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# Slim curves, limit sets and spherical CR uniformisations

Elisha Falbel, Antonin Guilloux, and Pierre Will

**Abstract.** We consider the 3-sphere $\mathbb{S}^3$ seen as the boundary at infinity of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. It comes equipped with a contact structure and two classes of special curves. First, $\mathbb{R}$-circles are boundaries at infinity of totally real totally geodesic subspaces and are tangent to the contact distribution. Second, $\mathbb{C}$-circles are boundaries of complex totally geodesic subspaces and are transverse to the contact distribution. We define a quantitative notion, called *slimness*, that measures to what extent a continuous path in the sphere $\mathbb{S}^3$ is near to being an $\mathbb{R}$-circle. We analyse the classical foliation of the complement of an $\mathbb{R}$-circle by arcs of $\mathbb{C}$-circles. Next, we consider deformations of this situation where the $\mathbb{R}$-circle becomes a slim curve. We apply these concepts to the particular case where the slim curve is the limit set of a quasi-Fuchsian subgroup of $\mathrm{PU}(2,1)$. As an application, we describe a class of spherical CR uniformisations of certain cusped 3-manifolds.

## 1. Introduction

The frame of this work is the study of quasi-Fuchsian deformations in complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, which can be thought of as the unit ball in $\mathbb{C}^2$. Using a projective model, the biholomorphism isometry group of $\mathbf{H}^2_{\mathbb{C}}$ can be identified with $\mathrm{PU}(2,1)$, the subgroup of $\mathrm{PGL}(3,\mathbb{C})$ corresponding to those transformations preserving a Hermitian form of signature $(2,1)$.

Complex hyperbolic space is a rank one Hermitian symmetric space and as such, it is a Kähler manifold with negative $\frac{1}{4}$-pinched curvature. Totally geodesic real planes and complex lines realise the extremal values of the sectional curvature (namely, $-1$ for complex lines and $-\frac{1}{4}$ for real planes). The boundary at infinity of $\mathbf{H}^2_{\mathbb{C}}$ can be seen as the 3-sphere $\mathbb{S}^3$. Complex lines and totally geodesic real planes give rise to two distinguished classes of curves in $\mathbb{S}^3$: $\mathbb{C}$-circles and $\mathbb{R}$-circles, respectively (see [23]). The sphere $\mathbb{S}^3$ inherits a CR structure from its embedding into $\mathbb{C}^2$. This CR structure defines a contact structure for which $\mathbb{C}$-circles are everywhere transverse (they are the chains of the CR structure) and $\mathbb{R}$-circles are Legendrian. We review these structures in Section 2.

We first consider $\mathrm{PO}(2,1)$ seen as the stabiliser of a totally real totally geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$. These subspaces are often called real planes for short, and the typical example is $\mathbf{H}^2_{\mathbb{R}} \subset \mathbf{H}^2_{\mathbb{C}}$, which, in coordinates, is the set of real points of the complex unit ball. This embedding $\mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ gives isometric actions of Fuchsian subgroups

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of $\mathrm{PO}(2,1)$ preserving $\mathbf{H}^2_{\mathbb{R}}$. Such subgroups are called $\mathbb{R}$-Fuchsian. The main theme we address here is to study deformations of $\mathbb{R}$-Fuchsian subgroups of $\mathrm{PU}(2,1)$.

The complex hyperbolic plane has another type of totally geodesic subspaces: complex lines, which give rise to the notion of $\mathbb{C}$-Fuchsian subgroups of $\mathrm{PU}(2,1)$. But, contrary to the $\mathbb{R}$-Fuchsian case, any deformation of a cocompact $\mathbb{C}$-Fuchsian subgroup of $\mathrm{PU}(2,1)$ is still $\mathbb{C}$-Fuchsian (see [39] for this rigidity result and [29] for a review and further generalisations).

For a discrete subgroup of $\mathrm{PU}(2,1)$, a most natural object to consider is its limit set in $\mathbb{S}^3$, which is a topological circle in the quasi-Fuchsian case. We aim at understanding the relative position of the limit set of a quasi-Fuchsian group and $\mathbb{C}$-circles in $\mathbb{S}^3$.

### 1.1. Horizontality, hyperconvexity and slimness in the sphere

We consider three related notions for subsets of $\mathbb{S}^3$. For the definition of these notions, we use the Cartan invariant $\mathbb{A}$ of triples of points in $\mathbb{S}^3$. It is a numerical invariant that classifies oriented triples up to the action of $\mathrm{PU}(2,1)$. For now, let us only mention that the Cartan invariant takes all values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and that a triple (of pairwise distinct points) $(p_1, p_2, p_3)$ is contained in an $\mathbb{R}$-circle (resp. a $\mathbb{C}$-circle) if and only if $\mathbb{A}(p_1, p_2, p_3) = 0$ (resp. $\mathbb{A}(p_1, p_2, p_3) = \pm\frac{\pi}{2}$). In particular, we note that if $|\mathbb{A}(p_1, p_2, p_3)| < \frac{\pi}{2}$ and the three points are distinct, then the triangle $(p_1, p_2, p_3)$ does not belong to any $\mathbb{C}$-circle. A more detailed presentation is given in Section 2.3.

Though our initial interest was for limit sets, we will first drop the invariance by a group assumption. In Section 3, we work with arbitrary compact subsets $E$ of $\mathbb{S}^3$ and consider the following three properties:

- *Horizontality.* This is an extension for arbitrary compact subsets of $\mathbb{S}^3$ of the concept of Legendrian submanifolds and is a local property. It is defined in Definition 3.1. It amounts to asking that convergence $p_n \to p$ in $E$ only happens tangentially to the contact structure; see Lemma 3.5. We describe in Section 3.2 some horizontal orbits of one-parameter subgroups of $\mathrm{PU}(2,1)$.
- *Hyperconvexity.* A subset of $\mathbb{S}^3$ is called hyperconvex if its intersection with any $\mathbb{C}$-circle contains at most two points. This notion is a version of a central notion in the theory of Anosov representations, stemming from [30] and, in a context similar to this paper, [35].
- *Slimness.* This is a quantitative notion that implies hyperconvexity and horizontality. For a closed subset $E$ of $\mathbb{S}^3$, we define

$$\mathbb{A}(E) = \sup\left\{\left|\mathbb{A}(p,q,r)\right|, p,q,r \in E\right\}.$$

  We say that $E$ is *$\alpha$-slim* whenever $\mathbb{A}(E) \leqslant \alpha < \pi/2$; see Definition 3.9. It directly implies hyperconvexity, from the above mentioned properties of $\mathbb{A}$. But it also implies horizontality, as proven in Proposition 3.11. In case $E$ is the limit set of a representation of a surface group, the quantity $\mathbb{A}(E)$ can be interpreted using bounded cohomology as a Gromov norm of a cohomology class; see Remark 3.10 (5).

We give geometric interpretations of slimness in Section 3.4. The simplest examples to study these three properties and their consequences are $\mathbb{R}$-circles. They are Legendrian, hyperconvex and 0-slim since any triple of distinct points in an $\mathbb{R}$-circle has vanishing Cartan invariant. We will describe other families of examples and non-examples in Section 3.5. In particular, we show that slim deformations of $\mathbb{R}$-circles do exist. We define *bent* $\mathbb{R}$-*circles*; see Section 3.5.1. In Heisenberg coordinates ($\{[z,t],\ z \in \mathbb{C},\ t \in \mathbb{R}\}$), for each $0 < \theta < \pi$, the set

$$E_\theta = \left\{[r,0], r \in \mathbb{R}_+\right\} \cup \left\{[re^{i\theta},0], r \in \mathbb{R}_+\right\} \cup \{\infty\}$$

is slim; see Proposition 3.19. Note that $E_\pi$ is in fact an $\mathbb{R}$-circle.

Moreover, as explained in Section 3.5.4, if $\Gamma \subset \mathrm{PO}(2,1)$ is a cocompact $\mathbb{R}$-Fuchsian group, then it can be deformed in $\mathrm{PU}(2,1)$ and the limit sets will be slim along this deformation, at least locally. This remark is essentially borrowed from [35].

### 1.2. A foliation on the complement of an $\mathbb{R}$-circle

We relate the three properties above and a known identification between the complement of an $\mathbb{R}$-circle and the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$, as studied in [11]. Assume $\Lambda_0$ is the $\mathbb{R}$-circle $\partial_\infty \mathbf{H}^2_{\mathbb{R}}$ and denote by $\Omega_0$ its complement in $\mathbb{S}^3$. Then, for any pair of distinct points $p \neq q$ in $\Lambda_0$, denote by $\mathcal{L}(p,q) \subset \mathbf{H}^2_{\mathbb{C}}$ the unique complex line containing $p$ and $q$. The $\mathbb{C}$-circle $\partial_\infty \mathcal{L}(p,q) \subset \mathbb{S}^3$ is naturally oriented by the complex structure of $\mathcal{L}(p,q)$. Moreover, it intersects $\Lambda_0$ only at $p$ and $q$ since $\Lambda_0$ is hyperconvex. It is therefore divided into two connected components, which are oriented intervals. We will denote these intervals by $p \curvearrowright q$ and $q \curvearrowright p$. An important result for our work is the following proposition [11, Proposition 6.7].

**Proposition.** *The open set $\Omega_0$ is homeomorphic to the unit tangent bundle of $\mathbf{H}^2_{\mathbb{R}}$. In this homeomorphism, the arcs $p \curvearrowright q$ correspond to the orbits of the geodesic flow on $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$.*

We refer to Section 2.6 for more details. This proposition also tells us that $\Omega_0$ is foliated by the arcs $p \curvearrowright q$. All along this paper, we reinterpret it in various ways; see Corollary 2.16, Proposition 2.17, Corollary 4.2, Proposition 4.7.

If $\mathbf{H}^2_{\mathbb{R}}$ is acted on by an $\mathbb{R}$-Fuschsian subgroup of $\mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$, then so is $\Omega_0$ and the above homeomorphism descends to a homeomorphism between $\Gamma\backslash\Omega_0$ and the unit tangent bundle $\mathrm{UT}(\Gamma\backslash\mathbf{H}^2_{\mathbb{R}})$ of $\Gamma\backslash\mathbf{H}^2_{\mathbb{R}}$ where orbits of the geodesic flow correspond to projection of arcs.

### 1.3. Deforming the foliation

Describing deformations of this foliation when deforming $\Lambda_0$ is one of the main points of this article (Section 4). As explained before, there exist Hausdorff-continuous deformations $(\Lambda_t)$ of $\Lambda_0$ such that all $\Lambda_t$ are slim. Denote by $\Omega_t$ the complement of $\Lambda_t$. First, we prove that arcs of $\mathbb{C}$-circles sweep out $\Omega_t$.

**Theorem** (First point of Theorem 4.4). *Let $\Lambda_t$ be a Hausdorff-continuous family of slim circles, with $\Lambda_0$ an $\mathbb{R}$-circle. Then, for all $t$, $\Omega_t$ is the union of the family of arcs $\{p \curvearrowright q, p, q \in \Lambda_t,\ p \neq q\}$.*

The strategy to prove this theorem is interesting per se. We first prove in Section 4.2 that a horizontal and hyperconvex circle $\Lambda$ can be continuously extended *outside* the complex hyperbolic space: there is an explicit continuous embedding of the Möbius strip in $\mathbb{CP}^2 \setminus \mathbf{H}^2_{\mathbb{C}}$ whose intersection with $\partial_\infty \mathbf{H}^2_{\mathbb{C}}$ is exactly $\Lambda$. Our construction is flexible enough to prove that, under deformations of $\Lambda$, the Möbius strips deform by homotopy; see Section 4.3. One can then apply an argument of intersection in homology to prove the theorem.

Thanks to this theorem, we can exhibit an actual deformation $\Lambda_t$ such that arcs of $\mathbb{C}$-circles define a foliation of $\Omega_t$.

**Theorem** (Theorem 4.16). *For any $\theta \in [\pi/2, 3\pi/2]$, the set of arcs of $\mathbb{C}$-circles with endpoints in $E_\theta$ defines a foliation of $\mathbb{S}^3 \setminus E_\theta$.*

A caveat is necessary here: not all bent $\mathbb{R}$-circles give rise to a foliation. Indeed, if the bending is too strong ($|\pi - \theta| > \frac{\pi}{2}$), then some arcs do intersect.

However, it is hard to deform an $\mathbb{R}$-circle into a slim circle $\Lambda$ invariant under a group and such that the family of arcs $p \curvearrowright q$ (for $p, q \in \Lambda$) defines a foliation. Indeed, the invariance by a single non-real loxodromic element implies that some arcs intersect. Recall that a loxodromic element of $\mathrm{PU}(2,1)$ is not real if the trace of its cube – which is well defined – is not real.

**Theorem** (Second point of Theorem 4.4). *Let $\Lambda$ be a slim circle, which is invariant by a non-real loxodromic transformation. Then, there are arcs $p \curvearrowright q$, with $p \neq q \in \Lambda$, that intersect in the complement $\Omega$ of $\Lambda$.*

We get as a corollary that no non-$\mathbb{R}$-Fuchsian deformation of a lattice in $\mathrm{PO}(2,1)$ determines a foliation of the complement of its limit set by arcs of $\mathbb{C}$-circles. This can also be interpreted as the following rigidity theorem.

**Theorem** (See Theorem 4.20). *Let $\Gamma$ be a cocompact lattice in $\mathrm{PO}(2,1)$ and $\rho : \Gamma \to \mathrm{PU}(2,1)$ a small deformation of the inclusion. Let $\Lambda$ be its limit set and $\Omega$ its complement.*

*If $\Omega$ is foliated by the arcs $p \curvearrowright q$, for $p \neq q \in \Lambda$, then $\rho$ is $\mathbb{R}$-Fuchsian.*

### 1.4. Drilling and crown-type uniformisations

We will call here CR-spherical uniformisation of a manifold $M$ a homeomorphism $M \simeq \Omega/\rho(\pi_1(M))$, where $\Omega$ is an open subset of the sphere on which $\rho(\pi_1(M))$ acts properly discontinuously (see [28]). One should be careful with this definition as, sometimes, uniformisation refers to the case where $\Omega$ is assumed to be the domain of discontinuity of $\rho(\pi_1(M))$. This is for instance the definition taken by Deraux in [15, Definition 1.3]. In particular, when $\Omega$ is the domain of discontinuity of $\rho(\pi_1(M))$, then the 3-manifold that is uniformised appears as the boundary at infinity of a quotient of the complex hyperbolic plane. This happens for most of the examples of CR-spherical uniformisations of hyperbolic 3-manifolds that have been constructed (see for instance [16, 34, 37]), but we will consider here examples where it is not the case. Note also that $\Omega$ needs not be simply connected – and is not in our examples. As a consequence, $\rho$ is not injective in general.

