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Inequalities about the area bounded by three cevian lines of a triangle

Yagub N. Aliyev

Yagub Aliyev is assistant professor and program director at ADA University (Baku, Azerbaijan). His research interests lie between Sturm–Liouville theory, elementary number theory, and geometric inequalities. Besides writing papers in these fields, he also likes to compose new problems and many of them were published in magazines such as Crux Mathematicorum, American Mathematical Monthly, and Elemente der Mathematik. Apart from mathematics, he enjoys learning languages (German, French, Arabic), doing sports, and travelling with his wife and five children.

1 Introduction

Consider cevians AD, BE, and CF of a triangle ABC (see Figure 1). Denote

$$\frac{|BD|}{|DC|} = \lambda_1, \quad \frac{|CE|}{|EA|} = \lambda_2, \quad \text{and} \quad \frac{|AF|}{|FB|} = \lambda_3.$$

Die geometrischen Ungleichungen und die neue Geometrie des Dreiecks sind Fortsetzungen der klassischen euklidischen Geometrie bis in unsere heutige Zeit. Diese beiden Zweige der Geometrie treffen manchmal aufeinander, haben aber grösstenteils unterschiedliche Ansätze entwickelt. Die geometrischen Ungleichungen konzentrieren sich vermehrt auf Extremalprobleme, bei denen die Verwendung von algebraischen Ungleichungen und Differentialrechnung im Vordergrund steht. Die neue Geometrie des Dreiecks befasst sich vorwiegend mit Fragen der Kollinearität und Kopunktalität, insbesondere mit bemerkenswerten Dreieckszentren. In der vorliegenden Arbeit wird eine Verbindung zwischen diesen beiden Zweigen der Geometrie gezeigt, indem eine Verallgemeinerung der Theoreme von Schlömilch und Zetel über kopunktale Geraden im Dreieck mithilfe einer scharfen geometrischen Ungleichung über das Verhältnis der Dreiecksflächen bewiesen wird. Die Ungleichung selbst wird mithilfe der diskreten Version der Hölderschen Ungleichung und des Steiner-Routh-Theorems bewiesen. Ausserdem wird eine neue scharfe Verfeinerung der Ungleichung von J. F. Rigby bewiesen, welche ihrerseits das Möbius-Theorem über die Flächen von Dreiecken verallgemeinert, die durch Cevane des Dreiecks gebildet werden.

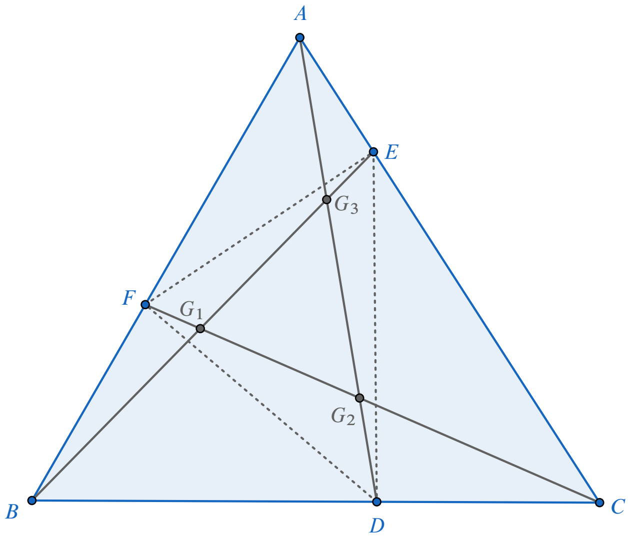

Figure 1. Steiner–Routh's theorem.

