Nodal quintic surfaces and lines on cubic fourfolds

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(with an appendix by John Christian OTTEM)

Abstract. We study nodal quintic surfaces with an even set of 16 nodes as analogues of singular Kummer surfaces. The interpretation of the natural double cover of an even 16-nodal quintic as a certain Fano variety of lines could be viewed as a replacement for the additive structure of the cover of a singular Kummer surface by its associated abelian surface.

Most of the results in this article can be seen as refinements of known facts and our arguments rely heavily on techniques developed by Beauville (1979), Murre (1972), and Voisin (1986), Results due to Shen (2012, 2014) are particularly close to some of the statements. In this sense, the text is mostly expository (but with complete proofs), although our arguments often differ substantially from the original sources.

1. Introduction

A nodal quartic surface $S \subset \mathbb{P}^3$ can have at most 16 nodes. When the maximum is attained, the minimal resolution $\tau \colon \widetilde{S} \to S$ is a K3 surface and, more precisely, a Kummer surface. In other words, S is then a singular Kummer surface, so it is isomorphic to the quotient A/ι of an abelian surface A by its natural involution ι. Ultimately, the link to abelian surfaces relies on the fact that the set $\{E_i\}$ of the 16 exceptional lines of the resolution τ is even, i.e., $\sum E_i = 0$ in $H^2(\widetilde{S}, \mathbb{Z}/2\mathbb{Z})$. Then the double cover $\widetilde{A} \to \widetilde{S}$ ramified along $\bigcup E_i \subset \widetilde{S}$ is the blow-up of A in its two-torsion points. The situation was first studied by Kummer and marks the beginning of the theory of K3 surfaces. The situation for nodal quadrics and nodal cubics is also well understood. For example, a nodal cubic can have at most four nodes and in this case it is the Cayley cubic.

The maximal number of nodes of a nodal quintic surface $D \subset \mathbb{P}^3$ was determined by Beauville [6]. It is 31, which is realized by the Togliatti surface. However, in this

case the set $\{E_i\}$ is not even. As Beauville showed [6, Section 2], only a set of 16 or of 20 nodes can lead to an even set of exceptional lines $\{E_i\}$. The two situations have been studied by Beauville [5] and Catanese [11]. In this article, we are concerned with even 16-nodal quintic surfaces, i.e., nodal quintic surfaces with exactly 16 nodes such that the integral cohomology class $\sum E_i$ on the minimal resolution is divisible by two. In this case, $D \simeq F/\iota$, where F is a smooth surface of general type endowed with an involution ι with 16 fixed points. Its blow-up in the fixed points can be viewed as the double cover $\widetilde{F} \to \widetilde{D}$ of the minimal resolution $\widetilde{D} \to D$ ramified along the union $\bigcup E_i \subset \widetilde{D}$ of all exceptional lines. In this sense, D is the analogue of a singular Kummer surface and F is the quintic analogue of the abelian surface A. It is now tempting to try to develop a theory that parallels the classical theory of abelian surfaces and their associated (singular) Kummer surfaces. But how far can one push this analogy? The following picture already includes the objects that we will study:

$$
\begin{array}{ccc}
\widetilde{A} & \longrightarrow & \widetilde{S} \\
\downarrow & & \downarrow \\
\iota \circlearrowleft A & \longrightarrow & S \subset \mathbb{P}^3
\end{array}
\qquad
\begin{aligned}
H^*(A, \mathbb{Z})^- &= H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z}), \\
\mathrm{CH}(A)^- &= A \times \widehat{A}, \\
\mathfrak{h}(A)^- &= \mathfrak{h}^1(A) \oplus \mathfrak{h}^3(A),
\end{aligned}
$$

$$
\begin{array}{ccc}
\widetilde{F} & \longrightarrow & \widetilde{D} \\
\downarrow & & \downarrow \\
\iota \circlearrowleft F & \longrightarrow & D \subset \mathbb{P}^3
\end{array}
\qquad
\begin{aligned}
H^*(F, \mathbb{Z})^- &= H^2(F, \mathbb{Z})^-, \\
\mathrm{CH}(F)^- &= \mathrm{CH}_0(F)^- \oplus \mathrm{CH}_1(F)^-, \\
\mathfrak{h}(F)^- &= \mathfrak{h}^2(F)^-.
\end{aligned}
$$

The first step is to find the correct analogue of the additional geometric structure of the double cover $A \to S$ provided by the interpretation of A as an abelian surface. Already the work of Fano [14] and Togliatti [37] suggests where to look. For any smooth cubic fourfold $X \subset \mathbb{P}^5$ the variety of lines $L' \subset X$ intersecting a fixed generic line $L \subset X$ is a smooth surface F_L endowed with a natural involution ι mapping L' to the residual line of $L \cup L' \subset X \cap \overline{LL'}$. By mapping $L' \in F_L$ to the point of intersection of the plane $\mathbb{P}^2 \simeq \overline{LL'}$ with a generic $\mathbb{P}^3 \subset \mathbb{P}^5$, the quotient $D_L := F_L/\iota$ is realized as an even 16-nodal quintic surface with the nodes corresponding to the fixed points of ι. From this perspective, the surfaces F and D have been studied by Voisin [41]. Understanding their geometry is a crucial step in her proof of the global Torelli theorem for cubic fourfolds. Viewing the double cover $F \to D$ of an even 16-nodal quintic as $F \simeq F_L$ can be seen as the analogue of viewing the double cover $A \to S$ as an abelian surface.

There are clear limitations to the analogy between the double covers $A \to S$ of a singular Kummer quartic surface and the natural double covers $F \to D$ of an even 16-nodal quintic surface. For one, F is a surface of general type with ample canonical bundle ω_F while the abelian surface A has of course trivial canonical bundle. Also, from a Hodge theoretic perspective, A and F are quite different. For example, the Hodge structure $H^1(F, \mathbb{Z})$ of weight one is trivial, although the minimal resolutions $\widetilde{S}$ and $\widetilde{D}$ are both simply connected. Although this phenomenon is well known, a clear topological reason for the different behaviour of the corresponding double covers $\widetilde{A} \to \widetilde{S}$ and $\widetilde{F} \to \widetilde{D}$ seems missing.

This article naturally splits in two interwoven parts: The geometry of even 16-nodal quintic surfaces $D \subset \mathbb{P}^3$ together with their double covers $F \to D$ and the geometry of Fano varieties of lines on cubic fourfolds.

1.1. In the first section of this article we collect the known facts about even 16-nodal quintic surfaces $D \subset \mathbb{P}^3$ and their natural double cover $F \to D$. We summarize the information about F in the following theorem. Note that (i) is due to Voisin [41] and (iii) strengthens a result of Beauville [6]. I presume that the other assertions are also more or less well known to the experts.

Theorem 1.1. *Let $D \subset \mathbb{P}^3$ be an even 16-nodal quintic surface written as a quotient $D \simeq F/\iota$.*

(i) *The numerical invariants of F are as follows:*

$$\chi(F, \mathcal{O}_F) = 6, \qquad q(F) = 0, \qquad p_g(F) = 5,$$
$$e(F) = 62, \quad b_2(F) = 60, \quad c_1^2(F) = 10.$$

(ii) *The surface F is algebraically simply connected.[1]*

(iii) *The integral cohomology $H^2(F, \mathbb{Z})$ is torsion free.*

(iv) *The anti-invariant part $H^2(F, \mathbb{Z})^-$ is a K3 Hodge structure, i.e., a Hodge structure of weight two with a 1-dimensional $(2, 0)$-part.*

(v) *The intersection form on $H^2(F, \mathbb{Z})^-$ is even and definite of signature $(2, 21)$.*

For the full cohomology one finds

$$H^*(F, \mathbb{Z})^- = H^2(F, \mathbb{Z})^-,$$

while for a singular Kummer surfaces and the covering abelian surface, we have

$$H^*(A, \mathbb{Z})^- = H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z}).$$

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[1] In the appendix by J. Ottem, the Fano perspective will be used to show that F is also topologically simply connected.

1.2. Even 16-nodal quintics naturally occur in the study of smooth cubic fourfolds $X \subset \mathbb{P}^5$. Recall that the Fano variety $F(X)$ of lines contained in X is a hyperkähler fourfold [7]. For a generic line $L \subset X$ the variety $F_L \subset F(X)$ of all lines intersecting L is a smooth surface with a natural involution ι. Its quotient $D_L := F_L/\iota$ is an even 16-nodal quintic.

Due to a classical result of Beauville and Donagi [7], there exists a Hodge isometry

$$\left(H^4(X, \mathbb{Z})_{\mathrm{pr}}(1), -(\,.\,)\right) \simeq \left(H^2(F(X), \mathbb{Z})_{\mathrm{pr}}, q\right),$$

where q is the Beauville–Bogomolov–Fujiki pairing restricted to the Plücker primitive cohomology. The next theorem complements this result by another Hodge isometry which refines results due to Izadi [23] and Shen [31, 33]; see Section 4.3 for a detailed comparison.

Theorem 1.2. *Let $L \in F(X)$ be a generic line contained in a smooth cubic fourfold $X \subset \mathbb{P}^5$ and let $F_L \subset F(X)$ be the smooth surface of all lines intersecting L. Then there exist natural isometries of Hodge structures*

$$\left(H^4(X, \mathbb{Z})_{\mathrm{pr}}(1), -(\,.\,)\right) \simeq \left(H^2(F(X), \mathbb{Z})_{\mathrm{pr}}, q\right) \simeq \left(H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-, (1/2)(\,.\,)\right).$$

On the right hand side, $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}$ denotes the primitive part with respect to the restriction of the Plücker polarization and $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^- \subset H^2(F_L, \mathbb{Z})_{\mathrm{pr}}$ is its ι-anti-invariant part. The standard intersection form $(\,.\,)$ on the surface F_L is scaled by the factor $(1/2)$.

The following geometric global Torelli-type result is an analogue of a well-known result for cubic threefolds; for a more complete version, see Section 4.2.

Corollary 1.3. *Assume $X, X' \subset \mathbb{P}^5$ are two smooth cubic fourfolds and let $L \subset X$ and $L' \subset X'$ be two generic lines. Then $X \simeq X'$ if and only if there exists an isometry of Hodge structures*

$$\left(H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-, (\,.\,)\right) \simeq \left(H^2(F_{L'}, \mathbb{Z})_{\mathrm{pr}}^-, (\,.\,)\right).$$

The assumptions on L and L' simply mean that the two Fano varieties $F_L \subset F(X)$ and $F_{L'} \subset F(X')$ of lines intersecting $L \subset X$ and $L' \subset X'$ are smooth surfaces.

A more careful analysis of the situation allows one to upgrade the Hodge isometry in Theorem 1.2 to the level of Chow groups and rational Chow motives. The first assertion of the next theorem was first proved by Shen; see [33, Theorem 1.5] and [31, Theorem 4.7]. The second part, a direct consequence of the first, formulated in terms of transcendental motives, is due to Bolognesi and Pedrini [9, Proposition 2.7]. See Section 5.5 for a comparison of the techniques.

Theorem 1.4. *The Fano correspondence induces an isomorphism*

$$\mathrm{CH}_0(F_L)^-_{\mathrm{hom}} \simeq \mathrm{CH}_1(X)_{\mathrm{hom}}$$

of the homologically trivial integral Chow groups and an isomorphism of rational Chow motives

$$\mathfrak{h}^2(F_L)^-_{\mathrm{pr}} \simeq \mathfrak{h}^4(X)_{\mathrm{pr}}(1).$$

To compare the result with the case of Kummer surfaces, observe that $\mathfrak{h}^2(F_L)^-$ is the quintic analogue of the motive $\mathfrak{h}^1(A) \oplus \mathfrak{h}^3(A)$ of an abelian surface.

