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Convex co-compact groups with one-dimensional boundary faces

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Abstract. In this paper, we consider convex co-compact subgroups of the projective linear group. We prove that such a group is relatively hyperbolic with respect to a collection of virtually Abelian subgroups of rank 2 if and only if each open face in the ideal boundary has dimension at most one. We also introduce the "coarse Hilbert dimension" of a subset of a convex set and use it to characterize when a naive convex co-compact subgroup is word hyperbolic or relatively hyperbolic with respect to a collection of virtually Abelian subgroups of rank 2.

1. Introduction

In this paper, we consider the class of (naive) convex co-compact subgroups of $\mathrm{PGL}_d(\mathbb{R})$, as defined in [13]. In earlier work [19], we proved a general, geometric characterization of when such a group is relatively hyperbolic with respect to a (possibly empty) collection of virtually Abelian subgroups of rank at least 2. In this paper, we specialize to the case of virtually Abelian subgroups of rank exactly 2 and provide a very simple (to state) characterization in terms of the ideal boundary of the associated convex hull. There are many examples of such convex co-compact groups coming from Coxeter groups and also from deformations of hyperbolic structures on certain cusped 3-manifolds followed by a doubling construction (see [2, 4], and [13, Section 12.2]).

To state our results precisely, we need to introduce some terminology. Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, the *automorphism group of* Ω is defined to be

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}_d(\mathbb{R}) : g\Omega = \Omega\}.$$

Then for a subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$, the *full orbital limit set of* Γ *in* Ω is defined to be

$$\mathcal{L}_\Omega(\Gamma) := \bigcup_{p \in \Omega} \big(\overline{\Gamma \cdot p} \setminus \Gamma \cdot p\big).$$

Next, let $\mathcal{C}_\Omega(\Gamma)$ denote the convex hull of $\mathcal{L}_\Omega(\Gamma)$ in Ω. Then, convex co-compact subgroups can be defined as follows.

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Definition 1.1 ([13, Definition 1.10]). (1) Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, then an infinite discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ is called *convex co-compact* when $\mathcal{C}_\Omega(\Gamma)$ is non-empty and Γ acts co-compactly on $\mathcal{C}_\Omega(\Gamma)$.

(2) A subgroup $\Gamma \subset \mathrm{PGL}_d(\mathbb{R})$ is *convex co-compact* if there exists a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ where $\Gamma \subset \mathrm{Aut}(\Omega)$ is a convex co-compact subgroup.

When Γ is word hyperbolic there is a close connection between this class of discrete groups in $\mathrm{PGL}_d(\mathbb{R})$ and Anosov representations (see [13] for details and [12,24] for related results). Further, by adapting an argument of Benoist [3], Danciger–Guéritaud–Kassel established a characterization of hyperbolicity in terms of the geometry of $\mathcal{C}_\Omega(\Gamma)$. To state their result, we need some more definitions.

Definition 1.2. A subset $S \subset \mathbb{P}(\mathbb{R}^d)$ is a *simplex* if there exist $g \in \mathrm{PGL}_d(\mathbb{R})$ and $0 \leq k \leq d-1$ such that

$$gS = \left\{[x_1 : \cdots : x_{k+1} : 0 : \cdots : 0] \in \mathbb{P}(\mathbb{R}^d) : x_1 > 0, \dots, x_{k+1} > 0\right\}.$$

Then the *dimension of* S, denoted $\dim(S)$, is k (notice that S is homeomorphic to $\mathbb{R}^k$) and the $(k+1)$ points

$$g^{-1}\{[1 : 0 : \cdots : 0], [0 : 1 : 0 : \cdots : 0], \dots, [0 : \cdots : 0 : 1 : 0 : \cdots : 0]\} \subset \partial S$$

are the *vertices* of S.

Definition 1.3. Suppose $A \subset B \subset \mathbb{P}(\mathbb{R}^d)$. Then, A is *properly embedded in* B if the inclusion map $A \hookrightarrow B$ is a proper map (relative to the subspace topology).

Finally, given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, let d_Ω denote the Hilbert metric on Ω (see Section 2.2 for the definition).

Theorem 1.4 (Danciger–Guéritaud–Kassel [13, Theorem 1.15]). *Suppose* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain,* $\Gamma \subset \mathrm{Aut}(\Omega)$ *is convex co-compact, and* $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$*. Then the following are equivalent:*

(1) *Every point in* $\overline{\mathcal{C}} \cap \partial\Omega$ *is an extreme point of* Ω*.*

(2) $\mathcal{C}$ *does not contain a properly embedded simplex with dimension at least* 2*.*

(3) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is Gromov hyperbolic.*

(4) Γ *is word hyperbolic.*

Remark 1.5. In the special case when Γ acts co-compactly on Ω, Theorem 1.4 is due to Benoist [3] and the proof in [13] follows similar arguments.

In this paper, we establish a similar theorem for groups, which are relatively hyperbolic with respect to a collection of virtually Abelian subgroups of rank 2. To state our main result precisely, we introduce the following notation: given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and $x \in \overline{\Omega}$ let $F_\Omega(x)$ denote the *(open) face* of x, that is

$$F_\Omega(x) = \{x\} \cup \{y \in \overline{\Omega} : \exists \text{ an open line segment in } \overline{\Omega} \text{ containing } x \text{ and } y\}.$$

When $x \in \partial\Omega$, we say that $F_\Omega(x)$ is a *boundary face* of $\partial\Omega$. Notice that $F_\Omega(x) = \Omega$ when $x \in \Omega$ and $F_\Omega(x) = \{x\}$ when $x \in \partial\Omega$ is an extreme point.

Theorem 1.6 (See Section 6). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \subset \operatorname{Aut}(\Omega)$ is convex co-compact, and $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$. Then the following are equivalent:*

(1) *Every boundary face of Ω which intersects $\overline{\mathcal{C}}$ has dimension at most* 1.

(2) *The collection of all properly embedded simplices in $\mathcal{C}$ with dimension* 2 *is closed and discrete in the local Hausdorff convergence topology induced by* d_Ω.

(3) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to a (possibly empty) collection of two-dimensional properly embedded simplices.*

(4) Γ *is a relatively hyperbolic group with respect to a (possibly empty) collection of virtually Abelian subgroups of rank* 2*.*

Remark 1.7. The implications (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (1) follow easily from the general results in [19] and so the difficulty is showing that (1) $\Rightarrow$ (2/3/4).

Remark 1.8. There are a number of other results in the literature concerning relatively hyperbolic groups acting on properly convex domains; see, for instance, [6–8, 10, 11, 23] (we note the authors of [11] are currently preparing an erratum for their paper). With the exception of [23], these results consider the case when $\Gamma\backslash\mathcal{C}$ is non-compact and Γ is relatively hyperbolic with respect to the fundamental groups of the ends (under some geometric assumptions on the ends and $\mathcal{C}$). There is some similarity between Theorem 1.6 and the statements in [6–8], but to the best of our knowledge, there is no non-trivial mathematical overlap between the results.

Theorem 1.6 can be viewed as an extension of the following result of Benoist.

Theorem 1.9 (Benoist [4]). *If M is a closed irreducible orientable* 3*-manifold and M admits a convex real projective structure, then either*

(1) *M is geometric with geometry $\mathbb{R}^3$, $\mathbb{R} \times \mathbb{H}^2$, or $\mathbb{H}^3$ or*

(2) *M is non-geometric and every component in the geometric decomposition is hyperbolic.*

Using Benoist's theorem, one can deduce the following special case of Theorem 1.6.

Corollary 1.10 (To Benoist's result). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ is a properly convex domain and $\Gamma \subset \operatorname{Aut}(\Omega)$ is a discrete group which acts co-compactly on Ω. If every boundary face of Ω has dimension at most* 1*, then Γ is relatively hyperbolic with respect to a (possibly empty) collection of virtually Abelian subgroups of rank* 2*.*

In fact, using the theory of 3-manifolds and relatively hyperbolic groups, one can deduce Benoist's theorem from the above corollary and so Theorem 1.6 can be viewed as an extension of this restated version of Benoist's theorem.

Theorem 1.6 also provides a partial answer to a question asked by Choi–Lee–Marquis.

Question 1.11 ([9, Remark 1.11]). Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a discrete group which acts co-compactly on Ω. If Ω is irreducible and non-symmetric, is Γ relatively hyperbolic with respect to a (possibly empty) collection of virtually Abelian subgroups of rank at least 2?

Theorem 1.6 says the answer is yes when every boundary face of Ω has dimension at most 1.

1.1. Naive convex co-compact subgroups

We will also prove a version of Theorem 1.6 for naive convex co-compact subgroups. This is a larger class of groups and as such the result is, by necessity, more technical.

Definition 1.12. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. An infinite discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ is called *naive convex co-compact* if there exists a non-empty closed convex subset $\mathcal{C} \subset \Omega$ such that

(1) $\mathcal{C}$ is Γ-invariant, that is, $g\mathcal{C} = \mathcal{C}$ for all $g \in \Gamma$.

(2) Γ acts co-compactly on $\mathcal{C}$.

In this case, we say that $(\Omega, \mathcal{C}, \Gamma)$ is a *naive convex co-compact triple*.

It is straightforward to construct examples where $\Gamma \subset \mathrm{Aut}(\Omega)$ is naive convex co-compact, but not convex co-compact (see, for instance, [19, Section 2.3]). In these cases, the convex subset $\mathcal{C}$ in Definition 1.12 is a strict subset of $\mathcal{C}_\Omega(\Gamma)$.

One key difference between convex co-compact and naive convex co-compact subgroups is the following: If $\Gamma \subset \mathrm{Aut}(\Omega)$ is a convex co-compact subgroup and $\overline{\mathcal{C}_\Omega(\Gamma)}$ intersects an open boundary face F of $\partial\Omega$, then $F \subset \overline{\mathcal{C}_\Omega(\Gamma)}$ (see, for instance, [13, Section 4]). However, if $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple, then it is possible for $\overline{\mathcal{C}}$ to intersect a boundary face without containing it entirely; see the following example.

Example. Consider $\Omega := \{[x_1 : x_2 : x_3] : x_1, x_2, x_3 > 0\}$, $\mathcal{C} := \{[x_1 : y : y] : x_1, y > 0\}$, and $\Gamma := \left\langle \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$. Then, $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple. Further, $F_\Omega([0:1:1]) \cap \overline{\mathcal{C}} = \{[0:1:1]\}$, while

$$F_\Omega([0:1:1]) = \{[0 : x_2 : x_3] : x_2, x_3 > 0\} \not\subset \overline{\mathcal{C}}.$$

So when studying naive convex co-compact subgroups, it is not enough to consider the dimension of the boundary faces of Ω which intersect the closure of convex subset $\mathcal{C}$, but the "size" of $\overline{\mathcal{C}}$ in each boundary face.

To make "size" precise we introduce the following definition.

Definition 1.13. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is properly convex and open in its span. Then, the *coarse dimension* of a non-empty subset $A \subset \Omega$, denoted by $\text{c-dim}_\Omega(A)$, is the smallest integer $k \geq 0$ such that there exist $R > 0$ and a k-dimensional convex subset $B \subset \Omega$ such that

$$A \subset \mathcal{N}_\Omega(B; R) := \{p \in \Omega : \mathrm{d}_\Omega(p, B) < R\},$$

where d_Ω is the Hilbert metric on Ω. In the extremal case when Ω is a point, we define $\text{c-dim}_\Omega(\Omega) := 0$.

Example. Suppose Ω and $\mathcal{C}$ are as in the previous example. Then, for any $r > 0$, $\text{c-dim}_\Omega(\mathcal{N}_\Omega(\mathcal{C}; r)) = 1$ and

$$\text{c-dim}_{F_\Omega([0:1:1])}\big(\overline{\mathcal{N}_\Omega(\mathcal{C}; r)} \cap F_\Omega([0:1:1])\big) = 0.$$

We will show that the coarse dimension of boundary faces can be used to characterize word hyperbolic naive convex co-compact subgroups.

Theorem 1.14 (See Section 5). *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. Then the following are equivalent:*

(1) $\text{c-dim}_{F_\Omega(x)}(\overline{\mathcal{C}} \cap F_\Omega(x)) = 0$ *for all* $x \in \overline{\mathcal{C}} \cap \partial\Omega$.

(2) $\mathcal{C}$ *does not contain a properly embedded simplex with dimension at least* 2.

(3) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is Gromov hyperbolic.*

(4) Γ *is a word hyperbolic group.*

Remark 1.15. Recall, if $x \in \partial\Omega$ is an extreme point, then $F_\Omega(x) = \{x\}$ and so $\dim F_\Omega(x) = 0$. Hence, Theorem 1.14 is a naive convex co-compact analog of Theorem 1.4.

For naive convex co-compact subgroups, we also prove the following analog of Theorem 1.6.

Theorem 1.16 (See Section 7). *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. Then the following are equivalent:*

(1) $\text{c-dim}_{F_\Omega(x)}(\overline{\mathcal{C}} \cap F_\Omega(x)) \leq 1$ *for all* $x \in \overline{\mathcal{C}} \cap \partial\Omega$.

(2) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to a (possibly empty) collection of two-dimensional properly embedded simplices.*

(3) Γ *is a relatively hyperbolic group with respect to a (possibly empty) collection of virtually Abelian subgroups of rank* 2.

2. Preliminaries

2.1. Convexity

In this section we recall some standard definitions related to convexity in real projective space.

