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Structure of singularities in the nonlinear nerve conduction problem

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Abstract. We give a characterization of the singular points of the free boundary $\partial\{u > 0\}$ for viscosity solutions of the nonlinear equation

$$F(D^2u) = -\chi_{\{u>0\}},$$

where F is a fully nonlinear elliptic operator and χ is the characteristic function. This equation models the propagation of a nerve impulse along an axon.

We analyze the structure of the free boundary $\partial\{u > 0\}$ near the singular points where u and ∇u vanish simultaneously. Our method uses the stratification approach developed in Dipierro and the author's 2018 paper.

In particular, when $n = 2$ we show that near a flat singular free boundary point, $\partial\{u > 0\}$ is a union of four C^1 arcs tangential to a pair of crossing lines.

1. Introduction

In this paper we study the free boundary problem

$$F(D^2u) = -\chi_{\{u>0\}} \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a given bounded domain with $C^{2,\alpha}$ boundary, $\chi_{\{u>0\}}$ is the characteristic function of $\{u > 0\}$, and F is a convex fully nonlinear elliptic operator satisfying some structural conditions. Equation (1.1) appears in a model of the nerve impulse propagation [10, 18, 19].

It comes from the following linearized diffusion system of FitzHugh:

$$\begin{cases} u_t = r(x)\Delta u + \mathcal{F}(u, \vec{v}), \\ \vec{v}_t = G(u, \vec{v}), \end{cases} \tag{1.2}$$

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where $u(x,t)$ is the voltage across the nerve membrane at distance x and time t, and the components of $\vec{v} = (v^1, \dots, v^k)$ model the conductance of the membrane to various ions [10]. A specific form for the interaction term $\mathcal{F}(u, \vec{v})$ was suggested by McKean—namely, $\mathcal{F}(u, \vec{v}) = -u + \chi_{\{u>0\}}$ [16]. Due to the homogeneity of the equation, the linear term in $\mathcal{F}$ disappears after quadratic scaling, so we neglect it.

The linearized steady state equation

$$\Delta u = -\chi_{\{u>0\}} \tag{1.3}$$

also arises in a solid combustion model [17] and the composite membrane problem; see [4] and also [6] for a variational formulation.

A chief difficulty is to analyze the free boundary near singular points where both u and ∇u vanish. The main technique used in [4, 6, 17] is a monotonicity formula, which is not available for the nonlinear equations. The aim of this paper is to use the boundary Harnack principles and anisotropic scalings to develop a new approach to circumvent the lack of the monotonicity formulas and obtain some of the main results from [17] and [15] for the fully nonlinear case. More precisely, in this paper we address the optimal regularity, uniqueness of blow-up at singular points, degeneracy, and the shape of the free boundary near the singular points.

One of the main results in [17] concerns the cross-shaped singularities in $\mathbb{R}^2$. It follows from the classification of homogeneous solutions and an application of the monotonicity formula introduced in [17]. For nonlinear equations, this method cannot be applied. We remark that the degenerate case (i.e., when $u(x) = o(|x - x_0|^2)$ near a free boundary point x_0) cannot be treated by the monotonicity formula introduced in [17] because it does not provide any qualitative information about u; see [17, Proposition 5.1].

It is well known that the strong solutions of (1.3) may not be $C^{1,1}_{\mathrm{loc}}$; see [5, Proposition 5.3.1]. However, if $F = \Delta$, then ∇u is always log-Lipschitz continuous; see [13, Lemma 2.1]. For general elliptic operators one can show that ∇u is C^α for every $\alpha \in (0, 1)$; see [3] and Remark 2.2. It appears that the natural scaling is quadratic, but the lack of compactness is one of the key difficulties we will have to deal with.

Problem (1.3) has some resemblance to the classical obstacle problem [2], and can be extended for fully nonlinear operators [14].

The paper is organized as follows: In Section 2 we state some technical results. In Section 3 we prove the existence of viscosity solutions using a penalization argument. We also show the existence of a maximal solution and establish its nondegeneracy. Section 4 contains the proof of the following dichotomy: either the free boundary points are flat or the solution has quadratic growth. As a consequence, we show that if $n = 2$, then near a flat point the free boundary is a union of four C^1 curves tangential to a pair of crossing lines. This is done in Section 6.

2. Technical results

Throughout this paper $B_r(x)$ denotes the open ball of radius r centered at $x \in \mathbb{R}^n$ and we write $B_r = B_r(0)$. For a continuous function u, we let $u = u^+ - u^-$, $u^+ = \max(0, u)$, $\Omega^+(u) = \{u > 0\}$, and $\Omega^-(u) = \{u < 0\}$, and let $\partial_{\text{sing}}\{u > 0\}$ be the singular subset of the free boundary $\partial\{u > 0\}$, where $u = |\nabla u| = 0$.

We shall now make two standing assumptions on the operators under consideration. To formulate them we let $\mathcal{S}$ be the space of $n \times n$ symmetric matrices and $\mathcal{S}^+(\lambda, \Lambda)$ positive definite symmetric matrices with eigenvalues bounded between two positive constants λ and Λ.

F1° The operator $F : \mathcal{S} \subset \mathbb{R}^{n\times n} \to \mathbb{R}$ is uniformly elliptic, that is, there are two positive constants λ, Λ such that

$$\lambda\|N\| \le F(M + N) - F(M) \le \Lambda\|N\|, \quad M \in \mathcal{S}, \tag{2.1}$$

for every nonnegative matrix N.

F2° F is smooth except at the origin and homogeneous of degree one. In addition, $F(tM) = tF(M), t \in \mathbb{R}$ and $F(0) = 0$.

For smooth F, hypothesis **F1°** is equivalent to

$$\lambda|\xi|^2 \le F_{ij}(M)\xi_i\xi_j \le \Lambda|\xi|^2,$$

where $F_{ij}(S) = \frac{\partial F(S)}{\partial s_{ij}}$, $S = [s_{ij}]$.

Typically, $F(M) = \sup_{t\in\mathcal{I}} A_{ij,t}M_{ij}$, where $\mathcal{I}$ is the index set and $A_{ij,t} \in \mathcal{S}^+(\lambda, \Lambda)$ is such that $\lambda|\xi|^2 \le A_{ij,t}\xi_i\xi_j \le \Lambda|\xi|^2$. Notice that if $w_t(x) = w(A_t^{\frac{1}{2}}x)$, then we have $\Delta w_t = A_{ij,t}w_{ij}$.

