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Acylindrical hyperbolicity of Artin groups associated with graphs that are not cones

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Abstract. Charney and Morris-Wright showed acylindrical hyperbolicity of Artin groups of infinite type associated with graphs that are not joins, by studying clique-cube complexes and the actions on them. In this paper, by developing their study and formulating some additional discussion, we demonstrate that acylindrical hyperbolicity holds for more general Artin groups. Indeed, we are able to treat Artin groups of infinite type associated with graphs that are not cones.

1. Introduction

Artin groups, also called Artin–Tits groups, have been widely studied since their introduction by Tits [30]. In particular, Artin groups are important examples in geometric group theory. For various nonpositively curved or negatively curved properties on discrete groups, Artin groups are interesting targets. In this paper, we consider acylindrical hyperbolicity of Artin groups.

Let Γ be a finite simple graph with the vertex set $V = V(\Gamma)$ and the edge set $E = E(\Gamma)$. Each edge e has two end vertices, which we denote by s_e and t_e. We suppose that any edge e is labeled by an integer $\mu(e) \geq 2$. The *Artin group* A_Γ associated with Γ is defined by the following presentation:

$$A_\Gamma = \langle V(\Gamma) \mid \underbrace{s_e t_e s_e t_e \cdots}_{\text{length } \mu(e)} = \underbrace{t_e s_e t_e s_e \cdots}_{\text{length } \mu(e)} \text{ for all } e \in E(\Gamma)\rangle. \tag{1.1}$$

Free abelian groups, free groups, and braid groups are typical examples of Artin groups. Adding the relation $v^2 = 1$ for all $v \in V(\Gamma)$ to (1.1) produces the associated *Coxeter group* W_Γ. In terms of the properties of W_Γ, we can define several important classes of Artin groups. The Artin group A_Γ is said to be *of finite type* if W_Γ is finite. Otherwise, it is said to be *of infinite type*. The Artin group A_Γ is said to be *irreducible* if W_Γ is *irreducible*, that is, the defining graph Γ cannot be decomposed as a join of two subgraphs such that all edges between them are labeled by 2. It is well known that an infinite Coxeter group W_Γ is irreducible if and only if W_Γ cannot be directly decomposed into two nontrivial subgroups [24, 28]. However, it is unclear whether A_Γ is irreducible if and only if A_Γ cannot

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be directly decomposed into two nontrivial subgroups. In general, Coxeter groups are well understood, but many basic questions for Artin groups remain open (refer to [10, 16]).

We consider nonpositively curved or negatively curved properties on Artin groups. The following is one of the most important open problems [10, Problem 4].

Problem 1.1. Which Artin groups are CAT(0) *groups*, that is, groups acting geometrically on CAT(0) spaces?

Here, CAT(0) *spaces* are geodesic spaces in which every geodesic triangle is not fatter than the comparison triangle in the Euclidean plane (see [5] for the precise definition). A group action is said to be *geometric* if the action is proper, cocompact, and isometric. In recent studies on geometric group theory, various properties besides the CAT(0) property have been actively investigated, such as systolic property and the Helly property (see, for example, [19, 20]).

In this paper, we consider the following problem [17, Conjecture B].

Problem 1.2. Are irreducible Artin groups of infinite type acylindrically hyperbolic?

The definition of acylindrical hyperbolicity is given in Section 2. There are many applications of acylindrical hyperbolicity (see, for example, [13, 25], and [26]).

Remark 1.3. (1) Reducible Artin groups can be directly decomposed into two infinite subgroups. However, acylindrical hyperbolic groups cannot be directly decomposed into two infinite subgroups [25, Corollary 7.3]. Hence, such Artin groups are not acylindrically hyperbolic.

(2) Irreducible Artin groups of finite type have infinite cyclic centers [6, 14]. Because acylindrical hyperbolic groups do not permit infinite centers [25, Corollary 7.3], such Artin groups are not acylindrically hyperbolic. We remark that the central quotients for irreducible Artin groups of finite type are acylindrically hyperbolic (see [2, 3, 18] for braid groups, and [8] for the general case).

Many affirmative partial answers for Problem 1.2 are known. Indeed, the following irreducible Artin groups of infinite type are known to be acylindrically hyperbolic:

- right-angled Artin groups [9, 22];
- two-dimensional Artin groups such that the associated Coxeter groups are hyperbolic [23];
- Artin groups of XXL-type [17];
- Artin groups of type FC such that the defining graphs have diameter greater than two [12];
- Artin groups that are known to be CAT(0) groups according to the result of Brady and McCammond [4] (see also [21]);
- Euclidean Artin groups [7].

Actually, except for Euclidean Artin groups, all of these Artin groups are regarded as special cases of the following irreducible Artin groups of infinite type, which are known to be acylindrically hyperbolic:

- Artin groups associated with graphs that are not joins [11];
- *two-dimensional* Artin groups, that is, Artin groups such that every triangle with three vertices v_1, v_2, v_3 of the defining graphs satisfies (see [32])

$$\frac{1}{\mu((v_1, v_2))} + \frac{1}{\mu((v_2, v_3))} + \frac{1}{\mu((v_3, v_1))} \leq 1.$$

Charney and Morris-Wright [11] showed acylindrical hyperbolicity of Artin groups of infinite type associated with graphs that are not joins, by studying clique-cube complexes, which are CAT(0) cube complexes, and the isometric actions on them. In fact, they constructed a WPD (weak properly discontinuous) contracting element of such an Artin group with respect to the isometric action on the clique-cube complex. In this paper, we generalize this result by developing their study and formulating some additional discussion. Our main theorem can be stated as follows.

Theorem 1.4. *Let A_Γ be an Artin group associated with Γ, where Γ has at least three vertices. Suppose that Γ is not a cone. Then, the following are equivalent:*

(1) *A_Γ is irreducible, that is, Γ cannot be decomposed as a join of two subgraphs such that all edges between them are labeled by* 2*;*

(2) *A_Γ has a WPD contracting element with respect to the isometric action on the clique-cube complex;*

(3) *A_Γ is acylindrically hyperbolic;*

(4) *A_Γ is directly indecomposable, that is, it cannot be decomposed as a direct product of two nontrivial subgroups.*

Remark 1.5. When an Artin group A_Γ is irreducible and the defining graph Γ is not a cone, the center $Z(A_\Gamma)$ is known to be trivial. This fact is shown in [11]. We present an alternative proof based on Theorem 1.4 (see Remark 6.6).

From Theorem 1.4, we find that many irreducible Artin groups of infinite type are acylindrically hyperbolic, e.g., the Artin groups associated with the defining graphs in Figure 1.

The remainder of this paper is organized as follows. Section 2 contains some preliminaries regarding acylindrically hyperbolic groups, WPD contracting elements, and CAT(0) cube complexes. Section 3 presents preliminaries on defining graphs of Artin groups and joins of graphs. In Section 4, we treat clique-cube complexes and the actions on them by Artin groups following [11]. In Section 5, we study the local geometry of clique-cube complexes. Section 6 gives a proof of Theorem 1.4. Our main task is to construct a candidate WPD contracting element and show that it really is a WPD contracting element.

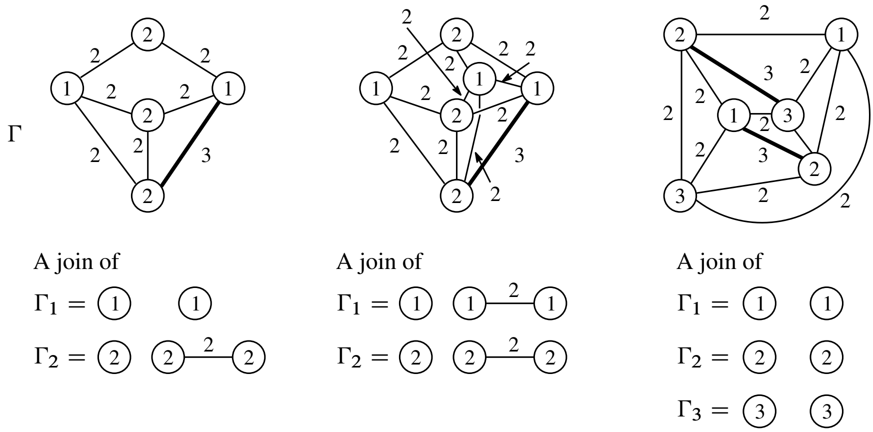

Figure 1. Defining graphs Γ of new examples A_Γ.

2. Acylindrical hyperbolicity, weak properly discontinuous contracting elements, and CAT(0) cube complexes

In this section, we collect some definitions and properties related to acylindrical hyperbolicity, WPD contracting elements, and CAT(0) cube complexes that will be used later in the paper. See [15] and the references therein for details.

First, we recall the definition of acylindrical hyperbolicity (see [25]).

Definition 2.1. A group G is *acylindrically hyperbolic* if it admits an isometric action on a hyperbolic space Y that is *non-elementary* (i.e., with an infinite limit set) and *acylindrical* (i.e., for every $D \geq 0$, there exist some $R, N \geq 0$ such that, for all $y_1, y_2 \in Y$, $d_Y(y_1, y_2) \geq R$ implies $\#\{g \in G \mid d_Y(y_1, g(y_1)), d_Y(y_2, g(y_2)) \leq D\} \leq N$).

Next, we recall the definition of a WPD contracting element.

Definition 2.2. Let a group G act isometrically on a metric space X. For $\gamma \in G$, we say that

- γ is *WPD* if, for every $D \geq 0$ and $x \in X$, there exists some $M \geq 1$ such that
$$\#\{g \in G \mid d_X(x, g(x)), d_X(\gamma^M(x), g\gamma^M(x)) \leq D\} < \infty;$$
- γ is *contracting* if γ is *loxodromic*, that is, there exists $x_0 \in X$ such that $\mathbb{Z} \to X$, $n \mapsto \gamma^n(x_0)$ is a quasi-isometry onto the image $\gamma^{\mathbb{Z}} x_0 := \{\gamma^n(x_0) \mid n \in \mathbb{Z}\}$, and $\gamma^{\mathbb{Z}} x_0$ is *contracting*, that is, there exists $B \geq 0$ such that the diameter of the nearest-point projection of any ball that is disjoint from $\gamma^{\mathbb{Z}} x_0$ onto $\gamma^{\mathbb{Z}} x_0$ is bounded by B.

The following is a consequence of [1].

Theorem 2.3. *Let a group G act isometrically on a geodesic metric space X. Suppose that G is not virtually cyclic. If there exists a WPD contracting element $\gamma \in G$, then G is acylindrically hyperbolic.*

CAT(0) cube complexes are considered as generalized trees in higher dimensions. The following is a precise definition (see [5, p. 111]).