Definition 2.1. (1) A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is *convex* if there exists an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a convex subset.

(2) A subset $C \subset \mathbb{P}(\mathbb{R}^d)$ is *properly convex* if there exists an affine chart $\mathbb{A}$ of $\mathbb{P}(\mathbb{R}^d)$ where $C \subset \mathbb{A}$ is a bounded convex subset.

(3) When C is a properly convex set which is open in $\mathbb{P}(\mathbb{R}^d)$, we say that C is a *properly convex domain*.

Notice that if $C \subset \mathbb{P}(\mathbb{R}^d)$ is convex, then C is a convex subset of every affine chart that contains it.

A *line segment* in $\mathbb{P}(\mathbb{R}^d)$ is a connected subset of a projective line. Given two points $x, y \in \mathbb{P}(\mathbb{R}^d)$ there is no canonical line segment with endpoints x and y, but we will use the following convention: If $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set and $x, y \in \overline{C}$, then (when the context is clear) we will let $[x, y]$ denote the closed line segment joining x to y which is contained in $\overline{C}$. In this case, we will also let $(x, y) = [x, y] \setminus \{x, y\}$, $[x, y) = [x, y] \setminus \{y\}$, and $(x, y] = [x, y] \setminus \{x\}$.

Along similar lines, given a properly convex subset $C \subset \mathbb{P}(\mathbb{R}^d)$ and a subset $X \subset C$, we will let

$$\mathrm{ConvHull}_C(X)$$

denote the smallest convex subset of C which contains X.

If $V \subset \mathbb{R}^d$ is a non-zero linear subspace, we will let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{R}^d)$ denote its projectivization. For a non-empty set $X \subset \mathbb{P}(\mathbb{R}^d)$, $\mathbb{P}(\mathrm{Span}(X))$ is the projectivization of the linear span of X.

We also make the following topological definitions.

Definition 2.2. Suppose $C \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex set. The *relative interior of* C, denoted by $\mathrm{relint}(C)$, is the interior of C in $\mathbb{P}(\mathrm{Span}\, C)$. In the case that $C = \mathrm{relint}(C)$, then C is said to be *open in its span*. The *boundary of* C is $\partial C := \overline{C} \setminus \mathrm{relint}(C)$, and the *ideal boundary of* C is

$$\partial_{\mathrm{i}} C := \partial C \setminus C.$$

Finally, we define $\dim C$ to be the dimension of $\mathrm{relint}(C)$ (notice that $\mathrm{relint}(C)$ is homeomorphic to $\mathbb{R}^{\dim C}$).

Recall from Definition 1.3 that a subset $A \subset B \subset \mathbb{P}(\mathbb{R}^d)$ is properly embedded if the inclusion map $A \hookrightarrow B$ is proper. If B is a properly convex set, then we have another characterization of properly embedded subsets using the notation in Definition 2.2 : $A \subset B$ is properly embedded if and only if $\partial_i A \subset \partial_i B$.

2.2. The Hilbert metric and faces

Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. For distinct points $x, y \in \Omega$, let $\overline{xy}$ be the projective line containing them and let a, b be the two points in $\overline{xy} \cap \partial\Omega$ ordered a, x, y, b along $\overline{xy}$. Then, the *Hilbert distance* between x and y is defined to be

$$\mathrm{d}_\Omega(x, y) = \frac{1}{2}\log[a, x, y, b],$$

where

$$[a, x, y, b] = \frac{|x-b||y-a|}{|x-a||y-b|}$$

is the projective cross ratio. It is a complete $\mathrm{Aut}(\Omega)$-invariant proper metric on Ω generating the standard topology on Ω. Moreover, if $x, y \in \Omega$, the projective line segment $[x, y]$ is a geodesic joining x and y.

For $x \in \Omega$ we will let

$$\mathcal{B}_\Omega(x; r) := \{y \in \Omega : \mathrm{d}_\Omega(y, x) < r\},$$

and for $A \subset \Omega$ we will let

$$\mathcal{N}_\Omega(A; r) := \{y \in \Omega : \mathrm{d}_\Omega(y, A) < r\}.$$

Recall (from the introduction) that given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ and $x \in \overline{\Omega}$ the open face of x is

$$F_\Omega(x) = \{x\} \cup \{y \in \overline{\Omega} : \exists \text{ an open line segment in } \overline{\Omega} \text{ containing } x \text{ and } y\}.$$

Given a subset $X \subset \overline{\Omega}$, we then define

$$F_\Omega(X) := \bigcup_{x \in X} F_\Omega(x).$$

The following observations follow immediately from convexity and the definitions (also see Appendix A).

Observation 2.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.*

(1) *$F_\Omega(x)$ is convex and open in its span.*

(2) *$y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$.*

(3) *If* $y \in \partial F_\Omega(x)$*, then* $F_\Omega(y) \subset \partial F_\Omega(x)$*.*

(4) *If* $x, y \in \overline{\Omega}$*,* $z \in (x, y)$*,* $p \in F_\Omega(x)$*, and* $q \in F_\Omega(y)$*, then*

$$(p, q) \subset F_\Omega(z).$$

In particular, $(p, q) \subset \Omega$ *if and only if* $(x, y) \subset \Omega$*.*

Directly from the definition of the Hilbert metric, one obtains the following.

Proposition 2.4. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain,* $(x_n)_{n\geq 1}$ *is a sequence in* Ω*, and* $\lim_{n\to\infty} x_n = x \in \overline{\Omega}$*. If* $(y_n)_{n\geq 1}$ *is another sequence in* Ω*,* $\lim_{n\to\infty} y_n = y \in \overline{\Omega}$*, and*

$$\liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n) < +\infty,$$

then $y \in F_\Omega(x)$ *and*

$$\mathrm{d}_{F_\Omega(x)}(x, y) \leq \liminf_{n\to\infty} \mathrm{d}_\Omega(x_n, y_n).$$

2.3. The center of mass of a compact subset

It is possible to define a "center of mass" for a compact set in a properly convex domain. Let $\mathcal{K}_d$ denote the set of all pairs (Ω, K) where $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $K \subset \Omega$ is a compact subset.

Proposition 2.5. *There exists a function*

$$(\Omega, K) \in \mathcal{K}_d \mapsto \mathrm{CoM}_\Omega(K) \in \mathbb{P}(\mathbb{R}^d)$$

such that

(1) $\mathrm{CoM}_\Omega(K) \in \mathrm{ConvHull}_\Omega(K)$*,*

(2) $\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_\Omega(\mathrm{ConvHull}_\Omega(K))$*, and*

(3) *if* $g \in \mathrm{PGL}_d(\mathbb{R})$*, then* $g\mathrm{CoM}_\Omega(K) = \mathrm{CoM}_{g\Omega}(gK)$*,*

for every $(\Omega, K) \in \mathcal{K}_d$*.*

Proof. There are several constructions of such a center of mass (see, for instance, [20, Lemma 4.2] or [18, Proposition 4.5]). The approach in [18] is based on an argument of Frankel [16, Section 12] in several complex variables. ∎

2.4. The Hausdorff distance

Recall that when (X, d) is a metric space, the *Hausdorff pseudo-distance* between two subsets $A, B \subset X$ is defined by

$$\mathrm{d}^{\mathrm{Haus}}(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \mathrm{d}(a, b), \sup_{b\in B} \inf_{a\in A} \mathrm{d}(a, b)\right\}$$

when A and B are both non-empty, and $\mathrm{d}^{\mathrm{Haus}}(A, B) = \infty$ otherwise.

The Hausdorff pseudo-distance is very useful when considering compact subsets: When (X, d) is a complete metric space, d^{Haus} is a complete metric on the set of non-empty compact subsets of X. This pseudo-distance is less useful when dealing with closed sets, as the next example demonstrates.

Example 2.6. Consider $\mathbb{R}^2$ with the Euclidean distance. Let $B_n := \overline{\mathcal{B}_{\mathbb{R}^2}((0, n); n)}$ be the closed ball of radius n centered at $(0, n)$ and let $H := \{p = (x, y) \in \mathbb{R}^2 : y \geq 0\}$ be the closed upper half plane. In any reasonable topology on closed sets one would like the sequence B_n to converge to H. Unfortunately, with respect to the Hausdorff pseudo-distance one has, for all n,

$$\mathrm{d}^{\mathrm{Haus}}(B_n, H) = \infty.$$

2.5. Local Hausdorff convergence topology

In this section we recall a useful topology on the set of non-empty closed subsets of a metric space. This can be interpreted as a localization of the Hausdorff pseudo-distance that we discussed above. The topology we describe is a natural extension of the topology on compact subsets determined by the Hausdorff distance and has been used extensively in different areas of mathematics (e.g., see Hruska–Kleiner's [17] work in CAT(0) geometry or Frankel's work in several complex variables [16]).

Let $\mathcal{C}(X)$ denote the set of all non-empty closed subset of a metric space (X, d). For any $x \in X$ and $r > 0$, we will denote the metric r-neighborhood of x by

$$\mathcal{B}_X(x; r) = \{y \in X : \mathrm{d}(x, y) < r\}.$$

Definition 2.7. For a closed set $C_0 \subset X$, a base point $x_0 \in X$, and $r_0, \varepsilon_0 > 0$ define the set $U(C_0, x_0, r_0, \varepsilon_0)$ to consist of all closed subsets $C \subset X$ where

$$\mathrm{d}^{\mathrm{Haus}}(C_0 \cap \mathcal{B}_X(x_0; r_0), C \cap \mathcal{B}_X(x_0; r_0)) < \varepsilon_0.$$

The *local Hausdorff convergence topology on* $\mathcal{C}(X)$ (induced by the metric d on X) is the topology generated by the sets $U(\cdot, \cdot, \cdot, \cdot)$.

When the metric space (X, d) is clear from context, we will often simply refer to this as the *local Hausdorff topology induced by* d for brevity.

Remark 2.8. There are other well-known topologies on the space of non-empty closed subsets of a metric space, for instance the Chabauty topology [1, 5].

Example 2.9. Assume the same set-up and notation as in Example 2.6. Then, B_n converges to H in the local Hausdorff convergence topology on $\mathcal{C}(\mathbb{R}^2)$ (see Corollary 2.12 below).

We note that when the metric space (X, d) is proper, the local Hausdorff convergence topology is second countable.

Observation 2.10. *If (X, d) is a proper metric space, then the local Hausdorff convergence topology on $\mathcal{C}(X)$ is second countable.*

Proof. Since (X, d) is proper, it has a countable dense subset $A \subset X$. Fix an enumeration $\mathbb{Q} \cap (0, \infty) = \{r_n\}$. Then, for each $n \in \mathbb{N}$ and $a \in A$, the set

$$\mathcal{C}_{n,a} := \{K : K \text{ compact and } K \subset \mathcal{B}_X(a; r_n)\}$$

endowed with the Hausdorff distance is a compact metric space. Hence, $\mathcal{C}_{n,a}$ has a countable dense subset $B_{n,a}$. Then,

$$\{U(C, a, r_n, m^{-1}) : a \in A,\ n, m \in \mathbb{N},\ C \in B_{n,a}\}$$

is a countable basis for the local Hausdorff convergence topology. ∎

Based on the definition of the topology, one might expect that $C_n \to C$ if and only if

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_{\mathbb{R}}(x_0; r), C \cap \mathcal{B}_{\mathbb{R}}(x_0; r)) = 0$$

for all $x_0 \in X$ and $r > 0$. However, the next example demonstrates that one has to be careful with the choice of $x_0 \in X$ and $r > 0$.

Example. Consider $\mathbb{R}$ with the Euclidean distance. Let $C_n := \{1/n\} \subset \mathbb{R}$ and $C := \{0\}$. One can show that $C_n \to C$ in the local Hausdorff convergence topology (see Corollary 2.12); however, if $x_0 = 1$ and $r = 1$, then

$$\mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_{\mathbb{R}}(x_0; r), C \cap \mathcal{B}_{\mathbb{R}}(x_0; r)) = \mathrm{d}^{\mathrm{Haus}}(C_n, \emptyset) = \infty$$

for all $n \geq 1$.

The next observation makes this naive characterization of convergence precise.

Observation 2.11. *Suppose (X, d) is a proper metric space and $(C_n)_{n\geq 1}$ is a sequence of closed sets in X. Then, $C_n \to C$ in the local Hausdorff convergence topology if and only if*

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_X(x_0; r), C \cap \mathcal{B}_X(x_0; r)) = 0$$

for all $x_0 \in X$ and $r > 0$, where $C \cap \mathcal{B}_X(x_0, r) \neq \emptyset$.

Proof. ($\Leftarrow$): Fix $x_0 \in X$ and $r > 0$ such that $C \cap \mathcal{B}_X(x_0, r) \neq \emptyset$. Then fix $\varepsilon > 0$. Since

$$C \in U(C, x_0, r, \varepsilon),$$

there exists $N \geq 1$ such that $C_n \in U(C, x_0, r, \varepsilon)$ for all $n \geq N$. Then,

$$\limsup_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_X(x_0; r), C \cap \mathcal{B}_X(x_0; r)) \leq \varepsilon$$

by the definition of $U(C, x_0, r, \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we see that

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_X(x_0; r), C \cap \mathcal{B}_X(x_0; r)) = 0.$$

($\Rightarrow$): Fix an open set $\mathcal{U}$, in the local Hausdorff convergence topology, that contains C. Then, by the definition of the topology, there exist $x_0 \in X$ and $r_0, \varepsilon_0 > 0$ such that

$$C \in U(C, x_0, r_0, \varepsilon_0) \subset \mathcal{U}.$$

In particular, $C \cap \mathcal{B}_X(x_0; r_0) \neq \emptyset$. Thus, by hypothesis,

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_X(x_0; r_0), C \cap \mathcal{B}_X(x_0; r_0)) = 0.$$

Then for n sufficiently large, we have

$$C_n \in U(C, x_0, r_0, \varepsilon_0) \subset \mathcal{U}.$$

Thus, $C_n \to C$. ∎

As a corollary to this observation, we have the following.