Going back to deformation of $\mathbb{R}$-Fuchsian surface groups, general arguments about geometric structures, namely, the Ehresmann–Thurston principle and work by Guichard–Wienhard [25], imply that, when deforming $\Gamma$ to a representation $\rho$ close enough to the inclusion, the complement $\Omega$ of the limit set $\Lambda$ of $\rho(\Gamma)$ still uniformizes $\mathrm{UT}\Sigma$. We recall these arguments in Proposition 5.9.

We can drill along closed orbits of the geodesic flow in $\mathrm{UT}\Sigma$. For an oriented closed geodesic $\lambda$, denote by $\mathrm{UT}\Sigma(\lambda)$ the unit tangent bundle drilled out along the natural lift of $\lambda$. The uniformisations of $\mathrm{UT}\Sigma$ described above naturally give uniformisations of $\mathrm{UT}\Sigma(\lambda)$. The manifolds constructed in this way cover in particular a number of hyperbolic cusped manifolds. We say that $\lambda$ is filling if its complement in $\Sigma$ is a union of discs. Then, by [21], as soon as $\lambda$ is *filling*, the drilled out unit tangent bundle is hyperbolic. We sum up this discussion in the proposition.

**Proposition 1.1** (Corollary 5.5). *Every manifold obtained by drilling a closed orbit of the geodesic flow in the unit tangent bundle of a hyperbolic surface admits a family of CR-spherical uniformisations.*

*An infinite number of cusped hyperbolic* 3*-manifolds can be obtained this way.*

We use the work done in the previous sections to describe explicitly these uniformisations. Having fixed a small deformation $\rho$, we want to describe an open subset whose quotient by $\rho(\Gamma)$ is homeomorphic to $\mathrm{UT}\Sigma(\lambda)$. To achieve that goal, we consider an element $\gamma$ in $\Gamma$ whose oriented axis lifts $\lambda$. A small deformation $\rho$ satisfies that $\delta := \rho(\gamma)$ is still a loxodromic transformation in $\Delta := \rho(\Gamma)$. So it has a repelling and an attracting fixed point, denoted by $\delta_-$ and $\delta_+$, both belonging to the limit set $\Lambda$ of $\Delta$. We call the *axis at infinity* of $\delta$ the arc $\alpha_\delta = \delta_- \curvearrowright \delta_+$. Then, we define in Section 5 the *crown*:

$$\mathrm{Crown}_{\Delta,\delta} = \Lambda \cup \Big( \bigcup_{g\in\Gamma} \rho(g)\cdot\alpha_\delta \Big).$$

The crown is a closed set containing the limit set and is invariant under the action of $\Delta$. We denote by $\Omega_{\Delta,\delta}$ its complement. We describe the following explicit family of uniformisations of $\mathrm{UT}\Sigma(\lambda)$.

**Theorem** (See Theorem 5.7). *For a small enough deformation $\rho$, the quotient $\Delta\backslash\Omega_{\Delta,\delta}$ is homeomorphic to* $\mathrm{UT}\Sigma(\lambda)$.

The proof works by deformation: if $\rho$ is $\mathbb{R}$-Fuchsian, this proposition is only a rephrasing of the foliation property. By small deformations, everything varies continuously and the family of axes $\rho(g)\cdot\alpha_\delta$ do not intersect.

### 1.5. Further questions and open problems

As mentioned above, many of the previously known examples of spherical CR uniformisations of hyperbolic 3-manifolds have been constructed as quotients of the whole discontinuity region of a discrete subgroup of PU(2,1). Many of them also share another common feature: the holonomy groups of the structures appear as degenerations of quasi-Fuchsian deformations of discrete subgroups of PO(2, 1), typically $(p, q, r)$-triangle groups. Note that other uniformisations have been obtained by applying Dehn-filling techniques to uniformisations obtained from these degenerations [1, 3, 38]. The typical situation observed is the following.

Let $\Gamma$ be a Fuchsian group and $\rho_0 : \Gamma \to \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ an $\mathbb{R}$-Fuchsian representation. For a variety of examples of 1-parameter families of deformations $\rho_t$ of $\rho_0$, there exists a word $w$ in $\Gamma$ which becomes parabolic for a critical value $t_w$ (for any $t < t_w$, all words are mapped to loxodromic transformations). The representation $\rho_t$ is discrete and faithful on the interval $[0, t_w]$ and is either non-discrete or non-faithful for $t > t_w$. It is in particular the conjectured situation when $\Gamma$ is a triangle group. Indeed, the Schwartz conjectures [36] predict precisely which word $w$ should become parabolic. In all cases where a detailed study of the long-time deformations of a triangle group has been achieved, the manifold at infinity for the critical value $t = t_w$ is a hyperbolic knot or link complement [16,27,31,34,37]. Note that in the case of triangle groups, the character variety has dimension 1; thus, the situation is relatively simple algebraically. However, even in this simpler case, doing a complete analysis is a difficult and very technical task based on the construction of fundamental domains. Also, it is not completely clear to this day if one can predict which 3-manifold is likely to appear as degeneration of a given triangle group deformation (see for instance the ubiquity phenomenon described by Deraux in [15, Theorem 1.5] and extended recently by Alexandre in [5]). The Schwartz conjectures have been generalised to some extent for quasi-Fuchsian deformations of surface groups by Parker and Platis (see [32, Problem 6.2]). We hope that this work could be a step toward a better understanding of these long time deformations.

Let us describe the situation of the (3, 3, 4)-triangle group, generated by three reflections $\iota_1, \iota_2, \iota_3$; see Example 5.6 for precise notation. It is a known fact that the degeneration of the (3, 3, 4)-triangle group corresponds to the word $w = \iota_3\iota_2\iota_1\iota_2$ becoming parabolic and yields a uniformisation of the figure eight knot complement by the even subgroup of the triangle group (see [16, 33]). The trace of the image of $w$, denoted by $\tau$, can be used (up to a 2-fold covering) as a coordinate for the deformation space. We thus have a 1-parameter family of representations $\rho_\tau$ of the (3, 3, 4)-triangle group in PU(2, 1).

The $\mathbb{R}$-Fuchsian representation corresponds to the value $\tau = 2 + 2\sqrt{2}$, whereas the degeneration corresponds to $\tau = 3$ (in that case, $w$ is mapped to a unipotent parabolic). It can happen that $\tau$ becomes smaller than 3, in which case $\rho_\tau(w)$ is elliptic, and the representation is either non-discrete or non-faithful in that case. One can estimate the supremum of Cartan invariants $\mathbb{A}(\Lambda_\rho)$ for the limit sets of these representations. Numerical experimentations indicate that the supremum is strictly increasing from 0 to $\pi/2$ as $\tau$ decreases

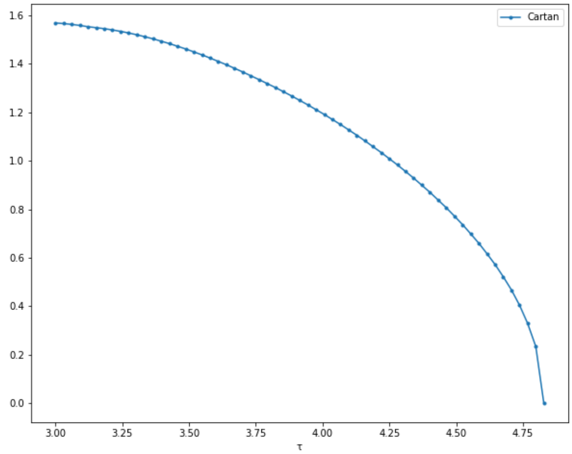


**Figure 1.** Estimation of the supremum of Cartan invariant for the (3, 3, 4)-triangle groups.

from $2 + 2\sqrt{2} \sim 4.828$ to 3, with $\pi/2$ being attained for the degeneration (see Figure 1, where the horizontal coordinate is $\tau$). In other words, the limit sets of the representations $\rho_\tau$ seem to remain slim until the degeneration.

Applying techniques using fundamental domains (see [16, 33]), it is possible to prove that the manifold uniformised by the action of the even subgroup of the (3, 3, 4)-triangle group on its discontinuity region is as follows.

- For $\tau_0 = 2 + 2\sqrt{2}$, the group is $\mathbb{R}$-Fuchsian, and the 3-manifold uniformised by the action of the even subgroup of the (3, 3, 4)-triangle group on its discontinuity region is the unit tangent bundle of the (3, 3, 4)-orbisurface.
- For $\tau \in ]3, 2 + 2\sqrt{2}[$, the image of the group remains discrete and isomorphic to the (3, 3, 4)-triangle group. The manifold at infinity remains the same.
- For $\tau = 3$, the word $w$ becomes unipotent parabolic. This implies a pinching of the limit set (the attracting and repelling fixed points of $\rho_\tau(w)$ and of its conjugates coalesce), and the manifold at infinity changes: it is the figure eight knot complement.

However, at the initial value $\tau_0 = 2 + 2\sqrt{2}$, the action of the group on the complement of the *crown* associated with $\rho_{\tau_0}(w)$ already uniformizes the figure eight knot complement (this follows from [14]). Here, the open subset giving the uniformisation is smaller than the discontinuity region. So we conjecture that all along this deformation, the crowns remain embedded and we have a family of uniformisations of the figure eight knot complement, with the last one being by the actual domain of discontinuity of the represented group.

Our results show that, in general, the topological type of the uniformised 3-manifold remains constant close to the $\mathbb{R}$-Fuchsian crown-type uniformisation, without considering explicit fundamental domains.

One can also consider larger deformation spaces. The website [4] presents experimentations about the even subgroup of the (3, 3, 4)-triangle group, which has a 2-parameter family of deformations around the $\mathbb{R}$-Fuchsian one, together with an estimation of the supremum of the Cartan invariant. The following questions seem very natural, for any $\mathbb{R}$-Fuchsian group:

- Does the whole connected component of convex-cocompact (or, equivalently, Anosov) deformations of the $\mathbb{R}$-Fuchsian representations consist of slim ones (or, equivalently, hyperconvex Anosov)?
- Which words in the group can become parabolic at the boundary of slim convex-cocompact representations?
- In the case of a representation at the boundary of slim convex-cocompact deformations with a finite set of classes of words having become parabolic, is the topology of the uniformised manifold related to the topology of a crown for the $\mathbb{R}$-Fuchsian representation?

## 2. PU(2, 1)-geometry of $\mathbb{C}\mathbb{P}^2$

One of the main thrusts behind this paper is that the geometry of some convex-cocompact representations of surface groups in PU(2, 1) is best understood considering the natural action on the complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ and its 3-sphere at infinity $\partial\mathbf{H}^2_{\mathbb{C}}$ as well as on its complement $\mathbf{H}^{1,1}_{\mathbb{C}}$ in $\mathbb{C}\mathbb{P}^2$. We hope to illustrate how the whole PU(2, 1)-geometry of $\mathbb{C}\mathbb{P}^2$ helps understand these representations. In this section, we review necessary material about this geometry.

We will constantly use points in the projective space $\mathbb{C}\mathbb{P}^2$ and lifts to $\mathbb{C}^3$. In this situation, we will denote the point and its lift by the same letter, but bolded for the lift. For example, if $p$ is a point (resp. $A$ is a projective transformation), $\boldsymbol{p}$ is a lift of $p$ (resp. $\boldsymbol{A}$ is a matrix lift of $A$). We denote by PU(2, 1) the projective unitary group associated with a Hermitian form $\langle\cdot,\cdot\rangle$ of signature (2, 1) on $\mathbb{C}^3$. At this stage, we do not specify this form.

### 2.1. Action of PU(2, 1) on $\mathbb{C}\mathbb{P}^2$

The action of PU(2, 1) on $\mathbb{C}\mathbb{P}^2$ has three orbits which are the projections to $\mathbb{C}\mathbb{P}^2$ of the three cones in $\mathbb{C}^3$ defined by

$$\begin{aligned} V^- &= \{Z \in \mathbb{C}^3, \langle Z, Z\rangle < 0\},\\ V^+ &= \{Z \in \mathbb{C}^3, \langle Z, Z\rangle > 0\},\\ V^0 &= \{Z \in \mathbb{C}^3, \langle Z, Z\rangle = 0\}. \end{aligned} \tag{1}$$

Clearly the two orbits $\mathbb{P}(V^\pm)$ are open, and $\mathbb{P}(V^0)$ is closed. We will say that a point $p \in \mathbb{C}\mathbb{P}^2$ has *negative, null or positive type* when it belongs respectively to $\mathbb{P}(V^-)$, $\mathbb{P}(V^0)$ or $\mathbb{P}(V^+)$. As sets, the two open orbits identify respectively to the homogeneous spaces

$$\begin{aligned} \mathbb{P}(V^-) &\sim \mathrm{PU}(2,1)/\mathrm{P}\big(\mathrm{U}(2)\times \mathrm{U}(1)\big) = \mathbf{H}^2_{\mathbb{C}},\\ \mathbb{P}(V^+) &\sim \mathrm{PU}(2,1)/\mathrm{P}\big(\mathrm{U}(1)\times \mathrm{U}(1,1)\big) = \mathbf{H}^{1,1}_{\mathbb{C}}. \end{aligned} \tag{2}$$

We thus view these two homogeneous spaces as subsets of $\mathbb{CP}^2$, where each of them appears as the complement of the closure of the other. These two spaces can be equipped with metrics: a Hermitian one for $\mathbf{H}^2_{\mathbb{C}}$ and pseudo-Hermitian one for $\mathbf{H}^{1,1}_{\mathbb{C}}$. Let us describe these metrics (see also [40]). First, whenever $p \in \mathbb{CP}^2$ does not have null type, we use the identification of the tangent space at $p$ given by

$$T_p\mathbb{CP}^2 = \mathrm{Hom}(\mathbb{C}\boldsymbol{p}, \boldsymbol{p}^{\perp}). \tag{3}$$

Now, if $\alpha, \beta$ are two linear maps $\mathbb{C}\boldsymbol{p} \to \boldsymbol{p}^{\perp}$, the metric is given by

$$h_p(\alpha, \beta) = -4\frac{\langle \alpha(\boldsymbol{p}), \beta(\boldsymbol{p})\rangle}{\langle \boldsymbol{p}, \boldsymbol{p}\rangle}. \tag{4}$$

Choosing a lift $\boldsymbol{p}$ of $p$ so that $\langle \boldsymbol{p}, \boldsymbol{p}\rangle = -1$ if $p \in \mathbf{H}^2_{\mathbb{C}}$ and $\langle \boldsymbol{p}, \boldsymbol{p}\rangle = +1$ if $p \in \mathbf{H}^{1,1}_{\mathbb{C}}$, and identifying $\alpha$ and $\beta$ with the images of $\boldsymbol{p}$ denoted by $\alpha(\boldsymbol{p}) = u$, $\beta(\boldsymbol{p}) = v$, we obtain

$$\begin{aligned} h_p(u, v) &= 4\langle u, v\rangle \quad \text{if } p \in \mathbf{H}^2_{\mathbb{C}},\\ h_p(u, v) &= -4\langle u, v\rangle \quad \text{if } p \in \mathbf{H}^{1,1}_{\mathbb{C}}. \end{aligned} \tag{5}$$

If $p \in \mathbf{H}^2_{\mathbb{C}}$, the direction $\mathbb{C}\boldsymbol{p}$ has negative type, and the restriction of $\langle\cdot,\cdot\rangle$ to $(\mathbb{C}\boldsymbol{p})^{\perp}$ has signature $(+,+)$. Thus, in this case, $h$ is a Hermitian metric on $T\mathbf{H}^2_{\mathbb{C}}$, whose real part is Riemannian. This is the complex hyperbolic metric. The factor 4 in (4) corresponds to normalising the sectional curvature of $\mathbf{H}^2_{\mathbb{C}}$ as being pinched between $-1$ and $-\frac{1}{4}$. If $p \in \mathbf{H}^{1,1}_{\mathbb{C}}$, the direction $\mathbb{C}\boldsymbol{p}$ has positive type, and the restriction of $\langle\cdot,\cdot\rangle$ to $(\mathbb{C}p)^{\perp}$ has signature $(+,-)$. Therefore, in this case, $h$ is a pseudo-Hermitian metric on $T\mathbf{H}^{1,1}_{\mathbb{C}}$ with (Hermitian) signature $(1,1)$, whose real part is pseudo-Riemannian with signature $(2,2)$.