Denote also $BE \cap CF = G_1$, $AD \cap CF = G_2$, and $AD \cap BE = G_3$. There is a result in geometry known as Steiner–Routh's theorem which says that

$$\frac{\text{Area}(\triangle G_1 G_2 G_3)}{\text{Area}(\triangle ABC)} = \frac{(\lambda_1\lambda_2\lambda_3 - 1)^2}{(\lambda_1\lambda_2 + \lambda_1 + 1)(\lambda_2\lambda_3 + \lambda_2 + 1)(\lambda_3\lambda_1 + \lambda_3 + 1)}. \tag{1}$$

Steiner–Routh's theorem which is sometimes called just Routh's theorem was discussed in many papers and books. See [1, 2], [7, p. 276], [8], [10, pp. 211, 212], [11, pp. 41–42], [13, p. 33], [15, p. 382], [16, 20, 23], [29, p. 89], [30–32], [33, p. 166], [36, 37], and their references. Steiner–Routh's formula was generalized in many different directions [6, 9, 18, 19, 35, 40]. There is a peculiar special case called *One seventh area triangle* or *Feynman's triangle* which corresponds to case $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and attracted much attention because it can also be proved using dissections (see e.g. [17, 20], [34, p. 9]). Some of these sources also mention the following formula:

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABC)} = \frac{\lambda_1\lambda_2\lambda_3 + 1}{(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1)}. \tag{2}$$

Formulas (1) and (2) generalize Ceva's ($\lambda_1\lambda_2\lambda_3 = 1$) and Menelaus' ($\lambda_1\lambda_2\lambda_3 = -1$) theorems, respectively. In these cases, the areas of $\triangle G_1 G_2 G_3$ and $\triangle DEF$ are equal to zero, which is equivalent to say that cevians AD, BE, and CF are concurrent, and points D, E, and F are collinear, respectively. In general, the vertices of a triangle do not necessarily coincide if its area is zero. It is possible that the vertices of the triangle are just collinear. But this is not possible for $\triangle G_1 G_2 G_3$, because otherwise points A, B, and C would also be collinear. In the paper, we will apply this idea to find a new proof for the following theorem and its generalization.

Schlömilch's theorem. *The lines connecting the midpoints of the sides of a triangle and the midpoints of the corresponding altitudes are concurrent.*

O. Schlömilch's theorem was discussed in many papers and books. See for example [3, pp. 256, 304], [12, p. 133], [24], [26, pp. 34, 37], [38, 41]. In [14, p. 215 (Corollary)], [27, Problem 5.135], it was mentioned that the point of concurrency in Schlömilch's theorem is the Lemoine (symmedian) point of the triangle. In [39], S. I. Zetel generalized the result by Schlömilch as follows (see Figure 2).

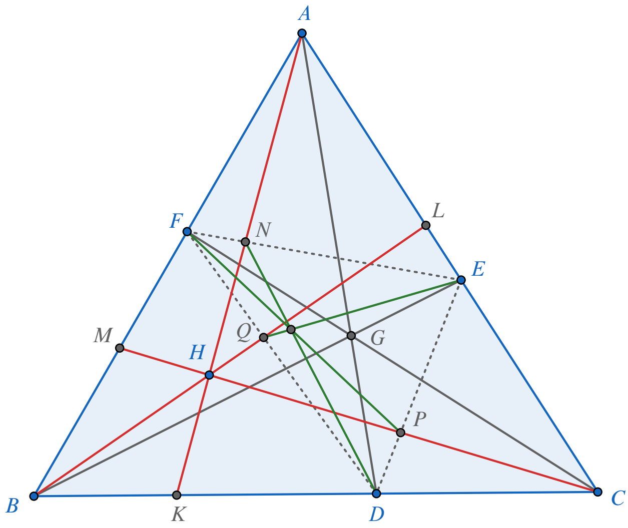

Figure 2. Zetel's generalization of Schlömilch's theorem.