Definition 2.4. A *cube complex* is a CW complex constructed by gluing together cubes of arbitrary (finite) dimension by isometries along their faces. Furthermore, the cube complex is *nonpositively curved* if the link of any of its vertices is a *simplicial flag* complex (i.e., $n+1$ vertices span an n-simplex if and only if they are pairwise adjacent), and CAT(0) if it is nonpositively curved and simply connected.

Definition 2.5. Let X be a CAT(0) cube complex. We define an equivalence relation for the edges of X as the transitive closure of the relation identifying two parallel edges of a square. For an equivalence class, a *hyperplane* is defined as the union of the midcubes transverse to the edges belonging to the equivalence class. Then, for any edge belonging to the equivalence class, the hyperplane is said to be *dual to* the edge.

For a hyperplane J, we denote the union of the cubes intersecting J by $N(J)$, that is, the smallest subcomplex of X containing J. We denote the union of the cubes not intersecting J by $X \backslash\backslash J$, that is, the largest subcomplex of X not intersecting J.

See [29] for the following.

Theorem 2.6. *Let X be a* CAT(0) *cube complex and J be a hyperplane. Then, $X \backslash\backslash J$ has exactly two connected components.*

The two connected components of $X \backslash\backslash J$ are often denoted by J^+ and J^-.

For convenience, we prepare the following for the proof of Theorem 1.4.

Definition 2.7. Let X be a CAT(0) cube complex. For two vertices x and x' in X, we call a sequence of hyperplanes $P_1, \dots, P_M$ a *sequence of separating hyperplanes from x to x'* if the sequence satisfies

$$x \in P_1^-, \quad P_1^+ \supsetneq P_2^+ \supsetneq \cdots \supsetneq P_{M-1}^+ \supsetneq P_M^+ \ni x'$$

for some connected components P_i^+ of $X \backslash\backslash P_i$ for all $i \in \{1, \dots, M\}$.

For two hyperplanes J and J' in X, we call a sequence of hyperplanes $P_1, \dots, P_M$ a *sequence of separating hyperplanes from J to J'* if the sequence satisfies

$$J^+ \supsetneq P_1^+ \supsetneq P_2^+ \supsetneq \cdots \supsetneq P_{M-1}^+ \supsetneq P_M^+ \supsetneq J'^+$$

for some connected components J^+ of $X \backslash\backslash J$, J'^+ of $X \backslash\backslash J'$, and P_i^+ of $X \backslash\backslash P_i$ for all $i \in \{1, \dots, M\}$.

Remark 2.8. When $P_1, \dots, P_M$ is a sequence of separating hyperplanes from J to J', for two vertices $x \in J^- \cup N(J)$ and $x' \in J'^+ \cup N(J')$, $P_1, \dots, P_M$ is a sequence of separating hyperplanes from x to x'.

The following is a part of [15, Theorem 3.3] and is used in the proof of Theorem 1.4.

Theorem 2.9. *Let a group G act isometrically on a* CAT(0) *cube complex X. Then, $\gamma \in G$ is a WPD contracting element if there exist two hyperplanes J and J' satisfying the following:*

(i) *J and J' are strongly separated, that is, no hyperplane can intersect both J and J';*

(ii) *γ skewers J and J', that is, we have connected components J^+ of $X \backslash\backslash J$ and J'^+ of $X \backslash\backslash J'$ such that $\gamma^n(J^+) \subsetneq J'^+ \subsetneq J^+$ for some $n \in \mathbb{N}$;*

(iii) $\mathrm{stab}(J) \cap \mathrm{stab}(J')$ *is finite, where* $\mathrm{stab}(J) = \{g \in G \mid g(J) = J\}$ *and* $\mathrm{stab}(J') = \{g \in G \mid g(J') = J'\}$.

3. Defining graphs of Artin groups and joins

3.1. Defining graphs of Artin groups

We now present a precise description of the defining graph of an Artin group and introduce some related graphs.

Let V be a finite set. Denote the diagonal set by $\mathrm{diag}(V \times V) := \{(v, w) \in V \times V \mid v = w\}$. We consider the involution on the off-diagonal set

$$\iota\colon V \times V \setminus \mathrm{diag}(V \times V) \ni (v, w) \mapsto (w, v) \in V \times V \setminus \mathrm{diag}(V \times V).$$

Any $e \in V \times V \setminus \mathrm{diag}(V \times V)$ is often presented as (s_e, t_e). Then, for any $e \in V \times V \setminus \mathrm{diag}(V \times V)$, we have $s_{\iota(e)} = t_e$ and $t_{\iota(e)} = s_e$. We take a symmetric map

$$\widetilde{\mu}\colon V \times V \setminus \mathrm{diag}(V \times V) \to \mathbb{Z}_{\geq 2} \cup \{\infty\}.$$

Here, '*symmetric*' means that $\widetilde{\mu} \circ \iota = \widetilde{\mu}$ is satisfied. Set $E_m := \widetilde{\mu}^{-1}(m)$ for any $m \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Then, we have

$$V \times V \setminus \mathrm{diag}(V \times V) = \bigsqcup_{m \in \mathbb{Z}_{\geq 2} \cup \{\infty\}} E_m.$$

We now have a finite simple labeled graph Γ with the vertex set $V(\Gamma) = V$, the edge set $E(\Gamma) = \bigsqcup_{m \in \mathbb{Z}_{\geq 2}} E_m$, and the labeling $\mu := \widetilde{\mu}|_{E(\Gamma)}$. The Artin group A_Γ associated with Γ is then defined by presentation (1.1), and Γ is called the *defining graph* of A_Γ.

For convenience, we define two other finite simple graphs Γ^c and Γ^t as follows. The graph Γ^c is the finite simple graph with the vertex set $V(\Gamma^c) = V$ and the edge set $E(\Gamma^c) = E_\infty$. This is the so-called *complement graph* of Γ. The graph Γ^t is the finite simple graph with the vertex set $V(\Gamma^t) = V$ and the edge set $E(\Gamma^t) = \bigsqcup_{m \in \mathbb{Z}_{\geq 3} \cup \{\infty\}} E_m$. See Figures 1 and 2.

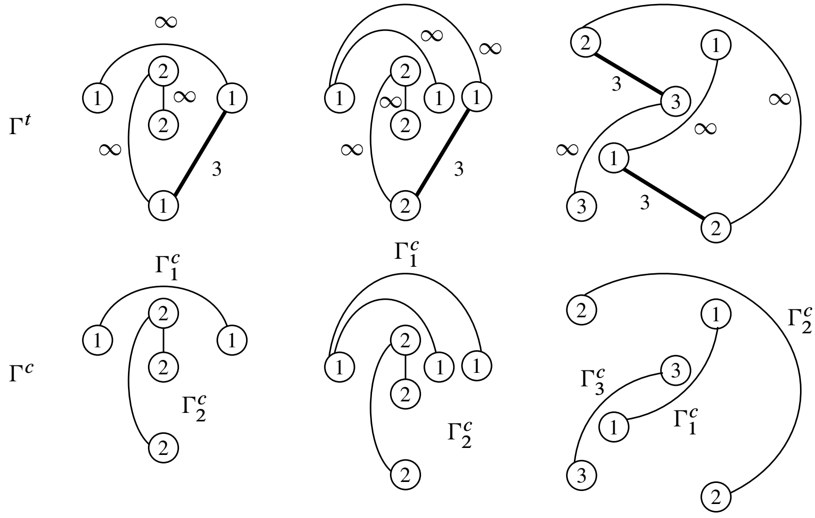

Figure 2. The graphs Γ^t and Γ^c with respect to Γ in Figure 1.

Remark 3.1. In research related to Coxeter groups and some traditional treatments of Artin groups, Γ^t is used with the label $\widetilde{\mu}|_{E(\Gamma^t)}$. We mainly use Γ in accordance with many recent studies on Artin groups. We only use Γ^t as an aid in this paper.

3.2. Joins

Definition 3.2. Let $\Lambda \neq \emptyset$ be an index set. The *join* $*_{\alpha\in\Lambda}\Gamma_\alpha$ of simple graphs Γ_α, $\alpha \in \Lambda$, is defined as a simple graph with the vertex set

$$V(*_{\alpha\in\Lambda}\Gamma_\alpha) := \bigsqcup_{\alpha\in\Lambda} V(\Gamma_\alpha)$$

and the edge set

$$E(*_{\alpha\in\Lambda}\Gamma_\alpha) := \bigsqcup_{\alpha\in\Lambda} E(\Gamma_\alpha) \sqcup \bigsqcup_{\alpha,\beta\in\Lambda,\alpha\neq\beta} \{(v_\alpha, v_\beta) \mid v_\alpha \in V(\Gamma_\alpha), v_\beta \in V(\Gamma_\beta)\}.$$

A simple graph Γ is said to be *decomposable* (as a join) if there exist an index set Λ with $\#\Lambda \geq 2$ and subgraphs Γ_α, $\alpha \in \Lambda$, of Γ such that $\Gamma = *_{\alpha\in\Lambda}\Gamma_\alpha$. This is called a *join decomposition* of Γ into factors Γ_α, $\alpha \in \Lambda$. The graph Γ is said to be *indecomposable* (as a join) if it is not decomposable.

A simple decomposable graph Γ is called a *cone* if Γ has a join decomposition into a subgraph consisting of only one vertex v_0 and a subgraph Γ'

$$\Gamma = \{v_0\} * \Gamma'.$$

Remark 3.3. Any simple graph Γ is indecomposable as a join if and only if its complement graph Γ^c is connected.

The following is a well-known fact. See Figures 1 and 2.

Lemma 3.4. *Let Γ be a simple graph. Suppose that Γ is decomposable. Then, Γ has a unique join decomposition into indecomposable factors*

$$\Gamma = *_{\alpha \in \Lambda} \Gamma_\alpha .$$

Proof. Consider the decomposition of Γ^c into connected components

$$\Gamma^c = \bigsqcup_{\alpha \in \Lambda} (\Gamma^c)_\alpha .$$

Set $V_\alpha := V((\Gamma^c)_\alpha)$ and define Γ_α as the subgraph of Γ spanned by V_α. Then, $(\Gamma_\alpha)^c = (\Gamma^c)_\alpha$. Additionally, we have a join decomposition $\Gamma = *_{\alpha \in \Lambda} \Gamma_\alpha$. ■

Remark 3.5. Any decomposable graph is not a cone if and only if each of its indecomposable factors has at least two vertices.

4. Clique-cube complexes and actions on them

In this section, we consider clique-cube complexes and the actions on them following [11]. Let A_Γ be an Artin group associated with a defining graph Γ as in Section 3.1. By the theorem of van der Lek (see [27, 31]), for any subset $U \subset V = V(\Gamma)$, the subgroup of A_Γ generated by U is itself an Artin group associated with the full subgraph of Γ spanned by U. We denote this subgroup by A_U. When U is empty, we define $A_\emptyset = \{1\}$. We say that U *spans a clique* in Γ if any two elements of U are joined by an edge in Γ.

