 ds \\
&\quad \le \frac{\widehat{C}}{2\gamma N|B|} X_n^{\frac{d+2}{d}} \le \frac{\widehat{C}}{2\gamma N|B|} \frac{2^{\frac{d+2}{d}n + \frac{4(d+2)}{d}} \delta^{\frac{4(d+2)}{d}}}{\widetilde{\tau}^{\frac{d+2}{d}}} y_n^{\frac{d+2}{d}}.
\end{aligned}
$$

Similarly, using (4.44) once more, we have

$$
\widehat{C} \int_{I_{n+1}} \|u_{\sigma,\delta}^{N-m,n}\|^{\frac{2d+4}{d}} ds \le 2\widehat{C}\widetilde{\tau} X_n^{\frac{d+2}{d}} = \widehat{C} \frac{2^{\frac{d+2}{d}n + \frac{5d+8}{d}} \delta^{\frac{4d+8}{d}}}{\widetilde{\tau}^{\frac{2}{d}}} y_n^{\frac{d+2}{d}}.
$$

Therefore, we infer from (4.45) that

$$
\begin{aligned}
\Big(\frac{\delta^2}{2^{n+1}}\Big)^3 y_{n+1} &\le \Big(\int_{I_{n+1}} \int_{\Omega} |u_{\sigma,\delta}^{N-m,n}|^{\frac{2d+4}{d}} dxds\Big)^{\frac{3d}{2d+4}} \Big(\int_{I_{n+1}} \int_{\mathcal{A}_n(s)} dxds\Big)^{\frac{4-d}{2d+4}} \\
&\le \delta^6 \frac{2^{\frac{3}{2}n+6} \widehat{C}^{\frac{3d}{2d+4}}}{\widetilde{\tau}^{\frac{3}{2}}} \Big(\frac{1}{2\gamma N|B|} + 2\widetilde{\tau}\Big)^{\frac{3d}{2d+4}} y_n^{\frac{5+d}{2+d}}.
\end{aligned}
$$

In conclusion, we end up with

$$
y_{n+1} \le \frac{2^{\frac{9}{2}n+9} \widehat{C}^{\frac{3d}{2d+4}}}{\widetilde{\tau}^{\frac{3}{2}}} \Big(\frac{1}{2\gamma N|B|} + 2\widetilde{\tau}\Big)^{\frac{3d}{2d+4}} y_n^{\frac{5+d}{2+d}}, \quad \forall n \ge 0.
$$

Thus, we can apply Lemma 4.5. In particular, we have

$$b = 2^{\frac{9}{2}} > 1, \quad C = \frac{2^6 \widehat{C}^{\frac{3d}{2d+4}}}{\widetilde{\tau}^{\frac{3}{2}}} \Big(\frac{1}{2\gamma N |B|} + 2\widetilde{\tau} \Big)^{\frac{3d}{2d+4}} > 0, \quad \varepsilon = \frac{3}{d+2},$$

to get that $y_n \to 0$, as long as

$$y_0 \le C^{-\frac{d+2}{3}} b^{-\frac{(d+2)^2}{9}},$$

that is,

$$y_0 \le \frac{2^{-[2(d+2)-\frac{(d+2)^2}{2}]} \widetilde{\tau}^{\frac{d+2}{2}}}{\widehat{C}^{\frac{d}{2}} \left(\frac{1}{2\gamma N|B|} + 2\widetilde{\tau} \right)^{\frac{d}{2}}}. \tag{4.46}$$

On the other hand, owing to (3.19), we know that $\|\psi'(u_j)\|_{L^\infty(\frac{\tau}{2},\infty;L^1(\Omega))} \le C(\tau)$ for any $j = 1, \dots, N$ and ψ' is monotone in a neighborhood of 0^+. Then, we get, for δ sufficiently small,

$$\begin{aligned} y_0 &= \int_{I_0} \int_{\mathcal{A}_0(s)} 1 dx ds \le \int_{I_0} \int_{\{x \in \Omega:\, \sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i(x,t) \le 2\delta^2\}} 1 dx ds \\ &\le \int_{I_0} \int_{\mathcal{A}_0(s)} \frac{1}{N-m} \sum_{i \in \mathcal{P}_\sigma^{N-m}} \frac{|\psi'(u_i)|}{-\psi'(2\delta^2)} dx ds \le -\frac{3C(\tau)\widetilde{\tau}}{\psi'(2\delta^2)(N-m)}. \end{aligned}$$