1.3. Here is an outline of the content. Section 2 deals with the topological and Hodge theoretic invariants of an even 16-nodal quintic and its natural double cover. It contains the proofs of the various parts of Theorem 1.1. With one minor exception in the proof of Lemma 2.6 and for convenience reasons only, we do not resort to the interpretation in terms of Fano varieties of lines on cubic fourfolds.

Section 3 discusses the natural occurrence of double covers of even 16-nodal quintics as Fano varieties of lines contained in a cubic fourfold intersecting a fixed line. This interpretation, allows one to consider the double cover endowed with the restriction of the Plücker polarization which is not canonical from the point of view of the quintic. This leads to the K3 Hodge structure of rank 22 that appears in Theorem 1.2, the proof of which is postponed to the short Section 4.

Chow groups with integral coefficients and rational Chow motives are discussed in Section 5. The proofs of the two parts of Theorem 1.4 can be found in Sections 5.4 and 5.7. The appendix contains the proof of the simple connectedness of F_L.

1.4. As should be clear from the introduction, the paper builds heavily on earlier work of Beauville, Murre, Voisin, and others. Many of the results can be found in a similar form in the literature, especially in the work of Shen [31,33], cf. Section 4.3. It seemed worthwhile to add further observations and refinements to their results and to present a coherent and streamlined picture of what is currently known. The paper was originally intended to become part of [22], but it grew out of proportion and it seemed more appropriate to publish it separately.

2. The topology of an even 16-nodal quintic and its double cover

In the following $D \subset \mathbb{P}^3$ will always denote a quintic surface with 16 nodes $x_1, \ldots, x_{16} \in D$ as its only singularities. By the adjunction formula, its canonical bundle is $\omega_D \simeq \mathcal{O}_D(1)$.

2.1. The minimal resolution $\tau\colon \widetilde{D} \to D$ with canonical bundle $\omega_{\widetilde{D}} \simeq \tau^*\mathcal{O}_D(1)$ replaces the nodes x_i by the (-2)-curves $E_i \subset \widetilde{D}$. The surface D is called an *even 16-nodal quintic* if in addition $\sum E_i = 0$ in $H^2(\widetilde{D}, \mathbb{Z}/2\mathbb{Z})$ or, equivalently, if the line bundle $\mathcal{O}(\sum E_i)$ admits a square root. Note that the line bundle $\mathcal{L}$ with $\mathcal{L}^2 \simeq \mathcal{O}(\sum E_i)$ is unique, as $\mathrm{Pic}(\widetilde{D})$ is torsion free due to $\pi_1(\widetilde{D}) = \{1\}$; see below. Unlike the case of 16-nodal quartics, cf. [20, Remark 14.3.19], it seems unclear whether a 16-nodal quintic is automatically even.[2]

Similar to the well-known smoothing of ordinary double points on a singular quartic Kummer surface, the surface $\widetilde{D}$ is diffeomorphic to a smooth quintic surface, cf. [6, Section 2] or for more general results [24]. (Ideally, one would like $\widetilde{D}$ to be deformation equivalent to a smooth quintic, which would need the nodes to be independent [10].) In particular, by the Lefschetz theorem, $\widetilde{D}$ is simply connected with

$$b_2(\widetilde{D}) = 53 \quad \text{and} \quad e(\widetilde{D}) = 55.$$

Hence, for the 16-nodal quintic D one has

$$b_2(D) = 37 \quad \text{and} \quad e(D) = 39.$$

2.2. Assume now that D is an even 16-nodal quintic and let $\widetilde{\pi}\colon \widetilde{F} \to \widetilde{D}$ be the double cover of $\widetilde{D}$ ramified along $\bigcup E_i \subset \widetilde{D}$, i.e., $\widetilde{\pi}$ is the cyclic double cover associated with the line bundle $\mathcal{L}$ and the section of $\mathcal{L}^2$ defining $\bigcup E_i$. We will use the same notation for the reduced pre-images of the (-2)-curves and write $E_i \subset \widetilde{F}$. These (-1)-curves can also be viewed as the fixed components of the natural involution $\widetilde{\iota}$ of $\widetilde{F}$. Then

$$\omega_{\widetilde{F}} \simeq \mathcal{O}\left(\sum E_i\right) \otimes \widetilde{\pi}^*\omega_{\widetilde{D}} \simeq \mathcal{O}\left(\sum E_i\right) \otimes \widetilde{\pi}^*\tau^*\mathcal{O}_D(1).$$

The blow-down $\widetilde{\tau}\colon \widetilde{F} \to F$ of the (-1)-curves $E_i \subset \widetilde{F}$ fits into the commutative diagram:

$$
\begin{array}{ccc}
\widetilde{F} & \xrightarrow{\;\widetilde{\pi}\;} & \widetilde{D} \\
{\scriptstyle\widetilde{\tau}}\downarrow & & \downarrow{\scriptstyle\tau} \\
F & \xrightarrow{\;\pi\;} & D.
\end{array}
$$

The smooth surface F with its canonical bundle $\omega_F \simeq \pi^*\omega_D \simeq \pi^*\mathcal{O}_D(1)$ comes with a covering involution ι with 16 fixed points over the nodes x_i and its quotient is $F/\iota \simeq D$. Throughout, we will write x_i for the fixed points in F and for their images in D.

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[2]F. Catanese informed the author that there are 16-nodal quintics which are not even and that this follows from a forthcoming paper of his.

The numerical invariants of the surface F are as follows:

$$\chi(F, \mathcal{O}_F) = 6, \qquad q(F) = 0, \qquad p_g(F) = 5,$$
$$e(F) = 62, \quad b_2(F) = 60, \quad c_1^2(F) = 10.$$

These facts are either classical [14, 37] or have been proved by Voisin [41]; see also [22, Section 6.4.5] for a variant of the proof. The most surprising fact is probably the regularity of the surface F, which can equivalently be phrased as $H^1(F, \mathbb{Z}) = 0$. This is in stark contrast to $H^1(A, \mathbb{Z}) \neq 0$ for the natural double cover of a 16-nodal quartic surface. To prove this, Voisin [41, Section 3, Lemma 3] uses the Fano description of F, cf. Remark 2.4 below or [22, Section 6.4.5] for a more classical argument.

The fact that F is regular in particular shows

$$\operatorname{Pic}(F) \subset H^2(F, \mathbb{Z}) \quad \text{and} \quad \operatorname{Tors}\operatorname{Pic}(F) = \operatorname{Tors} H^2(F, \mathbb{Z}).$$

2.3. The action of the involution ι on F leads to an eigenspace decomposition

$$H^0(F, \omega_F) = H^0(F, \omega_F)^+ \oplus H^0(F, \omega_F)^-$$

into a 4-dimensional invariant part $H^0(D, \omega_D) \simeq H^0(F, \omega_F)^+$ and a 1-dimensional anti-invariant part $H^0(F, \omega_F)^-$

Proof. Indeed, as $\omega_D \simeq \mathcal{O}_D(1)$ for the quintic D, we have $h^0(D, \omega_D) = 4$. Since $F \twoheadrightarrow D$ is an étale double cover over the complement of the nodes x_i, we know

$$\pi^* \colon H^0(D, \omega_D) \xrightarrow{\sim} H^0(F, \omega_F)^+.$$

Eventually, use $h^0(F, \omega_F) = p_g(F) = 5$. ∎

Clearly, the composition $\pi \colon F \to D \subset \mathbb{P}^3$ is the morphism associated with the invariant linear system $|\omega_F|^+ \subset |\omega_F|$.

2.4. The invariant and anti-invariant parts $H^2(F, \mathbb{Z})^\pm \subset H^2(F, \mathbb{Z})$ of the induced action of the involution ι satisfy

$$\operatorname{rk} H^2(F, \mathbb{Z})^+ = 37 \quad \text{and} \quad \operatorname{rk} H^2(F, \mathbb{Z})^- = 23.$$

In the following, $H^2(F, \mathbb{Z})^-$ is considered as a Hodge structure of K3 type (as we will see, torsion free). Furthermore, the intersection pairing on $H^2(F, \mathbb{Z})^-$ has signature $(2, 21)$.

Proof. In order to compute the ranks, we can work with rational cohomology. Then

$$H^2(D, \mathbb{Q}) \simeq H^2(F, \mathbb{Q})^+$$

by a standard spectral sequence argument [18, Proposition 5.2.3]. Combining $e(D) = 39$ with $H^1(D, \mathbb{Q}) \simeq H^1(F, \mathbb{Q}) = 0$ to deduce

$$\mathrm{rk}\, H^2(F, \mathbb{Z})^+ = b_2(D) = 37,$$

we can conclude by

$$\mathrm{rk}\, H^2(F, \mathbb{Z})^- = b_2(F) - 37 = 23.$$

Alternatively,[3] one can combine the Lefschetz fixed point formula $16 = 2 + b_2^+ - b_2^-$ with $62 = e(F) = 2 + b_2 = 2 + b_2^+ + b_2^-$ to compute the dimensions of $H^2(F, \mathbb{Q})^\pm$.

The last assertion follows from the orthogonal decomposition

$$H^2(F, \mathbb{Q}) = H^2(F, \mathbb{Q})^+ \oplus H^2(F, \mathbb{Q})^-$$

and the Hodge index theorem on $H^2(D, \mathbb{Q}) \simeq H^2(F, \mathbb{Q})^+$. ∎

Note that the result is in contrast to $H^2(A, \mathbb{Z}) = H^2(A, \mathbb{Z})^+$ and $H^2(A, \mathbb{Z})^- = 0$ for the natural involution of an abelian surface. However,

$$H^*(A, \mathbb{Z})^- = H^1(A, \mathbb{Z}) \oplus H^3(A, \mathbb{Z}).$$

2.5. It is known that not only $\widetilde{D}$ but also the nodal quintic D is simply connected, cf. [13, Section 5.2]. In particular, $H_1(D, \mathbb{Z}) = 0$ and by the universal coefficient theorem $H^2(D, \mathbb{Z})$ is torsion free. The same holds for F, see also Remark 2.4.

Lemma 2.1. *The integral cohomology $H^2(F, \mathbb{Z})$ is torsion free.*

Proof. We first show that $H^2(F, \mathbb{Z})$ has no two-torsion, i.e., $H^2(F, \mathbb{Z})[2] = 0$ or, equivalently, $\mathrm{Pic}(F)[2] = 0$. This is an immediate consequence of results of Beauville. According to [6, Lemma 2], one has

$$\mathrm{Pic}(F)[2] \simeq \ker\big((\mathbb{Z}/2\mathbb{Z})^{\oplus 16} \to \mathrm{Pic}(\widetilde{D}) \otimes \mathbb{Z}/2\mathbb{Z}\big)/\, \mathbb{Z}e.$$

Here, $e = (1, \ldots, 1)$ and the map is given by the classes $\mathcal{O}(E_i) \in \mathrm{Pic}(\widetilde{D})$. Furthermore, by virtue of [6, Proposition §2], if a collection $E_i, i \in I$, is even, i.e., $\mathcal{O}(\sum_{i \in I} E_i)$ has a square root, then $|I| = 16$ or 20. Hence, $\mathrm{Pic}(F)[2] = 0$.

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[3]Thanks to A. Kuznetsov for suggesting this.

To conclude, one can evoke a result by Ciliberto and Mendes Lopes [12, Theorem A]: The torsion of $\mathrm{Pic}(X)$ of a regular, minimal surface of general type with $c_1^2(X) = 2\chi(\mathcal{O}_X) - 2$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$. The assumptions are satisfied for the surface F. For the reader's convenience and later use, we recall the key argument. First, one shows $|\pi_1^{\mathrm{alg}}| \leq 3$ by the following argument originally due to Bombieri: Assume $F_0 \to F$ is an irreducible étale cover of degree d, then

$$c_1^2(F_0) = d \cdot c_1^2(F) = d \cdot (2\chi(\mathcal{O}_F) - 2)$$
$$= d \cdot ((2/d) \cdot \chi(\mathcal{O}_{F_0}) - 2) = 2\chi(\mathcal{O}_{F_0}) - 2d.$$

Noether's inequality

$$2p_g(F_0) \leq c_1^2(F_0) + 4,$$

together with the obvious $2\chi(\mathcal{O}_{F_0}) - 2 \leq 2p_g(F_0)$, then gives

$$c_1^2(F_0) + 2(d - 1) \leq c_1^2(F_0) + 4,$$

which proves $d \leq 3$.