Corollary 2.12. *Suppose (X, d) is a proper metric space and $C_n \to C$ in the local Hausdorff convergence topology. If $p \in X$, then the following are equivalent:*

(1) $p \in C$.

(2) *There exists a sequence $(p_n)_{n\geq 1}$ in X such that $p_n \in C_n$ for all n and $p_n \to p$.*

Proof. Fix $r > 0$ such that $C \cap \mathcal{B}_X(p, r) \neq \emptyset$. Then, by Observation 2.11,

$$\lim_{n\to\infty} \mathrm{d}^{\mathrm{Haus}}(C_n \cap \mathcal{B}_X(p; r), C \cap \mathcal{B}_X(p; r)) = 0,$$

which implies the desired equivalence. ∎

Besides the properties mentioned above, the only other property of the local Hausdorff convergence topology we will use in this paper is the following.

Proposition 2.13. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Then, the set of properly embedded simplices in Ω of dimension at least 2 is closed in the local Hausdorff convergence topology induced by d_Ω.*

Proof. This follows from [19, Observation 3.20], but we provide a proof for the reader's convenience.

Suppose $(S_n)_{n\geq 1}$ is a sequence of properly embedded simplices in Ω of dimension at least 2 which converges to a closed subset S in the local Hausdorff convergence topology induced by d_Ω. Passing to a subsequence we can suppose that $\dim S_n = k$ for all n.

Let $v_1^{(n)}, \dots, v_k^{(n)}$ be the vertices of S_n. Passing to a subsequence we can suppose that $v_j^{(n)} \to v_j$ for all j. To show that S is a properly embedded simplex of dimension k it suffices to show that

(a) $v_1, \dots, v_k$ are linearly independent.

(b) $S = \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(\{v_1, \dots, v_k\})$.

(c) $\Omega \cap \mathbb{P}(\operatorname{Span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}) = \emptyset$ for all $j = 1, \dots, k$.

First, we verify (c). Since each S_n is a properly embedded simplex,

$$\Omega \cap \mathbb{P}\Big(\operatorname{Span}\big\{v_1^{(n)}, \dots, v_{j-1}^{(n)}, v_{j+1}^{(n)}, \dots, v_k^{(n)}\big\}\Big) = \emptyset$$

for all $j = 1, \dots, k$ and $n \geq 1$. So sending $n \to \infty$ and using the fact that Ω is open, we see that

$$\Omega \cap \mathbb{P}(\operatorname{Span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}) = \emptyset \tag{1}$$

for all $j = 1, \dots, k$. This verifies (c).

Since each S_n is a properly embedded simplex,

$$S_n = \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}\Big(\big\{v_1^{(n)}, \dots, v_k^{(n)}\big\}\Big).$$

So taking limits and using Corollary 2.12, we see that

$$S = \Omega \cap \operatorname{ConvHull}_{\overline{\Omega}}(\{v_1, \dots, v_k\}). \tag{2}$$

Next, we verify (a). Suppose $v_1, \dots, v_k$ are not linearly independent. Then, $v_j \in \mathbb{P}(\operatorname{Span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\})$ for some j. Then, using equations (1) and (2), we have

$$S \subset \operatorname{ConvHull}_{\overline{\Omega}}(\{v_1, \dots, v_k\}) \subset \mathbb{P}(\operatorname{Span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}) \subset \mathbb{P}(\mathbb{R}^d) \setminus \Omega,$$

which is a contradiction. Thus, (a) is true.

Finally, we verify (b). By equation (2), it suffices to show that

$$\operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}\big(\{v_1, \dots, v_k\}\big) \subset \Omega.$$

Suppose not. Then, there exists

$$x \in \big(\operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(\{v_1, \dots, v_k\})\big) \setminus \Omega.$$

Since Ω is convex, there exists a projective hyperplane H such that $x \in H$ and $H \cap \Omega = \emptyset$. Equation (2) implies that H intersects $\mathbb{P}(\operatorname{Span}\{v_1, \dots, v_k\})$ transversally, that is, $\mathbb{P}(\mathrm{Span}\{v_1, \dots, v_k\}) \not\subset H$ (otherwise $S \subset \mathbb{P}(\mathrm{Span}\{v_1, \dots, v_k\}) \subset H \subset \mathbb{P}(\mathbb{R}^d) - \Omega$, a contradiction).

On the other hand, $v_j^{(n)} \to v_j$, the lines $v_1, \dots, v_k$ are linearly independent, and $x \in \mathrm{relint}\, \mathrm{ConvHull}_{\overline{\Omega}}\{v_1, \dots, v_k\} \cap H$. Thus, the hyperplane H non-trivially intersects

$$S_n = \mathrm{relint}\, \mathrm{ConvHull}_{\overline{\Omega}}\Big(\big\{v_1^{(n)}, \dots, v_k^{(n)}\big\}\Big)$$

for n large. This is impossible since $S_n \subset \Omega$ and thus (b) is true. ∎

2.6. Properly embedded simplices

In this section we record some basic facts about properly embedded simplices in a properly convex domain.

The following result is a simple consequence of any of the explicit formulas for the Hilbert metric on a simplex (see [21, Proposition 1.7], [14], or [22]).

Proposition 2.14. *If $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex, then (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$.*

The faces of a properly embedded simplex are themselves properly embedded simplices in the boundary faces that contain them.

Observation 2.15. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex. If $x \in \partial S$, then*

(a) *$F_S(x)$ is a properly embedded simplex in $F_\Omega(x)$.*

(b) *$F_S(x) = \overline{S} \cap F_\Omega(x)$.*

Proof. See, for instance, [19, Observation 5.4]. ∎

Definition 2.16. Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. Two properly embedded simplices $S_1, S_2 \subset \Omega$ are called *parallel* if $\dim S_1 = \dim S_2$ and there is a labeling $v_1, \dots, v_p$ of the vertices of S_1 and a labeling $w_1, \dots, w_p$ of the vertices of S_2 such that $F_\Omega(v_k) = F_\Omega(w_k)$ for all $1 \le k \le p$.

The following lemma allows us to “wiggle” the vertices of a properly embedded simplex and obtain a new parallel properly embedded simplex.

Lemma 2.17. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $S \subset \Omega$ is a properly embedded simplex with vertices $v_1, \dots, v_p$. If $w_j \in F_\Omega(v_j)$ for $1 \le j \le p$, then*

$$S' := \Omega \cap \mathbb{P}(\mathrm{Span}\{w_1, \dots, w_p\}) = \mathrm{relint}\, \mathrm{ConvHull}_{\overline{\Omega}}(w_1, \dots, w_p)$$

is a properly embedded simplex with vertices $w_1, \dots, w_p$. Moreover,

$$\mathrm{d}_\Omega^{\mathrm{Haus}}(S, S') \le \max_{1 \le j \le p} \mathrm{d}_{F_\Omega(v_j)}(v_j, w_j).$$

Proof. See, for instance, [19, Lemma 3.18]. ∎

3. Relative hyperbolic convex co-compact groups

In this section we recall some properties of general relatively hyperbolic spaces/groups and also recall some of the results from [19].

3.1. General relatively hyperbolic groups

We define relative hyperbolic spaces and groups in terms of Druţu and Sapir's tree-graded spaces (see [15, Definition 2.1]).

Definition 3.1. (1) A complete geodesic metric space (X, d) is *relatively hyperbolic with respect to a collection of subsets* $\mathcal{S}$ if all its asymptotic cones, with respect to a fixed non-principal ultrafilter, are tree-graded with respect to the collection of ultralimits of the elements of $\mathcal{S}$.

(2) A finitely generated group G is *relatively hyperbolic with respect to a family of subgroups* $\{H_1, \dots, H_k\}$ if the Cayley graph of G with respect to some (hence any) finite set of generators is relatively hyperbolic with respect to the collection of left cosets $\{gH_i : g \in G, i = 1, \dots, k\}$.

Remark 3.2. These are one among several equivalent definitions of relatively hyperbolic spaces/groups; see [15] and the references therein for more details.

If (X, d) is a metric space, we will use the following notation for metric tubular neighborhoods: if $A \subset X$ and $r > 0$, then

$$\mathcal{N}_X(A; r) := \{x \in X : \mathrm{d}(x, A) < r\}.$$

We will frequently use the following property of relatively hyperbolic spaces.

Theorem 3.3 (Druţu–Sapir [15, Corollary 5.8]). *Suppose (X, d) is relatively hyperbolic with respect to $\mathcal{S}$. Then, for any $A \geq 1$ and $B \geq 0$, there exists $M = M(A, B)$ such that if $k \geq 2$ and $f : \mathbb{R}^k \to X$ is an (A, B)-quasi-isometric embedding, then there exists some $S \in \mathcal{S}$ such that*

$$f(\mathbb{R}^k) \subset \mathcal{N}_X(S; M).$$

3.2. Convex co-compact relatively hyperbolic groups

Next we recall some of the results in [19] describing the structure of (naive) convex co-compact groups which are relatively hyperbolic with respect to a collection of virtually Abelian subgroups of rank at least 2.

In the convex co-compact case we have the following characterization and structural results.

Theorem 3.4 ([19, Theorems 1.7 and 1.8]). *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \leq \mathrm{Aut}(\Omega)$ is convex co-compact, and $\mathcal{S}_{\max}$ is the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ with dimension at least 2. Then the following are equivalent:*

(1) *$\mathcal{S}_{\max}$ is closed and discrete in the local Hausdorff convergence topology induced by d_Ω.*

(2) *Γ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank at least 2.*

(3) *$(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ is a relatively hyperbolic space with respect to $\mathcal{S}_{\max}$.*

(4) *$(\mathcal{C}_\Omega(\Gamma), \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a collection of properly embedded simplices of dimension at least 2.*

Moreover, when $\mathcal{S}_{\max}$ is closed and discrete in the local Hausdorff convergence topology induced by d_Ω, then:

(a) *Γ has finitely many orbits in $\mathcal{S}_{\max}$.*

(b) *If $S \in \mathcal{S}_{\max}$, then $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.*

(c) *If $\{S_1, \dots, S_m\}$ are representatives of the Γ-orbits in $\mathcal{S}_{\max}$, then Γ is a relatively hyperbolic group with respect to $\{\mathrm{Stab}_\Gamma(S_1), \dots, \mathrm{Stab}_\Gamma(S_m)\}$.*

(d) *If $A \leq \Gamma$ is an infinite Abelian subgroup with rank at least 2, then there exists a unique $S \in \mathcal{S}_{\max}$ with $A \leq \mathrm{Stab}_\Gamma(S)$.*

(e) *If $S \in \mathcal{S}_{\max}$ and $x \in \partial S$, then $F_\Omega(x) = F_S(x)$.*

(f) *If $S_1, S_2 \in \mathcal{S}_{\max}$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and $\partial S_1 \cap \partial S_2 = \emptyset$.*

(g) *For any $r > 0$ there exists $D(r) > 0$ such that if $S_1, S_2 \in \mathcal{S}_{\max}$ are distinct, then*

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) \leq D(r).$$

(h) *If $\ell \subset \partial_{\mathrm{i}} \mathcal{C}_\Omega(\Gamma)$ is a non-trivial line segment, then there exists $S \in \mathcal{S}_{\max}$ with $\ell \subset \partial S$.*

In the naive convex co-compact case, we established a similar characterization and structural results. However, they are much more technical. The main issue is that there can exist bounded families of parallel properly embedded simplices (see, for instance, [19, Section 2.3]). So the group being relative hyperbolic with respect to a family of virtually Abelian subgroups of rank at least 2 is not equivalent to the set of *all* properly embedded simplices being closed and discrete. Instead, it is equivalent to the existence of a Γ-invariant family of properly embedded simplices which is closed, discrete, and which coarsely contains every properly embedded simplex. This is made precise in the next definition.

Definition 3.5 ([19, Definition 1.11]). Suppose $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple. A family $\mathcal{S}$ of maximal properly embedded simplices in $\mathcal{C}$ of dimension at least 2 is called:

(1) *Isolated*, if $\mathcal{S}$ is closed and discrete in the local Hausdorff convergence topology induced by d_Ω.

(2) *Coarsely complete*, if any properly embedded simplex in $\mathcal{C}$ is contained in a uniformly bounded tubular neighborhood of some properly embedded simplex in $\mathcal{S}$.

(3) Γ-*invariant*, if $g \cdot S \in \mathcal{S}$ for all $S \in \mathcal{S}$ and $g \in \Gamma$.

We say that $(\Omega, \mathcal{C}, \Gamma)$ has *coarsely isolated simplices* if there exists an isolated, coarsely complete, and Γ-invariant family of maximal properly embedded simplices.

We then have the following characterization of relative hyperbolicity (with respect to a family of virtually Abelian subgroups of rank at least 2) in the naive convex co-compact case.