Definition 4.1 ([11, Definition 2.1]). Consider the set

$$\Delta_\Gamma = \{U \subset V \mid U \text{ spans a clique in } \Gamma \text{ or } U = \emptyset\}.$$

The *clique-cube complex* C_Γ is the cube complex whose vertices (i.e., 0-dimensional cubes) are cosets gA_U, $g \in A_\Gamma$, $U \in \Delta_\Gamma$, where two vertices gA_U and $hA_{U'}$ are joined by an edge (i.e., a 1-dimensional cube) in C_Γ if and only if $gA_U \subset hA_{U'}$ and U and U' differ by a single generator. Note that, in this case, we can always replace h by g, that is, $hA_{U'} = gA_{U'}$. More generally, two vertices gA_U and $gA_{U'}$ with $gA_U \subset gA_{U'}$ span a $\#(U' \setminus U)$-dimensional cube $[gA_U, gA_{U'}]$ in C_Γ.

The group A_Γ acts on the clique-cube complex C_Γ by left multiplication, $h \cdot gA_U = (hg)A_U$. This action preserves the cubical structure and is isometric. The action is also co-compact with a fundamental domain $\bigcup_{U \in \Delta_\Gamma} [A_\emptyset, A_U]$, where $[A_\emptyset, A_U]$ is a $\#U$-dimensional cube spanned by two vertices $A_\emptyset$ and A_U in C_Γ. However, the action is not proper. In fact, the stabilizer of a vertex gA_U is the conjugate subgroup $gA_U g^{-1}$, so all vertices except translations of $A_\emptyset$ have infinite stabilizers. We also note that C_Γ is not a proper metric space because it contains infinite valence vertices. Additionally, C_Γ has infinite diameter if and only if Γ itself is not a clique.

Remark 4.2 ([11, Section 2]). Each edge in C_Γ can be labeled with a generator in V. For example, the edge between gA_U and $gA_{U\sqcup\{v\}}$ is labeled by v. Any two parallel edges in a cube have the same label, so we can also label the hyperplane dual to such an edge by v and say that such a hyperplane is *of v-type*. Every hyperplane of v-type is the translation of a hyperplane dual to the edge between $A_\emptyset$ and $A_{\{v\}}$. If a hyperplane of v-type crosses another hyperplane of v'-type, then $(v, v') \in E(\Gamma)$. In particular, two different hyperplanes of the same type do not cross each other.

Theorem 4.3 ([11, Theorem 2.2]). *The clique-cube complex C_Γ is* CAT(0) *for any graph Γ.*

Lemma 4.4 ([11, Lemma 2.3]). *In the clique-cube complex C_Γ, the link of the vertex $A_\emptyset$ is isomorphic to the flag simplicial complex whose* 1*-skeleton is Γ.*

Lemma 4.5 ([11, Lemma 2.4]). *If the clique-cube complex C_Γ is reducible, that is, decomposable as a product of two subcomplexes, then Γ is decomposable (as a join). In particular, if Γ is indecomposable, then C_Γ is irreducible.*

More strongly, we can show the following. This proposition is not directly used in the proof of Theorem 1.4, but is of independent interest.

Proposition 4.6. *The following are equivalent:*

(1) *C_Γ is reducible.*

(2) *A_Γ is reducible, that is, Γ^t is connected. In other words, Γ can be decomposed as a join of two subgraphs such that all edges between them are labeled by* 2*.*

(3) *In addition to* (2)*, C_Γ is a direct product of $C_{\Gamma'}$ and $C_{\Gamma''}$ when Γ is decomposed as a join of two subgraphs Γ' and Γ'' such that all edges between them are labeled by* 2*.*

Proof. (3) $\Rightarrow$ (1) is obvious. We show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

We first consider (1) $\Rightarrow$ (2). Suppose that C_Γ is reducible. First, we show that Γ can be decomposed as a join of two subgraphs. We fix two subcomplexes C' and C'' of C_Γ satisfying $C_\Gamma = C' \times C''$. Then, for any vertex $v = (v', v'')$ of $C_\Gamma = C' \times C''$, $\mathrm{Lk}_{C_\Gamma} v$ is the join of $\mathrm{Lk}_{C'} v'$ and $\mathrm{Lk}_{C''} v''$. Let Γ_v, Γ'_v, and Γ''_v be 1-skeletons of $\mathrm{Lk}_C v$, $\mathrm{Lk}_{C'} v'$, and $\mathrm{Lk}_{C''} v''$, respectively. Then, Γ_v is the join of Γ'_v and Γ''_v. In particular, $\Gamma_{A_\emptyset}$ is the join of $\Gamma'_{A_\emptyset}$ and $\Gamma''_{A_\emptyset}$. We set

$$\Gamma' := \Gamma'_{A_\emptyset} \quad \text{and} \quad \Gamma'' := \Gamma''_{A_\emptyset}.$$

Because Γ is isomorphic to $\Gamma_{A_\emptyset}$ by Lemma 4.4, Γ can be regarded as the join of Γ' and Γ''. Thus far, the argument is based on the proof of Lemma 4.4 in [11].

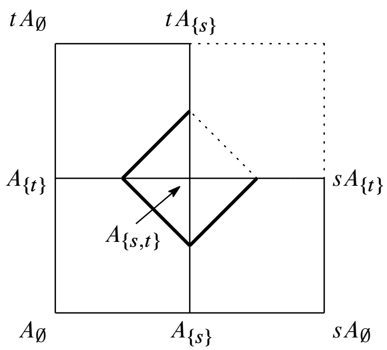

Figure 3. A 3-line full subgraph of $\Gamma_{A_{\{s,t\}}}$.

Next, we show that all edges between Γ' and Γ'' are labeled by 2. Assume that we have $s \in V(\Gamma')$ and $t \in V(\Gamma'')$ such that $e = (s,t)$ is an edge of Γ with label $m > 2$. Let us consider three squares $[A_\emptyset, A_{\{s,t\}}]$, $[sA_\emptyset, A_{\{s,t\}}]$, and $[tA_\emptyset, A_{\{s,t\}}]$ around $A_{\{s,t\}}$. Then, we have a 3-line subgraph of $\Gamma_{A_{\{s,t\}}}$ corresponding to these three squares (see Figure 3). The 3-line subgraph is a full subgraph of $\Gamma_{A_{\{s,t\}}}$, because it follows from Lemma 5.1 (see the next section) that there is no square in C_Γ containing both edges $[A_{\{s,t\}}, tA_{\{s\}}]$ and $[A_{\{s,t\}}, sA_{\{t\}}]$. Because C_Γ is $C' \times C''$ and edges $[A_\emptyset, A_{\{s\}}]$ and $[A_\emptyset, A_{\{t\}}]$ of C_Γ correspond to edges of $C' \times \{A_\emptyset\}$ and $\{A_\emptyset\} \times C''$, respectively, edges $[A_{\{t\}}, A_{\{s,t\}}]$ and $[A_{\{s\}}, A_{\{s,t\}}]$ of C_Γ must correspond to edges of $C' \times \{A_{\{t\}}\}$ and $\{A_{\{s\}}\} \times C''$, respectively. Thus, the two middle vertices of the 3-line full subgraph of $\Gamma_{A_{\{s,t\}}} = \Gamma'_{A_{\{s,t\}}} * \Gamma''_{A_{\{s,t\}}}$ belong to $\Gamma'_{A_{\{s,t\}}}$ and $\Gamma''_{A_{\{s,t\}}}$, respectively. This contradicts the fact that any 3-line full subgraph of a join of two graphs is contained in either of the join factors.

We now show that (2) $\Rightarrow$ (3). Suppose that Γ is decomposed as a join of two subgraphs Γ' and Γ'' such that all edges between them are labeled by 2. Then, we have a bijection

$$\Delta_{\Gamma'} \times \Delta_{\Gamma''} \to \Delta_\Gamma, \quad (T', T'') \mapsto T' \sqcup T''. \tag{4.1}$$

In addition, because A_Γ is a direct product of subgroups $A_{\Gamma'}$ and $A_{\Gamma''}$, we have a group isomorphism

$$A_{\Gamma'} \times A_{\Gamma''} \to A_\Gamma, \quad (g', g'') \mapsto g'g''. \tag{4.2}$$

Clearly, (4.1) and (4.2) imply a bijection from vertices of $C_{\Gamma'} \times C_{\Gamma''}$ to vertices of C_Γ

$$\phi^0\colon C^0_{\Gamma'} \times C^0_{\Gamma''} \to C^0_\Gamma, \quad (g'A_{T'}, g''A_{T''}) \mapsto g'g''A_{T' \sqcup T''}.$$

This can be extended to the cubical isomorphism

$$\phi\colon C_{\Gamma'} \times C_{\Gamma''} \to C_\Gamma$$

such that $\phi([g'A_{T'}, g'A_{U'}] \times [g''A_{T''}, g''A_{U''}]) = [g'g''A_{T' \cup T''}, g'g''A_{U' \cup U''}]$ for any $(T', T'') \in \Delta_{\Gamma'} \times \Delta_{\Gamma''}$ and $(g', g'') \in A_{\Gamma'} \times A_{\Gamma''}$. ∎

Lemma 4.7 ([11, Lemma 3.2]). *Suppose that Γ is not a cone. Then, the action of A_Γ on C_Γ is minimal. That is, for any point $x \in C_\Gamma$, we have* $\mathrm{Hull}(A_\Gamma x) = C_\Gamma$ *(the convex hull of the orbit of x is all of C_Γ).*

Proposition 4.8. *Suppose that Γ is not a cone. Then, a finite normal subgroup of A_Γ is trivial. In particular, a finite center of A_Γ is trivial. Also, if A_Γ is isomorphic to a direct product $A_1 \times A_2$ and A_1 is finite, then A_1 is trivial.*

Proof. Let N be a finite normal subgroup of A_Γ. Set

$$\mathrm{Fix}(N) := \{x \in C_\Gamma \mid nx = x \text{ for any } n \in N\}.$$

Because N is finite and C_Γ is a complete CAT(0) space, we have $\mathrm{Fix}(N) \neq \emptyset$. Take any $x \in \mathrm{Fix}(N)$. Then, $A_\Gamma x \subset \mathrm{Fix}(N)$. Indeed, the normality of N implies that, for any $g \in A_\Gamma$ and $n \in N$, there exists $n' \in N$ such that $ng = gn'$. Therefore, we have $ngx = gn'x = gx$. Because C_Γ is CAT(0), $\mathrm{Fix}(N)$ is convex. Hence, we have $\mathrm{Hull}(A_\Gamma x) \subset \mathrm{Fix}(N)$. By Lemma 4.7, we have $\mathrm{Hull}(A_\Gamma x) = C_\Gamma$. Hence, $\mathrm{Fix}(N) = C_\Gamma$. In particular, $A_\emptyset \in \mathrm{Fix}(N)$. In general, $A_\emptyset$ is not fixed by any nontrivial element of A_Γ. Hence, N must be trivial. ∎

5. Lemmas on local geometry of clique-cube complexes

In this section, we state two lemmas related to the local geometry of clique-cube complexes. The first one is used in the proof of Proposition 4.6. The second is used in the proof of Theorem 1.4.