The complex hyperbolic distance on $\mathbf{H}^2_{\mathbb{C}}$ can be expressed in Hermitian terms by

$$\cosh^2\left(\frac{d(p,q)}{2}\right) = \frac{\langle \boldsymbol{p}, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \boldsymbol{p}\rangle}{\langle \boldsymbol{p}, \boldsymbol{p}\rangle\langle \boldsymbol{q}, \boldsymbol{q}\rangle}. \tag{6}$$

The third orbit $\mathbb{P}(V^0)$ of the $\mathrm{PU}(2,1)$-action on $\mathbb{CP}^2$, the closed one, is the projection to $\mathbb{CP}^2$ of the quadric $\{Z \in \mathbb{C}^3, \langle Z, Z\rangle = 0\}$. This orbit identifies with the boundary at infinity of $\mathbf{H}^2_{\mathbb{C}}$, and we will denote it by $\partial\mathbf{H}^2_{\mathbb{C}}$ (it is of course the boundary of $\mathbf{H}^{1,1}_{\mathbb{C}}$ as well). It is a 3-sphere and we will also often denote it simply by $\mathbb{S}^3$. Once a lift $\boldsymbol{p}$ of $p$ is chosen, the tangent space $T_p\partial\mathbf{H}^2_{\mathbb{C}}$ can be identified with the 3-dimensional real vector subspace of $T_p\mathbb{CP}^2$ defined by the projection of $\{Z \in \mathbb{C}^3,\ \mathrm{Re}(\langle Z, \boldsymbol{p}\rangle) = 0\}$. This tangent space contains the complex 1-dimensional subspace which is the projection of $\ker(\langle\cdot, \boldsymbol{p}\rangle)$. This defines a $CR$-structure on $\partial\mathbf{H}^2_{\mathbb{C}}$, which is the homogeneous CR structure given by the field of tangent complex lines $(\ker\langle\cdot, \boldsymbol{p}\rangle)_{p\in\partial\mathbf{H}^2_{\mathbb{C}}}$; see [11]. The contact structure defined by this field of planes allows one to define horizontal submanifolds

**Definition 2.1.** A smooth submanifold of $\partial\mathbf{H}^2_{\mathbb{C}}$ is *horizontal* if at each point its tangent space is contained in the contact plane.

Such a manifold, if connected, can only be a point or a Legendrian curve. One of the main points of Section 3 will be to extend this notion to non-smooth locally closed sets.

### 2.2. Coordinate systems

Let us describe the objects considered in the previous section with the following two special choices of Hermitian forms:

$$H_B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \quad \text{or} \quad H_S = \begin{bmatrix} & & 1 \\ & 2 & \\ 1 & & \end{bmatrix}. \tag{7}$$

Using the Hermitian form given by $H_B$ leads to the so-called *ball model* of $\mathbf{H}^2_{\mathbb{C}}$. With this choice of coordinates, $\mathbf{H}^2_{\mathbb{C}}$ identifies with the unit ball of $\mathbb{C}^2$, where $\mathbb{C}^2$ itself is seen as the affine chart $z_3 = 1$ of $\mathbb{CP}^2$. Any point in $\mathbf{H}^2_{\mathbb{C}}$ can be lifted to $\mathbb{C}^3$ in a unique way as a vector $[z_1, z_2, 1]^T$, where $z_i \in \mathbb{C}$ and $|z_1|^2 + |z_2|^2 < 1$. In this model, the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ is just the 3-sphere $\mathbb{S}^3$ defined by $|z_1|^2 + |z_2|^2 = 1$. In turn, $\mathbf{H}^{1,1}_{\mathbb{C}}$ identifies with the complement in $\mathbb{CP}^2$ of the closed ball $\mathbf{H}^2_{\mathbb{C}} \cup \mathbb{S}^3$.

On the other hand, if one uses the form $H_S$, then the projection of $V^- \cup V_0$ to $\mathbb{CP}^2$ is contained in the affine chart $\{Z_3 = 1\}$, except for the projection of $[1, 0, 0]^T$, which is at infinity. Thus, any point in the closure of $\mathbf{H}^2_{\mathbb{C}}$ admits a unique lift to $\mathbb{C}^3$ which is given by

$$v_{(z,t,u)} = \begin{bmatrix} -|z|^2 - u + it \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

where $z \in \mathbb{C}$, $t \in \mathbb{R}$ and $u \geqslant 0$. These coordinates are often called *horospherical coordinates* since the level sets of $u > 0$ are the horospheres centred at $\infty$. When necessary, we will call the vector given in (8) the *standard lift* of a point in $\mathbf{H}^2_{\mathbb{C}}$. We will denote by $[z, t]$ the point in $\partial\mathbf{H}^2_{\mathbb{C}}$ which is the projection of $v_{z,t,0}$. Note that

$$\langle v_{(z,t,u)}, v_{(z,t,u)} \rangle = -2u,$$

so that the vectors $v_{(z,t,u)}$ for which $u < 0$ are lifts of those points of $\mathbf{H}^{1,1}_{\mathbb{C}}$ that belong to the affine chart $\{Z_3 = 1\}$. The line at infinity is the projection to $\mathbb{CP}^2$ of $\ker(\langle \cdot, \infty \rangle)$. It can be identified with the tangent complex line at $\infty$. Similarly, the tangent complex line $\ker(\langle , \boldsymbol{p} \rangle)$ at points $p = [x + iy, t] \in \partial\mathbf{H}^2_{\mathbb{C}}$ is easily seen to be the kernel of the 1-form

$$\alpha = dt - 2xdy + 2ydx. \tag{9}$$

The 1-form $\alpha$ is the contact form of the Heisenberg group. A $C^1$ curve $\gamma$ in $\partial\mathbf{H}^2_{\mathbb{C}}$ is horizontal, or Legendrian, if and only if its velocity belongs to the contact plane. This condition can be written with lifts in a simple way: $\gamma$ is horizontal if and only if it satisfies

$$\forall s \in \mathbb{R}, \quad \langle \dot{\boldsymbol{\gamma}}(s), \boldsymbol{\gamma}(s) \rangle = 0, \tag{10}$$

where $\boldsymbol{\gamma}(s)$ is the standard lift of $\gamma(s)$.

### 2.3. Totally geodesic subspaces and the Cartan invariant

The maximal proper totally geodesic spaces of $\mathbf{H}^2_{\mathbb{C}}$ come in the following two types.

(1) The complex lines of $\mathbf{H}^2_{\mathbb{C}}$ are the non-empty intersections with $\mathbf{H}^2_{\mathbb{C}}$ of projective lines in $\mathbb{CP}^2$. Note that a projective line intersects $\mathbf{H}^2_{\mathbb{C}}$ if and only if it is the projectivisation of a hyperbolic 2-plane of $\mathbb{C}^{2,1}$ (that is, those where the restriction of $\langle\cdot,\cdot\rangle$ has signature $(+,-)$). The sectional curvature along a complex line is constant and equal to $-1$. Typical examples are the complex axes of coordinates in the ball model of $\mathbf{H}^2_{\mathbb{C}}$.

(2) The real planes of $\mathbf{H}^2_{\mathbb{C}}$ are the non-empty intersections with $\mathbf{H}^2_{\mathbb{C}}$ of real projective planes. Real projective planes intersecting $\mathbf{H}^2_{\mathbb{C}}$ can be described as projectivisations of totally real subspaces of $\mathbb{C}^{2,1}$, that is, 3-dimensional real subspaces of $\mathbb{C}^{2,1}$ for which the restriction of $\langle\cdot,\cdot\rangle$ is real. These real planes realise the other bound $-1/4$ of the sectional curvature.

When clear from the context, we will often use the words complex line or real plane for the complex hyperbolic or projective objects. When necessary, we will specify complex hyperbolic lines or real hyperbolic planes, as opposed to complex projective lines and real projective planes.

We will make a constant use of the curves defined in $\partial\mathbf{H}^2_{\mathbb{C}}$ by intersecting complex lines and real planes with $\mathbf{H}^2_{\mathbb{C}}$.

**Definition 2.2.** A $\mathbb{C}$-*circle* in $\partial\mathbf{H}^2_{\mathbb{C}}$ is the intersection of a complex line of $\mathbf{H}^2_{\mathbb{C}}$ with $\partial\mathbf{H}^2_{\mathbb{C}}$. Similarly, an $\mathbb{R}$-*circle* in $\partial\mathbf{H}^2_{\mathbb{C}}$ is the intersection of a real plane of $\mathbf{H}^2_{\mathbb{C}}$ with $\partial\mathbf{H}^2_{\mathbb{C}}$.

**Example 2.3.** Examples of $\mathbb{R}$- and $\mathbb{C}$-circles in the Heisenberg space are depicted in Figures 2 and 3. Their description is as follows:

(1) In Heisenberg coordinates, the two axes of coordinates in the plane $\mathbb{C}\times\{0\}$ are examples of $\mathbb{R}$-circles, and more generally, so is any line through the origin in that plane. The axis $\{[0,t], t\in\mathbb{R}\}$ is a $\mathbb{C}$-circle. More generally, the $\mathbb{R}$-circles that contain the point $\infty$ are the lines through a point $p$ that are contained in the contact plane at $p$. The $\mathbb{C}$-circles through $\infty$ are the vertical lines.

(2) The $\mathbb{R}$-circles that do not contain the point $\infty$ are (compact) topological circles whose projection onto $\mathbb{C}$ is a square lemniscate (the tangents at the double point of the projection are orthogonal). The $\mathbb{C}$-circles not containing $\infty$ are ellipses contained in contact planes, which are centred at the contact point.

Note in particular that $\mathbb{R}$-circles are horizontal, whereas $\mathbb{C}$-circles are everywhere transverse to the contact distribution. The latter facts are clear in the situation where the considered $\mathbb{R}$ or $\mathbb{C}$-circle contains $\infty$, and they follow from the transitivity of the action of $\mathrm{PU}(2,1)$ on the two families of complex hyperbolic lines and real hyperbolic planes.

Another notable difference between $\mathbb{C}$-circles and $\mathbb{R}$-circles is that $\mathbb{C}$-circles have a natural orientation which is induced by the complex structure of the complex line they bound, whereas $\mathbb{R}$-circles do not have a natural $\mathrm{PU}(2,1)$-invariant orientation.

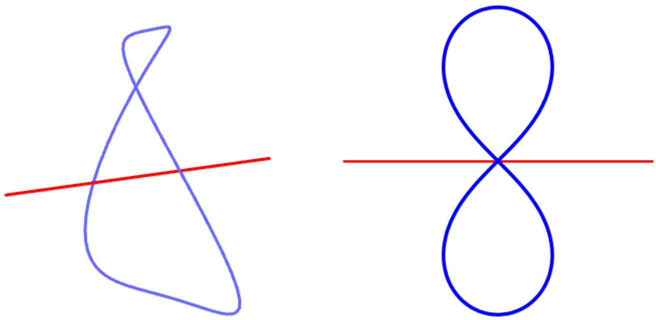

**Figure 2.** Two $\mathbb{R}$-circles: the red line is the $x$ axis of the Heisenberg coordinates. It is the boundary of $\mathbf{H}^2_{\mathbb{R}} = \mathbb{R}^2 \cap \mathbf{H}^2_{\mathbb{C}}$. The blue curve is the boundary of a real plane orthogonal to $\mathbf{H}^2_{\mathbb{R}}$. The left picture is a view in perspective in Heisenberg space, and the right picture is the vertical projection of the two $\mathbb{R}$-circles on $\mathbb{C}$.

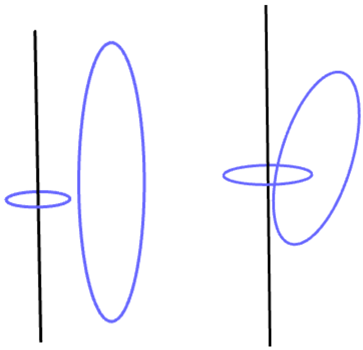

**Figure 3.** Examples of $\mathbb{C}$-circles in Heisenberg space. On both pictures, the black line is a $\mathbb{C}$-circle passing through $\infty$. Note that the pair of blue $\mathbb{C}$-circles on the left is unlinked, whereas in the right picture they are linked.

The Cartan invariant will play an important role from Section 3.3 on. It gives a simple characterisation of triples of points that lie in a $\mathbb{C}$-circle or in an $\mathbb{R}$-circle.

**Definition 2.4.** Let $(p, q, r)$ be a triple of points in $\partial\mathbf{H}^2_{\mathbb{C}}$. If the points are pairwise distinct, we define the *Cartan invariant* of the triple $(p, q, r)$ to be

$$\mathbb{A}(p, q, r) = \arg\big(-\langle \boldsymbol{p}, \boldsymbol{q}\rangle\langle \boldsymbol{q}, \boldsymbol{r}\rangle\langle \boldsymbol{r}, \boldsymbol{p}\rangle\big). \tag{11}$$

If at least two of the points coincide, we define it to be $\mathbb{A}(p, q, r) = 0$.