Theorem 3.6 ([19, Theorem 1.13]). *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. Then the following are equivalent:*

(1) $(\Omega, \mathcal{C}, \Gamma)$ *has coarsely isolated simplices.*

(2) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is a relatively hyperbolic space with respect to a family of properly embedded simplices in* $\mathcal{C}$ *of dimension at least* 2*.*

(3) Γ *is a relatively hyperbolic group with respect to a family of virtually Abelian subgroups of rank at least* 2*.*

The naive convex co-compact case has one more delicate point: If $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple and $\mathcal{S}$ is a family of properly embedded simplices satisfying Definition 3.5, then it is not always true that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}$ (see [19, Observation 2.10] for examples).

Instead one requires an even stricter isolation property: We say that a family of simplices $\mathcal{S}$ in a properly convex domain Ω is *strongly isolated*, if for every $r > 0$, there exists $D(r) > 0$ such that if $S_1, S_2 \in \mathcal{S}$ are distinct, then

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S_1; r) \cap \mathcal{N}_\Omega(S_2; r)) \leq D(r).$$

It is straightforward to see that a *strongly isolated* family of simplices is indeed *isolated*. However, the converse is not true in general (see Section 2.3, mainly Observation 2.10, in [19]). But we proved in [19] that one can modify a coarsely isolated family of simplices to construct a strongly isolated family of simplices.

Theorem 3.7 ([19, Theorem 1.17]). *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple with coarsely isolated simplices. Then, there exists a strongly isolated, coarsely* *complete, and Γ-invariant collection of properly embedded simplices in $\mathcal{C}$ of dimension at least* 2*.*

We also proved the following.

Theorem 3.8 ([19, Theorems 1.18 and 1.19]). *Suppose $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple with coarsely isolated simplices. If $\mathcal{S}$ is a strongly isolated, coarsely complete, and Γ-invariant collection of properly embedded simplices in $\mathcal{C}$ of dimension at least* 2*, then*

(1) *$(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to $\mathcal{S}$.*
(2) *If $S \in \mathcal{S}$, then $\mathrm{Stab}_\Gamma(S)$ acts co-compactly on S and contains a finite index subgroup isomorphic to $\mathbb{Z}^k$ where $k = \dim S$.*
(3) *Γ has finitely many orbits in $\mathcal{S}$.*
(4) *If $\{S_1, \dots, S_m\}$ are representatives of the Γ-orbits in $\mathcal{S}$, then Γ is a relatively hyperbolic group with respect to $\{\mathrm{Stab}_\Gamma(S_1), \dots, \mathrm{Stab}_\Gamma(S_m)\}$.*
(5) *If $A \leq \Gamma$ is an Abelian subgroup with rank at least* 2*, then there exists a unique $S \in \mathcal{S}$ with $A \leq \mathrm{Stab}_\Gamma(S)$.*
(6) *There exists $D > 0$ such that if $S \in \mathcal{S}$ and $x \in \partial S$, then*
$$\mathrm{d}^{\mathrm{Haus}}_{F_\Omega(x)}(\overline{\mathcal{C}} \cap F_\Omega(x), F_S(x)) \leq D.$$
(7) *If $S_1, S_2 \in \mathcal{S}$ are distinct, then $\#(S_1 \cap S_2) \leq 1$ and*
$$\left(\bigcup_{x \in \partial S_1} F_\Omega(x) \right) \cap \left(\bigcup_{x \in \partial S_2} F_\Omega(x) \right) = \emptyset.$$

4. Properties of coarse dimension

In this section we make some basic observations about the coarse dimension (see Definition 1.13).

Observation 4.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain. If $S \subset \Omega$ is a properly embedded simplex, then*
$$\text{c-dim}_{F_\Omega(x)}(F_S(x)) = \dim F_S(x)$$
for all $x \in \overline{S}$.

Proof. If $F_S(x)$ is a point, then there is nothing to prove. So we can assume $\dim F_S(x) = k > 0$. Then, $F_S(x)$ is a properly embedded simplex in $F_\Omega(x)$ by Observation 2.15.

Suppose $D \subset F_\Omega(x)$ is a convex subsets with
$$F_S(x) \subset \mathcal{N}_{F_\Omega(x)}(D; R)$$

for some $R > 0$. Let $v_1, \dots, v_{k+1} \in \partial F_S(x)$ denote the vertices of S in $\overline{F_S(x)}$. Then, by Proposition 2.4, for each $j \in \{1, \dots, k+1\}$ there exists

$$w_j \in F_\Omega(v_j) \cap \overline{D}.$$

By Lemma 2.17,

$$S' := \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(w_1, \dots, w_{k+1}) \subset D$$

is a k-dimensional properly embedded simplex in $F_\Omega(x)$ and so

$$\dim D \geq \dim S' \geq k = \dim F_S(x). \qquad \blacksquare$$

Observation 4.2. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\mathcal{C} \subset \Omega$ is a convex subset. If $\partial_{\rm i} \mathcal{C} \neq \emptyset$, then*

$$\text{c-dim}_\Omega(\mathcal{C}) \geq 1 + \max_{x \in \partial_{\rm i} \mathcal{C}} \text{c-dim}_{F_\Omega(x)}(\partial_{\rm i} \mathcal{C} \cap F_\Omega(x)).$$

Proof. Suppose $D \subset \Omega$ is a convex subset with $\dim D = \text{c-dim}_\Omega(\mathcal{C})$ and

$$\mathcal{C} \subset \mathcal{N}_\Omega(D; R)$$

for some $R > 0$. Fix $x \in \partial_{\rm i} \mathcal{C}$ and let $D_x := \overline{D} \cap F_\Omega(x)$. Proposition 2.4 implies that

$$\partial_{\rm i} \mathcal{C} \cap F_\Omega(x) \subset \mathcal{N}_{F_\Omega(x)}(D_x; R+1)$$

and hence, by definition,

$$\text{c-dim}_{F_\Omega(x)}(\partial_{\rm i} \mathcal{C} \cap F_\Omega(x)) \leq \dim D_x \leq -1 + \dim D = -1 + \text{c-dim}_\Omega(\mathcal{C}). \qquad \blacksquare$$

Observation 4.3. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain and $\mathcal{C} \subset \Omega$ is a convex subset. If there exist $x_1, x_2, x_3 \in \partial_{\rm i} \mathcal{C}$ such that $F_\Omega(x_1), F_\Omega(x_2), F_\Omega(x_3)$ are pairwise distinct, then*

$$\text{c-dim}_\Omega(\mathcal{C}) \geq 2.$$

Proof. Suppose not. Then, there exists a convex subset $D \subset \Omega$ where $\dim D \leq 1$ and

$$\mathcal{C} \subset \mathcal{N}_\Omega(D; R)$$

for some $R > 0$. Then, by Proposition 2.4, for each $j \in \{1, 2, 3\}$ there exists

$$y_j \in F_\Omega(x_j) \cap \overline{D} \subset \partial_{\rm i} D$$

By assumption y_1, y_2, y_3 are pairwise distinct. However, since $\dim D \leq 1$ the set $\partial_{\rm i} D$ contains at most two points. So we have a contradiction. $\blacksquare$

Next we show that a certain configuration of points in the ideal boundary of a naive convex co-compact triple implies that the boundary contains a face with coarse dimension at least 2.

Proposition 4.4. *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. If there exist distinct points* $x, y_1, y_2, y_3 \in \partial_{\mathrm{i}} \mathcal{C}$ *such that*

$$[x, y_1] \cup [x, y_2] \cup [x, y_3] \subset \partial\Omega$$

and

$$(y_1, y_2) \cup (y_2, y_3) \cup (y_3, y_1) \subset \Omega,$$

then there exists $w \in \partial_{\mathrm{i}} \mathcal{C}$ *with*

$$\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) \geq 2.$$

Proof. Fix $u \in \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(y_1, y_2, y_3)$, $a \in (y_1, y_2)$, $b \in (y_2, y_3)$, and $c \in (y_3, y_1)$. Then, we can find sequences $u_n \in (x, u)$, $a_n \in (x, a)$, $b_n \in (x, b)$, and $c_n \in (x, c)$ all converging to x such that

$$\mathrm{d}_\Omega(u_n, a_n) \leq \mathrm{d}_\Omega(u, a), \quad \mathrm{d}_\Omega(u_n, b_n) \leq \mathrm{d}_\Omega(u, b), \quad \text{and} \quad \mathrm{d}_\Omega(u_n, c_n) \leq \mathrm{d}_\Omega(u, c)$$

for all n.

By passing to a subsequence we can find $\gamma_n \in \Gamma$ such that $\gamma_n u_n \to \hat{u} \in \mathcal{C}$ and

$$\gamma_n a_n, \gamma_n b_n, \gamma_n c_n, \gamma_n x, \gamma_n y_1, \gamma_n y_2, \gamma_n y_3 \to \hat{a}, \hat{b}, \hat{c}, \hat{x}, \hat{y}_1, \hat{y}_2, \hat{y}_3.$$

Then

$$[\hat{x}, \hat{y}_1] \cup [\hat{x}, \hat{y}_2] \cup [\hat{x}, \hat{y}_3] \subset \partial_{\mathrm{i}} \mathcal{C}$$

and by our choice of sequences $\hat{a}, \hat{b}, \hat{c} \in \mathcal{C}$. Also, since $u_n \to x \in \partial\Omega$, we have

$$\lim_{n \to \infty} \mathrm{d}_\Omega\big(u_n, \Omega \cap \operatorname{ConvHull}_{\overline{\Omega}}(y_1, y_2, y_3)\big) = \infty$$

and so

$$\operatorname{ConvHull}_{\overline{\Omega}}(\hat{y}_1, \hat{y}_2, \hat{y}_3) \subset \partial_{\mathrm{i}} \mathcal{C}.$$

Fix $w \in \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(\hat{y}_1, \hat{y}_2, \hat{y}_3) \subset \partial_{\mathrm{i}} \mathcal{C}$.

Claim 1: $(\hat{x}, w) \subset \Omega$.

Since

$$\hat{u} \in \Omega \cap \operatorname{ConvHull}_{\overline{\Omega}}(\hat{x}, \hat{y}_1, \hat{y}_2, \hat{y}_3)$$

convexity implies that $(\hat{x}, w) \subset \Omega$.

Claim 2: $\hat{y}_1, \hat{y}_2, \hat{y}_3 \in \partial F_\Omega(w)$.

By construction, $\hat{y}_1, \hat{y}_2, \hat{y}_3 \in \overline{F_\Omega(w)}$. Fix $j \in \{1, 2, 3\}$. Since $(\hat{x}, w) \subset \Omega$ and $[\hat{x}, \hat{y}_j] \subset \partial\Omega$, Observation 2.3 part (4) implies that $\hat{y}_j \notin F_\Omega(w)$. So $\hat{y}_j \in \partial F_\Omega(w)$.

Claim 3: $F_\Omega(\hat{y}_1), F_\Omega(\hat{y}_2), F_\Omega(\hat{y}_3)$ are pairwise distinct.

By symmetry it is enough to show that $F_\Omega(\hat{y}_1)$ and $F_\Omega(\hat{y}_2)$ are distinct. If not, then

$$[\hat{y}_1, \hat{y}_2] \subset F_\Omega(\hat{y}_2).$$

Since $[\hat{x}, \hat{y}_1] \subset \partial\Omega$, Observation 2.3 part (4) then implies that

$$\mathrm{ConvHull}_{\overline{\Omega}}(\hat{x}, \hat{y}_1, \hat{y}_2) \subset \partial\Omega.$$

However, $\hat{a} \in \mathcal{C} \subset \Omega$ is contained in this convex hull and hence we have a contradiction.

Claim 4: $\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}}\mathcal{C} \cap F_\Omega(w)) \geq 2$.

This follows immediately from Claim 2, Claim 3, and Observation 4.3. ■

5. Proof of Theorem 1.14

In this section we prove the following extension of Theorem 1.14.

Theorem 5.1. *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. Then the following are equivalent:*

(1) *There exists* $R > 0$ *such that* $\mathrm{diam}_{F_\Omega(x)}(\partial_{\mathrm{i}}\mathcal{C} \cap F_\Omega(x)) \leq R$ *for all* $x \in \partial_{\mathrm{i}}\mathcal{C}$.

(2) $\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}}\mathcal{C} \cap F_\Omega(x)) = 0$ *for all* $x \in \partial_{\mathrm{i}}\mathcal{C}$.

(3) $\mathcal{C}$ *does not contain a properly embedded simplex with dimension at least* 2.

(4) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is Gromov hyperbolic.*

(5) Γ *is a word hyperbolic group.*

By definition (1) ⇒ (2), by Observation 4.1 (2) ⇒ (3), by Proposition 2.14 (4) ⇒ (3), and by the Švarc–Milnor lemma (4) ⇔ (5). We will complete the proof by showing that (3) ⇒ (1) and (2) ⇒ (4).

In the convex co-compact case, it is well known that a line segment in the ideal boundary implies the existence of a properly embedded simplex. This is given explicitly in [13, Lemma 6.2] using a proof nearly identical to [3, Proposition 2.5] and [4, Lemma 3.9]. Unfortunately, simple examples show that this observation fails in the naive co-convex co-compact case (see [19, Section 2.3]). The next lemma uses Benoist's argument to establish a more technical condition to guarantee that the existence of a properly embedded simplex.