Recall that the dihedral group for any $r \in \mathbb{N}$ is defined as

$$I_2(r) := \begin{cases} \langle s, t \mid s^2 = 1,\ t^2 = 1,\ st\cdots s = ts\cdots t \text{ (length } r)\rangle & \text{if } r \text{ is odd,} \\ \langle s, t \mid s^2 = 1,\ t^2 = 1,\ st\cdots t = ts\cdots s \text{ (length } r)\rangle & \text{if } r \text{ is even.} \end{cases}$$

It is well known that $\# I_2(r) = 2r$.

Let A_Γ be an Artin group associated with a defining graph Γ as in Section 3.1.

Lemma 5.1. *Let $e = (s, t)$ be an edge of Γ with label m greater than 2. Then, there exists no square in C_Γ containing both edges $[A_{\{s,t\}}, tA_{\{s\}}]$ and $[A_{\{s,t\}}, sA_{\{t\}}]$, that is, there exist no $p, q \in \mathbb{Z}$ such that $ts^p = st^q$ in A_Γ.*

See Figure 3.

Proof. Assume that we have a square in C_Γ containing both edges $[A_{\{s,t\}}, tA_{\{s\}}]$ and $[A_{\{s,t\}}, sA_{\{t\}}]$. Then, the square has four vertices $A_{\{s,t\}}$, $tA_{\{s\}}$, $sA_{\{t\}}$, and $ts^pA_\emptyset = st^qA_\emptyset$, where p, q are some integers. Then, we have $p = q$. Indeed, if we consider the projection $A_{\{s,t\}} \to A_{\{s\}} \cong \mathbb{Z}$ defined by $s \mapsto s$ and $t \mapsto s$, then $ts^p = st^q$ in $A_{\{s,t\}}$ implies $s^{1+p} = s^{1+q}$ in $A_{\{s\}}$, and thus $p = q$. Hence, we have $ts^p = st^p$ in $A_{\{s,t\}}$. By the natural projection $A_{\{s,t\}} \to W_{\{s,t\}}$, $ts^p = st^p$ in $A_{\{s,t\}}$ implies $ts = st$ in $W_{\{s,t\}}$ if p is odd and $t = s$ in $W_{\{s,t\}}$ if p is even. Both cases contradict the fact that $ts \neq st$ and $s \neq t$ in $W_{\{s,t\}}$ in the case where $m > 2$. ∎

Lemma 5.2. *Let $e = (s,t)$ be an edge of Γ with label m greater than 2. Let τ be an alternating word*

$$\tau := st\cdots s \begin{cases} \text{of length } m & \text{if } m \text{ is odd,} \\ \text{of length } m+1 & \text{if } m \text{ is even.} \end{cases}$$

Let $U \in \Delta_\Gamma$ with $s, t \in U$. Then, there exists no square in C_Γ containing both edges $[A_U, A_{U\setminus\{s\}}]$ and $[A_U, \tau A_{U\setminus\{t\}}]$, that is, there exists no $g \in A_{U\setminus\{s\}}$ such that $gA_{U\setminus\{t\}} = \tau A_{U\setminus\{t\}}$. Additionally, there exists no square in C_Γ containing both edges $[A_U, A_{U\setminus\{t\}}]$ and $[A_U, \tau A_{U\setminus\{s\}}]$, that is, there exists no $g \in A_{U\setminus\{t\}}$ such that $gA_{U\setminus\{s\}} = \tau A_{U\setminus\{s\}}$.

See Figure 4.

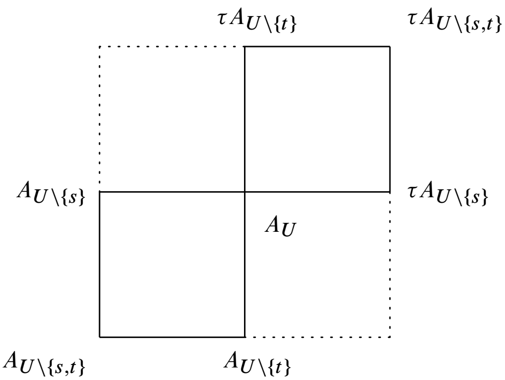

Figure 4. Squares around A_U.

Proof. We prove this lemma by induction on $k = \#U$.

(1) The base case where $k = 2$. In this case, $U = \{s,t\}$. It is sufficient to show that for any $p \in \mathbb{Z}$,

$$t^p A_{U\setminus\{t\}} \neq \tau A_{U\setminus\{t\}} \quad \text{and} \quad s^p A_{U\setminus\{s\}} \neq \tau A_{U\setminus\{s\}}.$$

(1-1) Suppose that m is odd. Note that $\tau = st\cdots s$ is equal to an alternating word $ts\cdots t$ of length m in

$$A_{\{s,t\}} = \langle s,t \mid st\cdots s = ts\cdots t \text{ (length } m) \rangle = A_U \subset A_\Gamma.$$

We assume that there exists $p \in \mathbb{Z}$ such that $t^p A_{U\setminus\{t\}} = \tau A_{U\setminus\{t\}}$. Then, $t^{-p}\tau \in A_{U\setminus\{t\}}$. In contrast, we clearly have that $t^{-p}\tau \in A_{\{s,t\}}$. Because $U \setminus \{t\} = \{s\}$, there exists $q \in \mathbb{Z}$

such that $t^p s^q = \tau$ in $A_{\{s,t\}}$. Then, we have $p+q=m$. Indeed, if we consider the projection $A_{\{s,t\}} \to A_{\{s\}} \cong \mathbb{Z}$ defined by $s \mapsto s$ and $t \mapsto s$, then $t^p s^q = \tau$ in $A_{\{s,t\}}$ implies $s^{p+q} = s^m$ in $A_{\{s\}}$, and thus $p+q=m$. Hence, we have $t^p s^{m-p} = st\cdots s = ts\cdots t$ in $A_{\{s,t\}}$.

(1-1-1) We treat the case where p is odd. By the natural projection $A_{\{s,t\}} \to W_{\{s,t\}}$, $t^p s^{m-p} = ts\cdots t$ in $A_{\{s,t\}}$ and $t^p s^{m-p} = t$ in $W_{\{s,t\}}$ imply $t = ts\cdots t$ in $W_{\{s,t\}}$. Thus, we have $1 = s\cdots t$ in $W_{\{s,t\}}$. This means that we have a projection $I_2((m-1)/2) \to W_{\{s,t\}} \cong I_2(m)$ defined by $s \mapsto s$ and $t \mapsto t$. Thus, we have $m-1 = \#I_2((m-1)/2) \geq \#I_2(m) = 2m$. This contradicts $m > 2$.

(1-1-2) We treat the case where p is even. In this case, $t^p s^{m-p} = st\cdots s$ in $A_{\{s,t\}}$ and $t^p s^{m-p} = s$ in $W_{\{s,t\}}$ imply $s = st\cdots s$ in $W_{\{s,t\}}$. Thus, we have $1 = s\cdots t$ in $W_{\{s,t\}}$. This means that we have a projection $I_2((m-1)/2) \to W_{\{s,t\}} \cong I_2(m)$ defined by $s \mapsto s$ and $t \mapsto t$. Thus, we have $m-1 = \#I_2((m-1)/2) \geq \#I_2(m) = 2m$. This contradicts $m > 2$.

By (1-1-1) and (1-1-2), for any $p \in \mathbb{Z}$, we have $t^p A_{U\setminus\{t\}} \neq \tau A_{U\setminus\{t\}}$. By the same argument, for any $p \in \mathbb{Z}$, we have $s^p A_{U\setminus\{s\}} \neq \tau A_{U\setminus\{s\}}$.

(1-2) Suppose that m is even. Note that alternating words $st\cdots t$ and $ts\cdots s$ of length m are equal in

$$A_{\{s,t\}} = \langle s,t \mid st\cdots t = ts\cdots s \text{ (length } m) \rangle = A_U \subset A_\Gamma.$$

We assume that there exists $p \in \mathbb{Z}$ such that $t^p A_{U\setminus\{t\}} = \tau A_{U\setminus\{t\}}$. Then, from the same argument as in (1-1), there exists $q \in \mathbb{Z}$ such that $t^p s^q = \tau$ in $A_{\{s,t\}}$. Thus, we have $p+q = m+1$. Indeed, if we consider the projection $A_{\{s,t\}} \to A_{\{s\}} \cong \mathbb{Z}$ defined by $s \mapsto s$ and $t \mapsto s$, then $t^p s^q = \tau$ in $A_{\{s,t\}}$ implies $s^{p+q} = s^{m+1}$ in $A_{\{s\}}$, and thus $p+q = m+1$. Hence, we have $t^p s^{m+1-p} = st\cdots s$ in $A_{\{s,t\}}$. This implies $t^p s^{m-p} = st\cdots t$ in $A_{\{s,t\}}$. Thus, we have $t^p s^{m-p} = st\cdots t = ts\cdots s$ in $A_{\{s,t\}}$.

(1-2-1) We treat the case where p is odd. By the natural projection $A_{\{s,t\}} \to W_{\{s,t\}}$, $t^p s^{m-p} = ts\cdots s$ in $A_{\{s,t\}}$ and $t^p s^{m-p} = ts$ in $W_{\{s,t\}}$ imply $ts = ts\cdots s$ in $W_{\{s,t\}}$. Thus, 1 and an alternating word $s\cdots t$ of length $m-2$ are equal in $W_{\{s,t\}}$. This means that we have a projection $I_2((m-2)/2) \to W_{\{s,t\}} \cong I_2(m)$ defined by $s \mapsto s$ and $t \mapsto t$. Thus, we have $m-2 = \#I_2((m-2)/2) \geq \#I_2(m) = 2m$. This contradicts $m > 2$.

(1-2-2) We treat the case where p is even. In this case, $t^p s^{m-p} = st\cdots t$ in $A_{\{s,t\}}$ and $t^p s^{m-p} = 1$ in $W_{\{s,t\}}$ implies $1 = st\cdots t$ in $W_{\{s,t\}}$. This means that we have a projection $I_2(m/2) \to W_{\{s,t\}} \cong I_2(m)$ defined by $s \mapsto s$ and $t \mapsto t$. Thus, we have $m = \#I_2(m/2) \geq \#I_2(m) = 2m$. This contradicts $m > 2$.