Zetel's theorem. *Let the trio of cevians AD, BE, and CF of a triangle ABC be concurrent at point G. Let another trio of cevians AK, BL, and CM of triangle ABC be concurrent at point H. Denote $AK \cap EF = N$, $BL \cap DF = Q$, and $CM \cap DE = P$. Then lines DN, EQ, and FP are concurrent.*

From the point of view of projective geometry, this generalization is equivalent to Schlömilch's theorem. Indeed, by Desarques' theorem, the intersection points $EF \cap BC$, $DF \cap AC$, and $DE \cap AB$ are on a line. Let us apply a projective transformation sending this line to infinity. We will continue to use the original notation for their images under these transformations. This transformation forces $EF \parallel BC$, $DF \parallel AC$, $DE \parallel AB$, and therefore points D, E, F are the midpoints of sides BC, AC, AB, respectively. Then apply affine transformations changing AK and BL to the corresponding altitudes of $\triangle ABC$. Then CM is also the altitude of $\triangle ABC$, and therefore we return to Schlömilch's theorem. In the current paper, we will obtain Zetel's generalization of Schlömilch's theorem and other similar theorems as corollaries of inequalities about triangular areas in the corresponding configurations. We will prove some of these inequalities using a discrete version of Hölder's inequality [5, p. 20].

Hölder's inequality (Discrete case). *If x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$ are non-negative numbers, $p_j > 1$, and $\sum_{j=1}^{m} \frac{1}{p_j} = 1$, then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} x_{ij} \le \prod_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \tag{3}$$

Hölder's inequality made it possible to prove the inequalities in the current paper without any use of calculus.

We also considered the following result by J. F. Rigby [28] (see also [21, p. 340]).

Rigby's inequality. *Let p, q, r, x, and y denote the areas of $\triangle AEF$, $\triangle BFD$, $\triangle CDE$, $\triangle DEF$, and $\triangle G_1G_2G_3$ (Figure 1). Then*

$$x^3 + (p + q + r)x^2 - 4pqr \ge 0, \tag{4}$$

with equality if and only if cevians AD, BE, and CF are concurrent.

The equality case is known as Möbius' theorem [22, p. 198] (see also [4, p. 95]).

Möbius' theorem. *If cevians AD, BE, and CF are concurrent, then*

$$x^3 + (p + q + r)x^2 - 4pqr = 0.$$

In the current paper, we will prove the following refinement of inequality (4):

$$x^3 + (p + q + r)x^2 - 4pqr \ge x^2 y. \tag{5}$$

Interesting inequalities involving the areas in these configurations also appeared in [25].

2 Main results

First, a general sharp inequality about the areas of triangles formed by cevians of a triangle will be proved. After the proof, its special cases corresponding to concurrent cevians will be discussed.

Theorem 2.1. *Let D and K, E and L, F and M be arbitrary points on sides BC, AC, and AB, respectively, of a triangle ABC. Denote $AK \cap EF = N$, $BL \cap DF = Q$, $CM \cap DE = P$, $DN \cap EQ = R$, $FP \cap EQ = S$, and $FP \cap DN = T$. Denote also*

$$\frac{|BD|}{|DC|} = \lambda_1, \quad \frac{|CE|}{|EA|} = \lambda_2, \quad \frac{|AF|}{|FB|} = \lambda_3, \quad \frac{|BK|}{|KC|} = u, \quad \frac{|CL|}{|LA|} = v, \quad \frac{|AM|}{|MB|} = w.$$

Then

$$\frac{\text{Area}(\triangle RST)}{\text{Area}(\triangle DEF)} \le \frac{(\lambda_1\lambda_2\lambda_3 uvw - 1)^2}{\left(\sqrt[3]{(\lambda_1\lambda_2\lambda_3 uvw)^2} + \sqrt[3]{\lambda_1\lambda_2\lambda_3 uvw} + 1\right)^3}. \tag{6}$$

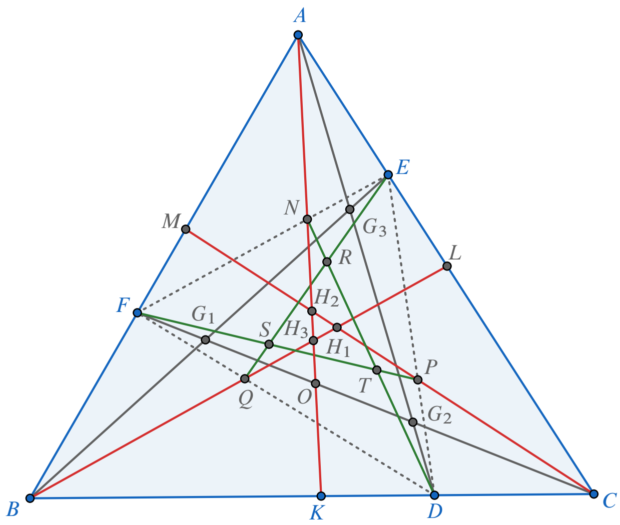

Figure 3. Inequality about the ratio of areas of $\triangle RST$ and $\triangle DEF$.