Hence, if we ensure that

$$-\frac{3C(\tau)\widetilde{\tau}}{\psi'(2\delta^2)(N-m)} \le \frac{2^{-[2(d+2)-\frac{(d+2)^2}{2}]} \widetilde{\tau}^{\frac{d+2}{2}}}{\widehat{C}^{\frac{d}{2}} \left(\frac{1}{2\gamma N|B|} + 2\widetilde{\tau} \right)^{\frac{d}{2}}},$$

then (4.46) holds. This is equivalent to

$$\frac{3C(\tau) 2^{[2(d+2)-\frac{(d+2)^2}{2}]} \widehat{C}^{\frac{d}{2}} \left(\frac{1}{2\gamma N|B|} + 2\widetilde{\tau} \right)^{\frac{d}{2}}}{\widetilde{\tau}^{\frac{d}{2}} (N-m)} \le -\psi'(2\delta^2).$$

Having fixed $\widetilde{\tau}$ such that (4.40) holds, we obtain the result by choosing δ sufficiently small, since $-\psi'(2\delta^2) \to +\infty$ as $\delta \to 0$ by assumption (**E1**). Notice that $\delta > 0$ is fixed and not infinitesimal.

In the end, passing to the limit in y_n as $n \to \infty$, we have obtained that

$$\Big\| \Big(\sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i - \delta^2 \Big)^- \Big\|_{L^\infty(\Omega \times (T-\widetilde{\tau},T))} = 0,$$

by uniqueness of the limit, since as $n \to \infty$,

$$y_n \to \Big| \Big\{ (x,t) \in \Omega \times [T-\widetilde{\tau}, T] : \sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i \le \delta^2 \Big\} \Big|_d;$$

and, on the other hand, $y_n \to 0$. Notice that, due to the choice of T, we have (see (4.41)) $T - \widetilde{\tau} = \frac{\tau}{2} + \frac{(m+1)\tau}{2N} \le \tau$, therefore we can repeat the same procedure on the interval $(T, T + \widetilde{\tau})$ (the new starting time will be $t_{-1} = T - 2\widetilde{\tau} \ge \frac{\tau}{2}$) and so on, thus eventually reaching the entire interval $[\frac{\tau}{2} + \frac{(m+1)\tau}{2N}, +\infty)$. Clearly δ and $\widetilde{\tau}$ do not change, since the estimates are independent of T. Therefore, since $\sigma = 1, \dots, \binom{N}{N-m}$ is arbitrary, we have obtained that there exists a $0 < \delta_{N-m} \le \delta_{N-m+1} \le \frac{1}{N}$ such that, for any possible $\mathcal{P}_\sigma^{N-m}$, with $\sigma = 1, \dots, \binom{N}{N-m}$,

$$\sum_{i \in \mathcal{P}_\sigma^{N-m}} u_i(t) \ge \delta > 0 \quad \text{a.e. in } \Omega \times \left[\frac{\tau}{2} + \frac{(m+1)\tau}{2N}, +\infty\right), \quad \forall\, \delta \in (0, \delta_{N-m}]. \tag{4.47}$$

Recalling Remark 3.11, we can deduce that (4.47) actually holds everywhere in

$$\overline{\Omega} \times \left[\frac{\tau}{2} + \frac{(m+1)\tau}{2N}, +\infty\right).$$

We can thus repeat the procedure, increasing m for a finite number of times, until each set $\mathcal{P}_\sigma$ is a singleton (as in the case discussed in Section 4.1). This entails that there exists $0 < \delta \le \frac{1}{N}$ such that, for any $i = 1 \dots, N$,

$$u_i \ge \delta > 0 \quad \text{a.e. in } \Omega \times [\tau, +\infty), \tag{4.48}$$

thus concluding the proof. Notice that the quantity δ depends on the initial datum only through the initial energy $\mathcal{E}(0)$ and $\overline{\mathbf{u}}_0$, since all the estimates involved in this proof are the ones mentioned in Theorem 3.10.