By (1-2-1) and (1-2-2), for any $p \in \mathbb{Z}$, we have $t^p A_{U\setminus\{t\}} \neq \tau A_{U\setminus\{t\}}$. By the same argument, for any $p \in \mathbb{Z}$, we have $s^p A_{U\setminus\{s\}} \neq \tau A_{U\setminus\{s\}}$.

(2) Suppose that $k > 2$ and the statement is true for $k-1$. Let $U = \{u_1, \ldots, u_{k-2}, s, t\}$, where $\#U = k$. Assume that there exists $g \in A_{U\setminus\{s\}}$ such that $gA_{U\setminus\{t\}} = \tau A_{U\setminus\{t\}}$. Then, we have a square $[gA_{U\setminus\{s,t\}}, A_U]$ in C_Γ. Its vertices are A_U, $\tau A_{U\setminus\{t\}}$, $gA_{U\setminus\{s,t\}}$, and $A_{U\setminus\{s\}}$. Because C_Γ is CAT(0), this square spans a cube together with other two squares $[A_{U\setminus\{u_1,s\}}, A_U]$ and $[\tau A_{U\setminus\{u_1,t\}}, A_U]$. See Figure 5.

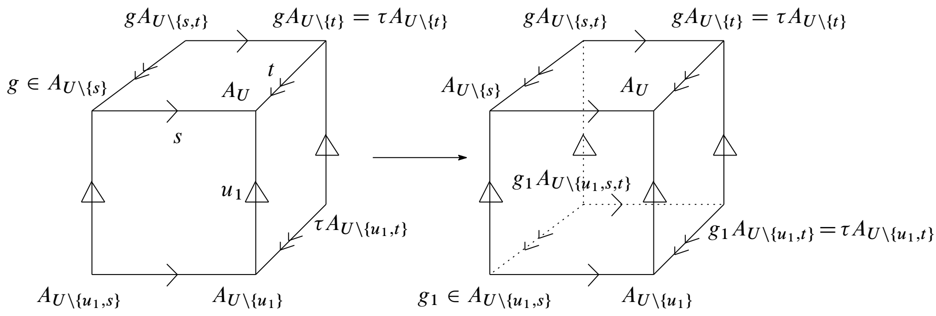

Figure 5. Three squares around A_U span a cube.

This cube contains another square $[g_1 A_{U\setminus\{u_1,s,t\}}, A_{U\setminus\{u_1\}}]$ as a face, where $g_1 \in A_{U\setminus\{u_1,s\}} = A_{(U\setminus\{u_1\})\setminus\{s\}}$. Then, we have

$$g_1 A_{(U\setminus\{u_1\})\setminus\{t\}} = \tau A_{(U\setminus\{u_1\})\setminus\{t\}}.$$

Because $\#(U \setminus \{u_1\}) = k - 1$, this contradicts the inductive assumption. ∎

6. Proof of Theorem 1.4

In this section, let A_Γ be an Artin group associated with a graph Γ that has at least three vertices, and suppose that Γ is not a cone. We show our main theorem (Theorem 1.4), that is, the following are equivalent:

(1) A_Γ is irreducible, that is, Γ cannot be decomposed as a join of two subgraphs such that all edges between them are labeled by 2;

(2) A_Γ has a WPD contracting element with respect to the isometric action on the clique-cube complex;

(3) A_Γ is acylindrically hyperbolic;

(4) A_Γ is directly indecomposable, that is, it cannot be decomposed as a direct product of two nontrivial subgroups.

6.1. Proof of (2) ⇒ (3), (3) ⇒ (4), (4) ⇒ (1) in Theorem 1.4

We show that (2) ⇒ (3), (3) ⇒ (4), (4) ⇒ (1) in Theorem 1.4.

First, (2) ⇒ (3) follows from Theorem 2.3.

Next, we show (3) ⇒ (4). Let A_Γ be acylindrically hyperbolic. If A_Γ is isomorphic to a direct product $A_1 \times A_2$, then either A_1 or A_2 is finite by acylindrical hyperbolicity [25, Corollary 7.3]. Proposition 4.8 implies that the finite one is trivial. Hence, A_Γ is directly indecomposable.

Finally, (4) ⇒ (1) is clear.

6.2. Proof of (1) $\Rightarrow$ (2) in Theorem 1.4

In this subsection, we give a proof of (1) $\Rightarrow$ (2) in Theorem 1.4. Suppose that A_Γ is irreducible. We will construct an element $\gamma \in A_\Gamma$ and show that γ is a WPD contracting element with respect to the action on the clique-cube complex. When Γ is indecomposable, that is, not a join, a WPD contracting element is already given in [11, Remark 4.5]. Hence, it is sufficient to treat the case where Γ is decomposable. From now on, we suppose that Γ is decomposable.

We consider a unique join decomposition $\Gamma = *_{1 \leq i \leq k} \Gamma_i$ into indecomposable factors (see Lemma 3.4). We set $V_i = V(\Gamma_i)$ and $E_i = E(\Gamma_i)$ for each $i \in \{1, \ldots, k\}$. Then, for every $i \in \{1, \ldots, k\}$, the complement graph $(\Gamma_i)^c$ of Γ_i is connected, and the complement graph Γ^c of Γ is the disjoint union of connected components $\Gamma^c = \bigsqcup_{1 \leq i \leq k} (\Gamma_i)^c$.

We define $Q(\Gamma)$ as a finite simple graph with

- the vertex set $V(Q(\Gamma)) = \{V_i\}_{1 \leq i \leq k}$, and
- the edge set $E(Q(\Gamma)) = \{(V_i, V_j), (V_j, V_i) \mid 1 \leq i < j \leq k, (\Gamma_i * \Gamma_j)^t \text{ is connected}\}$.

Note that $(\Gamma_i)^t$ is connected because $(\Gamma_i)^c$ is connected, $V((\Gamma_i)^c) = V((\Gamma_i)^t) = V_i$, and $E((\Gamma_i)^c) \subset E((\Gamma_i)^t)$. Note that the following are equivalent for different $i, j \in \{1, \ldots, k\}$:

- $(\Gamma_i * \Gamma_j)^t$ is connected;
- there exists an edge with label greater than 2 between a vertex of Γ_i and a vertex of Γ_j.

We confirm our setting:

(1) Γ is decomposable, i.e., Γ^c is not connected, that is, $k \geq 2$;

(2) Γ is not a cone, i.e., every connected component of Γ^c has at least two vertices, that is, $\#\Gamma_i \geq 2$ for any $i \in \{1, \ldots, k\}$;

(3) A_Γ is irreducible, i.e., Γ^t is connected, that is, $Q(\Gamma)$ is connected.

We take a spanning tree T of $Q(\Gamma)$. We regard T as a rooted tree with the root V_1. By trading indices of $V_2, \ldots, V_k$ if necessary, we suppose that $i < j$ only if V_i is not farther than V_j from V_1 in T. For each i, j with $i < j$ and $(V_i, V_j) \in E(T)$, take an edge $e_{i,j} \in E(\Gamma)$ with label $m_{i,j} = \mu(e_{i,j})$ greater than 2 with $s_{i,j} := s_{e_{i,j}} \in V(\Gamma_i)$ and $t_{i,j} := t_{e_{i,j}} \in V(\Gamma_j)$, and set $e_{j,i} := \iota(e_{i,j})$. For any $i, j \in \{1, \ldots, k\}$ with $(V_i, V_j) \in E(T)$, let $\tau_{i,j}$ be an alternating word

$$\tau_{i,j} = s_{i,j} t_{i,j} \cdots s_{i,j}$$

of length $m_{i,j}$ if $m_{i,j}$ is odd, and let $\tau_{i,j}$ be an alternating word

$$\tau_{i,j} = s_{i,j} t_{i,j} \cdots s_{i,j}$$

of length $m_{i,j} + 1$ if $m_{i,j}$ is even.

Lemma 6.1. *There exist* $n\in\mathbb{N}$, *for each* $i\in\{1,\dots,k\}$, *a closed path* $(v_{i,1},\dots,v_{i,n},v_{i,n+1})$ *with* $v_{i,1}=v_{i,n+1}$ *on* $(\Gamma_i)^c$ *passing through every vertex at least once, and for any* $i,j\in\{1,\dots,k\}$ *with* $(V_i,V_j)\in E(T)$, $l(i,j)\in\{1,\dots,n\}$ *such that* $(v_{i,l(i,j)},v_{j,l(i,j)})=(s_{i,j},t_{i,j})\ (=e_{i,j})$ *and* $l(j,i)=l(i,j)$.

Proof. For any $i\in\{1,\dots,k\}$, we consider the minimum length n_i of closed paths on $(\Gamma_i)^c$ passing through every vertex at least once. Set $n=\prod_{1\le i\le k}n_i$. Then, for any $i\in\{1,\dots,k\}$, by concatenating n/n_i copies of a closed path of length n_i on $(\Gamma_i)^c$ passing through every vertex at least once, we have a closed path $(v'_{i,1},\dots,v'_{i,n},v'_{i,n+1})$ with $v'_{i,1}=v'_{i,n+1}$ on $(\Gamma_i)^c$ passing through every vertex at least n/n_i times (in particular, at least once).