Proof. Let O be intersection point of lines AK and CF (see Figure 3). By Menelaus' theorem,

$$\frac{|BK|}{|KC|} \cdot \frac{|CO|}{|OF|} \cdot \frac{|FA|}{|AB|} = 1.$$

Then

$$\frac{|CO|}{|OF|} = \frac{1+\lambda_3}{\lambda_3} \cdot \frac{1}{u}.$$

Similarly, by Menelaus' theorem,

$$\frac{|CO|}{|OF|} \cdot \frac{|FN|}{|NE|} \cdot \frac{|EA|}{|AC|} = 1.$$

Then

$$\frac{|FN|}{|NE|} = \alpha := \frac{u\lambda_3(1+\lambda_2)}{1+\lambda_3}.$$

Similarly,

$$\frac{|DQ|}{|QF|} = \beta := \frac{v\lambda_1(1+\lambda_3)}{1+\lambda_1}, \qquad \frac{|EP|}{|PD|} = \gamma := \frac{w\lambda_2(1+\lambda_1)}{1+\lambda_2}.$$

By applying formula (1) to $\triangle DEF$ and points N, Q, P on its sides, and noting that orientation has changed, we obtain

$$\frac{\text{Area}(\triangle RST)}{\text{Area}(\triangle DEF)} = \frac{(\alpha\beta\gamma - 1)^2}{(\alpha\gamma + \alpha + 1)(\beta\alpha + \beta + 1)(\gamma\beta + \gamma + 1)}. \tag{7}$$

By Hölder's inequality (3),

$$(\alpha\gamma + \alpha + 1)(\beta\alpha + \beta + 1)(\gamma\beta + \gamma + 1) \geq \left(\sqrt[3]{(\alpha\beta\gamma)^2} + \sqrt[3]{\alpha\beta\gamma} + 1\right)^3, \tag{8}$$

with equality only when $\alpha = \beta = \gamma$. From (7) and (8), it follows that

$$\frac{\text{Area}(\triangle RST)}{\text{Area}(\triangle DEF)} \leq \frac{(\alpha\beta\gamma - 1)^2}{\left(\sqrt[3]{(\alpha\beta\gamma)^2} + \sqrt[3]{\alpha\beta\gamma} + 1\right)^3}. \tag{9}$$

Since $\alpha\beta\gamma = \lambda_1\lambda_2\lambda_3 uvw$, (6) follows from (9). The equality case in (6) holds true when

$$\frac{u\lambda_3(1+\lambda_2)}{1+\lambda_3} = \frac{v\lambda_1(1+\lambda_3)}{1+\lambda_1} = \frac{w\lambda_2(1+\lambda_1)}{1+\lambda_2}. \qquad \blacksquare$$

In particular, if $\lambda_1\lambda_2\lambda_3 uvw = 1$ in (6), then $\text{Area}(\triangle RST) = 0$ and therefore lines DN, EQ, and FP are concurrent. This generalizes Schlömilch's theorem even further (Figure 4).

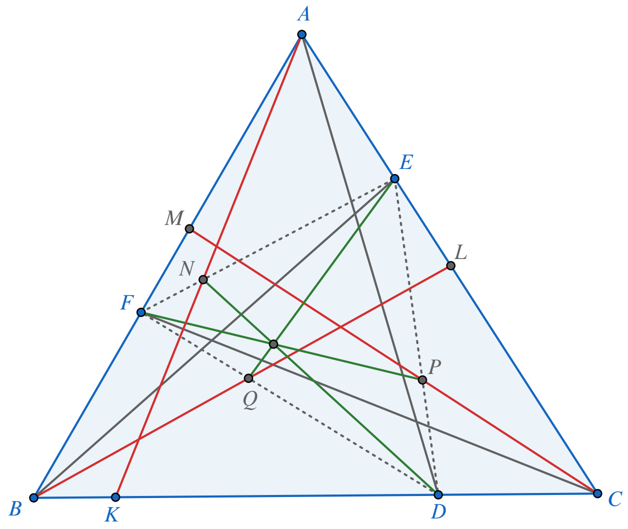

Figure 4. Generalization of Zetel's theorem.