5. Proof of Theorem 3.13

Proof of Theorem 3.13. By Remark 3.14, we only need to show the existence of a compact absorbing set. From Theorem 3.9, we deduce that for any $\mathbf{u}_0 \in \mathcal{V}_\mathbf{M}$, there exist constants $C_3, C_4 > 0$ such that

$$\|S(t)\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} \le C_3 e^{-\omega t} \|\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} + C_4, \quad \forall t \ge 0.$$

Indeed, since Ψ is bounded on $[0, 1]$ and $0 \le \mathbf{u}_0 \le 1$, it holds that

$$\frac{1}{2}\|\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} - C \le \mathcal{E}(0) \le \frac{1}{2}\|\mathbf{u}_0\|^2_{\mathcal{V}_\mathbf{M}} + C,$$

for some $C > 0$ independent of the initial datum $\mathbf{u}_0$. This means that the set

$$\widetilde{\mathcal{B}}_0 := \left\{\mathbf{u} \in \mathcal{V}_\mathbf{M} : \|\mathbf{u}\|_{\mathcal{V}_\mathbf{M}} \le \sqrt{\frac{C_3}{2} + C_4} := R_0\right\}$$

is an absorbing set, that is, for any bounded set $B \subset \mathcal{V}_\mathbf{M}$, there exists $t_e > 0$ depending on $\sup_{\mathbf{u}_0 \in B} \|\mathbf{u}_0\|_{\mathcal{V}_\mathbf{M}}$ such that $S(t)B \subset \widetilde{\mathcal{B}}_0$ for any $t \ge t_e$.

On account of (4.38) and (4.48), we can find $\delta = \delta(R_0) > 0$ and a bounded set

$$\mathcal{B}_0 := \left\{ \mathbf{u} \in \widetilde{\mathcal{B}}_0 : \|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} \le C_0, \ \mathbf{u} \ge \delta \text{ in } \overline{\Omega}, \ \partial_{\mathbf{n}}\mathbf{u} = 0 \quad \text{a.e. on } \partial\Omega \right\}, \tag{5.1}$$

for some $C_0 = C_0(R_0) > 0$ and a time t_{R_0}, depending only on R_0, such that $S(t)\widetilde{\mathcal{B}}_0 \subset \mathcal{B}_0$ for any $t \ge t_{R_0}$. Note that we can state this *for any* $t \ge t_{R_0}$ instead of *for almost any* t (see Remark 3.11). This clearly implies that $\mathcal{B}_0$ is a compact absorbing set, and thus the proof is complete. ∎

6. Proof of Theorem 3.15

We need some preliminary lemmas. First, recalling (5.1), we know that there exists $\widetilde{t} = \widetilde{t}(R_0, \mathbf{M}) > 0$ (with $\mathbf{M}$ fixed) such that $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$, for any $t \ge \widetilde{t}$. We then introduce the set

$$\mathbb{B} := \overline{\bigcup_{t \ge \widetilde{t}} S(t)\mathcal{B}_0}^{\mathcal{V}_{\mathbf{M}}},$$

which is compact, positively invariant, and absorbing. Let us prove the next lemma.