The quantity (11) does not depend on the choices made for lifts and is $\mathrm{PU}(2,1)$-invariant. The following statement sums up the main features of this invariant (see [23, Chapter 7] for proofs).

**Proposition 2.5.** *The Cartan invariant enjoys the following properties.*

(1) *For any triple* $(p, q, r)$*,* $\mathbb{A}(p, q, r) \in [-\pi/2, \pi/2]$.

(2) *Two triples of pairwise distinct points* $(p_1, p_2, p_3)$ *and* $(q_1, q_2, q_3)$ *have the same Cartan invariant if and only if there exists a map* $g \in \mathrm{PU}(2,1)$ *such that* $g(p_i) = q_i$ *for* $i = 1, 2, 3$.

(3) *For a triple of distinct points,* $|\mathbb{A}(p,q,r)| = \pi/2$ *if and only if the triple* $(p,q,r)$ *lies on the boundary of a complex line.*

(4) $\mathbb{A}(p,q,r) = 0$ *if and only if the triple* $(p,q,r)$ *lies on the boundary of a real plane.*

(5) $\mathbb{A}$ *is a* 3*-cocycle. In particular, if* $p, q, r, s$ *are four points, we have*

$$\mathbb{A}(p,q,r) - \mathbb{A}(p,q,s) + \mathbb{A}(p,r,s) - \mathbb{A}(q,r,s) = 0. \tag{12}$$

## 2.4. The line map and the duality between $\mathbf{H}^2_{\mathbb{C}}$ and $\mathbf{H}^{1,1}_{\mathbb{C}}$

We will often work with projective lines in $\mathbb{CP}^2$. The set of lines in $\mathbb{CP}^2$ can be described as the dual projective space, denoted by $\mathbb{CP}^{2*}$. The Hermitian form gives a natural identification between $\mathbb{CP}^2$ and $\mathbb{CP}^{2*}$, which in turn gives a polarity between points and lines of $\mathbb{CP}^2$. We review here some basic properties of this notion of polarity.

**Definition 2.6.** Let $p$ be a point in $\mathbb{CP}^2$ and $L$ a projective complex line. We say that $p$ is *polar* to $L$ if $L = \mathbb{P}(\boldsymbol{p}^{\perp})$.

The restriction of the Hermitian form on planes can be of signature $(+,+)$, $(-,+)$ or degenerate. Using polarity, we can describe the situation as follows:

- Positive type directions are orthogonal to 2-planes with signature $(+,-)$. This means that points in $\mathbf{H}^{1,1}_{\mathbb{C}}$ are polar to complex lines that intersect $\mathbf{H}^2_{\mathbb{C}}$.
- Negative type directions are orthogonal to 2-planes with signature $(+,+)$. Thus, points of $\mathbf{H}^2_{\mathbb{C}}$ are polar to complex lines contained in $\mathbf{H}^{1,1}_{\mathbb{C}}$.
- Null type directions are orthogonal to 2-planes with signature $(0,+)$. In fact, a point $p$ in $\partial\mathbf{H}^2_{\mathbb{C}}$ is polar to its orthogonal complex line $p^{\perp}$ tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at $p$. It is the only case where $p$ belongs to its polar line.

In particular, we observe that $\mathbf{H}^{1,1}_{\mathbb{C}}$ is in bijection with the Grassmanian of complex lines of $\mathbf{H}^2_{\mathbb{C}}$. This is indeed another usual definition of $\mathbf{H}^{1,1}_{\mathbb{C}}$.

Let us denote by $\Delta$ the diagonal of $\mathbb{CP}^2 \times \mathbb{CP}^2$, and by $\Delta_{\mathbb{S}^3}$ the subset $\mathbb{S}^3 \times \mathbb{S}^3 \subset \Delta$. For any pair $a, b$ of distinct points in $\mathbb{CP}^2$, we call $\mathcal{L}(a,b)$ the (unique) complex line containing $a$ and $b$. We note that this defines a PGL$(3,\mathbb{C})$-equivariant map on $(\mathbb{CP}^2 \times \mathbb{CP}^2) \setminus \Delta$, where PGL$(3,\mathbb{C})$ acts diagonally on $\mathbb{CP}^2 \times \mathbb{CP}^2$. We can extend this map to $\Delta_{\mathbb{S}^3}$ in a PU(2, 1)-equivariant way by defining $\mathcal{L}(a,a)$ to be the complex line tangent to $\mathbb{S}^3$ at $a$; that is, $\mathcal{L}(a,a) = \mathbb{P}(\ker(\langle , \boldsymbol{a}\rangle))$. This is the largest PU(2, 1)-equivariant extension of $\mathcal{L}$ to a subset of $\mathbb{CP}^2 \times \mathbb{CP}^2$ containing $(\mathbb{CP}^2 \times \mathbb{CP}^2) \setminus \Delta$.

**Definition 2.7.** The map

$$\mathcal{L} : \big((\mathbb{CP}^2 \times \mathbb{CP}^2) \setminus \Delta\big) \cup \Delta_{\mathbb{S}^3} \longrightarrow \mathbb{CP}^{2*} \tag{13}$$

defined above is called the *line map*.

The following proposition will play an important role in our work.

**Proposition 2.8.** *The line map is continuous on* $\mathbb{CP}^2 \times \mathbb{CP}^2 \setminus \Delta$, *but it is not continuous at any point of* $\Delta_{\mathbb{S}^3}$.

*Proof.* Observe that for any neighbourhood $U$ of a point $p$ in $((\mathbb{CP}^2 \times \mathbb{CP}^2) \setminus \Delta) \cup \Delta_{\mathbb{S}^3}$, $\mathcal{L}(p, U) = \mathbb{CP}^1$ identified to the set of all lines passing through $p$. But $\mathcal{L}(p, p)$ is a point in $\mathbb{CP}^{2*}$. This shows that the line map is not continuous at diagonal points. ■

As explained before, we can identify $\mathbb{CP}^{2*}$ to $\mathbb{CP}^2$ using the Hermitian form. It leads to a variant of the line map.

**Definition 2.9.** For any pair $(a, b)$ of distinct points in $\mathbb{CP}^2$, the point $a \boxtimes b$ is the intersection of the polar lines to $a$ and $b$.

As a direct consequence of the above discussion and definitions, we have the following.

**Lemma 2.10.** *The line map* $\mathcal{L}$ *enjoys the following properties.*

(1) *For any pair* $(a, b)$ *of distinct points of* $\mathbb{CP}^2$, *the line* $\mathcal{L}(a, b)$ *is polar to* $a \boxtimes b$.

(2) *For any point* $a$ *in* $\mathbb{S}^3$, *the line* $\mathcal{L}(a, a)$ *is polar to* $a$.

Given $\boldsymbol{a}$ and $\boldsymbol{b}$ as two vectors in $\mathbb{C}^3$, we denote by $\boldsymbol{a} \wedge \boldsymbol{b}$ the vector in $\mathbb{C}^3$ obtained by the usual formula of the vector product of two vectors. By twisting the vector product, one obtains a useful way of computing $a \boxtimes b$ (see also [23, Section 2.2.7]). This will allow us, later on, to explicitly compute when working with the line map.

**Definition 2.11.** Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors in $\mathbb{C}^3$, and let $J$ be the matrix of the Hermitian form in the canonical basis. We denote by $\boldsymbol{a} \boxtimes \boldsymbol{b}$ (the box product) the vector $\overline{J^{-1}\boldsymbol{a} \wedge \boldsymbol{b}}$.

Remark that the vector $\boldsymbol{a} \boxtimes \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$: this follows directly from

$$\langle X, \boldsymbol{a} \boxtimes \boldsymbol{b} \rangle = X^T J \cdot J^{-1}\boldsymbol{a} \wedge \boldsymbol{b}, \tag{14}$$

which is clearly vanishing if $X = \boldsymbol{a}$ or $X = \boldsymbol{b}$. The vector $\boldsymbol{a} \boxtimes \boldsymbol{b}$ vanishes if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are proportional. If $a \neq b$, then $\boldsymbol{a} \boxtimes \boldsymbol{b}$ is a lift of $a \boxtimes b$.

Computing with the box product is made easier by the following relations, where all come from standard identities for the usual exterior product. For any vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{C}^3$, we have (see also [23, Section 2.2.7])

$$\langle \boldsymbol{a} \boxtimes \boldsymbol{b}, \boldsymbol{c} \boxtimes \boldsymbol{d} \rangle = \langle \boldsymbol{d}, \boldsymbol{a} \rangle \langle \boldsymbol{c}, \boldsymbol{b} \rangle - \langle \boldsymbol{c}, \boldsymbol{a} \rangle \langle \boldsymbol{d}, \boldsymbol{b} \rangle, \tag{15}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \boxtimes \boldsymbol{c} \rangle = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}), \tag{16}$$

$$(\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{a} \boxtimes \boldsymbol{c}) = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \cdot \boldsymbol{a}. \tag{17}$$

## 2.5. Geometry of $\mathbf{H}^{1,1}_{\mathbb{C}}$

We will be most interested in this paper by $\mathbb{C}$-circles, which are intersections of a projective line meeting $\mathbf{H}^2_{\mathbb{C}}$ with its sphere at infinity. As stated before, these lines are polar to points in $\mathbf{H}^{1,1}_{\mathbb{C}}$. We thus describe here some properties of the geometry of $\mathbf{H}^{1,1}_{\mathbb{C}}$ about $\mathbb{C}$-circles and polarity that we will need later .

First, we can understand when two $\mathbb{C}$-circles meet, using polarity.

**Lemma 2.12.** *Let $x \neq y$ be two points of $\mathbf{H}^{1,1}_{\mathbb{C}}$. Then, the $\mathbb{C}$-circles polar to $x$ and $y$ meet if and only if $x \boxtimes y \in \mathbb{S}^3$. Their intersection is then the point $x \boxtimes y$.*

*In particular, if $a \neq b$ and $c \neq d$ are four points in $\mathbb{S}^3$, not belonging to the same complex line, the $\mathbb{C}$-circle through $a, b$ and the one through $c, d$ meet if and only if $(a \boxtimes b) \boxtimes (c \boxtimes d) \in \mathbb{S}^3$.*

*Proof.* The lines polar to $x$ and $y$ meet exactly at the point $x \boxtimes y$. So the $\mathbb{C}$-circles meet if and only if this point belongs to the sphere. The second part follows readily. ∎

We denote by $\mathbb{RP}^2 \subset \mathbb{CP}^2$ the projection to $\mathbb{CP}^2$ of $\mathbb{R}^3 \subset \mathbb{C}^3$: it is the set of points in $\mathbb{CP}^2$ fixed by complex conjugation. As we have seen above, the intersection of $\mathbb{RP}^2$ with $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbf{H}^2_{\mathbb{R}}$, and its intersection with $\mathbb{S}^3$ is the $\mathbb{R}$-circle $\partial\mathbf{H}^2_{\mathbb{R}}$.

Restricted to $\mathbb{R}^3$, the Hermitian form gives a scalar product. This comes with a notion of polarity. These two notions are coherent.

**Lemma 2.13.** *The following are equivalent:*

(1) *A point $m \in \mathbf{H}^{1,1}_{\mathbb{C}}$ belongs to $\mathbb{RP}^2$.*

(2) *The complex line $L_m$ polar to $m$ intersects $\mathbf{H}^2_{\mathbb{R}}$ along a geodesic.*

(3) *The $\mathbb{C}$-circle $\partial L_m$ intersects $\partial\mathbf{H}^2_{\mathbb{R}}$ in exactly two points.*

*Proof.* The last two items are equivalent since $\mathbf{H}^2_{\mathbb{R}}$ is totally geodesic.

Assume $L_m$ intersects $\mathbf{H}^2_{\mathbb{R}}$ along a geodesic $\gamma$, and pick two points $p, q \in \gamma$. Then, $m = p \boxtimes q$ by definition. As $p$ and $q$ belong to $\mathbb{RP}^2$, they are fixed by complex conjugation and so does $m$. So $m$ belongs to $\mathbb{RP}^2$.

Conversely, assume that $m \in \mathbb{RP}^2$, and pick a lift $\boldsymbol{m}$ with real coefficients. Then, the orthogonal $\boldsymbol{m}^\perp$ intersects $\mathbb{R}^3$ along a 2-dimensional (real) vector subspace $V$. The projection of this subspace to $\mathbb{CP}^2$ intersects $\mathbf{H}^2_{\mathbb{C}}$ along a geodesic which is contained in $\mathbf{H}^2_{\mathbb{R}}$. ∎

We can look at intersections of tangent lines to the sphere with $\mathbb{RP}^2$. Elementary projective geometry gives the following.

**Proposition 2.14.** *Let $p \in \mathbb{S}^3 \setminus \partial\mathbf{H}^2_{\mathbb{R}}$. Then, we have the following:*

(1) *The complex line $\mathcal{L}(p, p)$ tangent to $\mathbb{S}^3$ at $p$ intersects $\mathbb{RP}^2$ in exactly one point.*

(2) *The point $m = \mathcal{L}(p, p) \cap \mathbb{RP}^2$ is polar to a complex line whose $\mathbb{C}$-circle intersects $\partial\mathbf{H}^2_{\mathbb{R}}$ twice and contains $p$.*

*Proof.* Let $p$ be a point in $\mathbb{S}^3 \setminus \partial\mathbf{H}^2_{\mathbb{R}}$. It is not real, so distinct from its complex conjugate $\bar{p}$. One gets that

$$\mathcal{L}(p, p) \cap \mathbb{RP}^2 = \mathcal{L}(p, p) \cap \mathcal{L}(\bar{p}, \bar{p}).$$

These two distinct complex lines intersect in exactly one point $m$, proving the first item.

For the second item, observe that $m$ is orthogonal to $p$ and $\bar{p}$ by construction. Hence, the line $\mathcal{L}(p,\bar{p})$ is polar to $m \in \mathbb{RP}^2$. The associated $\mathbb{C}$-circle $\mathcal{L}(p,\bar{p}) \cap \mathbb{S}^3$ goes through $p$. The previous proposition gives that it intersects twice $\partial \mathbf{H}^2_{\mathbb{R}}$. ■

We call an *arc of* $\mathbb{C}$*-circle* any connected component of the complement of two distinct points in a $\mathbb{C}$-circle. For $a \neq b$ in $\mathbb{S}^3$, the unique $\mathbb{C}$-circle through $a$ and $b$ decomposes into two arcs with endpoints $a$ and $b$. Note that these two arcs are naturally oriented by the orientation of $C$. This leads to the following notation.