We also define Pucci's extremal operators

$$\mathcal{M}^-(M, \lambda, \Lambda) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \quad \mathcal{M}^+(M, \lambda, \Lambda) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where $e_1 \le e_2 \le \cdots \le e_n$ are the eigenvalues of $M \in \mathcal{S}$.

Definition 2.1. A continuous function u is said to be a viscosity solution of the equation $F(D^2u) = -\chi_{\{u>0\}}$ if the equation $F(D^2v(x_0)) = -\chi_{\{u>0\}}$ holds pointwise, whenever at $(x_0, u(x_0))$ the graph of u can be touched from above and below by paraboloids v.

Remark 2.2. We will be using some well-known estimates for the viscosity solutions. If F is convex or concave and u is a viscosity solution of $F(D^2u) = 0$ in B_1, then

$$\|u\|_{C^{2,\alpha}(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_1)} + |F(0)|), \tag{2.2}$$

where $0 < \alpha < 1$ and C are universal constants; see [3, Theorem 6.6]. Moreover, if F is convex or concave, then for the viscosity solutions of $F(D^2u) = 0$, we still have the local estimate

$$\|u\|_{C^{1,1}(B_{1/11})} \le C\|u\|_{L^\infty(B_1)}$$

(see [3, page 60, (6.14) and Remark 1]).

Under assumptions **F1°–F2°**, the classical weak and strong comparison principles are valid for the viscosity solutions [3]. Moreover, we have the strong and Hopf's comparison principles.

Lemma 2.3 (Strong comparison principle; [12, Theorem 3.1]). *Suppose $v \in C^2(D)$, $w \in C^1(D)$, and $\nabla v \not\equiv 0$ in a bounded domain D. Let $F(D^2v) \ge 0 \ge F(D^2w)$ in $D \subset \mathbb{R}^n$ in the viscosity sense and $v \le w$ where v, w are not identical. Then,*

$$v < w \quad \text{in } D. \tag{2.3}$$

Lemma 2.4 (Hopf's comparison principle; [12, Theorem 4.1]). *Let B be a ball contained in D and assume that $w \in C^1(D)$, $v \in C^2(D)$ and that $\nabla v \ne 0$ in B. Let v and w be a viscosity subsolution and a supersolution of $F(D^2u) = 0$, respectively. Moreover, suppose that $v < w$ in B, and that $v(x_0) = w(x_0)$, for some $x_0 \in \partial B$. Then, $\nabla v(x_0) \ne \nabla w(x_0)$.*

One of the main tools in our analysis is the boundary Harnack principle. As before, we assume that F is smooth, homogeneous of degree 1, and uniformly elliptic with ellipticity constants λ and Λ, and that $F(0) = 0$. We use the following notation: $f(x')$, $x' \in B'_1 \subset \mathbb{R}^{n-1}$ is a Lipschitz continuous function with Lipschitz constant $M > 1$; $f(0) = 0$; $\Omega_r = B'_r \times [-rM, rM] \cap \{x_n > f(x')\}$; $\Delta_r = B'_r \times [-rM, rM] \cap \{x_n = f(x')\}$; and $A = e_n M/2$, where e_n is the unit direction of the x_n axis.

Then, we have the following Harnack principle (see [20]):

Theorem 2.5. *Assume* **F1°–F2°** *hold and F is either concave or convex. Let u, v be two nonnegative solutions of $F(D^2u) = 0$ in Ω_1 that equal 0 along the Lipschitz bottom of Δ_1. Suppose also that $v \not\equiv 0$, $u - \sigma v \ge 0$ in Ω_1 for some $\sigma \ge 0$. Then, for some constant C depending only on λ, Λ, n, and the Lipschitz character of Ω_1, we have in $\Omega_{\frac{1}{2}}$*

$$C^{-1}\frac{u(A) - \sigma v(A)}{v(A)} \le \frac{u - \sigma v}{v} \le C\frac{u(A) - \sigma v(A)}{v(A)}. \tag{2.4}$$

Furthermore, as in [1] (see also [20, Section 2]) one can show that the nonnegative solutions in Ω_1 are monotone in Ω_{δ_0} for some universal δ_0. We state this only in two spatial dimensions.

Theorem 2.6. *Let w be a viscosity solution of $F(D^2 w) = 0$; $w \geq 0$ in $D = \{|x_1| \leq 1, f(x_1) < x_2 \leq M\}$, $M = \|f\|_{C^{0,1}}$; $w = 0$ on $f(x_1) = x_2$. Assume* **F1°–F2°** *hold and F is either concave or convex. Then, there is $\delta = \delta(M)$ such that*

$$\partial_2 w \geq 0 \quad \text{in } D_\delta = \big\{|x_1| \leq \delta,\, f(x_1) < x_2 \leq M\delta\big\}.$$

In [20], Theorem 2.6 is stated for concave operator F; however, the concavity is needed only to assure that the viscosity solutions of the homogeneous equation are locally $C^{2,\alpha}$ regular; see [20, Remark 1.2]. Seeing that in the proofs of [20, Lemmata 2.1–2.5] one needs only $C^{1,\alpha}$ regularity of the solutions, in view of Remark 2.2 we see that Theorem 2.6 continues to hold for convex F; see [9].

3. Existence and nondegeneracy

In this section we prove the existence of viscosity solutions and the nondegeneracy of maximal solutions.

3.1. Existence of viscosity solutions

Definition 3.1. A continuous function u is said to be a viscosity subsolution of the equation $F(D^2 u) = -\chi_{\{u>0\}}$ if the inequality $F(D^2 v(x_0)) \geq -\chi_{\{u>0\}}$ holds pointwise, whenever at $(x_0, u(x_0))$ the graph of u can be touched from below by a paraboloid v. Moreover, u is said to be a strict subsolution if the inequality above is strict.

Definition 3.2. A viscosity solution u of $F(D^2 u) = -\chi_{\{u>0\}}$ is said to be maximal in D if for every strong subsolution v satisfying $v \leq u$ on $\partial D'$ for some subdomain $D' \subset D$, we have $v \leq u$ in D'.