Lemma 6.1. *For any $T \ge 0$, there exists $C = C(T) > 0$ such that, given $\mathbf{u}_{0,1}, \mathbf{u}_{0,2} \in \mathbb{B}$, we have*

$$\begin{aligned} &\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|_{\mathcal{V}_{\mathbf{M}}}^2 + \int_0^t \|\partial_s S(s)\mathbf{u}_{0,1} - \partial_s S(s)\mathbf{u}_{0,2}\|^2 ds \\ &\quad \le C(T)\|\mathbf{u}_{0,1} - \mathbf{u}_{0,2}\|_{\mathcal{V}_{\mathbf{M}}}^2, \quad \forall t \in [0, T], \end{aligned} \tag{6.1}$$

and

$$\|S(t)\mathbf{u}_{0,1} - S(t)\mathbf{u}_{0,2}\|_{\mathbf{H}^2(\Omega)}^2 \le \frac{C(T)(1 + t^2)}{t^2}\|\mathbf{u}_{0,1} - \mathbf{u}_{0,2}\|_{\mathcal{V}_{\mathbf{M}}}^2, \quad \forall t \in (0, T]. \tag{6.2}$$

Proof. The following computations are formal, but they can be performed within a suitable approximating scheme, like the one used in the proof of Theorem 3.1. In particular, leaning on the strict separation property, which holds uniformly (depending only R_0 and $\mathbf{M}$, this last one being fixed; see Remark 3.7) if the initial data belong to $\mathbb{B}$ (see Theorem 3.1), then we are able to interpret, by uniqueness, the solutions to problem (1.6) as the solutions to a similar problem where ψ is replaced by a suitable regular potential (i.e., obtained by extending ψ outside $[\delta, 1 - (N-1)\delta]$ in a smooth way).

We start by observing that there exists $\delta > 0$ (possibly smaller than the one in the definition of $\mathbb{B}$) such that (see (3.13) and (3.20))

$$S(t)\mathbf{u}_0 \ge \delta \quad \text{in } \overline{\Omega}, \quad \forall\, t \ge 0, \ \forall\, \mathbf{u}_0 \in \mathbb{B}. \tag{6.3}$$

Now set $\mathbf{u}^i = S(t)\mathbf{u}_{0,i}$, with $\mathbf{u}_{0,i} \in \mathbb{B}$, $i = 1, 2$. Then, taking the difference between the equations satisfied by $\mathbf{u}^1$ and $\mathbf{u}^2$, multiplying it by $\partial_t \mathbf{u}$, where $\mathbf{u} = \mathbf{u}^1 - \mathbf{u}^2$, and integrating over Ω, after an integration by parts, we get

$$\frac{\gamma}{2}\frac{d}{dt}(\boldsymbol{\alpha}\nabla\mathbf{u}, \nabla\mathbf{u}) - \sum_{i,j=1}^{N}(\alpha_{ij}(\psi'(u_j^1) - \psi'(u_j^2)), \partial_t u_i) + \sum_{i,j=1}^{N}(\alpha_{ij}(\mathbf{A}\mathbf{u})_j, \partial_t u_i) + \|\partial_t \mathbf{u}\|^2 = 0,$$

where we exploited the following facts: $\overline{\partial_t \mathbf{u}} \equiv 0$, $\mathbf{P}(\partial_t \mathbf{u}) = \partial_t \mathbf{u}$, and the property $\boldsymbol{\alpha}(\mathbf{P}\boldsymbol{\xi}) = \boldsymbol{\alpha}\boldsymbol{\xi}$ for any $\boldsymbol{\xi} \in \mathbb{R}^N$.

Thanks to (6.3), we have $\|\psi''(su_j^1 + (1-s)u_j^2)\|_{L^\infty(\Omega)} \leq C$ for any $j = 1, \ldots, N$, so that, by standard inequalities,

$$\begin{aligned}
&\sum_{i,j=1}^{N}(\alpha_{ij}(\psi'(u_j^1) - \psi'(u_j^2)), \partial_t u_i) \\
&\quad = \sum_{i,j=1}^{N}\int_\Omega \int_0^1 \psi''(su_j^1 + (1-s)u_j^2)(u_j^1 - u_j^2)\alpha_{ij}\partial_t u_i \, ds dx \\
&\quad \leq C\|\mathbf{u}\|\|\partial_t \mathbf{u}\| \leq C\|\mathbf{u}\|^2 + \frac{1}{4}\|\partial_t \mathbf{u}\|^2 \leq C\|\nabla\mathbf{u}\|^2 + \frac{1}{4}\|\partial_t \mathbf{u}\|^2. \qquad (6.4)
\end{aligned}$$