Lemma 5.2 ((3) ⇒ (1)). *If* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple and*

$$\sup_{x \in \partial_{\mathrm{i}}\mathcal{C}} \mathrm{diam}_{F_\Omega(x)}(\partial_{\mathrm{i}}\mathcal{C} \cap F_\Omega(x)) = +\infty, \tag{3}$$

then $\mathcal{C}$ *contains a properly embedded two-dimensional simplex.*

Remark 5.3. In the convex co-compact case, one can show that if $\partial_i \mathcal{C} \cap F_\Omega(x) \neq \emptyset$, then $F_\Omega(x) \subset \partial_i \mathcal{C}$ (see, for instance, [13, Section 4]). So in this special case, equation (3) is equivalent to the condition that $\partial_i \mathcal{C}$ contains a line segment.

Proof. Fix a sequence $(x_n)_{n\geq 1}$ in $\partial_i \mathcal{C}$ such that

$$\operatorname{diam}_{F_\Omega(x_n)}(\partial_i \mathcal{C} \cap F_\Omega(x_n)) > n$$

for all n. Then fix $a_n, b_n \in \partial_i \mathcal{C} \cap F_\Omega(x_n)$ with

$$\mathrm{d}_{F_\Omega(x_n)}(a_n, b_n) > n.$$

We can assume that x_n is the midpoint of $[a_n, b_n]$ relative to the Hilbert distance $\mathrm{d}_{F_\Omega(x_n)}$. Also fix some $p_0 \in \mathcal{C}$.

Claim: For each n there exists $y_n \in [p_0, x_n) \subset \mathcal{C}$ such that

$$\min\{\mathrm{d}_\Omega(y_n, [p_0, a_n)), \mathrm{d}_\Omega(y_n, [p_0, b_n))\} > n/2.$$

Fix n and suppose not. Then, we can find $x_{n,m} \in [p_0, x_n)$, $a_{n,m} \in [p_0, a_n)$, and $b_{n,m} \in [p_0, b_n)$ such that $\lim_{m\to\infty} x_{n,m} = x_n$ and

$$\mathrm{d}_\Omega(x_{n,m}, \{a_{n,m}, b_{n,m}\}) \leq n/2 \quad \text{for all } m.$$

By passing to a subsequence and possibly relabeling a_n, b_n, we can assume that

$$\mathrm{d}_\Omega(x_{n,m}, \{a_{n,m}, b_{n,m}\}) = \mathrm{d}_\Omega(x_{n,m}, a_{n,m}) \leq n/2 \quad \text{for all } m.$$

Then, we must have $\lim_{m\to\infty} a_{n,m} = a_n$ and by Proposition 2.4

$$n/2 \geq \limsup_{m\to\infty} \mathrm{d}_\Omega(x_{n,m}, a_{n,m}) \geq \mathrm{d}_{F_\Omega(x_n)}(x_n, a_n) > n/2.$$

So we have a contradiction and hence the claim is established.

Next let $(\gamma_n)_{n\geq 1}$ be a sequence in Γ such that $\{\gamma_n y_n : n \geq 1\}$ is relatively compact in $\mathcal{C}$. By passing to subsequences, we can suppose that

$$\gamma_n y_n, \gamma_n a_n, \gamma_n b_n, \gamma_n p_0 \to y, a, b, p \in \overline{\mathcal{C}}.$$

Then $y \in \mathcal{C}$, by construction $[a, b] \subset \partial_i \mathcal{C}$, and by the claim

$$[b, p] \cup [p, a] \subset \partial_i \mathcal{C}.$$

So a, b, p are the vertices of a properly embedded simplex $S \subset \mathcal{C}$ which contains y. ∎

To show that (2) ⇒ (4), we will use the following sufficient condition for a metric to be Gromov hyperbolic.

Proposition 5.4. *Suppose* (X, d) *is a proper geodesic metric space,* $\delta > 0$*, and there exists a map*

$$(x, y) \in X \times X \mapsto \sigma_{x,y} \in C\big([0, \mathrm{d}(x, y)], X\big),$$

where $\sigma_{x,y}$ *is a geodesic joining* x *to* y*. If for every* $x, y, z \in X$ *distinct, the geodesic triangle formed by* $\sigma_{x,y}, \sigma_{y,z}, \sigma_{z,x}$ *is* δ*-thin, then* (X, d) *is Gromov hyperbolic.*

Proof. This proposition is a straightforward and well-known consequence of the Gromov product definition of Gromov hyperbolicity (see, for instance, [24, Proposition 2.2] for a detailed proof). ■

Lemma 5.5 ((2) ⇒ (4)). *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. If*

$$\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) = 0$$

for all $x \in \partial_{\mathrm{i}} \mathcal{C}$*, then* $(\mathcal{C}, \mathrm{d}_\Omega)$ *is Gromov hyperbolic.*

Proof. By Proposition 5.4 it suffices to show that there exists $\delta > 0$ such that every geodesic triangle in $(\mathcal{C}, \mathrm{d}_\Omega)$ whose sides are line segments is δ-thin. Suppose not. Then, for every $n \geq 0$, there exist $a_n, b_n, c_n \in \mathcal{C}$, and $u_n \in [a_n, b_n]$ such that

$$\mathrm{d}_\Omega(u_n, [a_n, c_n] \cup [c_n, b_n]) > n. \tag{4}$$

By translating by Γ and passing to a subsequence, we can suppose that $u_n \to u \in \mathcal{C}$ and

$$a_n, b_n, c_n \to a, b, c \in \overline{\mathcal{C}}.$$

By equation (4) we must have

$$[a, c] \cup [c, b] \subset \partial_{\mathrm{i}} \mathcal{C}$$

and by construction we have $u \in [a, b]$. Then, $(a, b) \subset \Omega$ since $u \in \Omega$.

Since $[a, c] \cup [c, b] \subset \partial\Omega$ and $(a, b) \subset \Omega$, Observation 2.3 part (4) implies that $c \in \partial F_\Omega(a)$. Then, Observation 4.2 implies that

$$\text{c-dim}_{F_\Omega(a)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a)) \geq 1$$

and we have a contradiction. ■

6. Proof of Theorem 1.6

In this section we prove Theorem 1.6 which we restate here.

Theorem 6.1. *Suppose* $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ *is a properly convex domain,* $\Gamma \subset \mathrm{Aut}(\Omega)$ *is convex co-compact, and* $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$*. Then the following are equivalent:*

(1) *Every boundary face of* Ω *which intersects* $\overline{\mathcal{C}}$ *has dimension at most* 1.

(2) *The collection of all properly embedded simplices in* $\mathcal{C}$ *with dimension* 2 *is closed and discrete in the local Hausdorff convergence topology induced by* d_Ω.

(3) $(\mathcal{C}, \mathrm{d}_\Omega)$ *is relatively hyperbolic with respect to a collection of two-dimensional properly embedded simplices.*

(4) Γ *is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank* 2.

For the rest of the section suppose that $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ is convex co-compact, and $\mathcal{C} := \mathcal{C}_\Omega(\Gamma)$.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2) are easy applications of Theorem 3.4.

6.1. Proof of implication (2) ⇒ (3)

Suppose that the collection of all properly embedded simplices in $\mathcal{C}$ with dimension 2 is closed and discrete in the local Hausdorff convergence topology induced by d_Ω.

Then, every properly embedded simplex in $\mathcal{C}$ has dimension at most 2. So the collection of all properly embedded simplices in $\mathcal{C}$ with dimension at least 2 coincides with the collection of all properly embedded simplices in $\mathcal{C}$ with dimension 2. So Theorem 3.4 implies that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a collection of two-dimensional properly embedded simplices.

6.2. Proof of implication (3) ⇒ (4)

Suppose that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a collection $\mathcal{S}$ of two-dimensional properly embedded simplices.

We claim that every properly embedded simplex in $\mathcal{C}$ has dimension at most 2. Suppose that $S \subset \mathcal{C}$ is a properly embedded simplex with dimension at least 2. Then, (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$ (see Proposition 2.14). So by Theorem 3.3 there exist $S' \in \mathcal{S}$ and $R > 0$ such that $S \subset \mathcal{N}_\Omega(S'; R)$. Since (S', d_Ω) is quasi-isometric to $\mathbb{R}^2$ we must have $\dim S = 2$.

Then, by Theorem 3.4 part (c), Γ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank 2.

6.3. Proof of implication (4) ⇒ (1)

Suppose that Γ is a relatively hyperbolic group with respect to $\{H_1, \dots, H_m\}$ where each H_j is a virtually Abelian subgroup of rank 2.

Let $\mathcal{S}_{\max}$ denote the family of all maximal properly embedded simplices in $\mathcal{C}_\Omega(\Gamma)$ of dimensional at least 2.

Fix $w \in \partial_{\mathrm{i}} \mathcal{C}$. We will show that $\dim F_\Omega(w) \leq 1$. It suffices to consider the case when $\dim F_\Omega(w) > 0$. Then, Theorem 3.4 parts (e) and (h) imply that there exists a simplex

$S \in \mathcal{S}_{\max}$ such that $F_\Omega(w) \subset \partial S$. Notice that $\dim S \geq 1 + \dim F_\Omega(w)$ and (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$, see Proposition 2.14.

Fix some $p \in \mathcal{C}$. By the Švarc–Milnor lemma and Theorem 3.3, there exists a coset gH_j such that S is contained in a bounded neighborhood of $gH_j \cdot p$ in $(\mathcal{C}, d_\Omega)$. Since H_j is virtually isomorphic to $\mathbb{Z}^2$, we must have $\dim S = 2$. Thus,

$$\dim F_\Omega(w) \leq -1 + \dim S = 1.$$

Since w was an arbitrary point in $\partial_i \mathcal{C}$, every boundary face of Ω which intersects $\overline{\mathcal{C}}$ has dimension at most 1.

6.4. Proof of implication (1) ⇒ (2)

Suppose that every boundary face of Ω which intersects $\overline{\mathcal{C}}$ has dimension at most 1.

Then, $\mathcal{C}$ does not contain any properly embedded simplices with dimension 3 or more. Hence, using Theorem 3.4, it is enough to show that the collection of all properly embedded two-dimensional simplices in $\mathcal{C}$ is closed and discrete in the local Hausdorff convergence topology induced by d_Ω.

Lemma 6.2. *If $\ell \subset \partial_i \mathcal{C}$ is a line segment, $S \subset \mathcal{C}$ is a properly embedded two-dimensional simplex, and $\ell \cap \partial S \neq \emptyset$, then $\ell \subset \partial S$.*

Proof. Suppose for a contradiction that there exists a line segment $\ell \subset \partial_i \mathcal{C}$ and a properly embedded two-dimensional simplex $S \subset \mathcal{C}$ such that $\ell \cap \partial S \neq \emptyset$, but ℓ is not contained in ∂S. By replacing ℓ with a subinterval we can suppose that ℓ intersects ∂S at a single point x.

If x is in a one-dimensional boundary face F of S, then the convex hull of ℓ and F provides a face in $\partial_i \mathcal{C}$ with dimension at least 2. So x must be a vertex of S.

Let $F_1, F_2 \subset \partial S$ be the edges adjacent to x. Then, pick $y_1 \in F_1$, $y_2 \in F_2$, and $y_3 \in \mathrm{relint}(\ell)$. Then, $(y_1, y_2) \subset S \subset \Omega$. If we had $[y_1, y_3] \subset \partial\Omega$, then the convex hull o F_1 and ℓ provides a face in $\partial_i \mathcal{C}$ with dimension at least 2. So $(y_1, y_3) \subset \Omega$. For the same reasons, $(y_2, y_3) \subset \Omega$. Then, Proposition 4.4 implies that there exists a face $\partial_i \mathcal{C}$ with dimensional at least 2. So we have a contradiction. ∎

Lemma 6.2 has the following consequences.

Lemma 6.3. *If $S_1, S_2 \subset \mathcal{C}$ are distinct properly embedded two-dimensional simplices, then $\partial S_1 \cap \partial S_2 = \emptyset$.*

Lemma 6.4. *If $S \subset \mathcal{C}$ is a properly embedded two-dimensional simplex, then*

$$\partial S = \bigcup_{x \in \partial S} F_\Omega(x).$$

We complete the proof of (1) ⇒ (2) by showing the following.

Lemma 6.5. *The collection of properly embedded two-dimensional simplices in $\mathcal{C}$ is closed and discrete in the local Hausdorff convergence topology.*

Proof. By Proposition 2.13 the collection of properly embedded two-dimensional simplices in $\mathcal{C}$ is closed in the local Hausdorff convergence topology. So we just have to verify discreteness.

Suppose that $S_n \to S$ in the local Hausdorff convergence topology. We need to show that $S_n = S$ for n sufficiently large. Suppose not, then by passing to a subsequence we can assume that $S_n \neq S$ for all n.

Fix $p_0 \in S$. Then for $n \geq 0$ let

$$R_n := \sup\left\{r \geq 0 : S \cap \mathcal{B}_\Omega(p_0; r) \subset \overline{\mathcal{N}_\Omega(S_n; 1)}\right\}.$$

If $R_n = \infty$ for some n, then

$$S \subset \overline{\mathcal{N}_\Omega(S_n; 1)}.$$

So by Proposition 2.4 and Lemma 6.4,

$$\partial S \subset \bigcup_{x \in \partial S_n} F_\Omega(x) = \partial S_n.$$

So Lemma 6.3 implies that $S = S_n$. Thus, we can assume that $R_n < \infty$ for all n. Further, since $S_n \to S$ in the local Hausdorff convergence topology, we see that $R_n \to \infty$ (see Observation 2.11).