**Definition 2.15.** Let $a \neq b$ be two points of $\mathbb{S}^3$. We call (open) *arc from* $a$ *to* $b$, denoted by $a \curvearrowright b$, the portion of the $\mathbb{C}$-circle through $a$ and $b$ oriented from $a$ to $b$.

These arcs are chains in the CR-geometry. They interact with $\mathbb{R}$-circles to define a nice foliation, as we now review.

### 2.6. A foliation by arc of $\mathbb{C}$-circles and unit tangent bundles

The geometry explained above may be used to describe the foliation of $\mathbb{S}^3 \setminus \partial \mathbf{H}^2_{\mathbb{R}}$ by arcs of $\mathbb{C}$-circles [11, Proposition 6.7] – see Figure 5.

**Corollary 2.16.** *Let $R$ be an $\mathbb{R}$-circle in $\mathbb{S}^3$. The set of arcs $a \curvearrowright b$ of $\mathbb{C}$-circles whose endpoints $a, b$ belong to $R$ defines a foliation of $\mathbb{S}^3 \setminus R$.*

*Proof.* Since $\mathrm{PU}(2,1)$ acts transitively on the set of real planes of $\mathbf{H}^2_{\mathbb{C}}$, we may assume that $R = \partial \mathbf{H}^2_{\mathbb{R}}$. Now, let $p \in \mathbb{S}^3 \setminus \partial \mathbf{H}^2_{\mathbb{R}}$. A complex line contains $p$ if and only if it is polar to a point $n$ in the tangent complex line $\mathcal{L}(p,p)$ and intersects $\partial \mathbf{H}^2_{\mathbb{R}}$ in two points if and only if $n \in \mathbb{RP}^2$. The result thus follows directly from Proposition 2.14.

Moreover, if two such arcs $a \curvearrowright b$ and $c \curvearrowright d$ meet at $p \in \mathbb{S}^3 \setminus R$, then the $\mathbb{C}$-circles supporting these arcs also meet at $\bar{p}$. Since $p \neq \bar{p}$, they have two common points, so the $\mathbb{C}$-circles are indeed the same one. This $\mathbb{C}$-circle has only two intersection point with $R$, by Proposition 2.14. So $\{a,b\} = \{c,d\}$. As two opposite arcs of a single $\mathbb{C}$-circle are disjoint, we have $(a,b) = (c,d)$. ■

The foliation given by Corollary 2.16 gives a natural homeomorphism between $\mathbb{S}^3 \setminus \partial \mathbf{H}^2_{\mathbb{R}}$ and the unit tangent bundle of $\mathbf{H}^2_{\mathbb{R}}$, which we now describe. Denote by $J$ the complex structure on $\mathbf{H}^2_{\mathbb{C}}$. Viewing $\mathbf{H}^2_{\mathbb{R}}$ as a subspace of $\mathbf{H}^2_{\mathbb{C}}$ gives an embedding $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}} \subset \mathrm{UT}\mathbf{H}^2_{\mathbb{C}}$ and $J$ acts on it. The homeomorphism is given by

$$\begin{aligned} \varphi : \quad \mathrm{UT}\mathbf{H}^2_{\mathbb{R}} &\longrightarrow \mathbb{S}^3 \setminus \mathbf{H}^2_{\mathbb{R}} \\ (p,\vec{u}) &\longmapsto \gamma(p, -J_p\vec{u}, +\infty), \end{aligned}$$

where, for any $\vec{v}$ tangent at $p$ to $\mathbf{H}^2_{\mathbb{C}}$, $\gamma(p,\vec{v},+\infty)$ is the point at infinity of the geodesic ray from $p$ in the direction $\vec{v}$ (see Figure 4). Note that $\varphi$ is in fact defined on the whole $\mathrm{UT}\mathbf{H}^2_{\mathbb{C}}$. If $(p,\vec{u}) \in \mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$, and $\gamma$ is the oriented spanned geodesic, the image of the orbit of the geodesic flow is obtained by applying $J$ to all unit tangent vectors along $\gamma$. In particular, it is contained in the boundary of the complex line of $\mathbf{H}^2_{\mathbb{C}}$ spanned by $p$ and $\vec{u}$.

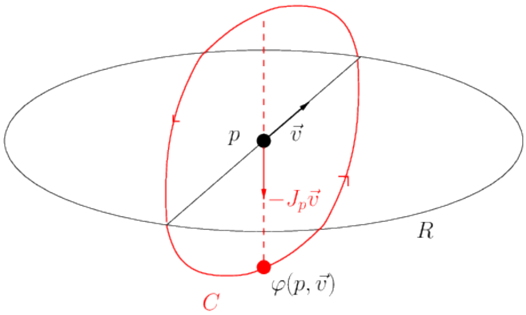


**Figure 4.** Identification between the complement in $\mathbb{S}^3$ of an $\mathbb{R}$-circle $R$ and the unit tangent bundle of $\mathbf{H}^2_{\mathbb{R}}$.

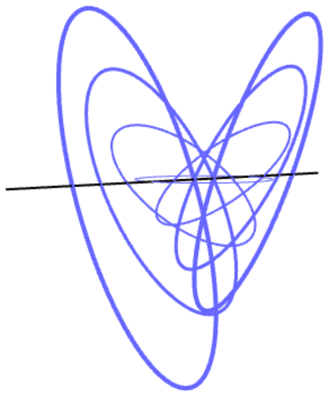

**Figure 5.** A few leaves of the foliation of $\mathbb{S}^3 \setminus \partial\mathbf{H}^2_{\mathbb{R}}$ by arcs of $\mathbb{C}$-circles (see Corollary 2.16).

In fact, it is exactly the arc of $\mathbb{C}$-circle connecting $\gamma(p, -J_p\vec{u}, +\infty)$ and $\gamma(p, J_p\vec{u}, +\infty)$ on which the natural orientation (given by the complex structure) coincides with the one given by $\vec{u}$. On Figure 4, this arc is the "lower" one.

This foliation and its link to the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ may in turn be used to understand CR-spherical structures on unit tangent bundles of hyperbolic surfaces. Let us consider a cocompact $\mathbb{R}$-Fuchsian group $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ acting on $\mathbf{H}^2_{\mathbb{C}}$ (preserving $\mathbf{H}^2_{\mathbb{R}}$). We note that the limit set of such a group is $\Lambda_\Gamma = \partial\mathbf{H}^2_{\mathbb{R}}$, and its discontinuity region is $\Omega_\Gamma = \mathbb{S}^3 \setminus \partial\mathbf{H}^2_{\mathbb{R}}$. The homeomorphism $\varphi$ constructed above is clearly $\mathrm{PO}(2,1)$-equivariant; thus, $\varphi$ descends to the quotient, and one easily obtains the following classical result – see for instance [11, Proposition 6.7] and [24, Proposition 2.7], where the same result is obtained by considering Euler numbers of circle bundles.

**Proposition 2.17.** *Let $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ be an $\mathbb{R}$-Fuchsian group. Then, $\varphi$ induces a homeomorphism $\overline{\varphi}$ between the unit tangent bundle $\mathrm{UT}(\Gamma\backslash\mathbf{H}^2_{\mathbb{R}})$ and the quotient of $\Omega_\Gamma$ by the action of $\Gamma$.*

*Moreover, the map $\varphi$ sends orbits of the geodesic flows in $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ to arcs $a \curvearrowright b$ where $a \neq b \in \partial\mathbf{H}^2_{\mathbb{R}}$.*

The main goal of this paper is first to understand what happens to this foliation when deforming the $\mathbb{R}$-circle and second, in the presence of a group action, to understand how the map $\overline{\varphi}$ deforms. But, if we consider arbitrary deformation, no meaningful description can be given. So, we first define in Section 3 a notion of horizontality for non-smooth curves and a quantitative version called *slimness*. Under these conditions, we will be able to understand how the foliation deforms in Section 4 and understand better the equivariant case in Section 5.

## 3. Horizontality and slimness in the CR-sphere

From now on, we focus especially on the sphere at infinity $\mathbb{S}^3 = \partial\mathbf{H}^2_{\mathbb{C}}$. So we will rather work with $\mathbb{R}$- and $\mathbb{C}$-circles than complex hyperbolic lines and real hyperbolic planes. All properties stated with $\mathbb{R}$ and $\mathbb{C}$-circles can be equivalently stated with their supporting complex lines and real planes.

Recall from Section 2.1 that the sphere comes with its contact structure and the notion of *horizontality* for smooth submanifolds; see Definition 2.1. However, the typical sets we want to describe are limit sets of discrete subgroups of $\mathrm{PU}(2,1)$. Those subsets are not usually smooth: in fact they are smooth only when they are the whole sphere or the group is not Zariski-dense. So we have to devise a notion of horizontality suitable for general non-smooth subset of $\mathbb{S}^3$.

### 3.1. Horizontality for non-smooth subsets of $\mathbb{S}^3 \subset \mathbb{CP}^2$

One property of horizontal submanifolds can be expressed in the following way: any $\mathbb{C}$-circle through two nearby points is entirely contained in a small neighbourhood of the two points. We could try and write a definition using the topology on the set of $\mathbb{C}$-circles. The problem is that this set is not compact, as $\mathbb{C}$-circles can degenerate to a single point. And we exactly want to use this degeneration: the $\mathbb{C}$-circle between two points in a horizontal submanifold degenerates as the two points collapse. With the previous section in mind, it appears more natural work instead in $\mathbb{CP}^2$ and use the line map. Indeed, a $\mathbb{C}$-circle is defined by a unique line in $\mathbb{CP}^2$. Moreover, when a family of $\mathbb{C}$-circles converges to a single point $p \in \mathbb{S}^3$, then the family of associated lines converges to the line $p^\perp$ in $\mathbb{CP}^{2*}$ and their polar points converge to $p$.

Recall that the restriction of the line map $\mathcal{L}$ to $\mathbb{S}^3 \times \mathbb{S}^3$ is defined in Definition 2.7 by the following: for $e \neq f \in \mathbb{S}^3$, $\mathcal{L}(e, f)$ is the line through $e$ and $f$ and $\mathcal{L}(e, e)$ is the line $e^\perp$. Using polarity, we will write equivalently $\mathcal{L}(e, f) = e \boxtimes f$ and $\mathcal{L}(e, e) = e$. Note in particular that this map is not continuous at the diagonal; see Proposition 2.8. We thus propose the following definition.

**Definition 3.1.** Let $E \subset \mathbb{S}^3$ be a closed subset. We say that $E$ is *CR-horizontal* if the restriction $\mathcal{L}_E$ of the line map $\mathcal{L}$ to $E \times E$ is continuous.

Away from the diagonal, the map $\mathcal{L}_E$ is always continuous. So the previous definition is in fact a local property. One could restate the continuity hypothesis by asking that for any $e \in E$, for any sequences $(e_n \neq f_n)$ converging to $e$, the sequence of lines $(e_n f_n)$ converges to $e^{\perp}$.

We recover in the smooth case the usual definition.

**Proposition 3.2.** *A submanifold $E$ is CR-horizontal if and only if it is horizontal.*

*Proof.* Consider the tangent space to the submanifold $E$ at some point $e \in E$. It contains a vector $v$ iff there are sequences $(e_n), (f_n)$ of points in $E$, which converge to $e$ and such that the real line $(e_n f_n)$ converges to the real line $\ell$ through $e$ containing $v$. The vector $v$ belongs to the contact plane if and only if the real line $\ell$ is included in the complex line $p^{\perp}$. After tensorising by $\mathbb{C}$, $v$ belongs to the contact plane if and only if the complex lines $(e_n f_n)$ converge to $p^{\perp}$. This proves the proposition. ■

**Remark 3.3.** In contact geometry and, more generally, in CR-geometry, horizontal paths are usually defined as absolutely continuous paths with tangent vectors in a fixed distribution. Here, we do not need existence of derivatives. On the other hand, we are using the extrinsic geometry of the CR-structure of the sphere embedded in $\mathbb{C}\mathbb{P}^2$. An intrinsic way to define horizontality for arbitrary CR-structures is to use the special paths called chains. We are saying that $E$ is horizontal if for any converging sequence of points in $E$ the directions defined by chains between the limit and the points in the sequence converges to a direction in the contact plane.

**Remark 3.4.** Thanks to the previous definition, our notion of CR-horizontal manifold is an extension of the usual notion of horizontality to non-smooth sets. So from now on, we will drop the specification of "CR-" and just call this notion *horizontality*.

The following lemma translates in coordinates the local condition at a point $p$: to first order, points arrive at $p$ along the orthogonal line $\mathcal{L}(p, p)$ or equivalently tangentially to the contact structure. Recall from Definition 2.11 that $\boldsymbol{p} \boxtimes \boldsymbol{q}$ is a specific lift of $p \boxtimes q$, even if in the following statement any lift could be used.

**Lemma 3.5.** *Let $E$ be a horizontal subset of $\mathbb{S}^3$ containing a point $p$. Let $q$ be another point in $\mathbb{S}^3$. Fix lifts $\boldsymbol{p}$ and $\boldsymbol{q}$ for $p$ and $q$. Then, any point $a \in E \setminus \{q\}$ admits a unique lift to $\mathbb{C}^3$ of the form*

$$\boldsymbol{a} = \boldsymbol{p} + x\,\boldsymbol{p} \boxtimes \boldsymbol{q} + y\boldsymbol{q}, \quad \textit{for some } x, y \textit{ in } \mathbb{C}.$$

*Moreover, $y = o(x)$ in a neighbourhood of $p$.*

*Proof.* Any point in $\mathbb{C}\mathbb{P}^2$ has a unique lift of this kind but for points in the line through $q$ and $p \boxtimes q$. This line is orthogonal to $q$, so is $\mathcal{L}(q, q)$. Its intersection with the sphere is $q$, so the first point of the lemma is proven.

