

Corrigendum to “Morse boundaries of proper geodesic metric spaces”

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Abstract. We introduce refined Morse gauges to correct the proof and statement of Lemma 2.10 of [Groups Geom. Dyn. 11, 1281–1306 (2017)] written by the first author.

1. Introduction

We adopt the notation of [1]. In [1, Lemma 2.10], the first author claimed that if a sequence of N -Morse geodesic rays converged uniformly on compact sets to a geodesic ray, then that ray would be N -Morse. This does not follow and we provide a counterexample below (Example 2.3). This corrigendum will do three things: We first introduce the notion of refined Morse gauge and (quasi)-geodesics and use them to prove a corrected version of [1, Lemma 2.10]. We then reprove [1, Corollary 3.2] without using [1, Lemma 2.10] of the same paper so that one can define the Morse boundary without reference to refined Morse gauges. Finally, we show that for any proper geodesic metric space, the Morse boundary defined with Morse gauges and the refined Morse boundary, defined by refined Morse gauges only, are homeomorphic. A key step in this is Lemma 2.4, which associates with any Morse gauge N and a refined Morse gauge \hat{N} such that all N -Morse geodesics are \hat{N} -Morse; essentially, the lemma says that one can always replace Morse gauges with refined Morse gauges.

A major consequence of the incorrectness of [1, Lemma 2.10] is that the N -Morse strata, $\partial_M^N X$, are not necessarily compact. Many papers which use the Morse boundary rely on [1, Lemma 2.10] either directly or by using [2, Theorem 3.14]. Thankfully, by Theorem 2.11 anywhere where these results are called upon, unless relying specifically on Morse gauges that are not refined, one may simply substitute Morse gauges for refined Morse gauges and the same conclusion will follow. For instance, [2, Theorem 3.14] holds if one considers only refined Morse gauges.

2. Refined Morse boundary

We now introduce the notion of refined Morse gauge and geodesic.

Definition 2.1. A refined Morse gauge is a Morse gauge $N: \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with the following additional properties:

- (1) N is non-decreasing.
- (2) N is continuous in the second coordinate.

We denote the collection of refined Morse gauges $\widehat{\mathcal{M}}$.

To correct the proof of [1, Lemma 2.10], we do not actually need that a Morse gauge N is non-decreasing, but it is generally useful. In particular, given a Morse gauge N , one has an associated constant δ_N (to be thought of as a hyperbolicity constant) which is useful in several ways. If $N \leq N'$ are non-decreasing Morse gauges, then $\delta_N \leq \delta_{N'}$, which does not hold in general otherwise (unless one redefines δ_N as proposed in Remark 2.8). Furthermore, it is an intermediate step in the proof of Lemma 2.4. We now define a refined Morse quasi-geodesic.

Definition 2.2. Let X be a metric space, N a refined Morse gauge, and I a closed interval of \mathbb{R} . The quasi-geodesic $\gamma: I \rightarrow X$ is a *refined N -Morse geodesic* if for any (λ, ε) -quasi-geodesic σ with endpoints on γ , the image of σ is contained in the closed $N(\lambda, \varepsilon)$ -neighborhood of γ .

There are two differences between the definition of Morse geodesics and refined-Morse geodesics. The first is that we require the use of a *refined* Morse gauge. The second is that we ask for the quasi-geodesic σ to be in the *closed* neighborhood of γ .

We now present the counterexample to [1, Lemma 2.10].

Example 2.3. Consider the space X formed by taking the hyperbolic plane \mathbb{H}^2 and gluing in a line segment connecting two points $p, q \in \mathbb{H}^2$ of length $d(p, q)$. Thus in X there are two geodesics between p and q , one in \mathbb{H}^2 and the other along the line segment. Let α be a geodesic ray with basepoint p passing through q whose image is in \mathbb{H}^2 and let $\{\alpha_i\}$ be a collection of geodesic rays with image in \mathbb{H}^2 and basepoint p converging to α . Since X is hyperbolic, every geodesic is uniformly N -Morse for some N , and because of the extra segment we glued on, $N(1, 0) \neq 0$. Define

$$N'(\lambda, \varepsilon) = \begin{cases} 0 & \lambda = 1, \varepsilon = 0, \\ N(\lambda, \varepsilon) & \text{else.} \end{cases}$$

Since X is uniquely geodesic except for geodesics passing through p and q , we know that the α_i are N' -Morse. Since α is not N' -Morse, we have our counterexample.