The case $d = 3$ is excluded by [12, (1.2)]. Alternatively, one can observe that for $d = 3$ the two inequalities above are equalities. Hence, the surface F_0 is on the Noether line with $q(F_0) = 0$, $p_g = 17$, and $c_1^2(F_0) = 30$. According to results of Horikawa, see [3, Section VII,9], the minimal resolution of the canonical model of F_0 is then a Hirzebruch surface. As the canonical model of F_0 dominates the canonical model of F which in turn maps onto D, this results in a contradiction. ∎

Lemma 2.2. *The surface F is algebraically simply connected, i.e., $\pi_1^{\mathrm{alg}}(F) = \{1\}$.*

Proof. According to [44, Corollary 4.4], every minimal surface of general type satisfying $c_1^2 < (8/3)(\chi(\mathcal{O}) - 2)$ is algebraically simply connected. As the inequality holds for the surface F, one indeed has $\pi_1^{\mathrm{alg}}(F) = \{1\}$.

Alternatively, one may use the arguments in the proof of Lemma 2.1. There we saw that irreducible étale covers $F_0 \to F$ are of degree at most two. However, an étale double cover corresponds to a line bundle of order two and thus defines a non-trivial two-torsion class in $H^2(F, \mathbb{Z})$, the existence of which is excluded by the previous lemma or rather by its first step showing $\mathrm{Pic}(F)[2] = 0$. ∎

The following immediate consequence is originally due to Shen [31, Lemma 4.5 and Section 5]. The original proof is rather involved and makes heavy use of the Fano interpretation of F.

Corollary 2.3. *The surface F satisfies $H_1(F, \mathbb{Z}) = 0$.* ∎

Remark 2.4. (i) Lemmas 2.1 and 2.2 are essentially equivalent. Clearly, Lemma 2.2 and its consequence Corollary 2.3 combined with the universal coefficient theorem imply that $H^2(F, \mathbb{Z})$ is torsion free. Conversely, the torsion freeness was used to show that F is algebraically simply connected. Also note that $H_1(F, \mathbb{Z}) = 0$ implies the regularity of F, which gives an alternative argument to Voisin's original proof [41].

(ii) From the above, it is not clear whether also the topological fundamental group $\pi_1(F)$ is trivial.[4] Since

$$F' := F \setminus \{x_i\} \to D' := D \setminus \{x_i\}$$

is an étale cover, $\pi_1(F) \simeq \pi_1(F')$ is the kernel of the surjection $\pi_1(D') \to \mathbb{Z}/2\mathbb{Z}$ mapping a simple loop γ_i in D around $x_i \in D$ (or, equivalently, in $\widetilde{D}$ around $E_i \subset \widetilde{D}$) to the generator. Here, the fundamental group $\pi_1(D')$ is viewed as a quotient of $(\mathbb{Z}/2\mathbb{Z})^{*16}$. Since by the above the pro-finite completion $\pi_1^{\mathrm{alg}}(F)$ of $\pi_1(F)$ is trivial, F is simply connected if and only if its fundamental group $\pi_1(F)$ is residually finite.

Remark 2.5. A nodal quintic $D \subset \mathbb{P}^3$ with an even set of 20 nodes gives rise to a double cover $F \to D$ satisfying $q(F) = 0$, $p_g(F) = 4$, and $c_1^2(F) = 10$. In particular, the composition $F \to \mathbb{P}^3$ is the canonical map. Examples were first constructed by Beauville [5, Proposition 3.6] and a construction due to Gallarati was studied in detail by Catanese [11]. A special example was described by van der Geer and Zagier [39, Section 4] as the minimal model of a Hilbert modular surface associated with $\mathbb{Q}(\sqrt{21})$.

It turns out that quintics with an even set of 20 nodes form an irreducible family [5, Remark 3.7 (2)]. However, without the condition on the 20 nodes to be even the moduli space is reducible [11, Proposition 3.8]. The Hilbert modular surface is simply connected [38, Theorem 6.1], hence for any quintic with an even set of 20 nodes the double cover F is simply connected.

2.6. The Picard group $\mathrm{Pic}(D')$ of the open subset $D' = \widetilde{D} \setminus \bigcup E_i = D \setminus \{\overline{x}_i\}$ is generated by the image of the restriction map $\mathrm{Pic}(D) \to \mathrm{Pic}(D')$ and the two-torsion line bundle $\mathcal{L}|_{D'}$ corresponding to $F' \twoheadrightarrow D'$.

Proof. Indeed, the restriction map $\mathrm{Pic}(\widetilde{D}) \to \mathrm{Pic}(D')$ is surjective and its kernel is generated by the line bundles $\mathcal{O}(E_i)$. Furthermore, $\mathrm{Pic}(\widetilde{D})$ is generated by $\mathrm{Pic}(D)$, the line bundles $\mathcal{O}(E_i)$ and all line bundles $\mathcal{M}$ with

$$\mathcal{M}^2 \simeq \mathcal{O}\left(\sum_{i \in I} \mathcal{O}(E_i)\right)$$

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[4]Now proved in the appendix.

for some $I \subset \{1, \ldots, 16\}$. However, the kernel of the pull-back $\mathrm{Pic}(D') \to \mathrm{Pic}(F')$ is generated by $\mathcal{L}|_{D'}$; see, for example, [27, Lemma 3.1]. Thus, if $|I| < 16$, such an $\mathcal{M}$ would pull-back to a non-trivial two-torsion line bundle on F'. Since $\mathrm{Pic}(F') \simeq \mathrm{Pic}(F)$, this would contradict Lemma 2.1. ∎

2.7. The following assertion is analogous to the classical fact that for any étale double cover $C \to \bar{C}$ of a smooth curve the image of the map

$$1 - \iota^* \colon H^1(C, \mathbb{Z}) \to H^1(C, \mathbb{Z})$$

is the anti-invariant part $H^1(C, \mathbb{Z})^- \subset H^1(C, \mathbb{Z})$, see [22, Section 5.3.2] for references.

Lemma 2.6. *The image of the map*

(2.1) $$1 - \iota^* \colon H^2(F, \mathbb{Z}) \to H^2(F, \mathbb{Z})$$

is the anti-invariant part $H^2(F, \mathbb{Z})^- = \{\alpha \in H^2(F, \mathbb{Z}) \mid \iota^*\alpha = -\alpha\}$.

Proof. Clearly, the image of (2.1) is contained in the anti-invariant part. The difficult part is to show that every anti-invariant class $\alpha \in H^2(F, \mathbb{Z})^-$ is of the form $\alpha = \beta - \iota^*\beta$ for some integral class $\beta \in H^2(F, \mathbb{Z})$.

We first consider the class $\alpha = [E]$ of a divisor E on F, which typically is not effective nor irreducible. In this case, one can use the following argument, which is inspired by [4, Lemma 0.3.4], see also [27, Lemma 3.1]. By using the projectivity of F, we may assume that at least one of the fixed points $x_1, \ldots, x_{16} \in F$ is not contained in the support of E. Then $\iota^*E + E$ is a principal divisor (f) for some $f \in K(F)$. Hence,

$$(\iota^* f) = \iota^*(\iota^* E + E) = E + \iota^* E = (f)$$

and, therefore, $\iota^* f = \lambda \cdot f$ for some $\lambda \in \mathbb{C}^*$. By evaluating at one point x_i not contained in the support of E, one finds $\lambda = 1$, i.e., $\iota^* f = f$ or, equivalently, $f = \pi^* g$ for some $g \in K(D)$. Now write

$$E = E_1 - E_2 + \pi^{-1}(E_0),$$

where $E_1, E_2 \subset F$ are effective divisors without common irreducible components and $\pi^{-1}(E_0)$ is the ι-invariant part of E. In other words, if $E_i' \subset E_i$, $i = 1, 2$, is an irreducible component then $\iota^* E_i'$ is not an irreducible component of E_i.

Then $\iota^* E + E = (\pi^* g)$ can be rewritten as

$$\iota^* E_1 + E_1 = \iota^* E_2 + E_2 + \pi^{-1}(E_0'),$$

where $E'_0 = (g) - 2E_0$. By our assumptions on E_1 and E_2, this implies $E_1 = \iota^* E_2$ or, equivalently, $E_2 = \iota^* E_1$. Therefore,

$$E = E_1 - \iota^* E_1 + \pi^{-1}(E_0),$$

which passing to cohomology becomes

$$\alpha = [E] = [E_1] - \iota^*[E_1] + \pi^*[E_0].$$

Since the last summand is contained in the invariant part, it has to be trivial, which proves $\alpha = (1 - \iota^*)\beta$ for $\beta = [E_1]$.

As the assertion is purely topological, it is invariant under deformations. It thus suffices to argue that $H^2(F, \mathbb{Z})^-$ is generated by classes that are of type $(1, 1)$ on some deformation. This can be done directly or by using the Fano perspective. We defer the proof to Remark 4.3. ∎

Corollary 2.7. *For all classes $\alpha_1, \alpha_2 \in H^2(F, \mathbb{Z})^-$ the intersection pairing $(\alpha_1.\alpha_2)$ is even.*

Proof. Indeed, writing $\alpha_i = \beta_i - \iota^* \beta_i$, $i = 1, 2$, with $\beta_i \in H^2(F, \mathbb{Z})$ the assertion follows from

$$(\alpha_1.\alpha_2) = 2(\beta_1.\beta_2) - 2(\beta_1.\iota^* \beta_2),$$

which uses that ι^* is an isometric involution. ∎

2.8. The linear system of all quintics $|\mathcal{O}(5)|$ is of dimension 55. Dividing by the natural action of $\mathrm{PGL}(4)$ and imposing the existence of 16 nodes reduces the dimension to 24. For the details of this dimension count, in a more general setting, see [11, Proposition 2.26].

As we will recall below, the family of even 16-nodal quintics constructed from cubic fourfolds is also of dimension 24. In fact, due to results of Catanese [11, Theorem 3.3 and Remark 3.4], the space of even 16-nodal quintics is irreducible. Therefore, the generic even 16-nodal quintic $D \subset \mathbb{P}^3$ comes from a Fano variety of lines on a cubic fourfold, by the construction to be recalled in the next section. In particular, any additional topological information obtained in the Fano setting, holds for any even 16-nodal quintic and its natural double cover.

Remark 2.8. The situation is similar to the case of cubic threefolds. The generic smooth plane quintic curve is indeed of the form D_L. Note however that the generic étale double quotient $C \to D$ of a curve of genus 11 is not of the form $C_L \to D_L$ and the generic deformation of C alone will not have any quotient of this form. Similarly, for cubic fourfolds, the generic (infinitesimal) deformation of F will not be a double

cover of a nodal quintic. Indeed, a computation using the Hirzebruch–Riemann–Roch theorem reveals that $\chi(F, \mathcal{T}_F) = -40$, and, therefore, $h^1(F, \mathcal{T}_F) \geq 40$.[5]

3. The Fano perspective

In the following, $X \subset \mathbb{P}^5$ is a smooth cubic fourfold and $F(X)$ denotes its Fano variety of lines. According to Beauville and Donagi [7], $F(X)$ is a hyperkähler fourfold and we will use $\sigma \in H^{2,0}(F(X))$ to denote a non-degenerate holomorphic symplectic form on it.