Suppose now by contradiction that there is a sequence of points $a_n$ in $E$, converging to $p$, with lifts $\boldsymbol{a}_n = \boldsymbol{p} + x_n \boldsymbol{p} \boxtimes \boldsymbol{q} + y_n \boldsymbol{q}$, such that $y_n$ is not negligible compared to $x_n$. Up to passing to a subsequence, this implies that there exist $x_\infty \in \mathbb{C}$, $y_\infty \in \mathbb{C} \setminus \{0\}$ and a sequence $t_n \to 0$ of positive real numbers such that, to first order,

$$\boldsymbol{a}_n = \boldsymbol{p} + t_n(x_\infty \boldsymbol{p} \boxtimes \boldsymbol{q} + y_\infty \boldsymbol{q}) + o(t_n).$$

Then, the point $p \boxtimes a_n$ polar to the line $\mathcal{L}(p, a_n)$ can be computed with box-product:

$$\begin{aligned} \boldsymbol{p} \boxtimes \boldsymbol{a}_n &= \boldsymbol{p} \boxtimes \big(\boldsymbol{p} + t_n(x_\infty \boldsymbol{p} \boxtimes \boldsymbol{q} + y_\infty \boldsymbol{q})\big) + o(t_n) \\ &= t_n\big(x_\infty \boldsymbol{p} \boxtimes (\boldsymbol{p} \boxtimes \boldsymbol{q}) + y_\infty \boldsymbol{p} \boxtimes \boldsymbol{q}\big) + o(t_n). \end{aligned}$$

This implies that $p \boxtimes a_n$ converges to the projection of $x_\infty \boldsymbol{p} \boxtimes (\boldsymbol{p} \boxtimes \boldsymbol{q}) + y_\infty \boldsymbol{p} \boxtimes \boldsymbol{q}$. Note that $\boldsymbol{p} \boxtimes (\boldsymbol{p} \boxtimes \boldsymbol{q})$ is a multiple of $\boldsymbol{p}$. So $p \boxtimes a_n$ is a point in $p^\perp$, different from $p$ as $y_\infty \neq 0$. This contradicts the horizontality condition on $E$: $p \boxtimes a_n$ should converge to $p$ as $a_n$ goes to $p$. ■

The non-continuity of $\mathcal{L}$ at the diagonal has an interesting consequence in terms of Cartan invariants. We will make a repeated use of the following lemma.

**Lemma 3.6.** *Let $e$ be a point in $\mathbb{S}^3$, and let $(e_n)$, $(f_n)$ be two sequences of points both converging to $e$ and such that $\mathcal{L}(e_n, f_n)$ converge to some complex line $\ell \neq e^\perp$. Then, for any point $x \in \mathbb{S}^3$ distinct from $e$, it holds that*

$$\big|\mathbb{A}(x, e_n, f_n)\big| \longrightarrow \pi/2.$$

*Proof.* It suffices to prove the result for $e_n = e$. Indeed, one can make $x = \infty$ and $e_n = 0$ in Heisenberg coordinates by translating through $\mathrm{PU}(2,1)$. We have $f_n \to e$ and we suppose by contradiction that $|\mathbb{A}(x, e, f_n)|$ does not converge to $\pi/2$. There is a subsequence of $\mathbb{A}(x, e, f_n)$ converging to $A \in ]-\pi/2, \pi/2[$. Using standard lifts, one writes in coordinates the corresponding subsequence of $f_n$ as

$$\boldsymbol{f}_n = \begin{bmatrix} |\lambda_n|^2(-1 + i a_n) \\ \lambda_n \\ 1 \end{bmatrix},$$

where $a_n = \mathbb{A}(x, e, f_n) \to \tan A$ and $\lambda_n \to 0$ as $f_n \to e$. Now, the line between $e = 0$ and $f_n$ is generated by two points whose lifts are

$$\boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{f}_n = \begin{bmatrix} |\lambda_n|^2(-1 + i a_n) \\ \lambda_n \\ 0 \end{bmatrix} = \lambda_n \begin{bmatrix} \overline{\lambda}_n(-1 + i a_n) \\ 1 \\ 0 \end{bmatrix}.$$

The complex lines determined by these vectors converge to the line $e^\perp$, which is a contradiction with the assumption. ■

### 3.2. Smooth horizontal curves, characteristic foliations and one-parameter subgroups

We exhibit in this section smooth horizontal curves that are invariant under one-parameter subgroups of $\mathrm{PU}(2,1)$. A common general construction of smooth horizontal curves is through the characteristic foliation induced in an embedded surface in a contact manifold. One defines that foliation (generically with singular points) by the field of directions given by the intersection of the contact plane with the tangent space of the surface [17, Chapter I].

For example, if the surface is the complex plane through 0 in the Siegel model union $\infty$, which is a sphere, the foliation has two singularities, at 0 and $\infty$, where the contact plane is tangent to the surface. Closure of leaves of this foliation is exactly the half-lines between 0 and $\infty$, so half of $\mathbb{R}$-circles. Note that one can glue two such half-lines at the two singularities, obtaining what we call *bent* $\mathbb{R}$*-circles* (see Section 3.5.1).

Given a one-parameter subgroup $L$ of $\mathrm{PU}(2,1)$, one can consider a 2-dimensional abelian group $G$ containing $L$. Any 2-dimensional orbit of $G$ is foliated by orbits of $L$. The characteristic foliation coincides with this one if and only if the orbits of $L$ are horizontal. Observe that inside an orbit of $G$, $G$ acts transitively on the leaves of the characteristic foliation because it preserves the contact plane. Therefore, if one of the leaves is an orbit of $L$ of $G$, the same holds for every other leaf.

The following question is natural: for which pair $L \subset G$ is there an orbit of $G$ whose characteristic foliation is given by horizontal orbits of $L$? We postpone the complete answer to this question to a forthcoming paper. We only treat here the loxodromic case, which will lead to examples of slim curves; see Section 3.5.3. In this case, we prove that all characteristic leaves are indeed orbits of a one-parameter subgroup. Recall that if $L$ is a one-parameter loxodromic subgroup fixing two points, then its centraliser is the 2-dimensional group $G \simeq \mathbb{C}^*$ of transformations fixing exactly these two points. The orbits of $G$ are each of the fixed points, the two arcs of $\mathbb{C}$-circles between the two points and a family of surfaces. In the Siegel model, if the two points are 0 and $\infty$, the surfaces are the paraboloids $t = k|z|^2$, $(z,t) \neq (0,0)$, for $k$ real (including the horizontal plane $\mathbb{C}^*$).

**Proposition 3.7.** *Let $G$ be a 2-dimensional subgroup $G$ of* $\mathrm{PU}(2,1)$ *fixing exactly two points in $\mathbf{H}^2_{\mathbb{C}}$. Then, in any 2-dimensional orbit of $G$, the characteristic foliation is given by orbits of a one-parameter loxodromic subgroup $L$.*

*Conversely, for a given one-parameter subgroup $L$ of $G$, there is exactly one 2-dimensional orbit of $G$ where the characteristic foliation is given by orbits of $L$.*

*Proof.* Up to conjugation, we work in the Siegel model, with the two fixed points being 0 and $\infty$. The group $G$ consists of those elements of $\mathrm{PU}(2,1)$ that lift to diagonal matrices $\mathrm{diag}(\lambda, \overline{\lambda}/\lambda, 1/\overline{\lambda})$. It acts simply transitively on any sheet of any paraboloid $t = k|z|^2$, $(z,t) \neq (0,0)$, for $k$ real. Loxodromic 1-parameter subgroups are given by the following matrices, for some $\alpha \in \mathbb{C} \setminus i\mathbb{R}$:

$$L_\alpha : s \longmapsto \begin{bmatrix} e^{s\alpha} & & \\ & e^{s(\overline{\alpha}-\alpha)} & \\ & & e^{-s\overline{\alpha}} \end{bmatrix}. \tag{18}$$

Remark that multiplying alpha by a non-zero real factor amounts to changing the parametrisation of the 1-parameter subgroup, whereas a purely imaginary $\alpha$ corresponds to elements fixing pointwise the whole $\mathbb{C}$-circle through 0 and $\infty$.

Now, if $p_0 = [z_0, t_0]$ is a point in $\partial_\infty \mathbf{H}^2_{\mathbb{C}}$, represented by the null vector

$$\boldsymbol{p} = \left[ -|z_0|^2 + i t_0, z_0, 1 \right]^T,$$

then the orbit of $p_0$ under $L_\alpha$ is the curve $[z_0 e^{s(2\bar{\alpha}-\alpha)}, t_0 e^{2s\,\mathrm{Re}(\alpha)}]$ in the Heisenberg group, which is a spiral inscribed on the pararaboloid with equation $t|z_0|^2 = |z|^2 t_0$.

Next, we remark that if $p$ is a point in $\partial \mathbf{H}^2_{\mathbb{C}}$ and $s \mapsto \gamma(s)$ is the orbit of $p$ under a 1-parameter subgroup $(G_s)_s$ of PU(2, 1), then $\gamma$ is horizontal if and only if

$$\left\langle \frac{d}{ds}\Big|_{s=0} \gamma(s), \boldsymbol{p} \right\rangle = 0. \tag{19}$$

Indeed, the contact plane at $p$ is the kernel of $\langle \cdot, \boldsymbol{p} \rangle$ and horizontality is preserved by the action of the 1-parameter subgroup.

To conclude, consider the point in $\partial \mathbf{H}^2_{\mathbb{C}}$ given in Heisenberg coordinates by $p = [z, t]$. Then, using (19), we see that the orbit of $p$ under $L_\alpha$ is horizontal if and only if

$$\frac{t}{|z|^2} = 3 \frac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}. \tag{20}$$

So the orbits of $L_\alpha$ in the paraboloid $t = k|z|^2$ are horizontal if and only if we have $k = 3\frac{\mathrm{Im}(\alpha)}{\mathrm{Re}(\alpha)}$. This proves the proposition. ■

Note that in the case $k = 0$, $\alpha$ is real and one recovers the half-line between 0 and $\infty$.

### 3.3. Slimness in the sphere

The notion of horizontality captures a local property of subsets of $\mathbb{S}^3$. We would like a quantitative and global version of this property. We define the relevant notion in this section using the Cartan invariant to guarantee quantitatively that no $\mathbb{C}$-circle intersects the subset three times. We first recall this global property which is independent of horizontality. This is a particular case of one of the central notions of the theory of Anosov representations [30, 35].

**Definition 3.8.** We say a subset $E \subset S^3$ is hyperconvex if no three points in $E$ are contained in the same complex line.

**3.3.1. A quantitative version of horizontality.** We define now the central notion of this paper.

**Definition 3.9.** Let $0 \leqslant \alpha < \frac{\pi}{2}$. We call *$\alpha$-slim* any subset $E$ of $\mathbb{S}^3$ such that the absolute value of the Cartan invariant of any triple of points is bounded above by $\alpha$:

$$\sup \left\{ \left| \mathbb{A}(a, b, c) \right|, (a, b, c) \in E^3 \right\} \leqslant \alpha. \tag{21}$$

As a shortcut, we say that a subset $E$ is *slim* if it is $\alpha$-slim for some $\alpha < \frac{\pi}{2}$. Moreover, we denote by $\mathbb{A}(E)$ the supremum $\mathbb{A}(E) = \sup(|\mathbb{A}(a, b, c)|, (a, b, c) \in E^3)$.

This condition will prove to be a strong constraint on subsets of $\mathbb{S}^3$. A first point to be proven is that this assumption indeed implies horizontality – see Proposition 3.11. Some preliminary remarks, though, follow from the definition.

**Remark 3.10.**

(1) If a subset $E$ is 0-slim, then it is contained in an $\mathbb{R}$-circle.

(2) If a subset $E$ is slim, then its intersection with any $\mathbb{C}$-circle has cardinality at most 2: if $E$ had 3 points on a $\mathbb{C}$-circle, then this triple of points would have Cartan invariant $\pi/2$ and thus we would have $\mathbb{A}(E) = \pi/2$.

(3) One could likewise define the notion of $\alpha$-thickness by asking that every Cartan invariant of three distinct points has absolute value at least $\alpha$. A $\frac{\pi}{2}$-thick set is then included inside a $\mathbb{C}$-circle.

(4) If $t \to E_t$ is a Hausdorff-continuous family of closed subsets in $\mathbb{S}^3$, then $t \to \mathbb{A}(E_t)$ is upper semi-continuous. It is not continuous in general, cf. Section 3.5.5.

(5) The Cartan (measurable bounded) cocycle $\mathbb{A}$ determines a bounded cohomology class on PU(2, 1) which coincides with the continuous bounded Kähler class $\kappa$. Let $\rho : \Gamma \to \mathrm{PU}(2,1)$ be a representation and $\rho^*(\kappa) \in H^2(\Gamma, \mathbb{R})$ the corresponding bounded cohomology class of $\Gamma$. One should observe that if $E = \Lambda_\rho$ is the limit set of $\rho(\Gamma)$, we obtain that the Gromov norm of this class coincides with the supremum of the Cartan cocycle restricted to the limit set (see [9, Proposition 3.1], see also [10]). That is,
$$\left\|\rho^*(\kappa)\right\| = \mathbb{A}(E).$$

We now use Lemma 3.6 to prove that $\alpha$-slimness implies horizontality. However, in a different setting, the following proposition is almost the same as [35, Theorem B].

**Proposition 3.11.** *If $E$ is a slim subset of $\mathbb{S}^3$, then it is horizontal.*

*Proof.* Let $E$ be a slim set. Suppose that it is not horizontal. This means there exist a point $e$ and two sequences $e_n \neq f_n$ converging to $e$ such that the sequence of lines $(e_n f_n)$ in $\mathbb{CP}^2$ converges to a line $l$ with $l \neq e^\perp$.

Note that $E \cap l$ contains at most two points, as noted in the previous Remark 3.10 (2). So we fix an arbitrary $x \in E \setminus l$. By Lemma 3.6, we have
$$\left|\mathbb{A}(e_n, f_n, x)\right| \to \frac{\pi}{2}.$$
The assumption that $(e_n f_n) \to l \neq p^\perp$ implies therefore that the sup of Cartan invariant is $\frac{\pi}{2}$ and this contradicts the slimness assumption. ∎

For a submanifold, slimness implies that it is a Legendrian smooth curve.

**Corollary 3.12.** *A connected slim submanifold of $\mathbb{S}^3$ is a smooth Legendrian curve.*

This applies for absolutely continuous paths as well: their tangent vectors are horizontal wherever they are defined.

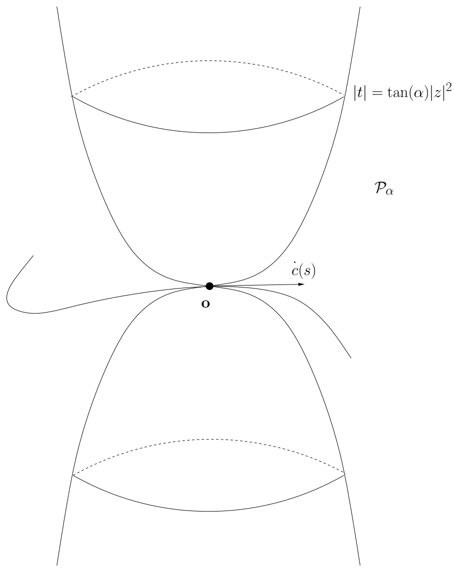


**Figure 6.** Local aspect of a $\alpha$-slim curve close to the point $[0,0]$.

**3.3.2. Local picture in Heisenberg space.** A first geometric way of seeing the horizontality of a slim subset is the following. Assume $E$ is $\alpha$-slim and the points $\boldsymbol{o} = [0,0]$ and $\infty$ belong to $E$. Then, any point in $E$ satisfies $|\mathbb{A}(\infty, \boldsymbol{o}, p)| \leqslant \alpha$. In Heisenberg coordinates, the point $p$ is $[z,t]$. Lifting the three points of $\mathbb{C}^3$, we have

$$\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{o} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} -|z|^2 + it \\ z \\ 1 \end{bmatrix} \quad \text{where } z \in \mathbb{C} \text{ and } t \in \mathbb{R}. \tag{22}$$

A straightforward computation leads directly to

$$\left|\mathbb{A}(\infty, \boldsymbol{o}, \boldsymbol{p})\right| \leq \alpha \iff |t| \leqslant \tan(\alpha)|z|^2. \tag{23}$$

This means that $E \setminus \{\boldsymbol{o}, \infty\}$ is contained in the complement of the union of the two open solid paraboloids defined by $|t| > \tan(\alpha)|z|^2$. We denote by $\mathcal{P}_\alpha$ this region.