Theorem 3.3. *Assume* **F1°–F2°** *hold. Let D be a bounded $C^{2,\alpha}$ domain and $g \in C^{2,\alpha}(\overline{D})$. There exists a maximal viscosity solution u to*

$$\begin{cases} F(D^2 u) = -\chi_{\{u>0\}} & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases} \tag{3.1}$$

such that $u \in W^{2,p}(D)$ for every $p \geq 1$.

Proof. We use a standard penalization argument (see [11, page 24, Lemma 3.1]). Let $\beta_\varepsilon(t), t \in \mathbb{R}$ be a family of C^∞ functions such that

$$\begin{cases} \beta_\varepsilon(t) \geq \chi_{\{t>0\}} & \text{on } \mathbb{R}, \\ \beta_{\varepsilon'}(t) \leq \beta_\varepsilon(t) & \text{if } \varepsilon' < \varepsilon, \\ \lim_{\varepsilon\to 0} \beta_\varepsilon(t) = \chi_{\{t>0\}} & t \in \mathbb{R}. \end{cases} \tag{3.2}$$

Given $\varepsilon > 0$, there is a solution v of

$$\begin{cases} F(D^2 v) = -\beta_\varepsilon(v) & \text{in } D, \\ v = g & \text{on } \partial D. \end{cases} \tag{3.3}$$

This follows from Schauder's fixed point theorem; see [11, page 24, Lemma 3.1]. Observe that Perron's method implies that for every $\varepsilon > 0$, the maximal solution u_ε exists. Furthermore, since β_ε are uniformly bounded, then $\|v\|_{W^{2,p}(D)} \leq C$ with some C independent of ε; see [3, Theorem 7.1] and Remark 2.2 above.

If v is a subsolution, that is, $F(D^2 v) \geq -\chi_{\{v>0\}}$, then by (3.2) we also have that $F(D^2 v) \geq -\beta_\varepsilon(v)$. Thus, for $\varepsilon > \varepsilon'$ (using (3.2)) we get

$$F(D^2 u_{\varepsilon'}) = -\beta_{\varepsilon'}(u_{\varepsilon'}) \geq -\beta_\varepsilon(u_{\varepsilon'}).$$

This shows that $u_{\varepsilon'}$ is a subsolution to (3.3). Since u_ε is the maximal solution, we then have

$$v \leq u_\varepsilon, \quad u_{\varepsilon'} \leq u_\varepsilon. \tag{3.4}$$

Thus, $u(x) = \lim_{\varepsilon\to 0} u_\varepsilon$ in $W^{2,p}$, and by (3.4), $u \geq v$ for every subsolution v. From the uniform convergence, it follows that $u(z) > 0$, implying that $u_\varepsilon > 0$ in some neighborhood of z. Thus, $F(D^2 u) = -1$ near z. Since $D^2 u = 0$ almost everywhere on $\{u = 0\}$, it follows that $F(D^2 u) = -\chi_{\{u>0\}}$. ∎

3.2. Nondegeneracy

Theorem 3.4. *Assume* **F1°–F2°** *hold and let u be the maximal solution. Then, there is a universal constant $c_{n,\gamma}$, depending only on dimension n and $\gamma = \frac{\Lambda(n-1)}{\lambda} - 1$, such that*

$$\inf_{B_r(x_0)} u > -c_{n,\gamma} r^2$$

implies that $u(x_0) > 0$.

Proof. Let us consider

$$b(x) = \begin{cases} C(1 - |x|^2) & \text{if } |x| \leq 1, \\ \phi(x) - \phi(1) & \text{if } |x| > 1, \end{cases}$$

where

$$\phi(x) = \begin{cases} -\log|x| & \text{if } n = 2, \\ \dfrac{1}{\gamma}|x|^{-\gamma} & \text{if } n \geq 3, \end{cases}$$

and the constant C is chosen so that $b(x)$ is C^1 regular. It is straightforward to compute $D^2 b$, and thus,

$$F(D^2 b) = \begin{cases} -2CF(\delta_{ij}) & \text{if } |x| \leq 1, \\ -\dfrac{1}{|x|^{\gamma+2}} F\Big(\delta_{ij} - (\gamma+2)\dfrac{x_i x_j}{|x|^2}\Big) & \text{if } |x| > 1. \end{cases}$$

From the ellipticity in (2.1), we get that

$$F\Big(\delta_{ij} - (\gamma+2)\frac{x_i x_j}{|x|^2}\Big) \leq \mathcal{M}^+\Big(\delta_{ij} - (\gamma+2)\frac{x_i x_j}{|x|^2}\Big) = 0, \quad |x| > 1.$$

Hence,

$$F(D^2 b) \geq -\frac{1}{|x|^{\gamma+2}} \mathcal{M}^+\Big(\delta_{ij} - (\gamma+2)\frac{x_i x_j}{|x|^2}\Big) = 0, \quad |x| > 1.$$

Consequently, we see that $\widehat{b}(x) = \frac{b(x)}{2CF(\delta_{ij})}$ is a subsolution.

Given r, choose ρ so that $\frac{2}{\rho} = r$. Then, for $|x| > \frac{1}{\rho}$, we have

$$\frac{1}{\rho^2}\widehat{b}(\rho x) = \frac{1}{\rho^{\gamma+2}\gamma}\Big[\frac{1}{|x|^\gamma} - \rho^\gamma\Big],$$

and consequently,

$$\begin{aligned} \frac{\widehat{b}(r)}{\rho^2} &= \frac{1}{\rho^2}\widehat{b}\Big(\frac{2}{\rho}\Big) = -\Big(1 - \frac{1}{2^\gamma}\Big)\frac{1}{\rho^2\gamma} \\ &= -\Big(1 - \frac{1}{2^{n-2}}\Big) r^2 \frac{1}{4\gamma} =: -c_{n,\gamma} r^2. \end{aligned}$$

Thus, $u(0) \geq \widehat{b}(0) > 0$. ∎

4. Dichotomy

In order to formulate the main result of this section, we first introduce the notion of flatness. Let P_2 be the set of all homogeneous normalized polynomials of degree two, that is,

$$P_2 := \Big\{p(x) = \sum a_{ij} x_i x_j, \text{ for any } x \in \mathbb{R}^n, \text{ with } \|p\|_{L^\infty(B_1)} = 1\Big\}, \tag{4.1}$$

where a_{ij} is a symmetric $n \times n$ matrix. For given $p \in P_2$ and $x_0 \in \mathbb{R}^n$, we set $p_{x_0}(x)$

$:= p(x - x_0)$ and consider the zero level set of translated polynomial p

$$S(p, x_0) := \{x \in \mathbb{R}^n : p_{x_0}(x) = 0\}. \tag{4.2}$$

By definition, $S(p, x_0)$ is a cone with a vertex at x_0.