Corollary 2.2. *Let D and K, E and L, F and M be points on sides BC, AC, and AB, respectively, of a triangle ABC. Set $AK \cap EF = N$, $BL \cap DF = Q$, $CM \cap DE = P$. If*

$$\frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} \cdot \frac{|AF|}{|FB|} \cdot \frac{|BK|}{|KC|} \cdot \frac{|CL|}{|LA|} \cdot \frac{|AM|}{|MB|} = 1,$$

then DN, EQ, and FP are concurrent.

The special case $uvw = 1$ of Theorem 2.1 is also of interest (Figure 5).

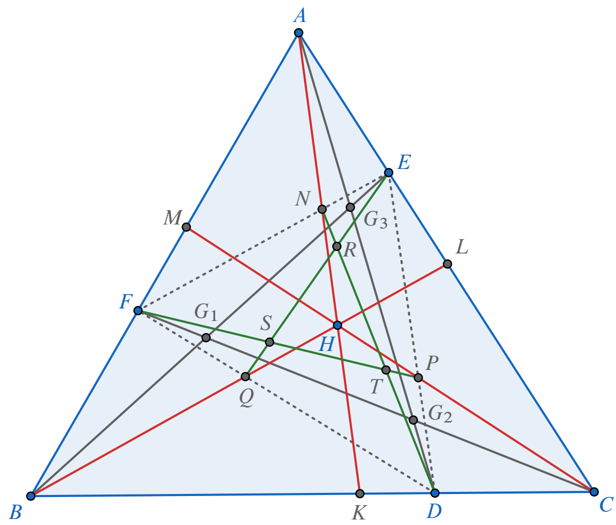

Figure 5. A new proof of Zetel's generalization of Schlömilch's theorem.

Corollary 2.3. *Let D, E, and F be arbitrary points on sides BC, AC, and AB, respectively, of a triangle ABC. Let cevians AK, BL, and CM of triangle ABC be concurrent at point H. Denote $AK \cap EF = N$, $BL \cap DF = Q$, $CM \cap DE = P$, $DN \cap EQ = R$, $FP \cap EQ = S$, and $FP \cap DN = T$. Denote also*

$$\frac{|BD|}{|DC|} = \lambda_1, \quad \frac{|CE|}{|EA|} = \lambda_2, \quad \text{and} \quad \frac{|AF|}{|FB|} = \lambda_3.$$

Then

$$\frac{\operatorname{Area}(\triangle RST)}{\operatorname{Area}(\triangle DEF)} \leq \frac{(\lambda_1\lambda_2\lambda_3 - 1)^2}{\left(\sqrt[3]{(\lambda_1\lambda_2\lambda_3)^2} + \sqrt[3]{\lambda_1\lambda_2\lambda_3} + 1\right)^3}.$$

Proof. Denote as before

$$\frac{|BK|}{|KC|} = u, \quad \frac{|CL|}{|LA|} = v, \quad \text{and} \quad \frac{|AM|}{|MB|} = w.$$

Since $uvw = 1$ (Ceva's theorem), $\alpha\beta\gamma = \lambda_1\lambda_2\lambda_3$, and therefore the inequality follows from (6). The equality case holds true when

$$u = \frac{1+\lambda_3}{1+\lambda_2}\sqrt[3]{\frac{\lambda_1\lambda_2}{\lambda_3^2}}, \quad v = \frac{1+\lambda_1}{1+\lambda_3}\sqrt[3]{\frac{\lambda_2\lambda_3}{\lambda_1^2}}, \quad w = \frac{1+\lambda_2}{1+\lambda_1}\sqrt[3]{\frac{\lambda_1\lambda_3}{\lambda_2^2}}. \qquad \blacksquare$$