Then, there exists a sequence $(q_n)_{n \geq 1}$ in S such that

(1) $\lim_{n\to\infty} \mathrm{d}_\Omega(q_n, p_0) = \infty$.

(2) $[q_n, p_0] \subset \overline{\mathcal{N}_\Omega(S_n; 1)}$.

(3) $\mathrm{d}_\Omega(q_n, S_n) = 1$.

Next pick $\gamma_n \in \Gamma$ such that $\{\gamma_n q_n : n \geq 0\}$ is a relatively compact set in $\mathcal{C}$. Then, by passing to a subsequence we can suppose that $\gamma_n q_n \to q \in \mathcal{C}$ and $\gamma_n p_0 \to p \in \partial_i \mathcal{C}$. Using Proposition 2.13 and passing to another subsequence, we can suppose that $\gamma_n S_n \to S'$ and $\gamma_n S \to S''$ where S' and S'' are both properly embedded two-dimensional simplices in $\mathcal{C}$. Further,

$$[q, p) \subset S'' \cap \overline{\mathcal{N}_\Omega(S'; 1)}.$$

Then, Proposition 2.4 implies that $p \in \partial S'' \cap \bigcup_{s' \in \partial S'} F_\Omega(s')$. Then, $p \in \partial S'' \cap \partial S'$ by Lemma 6.4. So $S'' = S'$ by Lemma 6.3. However, by construction $q \in S''$ and

$$\mathrm{d}_\Omega(q, S') = 1.$$

So we have a contradiction. ∎

7. Proof of Theorem 1.16

In this section we prove Theorem 1.16 which we restate here.

Theorem 7.1. *Suppose* $(\Omega, \mathcal{C}, \Gamma)$ *is a naive convex co-compact triple. Then the following are equivalent:*

(1) $\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) \leq 1$ *for all* $x \in \partial_{\mathrm{i}} \mathcal{C}$.

(2) $(\mathcal{C}, d_\Omega)$ *is relatively hyperbolic with respect to a collection of two-dimensional properly embedded simplices.*

(3) Γ *is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank* 2.

The proof is similar in structure to the proof of Theorem 1.6 in the previous section, but extending the argument to the naive convex co-compact case introduces a number of technicalities, especially in the proof that (1) $\Rightarrow$ (2).

Suppose for the rest of the section that $(\Omega, \mathcal{C}, \Gamma)$ is a naive convex co-compact triple. We also recall a notation that will be used frequently below: If $X \subset \overline{\Omega}$ is a subset, then

$$F_\Omega(X) = \bigcup_{x \in X} F_\Omega(x).$$

7.1. Proof of implication (2) ⇒ (3)

Suppose that $(\mathcal{C}, d_\Omega)$ is relatively hyperbolic with respect to a collection $\mathcal{S}$ of two-dimensional properly embedded simplices.

We claim that every properly embedded simplex in $\mathcal{C}$ has dimension at most 2. Suppose that $S \subset \mathcal{C}$ is a properly embedded simplex with dimension at least 2. Then, (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim S}$ (see Proposition 2.14). So by Theorem 3.3 there exist $S' \in \mathcal{S}$ and $R > 0$ such that $S \subset \mathcal{N}_\Omega(S'; R)$. Since (S', d_Ω) is quasi-isometric to $\mathbb{R}^2$ we must have $\dim S \leq 2$.

Then, by Theorem 3.8 part (4), Γ is a relatively hyperbolic group with respect to a collection of virtually Abelian subgroups of rank 2.

7.2. Proof of implication (3) ⇒ (2)

Suppose that Γ is a relatively hyperbolic group with respect to $\{H_1, \dots, H_m\}$ where each H_j is a virtually Abelian subgroup of rank 2. Then, Theorem 3.6 implies that $(\mathcal{C}, d_\Omega)$ is a relatively hyperbolic space with respect to a family $\mathcal{S}$ of properly embedded simplices of dimension at least 2. Thus, it is enough to show that if $S \in \mathcal{S}$, then $\dim(S) = 2$.

Fix $S \in \mathcal{S}$. Then, Proposition 2.14 implies that (S, d_Ω) is quasi-isometric to $\mathbb{R}^{\dim(S)}$. Next fix some $p \in \mathcal{C}$. By the Švarc–Milnor lemma and Theorem 3.3, there exists a coset gH_j such that S is contained in a bounded neighborhood of $gH_j \cdot p$ in $(\mathcal{C}, d_\Omega)$. Since H_j is virtually isomorphic to $\mathbb{Z}^2$, we must have $\dim S = 2$.

7.3. Proof of implication (2) ⇒ (1)

Suppose $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a collection $\mathcal{S}$ of two-dimensional properly embedded simplices. By Theorems 3.6 and 3.7, there exists a strongly isolated, coarsely complete, and Γ-invariant collection $\mathcal{S}_0$ of properly embedded simplices in $\mathcal{C}$ of dimension at least 2. By Proposition 2.14 and Theorem 3.3, each simplex in $\mathcal{S}_0$ is contained in a bounded neighborhood of a simplex in $\mathcal{S}$. Hence, each simplex in $\mathcal{S}_0$ is two-dimensional.

Fix $w \in \partial_{\mathrm{i}} \mathcal{C}$. We will show that $\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) \leq 1$. It suffices to consider the case when $\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) > 0$. Then,

$$\mathrm{diam}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) = +\infty$$

which implies that there exists

$$w' \in \partial_{\mathrm{i}} \mathcal{C} \cap \partial F_\Omega(w).$$

We first prove the following lemma showing that if we approach points on (w, w') non-tangentially (i.e., along a projective geodesic ray), then we are close to some properly embedded simplex. This can be viewed as a quantitative version of [3, Proposition 2.5] or [4, Lemma 3.9].

Lemma 7.2. *For any $r, \varepsilon > 0$, and $p \in \mathcal{C}$, there exist $w_0 \in (w, w')$ and $p_0 \in [p, w_0)$ such that if $x \in [p_0, w_0)$, then there exists a properly embedded simplex S_x in $\mathcal{C}$ of dimension at least 2 such that*

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(x; r) \subset \mathcal{N}_\Omega(S_x; \varepsilon).$$

Proof. Since $w' \in \partial F_\Omega(w)$, for each n we can find $w_n \in (w, w')$ such that

$$\mathrm{d}_{F_\Omega(w)}(w, w_n) = n.$$

Then $w_n \to w'$. Fix $r, \varepsilon > 0$, and $p \in \mathcal{C}$. Suppose that the lemma fails. So, in particular, it fails for each w_n. Then, for each $n \geq 1$, there exists a sequence $(q_{n,m})_{m \geq 1}$ in $[p, w_n)$ with $\lim_{m \to \infty} q_{n,m} = w_n$ and

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(q_{n,m}; r) \not\subset \mathcal{N}_\Omega(S; \varepsilon) \tag{5}$$

for any properly embedded simplex S in $\mathcal{C}$ of dimension at least 2. By Proposition 2.4,

$$\liminf_{m \to \infty} \mathrm{d}_\Omega(q_{n,m}, [p, w) \cup [p, w')) \geq \mathrm{d}_{F_\Omega(w)}(w_n, w) = n.$$

Then for each n, we choose m_n large enough such that

$$\mathrm{d}_\Omega(q_{n,m_n}, [p, w] \cup [p, w']) \geq n/2. \tag{6}$$

Set $q'_n := q_{n,m_n}$.

Since Γ acts co-compactly on $\mathcal{C}$, we can pass to a subsequence and choose $\gamma_n \in \Gamma$ such that $\gamma_n q'_n \to q'_\infty \in \mathcal{C}$. Up to passing to another subsequence, we can assume that

$$\gamma_n w', \gamma_n w, \gamma_n p \to w'_\infty, w_\infty, p_\infty \in \overline{\mathcal{C}}.$$

By construction and by equation (6),

$$[p_\infty, w'_\infty] \cup [w'_\infty, w_\infty] \cup [w_\infty, p_\infty] \subset \partial_{\mathrm{i}} \mathcal{C}\,.$$

Thus,

$$S := \mathrm{relint}(\mathrm{ConvHull}_{\overline{\Omega}}\{w_\infty, w'_\infty, p_\infty\})$$

is a properly embedded two-dimensional simplex in $\mathcal{C}$ which contains q'_∞. Then,

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(q'_n; r) \subset \mathcal{N}_\Omega(\gamma_n^{-1} S; \varepsilon)$$

for n sufficiently large, which contradicts equation (5) and concludes the proof of this lemma. ∎

We will now use Lemma 7.2 to show that there exists $S_0 \in \mathcal{S}_0$ such that $w \in F_\Omega(\partial S_0)$.

Since $\mathcal{S}_0$ is coarsely complete, there exists $R_0 \geq 0$ such that any properly embedded simplex of dimension at least 2 in $\mathcal{C}$ is contained in the R_0-tubular neighborhood of a simplex in $\mathcal{S}_0$. Fix $\varepsilon > 0$. Since $\mathcal{S}_0$ is strongly isolated, there exists $D_\varepsilon \geq 0$ such that if $S_1, S_2 \in \mathcal{S}_0$ are distinct, then

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S_1; \varepsilon + R_0) \cap \mathcal{N}_\Omega(S_2; \varepsilon + R_0)) \leq D_\varepsilon. \tag{7}$$

Fix $r := D_\varepsilon + 1$ and any point $p \in \mathcal{C}$. Apply Lemma 7.2 to r, ε, and p to get $w_0 \in (w, w')$ and $p_0 \in [p, w)$ satisfying the conclusions of the lemma. Then pick a sequence $(x_n)_{n\geq 1}$ in $[p_0, w_0)$ such that $x_n \to w_0$ and

$$\mathrm{d}_\Omega(x_n, x_{n+1}) = r$$

for all $n \geq 1$. By Lemma 7.2 and our choice of $R_0 > 0$, for each n there exists a properly embedded simplex $S_n \in \mathcal{S}_0$ such that

$$\mathbb{P}(\mathrm{Span}\{w, w', p\}) \cap \mathcal{B}_\Omega(x_n; r) \subset \mathcal{N}_\Omega(S_n; \varepsilon + R_0).$$

Then, if $n \geq 1$,

$$\begin{aligned}(x_n, x_{n+1}) &\subset \mathcal{B}_\Omega(x_n; r) \cap \mathcal{B}_\Omega(x_{n+1}; r) \cap \mathbb{P}(\mathrm{Span}\{w, w', p\}) \\ &\subset \mathcal{N}_\Omega(S_n; \varepsilon + R_0) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon + R_0).\end{aligned}$$

Thus,

$$\mathrm{diam}_\Omega(\mathcal{N}_\Omega(S_n; \varepsilon + R_0) \cap \mathcal{N}_\Omega(S_{n+1}; \varepsilon + R_0)) \geq \mathrm{d}_\Omega(x_n, x_{n+1}) = r > D_\varepsilon.$$

Then, equation (7) implies that $S_n = S_{n+1} =: S_0$ for all $n \geq 1$. Then, $\{x_n : n \in \mathbb{N}\} \subset \mathcal{N}_\Omega(S_0; \varepsilon + R_0)$ and so by Proposition 2.4,

$$w_0 = \lim_{n\to\infty} x_n \in F_\Omega(\partial S_0).$$

Then $w \in F_\Omega(\partial S_0)$ as $w_0 \in F_\Omega(w)$. By Theorem 3.8 part (6), there exists $D' > 0$ such that

$$\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w) \subset \mathcal{N}_{F_\Omega(w)}\big(F_{S_0}(w); D'\big),$$

that is,

$$\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) \leq \dim F_{S_0}(w) \leq \dim S_0 - 1 = 1.$$

This proves that for any $w \in \partial_{\mathrm{i}} \mathcal{C}$,

$$\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) \leq 1.$$

7.4. Proof of implication (1) ⇒ (2)

Suppose $\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) \leq 1$ for all $x \in \partial_{\mathrm{i}} \mathcal{C}$.

Let $\mathcal{S}_0$ denote the collection of all properly embedded simplices in $\mathcal{C}$ with dimension at least 2. By Observation 4.1, $\mathcal{C}$ does not contain any properly embedded simplices with dimension 3 or more. Hence, $\mathcal{S}_0$ consists of two-dimensional simplices.

We will construct a collection $\mathcal{S} \subset \mathcal{S}_0$ of properly embedded two-dimensional simplices which are isolated, coarsely complete, and Γ-invariant (see Definition 3.5). Then, Theorem 3.6 will imply that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a family of properly embedded simplices in $\mathcal{C}$.

We note that it is possible for $\mathcal{S}_0$ to have non-discrete families of parallel maximal properly embedded simplices (see Lemma 2.17 and [19, Section 2.3]) and hence the challenge in constructing $\mathcal{S}$ is to identify a "canonical" simplex in each family of parallel simplices. This is accomplished by using a center of mass construction, which is similar to the construction of $\mathcal{S}_{\text{core}}$ in the proof of Theorem 10.1 of [19].

Since the proof is lengthy, we provide a short outline of the steps involved. First we prove a technical result, Lemma 7.3, which implies that each family of parallel simplices is uniformly bounded (also see Lemma 2.17). This uniformity is key in the center of mass construction in equations (11) and (12). As mentioned above, this construction identifies one "canonical" simplex in each family of parallel simplices. Once this "canonical" set of simplices is constructed, the rest of the section (Lemmas 7.4–7.8) is devoted to verifying that this family is indeed isolated, coarsely complete, and Γ-invariant. These lemmas are analogs in the naive convex co-compact case of Lemma 6.2 through Lemma 6.5. The former lemmas play a similar role here as the latter lemmas did in the proof of (1) ⇒ (2) of Theorem 1.6.