For a line $L \subset X$ let

$$F_L := \{\, L' \mid L \cap L' \neq \emptyset \,\} \subset F(X)$$

be the Fano variety of all lines contained in X that intersect $L \subset X$.

3.1. From [41] we recall the following. For a generic line $L \subset X$ in a smooth cubic fourfold $X \subset \mathbb{P}^5$ the Fano variety F_L is a smooth surface of general type. Mapping $L' \in F_L$ to the intersection point of the plane $\overline{LL_1} \simeq \mathbb{P}^2$ with a fixed generic $\mathbb{P}^3$ defines a finite morphism

$$\pi \colon F_L \twoheadrightarrow D_L \subset \mathbb{P}^3$$

of degree two onto a quintic surface $D_L \subset \mathbb{P}^3$ with exactly 16 nodes $x_1, \ldots, x_{16} \in D_L$ as its only singularities. The covering involution $\iota \colon F_L \xrightarrow{\sim} F_L$ has exactly 16 fixed points $x_1, \ldots, x_{16} \in F_L$ mapping bijectively onto the nodes of D_L. In other words, D_L is an even 16-nodal quintic and $F_L \to D_L$ is its natural double cover.

Proof. Indeed, the surface D_L is the discriminant divisor of the linear projection $\mathrm{Bl}_L(X) \twoheadrightarrow \mathbb{P}^3$ of X from the line $L \subset X$. By construction, the covering involution maps a line $L' \in F_L$ to the residue line of $L \cup L' \subset X \cap \overline{LL'}$, i.e.,

$$L \cup L' \cup \iota(L') = X \cap \mathbb{P}^2$$

for a certain plane $\mathbb{P}^2 \subset \mathbb{P}^5$. For generic L the singularities of the discriminant surface D_L are nodes which correspond exactly to the fixed points of ι. The number of these points is computed by Porteous's formula; see the original [41] or [22, Section 6.4.5] for more details. ∎

By definition, the line L defines a point in F_L.[6] For a generic choice, this is not a fixed point of the involution ι.

---

[5]This comment was prompted by a question of G. Oberdieck.
[6]This is in contrast to the case of cubic threefolds, where for a generic line L the set all lines intersecting L splits off the point corresponding to L.

3.2. Observe that the restriction $\sigma|_{F_L} \in H^{2,0}(F_L)$ of the holomorphic symplectic form $\sigma \in H^{2,0}(F(X))$ is not zero, i.e., $F_L \subset F(X)$ is not Lagrangian.

Proof. Indeed, due to [41] one knows that $3[F_L]$ is the image of $h^3 = [\mathbb{P}^2] \in H^6(X, \mathbb{Z})$ under the Fano correspondence and, therefore,

$$[F_L] = (1/3)\big(g^2 - [F(Y)]\big).$$

Here, g denotes the Plücker polarization and $F(Y) \subset F(X)$ is the Fano surface of a generic hyperplane section $Y = X \cap \mathbb{P}^4$; see [2] or [22, Sections 2.5.1 and 6.4.1]. Using the positivity of g and the fact that $F(Y) \subset F(X)$ is Lagrangian [42], one concludes

$$\int_{F_L} (\sigma\bar\sigma)|_{F_L} = \frac{1}{3} \int_{F(X)} (\sigma\bar\sigma)g^2 \neq 0,$$

which in particular proves the assertion $\sigma|_{F_L} \neq 0$. ∎

3.3. The restriction of any primitive class $\alpha \in H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ to F_L is anti-invariant with respect to the action of ι on $H^2(F_L, \mathbb{Z})$, i.e.,

$$\iota^*(\alpha|_{F_L}) = -\alpha|_{F_L}.$$

Proof. The proof imitates the well-known argument for the analogous fact for cubic threefolds, cf. [22, Section 5.3.1] for an account and references. First note that the assertion is invariant under deformations. Next observe that the union of all

$$H^{2,0}(F(X')) \subset H^2(F(X), \mathbb{C})_{\mathrm{pr}},$$

for arbitrary smooth deformations $X' \subset \mathbb{P}^5$ of X, is Zariski dense. Thus, it suffices to show that $\iota^*(\sigma|_{F_L}) = -\sigma|_{F_L}$ or, dually, that the composition

$$H^{2,0}(D_L) \to H^{2,0}(F_L) \to H^{4,2}(F(X)) \to H^{3,1}(X)$$

is zero. This follows from the Bloch–Srinivas principle, see for example [43, Proposition 22.24], and the observation that the map $D_L \to \mathrm{CH}_1(X)$ is constant. To see the latter, consider $L' \in F_L$ and let $t := \pi(L') \in D_L$. Then,

$$L \cup L' \cup \iota(L') = X \cap \mathbb{P}^2$$

and the image of the point $t \in D_L$ is the constant class $[X \cap \mathbb{P}^2] - [L] \in \mathrm{CH}_1(X)$. ∎

3.4. Restriction defines an injection of Hodge structures of corank one

$$H^2(F(X), \mathbb{Z})_{\mathrm{pr}} \hookrightarrow H^2(F_L, \mathbb{Z})^-.$$

Proof. By Section 3.3, the restriction of any primitive class is indeed anti-invariant. To prove the injectivity of the restriction map one can use a standard deformation argument, cf. [22, Remark 2.5.7]. As the assertion is purely topological, we may assume that X is very general and, in particular, that $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ is an irreducible Hodge structure. Hence, the restriction map is either trivial or injective. However, as $F_L \subset F(X)$ is not Lagrangian by Section 3.2, i.e., $\sigma|_{F_L} \neq 0$, and

$$\sigma \in H^{2,0}(F(X)) \subset H^2(F(X), \mathbb{Z})_{\mathrm{pr}} \otimes \mathbb{C},$$

it is certainly not trivial. Alternatively, one can use Section 3.5 below.

The assertion on the corank follows from rk $H^2(F(X), \mathbb{Z}) = 23$ and Section 2.4.
∎

3.5. For any class $\alpha \in H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ and its restriction $\alpha|_{F_L} \in H^2(F_L, \mathbb{Z})$, one has

$$2\, q(\alpha) = (\alpha|_{F_L}.\alpha|_{F_L}).$$

Here, q is the Beauville–Bogomolov–Fujiki form on the hyperkähler fourfold $F(X)$ and $(\,.\,)$ denotes the intersection form on the surface F_L.

Proof. The result follows from a straightforward and well-known computation of certain natural cohomology classes on $F(X)$, see [22, Section 6.4.1] for an account and references, and a result of Voisin mentioned before: The natural class $\widetilde{q} \in H^4(F(X), \mathbb{Q})$ defined by the condition that $q(\alpha) = \int_{F(X)} \alpha^2 \cdot \widetilde{q}$ satisfies

$$30\, \widetilde{q} = c_2(\mathcal{T}_{F(X)}) = 15\,[F_L] - 3\,c_2(\mathcal{S}_F) = 15\,[F_L] - 3\,[F(Y)],$$

where as above $F(Y) \subset F(X)$ is the surface of all lines contained in a generic hyperplane section $Y = X \cap H$. Hence,

$$q(\alpha) = \int_{F(X)} \widetilde{q} \cdot \alpha^2 = \frac{1}{2} \int_{F_L} \alpha|_{F_L}^2 .$$

Here, the last equality follows from the vanishing $\alpha|_{F(Y)} = 0$, and hence

$$\int_{F(Y)} \alpha|_{F(Y)}^2 = 0,$$

for all primitive classes α, which is a consequence of $F(Y) \subset F(X)$ being Lagrangian [42, Example 3.7]. ∎

3.6. Let $g = c_1(\mathcal{O}(1)) \in H^2(F(X), \mathbb{Z})$ be the class of the Plücker polarization and let $g|_{F_L} = g^+ + g^-$ be the decomposition of its restriction in its invariant and anti-invariant part, i.e., $g^\pm \in H^2(F_L, \mathbb{Q})$ with $\iota^* g^\pm = \pm g^\pm$. Then both parts are non-zero, i.e., $g^\pm \neq 0$.

Proof. Since $g|_{F_L}$ is ample, so is $\iota^* g|_{F_L}$. Hence,

$$\iota^* g|_{F_L} \neq -g|_{F_L},$$

i.e., $g^+ \neq 0$. Suppose now that $g^- = 0$. Then g is ι-invariant. Since $\mathrm{Pic}(F_L) \subset H^2(F_L, \mathbb{Z})$ is torsion free, see Lemma 2.1, it means that the restriction $\mathcal{O}(1)$ of the Plücker polarization to F_L is ι-invariant. Hence, its restriction to $F_L' := F_L \setminus \{x_i\}$ descends to a line bundle on $D_L' = D_L \setminus \{x_i\}$, cf. [4, Lemma 0.3.4] or [27, Lemma 3.1]. Combining this with Section 2.6 and the fact that $\pi^*(\mathcal{L}|_{D_L'}) \simeq \mathcal{O}_{F_L'}$, one finds a line bundle $\mathcal{K}$ on D_L such that

$$\pi^*(\mathcal{K}|_{D_L'}) \simeq \mathcal{O}(1)|_{F_L'}$$

and, in fact, $\pi^* \mathcal{K} \simeq \mathcal{O}(1)|_{F_L}$, as $\mathrm{Pic}(F_L) \simeq \mathrm{Pic}(F_L')$. Thus, for $k := c_1(\mathcal{K})$, we have $\pi^* k = g|_{F_L}$ and, in particular,

$$(\pi^* k . \pi^* k) = (g|_{F_L} . g|_{F_L}),$$

which we will show to be impossible.

First note that $(\pi^* k . \pi^* k) = 2 \cdot (k.k)$ is even, as $F_L \twoheadrightarrow D_L$ is of degree two. On the other hand, $3[F_L]$ is the image of $h^3 \in H^6(X, \mathbb{Z})$ under the Fano correspondence and $[F_L] = (1/3)(g^2 - [F(Y)])$, cf. [22, Sections 2.5.1 and 6.4.1]. This implies

$$(g|_{F_L} . g|_{F_L}) = \int_{F(X)} [F_L] \cdot g^2 = \frac{1}{3} \left(\int_{F(X)} g^4 - \int_{F(Y)} g^2|_{F(Y)} \right)$$

$$= \frac{1}{3} \big(\deg(F(X)) - \deg(F(Y)) \big) = 21.$$

Thus, $(g|_{F_L} . g|_{F_L})$ is odd, which produces the desired contradiction. ∎

Remark 3.1. To show that g is not invariant, one can alternatively use that the linear system $\mathcal{O}(1)|_{F_L} \otimes \pi^* \mathcal{O}(-1)$ defines the rational map $F_L \dashrightarrow L$ that maps L' to its point of intersection with L, which is clearly not ι-invariant; see [41, Section 3, Lemma 2].

Remark 3.2. (i) The arguments in the proof also show that every ι-invariant line bundle on F_L is the pull-back of a line bundle on D_L. A priori, this is not clear, as a linearization of an invariant line bundle may act non-trivially on the fibre at one of the fixed points.

(ii) Also, it is not clear whether the same holds true for arbitrary cohomology classes. In other words, is $H^2(D_L, \mathbb{Z}) \to H^2(F_L, \mathbb{Z})^+$ surjective? One way to prove surjectivity would be to show that $H^2(F_L, \mathbb{Z})^+$ is generated by classes that become algebraic on some deformation of D_L as an even 16-nodal quintic. Note that in the analogous situation for cubic threefolds the map is indeed not surjective. Indeed, for an étale double cover $C \to \bar{C}$ of smooth curves the map $H^1(\bar{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z})^+$ has a cokernel of order two; see [22, Section 5.3.2] for references.

Consider the torsion free lattice

$$H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^- \subset H^2(F_L, \mathbb{Z})$$

of all classes that are primitive with respect to the restriction $g|_{F_L}$ of the Plücker polarization and anti-invariant with respect to the involution ι.