Definition 4.1. Let $\delta > 0$, $R > 0$, and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is (δ, R)-flat at x_0 if, for every $r \in (0, R]$, there exists $p \in P_2$ such that

$$\mathrm{HD}(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)) < \delta r.$$

Here HD denotes the Hausdorff distance defined as follows:

$$\mathrm{HD}(A, B) := \max\Big\{\sup_{a\in A} \mathrm{dist}(a, B), \ \sup_{b\in B} \mathrm{dist}(b, A)\Big\}. \tag{4.3}$$

Remark 4.2. In the previous definition p may depend on r. Later we will show that in the two-dimensional case, the limiting configurations at asymptotically flat points are unique.

Given $r > 0$, $x_0 \in \partial\{u > 0\}$, and $p \in P_2$, we let

$$h_{\min}(r, x_0, p, u) := \mathrm{HD}(\partial\{u > 0\} \cap B_r(x_0), S(p, x_0) \cap B_r(x_0)). \tag{4.4}$$

Then, we define the flatness at level $r > 0$ of $\partial\{u > 0\}$ at x_0 as follows:

Definition 4.3. Let $\delta > 0$, $r > 0$, and $x_0 \in \partial\{u > 0\}$. We say that $\partial\{u > 0\}$ is δ-flat at level r at x_0 if $h(r, x_0, u) < \delta r$, where

$$h(r, x_0, u) := \inf_{p\in P_2} h_{\min}(r, x_0, p, u). \tag{4.5}$$

Remark 4.4. In view of Definitions 4.1 and 4.3, we can say that $\partial\{u > 0\}$ is (δ, R)-flat at $x_0 \in \partial\{u > 0\}$ if and only if, for every $r \in (0, R]$, it is δ-flat at level r at x_0.

Theorem 4.5. *Let $n \geq 2$ and u be a viscosity solution of (1.1). Let $D \subset \Omega$, $\delta > 0$, and let $x_0 \in \partial\{u > 0\} \cap D$ such that $|\nabla u(x_0)| = 0$ and $\partial\{u > 0\}$ is not δ-flat at x_0 at any level $r > 0$. Then, u has at most quadratic growth at x_0 and is bounded from above in dependence on δ.*

Theorem 4.5 will follow from Proposition 4.6 below in a standard way; see [8]. Let us define $r_k = 2^{-k}$ and $M(r_k, x_0) = \sup_{B_{r_k}(x_0)} |u|$, where $x_0 \in \partial\{u > 0\} \cap \{|\nabla u| = 0\}$.

Proposition 4.6. *Let u be as in Theorem 4.5 and* $\sup |u| \leq 1$. *If*

$$h(r_k, x_0, u) > \delta r_k$$

for some $\delta > 0$, then there exists $C = C(\delta, n, \lambda, \Lambda)$ such that

$$M(r_{k+1}, x) \leq \max\Big(C r_k^2, \frac{1}{2^2} M(r_k, x), \dots, \frac{M(r_{k-m}, x)}{2^{2(m+1)}}, \dots, \frac{M(r_0, x)}{2^{2(k+1)}}\Big). \tag{4.6}$$

Proof. If (4.6) fails, then there are solutions $\{u_j\}$ of (1.1) with $\sup|u_j| \leq 1$, sequences $\{k_j\}$ of integers, and free boundary points $\{x_j\}$, $x_j \in B_1$ such that

$$M(r_{k_j}+1, x_j) > \max\Big(j r_{k_j}^2, \frac{1}{2^2} M(r_{k_j}, x_j), \dots, \frac{M(r_{k_j-m}, x_j)}{2^{2(m+1)}}, \dots, \frac{M(r_0, x_j)}{2^{2(k_j+1)}}\Big), \quad (4.7)$$

where with some abuse of notation we set $M(r_{k_j}, x_j) = \sup_{B_{r_{k_j}}(x_j)} |u_j|$. Since $M(r_{k_j}, x_j) \leq \sup_{B_1} |u_j| < \infty$, it follows that $k_j \to \infty$. Define the scaled functions

$$v_j(x) = \frac{u_j(x_j + r_{k_j} x)}{M(r_{k_j}+1, x_j)}.$$

By construction, we have

$$\begin{aligned} v_j(0) &= 0, \quad |\nabla v_j(x)| = 0, \\ \sup_{B_{\frac{1}{2}}} |v_j| &= 1, \\ h(0, 1, v_j) &> \delta, \\ v_j(x) &\leq 2^{2m-1}, \quad |x| \leq 2^m, \quad m < 2^{k_j}, \end{aligned} \quad (4.8)$$

where the last inequality follows from (4.7) after rescaling the inequality

$$\frac{M(r_{k_j-m}, x_j)}{M(r_{k_j+1}, x_j)} < 2^{2(m-1)}.$$

Utilizing the homogeneity of operator F and noting that

$$D^2_{x_\alpha x_\beta} v_j(x) = r_j^2 (D^2_{\alpha\beta} u_j)(x_j + r_{k_j} x),$$

it follows that

$$F(D^2 v_j(x)) = -\frac{r_{k_j}^2}{M(r_{k_j+1}, x_j)} \chi_{\{v_j>0\}} = -\sigma_j \chi_{\{v_j>0\}}, \quad (4.9)$$

where $\sigma_j = \frac{r_{k_j}^2}{M(r_{k_j+1}, x_j)}$. Observe that $\sigma_j < \frac{1}{j}$ in view of (4.7). Since under hypotheses **F1°–F2°** we have local $W^{2,p}$ bounds for all $p \geq 1$ (see [3, Theorem 7.1]), it follows that we can employ a customary compactness argument for the viscosity solutions to show that there is a function $v_0 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned} v_{k_j} &\to v_0 \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n), \\ v_0(0) &= |\nabla v_0(0)| = 0, \\ F(D^2 v_0) &= 0. \end{aligned}$$