Note that if $\lambda_1\lambda_2\lambda_3 = 1$, then Corollary 2.3 implies Zetel's generalization of Schlömilch's theorem. Denote $BE \cap CF = G_1$, $AD \cap CF = G_2$, and $AD \cap BE = G_3$. We can also observe that if $H \in \triangle G_1G_2G_3$, then $\triangle RST \subset \triangle G_1G_2G_3$, and therefore

$$\operatorname{Area}(\triangle RST) < \operatorname{Area}(\triangle G_1G_2G_3). \tag{10}$$

In general, (10) is not always true. For example, if $\lambda_1 = 1$, $\lambda_2 = 0.001$, $\lambda_3 = 1$, $u = \frac{1}{3}$, $v = \frac{3}{40}$, $w = 40$, then by (1), (2), and (7),

$$\begin{aligned}\frac{\text{Area}(\triangle RST)}{\text{Area}(\triangle G_1G_2G_3)} &= \frac{\lambda_1\lambda_2\lambda_3+1}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)} \\ &\quad \times \frac{(\lambda_1\lambda_2+\lambda_1+1)(\lambda_2\lambda_3+\lambda_2+1)(\lambda_3\lambda_1+\lambda_3+1)}{(\alpha\gamma+\alpha+1)(\beta\alpha+\beta+1)(\gamma\beta+\gamma+1)} \\ &\approx 1.079 > 1. \end{aligned} \tag{11}$$

By considering the limiting cases $\lambda_1 = \varepsilon$, $\lambda_2 = \varepsilon$, $\lambda_3 = \varepsilon^2$, $u = \varepsilon$, $v = \varepsilon$, $w = \frac{1}{\varepsilon^2}$, where $\varepsilon \to 0^+$ and $\varepsilon \to +\infty$, we can see that the ratio of areas in (11) can be arbitrarily small and arbitrarily large positive numbers, respectively.

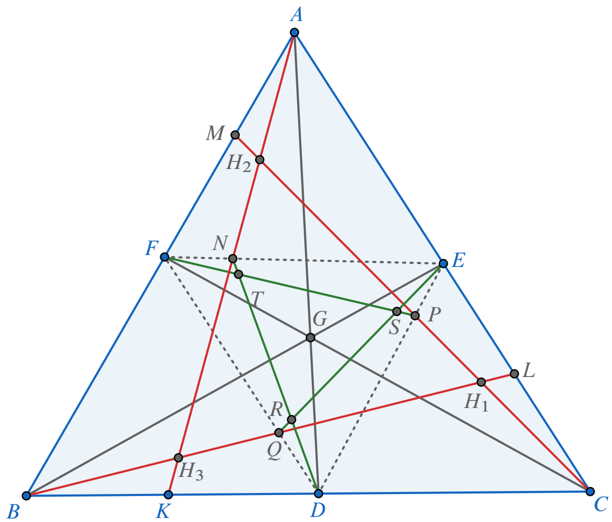

Figure 6. Comparison of areas of $\triangle RST$ and $\triangle H_1H_2H_3$.

Let us now consider the special case $\lambda_1\lambda_2\lambda_3 = 1$ of the configuration in Theorem 2.1 ($G = G_1 = G_2 = G_3$, Figure 6). From (7), we obtain

$$\frac{\text{Area}(\triangle RST)}{\text{Area}(\triangle DEF)} = \frac{(uvw-1)^2}{(\alpha\gamma+\alpha+1)(\beta\alpha+\beta+1)(\gamma\beta+\gamma+1)}. \tag{12}$$

Denote $BL \cap CM = H_1$, $AK \cap CM = H_2$, and $AK \cap BL = H_3$. By equality (1) for $\triangle H_1H_2H_3$,

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle H_1H_2H_3)} = \frac{(uv+u+1)(vw+v+1)(wu+w+1)}{(uvw-1)^2}. \tag{13}$$