We now being our proof. The key idea behind the proof of the next lemma is the following. If the lemma fails, we can use a re-scaling argument to construct a properly embedded two-dimensional simplex S with a vertex a such that $\text{c-dim}_{F_\Omega(a)} F_\Omega(a) \cap \partial_{\mathrm{i}} \mathcal{C} = 1$. We can then construct a boundary face of coarse dimension 2 and reach a contradiction.

Lemma 7.3. *There exists $R > 0$ such that if $S \in \mathcal{S}_0$ and $a \in \partial S$ is a vertex of S, then*

$$\operatorname{diam}_{F_\Omega(a)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a)) \leq R.$$

Proof. Suppose not. Then, for each $n \geq 1$, there exists a properly embedded two-dimensional simplex $S_n \subset \mathcal{C}$ with a vertex $a_n \in \partial S_n$, where

$$\operatorname{diam}_{F_\Omega(a_n)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a_n)) > n.$$

So there exists $a_n', a_n'' \in \partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a_n)$ with

$$\mathrm{d}_{F_\Omega(a_n)}(a_n', a_n'') \geq n.$$

Using Lemma 2.17 we can assume that a_n is the $\mathrm{d}_{F_\Omega(a_n)}$ Hilbert distance midpoint of $[a_n', a_n'']$.

Let $b_n, c_n \in \partial S_n$ be the other vertices of S_n. Then, Lemma 2.17 implies that

$$S_n' := \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(a_n', b_n, c_n)$$

and

$$S_n'' := \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(a_n'', b_n, c_n)$$

are properly embedded simplices in $\mathcal{C}$ with

$$\mathrm{d}_\Omega^{\mathrm{Haus}}(S_n, S_n') \leq \mathrm{d}_{F_\Omega(a_n)}(a_n, a_n')$$

and

$$\mathrm{d}_\Omega^{\mathrm{Haus}}(S_n, S_n'') \leq \mathrm{d}_{F_\Omega(a_n)}(a_n, a_n'').$$

Claim: For each $n \geq 1$ there exists $p_n \in S_n$ with

$$\min\{\mathrm{d}_\Omega(p_n, S_n'), \mathrm{d}_\Omega(p_n, S_n'')\} \geq n/2 - 1. \tag{8}$$

Fix n and a point $x_n \in (b_n, c_n)$. Then fix a sequence $(q_m)_{m \geq 1}$ in (a_n, x_n) converging to a_n. For each m, fix $q_m' \in S_n'$ with

$$\mathrm{d}_\Omega(q_m, S_n') = \mathrm{d}_\Omega(q_m, q_m').$$

Since $\mathrm{d}_\Omega^{\mathrm{Haus}}(S_n, S_n') \leq \mathrm{d}_{F_\Omega(a_n)}(a_n, a_n')$, we have

$$\mathrm{d}_\Omega(q_m, q_m') \leq \mathrm{d}_{F_\Omega(a_n)}(a_n, a_n')$$

for all $m \geq 1$.

Since $q_m \to a_n$, the above estimate and Proposition 2.4 imply that any limit point of $(q'_m)_{m\geq 1}$ is in $F_\Omega(a_n) \cap \partial S'_n = \{a'_n\}$. Thus, up to passing to a subsequence, $\lim_{m\to\infty} q'_m = a'_n$. Then, Proposition 2.4 implies that

$$\frac{n}{2} \leq \mathrm{d}_{F_\Omega(a_n)}(a_n, a'_n) \leq \liminf_{m\to\infty} \mathrm{d}_\Omega(q_m, q'_m) = \liminf_{m\to\infty} \mathrm{d}_\Omega(q_m, S'_n).$$

So for m sufficiently large $\frac{n}{2} - 1 \leq \mathrm{d}_\Omega(q_m, S'_n)$. The same reasoning shows that $\frac{n}{2} - 1 \leq \mathrm{d}_\Omega(q_m, S''_n)$ when m is large. So $p_n := q_m$ for m large enough satisfies the claim. This finishes the proof of this claim.

By passing to a subsequence and translating by Γ, we can assume that $p_n \to p \in \mathcal{C}$. Passing to further subsequences we can suppose that

$$a_n, a'_n, a''_n, b_n, c_n \to a, a', a'', b, c \in \partial_{\mathrm{i}} \mathcal{C}\,.$$

By construction $[a,b] \cup [b,c] \cup [c,a] \subset \partial_{\mathrm{i}} \mathcal{C}$ while $p \in \mathrm{relint}\,\mathrm{ConvHull}_{\overline{\Omega}}(a,b,c) \cap \mathcal{C}$. So

$$S := \mathrm{relint}\,\mathrm{ConvHull}_{\overline{\Omega}}(a,b,c) \subset \Omega \tag{9}$$

is a properly embedded simplex in $\mathcal{C}$. Equation (8) implies that

$$\mathrm{ConvHull}_{\overline{\Omega}}(a',b,c) \cup \mathrm{ConvHull}_{\overline{\Omega}}(a'',b,c) \subset \partial_{\mathrm{i}} \mathcal{C}\,. \tag{10}$$

By construction, $a_n \in [a'_n, a''_n]$ for all n and so $a \in [a', a'']$. Observation 2.3 part (4) and equations (9) and (10) imply that $a' \neq a'' \in \partial F_\Omega(a)$. So $L := (a', a'')$ is a properly embedded one-dimensional simplex in $\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a)$. Thus, Observation 4.1 implies that

$$\text{c-dim}_{F_\Omega(a)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a)) \geq \text{c-dim}_{F_\Omega(a)}(L) = \dim(L) = 1.$$

Now fix a point $x \in \partial S$ in the relative interior of an edge adjacent to a, then $a \in \partial F_\Omega(x)$ by Observation 2.15. So Observation 4.2 applied to $\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x) \subset F_\Omega(x)$ yields

$$\begin{aligned}\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) &\geq 1 + \text{c-dim}_{F_{F_\Omega(x)}(a)}(\partial_{\mathrm{i}} \mathcal{C} \cap \partial F_\Omega(x) \cap F_{F_\Omega(x)}(a)) \\ &= 1 + \text{c-dim}_{F_\Omega(a)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(a)) \\ &\geq 2.\end{aligned}$$

This is a contradiction to our hypothesis that $\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) \leq 1$ for all $x \in \partial_{\mathrm{i}} \mathcal{C}$. ∎

Next we define a map $\Phi : \mathcal{S}_0 \to \mathcal{S}_0$ which maps parallel simplices to a single simplex. Suppose $S \in \mathcal{S}_0$ has vertices v_1, v_2, v_3. By the above lemma, $\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(v_i)$ is a compact subset of $F_\Omega(v_i)$ for $i = 1, 2, 3$. Then, using the center of mass from Proposition 2.5, define

$$w_j := \mathrm{CoM}_{F_\Omega(v_j)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(v_j)) \tag{11}$$

and

$$\Phi(S) := \operatorname{relint} \operatorname{ConvHull}_{\overline{\Omega}}(w_1, w_2, w_3). \tag{12}$$

Then $\Phi(S)$ is a properly embedded two-dimensional simplex in $\mathcal{C}$ by Lemma 2.17. Then define

$$\mathcal{S} := \{\Phi(S) : S \in \mathcal{S}_0\}.$$

The next two lemmas verify that $\mathcal{S}$ is Γ-invariant and coarsely complete.

Lemma 7.4. *The set $\mathcal{S}$ is Γ-invariant.*

Proof. Since $\mathcal{S}_0$ is Γ-invariant, this follows from the equivariance of the center of mass. ■

Lemma 7.5. *If $S_1, S_2 \subset \mathcal{C}$ are properly embedded two-dimensional simplices and $\Phi(S_1) = \Phi(S_2)$, then*

$$d_\Omega^{\mathrm{Haus}}(S_1, S_2) \le R.$$

In particular, $\mathcal{S}$ is coarsely complete.

Proof. Note that $\Phi(S_1) = \Phi(S_2)$ implies that S_1 and S_2 are parallel simplices. The first assertion then follows immediately from Lemmas 7.3 and 2.17. For the in particular part, suppose $S \subset \mathcal{C}$ is a properly embedded two-dimensional simplex. Then, $\Phi(S) = \Phi(\Phi(S))$ and so by the first part

$$d_\Omega^{\mathrm{Haus}}(S, \Phi(S)) \le R.$$

Thus, $S \subset \mathcal{N}_\Omega(\Phi(S); R)$. ■

The proof that $\mathcal{S}$ is isolated is more involved and requires two preliminary lemmas.

Lemma 7.6. *If $\ell \subset \partial_i \mathcal{C}$ is a line segment, S is a properly embedded two-dimensional simplex, and $\ell \cap F_\Omega(\partial S) \neq \emptyset$, then $\ell \subset F_\Omega(\partial S)$.*

Proof. It is enough to consider the case where $\ell = [x, y] \subset \partial_i \mathcal{C}$ and S is a properly embedded two-dimensional simplex $S \subset \mathcal{C}$ with $x \in F_\Omega(\partial S)$. Using Observation 2.3 we may assume that $x \in \partial S$. Indeed, by definition there exists $x_0 \in \partial S$ such that $x \in F_\Omega(x_0)$. Then the projective line segment $\ell_0 := [x_0, y] \subset \partial_i \mathcal{C}$ also satisfies our assumptions and Observation 2.3 implies that $\ell \subset F_\Omega(\ell_0)$. Hence, without loss of generality, we will make the simplifying assumption that $x \in \partial S$.

Now suppose, for a contradiction, that ℓ is not contained in $F_\Omega(\partial S)$. Since ℓ is not contained in $F_\Omega(\partial S)$ we must have $y \notin F_\Omega(\partial S)$.

Recall that by hypothesis, $\text{c-dim}_{F_\Omega(x')}(\partial_i \mathcal{C} \cap F_\Omega(x')) \le 1$ for any $x' \in \partial_i \mathcal{C}$. Our proof will be a case-by-case analysis where we arrive at a contradiction in each case by finding a point in $\partial_i \mathcal{C}$ where the above hypothesis on coarse dimension fails. Since $x \in \partial S$, there are two cases to consider based on whether x is a vertex of S or x is contained in an edge of S.

Case 1: Assume x is contained in an edge of S.

In this case, fix some $m \in (x, y)$. Then, $m \in \partial_i \mathcal{C}$ and there are two sub-cases to consider depending on whether $x \in \partial F_\Omega(m)$ or $x \in F_\Omega(m)$.

Case 1 (a): Assume $x \in \partial F_\Omega(m)$. In this case, we will arrive at a contradiction by showing that the coarse dimension of $\partial_i \mathcal{C} \cap F_\Omega(m)$ is at least 2.

To this end, we first apply Observation 4.2 to the properly convex domain $F_\Omega(m)$ in $\mathbb{P}(\mathbb{R}^{d'})$, where $d' := \dim F_\Omega(m)$, and the non-empty convex subset $\partial_i \mathcal{C} \cap F_\Omega(m) \subset F_\Omega(m)$. Note that in this case, $\partial_i(\partial_i \mathcal{C} \cap F_\Omega(m)) = \partial_i \mathcal{C} \cap \partial F_\Omega(m)$. Thus, Observation 4.2 yields

$$\text{c-dim}_{F_\Omega(m)}(\partial_i \mathcal{C} \cap F_\Omega(m)) \ge 1 + \text{c-dim}_{F_{F_\Omega(m)}(x)}(\partial_i \mathcal{C} \cap \partial F_\Omega(m) \cap F_{F_\Omega(m)}(x)). \quad (13)$$

We now claim that

$$F_S(x) \subset \partial_i \mathcal{C} \cap \partial F_\Omega(m) \cap F_{F_\Omega(m)}(x).$$

To prove the claim, first observe that $F_{F_\Omega(m)}(x) = F_\Omega(x)$. Then, the only non-trivial part in the claim is to show that $F_S(x) \subset \partial F_\Omega(m)$. Indeed, since $x \in \partial F_\Omega(m)$, Observation 2.3 part (3) implies that $F_\Omega(x) \subset \partial F_\Omega(m)$. Since $\mathcal{S} \subset \Omega$ is properly embedded, $F_S(x) \subset F_\Omega(x)$ and thus $F_S(x) \subset \partial F_\Omega(m)$.

Then, the above claim and the inequality in (13) imply that

$$\text{c-dim}_{F_\Omega(m)}(\partial_i \mathcal{C} \cap F_\Omega(m)) \ge 1 + \text{c-dim}_{F_{F_\Omega(m)}(x)}(F_S(x)) = 1 + \text{c-dim}_{F_\Omega(x)}(F_S(x)).$$

By Observation 4.1, $\text{c-dim}_{F_\Omega(x)}(F_S(x)) = \dim(F_S(x)) = 1$. Thus,

$$\text{c-dim}_{F_\Omega(m)}(\partial_i \mathcal{C} \cap F_\Omega(m)) \ge 2$$

and we have a contradiction.

Case 1 (b): Assume $x \in F_\Omega(m)$ or equivalently $m \in F_\Omega(x)$. In this case again, we will arrive at a contradiction by showing that the coarse dimension of $\partial_i \mathcal{C} \cap F_\Omega(m)$ is at least 2.