Then, similarly,

$$\sum_{i,j=1}^{N}(\alpha_{ij}(\mathbf{A}\mathbf{u})_j, \partial_t u_i) \leq C\|\mathbf{u}\|^2 + \frac{1}{4}\|\partial_t \mathbf{u}\|^2,$$

so that, owing to Poincaré's inequality, we obtain

$$\frac{\gamma}{2}\frac{d}{dt}(\boldsymbol{\alpha}\nabla\mathbf{u}, \nabla\mathbf{u}) + \frac{1}{4}\|\partial_t \mathbf{u}\|^2 \leq C(\boldsymbol{\alpha}\nabla\mathbf{u}, \nabla\mathbf{u}), \qquad \text{for a.a. } t \in [0, T], \qquad (6.5)$$

where we exploited the fact that ($f_{,k} := \partial_{x_k} f$)

$$(\boldsymbol{\alpha}\nabla\mathbf{u}, \nabla\mathbf{u}) = \sum_{k=1}^{d}\sum_{i,j=1}^{N}(\alpha_{ij}u_{i,k}, u_{j,k}) = \sum_{k=1}^{d}(\boldsymbol{\alpha}\mathbf{u}_{,k}, \mathbf{u}_{,k}) \geq C\|\nabla\mathbf{u}\|^2,$$

by (2.1) (recall that $\mathbf{P}\mathbf{u}_{,k} = \mathbf{u}_{,k}$). Thus, (6.1) follows from (6.5) owing to Gronwall's lemma and Poincaré's inequality. Notice that the constant C, thanks to (6.3), does not depend on the specific $\mathbf{u}_{0,i} \in \mathbb{B}$.

Concerning (6.2), we write (3.4) for the difference (defined as $\mathbf{u}$) between $\mathbf{u}^1$ and $\mathbf{u}^2$ and we differentiate the resulting equation with respect to time. Then, we multiply it by $\partial_t \mathbf{u}$ and integrate over Ω. This gives, after an integration by parts, the identity

$$\frac{1}{2}\frac{d}{dt}\|\partial_t \mathbf{u}\|^2 + \sum_{i,j=1}^{N} (\alpha_{ij}(\psi''(u_j^1)\partial_t u_j^1 - \psi''(u_j^2)\partial_t u_j^2), \partial_t u_i)$$
$$- \sum_{i,j=1}^{N} (\alpha_{ij}(\mathbf{A}\partial_t \mathbf{u})_j, \partial_t u_i) + \gamma(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}) = 0,$$

where we exploited $\overline{\partial_t \mathbf{u}} \equiv 0$, $\mathbf{P}\partial_t \mathbf{u} = \partial_t \mathbf{u}$, and the properties of $\boldsymbol{\alpha}$. Using now (6.3) once more and standard inequalities, and on account of assumption $\psi \in C^3(0,1]$, we get

$$\left|\sum_{i,j=1}^{N} (\alpha_{ij}(\psi''(u_j^1)\partial_t u_j^1 - \psi''(u_j^2)\partial_t u_j^2), \partial_t u_i)\right|$$
$$= \left|\sum_{i,j=1}^{N} (\alpha_{ij}(\psi''(u_j^1) - \psi''(u_j^2))\partial_t u_j^1), \partial_t u_i)\right|$$
$$+ \left|\sum_{i,j=1}^{N} (\alpha_{ij}\psi''(u_j^2)(\partial_t u_j^1 - \partial_t u_j^2), \partial_t u_i)\right|$$
$$\leq \left|\sum_{i,j=1}^{N} \int_\Omega \int_0^1 \alpha_{ij}\psi'''(su_j^1 + (1-s)u_j^2)(u_j^1 - u_j^2)\partial_t u_j^1 \partial_t u_i \, ds dx\right| + C\|\partial_t \mathbf{u}\|^2$$
$$\leq C\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\partial_t \mathbf{u}^1\|\|\partial_t \mathbf{u}\|_{\mathbf{L}^4(\Omega)} + C\|\partial_t \mathbf{u}\|^2$$
$$\leq C\|\mathbf{u}\|_{\mathcal{V}_\mathbf{M}}\|\nabla\partial_t \mathbf{u}\| + \|\partial_t \mathbf{u}\|^2 \leq C(\|\mathbf{u}\|^2_{\mathcal{V}_\mathbf{M}} + \|\partial_t \mathbf{u}\|^2) + \frac{\gamma}{2}(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}),$$