We set $(v_{1,1},\dots,v_{1,n},v_{1,n+1}):=(v'_{1,1},\dots,v'_{1,n},v'_{1,n+1})$. For any $j\in\{2,\dots,k\}$, we define $(v_{j,1},\dots,v_{j,n},v_{j,n+1})$ inductively as follows. Take $j\in\{2,\dots,k\}$. Assume that $(v_{i,1},\dots,v_{i,n},v_{i,n+1})$ is defined for $i\in\{1,\dots,j-1\}$ with $(V_i,V_j)\in E(T)$. Then, we set $l(i,j)$ as the minimum l such that $v_{i,l}=s_{i,j}$. By a cyclic permutation of $v'_{j,1},\dots,v'_{j,n}$, we have $v_{j,1},\dots,v_{j,n}$ such that $v_{j,l(i,j)}=t_{i,j}$. By setting $v_{j,n+1}:=v_{j,1}$, we have $(v_{j,1},\dots,v_{j,n},v_{j,n+1})$. For any i,j with $1\le i<j\le k$ and $(V_i,V_j)\in E(T)$, we set $l(j,i):=l(i,j)$. ∎

We take $n\in\mathbb{N}$, $(v_{i,1},\dots,v_{i,n},v_{i,n+1})$ for each $i\in\{1,\dots,k\}$, and $l(i,j)\in\{1,\dots,n\}$ for any $i,j\in\{1,\dots,k\}$ with $(V_i,V_j)\in E(T)$ as in Lemma 6.1. For $1\le l\le n$, set

$$\lambda_l:=v_{1,l}v_{2,l}\cdots v_{k,l}.$$

For any $i,j\in\{1,\dots,k\}$ with $(V_i,V_j)\in E(T)$, we define $\lambda_l(i,j)$ as $\lambda_l(i,j):=\lambda_l$ if $l\ne l(i,j)$. In addition, we define $\lambda_{l(i,j)}(i,j)$ as

$$\begin{aligned}\lambda_{l(i,j)}(i,j):=\tau_{i,j}v_{1,l(i,j)}v_{2,l(i,j)}&\cdots v_{i-1,l(i,j)}v_{i+1,l(i,j)}\\ &\cdots v_{j-1,l(i,j)}v_{j+1,l(i,j)}\cdots v_{k,l(i,j)}\end{aligned}$$

if $i<j$, and $\lambda_{l(i,j)}(i,j):=\lambda_{l(j,i)}(j,i)$ if $i>j$, where we note that $l(j,i)=l(i,j)$. Moreover, we define $\gamma(i,j)$ as

$$\begin{aligned}\gamma(i,j):&=\lambda_1(i,j)\lambda_2(i,j)\cdots\lambda_n(i,j)\\ &=\lambda_1\lambda_2\cdots\lambda_{l(i,j)-1}\lambda_{l(i,j)}(i,j)\lambda_{l(i,j)+1}\cdots\lambda_n.\end{aligned}$$

Then, we have $\gamma(j,i)=\gamma(i,j)$.

Take a closed path

$$(V_{i_1},\dots,V_{i_r},V_{i_{r+1}}) \tag{6.1}$$

with $i_1=i_{r+1}=1$ on the spanning tree T of $Q(\Gamma)$ passing through every vertex at least once. We define γ as

$$\gamma:=\gamma(i_1,i_2)\gamma(i_2,i_3)\cdots\gamma(i_r,i_{r+1}). \tag{6.2}$$

We set

$$\gamma(0) := 1,\ \gamma(1) := \lambda_1(i_1, i_2),\ \gamma(2) := \lambda_1(i_1, i_2)\lambda_2(i_1, i_2),\ \ldots,\ \gamma(n) := \gamma(i_1, i_2),$$

and for $a \in \{2, \ldots, r\}$ and $l \in \{1, \ldots, n\}$, we set

$$\gamma((a-1)n + l) := \gamma(i_1, i_2) \cdots \gamma(i_{a-1}, i_a)(\lambda_1(i_a, i_{a+1}) \cdots \lambda_l(i_a, i_{a+1})).$$

In particular, we have $\gamma(rn) = \gamma$.

We confirm the following for convenience:

(1) k is the number of indecomposable factors of a unique join decomposition of Γ;

(2) n is the common length of the closed paths on $(\Gamma_i)^c$ for all $i \in \{1, \ldots, k\}$ taken in Lemma 6.1;

(3) r is the length of the closed path (6.1) on T taken above.

For $1 \le l \le n$, set

$$U_l := \{v_{1,l}, v_{2,l}, \ldots, v_{k,l}\}.$$

Then, U_l spans a clique in Γ, that is, $U_l \in \Delta_\Gamma$. Hence, we have a k-dimensional cube $[A_\emptyset, A_{U_l}]$ in C_Γ. The hyperplanes dual to edges of $[A_\emptyset, A_{U_l}]$ are of $v_{i,l}$-type, $i \in \{1, \ldots, k\}$. We denote such hyperplanes by $H_{i,l}$, $i \in \{1, \ldots, k\}$.

Lemma 6.2. *For $i \in \{1, \ldots, k\}$, $l \in \{1, \ldots, n\}$, and $a \in \{1, \ldots, r\}$, we have the following:*

(1) *$H_{i,l} \cap H_{i,l+1} = \emptyset$, where we set $H_{i,n+1} := H_{i,1}$;*

(2) *$H_{i,l} \cap \lambda_l H_{i,l} = \emptyset$;*

(3) *$H_{i,l(i_a,i_{a+1})} \cap \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i,l(i_a,i_{a+1})} = \emptyset$;*

(4) *$[A_\emptyset, A_{U_l}] \cap [A_\emptyset, A_{U_{l+1}}] = \{A_\emptyset\}$, where we set $U_{n+1} := U_1$;*

(5) *$[A_\emptyset, A_{U_l}] \cap \lambda_l [A_\emptyset, A_{U_l}] = \{A_{U_l}\}$;*

(6) *$[A_\emptyset, A_{U_{l(i_a,i_{a+1})}}] \cap \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1})[A_\emptyset, A_{U_{l(i_a,i_{a+1})}}] = \{A_{U_{l(i_a,i_{a+1})}}\}$.*

Proof. (1) $H_{i,l}$ and $H_{i,l+1}$ are of $v_{i,l}$-type and $v_{i,l+1}$-type, respectively. Note that $v_{i,l} \ne v_{i,l+1}$ implies $H_{i,l} \ne H_{i,l+1}$. Because $(v_{i,l}, v_{i,l+1}) \notin E(\Gamma)$, we have $\{v_{i,l}, v_{i,l+1}\} \notin \Delta_\Gamma$, and thus $H_{i,l} \cap H_{i,l+1} = \emptyset$.

(2) Because $[A_{U_l \setminus \{v_{i,l}\}}, A_{U_l}] \subset N(H_{i,l})$, we have $[\lambda_l A_{U_l \setminus \{v_{i,l}\}}, \lambda_l A_{U_l}] \subset \lambda_l N(H_{i,l})$. Note that $\lambda_l A_{U_l} = A_{U_l}$ and $\lambda_l A_{U_l \setminus \{v_{i,l}\}} = v_{1,l} \cdots v_{i,l} A_{U_l \setminus \{v_{i,l}\}}$. Now, assume that $H_{i,l} \cap \lambda_l H_{i,l} \ne \emptyset$. Because $H_{i,l}$ and $\lambda_l H_{i,l}$ are of $v_{i,l}$-type, we see that $H_{i,l} = \lambda_l H_{i,l}$ (see Remark 4.2). Then, we have

$$[\lambda_l A_{U_l \setminus \{v_{i,l}\}}, \lambda_l A_{U_l}] = [v_{1,l} \cdots v_{i,l} A_{U_l \setminus \{v_{i,l}\}}, A_{U_l}] \subset N(H_{i,l}).$$

Hence, we obtain $A_{U_l \setminus \{v_{i,l}\}} = v_{1,l} \cdots v_{i,l} A_{U_l \setminus \{v_{i,l}\}}$. This means that $v_{i,l} \in A_{U_l \setminus \{v_{i,l}\}}$. However, from [31], $\{v_{i,l}\} \cap (U_l \setminus \{v_{i,l}\}) = \emptyset$ implies that $A_{\{v_{i,l}\}} \cap A_{U_l \setminus \{v_{i,l}\}} = A_\emptyset = \{1\}$. This contradicts $v_{i,l} \ne 1$ in A_Γ.

(3) When $i \neq i_a, i_{a+1}$, by the same argument as in (2), we see that

$$H_{i,l(i_a,i_{a+1})} \cap \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i,l(i_a,i_{a+1})} = \emptyset.$$

Now, assume that $i = i_a$ or $i = i_{a+1}$ and

$$H_{i,l(i_a,i_{a+1})} \cap \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i,l(i_a,i_{a+1})} \neq \emptyset.$$

Note that $\lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) A_{U_{l(i_a,i_{a+1})}} = A_{U_{l(i_a,i_{a+1})}}$ and

$$\lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}} = \tau_{i_a,i_{a+1}} A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}}.$$

Because $H_{i,l(i_a,i_{a+1})}$ and $\lambda_{l(i_a,i_{a+1})} H_{i,l(i_a,i_{a+1})}$ are of $v_{i,l(i_a,i_{a+1})}$-type, we see that

$$H_{i,l(i_a,i_{a+1})} = \lambda_{l(i_a,i_{a+1})} H_{i,\lambda_{l(i_a,i_{a+1})}}.$$

Then, we have

$$\begin{aligned}
&[\lambda_{l(i_a,i_{a+1})} A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}}, \lambda_{l(i_a,i_{a+1})} A_{U_{l(i_a,i_{a+1})}}] \\
&\quad = [\tau_{i_a,i_{a+1}} A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}}, A_{U_{l(i_a,i_{a+1})}}] \subset N(H_{i,l(i_a,i_{a+1})}).
\end{aligned}$$

Hence, we have

$$\tau_{i_a,i_{a+1}} A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}} = A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i,l(i_a,i_{a+1})}\}}.$$

This contradicts Lemma 5.2.

Parts (4), (5), and (6) follow from (1), (2), and (3), respectively. ∎

The following is a key lemma.

Lemma 6.3. *For $a \in \{1, \dots, r\}$, we have*

$$H_{i_a,l(i_a,i_{a+1})} \cap \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i_{a+1},l(i_a,i_{a+1})} = \emptyset.$$

Proof. Let us note that the hyperplane $H_{i_a,l(i_a,i_{a+1})}$ is of $v_{i_a,l(i_a,i_{a+1})}$-type, and the hyperplane $\lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i_{a+1},l(i_a,i_{a+1})}$ is of $v_{i_{a+1},l(i_a,i_{a+1})}$-type. Note that $v_{i_a,l(i_a,i_{a+1})} \neq v_{i_{a+1},l(i_a,i_{a+1})}$ implies $H_{i_a,l(i_a,i_{a+1})} \neq \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) H_{i_{a+1},l(i_a,i_{a+1})}$. Additionally,

$$\begin{aligned}
\lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) A_{U_{l(i_a,i_{a+1})}} &= A_{U_{l(i_a,i_{a+1})}}, \\
\lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i_{a+1},l(i_a,i_{a+1})}\}} &= \tau_{l(i_a,i_{a+1})} A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i_{a+1},l(i_a,i_{a+1})}\}}.
\end{aligned}$$

Consider

$$[A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i_a,l(i_a,i_{a+1})}\}}, A_{U_{l(i_a,i_{a+1})}}] \subset N(H_{i_a,l(i_a,i_{a+1})})$$

and

$$\begin{aligned}
&[\tau_{i_a,i_{a+1}}(i_a, i_{a+1}) A_{U_{l(i_a,i_{a+1})} \setminus \{v_{i_{a+1},l(i_a,i_{a+1})}\}}, A_{U_{l(i_a,i_{a+1})}}] \\
&\quad \subset \lambda_{l(i_a,i_{a+1})}(i_a, i_{a+1}) N(H_{i_{a+1},l(i_a,i_{a+1})}).
\end{aligned}$$