The Morse gauge N' is not a refined Morse gauge because it is not continuous in the second coordinate. To see this, we note that $N'(1, \varepsilon) = N(1, \varepsilon)$ must be at least $\frac{1}{2}d(p, q)$ since a geodesic is a $(1, \varepsilon)$ -quasi-geodesic. So N' is discontinuous at 0.

The following lemma can be used in several contexts to harmlessly pass from Morse gauges to refined Morse gauges.

Lemma 2.4. *For every Morse gauge N , there exists a refined Morse gauge \widehat{N} with the property that every N -Morse geodesic is also an \widehat{N} -Morse geodesic.*

Proof. We will first construct a non-decreasing Morse gauge from N . This is done by setting

$$N'(\lambda, \varepsilon) = \inf_{\lambda', \varepsilon'} \{N(\lambda', \varepsilon') \mid \lambda' \geq \lambda, \varepsilon' \geq \varepsilon\} + 1.$$

We see that by construction N' is non-decreasing.

To arrange that N' is continuous in the second coordinate, we first note that since we have arranged that N' is non-decreasing, then if we fix a $\lambda \in \mathbb{R}_{\geq 1}$, the function $N'(\lambda, \varepsilon): \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is non-decreasing and thus (Riemann) integrable. We then set

$$\widehat{N}(\lambda, \varepsilon) = \int_{\varepsilon}^{\varepsilon+1} N'(\lambda, t) dt.$$

A standard calculation using that $\widehat{N}(\lambda, \cdot)$ is bounded on compact intervals shows that \widehat{N} is continuous in the second coordinate. We also note that, since N' is non-decreasing, we have $N' \leq \widehat{N}$.

Let α be an N -Morse geodesic and let σ be a (λ, ε) -geodesic with endpoints on α . We note that σ is also an $N(\lambda', \varepsilon')$ -quasi-geodesic for any $\lambda' \geq \lambda$ and $\varepsilon' \geq \varepsilon$. Thus by construction, we know that σ must be in the $N'(\lambda, \varepsilon)$ -neighborhood of α and since $N' \leq \widehat{N}$, we conclude that σ is in the $\widehat{N}(\lambda, \varepsilon)$ -neighborhood of α . ■

We now state and prove the corrected version of [1, Lemma 2.10].

Lemma 2.5 ([1, Lemma 2.10] redux). *Let X be a geodesic metric space and suppose that $\{\gamma_i: \mathbb{R}_{\geq 0} \rightarrow X\}$ is a sequence of refined N -Morse geodesic rays that converge uniformly on compact sets to a geodesic ray γ . Then γ is refined N -Morse.*

Proof. Let $\eta > 0$. Let σ be a (λ, ε) -quasi-geodesic with endpoints $\gamma(s)$ and $\gamma(t)$ on γ . Since the γ_i converge uniformly on compact sets and are refined N -Morse, there exists an $I \in \mathbb{N}$ such that for any $i \geq I$, we have $d(\gamma_i(t), \gamma(t)) \leq \eta$ on $\gamma|_{[s,t]}$. It follows that γ is N' -Morse where $N'(\lambda, \varepsilon) = N(\lambda, \varepsilon + \eta) + \eta$. Since this is true for all $\eta > 0$ and N is continuous in the second coordinate, we can conclude that σ is in the closed $N(\lambda, \varepsilon)$ neighborhood of $\gamma|_{[s,t]}$. It follows that γ is refined N -Morse. ■

We wish to reprove [1, Corollary 3.2] to avoid using [1, Lemma 2.10] from the same paper so that we may define the Morse boundary without reference to refined Morse geodesics.