From Liouville's theorem, it follows that v_0 is homogeneous quadratic polynomial p of degree two. Since (4.8) holds, we have for this particular p that

$$h_{\min}(0, 1, p, v_j) \geq h(0, 1, v_j) > \delta.$$

Consequently, there are points $z_j = y_j + \delta \hat{e}_j$ such that z_j's are outside of the δ neighborhood of $\{p = 0\}$ and $v_j(z_j) = 0$. We can extract a subsequence from z_j so that it converges to some z_0, and z_0 is at least δ away from $\{p = 0\}$. Moreover, $v_0(z_0) = 0$ by uniform convergence. This is a contradiction and, therefore, the proof is complete. ∎

Remark 4.7. In [15] the authors proved some partial results for the problem

$$F(D^2 u) = \chi_{\mathcal{D}} \quad \text{in } B_1, \qquad u = |\nabla u| = 0 \quad \text{in } B_1 \setminus \mathcal{D}. \tag{4.10}$$

For $F = \Delta$, this problem arises in the linear potential theory related to harmonic continuation of the Newtonian potential of $B_1 \cap \mathcal{D}$.

Analysis similar to that of the proof of Proposition 4.6 shows that the result is also valid for the solutions of (4.10).

Corollary 4.8. *Let u be a viscosity solution to* (4.10). *Then, the statement of Theorem* 4.5 *holds for u too.*

Remark 4.9. If in Proposition 4.6 we let $\delta \downarrow 0$, say $\delta = 1/k, k \uparrow \infty$ then either (4.6) remains valid uniformly or $C \to \infty$. For the first scenario, in the limit we get a degree 2 homogeneous solution U solving $F(D^2 U) = -\chi_{\{U>0\}}$. Such a solution does not exist for $F = \Delta$. Also, by a simple computation, one can check that for more general operators such a solution does not exist. Therefore, from now on we will assume that as $\delta \downarrow 0$, the constant in (4.6) $C \to \infty$. We conclude that at an asymptotically flat point x_0, that is, for vanishing δ, one has

$$\frac{\sup_{B_r(x_0)} |u|}{r^2} \to \infty. \tag{4.11}$$

5. Uniqueness of blow-up

In this section we prove that for $n = 2$, the blow-up configuration at the flat point is unique. The proof is based on an argument from [7]. Let

$$Q_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

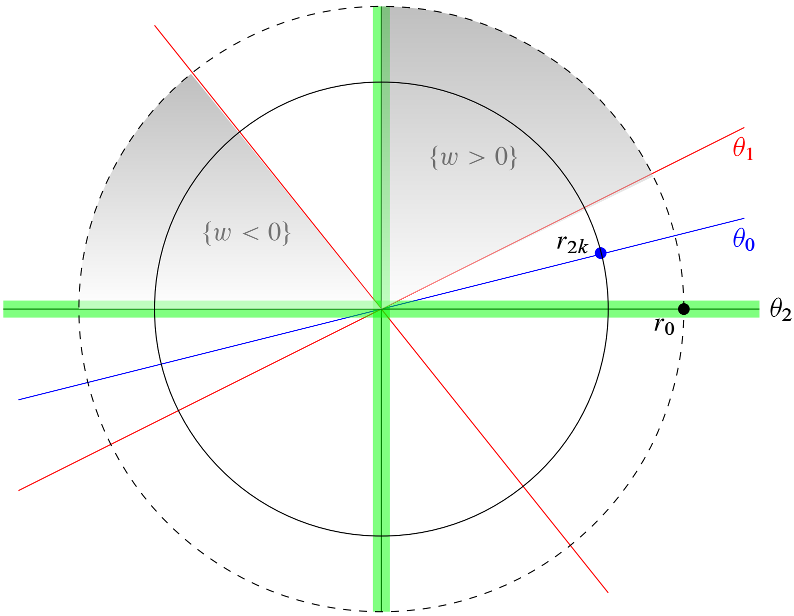

Figure 1. The uniqueness proof via a reflection principle.

be the counterclockwise rotation by θ. Suppose

$$U = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

solves $F(D^2U) = 0$ in $\mathbb{R}^2$. We consider two cases; first, if $a_{11} = a_{22} = 0$, then this means

$$F\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0.$$

The other case is when one of these coefficients is not zero, say a_{11}. Since F is homogeneous, without loss of generality we take $a_{11} > 0$. Then,

$$U = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2\right)^2 + \frac{a_{22}a_{11} - a_{12}^2}{a_{11}}x_2^2.$$

Observe that $a_{22}a_{11} - a_{12}^2 \neq 0$, since otherwise $U \geq 0$ is a solution to $F(D^2U) = 0$ with local minimum at the origin. Consequently, the matrix $a = [a_{ij}]$ is nonsingular, and the zero set of U is a pair of crossing lines.

The main result of this section is that for $F(M) = \sup_{t\in\mathcal{I}} A_{ij,t}M_{ij}$ and $n = 2$, for vanishing δ-flat points the approximating quadratic polynomial p is unique.

Theorem 5.1. *Suppose that* $n = 2$*,* $x_0 \in \partial_{\text{sing}}\{u > 0\}$*, and*

$$\lim_{r \downarrow 0} \frac{h(r, x_0, u)}{r} = 0.$$

Then, there is a unique $p \in P_2$ *and an* $r_0 > 0$ *such that*

$$h_{\min}(r, x_0, p, u) = o(r), \quad r < r_0.$$

Proof. For simplicity let us assume that $x_0 = 0$ and $F(M) = \sup_{t \in \mathcal{I}} A_{ij,t} M_{ij}$; see Section 2.