Corollary 3.3. *The lattice $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-$ is of rank 22 and signature $(2, 20)$. It is naturally endowed with a Hodge structure of K3 type.*

Proof. As rk $H^2(F_L, \mathbb{Z})^- = 23$ by Section 2.4, it suffices to show that $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^- \subset H^2(F_L, \mathbb{Z})^-$ is a proper sub-lattice, i.e., the linear form $(g.\ )$ on $H^2(F_L, \mathbb{Z})^-$ is non-zero. As the two sublattices $H^2(F_L, \mathbb{Z})^-$ and $H^2(F_L, \mathbb{Z})^+$ are orthogonal with respect to $(\ .\ )$, this follows from $g^- \neq 0$ proved above. By the Hodge index theorem, $(g^-.g^-) < 0$, which implies the claimed signature. That the Hodge structure is of K3 type follows from Section 2.3. ∎

4. Hodge isometries and global Torelli

The goal of this section is to prove Theorem 1.2.

4.1. As before, we consider a smooth cubic fourfold $X \subset \mathbb{P}^5$ and a generic line $L \subset X$.

Proposition 4.1. *The restriction map induces an isometry of Hodge structures of K3 type*

$$\left(H^2(F(X), \mathbb{Z})_{\mathrm{pr}}, q \right) \simeq \left(H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-, (1/2)(\ .\ ) \right).$$

Proof. By virtue of Sections 3.4 and 3.5, restriction embeds the Hodge structure $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ endowed with the Beauville–Bogomolov–Fujiki form q isometrically into $H^2(F_L, \mathbb{Z})^-$ viewed with the scaled intersection form $(1/2)(\ .\ )$. The latter is integral on $H^2(F_L, \mathbb{Z})^-$ due to Corollary 2.7.

This leads to a morphism of rational Hodge structures

$$H^2(F(X), \mathbb{Q})_{\mathrm{pr}} \hookrightarrow H^2(F_L, \mathbb{Q})^- = H^2(F_L, \mathbb{Q})_{\mathrm{pr}}^- \oplus \mathbb{Q} \cdot g^- \twoheadrightarrow \mathbb{Q} \cdot g^-,$$

which has to be trivial due to the irreducibility of the Hodge structure $H^2(F(X), \mathbb{Q})_{\mathrm{pr}}$ for the very general X. Hence, restriction defines an embedding of Hodge structures

$$(4.1) \qquad H^2(F(X), \mathbb{Z})_{\mathrm{pr}} \hookrightarrow H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-,$$

which is isometric with respect to the two symmetric forms q and $(1/2)(\ .\ )$. Note that both sides are of rank 22. Thus, restriction identifies $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ with a sub-lattice of $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-$ of finite index $m < \infty$. By a standard fact in lattice theory, the discriminants of the two lattices are related by the formula

$$\mathrm{discr}(q) = \mathrm{discr}((1/2)(\ .\ )) \cdot m^2.$$

Since by [7] there exists an isometry (up to a global sign)

$$H^2(F(X), \mathbb{Z})_{\mathrm{pr}} \simeq H^4(X, \mathbb{Z})_{\mathrm{pr}},$$

we know that $\mathrm{discr}(q) = 3$. This suffices to conclude that $m = 1$, i.e., (4.1) is bijective ∎

Together with the result of Beauville and Donagi [7] this proves Theorem 1.2. ∎

Remark 4.2. (i) The isomorphism types of the two lattices given by $H^2(F(X), \mathbb{Z})$ and $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ are known. The latter is the even lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1)$, which therefore also describes $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-$.

However, it is not clear to us how to determine the isomorphism type of the lattice $H^2(F_L, \mathbb{Z})^-$, which according to Section 2.4 has signature $(2, 21)$. It is tempting to conjecture the existence of an isometry (up to a global sign) $H^4(X, \mathbb{Z}) \simeq H^2(F_L, \mathbb{Z})^-$, but I have no further evidence for it.

(ii) A. Kuznetsov suggested that Proposition 4.1 could possibly be seen as a consequence of a result for general conic fibrations $\phi \colon \widetilde{X} \to \mathbb{P}^3$. Indeed, it seems feasible that one can establish a direct link between the Hodge structures $H^4(\widetilde{X}, \mathbb{Z})$ and $H^2(F, \mathbb{Z})^-$, where F is the natural double cover of the discriminant surface $D \subset \mathbb{P}^3$ of ϕ. Due to the dependence on the Plücker polarization, incorporating the quadratic forms and defining the primitive part of $H^2(F_L, \mathbb{Z})^-$ seem less obvious. In any case, in the context of a cubic fourfold X the approach via its Fano variety $F(X)$ has the advantage that the Hodge isometry constructed above is compatible with the inclusion $F = F_L \subset F(X)$.

Remark 4.3. General deformation theory for hyperkähler manifolds implies that the classes in $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ that are of type $(1, 1)$ for some deformation of X generate the full primitive cohomology. Hence, the same is true for $H^2(F_L, \mathbb{Z})^-$ and then for $H^2(F, \mathbb{Z})^-$ of the natural cover of an arbitrary even 16-nodal quintic. This was used in the proof of Lemma 2.6.

Remark 4.4. According to Hassett [19], there exists a countable union of divisors in the moduli space of cubic fourfolds for which $H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$ contains the Hodge structure $H^2(S, \mathbb{Z})_{\mathrm{pr}}$ of a polarized K3 surface. The correspondence is known to be algebraic [1]. Thus, along these divisors one finds algebraic correspondences between a K3 surface S and the surface of general type F_L. Can those be realized geometrically?

4.2. It may be worthwhile to spell out the various Hodge theoretic conditions and their geometric consequences. Let $L \subset X$ and $L' \subset X'$ be as above and let us consider the following statements:

(i)   There exists an isomorphism $F_L \simeq F_{L'}$ compatible with the natural involutions ι and ι'.

(ii) There exists an equivariant isometry

$$\left(H^2(F_L, \mathbb{Z})_{\mathrm{pr}}, (\,.\,), \iota^*\right) \simeq \left(H^2(F_{L'}, \mathbb{Z})_{\mathrm{pr}}, (\,.\,), \iota'^*\right)$$

of Hodge structures.

(iii) There exists an isometry of Hodge structures

$$\left(H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-, (\,.\,)\right) \simeq \left(H^2(F_{L'}, \mathbb{Z})_{\mathrm{pr}}^-, (\,.\,)\right).$$

(iv) There exists an isomorphism of polarized varieties $(F(X), g) \simeq (F(X'), g')$.

(v)   There exists an isomorphism $X \simeq X'$.

Then the following implications hold

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)}.$$

The equivalence of (iii) and (iv) follows from Theorem 1.2 and the global Torelli theorem for hyperkähler manifolds. All other implications are either obvious or well known. The first implication is not an equivalence and neither should be the second.

Corollary 1.3 is the combination (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). It is the analogue of the celebrated result of Clemens–Griffiths and Tyurin combined with Mumford's work on Prym varieties: For two smooth cubic threefolds Y and Y', one has

$$Y \simeq Y' \Leftrightarrow \mathrm{Prym}(C_L/D_L) \simeq \mathrm{Prym}(C_{L'}/D_{L'})$$

with an isomorphism of polarized abelian varieties on the right hand side. Here, $L \subset Y$ and $L' \subset Y'$ are generic lines and $C_L \to D_L$ and $C_{L'} \to D_{L'}$ are the analogues of $F_L \to D_L$ and $F_{L'} \to D_{L'}$; see [22, Chapter 5] for references.

4.3. Theorem 1.2 can be seen as a stronger and more precise version (in dimension four) of a result of Izadi [23, Theorem 3], which asserts the existence of an exact sequence

$$0 \to H^2(D_L, \mathbb{Z})_{\mathrm{pr}} \to H^2(F_L, \mathbb{Z})_{\mathrm{pr}} \to H^4(X, \mathbb{Z})_{\mathrm{pr}} \to 0,$$

where the ample class $g|_{F_L} + c_1(\pi^*\mathcal{O}(1))$ is used to define the primitive part. However, the proof of the surjectivity is incomplete. It uses the surjectivity of

$$H^2(F(X), \mathbb{Z}) \twoheadrightarrow H^2(F(X), \mathbb{Z})_{\mathrm{pr}}$$

claimed in [23, Section 4], which only exists with coefficients in $\mathbb{Q}$ or after some suitable localization, and, no argument is given for the surjectivity in the proof of [23, Lemma 5.12]. The problem is similar to showing that

$$H^2(F(X), \mathbb{Z})_{\mathrm{pr}} \hookrightarrow H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-$$

is bijective, see the proof of Proposition 4.1. Furthermore, the claim that $H^2(D_L, \mathbb{Z})$ is the invariant part of $H^2(F_L, \mathbb{Z})$, which again does hold for coefficients in $\mathbb{Q}$ and for algebraic classes, see Sections 2.4 and 3.6, is not adequately addressed and no proof is given for the torsion freeness of $H^2(F_L, \mathbb{Z})$.

Shen's result [33, Theorem 1.5 (2) and Remark 5.10] for the Fano surface of lines meeting a fixed general rational curve of degree at least two is similar to the above proposition. The proof of the surjectivity there relies on degeneration techniques developed in [34]. Also the image of $1 - \iota^*$ is used instead of $H^2(F_L, \mathbb{Z})_{\mathrm{pr}}^-$, which, however, by virtue of 2.7 eventually amounts to the same. Subsequently, Shen considered the case of lines. The results [31, Theorem 4.7 and Corollary 4.8] come closest to Proposition 4.1 and Theorem 1.2, although Shen defines the primitive anti-invariant part in terms of two classes in $H^2(F_L, \mathbb{Z})$, the Plücker polarization and the class of the fibre of the natural projection $F_L \dashrightarrow L$. Also, the convention for the pairings q and $(1/2)(\,.\,)$ on the two sides are different. In Section 5.5 we give a brief comparison of the techniques.

5. Chow groups and Chow motives

The goal of this section is to 'lift' the Hodge isometry constructed above to the level of integral Chow groups and rational Chow motives. As a first step, one needs to define properly the analogue of the Prym variety $\mathrm{Prym}(C_L/D_L)$ as a subgroup of the Chow group $\mathrm{CH}_0(F_L)$.

5.1. Let us first consider the general situation of an even 16-nodal quintic $D \subset \mathbb{P}^3$ and the natural double cover $\pi \colon F \twoheadrightarrow D$ with the covering involution ι.

Let $\mathrm{CH}(F)_{\mathrm{hom}} = \mathrm{CH}_0(F)_{\mathrm{hom}} \subset \mathrm{CH}_0(F)$ be the homologically trivial part of the Chow group (of 0-cycles) on the surface F. We consider it with the action induced by the involution

$$\iota^* \colon \mathrm{CH}_0(F)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{CH}_0(F)_{\mathrm{hom}}$$

and define the two subgroups

$$\mathrm{CH}_0(F)_{\mathrm{hom}}^{\pm} := \{\alpha \mid \iota^*\alpha = \pm\alpha\}.$$

The groups $\mathrm{CH}_0(D)_{\mathrm{hom}}$ and $\mathrm{CH}_0(F)_{\mathrm{hom}}$ are divisible and, since the Albanese of the regular surfaces D and F are trivial, also torsion free [8, 30]. In particular, this allows one to write any class $\alpha \in \mathrm{CH}_0(F)_{\mathrm{hom}}$ as

$$\alpha = \alpha^+ + \alpha^-$$

with integral $\alpha^{\pm} \in \mathrm{CH}_0(F)_{\mathrm{hom}}^{\pm}$. Explicitly, $\alpha^{\pm} := (1/2)(\alpha \pm \iota^*\alpha)$. In other words,

$$(5.1) \qquad \mathrm{CH}_0(F)_{\mathrm{hom}} = \mathrm{CH}_0(F)_{\mathrm{hom}}^+ \oplus \mathrm{CH}_0(F)_{\mathrm{hom}}^-.$$

The group $\mathrm{CH}_0(F)_{\mathrm{hom}}^+$ can be identified with $\mathrm{CH}_0(D)_{\mathrm{hom}}$ via

$$\pi^* \colon \mathrm{CH}_0(D)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{CH}_0(F)_{\mathrm{hom}}^+.$$

Indeed, π^* is injective, as $\pi_* \circ \pi^* = 2 \cdot \mathrm{id}$ and $\mathrm{CH}_0(D)_{\mathrm{hom}}$ is torsion free. The surjectivity follows from the well-known statement for Chow groups with coefficients in $\mathbb{Q}$, see [15, Example 1.7.6], and the divisibility of the Chow group. Note that the singularities of the surface D do not cause trouble, for $\mathrm{CH}_0(D)_{\mathrm{hom}} \simeq \mathrm{CH}_0(\widetilde{D})_{\mathrm{hom}}$ by [35, Corollary 9.8].