where we exploited the embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega)$, the bound $\|\partial_t \mathbf{u}^1\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \leq C$ with C depending only on R_0 (see part (2) of Theorem 3.1, Theorem 3.10, and (5.1)), Poincaré's inequality, and the fact that $(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}) \geq C\|\nabla\partial_t \mathbf{u}\|^2$. This last estimate comes from (2.1), since we have

$$(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}) = \sum_{k=1}^{d}\sum_{i,j=1}^{N} (\alpha_{ij}\partial_t u_{i,k}, \partial_t u_{j,k}) = \sum_{k=1}^{d} (\boldsymbol{\alpha}\partial_t \mathbf{u}_{,k}, \partial_t \mathbf{u}_{,k}) \geq C\|\nabla\partial_t \mathbf{u}\|^2,$$

with $\mathbf{P}\partial_t \mathbf{u}_{,k} = \partial_t \mathbf{u}_{,k}$. In conclusion, we have

$$\left|\sum_{i,j=1}^{N} (\alpha_{ij}(\mathbf{A}\partial_t \mathbf{u})_j, \partial_t u_i)\right| \leq C\|\partial_t \mathbf{u}\|^2.$$

We thus end up with

$$\frac{1}{2}\frac{d}{dt}\|\partial_t \mathbf{u}\|^2 + \frac{\gamma}{2}(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}) \leq C(\|\mathbf{u}\|^2_{\mathcal{V}_\mathbf{M}} + \|\partial_t \mathbf{u}\|^2),$$

and, multiplying both sides by $s^2 \in [0, T^2]$, we obtain

$$\frac{1}{2}\frac{d}{dt}s^2\|\partial_t \mathbf{u}\|^2 + \gamma\frac{s^2}{2}(\boldsymbol{\alpha}\nabla\partial_t \mathbf{u}, \nabla\partial_t \mathbf{u}) \le C(s^2\|\mathbf{u}\|^2_{\mathcal{V}_M} + (s^2 + s)\|\partial_t \mathbf{u}\|^2).$$

Integrating over $(0, t)$, recalling (6.1), and dividing by t^2, we deduce

$$\|\partial_t \mathbf{u}(t)\| \le \frac{C(T)}{t}\|\mathbf{u}_{0,1} - \mathbf{u}_{0,2}\|_{\mathcal{V}_M}, \quad \forall t \in (0, T]. \tag{6.6}$$

We now multiply the equation for $\mathbf{u}$ by $-\Delta\mathbf{u}$ and integrate over Ω. We get

$$-(\partial_t \mathbf{u}, \Delta\mathbf{u}) + \gamma(\boldsymbol{\alpha}\Delta\mathbf{u}, \Delta\mathbf{u}) - \sum_{i,j=1}^{N}(\alpha_{ij}(\psi'(u_i^1) - \psi'(u_i^2)), \Delta u_j) + \sum_{i,j=1}^{N}(\alpha_{ij}(\mathbf{A}\mathbf{u})_j, \Delta u_j) = 0, \tag{6.7}$$

where we used $\overline{\Delta\mathbf{u}} = 0$ and the properties of $\boldsymbol{\alpha}$. Now, since $\mathbf{P}\Delta\mathbf{u} = \Delta\mathbf{u}$, we have (see (2.1))

$$(\boldsymbol{\alpha}\Delta\mathbf{u}, \Delta\mathbf{u}) \ge C\|\Delta\mathbf{u}\|^2.$$