Suppose that the limiting configuration is not unique; in other words, there are two quadratic polynomials

$$p_i(x) = M_i^2 x_1^2 - x_2^2, \quad M_i > 0$$

such that for some $r_k \downarrow 0$, we have that $\partial\{u > 0\} \cap B_{r_{2m}}$ (up to a rotation) is close to $\partial\{p_2 > 0\}$ and $\partial\{u > 0\} \cap B_{r_{2m+1}}$ is close to $\partial\{p_1 > 0\}$. Let us define the rotated polynomials

$$\widetilde{p}_i(r, \theta) = p_i(Q_{\theta_i} x) = r^2(M_i^2 \cos^2(\theta - \theta_i) - \sin^2(\theta - \theta_i)),$$

and note that

$$\begin{aligned}
&\partial_\theta(\widetilde{p}_i(r, \theta) - \widetilde{p}_i(r, 2\theta_0 - \theta)) \\
&\quad = r^2(-2M_i^2 \cos(\theta - \theta_i) \sin(\theta - \theta_i) - 2 \sin(\theta - \theta_i) \cos(\theta - \theta_i)) \\
&\qquad - r^2(2M_i^2 \cos(2\theta_0 - \theta_i - \theta) \sin(2\theta_0 - \theta_i - \theta) \\
&\qquad + 2 \sin(2\theta_0 - \theta_i - \theta) \cos(2\theta_0 - \theta_i - \theta)).
\end{aligned}$$

At $\theta = \theta_0 := (\theta_1 + \theta_2)/2$, this gives

$$\begin{aligned}
\partial_\theta(\widetilde{p}_1(r, \theta) - \widetilde{p}_1(r, 2\theta_0 - \theta))\big|_{\theta=\theta_0} &= -r^2(4M_1^2 + 2) \sin(\theta_0 - \theta_1) \cos(\theta_0 - \theta_1) \\
&= -r^2(4M_1^2 + 2) \sin\Big(\frac{\theta_2 - \theta_1}{2}\Big) \cos\Big(\frac{\theta_2 - \theta_1}{2}\Big).
\end{aligned}$$

Similarly,

$$\partial_\theta(\widetilde{p}_2(r, \theta) - \widetilde{p}_2(r, 2\theta_0 - \theta))\big|_{\theta=\theta_0} = r^2\big(4M_2^2 + 2\big) \sin\Big(\frac{\theta_2 - \theta_1}{2}\Big) \cos\Big(\frac{\theta_2 - \theta_1}{2}\Big).$$

Introduce

$$w(r, \theta) = u(r, \theta) - u(r, 2\theta_0 - \theta);$$

then, since $u(x)/M(r_{2m})$ is close to $\widetilde{p}_2$ in $B_{r_{2m}}$ and $u(x)/M(r_{2m+1})$ is close to $\widetilde{p}_1$

in $B_{r_{2m+1}}$, it follows from the above computation that

$$\partial_\theta w(r_k, \theta_0)(-1)^k > 0, \quad k = 1, 2, \ldots. \tag{5.1}$$

This will be enough to get a contradiction, because after rescaling and using a customary compactness argument as in Proposition 4.6, we have

$$U_k = w(rr_k, \theta)/\mu(r_k) \to U^o, \quad \mu(r_k) = \frac{1}{r_k}\Big(\int_{B_{r_k}} w^2\Big)^{\frac{1}{2}}$$

with the properties that

$$U^o(r, \theta_0) = \partial_\theta U^o(r, \theta_0) = 0 \quad \text{and} \quad \|U^o\|_{L^2(B_1)} = 1, \tag{5.2}$$

provided that U^o solves an elliptic equation. To find this equation let us first note that $w(x) = u(x) - u(Qx)$ for some rotation Q; therefore,

$$D^2 w(x) = D^2 u(x) - Q^*(D^2 u)(Qx)Q.$$

By assumption,

$$F(D^2 u(x)) = \sup_{t \in \mathcal{I}} c_{ij,t} u_{ij}(x); \tag{5.3}$$

and moreover, from the homogeneity of F, we get that

$$F(M) = F_{ij}(M)M_{ij}, \quad F_{ij}(M) = \frac{\partial F(M)}{\partial M_{ij}}, \quad M \neq 0, \quad F_{ij}(M) \in \mathcal{S}^+(\lambda, \Lambda),$$

so that $F(D^2u(x)) \geq F_{ij}(D^2u(x))u_{ij}(x)$. Thus, taking $c_{ij}(x) = F_{ij}(D^2u(x))$ we have

$$c_{ij} w_{ij} = -\chi_{\{u>0\}} - c_{ij}(Q^*(D^2u)(Qx)Q)_{ij} \geq -\chi_{\{u>0\}} + \chi_{\{u(Qx)>0\}}. \tag{5.4}$$

By inspection, one can check that in the sector (θ_0, θ_1) both $u(x)$ and $u(Qx)$ are positive; see Figure 1. Using (5.1), near (r_{2m}, θ_0) it follows from (5.4) that $c_{ij} w_{ij} \geq 0$, and hence $\{w > 0\}$ has a nontrivial component on the line $\theta = \theta_0$ as part of its boundary; see Figure 1. Consequently, it follows from (5.4) that this component should propagate to the boundary of B_{r_0} for small r_0. A similar argument, with $\widetilde{c}_{ij} = F_{ij}(Q^* D^2(Qx)Q)$, shows that

$$\begin{aligned} \widetilde{c}_{ij} w_{ij} &= F_{ij}(Q^* D^2(Qx)Q)D^2u(x) - F(D^2u(Qx)) \\ &= F_{ij}(Q^* D^2(Qx)Q)D^2u(x) + \chi_{\{u(Qx)>0\}} \\ &\leq -\chi_{\{u>0\}} + \chi_{\{u(Qx)>0\}}; \end{aligned} \tag{5.5}$$

in other words, the component of $\{w < 0\}$ near (r_{2m+1}, θ_0) has a nontrivial component on the line $\theta = \theta_0$ as part of its boundary which propagates to the boundary of B_{r_0} for small r_0. This means that we cannot have infinitely many such components in view of the definition of p_1 and p_2.

Observe that

$$\frac{1}{\rho^4}\int_{B_\rho} w^2 = \frac{2}{\rho^4}\int_{B_\rho} u^2 - \frac{2}{\rho^4}\int_{B_\rho} u(x)u(Qx)dx \to \infty \tag{5.6}$$

as $\rho \to \infty$. Indeed, if it fails, then

$$\frac{\int_{B_\rho} u(x)u(Qx)dx}{\int_{B_\rho} u^2} \to 1, \tag{5.7}$$

but this is impossible since u changes sign and $\{u < 0\}$ is asymptotically a cone.