Lemma 2.6. *Let X be a geodesic metric space and let $\alpha: [0, A] \rightarrow X$ be an N -Morse geodesic and $\beta: [0, A] \rightarrow X$ be any geodesic. Also assume that $\alpha(0) = \beta(0)$. Then for $t \in [0, A - d(\alpha(A), \beta(A))]$ we have $d(\alpha(t), \beta(t)) < 4N(3, 0)$.*

Proof. Let $\alpha(x)$ be the closest point on α to $\beta(b)$. By the triangle inequality $x \geq A - d(\alpha(A), \beta(A))$. It follows as in the proof of Case 1 of [1, Proposition 2.4] that the concatenation $\sigma = \beta([0, A]) \cup [\beta(A), \alpha(x)]$ is a $(3, 0)$ -quasi-geodesic. Since α is N -Morse, we know that σ is in the $N(3, 0)$ -neighborhood of $\alpha([0, y])$, and by [1, Lemma 2.1] we can bound the Hausdorff distance between $\alpha[0, y]$ and σ by $2N(3, 0)$. By a standard argument we may conclude that for all $t \in [0, A - d(\alpha(A), \beta(A))]$, we have $d(\alpha(t), \beta(t)) < 4N(3, 0)$. ■

We now reprove [1, Corollary 3.2].

Lemma 2.7 (Corollary 3.2 redux). *Let N and N' be Morse gauges such that every N -Morse geodesic is also N' -Morse. Then the natural inclusion $i: \partial_M^N X_p \rightarrow \partial_M^{N'} X_p$ is continuous. In particular, if $N \leq N'$ then this condition is satisfied.*

Proof. Let U be an open set in $\partial_M^{N'} X_p$. We wish to show that $i^{-1}(U)$ is open. Let $x \in i^{-1}(U)$ and α_x a geodesic ray representing x . Since i is an inclusion and U is open in $\partial_M^{N'} X_p$, then there exists an $j \in \mathbb{N}$ so that $V_j^{N'}(\alpha_x) \subset U$. Let $k = j + 3\delta_N$. Applying Lemma 2.6 thinking of α_x as an N' -Morse geodesic, we know that for any $y \in V_k^N(\alpha_x)$ and any geodesic α_y representing y , that $d(\alpha_x(t), \alpha_y(t)) < 4N'(3, 0)$ for $t \in [0, j]$. So $i(V_k^N(\alpha_x)) \subset V_j^{N'}(\alpha_x) \subset U$. Since we can do this for any $x \in i^{-1}(U)$, we can conclude that $i^{-1}(U)$ is open. ■

Remark 2.8. We note that one can also prove this using [1, Corollary 2.5], but we prove Lemma 2.6 because it is a useful lemma of independent interest. In fact, in future research on the Morse boundary, one might wish to redefine δ_N to be $4N(3, 0)$.

Definition 2.9. Let X be a proper geodesic metric space and let $p \in X$. Let $\alpha: [0, \infty) \rightarrow X$ be an N -Morse geodesic ray with $\alpha(0) = p$ and for each positive integer n , let $V_n(\alpha)$ be the set of geodesic rays γ such that $d(\alpha(t), \gamma(t)) < \delta_N$ for all $t < n$. Then by [1, Lemma 3.1], this forms a fundamental system of neighborhoods of $[\alpha]$ in $\partial_M^N X_p$. We topologize $\partial_M^N X_p$ with the topology arising from this fundamental system of neighborhoods.

Let \mathcal{M} be the set of all Morse gauges with the usual partial order. The Morse boundary is

$$\partial_M X_p = \varinjlim_{\mathcal{M}} \partial_M^N X_p$$

with the induced direct limit topology.

Definition 2.10. The refined Morse boundary, denoted $\partial_{\widehat{\mathcal{M}}} X_p$, is defined the same way as the Morse boundary in [1] but rather than considering all Morse gauges \mathcal{M} , we consider only refined Morse gauges $\widehat{\mathcal{M}}$.

Theorem 2.11. *Let X be a proper geodesic metric space. Then the inclusion $\partial_{\widehat{\mathcal{M}}} X \hookrightarrow \partial_M X$ induces a homeomorphism.*

Proof. This follows from Lemmas 2.4 and 2.7, and the universal property of direct limits. ■

References

- [1] M. Cordes, [Morse boundaries of proper geodesic metric spaces](#). *Groups Geom. Dyn.* **11** (2017), no. 4, 1281–1306 Zbl [1423.20043](#) MR [3737283](#)
- [2] M. Cordes and D. Hume, [Stability and the Morse boundary](#). *J. Lond. Math. Soc. (2)* **95** (2017), no. 3, 963–988 Zbl [1475.20069](#) MR [3664526](#)

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