We are more interested in the anti-invariant part which admits several alternative descriptions.

Lemma 5.1. *Projection defines an isomorphism*

$$\mathrm{CH}_0(F)_{\mathrm{hom}}^- \simeq \mathrm{CH}_0(F)_{\mathrm{hom}} / \mathrm{CH}_0(F)_{\mathrm{hom}}^+ \simeq \mathrm{CH}_0(F)_{\mathrm{hom}} / \mathrm{CH}_0(D)_{\mathrm{hom}}.$$

Furthermore,

$$\mathrm{CH}_0(F)_{\mathrm{hom}}^- = \mathrm{Im}\bigl(1 - \iota^* \colon \mathrm{CH}_0(F)_{\mathrm{hom}} \to \mathrm{CH}_0(F)_{\mathrm{hom}}\bigr)$$

and

$$\mathrm{CH}_0(F)_{\mathrm{hom}}^- = \ker\bigl(\pi_* \colon \mathrm{CH}_0(F)_{\mathrm{hom}} \to \mathrm{CH}_0(D)_{\mathrm{hom}}\bigr).$$

In particular, $\mathrm{CH}_0(F)_{\mathrm{hom}}^-$ is generated by classes of the form $[t] - [\iota(t)] \in \mathrm{CH}_0(F)$.

Proof. The first two assertions follow from the discussion above. It remains to verify the description of $\mathrm{CH}_0(F)_{\mathrm{hom}}^-$ as $\ker(\pi_*)$.

As $\pi_*\alpha = \pi_*\iota^*\alpha = \pi_*(-\alpha)$ for all $\alpha \in \mathrm{CH}_0(F)_{\mathrm{hom}}^-$ and since $\mathrm{CH}_0(D)_{\mathrm{hom}}$ is torsion free, the inclusion $\mathrm{CH}_0(F)_{\mathrm{hom}}^- \subset \ker(\pi_*)$ is clear. Conversely, write a given $\alpha \in \ker(\pi_*)$ as above as $\alpha = \alpha^+ + \alpha^- = \pi^*\beta + \alpha^-$, which leads to $0 = \pi_*\alpha = \pi_*\pi^*\beta = 2\beta$ and, therefore, $\beta = 0$, i.e., $\alpha = \alpha^-$. ∎

Remark 5.2. The situation is very similar to the case of cubic threefolds where $\mathrm{Prym}(C_L/D_L)$ is replaced by $\mathrm{CH}_0(F_L)_{\mathrm{hom}}^-$. However, there are two notable differences caused by the torsion freeness of the Chow group.

First, the Prym variety $\mathrm{Prym}(C_L/D_L)$, defined as the image of

$$1 - \iota^* \colon \mathrm{Pic}^0(C_L) \to \mathrm{Pic}^0(C_L),$$

and thus isomorphic to the quotient $\mathrm{Pic}(C_L)/\mathrm{Pic}(C_L)^+$, is only one of the two connected components of the kernel of $\pi_* \colon \mathrm{Pic}^0(C_L) \to \mathrm{Pic}^0(D_L)$. Second,

$$\pi^* \colon \mathrm{Pic}^0(D_L) \to \mathrm{Pic}^0(C_L)$$

has a kernel of order two generated by the torsion line bundle defining the étale double cover $C_L \to D_L$. In other words, the analogue of (5.1) in the case of cubic threefolds is not a direct product decomposition of $\mathrm{Pic}^0(C_L)$ but the étale degree two map

$$\mathrm{Pic}^0(D_L) \times \mathrm{Prym}(C_L/D_L) \twoheadrightarrow \mathrm{Pic}^0(C_L).$$

5.2. The arguments to prove the first part of Theorem 1.4 follow closely ideas of Murre [25]. At the heart of the proof is the following geometric construction which has immediate consequences for the Chow group of X. Here, as before, we use L, L', etc. to denote a line in X as well as the corresponding point in F_L.

Consider a tangent direction v at a point $x \in L \subset X$, i.e., a line in $T_x X$, and let $L_v \subset \mathbb{P}^5$ be the unique line through x realizing this tangent direction. If L_v is not contained in X, then it intersects X in x with multiplicity at least two and, therefore, defines a unique point $y_v \in X$ such that $L_v \cap X = 2x + y_v$, cf. [22, Section 2.1.5]. This defines a dominant rational map $\mathbb{P}(\mathcal{T}_X|_L) \dashrightarrow X$ which is not defined at the points v with $L_v \subset X$. Note that by construction any $L_v \subset X$, $v \in \mathbb{P}(\mathcal{T}_X|_L)$, intersects L, and hence defines a point $L_v \in F_L$. Conversely, mapping a line $L' \in F_L$ distinct from L to its tangent direction at the point of intersection $L \cap L' = \{x\}$ defines a map

$$F_L \setminus \{L\} \hookrightarrow \mathbb{P}(\mathcal{T}_X|_L), \quad L' \mapsto (x, T_x L'),$$

which extends to a closed embedding

$$\mathrm{Bl}_L(F_L) \hookrightarrow \mathbb{P}(\mathcal{T}_X|_L)$$

of the blow-up $\mathrm{Bl}_L(F_L)$ of F_L in the point $L \in F_L$. The blow-up can also be described as $q^{-1}(L)$, where $F \xleftarrow{p} \mathbb{L} \xrightarrow{q} X$ denotes the universal family of lines in X. In particular, the exceptional curve in $\mathrm{Bl}_L(F_L)$ can be viewed as $p^{-1}L \subset \mathbb{L}$. The picture is as follows:

$$
\begin{array}{ccc}
\mathbb{L} & \longhookrightarrow & \mathbb{P}(\mathcal{T}_X) \\
\uparrow & & \uparrow \\
\mathrm{Bl}_L(F_L) \simeq q^{-1}(L) & \longhookrightarrow & \mathbb{P}(\mathcal{T}_X|_L) \\
\uparrow & & \uparrow \\
L \simeq p^{-1}L & \xrightarrow{\;\sim\;} & \mathbb{P}(\mathcal{T}_L).
\end{array}
$$

We now consider the blow-up

$$\tau \colon \widetilde{X} := \mathrm{Bl}_{\mathrm{Bl}_L(F_L)}(\mathbb{P}(\mathcal{T}_X|_L)) \twoheadrightarrow \mathbb{P}(\mathcal{T}_X|_L)$$

with the surface $\mathrm{Bl}_L(F_L)$ as its center. Then the dominant rational map $\mathbb{P}(\mathcal{T}_X|_L) \dashrightarrow X$ extends to a surjective morphism

$$\gamma \colon \widetilde{X} \twoheadrightarrow X.$$

Alternatively, $\widetilde{X}$ can be described as the incidence variety of all triples (x, y, L') consisting of a line $L' \subset \mathbb{P}^5$ and points $x \in L \cap L'$, $y \in X \cap L'$ such that $2x \subset X \cap L'$, i.e., L' is tangent to X at the point x.[7] Then

$$\tau(x, y, L') = (x, T_x L') \in \mathbb{P}(\mathcal{T}_X|_L) \quad \text{and} \quad \gamma(x, y, L') = y \in X.$$

For $L' \in F_L \setminus \{L\}$ thought of as the point $(x, T_x L') \in \mathbb{P}(\mathcal{T}_X|_L)$, the fibre $\tau^{-1}(L') \subset \widetilde{X}$ consists all triples (x, y, L'), where $\{x\} = L \cap L'$ and $y \in L'$ arbitrary. In particular,

$$\gamma \colon \tau^{-1}(L' = (x, T_x L')) \xrightarrow{\sim} L' \subset X.$$

Similarly, for $(x, T_x L) \in \mathbb{P}(\mathcal{T}_X|_L)$ the fibre under the blow-up is $\{(x, y, L) \mid y \in L\}$. Hence, the pre-image $\tau^{-1}(p^{-1}L)$ of the exceptional line $L \simeq p^{-1}L \subset \mathrm{Bl}_L(F_L)$ is naturally identified with $L \times L$ such that τ and γ correspond to the two-projections.

The morphism γ is generically of degree two, which is seen as follows. For any point $y \in X \setminus L$ consider the plane cubic curve $X \cap \overline{Ly}$. For a generic point y the residual conic Q_y of $L \subset X \cap \overline{Ly}$ does not contain L and, therefore, intersects L in

---

[7]A. Kuznetsov suggested to view $\widetilde{X}$ alternatively as the fibre product $\mathrm{Bl}_L(X) \times_{\mathbb{P}^3} E$, where $E \subset \mathrm{Bl}_L(X)$ denotes the exceptional divisor. Note that the restriction $\phi \colon E \to \mathbb{P}^3$ is generically finite of degree two and that the conic fibration given by the first projection $\widetilde{X} \to E$ comes with a section.

at most two points $x_1, x_2 \in L$. The tangent directions of the two lines connecting x_1 and x_2 with y are the pre-images of y; see [22, Corollary 2.1.21] for further details.

The induced rational covering involution

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\quad j \quad} & \widetilde{X} \\
& \searrow{\gamma} \quad \swarrow{\gamma} & \\
& X &
\end{array}
$$
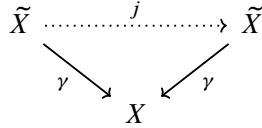
maps a generic point (x_1, y, L) to (x_2, y, L'), where L' is the line through y and the second point of intersection of $Q_y \cap L = \{x_1, x_2\}$.

5.3. Since $\tau \colon \widetilde{X} \to \mathbb{P}(\mathcal{T}_X|_L)$ is the blow-up in $\mathrm{Bl}_L(F_L) \subset \mathbb{P}(\mathcal{T}_X|_L)$, the Chow group $\mathrm{CH}_1(\widetilde{X})$ naturally splits as

$$
\begin{aligned}
\mathrm{CH}_1(\widetilde{X}) &\simeq \tau^* \mathrm{CH}_1(\mathbb{P}(\mathcal{T}_X|_L)) \oplus k_* \tau_0^* \mathrm{CH}_0(\mathrm{Bl}_L(F_L)) \\
&\simeq \mathrm{CH}_1(\mathbb{P}(\mathcal{T}_X|_L)) \oplus \mathrm{CH}_0(F_L).
\end{aligned}
$$

Here,

$$
\tau_0 \colon E := \tau^{-1}(\mathrm{Bl}_L(F_L)) \twoheadrightarrow \mathrm{Bl}_L(F_L) \quad \text{and} \quad k \colon \tau^{-1}(\mathrm{Bl}_L(F_L)) \hookrightarrow \widetilde{X}
$$

are the projection, a $\mathbb{P}^1$-bundle, and the natural closed embedding of the exceptional divisor.