At a singular point, we have from the weak Harnack inequality

$$\infty \leftarrow \frac{M(\rho)}{\rho^2} \lesssim o(1) + \frac{1}{\rho^2}\Big(\int_{B_\rho} u^2\Big)^{\frac{1}{2}}.$$

This together with (5.4) and (5.5) implies that at the limit,

$$c^o_{ij}U^o_{ij} \geq 0 \quad \text{and} \quad \tilde{c}^o_{ij}U^o_{ij} \leq 0. \tag{5.8}$$

Combining this with (5.2) and applying Hopf's lemma, we get a contradiction. ∎

6. Quadruple junctions

Throughout this section we assume that F is convex and satisfies **F1°–F2°** and that u is a viscosity solution; see Section 3.

Lemma 6.1. *Assume* **F1°–F2°** *hold and F is convex. Let $n = 2$ and $|\nabla u(0)| = 0$ and let $0 \in \partial\{u > 0\}$ be a δ-flat point such that the zero set of the polynomial $p(x) = M^2x_1^2 - x_2^2$, $M > 0$ approximates $\partial\{u > 0\}$ near 0. Assume further that u is nondegenerate at 0. Then, for every $\delta_0 > 0$, there is $r_0 = 2^{-k_0}$ (for some $k_0 \in \mathbb{N}$) such that $\partial_2 u^-(x + te_2) \geq 0$ whenever $x \in (B_{r_0} \setminus B_{\delta_0 r_0}) \cap \{x_2 \geq M|x_1|\}$ and $\delta_0 \leq t \leq 2$.*

Proof. Let $\theta_0 = \arctan M$ and denote $K^- = \{x_2 \geq M|x_1|\}$. After rotation of the coordinate system, we can assume that K^- contains $u < 0$ away from some small neighborhood of $x_2 = M|x_1|$ (the green cones in Figure 2 represent that neighborhood).

Suppose the claim fails; then, there is $\delta_0 > 0$ so that for every $r_k = 2^{-k} \to 0$ and some points $x_k \in B_{r_k} \setminus B_{\delta_0 r_k} \cap \Omega^-(u)$, we have

$$\partial_2 u^-(x_k + r_k t_k e_2) < 0 \quad \text{for some } \delta_0 \leq t_k \leq 2. \tag{6.1}$$

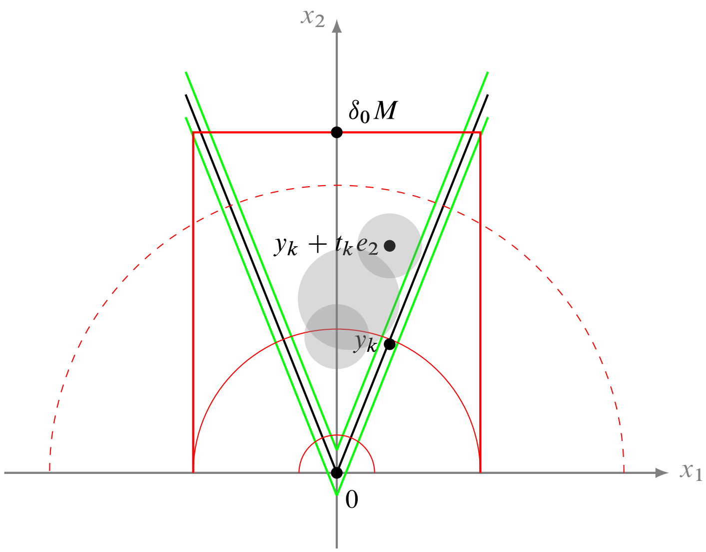

Figure 2. The geometric construction in the proof of Lemma 6.1. The shadowed balls are in the Harnack chain.

We can choose δ_0 so that for large k, we have $\delta_0 > \frac{h_k}{\cos\theta_0} \to 0$, where $h_k = h(2^{-k}, 0)$.

Introduce the scaled functions

$$v_k(x) = \begin{cases} \dfrac{u(r_k x)}{M(r_k)} & \text{if (4.6) is true for all } k \geq \widehat{k}, \text{ for some fixed } \widehat{k}, \\ \dfrac{u(r_k x)}{M(r_{k+1})} & \text{if there is a sequence } r_k = 2^{-k} \text{ such that (4.7) holds.} \end{cases} \tag{6.2}$$

Here we set $M(r_k) = M(r_k, 0)$. For both scalings, we have that v_k's are nondegenerate; for the first scaling it follows from Theorem 3.4 (our assumption on nondegeneracy), and for the second one it follows from the fact that $\sup_{B_{1/2}} |v_k| = 1$.

Moreover, by (6.1) there is $y_k \in (B_1 \setminus B_{\delta_0}) \cap \{v_k < 0\}$ such that

$$\partial_2 v_k^-(y_k + t_k e_2) < 0 \quad \text{for some } \ \delta_0 \leq t_k \leq 2. \tag{6.3}$$

Consequently, there is a subsequence $y_{k_j} + t_{k_j} e_2 \to y_0 + t_0 e_2 \in K^- \cap B_2$ and there is a Harnack chain $B^1, \ldots, B^N$ where $B^1 = B_{\cos\theta_0/2}(e_2)$ and $B^N = B_{\delta_0/2}(y_0)$, where N is independent of k_j. Let $\widetilde{K} = B_1 \cup \bigcup_{i=1}^N B^i$. Since under hypotheses **F1°–F2°** we have local $W^{2,p}$ bounds for all $p \geq 1$ (see [3, Theorem 7.1]), it follows that we can employ a customary compactness argument for viscosity solutions to infer that there is a function

$v_0 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ such that we have

$$\begin{aligned} v_k \to v_0 \quad & \text{in } W^{2,p}, \forall p \geq 1, \\ |F(D^2 v_k)| \leq C \quad & \text{uniformly}, \\ v_0 < 0 \quad & \text{in } K^-, \\ |v_k| \leq C \quad & \text{in Harnack chain domain } \widetilde{K}, \\ \partial_2 v_0^-(y_0 + t_0 e_2) \leq 0 \quad & \text{in view of (6.3)}. \end{aligned}$$

Applying Theorem 2.6 to $v_0^- \geq 0$, it follows that $\partial_2 v_0^-(y_0 + t_0 e_2) = 0$. Moreover, $w = \partial_2 v_0^-$ satisfies the equation $F_{ij} D_{ij} w = 0$ in $\widetilde{K}$; hence, from the strong maximum principle it follows that $w = 0$ in K^-. Consequently, v_0 depends only on x_1, implying that $\theta_0 = 0$ or $\theta_0 = \pi/2$, which is a contradiction. ∎