The paraboloids are illustrated in Figure 6. It makes visible that, if $E$ is a slim submanifold of $\mathbb{S}^3$, then it is CR-horizontal (see Proposition 3.11). Moreover, Condition (23) may be used to strengthen Lemma 3.5 under the hypothesis of slimness rather than horizontality: in this case and with the notation of the lemma, $y$ is easily seen to be in fact $O(|x|^2)$ instead of $o(x)$. We will not use this fact here, so we do not go into details.

### 3.4. Projections of slim subsets

Before going on with the properties of slim subsets, one can give a few geometric interpretations of the slimness condition.

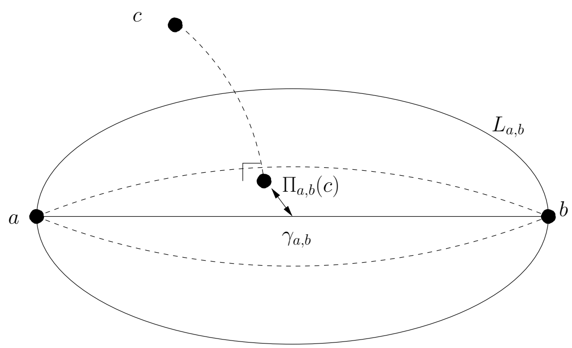


**Figure 7.** Orthogonal projection onto the complex line $L_{a,b}$. The region bounded by dashed lines in $L_{a,b}$ is a $k$-tubular neighbourhood of $\gamma_{a,b}$.

In this section, we fix an $\alpha$-slim subset $E$ of $\mathbb{S}^3 = \partial\mathbf{H}^2_{\mathbb{C}}$ and two points $a \neq b$ in $E$. Recall that the complex line through $a$ and $b$ is denoted by $\mathcal{L}(a,b)$, and that $\mathcal{L}(a,a)$ is the complex line tangent to $\mathbb{S}^3$ at the point $a$. We are now going to interpret the $\alpha$-slimness condition in terms of different projections on $\mathcal{L}(a,b)$ and $\mathcal{L}(a,a)$.

**3.4.1. Projections to $\mathcal{L}(a,b)$, $a \neq b$ in $E$.** (See Figure 8.) One can define two natural projections of $\mathbb{S}^3 \setminus \{a,b\}$ to $\mathcal{L}(a,b)$: the first one $\Pi_{a,b}$ is the orthogonal projection in hyperbolic geometry and lands inside the complex hyperbolic line $L_{a,b} := \mathcal{L}(a,b) \cap \mathbf{H}^2_{\mathbb{C}}$; the second one $\Pi^*_{a,b}$ using projective geometry and landing in $\mathcal{L}(a,b) \cap \mathbf{H}^{1,1}_{\mathbb{C}}$ is defined by

$$\begin{aligned} \Pi^*_{a,b} : \quad \mathbb{S}^3 \setminus \{a,b\} &\to \mathcal{L}(a,b) \\ e &\mapsto \mathcal{L}(e) \cap \mathcal{L}(a,b). \end{aligned} \tag{24}$$

This second projection is more adapted to projective geometry and inspired by considerations for *generalised Hilbert distance* in [18]. We will see shortly that our two projections $\Pi_{a,b}$ and $\Pi^*_{a,b}$ are closely related, but they serve different purposes.

The first projection explains the terminology: the maximum of the Cartan invariant is a measure of the width of the orthogonal projection of $E$ on hyperbolic discs defined by pairs of points in the set. Indeed, for any $c$ in $\mathbb{S}^3 \setminus \{a,b\}$, the Cartan invariant of $(a,b,c)$ satisfies the following relation (see [23, Theorem 7.1.2]):

$$\sinh^{-1}\left(\left|\tan\left(\mathbb{A}(a,b,c)\right)\right|\right) = d\left(\Pi_{a,b}(c), \gamma_{a,b}\right). \tag{25}$$

Denote by $k_\alpha$ the number $\sinh^{-1}(\tan(\alpha))$. Observe that $k_\alpha = \alpha + o(\alpha^2)$. We obtain directly the following proposition, illustrated in Figure 7.

**Proposition 3.13.** *Let $E$ be an $\alpha$-slim subset of $\partial\mathbf{H}^2_{\mathbb{C}}$. Then,*

$$\forall a,b \in E, \quad \Pi_{a,b}(E) \subset N^{k_\alpha}(\gamma_{a,b}), \tag{26}$$

*where $N^{k_\alpha}(\gamma_{a,b})$ is the $k_\alpha$-tubular neighbourhood in $L_{a,b}$ of the geodesic $\gamma_{a,b}$.*

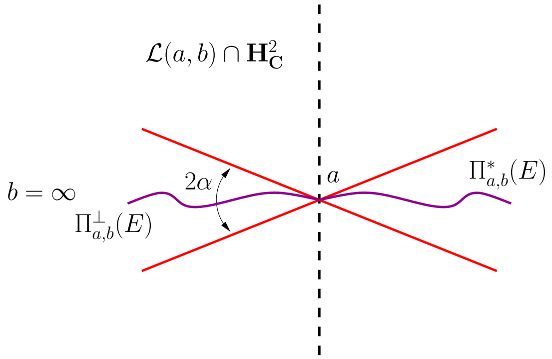


**Figure 8.** Projections of an $\alpha$-slim set $E$ on the complex line through two of its points.

In coordinates, we can always assume that $a = [0, 0]$ and $b = \infty$ in the Heisenberg group, so that the complex line through $a$ and $b$ corresponds to those vectors $[z, 0, 1]^T$, where $\mathrm{Re}(z) < 0$. The orthogonal projection of a point $c = [z_1, z_2, 1]^T$ in $\mathbb{S}^3$ is just $[z_1, 0, 1]^T$, and the Cartan invariant is $\mathbb{A}(a, b, c) = \arg(-z_1)$. Thus, the projection of $c$ belongs to the cone $\arg(z) \leqslant \alpha$.

One can use the same coordinates to write down the second projection. If $e = [z_1, z_2, 1]^T$ is a point in $\mathbb{S}^3$, the intersection of $\mathcal{L}(e)$ and $\mathcal{L}(a, b)$ is the point $[-\overline{z_1}, 0, 1]^T$. We obtain thus a link between $\Pi$ and $\Pi^*$ (see Figure 8).

**Lemma 3.14.** *We have* $-\Pi_{a,b} = \overline{\Pi^*_{a,b}}$.

The previous discussion translates into the following.

**Proposition 3.15.** *Let $E \subset \mathbb{S}^3$ be $\alpha$-slim for some $0 \leqslant \alpha < \frac{\pi}{2}$ and containing at least two distinct points $a$ and $b$. Then, in the line $(ab)$ equipped with the previous coordinates, both coordinates are contained in positive cones of angle $2\alpha$:*

- $\Pi^*_{a,b}(E)$ *is contained in* $\{z \in \mathbb{C}, |\arg(z)| \leq \alpha\}$.
- $\Pi_{a,b}(E)$ *is contained in* $\{z \in \mathbb{C}, |\arg(-z)| \leq \alpha\}$.

**3.4.2. Projection to a tangent line at $e \in E$.** This last geometric interpretation uses the line map and in fact only relies on hyperconvexity. Indeed, let $E$ be a hyperconvex subset of $\mathbb{S}^3$ and $e \in E$. We can project $E$ to the line $\mathcal{L}(e, e) = e^{\perp}$ via the map $\Pi_e$ defined on $\mathbb{S}^3$ by

$$\Pi_e(p) = p \boxtimes e \quad \text{if } p \neq e \text{ and } \Pi_e(e) = e.$$

Geometrically, for $p \neq e$, $\Pi_e(p)$ is the intersection point of the two lines $e^{\perp}$ and $p^{\perp}$. We have the following.

**Proposition 3.16.** *The map $\Pi_e : \mathbb{S}^3 \to \mathcal{L}(e, e)$ is surjective. Moreover, if $E \subset \mathbb{S}^3$ is hyperconvex, then $\Pi_e$ restricted to $E$ is injective.*

*Proof.* The preimage of any $x \in e^{\perp}$ by $\Pi_e$ is the intersection between the polar line to $x$ and $\mathbb{S}^3$. This preimage is non-empty for any point $x \in e^{\perp}$.

More precisely, the preimage of $e \in e^{\perp}$ in $\mathbb{S}^3$ is the singleton $\{e\}$. On the other hand, the preimage of $x = \Pi_e(q) \neq e$ is exactly the $\mathbb{C}$-circle through $e$ and $q$. When $E$ is hyperconvex, this $\mathbb{C}$-circle can intersect $E$ at most twice. And we already know two intersection points: $e$ and $q$. This proves that $\Pi_e : E \to e^{\perp}$ is injective. ■

The projections $\Pi_e(E)$ play the role of space-like geodesics on $\mathcal{L}(E, E)$. In the case where $E = \partial_\infty \mathbf{H}^2_{\mathbb{R}}$, they are exactly these geodesics.

**Remark 3.17.** Assume that $e = \infty$. Then, if a point $m \in E$ has Heisenberg coordinates $[z, t]$, it is easy to see that $\Pi_\infty(m)$ lifts to the vector $[2\bar{z}, 1, 0]^T$. In particular, the slimness of $E$ cannot be deduced from its projection on $\mathcal{L}(e, e)$ for $e \in E$, as the coordinate $t$ disappears. In the case where $e = \infty$, the map $\Pi_e$ corresponds up to a factor 2 and complex conjugation to the vertical projection onto $\mathbb{C}$, and the fact that it is one-to-one says that two points of $E$ cannot be vertically aligned if $E$ contains $\infty$.

## 3.5. Examples and non-examples

We describe here examples of slim or non-slim subsets. We also introduce limit sets of surface groups, on which we will further focus in the next sections.

**3.5.1. $\mathbb{R}$-circles and bent $\mathbb{R}$-circles.** As recalled in Proposition 2.5, three points are on a common $\mathbb{R}$-circle if and only if their Cartan invariant is 0. As such, any $\mathbb{R}$-circle is 0-slim. Those are maximal slim subsets.

**Proposition 3.18.** *Let $E$ be a slim subset of $\mathbb{S}^3$ containing an $\mathbb{R}$-circle $R$. Then, $E$ equals $R$.*

*Proof.* The set of arcs of $\mathbb{C}$-circles connecting two points of $R$ defines a foliation of the complement of $R$ in $\mathbb{S}^3$, as stated in Corollary 2.16.

Therefore, if $E$ contains a point outside $R$, this point belongs to (exactly) one of these arcs. This gives three points in $E$ that lie on a $\mathbb{C}$-circle, thus having Cartan invariant equal to $\pm\pi/2$. So if $E$ strictly contains $R$, $E$ is not slim. ■

A simple example of a slim set which is not an $\mathbb{R}$-circle is given by the union of two half $\mathbb{R}$-circles through 2 points; see the beginning of Section 3.2. In coordinates, we may write the following.

**Proposition 3.19.** *For all $0 < \theta < 2\pi$, the union*

$$E_\theta = \left\{[x, 0], x \in \mathbb{R}_+\right\} \cup \left\{[ye^{i\theta}, 0], y \in \mathbb{R}_+\right\}$$

*is $\alpha$-slim for $\alpha = |\frac{\pi-\theta}{2}|$.*

*Proof.* Let $R_1$ and $R_2$ be the two sets appearing in the union $E_\theta$. We want to compute the maximum of $\mathbb{A}(p, q, r)$ for $(p, q, r)$ in $E_\theta^3$. First, if they all belong to $R_1$ or all to $R_2$, as they are halves of $\mathbb{R}$-circles, this Cartan invariant is 0. So we may assume, up to permutations, that $p, q$ are in $R_1$ and $r$ in $R_2$. Denote by 0 the point $[0, 0]$.

Using the last point of Proposition 2.5 and the fact that $p, q$ and 0 belong to $R_1$, we have the equality:

$$\mathbb{A}(p,q,r) = \mathbb{A}(p,q,0) - \mathbb{A}(p,r,0) + \mathbb{A}(q,r,0) = \mathbb{A}(q,r,0) - \mathbb{A}(p,r,0).$$

We first estimate $\mathbb{A}(q,r,0)$. Write $q = [x,0]$ and $r = [ye^{i\theta},0]$ where $x$ and $y$ are positive. We can write $x + iy = \rho e^{it}$ with $0 < t < \pi/2$. Then, a direct computation in Heisenberg coordinates gives

$$\mathbb{A}(q,r,0) = \arg\big(-\langle \boldsymbol{q}, \boldsymbol{r}\rangle\big) = \arg\big(1 - \sin(2t)e^{-i\theta}\big).$$

Note that $0 < \sin(2t) \leq 1$. It is easily seen that $0 < \mathbb{A}(q,r,0) \leq \frac{\pi-\theta}{2}$ with the maximum attained at $t = \pi/4$ or equivalently $x = y$.

This implies that the difference

$$\mathbb{A}(p,q,r) = \mathbb{A}(q,r,0) - \mathbb{A}(p,r,0)$$

(where $p$ and $q$ are in $R_1$ and $r$ in $R_2$) is bounded between $\frac{\theta-\pi}{2}$ and $\frac{\pi-\theta}{2}$, proving the proposition. ∎

**3.5.2. Slim circles are unknotted.** We will from now on be especially interested in slim subsets homeomorphic to circles. We give a straightforward name to these sets.

**Definition 3.20.** A subset $E \subset \mathbb{S}^3$ is a slim circle if it is both homeomorphic to the circle and a slim subset of $\mathbb{S}^3$.

We remark here that slim circles are unknotted. This rules out a non-trivial knot in the sphere being slim. We prove it by constructing a diagram of the knot without self-intersection.