Lemma 5.3. *The homologically trivial part of $\mathrm{CH}(\widetilde{X})$ sits in dimension one and is naturally identified with the homologically trivial part of the center of the blow-up τ, i.e.,*

$$
\mathrm{CH}(\widetilde{X})_{\mathrm{hom}} = \mathrm{CH}_1(\widetilde{X})_{\mathrm{hom}} \simeq \mathrm{CH}_0(F_L)_{\mathrm{hom}} = \mathrm{CH}(F_L)_{\mathrm{hom}}.
$$

Proof. The assertion follows directly from the fact that the first summand in the above decomposition injects into cohomology. ∎

The component in $\mathrm{CH}_0(F_L)$ of a curve class $\alpha \in \mathrm{CH}_1(\widetilde{X})$ is computed by $-\tau_{0*}\alpha|_E$. For example, if $C \subset \widetilde{X}$ intersects only one fibre $\tau_0^{-1}(L')$ and does so transversally in only one point, then $[C]$ is mapped (up to the sign) to the class of that fibre $-[\tau_0^{-1}(L')] \in k_* \tau_0^* \mathrm{CH}_0(F_L)$.

Lemma 5.4. *The direct image $\gamma_* \colon \mathrm{CH}(\widetilde{X}) \to \mathrm{CH}(X)$ defines a surjective map*

$$
\mathrm{CH}_1(\widetilde{X})_{\mathrm{hom}} \twoheadrightarrow \mathrm{CH}_1(X)_{\mathrm{hom}}.
$$

Furthermore, $\mathrm{CH}_1(X)$ is generated by lines intersecting a fixed line L.

Proof. As γ is of degree two, we know that

$$\gamma_* \circ \gamma^* = 2 \cdot \text{id}.$$

As $\text{CH}_1(X)$ is generated by lines, see [28, 32] or the more general result [36, Theorem 1.6], homological equivalence equals algebraic equivalence. Thus, $\text{CH}_1(X)_{\text{hom}}$ is divisible and the assertion follows.

The second assertion follows from $\text{CH}_1(\widetilde{X})_{\text{hom}}$ being contained in the subgroup generated by classes of fibres $L' \simeq \tau_0^{-1}(L') \subset \widetilde{X}$ over $L' \in \text{Bl}_L(F_L)$. ∎

Lemma 5.5. *The pull-back* $\gamma^* \colon \text{CH}_1(X) \to \text{CH}_1(\widetilde{X})$ *maps the subgroup* $\text{CH}_1(X)_{\text{hom}}$ $\subset \text{CH}_1(X)$ *onto* $\text{CH}_0(F_L)^-_{\text{hom}} \subset \text{CH}_0(F_L)_{\text{hom}} \simeq \text{CH}_1(\widetilde{X})_{\text{hom}}$.

Proof. We have to show that the image of $\gamma^* \colon \text{CH}_1(X)_{\text{hom}} \to \text{CH}_0(F_L)$ is $\text{CH}_0(F_L)^-$. Clearly, the image is invariant under the covering involution j^* and we also know that $\text{CH}_1(X)$ is generated by the classes of lines $L' \in F_L$.

Fix a generic line $L' \in F_L$. As explained above, $\tau^{-1}(L') \subset \widetilde{X}$ consists of all triples (x', y, L') with $\{x'\} = L \cap L'$ and $y \in L'$ arbitrary. By definition,

$$j(x', y, L') = (x'', y, L''),$$

where x'' is the point of intersection of L with the residual line $\iota(L')$ of $L \cup L' \subset X \cap \overline{LL'}$, i.e., $L \cup L' \cup \iota(L') = X \cap \mathbb{P}^2$ and $L'' := \overline{x''y}$.

The second component of the class

$$[j(\tau^{-1}(L'))] \in \text{CH}_1(\widetilde{X}) = \text{CH}_1(\mathbb{P}(\mathcal{T}_X|_L)) \oplus \text{CH}_0(F_L)$$

is $-[\tau^{-1}(L'')]$, because $j(\tau^{-1}(L'))$ intersects the exceptional divisor $\tau^{-1}(\text{Bl}_L(F_L))$ transversally in the point $(x'', z, \iota(L'))$, where $\{z\} = L' \cap \iota(L')$. Therefore, the homologically trivial part of $\gamma^*[L']$ is nothing but

$$[\tau^{-1}(L')] + [j(\tau^{-1}(L'))] = [\tau^{-1}(L')] - [\tau^{-1}(\iota(L'))],$$

which corresponds to $[L'] - [\iota(L')] \in \text{CH}_0(F_L)^-$. As classes of this form generate $\text{CH}_0(F_L)^-$, this proves surjectivity. ∎

Lemma 5.6. *Consider a class* $\alpha \in \text{CH}_0(F_L)^- \subset \text{CH}_0(F_L) \subset \text{CH}_1(\widetilde{X})$. *Then, we have* $\gamma^*\gamma_*(\alpha) = 2\alpha$.

Proof. By the surjectivity of $\gamma^* \colon \text{CH}_1(X)_{\text{hom}} \twoheadrightarrow \text{CH}_0(F_L)^-$ proved in the previous lemma, it suffices to verify the assertion for classes of the form $\gamma^*\beta$, which follows from $\gamma_* \circ \gamma^* = 2 \cdot \text{id}$. ∎

5.4. We can now conclude the proof of the first part of Theorem 1.4, by adapting arguments of Murre [25] to the case of cubic fourfolds.

First observe that for $L \neq L' \in F_L$ the morphism $\gamma \colon \widetilde{X} \to X$ induces an isomorphism $\gamma \colon \tau^{-1}(L') \xrightarrow{\sim} L' \subset X$. Hence, for two arbitrary points $L', L'' \in F_L$ the map

(5.2) $$\mathrm{CH}_0(F_L)_{\mathrm{hom}} \simeq \mathrm{CH}_1(\widetilde{X})_{\mathrm{hom}} \xrightarrow{\gamma_*} \mathrm{CH}_1(X)_{\mathrm{hom}}$$

sends $[L'] - [L''] \in \mathrm{CH}_0(F_L)_{\mathrm{hom}}$ to the class $[L'] - [L''] \in \mathrm{CH}_1(X)_{\mathrm{hom}}$. In other words,

$$\mathrm{CH}_0(F_L)_{\mathrm{hom}} \xrightarrow{k_* \circ \tau_0^*} \mathrm{CH}_1(\widetilde{X})_{\mathrm{hom}} \xrightarrow{\gamma_*} \mathrm{CH}_1(X)_{\mathrm{hom}}$$

is indeed the map induced by the Fano correspondence.

Next recall from the proof in Section 3.3 the standard fact that we have

$$[L'] + [L''] = [X \cap \mathbb{P}^2] - [L] \equiv \mathrm{const}$$

in $\mathrm{CH}_1(X)$ for any $L' \in F_L$ and its image $L'' := \iota(L') \in F_L$ under the covering involution of $\pi \colon F_L \to D_L$. Combining this with Lemma 5.1, we find that the map (5.2) factors through a surjection

$$\mathrm{CH}_0(F_L)_{\mathrm{hom}}^- \twoheadrightarrow \mathrm{CH}_1(X)_{\mathrm{hom}}.$$

However, as by virtue of Lemma 5.6 we know that $\gamma^* \gamma_*(\alpha) = 2\alpha$ for all $\alpha \in \mathrm{CH}_0(F_L)^-$ and since $\mathrm{CH}_0(F_L)$ is torsion free, the map is in fact bijective.

5.5. The approaches to Theorem 1.4 here and in [31, Theorem 4.7] are different. Shen uses a family of correspondences of degree five for the Fano surfaces F_L, similar to the construction in [32, Section 3] for higher degree rational curves. The action of this correspondence is the composition of the Fano correspondence and its dual, which he identifies with the action $\iota^* - \mathrm{id}$. Our arguments instead rely on techniques due to Murre for cubic threefolds [25] and, at least at first glance, there does not seem to be any link between the two.

5.6. Consider the general situation of Section 5.1, i.e., $\pi \colon F \twoheadrightarrow D$ is a finite morphism of degree two between surfaces with vanishing irregularity and ι is the covering involution with its graph $\Gamma_\iota \subset F \times F$. Then the rational Chow motive $\mathfrak{h}(F) \in \mathrm{Mot}(\mathbb{C})$ decomposes into

$$\mathfrak{h}(F) \simeq \mathfrak{h}(F)^+ \oplus \mathfrak{h}(F)^-,$$

where $\mathfrak{h}(F) := (F, [\Delta_F])$ and $\mathfrak{h}(F)^\pm := (F, (1/2)([\Delta] \pm [\Gamma_\iota]))$. On the other hand,

$$\mathfrak{h}(F) \simeq \mathfrak{h}^0(F) \oplus \mathfrak{h}^2(F) \oplus \mathfrak{h}^4(F).$$

Note that due to $q(F) = 0$, the odd parts $\mathfrak{h}^1(F)$ and $\mathfrak{h}^3(F)$ are both trivial and recall that

$$\mathfrak{h}^0(F) := \bigl(F, p_0 = [x \times F]\bigr) \simeq \mathbb{L}^0,$$
$$\mathfrak{h}^4(F) := \bigl(F, p_4 = [F \times x]\bigr) \simeq \mathbb{L}^2,$$
$$\mathfrak{h}^2(F) := \bigl(F, [\Delta] - p_0 - p_4\bigr).$$

The two decompositions are compatible in the sense that

$$\mathfrak{h}(F)^+ \simeq \mathfrak{h}^0(F) \oplus \mathfrak{h}^2(F)^+ \oplus \mathfrak{h}^4(F) \quad \text{and} \quad \mathfrak{h}(F)^- \simeq \mathfrak{h}^2(F)^-.$$

Eventually, the choice of a polarization on F allows one to write

$$\mathfrak{h}^2(F) \simeq \mathfrak{h}^2(F)_{\mathrm{pr}} \oplus \mathbb{L} \quad \text{and} \quad \mathfrak{h}(F)^- = \mathfrak{h}^2(F)^- \simeq \mathfrak{h}^2(F)^-_{\mathrm{pr}} \oplus \mathbb{L}.$$

5.7. The Chow motive of a smooth cubic fourfold $X \subset \mathbb{P}^5$ naturally decomposes as

$$\mathfrak{h}(X) \simeq \bigoplus_{i=0}^{4} \mathbb{L}^i \oplus \mathfrak{h}^4(X)_{\mathrm{pr}}.$$

The first summand corresponds to the image of $H^*(\mathbb{P}^5, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ while the cohomology of the second is $H^4(X, \mathbb{Q})_{\mathrm{pr}}$. Recall that due to results of Bloch–Srinivas, cf. [43, Proposition 22.27], and Murre [26], the cycle class map

$$\mathrm{CH}^2(X) \otimes \mathbb{Q} \to H^4(X, \mathbb{Q})$$

is injective for the rationally connected variety X. Together with the Hodge conjecture for cubic fourfolds [41], this gives

$$\mathrm{CH}(\mathfrak{h}^4(X)_{\mathrm{pr}}) \otimes \mathbb{Q} = \bigl(\mathrm{CH}_2(X)_{\mathrm{pr}} \oplus \mathrm{CH}_1(X)_{\mathrm{hom}}\bigr) \otimes \mathbb{Q}$$
$$\simeq H^{2,2}(X, \mathbb{Q})_{\mathrm{pr}} \oplus \mathrm{CH}_1(X)_{\mathrm{hom}} \otimes \mathbb{Q}.$$

Now, the Fano correspondence induces a morphism

(5.3) $$\mathfrak{h}^2(F_L)^-_{\mathrm{pr}} \to \mathfrak{h}^4(X)_{\mathrm{pr}}(1)$$

in $\mathrm{Mot}(\mathbb{C})$. Taking Chow groups (with $\mathbb{Q}$-coefficients) gives

$$\mathrm{CH}_0(F_L)^- \oplus \mathrm{CH}_1(F_L)^-_{\mathrm{pr}} \to \mathrm{CH}_1(X)_{\mathrm{hom}} \oplus \mathrm{CH}_2(X)_{\mathrm{pr}}.$$

The first component is nothing but the isomorphism $\mathrm{CH}_0(F_L)^-_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{CH}_1(X)_{\mathrm{hom}}$ proved in Section 5.4. The second defines the map

$$H^{1,1}(F_L, \mathbb{Q})^-_{\mathrm{pr}} \simeq \mathrm{CH}_1(F_L)^-_{\mathrm{pr}} \otimes \mathbb{Q} \to \mathrm{CH}_2(X)_{\mathrm{pr}} \otimes \mathbb{Q} \xrightarrow{\sim} H^{2,2}(X, \mathbb{Q})_{\mathrm{pr}},$$

which according to Proposition 4.1 is an isomorphism.