Recall that $x \in S$ and $y \notin F_\Omega(\partial S)$. Since $(x, y) \subset \partial\Omega$, then we must have $y \in \partial F_\Omega(x)$. Let $v_1, v_2 \in \partial S$ be the vertices of the edge containing x. Then, by Observation 2.15

$$v_1, v_2 \in \partial_i \mathcal{C} \cap \partial F_\Omega(m)$$

and $F_\Omega(v_1)$ and $F_\Omega(v_2)$ are distinct. Further, since $y \notin F_\Omega(\partial S)$ the faces $F_\Omega(v_1)$, $F_\Omega(v_2)$, $F_\Omega(y)$ are all distinct.

Finally, we will apply Observation 4.3 to the properly convex domain $F_\Omega(m)$ in $\mathbb{P}(\mathbb{R}^{d'})$, where $d' := \dim F_\Omega(m)$, and the non-empty convex subset $\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(m) \subset F_\Omega(m)$. The three points in $\overline{F_\Omega(m)}$ that we consider are v_1, v_2, and y. Since $F_{F_\Omega(m)}(\cdot) = F_\Omega(\cdot)$ for any point in $\overline{F_\Omega(m)}$, the faces in $F_\Omega(m)$ of the three points v_1, v_2, and y are pairwise distinct. Note that in this case, $\partial_{\mathrm{i}}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(m)) = \partial_{\mathrm{i}} \mathcal{C} \cap \partial F_\Omega(m)$. Thus, by Observation 4.3, we have

$$\text{c-dim}_{F_\Omega(m)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(m)) \geq 2$$

and hence a contradiction.

Case 2: Assume x is a vertex of S. In this case, we will arrive at a contradiction by finding a point $w \in \partial_{\mathrm{i}} \mathcal{C}$ for which the coarse dimension of $\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)$ is at least 2. In particular, we will use Proposition 4.4 to find such a point w.

Let $y_1, y_2 \in \partial S$ be points on the edges adjacent to x. Then, $(y_1, y_2) \subset \Omega$. Then,

$$[x, y_1] \cup [x, y_2] \cup [x, y] \subset \partial_{\mathrm{i}} \mathcal{C}$$

and $(y_1, y_2) \subset \Omega$. We claim that $(y_1, y) \subset \Omega$. If not, then we could apply Case 1 to the line segment $\ell' := [y_1, y]$ and obtain a contradiction. So we must have $(y_1, y) \subset \Omega$. By symmetry we also have $(y_2, y) \subset \Omega$. But then by Proposition 4.4 there exists $w \in \partial_{\mathrm{i}} \mathcal{C}$ with

$$\text{c-dim}_{F_\Omega(w)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(w)) \geq 2.$$

So we have a contradiction. ∎

Lemma 7.7. *If $S_1, S_2 \subset \mathcal{C}$ are properly embedded two-dimensional simplices and $F_\Omega(\partial S_1) \cap F_\Omega(\partial S_2) \neq \emptyset$, then $\Phi(S_1) = \Phi(S_2)$.*

Proof. Lemma 7.6 implies that $F_\Omega(\partial S_1) = F_\Omega(\partial S_2)$. Suppose $v_1, v_2, v_3 \in \partial S_1$ are the vertices of S_1. Then, there exist $w_1, w_2, w_3 \in \partial S_2$ such that $F_\Omega(v_j) = F_\Omega(w_j)$. Then, Lemma 7.3 and Observation 2.15 imply that w_1, w_2, w_3 are the vertices of S_2. So by definition $\Phi(S_1) = \Phi(S_2)$. ∎

Lemma 7.8. *The set $\mathcal{S}$ is isolated, that is $\mathcal{S}$ is closed and discrete in the local Hausdorff convergence topology.*

Proof. By Proposition 2.13 the collection $\mathcal{S}_0$ of all properly embedded two-dimensional simplices in $\mathcal{C}$ is closed in the local Hausdorff convergence topology. So to show that $\mathcal{S}$ is closed and discrete in the local Hausdorff convergence topology, it is enough to fix a sequence $(S_n)_{n \geq 1}$ in $\mathcal{S}$ such that S_n converges in the local Hausdorff convergence topology to a properly embedded two-dimensional simplex S and then show that $S_n = S$ for n sufficiently large.

Suppose not, then by passing to a subsequence we can suppose that $S_n \neq S$ for all n. Fix $p_0 \in S$. Then, for $n \geq 0$, let

$$R_n := \sup\left\{r \geq 0 : S \cap \mathcal{B}_\Omega(p_0; r) \subset \overline{\mathcal{N}_\Omega(S_n; R+1)}\right\},$$

where $R > 0$ is as in the statements of Lemmas 7.3 and 7.5. After passing to a subsequence, we can consider the following two cases.

Case 1: Assume $R_n = \infty$ for all n. Then, for any n,

$$S \subset \overline{\mathcal{N}_\Omega(S_n; R+1)}$$

and so by Proposition 2.4

$$\partial S \subset F_\Omega(\partial S_n).$$

Then, Lemma 7.7 implies that $\Phi(S) = \Phi(S_n) = S_n$ for all n. Since $S_n \to S$, we then have $S = \Phi(S) = S_n$ for all n. So we have a contradiction.

Case 2: Assume $R_n < \infty$ for all n. Since $S_n \to S$ in the local Hausdorff convergence topology, we see that $R_n \to \infty$ (see Observation 2.11). Then, there exists a sequence $(q_n)_{n\geq 1}$ in S such that

(1) $\lim_{n\to\infty} \mathrm{d}_\Omega(q_n, p_0) = \infty$.

(2) $[q_n, p_0] \subset \overline{\mathcal{N}_\Omega(S_n; R+1)}$.

(3) $\mathrm{d}_\Omega(q_n, S_n) = R + 1$.

Next pick $\gamma_n \in \Gamma$ such that $\{\gamma_n q_n : n \geq 0\}$ is relatively compact in $\mathcal{C}$. Then by passing to a subsequence, we can suppose that $\gamma_n q_n \to q \in \mathcal{C}$ and $\gamma_n p_0 \to p \in \partial_{\mathrm{i}} \mathcal{C}$. Using Proposition 2.13 and passing to another subsequence, we can suppose that $\gamma_n S_n \to S'$ and $\gamma_n S \to S''$ where S' and S'' are properly embedded two-dimensional simplices in $\mathcal{C}$. Further,

$$[q, p) \subset S'' \cap \overline{\mathcal{N}_\Omega(S'; R+1)}.$$

Then Proposition 2.4 implies that $p \in \partial S'' \cap F_\Omega(S')$. So $\Phi(S') = \Phi(S'')$ by Lemma 7.7. However, by construction $q \in S''$ and $\mathrm{d}_\Omega(q, S') = R + 1$. So we have a contradiction with Lemma 7.5. ∎

Thus $\mathcal{S}$ is isolated, coarsely complete, and Γ-invariant by Lemmas 7.4, 7.5, and 7.8. Then, Theorem 3.6 implies that $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a family $\mathcal{S}_\diamond$ of properly embedded simplices in $\mathcal{C}$ of dimension at least 2. Note that $\mathcal{S}_\diamond$ is not necessarily $\mathcal{S}$; see the discussion following Theorem 3.6.

Since $\text{c-dim}_{F_\Omega(x)}(\partial_{\mathrm{i}} \mathcal{C} \cap F_\Omega(x)) \leq 1$ for all $x \in \partial_{\mathrm{i}} \mathcal{C}$, Observation 4.1 implies that $\mathcal{C}$ does not contain any properly embedded simplices with dimension 3 or more. So each simplex in $\mathcal{S}_\diamond$ is two-dimensional. Thus, $(\mathcal{C}, \mathrm{d}_\Omega)$ is relatively hyperbolic with respect to a collection of two-dimensional properly embedded simplices. This completes the proof of this direction.

A. Proof of Observation 2.3

At the request of one of the referees, we include a proof of Observation 2.3 which we restate here.

Observation A.1. *Suppose $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a properly convex domain.*

(1) *$F_\Omega(x)$ is convex and open in its span*

(2) *$y \in F_\Omega(x)$ if and only if $x \in F_\Omega(y)$ if and only if $F_\Omega(x) = F_\Omega(y)$,*

(3) *if $y \in \partial F_\Omega(x)$, then $F_\Omega(y) \subset \partial F_\Omega(x)$, and*

(4) *if $x, y \in \overline{\Omega}$, $z \in (x, y)$, $p \in F_\Omega(x)$, and $q \in F_\Omega(y)$, then*

$$(p, q) \subset F_\Omega(z).$$

In particular, $(p, q) \subset \Omega$ if and only if $(x, y) \subset \Omega$ (see Figure 1).

For the rest of the section, fix a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$.

Lemma A.2. *If $x \in \overline{\Omega}$ and $y \in F_\Omega(x)$, then $F_\Omega(x) = F_\Omega(y)$.*

Proof. We start by showing that $F_\Omega(x) \subset F_\Omega(y)$. To that end, fix $z \in F_\Omega(x)$ and let $V := \mathrm{Span}\{x, y, z\}$. If $\dim V \leq 2$, then it is clear that $z \in F_\Omega(y)$. So suppose that $\dim V = 3$. Then we can fix coordinates on V so that

$$x = [1 : 0 : 0], \quad y = [1 : 1 : 0], \quad z = [1 : 0 : 1].$$

Since $y, z \in F_\Omega(x)$, there exists $\varepsilon > 0$ such that

$$[1 : -\varepsilon : 0], [1 : 1 + \varepsilon : 0], [1 : 0 : -\varepsilon], [1 : 0 : 1 + \varepsilon] \in \overline{\Omega}.$$

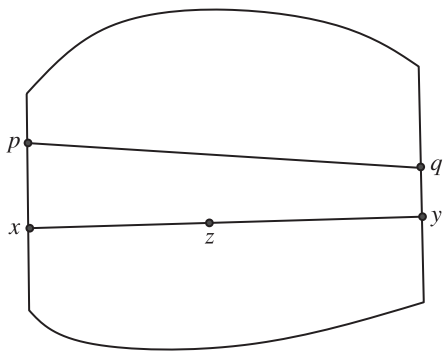

Figure 1. Figure for the proof of part (4) Case 4, when the convex hull of $\overline{\ell}_1 \cup \overline{\ell}_2$ is a two-dimensional 4-gon.

Since the convex hull of these points is in $\overline{\Omega}$, we see that $z \in F_\Omega(y)$. Hence, $F_\Omega(x) \subset F_\Omega(y)$.

Then, $x \in F_\Omega(x) \subset F_\Omega(y)$ and so the above argument implies that $F_\Omega(y) \subset F_\Omega(x)$. ■

Proof of (1). We first show that $F_\Omega(x)$ is convex. Fix $y, z \in F_\Omega(x)$. Then by the lemma, $z \in F_\Omega(y)$ and so $[y, z] \subset F_\Omega(y) = F_\Omega(x)$. So $F_\Omega(x)$ is convex. Then by definition $F_\Omega(x)$ is open in its span. ■

Proof of (2). This follows immediately from the lemma. ■

Proof of (3). Since $F_\Omega(y) \cap F_\Omega(x) = \emptyset$, it suffices to show that $F_\Omega(y) \subset \overline{F_\Omega(x)}$. To that end, fix $z \in F_\Omega(y)$ and let $V := \mathrm{Span}\{x, y, z\}$. If $\dim V \leq 2$, then $z = y \in \overline{F_\Omega(x)}$. So suppose that $\dim V = 3$. Then, we can fix coordinates on V so that

$$x = [1:0:0], \quad y = [1:1:0], \quad z = [1:1:1].$$

Since $F_\Omega(x)$ is open in its span and $z \in F_\Omega(y)$, there exists $\varepsilon > 0$ such that

$$[1:-\varepsilon:0], [1:1:-\varepsilon], [1:1:1+\varepsilon] \in \overline{\Omega}.$$

Since the convex hull of these points is in $\overline{\Omega}$, we see that $z \in \overline{F_\Omega(x)}$. Hence, $F_\Omega(y) \subset \overline{F_\Omega(x)}$. ■

Proof of (4). By symmetry it suffices to consider the following cases.

Case 1: Assume $F_\Omega(x) = F_\Omega(y)$. In this case, $(p, q) \subset F_\Omega(x)$ and $z \in F_\Omega(x)$. So by part (2), $(p, q) \subset F_\Omega(x) = F_\Omega(z)$.

Then, for the rest of the cases, we may assume that $F_\Omega(x) \cap F_\Omega(y) = \emptyset$.

Case 2: Assume $x = p$ and $y = q$. Then, $z \in (x, y) = (p, q)$ and so $(p, q) \subset F_\Omega(z)$.

Case 3: Assume $x = p$ and $y \neq q$. In this case, fix an open line segment $\ell \subset \overline{\Omega}$ with $y, q \in \ell$. Then, the convex hull of $\{x\} \cup \overline{\ell}$ in $\overline{\Omega}$ is a two-dimensional simplex whose relative interior contains (p, q) and z. Hence, $(p, q) \subset F_\Omega(z)$.

Case 4: Assume $x \neq p$ and $y \neq q$. In this case, fix open line segments $\ell_1, \ell_2 \subset \overline{\Omega}$ with $x, p \in \ell_1$, and $y, q \in \ell_2$. Then, the convex hull of $\overline{\ell_1} \cup \overline{\ell_2}$ in $\overline{\Omega}$ is either a two-dimensional 4-gon or a three-dimensional simplex. In either case, the relative interior of this convex hull contains (p, q) and z. Hence, $(p, q) \subset F_\Omega(z)$. ■

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