Moreover, like with (6.4), we have

$$\Big|\sum_{i,j=1}^{N}(\alpha_{ij}(\psi'(u_i^1) - \psi'(u_i^2)), \Delta u_j)\Big| \le C\|\mathbf{u}\|^2 + \frac{\gamma}{4}(\boldsymbol{\alpha}\Delta\mathbf{u}, \Delta\mathbf{u}),$$

and the Cauchy–Schwarz inequality and Young's inequality yield

$$\Big|\sum_{i,j=1}^{N}(\alpha_{ij}(\mathbf{A}\mathbf{u})_j, \Delta u_j)\Big| \le C\|\mathbf{u}\|^2 + \frac{\gamma}{4}(\boldsymbol{\alpha}\Delta\mathbf{u}, \Delta\mathbf{u}).$$

Therefore, from (6.7) and Poincaré's inequality, we deduce

$$C\|\Delta\mathbf{u}\|^2 \le C\|\nabla\mathbf{u}\|^2 + C\|\partial_t \mathbf{u}\|^2,$$

and combining it with (6.1) and (6.6), we infer (6.2). ∎

We can now continue the proof of Theorem 3.15, following [39]. By (3.10), given $\mathbf{u}(t) = S(t)\mathbf{u}_0$, with $\mathbf{u}_0 \in \mathbb{B}$ we have, for any given $T > 0$,

$$\|\mathbf{u}(t) - \mathbf{u}(s)\|_{\mathcal{V}_M} \le \int_s^t \|\partial_t \mathbf{u}(\tau)\|_{\mathcal{V}_M} d\tau \le |t - s|^{\frac{1}{2}}\Big(\int_s^t \|\partial_t \mathbf{u}(\tau)\|^2_{\mathcal{V}_M} d\tau\Big)^{\frac{1}{2}} \le C(T)|t - s|^{\frac{1}{2}} \tag{6.8}$$

for any $s, t \in [0, T]$, that is, $t \mapsto S(t)\mathbf{u}_0$ is $\frac{1}{2}$-Hölder continuous in $[0, T]$, with $C(T)$ depending only on R_0. Let us now fix $t^* > 0$. Thanks to smoothing property (6.2), valid at $t = t^* > 0$, the discrete dynamical system generated by the iterations of $S(t^*)$ possesses an exponential attractor $\mathcal{M}^* \subset \mathbb{B}$ (see, e.g., [39, Theorem 3.7]). Moreover, (6.1) and (6.8) entail

$$S : [0, t^*] \times \mathbb{B} \to \mathbb{B}, \quad S(t, \mathbf{u}_0) := S(t)\mathbf{u}_0$$

is Hölder continuous, when $\mathbb{B}$ is endowed with the $\mathcal{V}_{\mathbf{M}}$ topology. Therefore, we can define

$$\mathcal{M} := \bigcup_{t \in [0,t^*]} S(t)\mathcal{M}^* \subset \mathbb{B},$$

and, following [39], show that $\mathcal{M}$ is an exponential attractor for $S(t)$ on $\mathbb{B}$. Since $\mathbb{B}$ is also a compact absorbing set, the basin of exponential attraction of $\mathcal{M}$ is the whole phase space $\mathcal{V}_{\mathbf{M}}$. This means that $\mathcal{M}$ is an exponential attractor on $\mathcal{V}_{\mathbf{M}}$. The proof is finished.

Acknowledgements. The authors are grateful to the anonymous referees for their careful reading of the manuscript, as well as for their valuable comments.

Funding. The authors have been partially funded by MIUR-PRIN research grant no. 2020F3NCPX "Mathematics for Industry 4.0 (Math4I4)". The authors are also members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), Istituto Nazionale di Alta Matematica (INdAM). The present research is part of the activities of "Dipartimento di Eccellenza 2023–2027".

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Received 5 April 2023; revised 2 August 2023.

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