By multiplying equalities (2), (12), and (13), we obtain

$$\frac{\mathrm{Area}(\triangle RST)}{\mathrm{Area}(\triangle H_1H_2H_3)} = \frac{(uv+u+1)(vw+v+1)(wu+w+1)}{(\alpha\gamma+\alpha+1)(\beta\alpha+\beta+1)(\gamma\beta+\gamma+1)} \times \frac{2}{(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)}. \tag{14}$$

We observe that if $G \in \triangle H_1H_2H_3$, then $\triangle RST \subset \triangle H_1H_2H_3$, and therefore

$$\mathrm{Area}(\triangle RST) < \mathrm{Area}(\triangle H_1H_2H_3). \tag{15}$$

In general, (15) is not always true. For example, if $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$, $u = 0.01$, $v = 1$, $w = 20$, then by (14),

$$\frac{\mathrm{Area}(\triangle RST)}{\mathrm{Area}(\triangle H_1H_2H_3)} \approx 1.19 > 1. \tag{16}$$

By considering the limiting cases $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $u = \varepsilon$, $v = 1$, $w = \frac{1}{\varepsilon^2}$, where $\varepsilon \to 0^+$ and $\varepsilon \to +\infty$, we can see that the ratio of areas in (16) can be arbitrarily large and arbitrarily small positive numbers, respectively.

We will now return to the configuration in Figure 1. A. F. Möbius considered areas in the special case where AD, BE, and CF are concurrent [22, p. 198]. J. F. Rigby's inequality (4) generalized this result [28] (see also [21, p. 340]). The following theorem is a further generalization of these two results.

Theorem 2.4. *Let D, E, and F be arbitrary points on sides BC, AC, and AB, respectively, of a triangle ABC. Set $BE \cap CF = G_1$, $AD \cap CF = G_2$, and $AD \cap BE = G_3$. Let areas of $\triangle AEF$, $\triangle BFD$, $\triangle CDE$, $\triangle DEF$, and $\triangle G_1G_2G_3$ be p, q, r, x, and y (Figure 1). Then*

$$x^3 + (p+q+r)x^2 - 4pqr \geq x^2 y.$$

Proof. Denote

$$\frac{|BD|}{|DC|} = \lambda_1, \quad \frac{|CE|}{|EA|} = \lambda_2, \quad \text{and} \quad \frac{|AF|}{|FB|} = \lambda_3.$$

Then the left side of the inequality can be written as (see [28, p. 115])

$$x^3 + (p+q+r)x^2 - 4pqr = \frac{(\lambda_1\lambda_2\lambda_3 - 1)^2}{(\lambda_1+1)^2(\lambda_2+1)^2(\lambda_3+1)^2} \cdot \big(\mathrm{Area}(\triangle ABC)\big)^3.$$

By (1) and (2), this can also be written as

$$\begin{aligned} &x^3 + (p+q+r)x^2 - 4pqr \\ &\quad = x^2 y \cdot \frac{(\lambda_1\lambda_2+\lambda_1+1)(\lambda_2\lambda_3+\lambda_2+1)(\lambda_3\lambda_1+\lambda_3+1)}{(\lambda_1\lambda_2\lambda_3+1)^2}. \end{aligned}$$

The values of the last fraction change in the interval $(1, +\infty)$ (consider the limiting case $\lambda_1, \lambda_2 \to 0$, $0 < \lambda_3 < +\infty$), and therefore inequality (5) holds true. ∎

Furthermore, from this proof, it follows that $\lambda = 1$ is the best constant for the inequality

$$x^3 + (p+q+r)x^2 - 4pqr \geq \lambda x^2 y.$$

As a problem for further exploration, it would be interesting to prove that, in case $uvw = 1$ (Corollary 2.3, Figure 5), the vertices of the triangle formed by lines G_1S, G_2T, and G_3R are on lines AK, BL, CM.

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Yagub N. Aliyev
School of IT and Engineering
ADA University
Ahmadbey Aghaoglu str. 61
1008 Baku, Azerbaijan
yaliyev@ada.edu.az