To conclude the verification of the second part of Theorem 1.4, i.e., the isomorphism $\mathfrak{h}^2(F)_{\mathrm{pr}}^- \simeq \mathfrak{h}^4(X)_{\mathrm{pr}}(1)$, it is enough to observe that (5.3) induces isomorphisms between their Chow groups even after arbitrary base field extension. By Manin's identity principle, combine [16, Lemma 1] and [29, Lemma 3.2], see also [21, Lemma 1.1] and [40, Lemma 4.3], this implies that (5.3) is indeed an isomorphism. This last argument is rather standard and essentially identical to the proof in [9].

A. The topological fundamental group of F_L (by John Christian Ottem)

Let X be a smooth cubic fourfold and $L \subset X$ a line. In this appendix, we prove that the surface $F_L \subset F(X)$ of lines in X meeting L is simply connected.

Theorem A.1. *The surface F_L is simply connected.*

Let F' denote the blow-up of F_L in the point $[L] \in F_L \subset F(X)$. The starting point of the proof is the fact that F' embeds into the projective bundle $\mathbb{P}(\mathcal{T}_X|_L)$ over $L = \mathbb{P}^1$. Taking L general, we may assume that $\mathcal{T}_X|_L \simeq \mathcal{E}$, where $\mathcal{E}$ is the following vector bundle on $\mathbb{P}^1$:

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2).$$

Write $\rho \colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ for the bundle projection. Then the Picard group of $\mathbb{P}(\mathcal{E})$ is generated by $\rho^* \mathcal{O}(1)$ and the relative tautological $\mathcal{O}_\rho(1)$. We will be interested in the following two line bundles

$$\mathcal{L}_1 := \rho^* \mathcal{O}(3) \otimes \mathcal{O}_\rho(2) \quad \text{and} \quad \mathcal{L}_2 := \rho^* \mathcal{O}(3) \otimes \mathcal{O}_\rho(3).$$

The surface $F' \subset \mathbb{P}(\mathcal{E})$ is defined by a section of the vector bundle E appearing as the extension, cf. [20, Proposition 2.3.10]:

(A.1) $$0 \to \mathcal{L}_2 \to E \to \mathcal{L}_1 \to 0.$$

In particular, we have

(A.2) $$F' = V(s_2) \subset V(s_1) \subset \mathbb{P}(\mathcal{E}),$$

where $s_1 \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}_1)$ and $s_2 \in H^0(V(s_1), \mathcal{L}_2|_{V(s_1)})$.

We will prove Theorem A.1 by a Lefschetz-type argument. This is slightly delicate, because the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are not ample. However, as we will see, they still have just enough positivity to make the argument work.

By Corollary A.4 below, the surface F' is deformation equivalent to a surface $S = V(s) \subset D$ for a general divisor $D \in |\mathcal{L}_1|$ and a general section $s \in H^0(D, \mathcal{L}_2|_D)$.

It will be convenient to pick sections x_0, x_1, y_0, y_1, y_2, y_3 such that:

$$H^0(\mathbb{P}(\mathcal{E}), \rho^*\mathcal{O}(1)) = \langle x_0, x_1 \rangle,$$
$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_\rho(1)) = \langle y_0, y_1 \rangle,$$
$$H^0(\mathbb{P}(\mathcal{E}), \rho^*\mathcal{O}(1) \otimes \mathcal{O}_\rho(1)) = \langle x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1, y_2 \rangle,$$
$$H^0(\mathbb{P}(\mathcal{E}), \rho^*\mathcal{O}(2) \otimes \mathcal{O}_\rho(1)) = \langle x_0^2 y_0, x_0^2 y_1, x_0 x_1 y_0, x_0 x_1 y_1, x_0 y_2,$$
$$x_1^2 y_0, x_1^2 y_1, x_1 y_2, y_3 \rangle.$$

In terms of these sections, a basis of $H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}_1)$ is given by the following 25 monomials:

(A.3)
$$x_0^i x_1^{3-i} y_0^2, \quad x_0^i x_1^{3-i} y_0 y_1, \quad x_0^i x_1^{3-i} y_1^2, \quad x_0^i x_1^{2-i} y_0 y_2,$$
$$x_0^i x_1^{2-i} y_1 y_2, \quad x_i y_0 y_3, \quad x_i y_1 y_3, \quad x_i y_2^2, \quad y_2 y_3.$$

Similarly, $H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}_2)$ has a basis consisting of the 38 monomials:

(A.4)
$$x_0^i x_1^{3-i} y_0^3, \dots, x_0^i x_1^{3-i} y_1^3, \quad x_0^i x_1^{2-i} y_0^2 y_2, \dots, x_0^i x_1^{2-i} y_1^2 y_2,$$
$$x_i y_0^j y_1^{2-j} y_3, \quad x_i y_0 y_3, \quad x_i y_1 y_3, y_0 y_2 y_3, \quad y_1 y_2 y_3, \quad y_2^3.$$

Lemma A.2. *The base locus of both linear systems* $|\mathcal{L}_1|$ *and* $|\mathcal{L}_2|$ *is the curve*

$$Z = \mathbb{P}(\mathcal{O}(2)) = V(y_0, y_1, y_2) \subset \mathbb{P}(\mathcal{E}).$$

A general divisor in $|\mathcal{L}_1|$ *is non-singular; a general divisor of* $|\mathcal{L}_2|$ *has multiplicity two along* Z.

Proof. Both base loci must be contained in the base locus of $\rho^*\mathcal{O}(1) \otimes \mathcal{O}_\rho(1)$ which clearly equals Z, as we see using the basis above. Conversely, every monomial in (A.3) and (A.4) is divisible by either y_0, y_1 or y_2, so Z is exactly the base locus.

For the second claim, one easily checks that $x_0^3 y_0^2 + x_1^3 y_0 y_1 + x_0^2 x_1 y_1^2 + y_2 y_3 = 0$ defines a non-singular divisor in $\mathbb{P}(\mathcal{E})$. Similarly, $y_1 y_2 y_3$ has multiplicity two along Z and every other monomial of (A.4) vanishes to order at least two there. ∎

Remark A.3. It is natural to wonder whether the surface F' is actually a complete intersection of divisors in $|\mathcal{L}_1|$ and $|\mathcal{L}_2|$ on $\mathbb{P}(\mathcal{E})$ (or stronger, whether the sequence (A.1) splits). However, this is excluded by Lemma A.2, as any such complete intersection will be singular along Z, but F' is smooth.

In fact, for any $D \in |\mathcal{L}_1|$ the restriction $\mathcal{L}_2|_D$ admits a section which is not a restriction from $\mathbb{P}(\mathcal{E})$. For example, for $D = V(s_1)$ one can take the section defining $F' \subset V(s_1)$. Indeed, this follows from the exact sequence

(A.5)
$$0 \to \mathcal{O}_\rho(1) \to \mathcal{L}_2 \to \mathcal{L}_2|_{V(s_1)} \to 0$$

and

$$H^1(\mathbb{P}(\mathcal{E}), \mathcal{O}_\rho(1)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq \mathbb{C},$$

and

$$H^1(\mathbb{P}(\mathcal{E}), \mathcal{L}_2) \simeq H^1(\mathbb{P}^1, S^2(\mathcal{E}^\vee) \otimes \mathcal{O}_{\mathbb{P}^1}(3)) = 0.$$

By the sequence (A.5), the line bundle $\mathcal{L}_2|_D$ has a 37-dimensional space of global sections for every $D \in |\mathcal{L}_1|$. This implies that the parameter space of surfaces of the form $V(s_2) \subset V(s_1)$ as in (A.2) is a projective bundle over a projective space, hence it is a smooth projective variety.

Corollary A.4. *The parameter space of surfaces F' appearing as in a flag* (A.2) *is irreducible.*

Let $X := D \setminus Z$ and let $\pi \colon X \to \mathbb{P}^{36}$ be the morphism defined by the linear system $|\mathcal{L}_2|_D|$.

Lemma A.5. *The morphism π is generically injective, and contracts no divisor.*

Proof. The 38 sections in (A.4) define a morphism

$$\tau \colon \mathbb{P}(\mathcal{E}) \setminus Z \to \mathbb{P}^{37}.$$

We claim that τ contracts only the surface $V(y_0, y_1)$ to a point and is an embedding on the complement $\mathbb{P}(\mathcal{E}) \setminus V(y_0, y_1)$. Note first that τ is a toric morphism. This implies that the locus of points $z \in \mathbb{P}^{37}$ for which the fibre $\tau^{-1}(z)$ has positive dimension is a union of linear subvarieties of $\mathbb{P}^{37}$. Given the 38 monomials above, it is easy (for instance, in Macaulay2) to compute the fibres $\tau^{-1}(e_0), \ldots, \tau^{-1}(e_{37})$ over the standard coordinate points. Doing this, we find that there is a single point with positive-dimensional fibre, namely

$$\tau^{-1}\big([0 : \cdots : 0 : 1]\big) = V(y_0, y_1);$$

for the remaining points, the fibre $\tau^{-1}(e_k)$ is either empty or consists of a single point. By semicontinuity of fibre dimension, this implies that τ is injective on the complement $\mathbb{P}(\mathcal{E}) \setminus V(y_0, y_1)$.

Now, note that the intersection $\Gamma = D \cap V(y_0, y_1)$ is 1-dimensional. This follows by Bertini, because the base locus of $\mathcal{L}_1$ is Z and D is assumed to be general. It follows that the restriction

$$\tau|_{D \setminus Z} \colon D \setminus Z \to \mathbb{P}^{37}$$

contracts at most finitely many curves to points. This morphism is defined by a subsystem of $|\mathcal{L}_2|_D|$, so the same conclusion holds for π. ∎

Lemma A.6. *The generic $D \in |\mathcal{L}_1|$ is a smooth projective rational threefold, and hence simply connected.*

Proof. The generic $D \in |\mathcal{L}_1 = \rho^*(\mathcal{O}(3) \otimes \mathcal{O}_\rho(2))|$ is smooth, see Lemma A.2, and the projection $D \to \mathbb{P}^1$ is generically a quadric bundle over $\mathbb{P}^1$. Hence, D is rational. ∎

Proof of Theorem A.1. Let $H \subset \mathbb{P}^{36}$ denote a general hyperplane. Since π defines a small birational morphism, we may apply a theorem of Goresky–MacPherson [17, pp. 150–151] (where with the notation there $\hat{n} = 2$), and deduce that the natural map

$$\pi_i(\pi^{-1}(H)) \to \pi_i(X)$$

is an isomorphism for $i < 2$. However, $\pi_1(X) = 0$ by Lemma A.6, because D is simply connected and X is obtained from D by removing a closed subset of real codimension four.

Let now $S \in |\mathcal{L}_2|_D|$ be a general divisor. Then S is non-singular and irreducible, because F is of this form. From the previous paragraph, we know that $S \setminus Z$, which equals $\pi^{-1}(H)$ for a general H, is simply connected. As $\pi_1(S \setminus Z)$ surjects onto $\pi_1(S)$, we deduce that S is simply connected as well. Finally, we note that F' is deformation equivalent to S, so F' and hence F is simply connected. This completes the proof of Theorem A.1. ∎

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