Theorem 6.2. *Let u be as in Lemma 6.1 and let 0 be a flat free boundary point. Then, in some neighborhood of 0 the free boundary consists of four C^1 curves tangential to the zero set of the polynomial $M^2 x_1^2 - x_2^2$.*

Proof. Let $\partial_{\mathrm{sing}}\{u > 0\} = \partial\{u > 0\} \cap \{|\nabla u| = 0\}$. Clearly, it is enough to prove that there is r such that $\partial_{\mathrm{sing}}\{u > 0\} \cap B_r = \{0\}$. Suppose the claim fails. Then, there is a sequence $x_k \in \partial_{\mathrm{sing}}\{u > 0\}$, $x_k \to 0$. Let $M_k^- := M^-(2r_k\ell_0) = \sup_{B_{2r_k\ell_0}} u^-$, $r_k = |x_k|$ and consider

$$v_k(x) = \frac{u(r_k x)}{M^-(2r_k\ell_0)} \quad \text{where } \ell_0 = \sqrt{\frac{\Lambda}{\lambda}}. \tag{6.4}$$

Note that $F(D^2 v_k) = -\chi_{\{v_k>0\}} \frac{r_k^2}{M^-(2r_k\ell_0)}$, and therefore by nondegeneracy we have $|F(D^2 v_k)| \leq C$ for some $C > 0$ independent of k.

By construction, $\sup_{B_{2\ell_0}} |v_k^-| = 1$ and since $F(D^2 v_k) = 0$ in $\Omega^-(v_k) := \{v_k < 0\}$, it follows that there is $z_k \in \partial B_{2\ell_0} \cap \Omega^-(v_k)$ such that $v_k^-(z_k) = 1$. Consequently, $\mathrm{dist}(z_k + \delta_0 e_2, \{p = 0\}) \geq \delta_0/2$ and by Lemma 6.1,

$$v_k^-(z_k + \delta_0 e_2) \geq 1.$$

Claim 6.3. *With the notation above, we have*

$$M_k^+ \leq C M_k^-,$$

for some universal constant $C > 0$.

To check this, we first observe that $\mathrm{trace}(A_t D^2 v_k(x)) \leq F(D^2 v_k x)$ thanks to the convexity of F and $A_t \in \mathcal{S}_{\lambda,\Lambda}$. Now consider that if $w_{k,t}(x) = v_k(A_t^{\frac{1}{2}} x)$, then $\Delta w_{k,t}(x) =$

$\operatorname{trace}(A_t D^2 v_k(x)) \le F(D^2 v_k(x)) \le 0$. Since $w_{k,t}$ is continuous and $w_{k,t}(0) = 0$, then one can easily check that

$$\fint_{B_r} w_{k,t} = \int_0^r \frac{1}{t} \int_{B_t} \Delta w_{k,t} \le 0 \tag{6.5}$$

because of convexity of F and the estimate $F(D^2 v_k) \le 0$.

Note that

$$\int_{B_r} w_{k,t}(x)dx = \frac{1}{\sqrt{\det A_t}} \int_{|A^{-\frac{1}{2}} y|<r} v_k(y)dy \le 0.$$

Thus, from (6.5) it follows that

$$\begin{aligned}\frac{1}{\Lambda} \int_{B_{\frac{r}{\sqrt{\Lambda}}}} v_k^+ &\le \frac{1}{\sqrt{\det A_t}} \int_{|A^{-\frac{1}{2}} y|<r} v_k^+(y)dy \le \frac{1}{\sqrt{\det A_t}} \int_{|A^{-\frac{1}{2}} y|<r} v_k^-(y)dy \\ &\le \frac{1}{\lambda} \int_{B_{\frac{r}{\sqrt{\lambda}}}} v_k^-(y)dy, \quad r < 2.\end{aligned}$$

Consequently, we get

$$\int_{B_r} v_k^+(y)dy \le \ell_0^2 \int_{B_{r\ell_0}} v_k^-.$$

Let $\widehat{v}_k = v_k + \widehat{C}|x|^2$. Then,

$$F(D^2 \widehat{v}_k) \ge F(D^2 v_k) + 2\widehat{C}\lambda \ge 0,$$

provided that $\widehat{C}$ is sufficiently large.

We see that $\widehat{v}_k$, and hence $\widehat{v}_k^+$, is a subsolution. Consequently, applying the weak Harnack inequality [3], we get

$$\sup_{B_{\frac{4}{3}}} v_k^+ \le \sup_{B_{\frac{4}{3}}} \widehat{v}_k^+ \le c_0 \int_{B_2} (v_k^- + \widehat{C}|x|^2) \le c_0(1 + 2\pi \widehat{C}).$$

This completes the proof of the claim.

Thus, as in the proof of Lemma 6.1, we can employ a customary compactness argument in $W^{2,p}$ so that $y_k = x_k/r_k \to y_0 \in \{x_2 = M|x_1|\} \cap \partial B_1$ and

$$\nabla v_0(y_0) = 0, \quad v_0(z_0 + \delta_0 e_2) \ge 1,$$

by Harnack chain and $C^{1,\alpha}$ estimates in the Harnack chain domain (which joins $2\ell_0 e_2$ with $z_0 + \delta_0 e_2$). Since $y_0 \in \{x_2 = M|x_1|\}$, $y_0 \ne 0$, the free boundary at y_0 is a line. Therefore, we can apply Hopf's lemma to conclude that $v_0^- \equiv 0$, which is a contradiction.

It remains to show that the curves are C^1 up to the origin. Suppose this is not the case; then, there is a sequence $x_k \to 0$ of regular free boundary points such that the unit normal ν_k at x_k does not converge to the corresponding unit normal e of the component of $\{Mx_1^2 - x_2^2 = 0\}$. Using the same compactness argument for v_k as before, we can see that $|e - \nu_k| \geq \sigma$ for some fixed $\sigma > 0$ and large k, where ν_k is now the normal of some free boundary point of v_k with distance 1 from 0. But this is a contradiction, since v_k converge locally uniformly to some v_0 and its free boundary is exactly the zero set of the polynomial $Mx_1^2 - x_2^2$. ■

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