Now, assume that

$$H_{i_a,l(i_a,i_{a+1})} \cap \lambda_{l(i_a,i_{a+1})}(i_a,i_{a+1})H_{i_{a+1},l(i_a,i_{a+1})} \neq \emptyset.$$

Because

$$A_{U_{l(i_a,i_{a+1})}} \in N(H_{i_a,l(i_a,i_{a+1})}) \cap \lambda_{l(i_a,i_{a+1})}(i_a,i_{a+1})N(H_{i_{a+1},l(i_a,i_{a+1})}),$$

the two edges

$$[A_{U_{l(i_a,i_{a+1})}\setminus\{v_{i_a,l(i_a,i_{a+1})}\}}, A_{U_{l(i_a,i_{a+1})}}]$$

and

$$[\tau_{i_a,i_{a+1}}(i_a,i_{a+1})A_{U_{l(i_a,i_{a+1})}\setminus\{v_{i_{a+1},l(i_a,i_{a+1})}\}}, A_{U_{l(i_a,i_{a+1})}}]$$

must span a square. This contradicts Lemma 5.2. ∎

Noting Lemma 6.2, for any $i \in \{1,\dots,k\}$, we define a sequence of hyperplanes

$$\dots, J_{i,-1}, J_{i,0}, J_{i,1},\dots, J_{i,2rn}, J_{i,2rn+1},\dots,$$

a sequence of k-dimensional cubes

$$\dots, K_{-1}, K_0, K_1,\dots, K_{2rn}, K_{2rn+1},\dots,$$

and a sequence of vertices of C_Γ

$$\dots, w_{-1}, w_0, w_1,\dots, w_{2rn}, w_{2rn+1},\dots$$

as follows. First, for $a \in \{1,\dots,r\}$ and $l \in \{1,\dots,n\}$, we define

$$\begin{aligned}
J_{i,2((a-1)n+l)-1} &:= \gamma((a-1)n+l-1)H_{i,l},\\
J_{i,2((a-1)n+l)} &:= \gamma((a-1)n+l)H_{i,l},\\
K_{2((a-1)n+l)-1} &:= \gamma((a-1)n+l-1)[A_\emptyset, A_{U_l}],\\
K_{2((a-1)n+l)} &:= \gamma((a-1)n+l)[A_\emptyset, A_{U_l}],\\
w_{2((a-1)n+l)-1} &:= \gamma((a-1)n+l-1)A_{U_l},\\
w_{2((a-1)n+l)} &:= \gamma((a-1)n+l)A_\emptyset,
\end{aligned}$$

where we note that both $J_{i,2((a-1)n+l)-1}$ and $J_{i,2((a-1)n+l)}$ are of the same $v_{i,l}$-type.

Second, for any $b \in \{1,\dots,2rn\}$ and $c \in \mathbb{Z}$, we set

$$J_{i,2rnc+b} := \gamma^c J_{i,b}, \quad K_{2rnc+b} := \gamma^c K_b, \quad w_{2rnc+b} := \gamma^c w_b.$$

Then, we have

$$K_{2rnc+b} = [w_{2rnc+b-1}, w_{2rnc+b}].$$

Additionally, we have the two connected components $J^-_{i,2rnc+b}$ and $J^+_{i,2rnc+b}$ such that

$$C_\Gamma \setminus\setminus J_{i,2rnc+b} = J^-_{i,2rnc+b} \sqcup J^+_{i,2rnc+b},$$
$$w_{2rnc+b-1} \in J^-_{i,2rnc+b}, w_{2rnc+b} \in J^+_{i,2rnc+b}.$$

Then, Lemma 6.2 implies that, for any $i \in \{1,\dots,k\}$,

$$\cdots \subsetneq J^-_{i,-1} \subsetneq J^-_{i,0} \subsetneq J^-_{i,1} \subsetneq \cdots \subsetneq J^-_{i,2rn} \subsetneq J^-_{i,2rn+1} \subsetneq \cdots,$$
$$\cdots \supsetneq J^+_{i,-1} \supsetneq J^+_{i,0} \supsetneq J^+_{i,1} \supsetneq \cdots \supsetneq J^+_{i,2rn} \supsetneq J^+_{i,2rn+1} \supsetneq \cdots.$$

Note that

$$J^-_{i,0} \not\ni w_0 = A_\emptyset \in J^-_{i,1}, \quad J^+_{i,2rn} \ni w_{2rn} = \gamma A_\emptyset \notin J^+_{i,2rn+1}.$$

Let ℓ be a path from $w_0 = A_\emptyset$ to $w_{2rn} = \gamma A_\emptyset$ that diagonally penetrates each of the cubes $K_1,\dots,K_{2rn}$ in order. Then, the set of all hyperplanes intersecting the path ℓ is $\{J_{i,d}\}_{i\in\{1,\dots,k\},\, d\in\{1,\dots,2rn\}}$. Hence, we have the following.

Lemma 6.4. *The set $\{J_{i,d}\}_{i\in\{1,\dots,k\},\, d\in\{1,\dots,2rn\}}$ is the set of all hyperplanes separating $w_0 = A_\emptyset$ and $w_{2rn} = \gamma A_\emptyset$.*

We now state the final lemma required for the proof of Theorem 1.4, where we recall Definition 2.7 and Remark 2.8.

Lemma 6.5. (1) *For any $i \in \{1,\dots,k\}$, the sequence of hyperplanes*

$$J_{i,1},\dots,J_{i,2rn}$$

is a sequence of separating hyperplanes from $J_{i,0}$ to $J_{i,2rn+1}$, where $J_{i,0}$ and $J_{i,2rn+1}$ are of $v_{i,n}$-type and $v_{i,1}$-type, respectively (in particular, from $w_0 = A_\emptyset$ to $w_{2rn} = \gamma A_\emptyset$).

(2) *The sequence of hyperplanes*

$$\begin{aligned}
&J_{i_1,1}, J_{i_1,2},\dots,J_{i_1,2l(i_1,i_2)-1},\\
&J_{i_2,2l(i_1,i_2)}, J_{i_2,2l(i_1,i_2)+1},\dots,J_{i_2,2n},\\
&J_{i_2,2n+1}, J_{i_2,2(n+1)},\dots,J_{i_2,2(n+l(i_2,i_3))-1},\\
&\quad\vdots\\
&J_{i_r,2((r-2)n+l(i_{r-1},i_r))}, J_{i_r,2((r-2)n+l(i_{r-1},i_r))+1},\dots,J_{i_r,2(r-1)n},\\
&J_{i_r,2(r-1)n+1}, J_{i_r,2((r-1)n+1)},\dots,J_{i_r,2((r-1)n+l(i_r,i_{r+1}))-1},\\
&J_{i_{r+1},2((r-1)n+l(i_r,i_{r+1}))}, J_{i_{r+1},2((r-1)n+l(i_r,i_{r+1}))+1},\dots,J_{i_{r+1},2rn}
\end{aligned}$$

is a sequence of separating hyperplanes from $J_{1,0}$ to $J_{1,2rn+1}$, where $J_{1,0}$ and $J_{1,2rn+1}$ are of $v_{1,n}$-type and $v_{1,1}$-type, respectively (in particular, from $w_0 = A_\emptyset$ to $w_{2rn} = \gamma A_\emptyset$). (See Figure 6.)

Moreover, the sequence contains a hyperplane of v_i-type for any $i \in \{1,\dots,k\}$ and $v_i \in V(\Gamma_i)$.

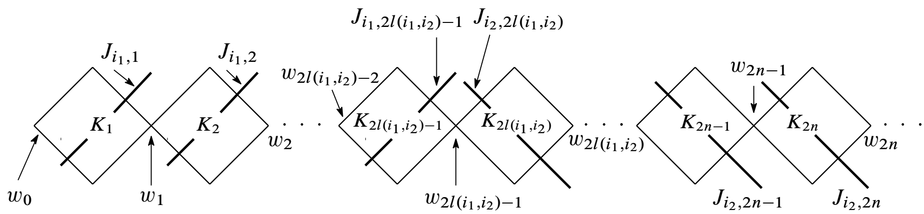

Figure 6. Part of a sequence of hyperplanes for the case where $\Gamma = \Gamma_1 * \Gamma_2$. Here $w_0 = \gamma(0)A_\emptyset = A_\emptyset$, $w_1 = \gamma(0)A_{U_1} = A_{U_1}$ $(= \gamma(1)A_{U_1})$, $w_{2l(i_1,i_2)-2} = \gamma(l(i_1,i_2)-1)A_\emptyset$, $w_{2l(i_1,i_2)-1} = \gamma(l(i_1,i_2)-1)A_{U_{l(i_1,i_2)}}$ $(= \gamma(l(i_1,i_2))A_{U_{l(i_1,i_2)}})$, $w_{2n} = \gamma(n)A_\emptyset$.

Proof. (1) The assertion is clear from parts (1) and (2) of Lemma 6.2.

(2) Parts (1), (2), and (3) of Lemma 6.2 and Lemma 6.3 imply that the sequence specified in the assertion is a sequence of separating hyperplanes from $J_{i_1,0}$ to $J_{i_{r+1},2rn+1}$. Note that $i_1 = 1$ and $i_{r+1} = 1$ by definition.

We show that the sequence contains a hyperplane of v_i-type for any $i \in \{1,\dots,k\}$ and $v_i \in V(\Gamma_i)$. Recall that $(V_{i_1},\dots,V_{i_r},V_{i_{r+1}})$ (6.1) is a closed path on the tree T. Thus, if an edge $(V_i,V_j) \in E(T)$ is contained in the closed path, then so is the inverse edge (V_j,V_i). In addition, because T is a spanning tree of $Q(\Gamma)$ and the closed path $(V_{i_1},\dots,V_{i_r},V_{i_{r+1}})$ passes through every vertex at least once, any $i \in \{1,\dots,k\}$, V_i is contained in the closed path as a vertex. Additionally, recall that T has the root V_1 and that $i < j$ only if V_i is not farther than V_j from V_1 in T.

Now, take any $i \in \{1,\dots,k\}$. We consider the two cases of $i = 1$ and $i \neq 1$.

First, suppose that $i = 1$. Note that $i_1 = 1$ and set $j = i_2$. Take $a \in \{1,2,\dots,r\}$ such that $a \neq 1$, $i_a = 1$, and $i_{a-1} = j$. Then, the sequence in the assertion contains the two subsequences

$$J_{i_1,1}, J_{i_1,2}, \dots, J_{i_1,2l(i_1,i_2)-1},$$

which are of $v_{1,1}$-type, $v_{1,1}$-type, $\dots$, $v_{1,l(1,j)}$-type, and

$$J_{i_a,2((a-2)n+l(i_{a-1},i_a))}, J_{i_a,2((a-2)n+l(i_{a-1},i_a))+1}, \dots, J_{i_a,2(a-1)n},$$

which are of $v_{1,l(j,1)}$-type, $v_{1,l(j,1)+1}$-type, $\dots$, $v_{1,n}$-type. Note that $l(j,1) = l(1,j)$ by Lemma 6.1. Take any vertex $v_1 \in V(\Gamma_1)$. Then, there exists some $l \in \{1,\dots,n\}$ such that $v_{1,l} = v_1$ by Lemma 6.1. Hence, the sequence in the assertion contains a hyperplane of v_1-type.

Next, suppose that $i \neq 1$. Take the smallest $a \in \{1,\dots,r\}$ such that $i_a = i$. Because $i_1 = 1$ and $i \neq 1$, we have $a \neq 1$. We set $j = i_{a-1}$. Then, we have $a' \in \{1,\dots,r\}$ such that $a \leq a'$, $i_{a'} = i$, and $i_{a'+1} = j$. The sequence contains the two subsequences

$$J_{i_a,2((a-2)n+l(i_{a-1},i_a))}, J_{i_a,2((a-2)n+l(i_{a-1},i_a))+1}, \dots, J_{i_a,2(a-1)n},$$

which are of $v_{i,l(j,i)}$-type, $v_{i,l(j,i)+1}$-type, $\ldots$, $v_{i,n}$-type, and

$$J_{i_{a'},2(a'-1)n+1}, J_{i_{a'},2((a'-1)n+1)}, \ldots, J_{i_{a'},2((a'-1)n+l(i_{a'},i_{a'+1}))-1},$$

which are of $v_{i,1}$-type, $v_{i,1}$-type, $\ldots$, $v_{i,l(j,i)}$-type. Note that

$$l(j,i) = l(i,j)$$

by Lemma 6.1. Take any vertex $v_i \in V(\Gamma_i)$. Then, there exists some $l \in \{1,\ldots,n\}$ such that $v_{i,l} = v_i$ by Lemma 6.1. Hence, the sequence in the assertion contains a hyperplane of v_i-type. ∎

We can now complete the proof of Theorem 1.4.

Proof of (1) $\Rightarrow$ (2) *in Theorem* 1.4. We show (1) $\Rightarrow$ (2) in Theorem 1.4. Consider $\gamma \in A_\Gamma$ defined by (6.2) and hyperplanes

$$J := J_{1,0} \quad \text{and} \quad J' := J_{1,2rn+1},$$

which are of $v_{1,n}$-type and $v_{1,1}$-type, respectively. We will confirm conditions (i), (ii), and (iii) in Theorem 2.9.

(i) γ skewers (J, J'). Indeed, it is clear that

$$J^+ \supsetneq (\gamma^{-1}(J'^+) \supsetneq \gamma(J^+) \supsetneq) J'^+ \supsetneq \gamma^2(J^+).$$

(ii) We show that J and J' are strongly separated. Take any hyperplane H with $J \cap H \neq \emptyset$. When H is of v_i-type for some i and $v_i \in V(\Gamma_i)$, take a hyperplane H' of v_i-type separating $A_\emptyset$ and $\gamma A_\emptyset$ such that $J \cap H' = \emptyset$ by part (2) of Lemma 6.5. Then, $H \cap H' = \emptyset$ by Remark 4.2. Because

$$J \subset H'^-, H'^+ \supset J', \quad J \cap H \neq \emptyset \quad \text{and} \quad H \cap H' = \emptyset,$$

we have $J' \cap H = \emptyset$.

(iii) We show $\mathrm{Stab}(J) \cap \mathrm{Stab}(J') = \{1\}$. Note that for any $i \in \{1,\ldots,k\}$ and any $v_i \in V(\Gamma_i)$, we have at least one sequence of separating hyperplanes $P'_1, \ldots, P'_{M'}$ from $A_\emptyset$ to $\gamma A_\emptyset$ such that P'_1 is of v_i-type and $P'_{M'}$ is of $v_{1,n}$-type. For example, we can take such a sequence by considering a subsequence of the sequence in part (2) of Lemma 6.5. For any $i \in \{1,\ldots,k\}$ and any $v_i \in V(\Gamma_i)$, we can take a longest sequence of separating hyperplanes $P_1, \ldots, P_M$ from $A_\emptyset$ to $\gamma A_\emptyset$ such that P_1 is of v_i-type and P_M is of $v_{1,n}$-type, where

$$A_\emptyset \in P_1^-, \quad P_1^+ \supsetneq P_2^+ \supsetneq \cdots \supsetneq P_{M-1}^+ \supsetneq P_M^+ \ni \gamma A_\emptyset$$

by taking decompositions by appropriate connected components

$$C_\Gamma \backslash\backslash P_1 = P_1^- \sqcup P_1^+, \ \ldots, \ C_\Gamma \backslash\backslash P_M = P_M^- \sqcup P_M^+.$$

Note that

$$P_1, \dots, P_M \in \{J_{j,d}\}_{j \in \{1,\dots,k\},\, d \in \{1,\dots,2rn\}}$$

by Lemma 6.4. By noting $P_M \in \{J_{1,d}\}_{d \in \{1,\dots,2rn\}}$ and part (1) of Lemma 6.5 for the case $i = 1$, we have $P_M^+ \supsetneq J'^+$.

Now assume that $\mathrm{Stab}(J) \cap \mathrm{Stab}(J') \neq \{1\}$. Take $g \in \mathrm{Stab}(J) \cap \mathrm{Stab}(J')$ with $g \neq 1$. Note that $g^{-1} \in \mathrm{Stab}(J) \cap \mathrm{Stab}(J')$. Then, we have a hyperplane H of v_i-type for some $i \in \{1, \dots, k\}$ and some $v_i \in V(\Gamma_i)$, separating $A_\emptyset$ and $gA_\emptyset$. We take a longest sequence of separating hyperplanes $P_1, \dots, P_M$ from $A_\emptyset$ to $\gamma A_\emptyset$ such that P_1 is of v_i-type and P_M is of $v_{1,n}$-type. Then, we have

$$g\gamma A_\emptyset,\ g^{-1}\gamma A_\emptyset \in P_M^+$$

by $P_M^+ \supsetneq J'^+$. We take a connected component H^- of $C_\Gamma \setminus\setminus H$ such that $A_\emptyset \in H^-$. Then, the other connected component H^+ satisfies $gA_\emptyset \in H^+$. Because $H \cap J \neq \emptyset$, we have $H \cap J' = \emptyset$ by (ii). Thus, H cannot separate $\gamma A_\emptyset$ and $g\gamma A_\emptyset$. Hence, we have either

(a) $A_\emptyset, \gamma A_\emptyset, g\gamma A_\emptyset \in H^-$ and $gA_\emptyset \in H^+$ or

(b) $A_\emptyset \in H^-$ and $gA_\emptyset, \gamma A_\emptyset, g\gamma A_\emptyset \in H^+$.

Assume that case (a) occurs. Then, H does not separate $A_\emptyset$ and $\gamma A_\emptyset$ and thus $H \notin \{J_{i,d}\}_{d \in \{1,\dots,2rn\}}$ by Lemma 6.4. Hence, H and P_1 are different hyperplanes of the same v_i-type, and thus they cannot intersect by Remark 4.2. Then, we have either $H^- \supsetneq P_1^+$ or $H^+ \supsetneq P_1^+$. However, $H^+ \supsetneq P_1^+$ cannot occur because $\gamma A_\emptyset \notin H^+$ and $\gamma A_\emptyset \in P_1^+$. Therefore,

$$H^- \supsetneq P_1^+.$$

Then, we have

$$gA_\emptyset \in H^+, \quad H^- \supsetneq P_1^+ \supsetneq P_2^+ \supsetneq \cdots \supsetneq P_{M-1}^+ \supsetneq P_M^+ \ni g\gamma A_\emptyset.$$

Then, $Q_1 = H, Q_2 = P_1, \dots, Q_{M+1} = P_M$ is a sequence of separating hyperplanes from $gA_\emptyset$ to $g\gamma A_\emptyset$ such that Q_1 is of v_i-type and Q_{M+1} is of $v_{1,n}$-type. Thus, $g^{-1}Q_1$, $g^{-1}Q_2, \dots, g^{-1}Q_{M+1}$ is a sequence of separating hyperplanes from $A_\emptyset$ to $\gamma A_\emptyset$ such that $g^{-1}Q_1$ is of v_i-type and $g^{-1}Q_{M+1}$ is of $v_{1,n}$-type, which contradicts the fact that the sequence $P_1, \dots, P_M$ is longest.

Next, assume that case (b) occurs. Then, H does not separate $gA_\emptyset$ and $g\gamma A_\emptyset$, that is, $g^{-1}H$ does not separate $A_\emptyset$ and $\gamma A_\emptyset$ and thus $g^{-1}H \notin \{J_{i,d}\}_{d \in \{1,\dots,2rn\}}$ by Lemma 6.4. Hence, $g^{-1}H$ and P_1 are different hyperplanes of the same v_i-type, and thus they cannot intersect by Remark 4.2. Then, we have either $g^{-1}H^+ \supsetneq P_1^+$ or $g^{-1}H^- \supsetneq P_1^+$. However, $g^{-1}H^- \supsetneq P_1^+$ cannot occur because $\gamma A_\emptyset \notin g^{-1}H^-$ and $\gamma A_\emptyset \in P_1^+$. Therefore,

$$g^{-1}H^+ \supsetneq P_1^+.$$

Then, we have

$$g^{-1}A_\emptyset \in g^{-1}H^-, \quad g^{-1}H^+ \supsetneq P_1^+ \supsetneq P_2^+ \supsetneq \cdots \supsetneq P_{M-1}^+ \supsetneq P_M^+ \ni g^{-1}\gamma A_\emptyset.$$

Then, $Q_1 = H, Q_2 = gP_1, \dots, Q_{M+1} = gP_M$ is a sequence of separating hyperplanes from $A_\emptyset$ to $\gamma A_\emptyset$ such that Q_1 is of v_i-type and Q_{M+1} is of $v_{1,n}$-type, which contradicts the fact that the sequence $P_1, \dots, P_M$ is longest.

We now see that $\mathrm{Stab}(J) \cap \mathrm{Stab}(J') = \{1\}$.

This completes the proof of Theorem 1.4. ∎

Remark 6.6. In [11, Theorem 3.3], it is shown that A_Γ is centerless under the setting in Theorem 1.4 with (1). This claim can be proved based on Theorem 1.4. Indeed, Theorem 1.4 (1) $\Rightarrow$ (2) implies that A_Γ is acylindrically hyperbolic. Therefore, the center of A_Γ is finite by acylindrical hyperbolicity [25, Corollary 7.3]. Proposition 4.8 then implies that the center is trivial. Hence, A_Γ is centerless.

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