**Proposition 3.21.** *A slim circle is unknotted.*

*Proof.* Let $E$ be a knot in $\mathbb{S}^3$ and $e$ a point in $E$. Without loss of generality, choose coordinates such that $e = \infty$. The projection $\Pi_e$ is then the one described in Remark 3.17, which is given by

$$[z,t] \longmapsto 2\bar{z}.$$

In these coordinates, the projection $\Pi_e$ appears as an affine (vertical) projection followed by a symmetry and a dilation by a factor of 2. In particular, $\Pi_e(E)$ is a (dilated) diagram of a symmetric of the image of the knot $E$ by $[z,t] \mapsto [\bar{z},t]$, which we denote by $\bar{E}$. By slimness and Proposition 3.16, $\Pi_e$ is injective on $E$, so $\Pi_e(E)$ has no crossings. In particular, $\bar{E}$ is trivial, and so is $E$. ∎

The following corollary is proven in the same way.

**Corollary 3.22.** *Suppose $E$ is a slim subset of $\mathbb{S}^3$. Let $F \subset E$ be a subset homeomorphic to a disjoint union of circles. Then, $F$ is an unknotted link.*

For example, if $E$ is an immersion of a circle with several double points, $E$ contains disjoint circles. They cannot be knotted nor linked.

**3.5.3. Slim orbits of 1-parameter subgroups.** Section 3.2 describes some very specific families of horizontal orbits of 1-parameter subgroups. Proposition 3.11 implies that among all 1-parameter orbits, those are the only ones that can be slim. We will prove that horizontal orbits of 1-parameter loxodromic subgroups are indeed slim. This gives concrete examples that are not $\mathbb{R}$-circles. On the contrary, horizontal orbits of 1-parameter parabolic subgroups are not slim unless the group is horizontal unipotent. In this last case, the orbit is an $\mathbb{R}$-circle.

Let us first look at the parabolic case. We indeed prove that invariance by a single parabolic transformation is compatible with slimness only if this transformation is horizontal unipotent, that is, a Heisenberg translation which is not in the centre of the Heisenberg group.

**Proposition 3.23.** *Let $E$ be a closed slim subset of $\mathbb{S}^3$ with at least two distinct points which is invariant under the action of a parabolic element $u$ of* $\mathrm{PU}(2,1)$. *Then, $u$ is horizontal unipotent.*

*Proof.* We prove it by a case disjunction. Note that in any case, $E$ contains the fixed point $p$ of $u$ and another point $q \in E$.

- If $u$ is vertical unipotent, then the orbit $u^n(q)$ completely lies inside the $\mathbb{C}$-circle through $p$ and $q$. So $\mathbb{A}(p,q,u(q)) = \pm\frac{\pi}{2}$, which prevents the slimness of $E$.
- If $u$ is ellipo-parabolic, then the orbit $u^n(q)$ is contained in a cylinder foliated by $\mathbb{C}$-circles. Let $L$ be the compact set of lines in $\mathbb{CP}^{2*}$ supported by these $\mathbb{C}$-circles. Note that $L$ does not contain $p^{\perp}$. We can extract a subsequence $q_j = u^{n_j}(q)$ such that the lines $(q_j q_{j+1})$ converge to one of the lines in $L$. Then, the sequence $(q_j)$ of points in $E$ converges to $p$ and the line $(q_j q_{j+1})$ does not converge to $p^{\perp}$. Lemma 3.6 proves that the supremum of Cartan invariants is $\frac{\pi}{2}$.

The only remaining case is that $u$ is horizontal parabolic. It is of course possible, as an $\mathbb{R}$-circle is invariant under some horizontal parabolic elements. ∎

A direct corollary reads the following.

**Corollary 3.24.** *A horizontal orbit of a* 1*-parameter parabolic subgroup is slim if and only if the subgroup is horizontal unipotent. In this case, it is an $\mathbb{R}$-circle.*

The proof of slimness in the loxodromic case, however, is more involved. Moreover, we are not able to estimate the parameter of slimness, leaving us with an indirect proof. We will just give a detailed sketch of the proof and spare some technicalities. Recall from Equation (18) that the one-parameter loxodromic subgroups can be parametrised by

$$L_\alpha : s \longmapsto \begin{bmatrix} e^{s\alpha} & & \\ & e^{s(\overline{\alpha}-\alpha)} & \\ & & e^{-s\overline{\alpha}} \end{bmatrix}, \quad \text{where } \alpha \in \mathbb{C}^* \text{ satisfies } \mathrm{Re}(\alpha) \neq 0.$$

Their horizontal orbits are described in Section 3.2. We now prove the following.

**Proposition 3.25.** *If an orbit of $L_\alpha$ is horizontal, then it is slim.*

*Proof.* Recall from Section 3.2 that the orbit $p_s = L_s \cdot p$ is horizontal if and only if the Heisenberg coordinates $[z, t]$ of $p$ satisfy condition (20); that is,

$$t = 3|z|^2 \frac{\operatorname{Im}(\alpha)}{\operatorname{Re}(\alpha)}.$$

Denote by $\mathscr{P}_\alpha$ the paraboloid defined by the above condition. It is a simple computation in Heisenberg coordinates to verify that a $\mathbb{C}$-circle is either contained in $P_\alpha$ or intersects $P_\alpha$ in at most two points. Moreover, $\mathbb{C}$-circles contained in $P_\alpha$ are all contained in horizontal planes (on which $t$ is constant). Since the orbits of $L_\alpha$ are never contained in such planes, this implies that the orbit we consider never intersects a $\mathbb{C}$-circle thrice; i.e., it is hyperconvex. Therefore, no triple of points in the orbit has Cartan invariant equal to $\pi/2$. This means in particular that if the orbit were not slim, then the supremum of the Cartan invariant (which would be equal to $\pi/2$) would not be attained.

Up to a reparametrisation of the 1-parameter subgroup, and conjugating $\alpha$ if necessary, we may assume that $\alpha = 1 + ia$ for some $a > 0$. Applying an element of the normaliser of $L_\alpha$, we may moreover assume that the $z$-coordinate of $p$ is equal to 1. The horizontality condition becomes then $p = [1, 3a]$. Denote by $p_- = [0, 0] = \lim_{s \to -\infty} p_s$. Taking lifts, we have

$$\boldsymbol{p}_- = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} -1 + 3ia \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}_s = \begin{bmatrix} e^s(-1 + 3ia) \\ e^{-3isa} \\ e^{-s} \end{bmatrix}. \tag{27}$$

We obtain then directly

$$\mathbb{A}(p_-, p, p_s) = \arg\left(e^s(1 + 3ia) + e^{-s}(1 - 3ia) - 2e^{3isa}\right).$$

Denote the latter quantity by

$$\mathscr{P}(s) = \arg\left(e^s(1 + 3ia) + e^{-s}(1 - 3ia) - e^{-isa}\right).$$

When $s \to 0$, $\mathscr{P}(s)$ goes to 0. When $s \to \pm\infty$, then $\mathscr{P}(s)$ goes to $\pm \arctan(3a)$. As the real part of the complex number inside the argument does not vanish, $|\mathscr{P}| < \frac{\pi}{2}$. So $|\mathscr{P}|$ admits a maximum $m_a < \frac{\pi}{2}$. Note moreover that $\mathscr{P}(-s) = -\mathscr{P}(s)$.

Recall from (12) the cocycle property for the Cartan invariant, with $p, q, r, s$ being four points in $\mathbb{S}^3$:

$$\mathbb{A}(p, q, r) + \mathbb{A}(p, q, s) + \mathbb{A}(p, r, s) + \mathbb{A}(q, r, s) = 0.$$

Let $s < x < t$ be three real numbers. We want to estimate the Cartan invariant $\mathbb{A}(p_s, p_x, p_t)$. By invariance of the Cartan invariant under the action of $L_\alpha(-s)$, one may assume that $x = 0$, so that $p_0 = p$. Using the cocycle equality to introduce $p_-$, we deduce that

$$\mathbb{A}(p_s, p, p_t) = -\mathscr{P}(-s) - \mathscr{P}(t - s) - \mathscr{P}(t).$$

To prove that the supremum of the Cartan invariant is strictly less than $\frac{\pi}{2}$, we have to control the boundary behaviour. So we are left to understand what happens when $s$ goes to 0 or $-\infty$ and/or $t$ to 0 or $+\infty$. In each case, one proves that at the limit, the absolute value of the Cartan invariant remains bounded above by $m_a$. ∎

**3.5.4. Deformations of $\mathbb{R}$-Fuchsian limit sets.** Limit sets $\Lambda_\rho$ of convex cocompact representations $\rho$ of surface groups $\Gamma$ give rise to topological circles in the sphere. A natural question is whether $\Lambda_\rho$ is slim. A first observation is the following.

**Lemma 3.26.** *Let $\Gamma$ be a subgroup of* $\mathrm{PU}(2, 1)$ *whose limit set $\Lambda$ is $\alpha$-slim for some* $0 < \alpha < \pi/2$*, but not* 0*-slim. Then, $\Gamma$ is Zariski-dense and discrete.*

*Proof.* A subgroup $\Gamma$ of $\mathrm{PU}(2, 1)$ is Zariski dense if and only if it is not contained in a proper closed Lie subgroup of $\mathrm{PU}(2, 1)$. This means that $\Gamma$ is Zariski dense if and only if it does not preserve a complex line or a real plane, and it does not have a global fixed point in the closure of $\mathbf{H}^2_{\mathbb{C}}$. Now, consider the following.

- If $\Gamma$ preserves a real plane $P$, then $\Lambda$ is contained in an $\mathbb{R}$-circle (the boundary of $P$), so it is 0-slim.
- If $\Gamma$ preserves a complex line $L$, then $\Lambda$ is contained in the boundary of $L$ and cannot be $\alpha$-slim for any $\alpha < \pi/2$.
- If $\Gamma$ fixes a point, then $\Lambda$ is empty.

Therefore, the condition that $\Lambda$ is $\alpha$-slim for some $0 < \alpha < \pi/2$ but not 0-slim implies that $\Gamma$ is Zariski dense.

On the other hand, a Zariski dense subgroup of $\mathrm{PU}(n, 1)$ is either dense or discrete (see for instance [13, Corollary 4.5.1] or [41, Theorem 6.1]). In case $\Gamma$ is dense, its limit set is the whole sphere and is thus not $\alpha$-slim for any $\alpha < \pi/2$. Therefore, $\Gamma$ is discrete. ∎

Note that a group $\Gamma$ with 0-slim limit set preserves a real plane since its limit set is contained in an $\mathbb{R}$-circle. It may or may not be discrete (consider for instance an $\mathbb{R}$-Fuschsian group, or the embedding in $\mathrm{PU}(2, 1)$ of a dense subgroup of $\mathrm{PO}(2, 1)$).

We next show that in the case of a convex-cocompact representation of a surface group, the sup of the Cartan invariant is actually a maximum.

**Proposition 3.27.** *Let $\Lambda_\rho$ be the limit set of a convex cocompact representation $\rho$ of the fundamental group $\Gamma$ of a compact hyperbolic surface. Then, the supremum of the Cartan invariant $\mathbb{A}(\Lambda_\rho)$ on the limit set $\Lambda_\rho$ is attained at a triple of distinct points.*

*Proof.* For any convex-cocompact representation $\rho$ as in the statement, there exists a unique $\rho$-equivariant homeomorphism

$$B_\rho : \partial_\infty\Gamma \xrightarrow{\sim} \Lambda_\rho$$

called the boundary map [6]. For any $\rho$, denote by $A_\rho$ the map defined on the set of triples of distinct points $\partial_\infty\Gamma^{(3)}$ by

$$A_\rho(p, q, r) = \mathbb{A}\big(B_\rho(p), B_\rho(q), B_\rho(r)\big).$$

As $B_\rho$ is a bijection between $\partial_\infty\Gamma$ and $\Lambda_\rho$, we deduce that

$$\mathbb{A}(\Lambda_\rho) = \sup_{(p,q,r)\in\partial_\infty\Gamma^{(3)}} A_\rho(p,q,r).$$

As moreover $B_\rho$ is $\Gamma$-equivariant, we have for all $\gamma \in \Gamma$ and for all $(p,q,r) \in \partial_\infty\Gamma^{(3)}$, the equality $A_\rho(p,q,r) = A_\rho(\gamma\cdot p, \gamma\cdot q, \gamma\cdot r)$. The action of $\Gamma$ on the set of triples of distinct points $\partial_\infty\Gamma^{(3)}$ is cocompact: the quotient $X$ may be identified with the unit tangent bundle to the surface (see for instance [8]).

To sum up, the map $A_\rho$ descends to a continuous map defined on the compact set $\Gamma\backslash\partial_\infty\Gamma^{(3)}$ and the supremum is attained. ■

As a corollary, if the limit set of a convex cocompact representation is hyperconvex, then it is slim: the attained supremum cannot be $\frac{\pi}{2}$ by hyperconvexity. The following proposition gives a rich family of examples of slim circles that are not $\mathbb{R}$-circles. It is proven by Pozzetti–Sambarino–Wienhard [35, Proposition 6.2]. They work with the notion of $(1,1,2)$-hyperconvex representations. But note, by the previous proposition, that the limit set $\Lambda$ of a convex-cocompact representation $\rho$ of $\Gamma$ is slim if and only if the representation is $(1,1,2)$-hyperconvex Anosov: if $x, y, z$ are in the limit set, then the projective complex line generated by $x, y$ does not contain the point $z$.

**Proposition 3.28** (Pozzetti–Sambarino–Wienhard). *Let $\Gamma$ be the fundamental group of a compact hyperbolic surface and let $\rho_0 : \Gamma \to \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ be a representation of $\Gamma$. Then, for any sufficiently small deformation $\rho$ of $\rho_0$, the limit set of $\rho(\Gamma)$ is a slim circle.*

We nevertheless give a proof of this proposition as it is important for our work. In fact, we prove the following proposition, which implies the previous one.

**Proposition 3.29.** *If $\Gamma$ is the fundamental group of a compact hyperbolic surface, then the sup of the Cartan invariant $\mathbb{A}(\Lambda_\rho)$ on the limit set $\Lambda_\rho$ of a convex cocompact representation $\rho$ varies continuously with $\rho$ in the set of convex-cocompact representations of $\Gamma$.*

*Proof.* [6, Lemma 5.5.4 and Remark 5.5.7] proves that the boundary map $B_\rho : \partial_\infty\Gamma \xrightarrow{\sim} \Lambda_\rho$ varies continuously for the compact-open topology with the convex-cocompact representation $\rho$. Therefore, the map $\rho \mapsto A_\rho$ is continuous, where $A_\rho$ is defined in the previous proof.

The max on the compact set $\Gamma\backslash\partial_\infty\Gamma^{(3)}$ of the continuous function $A_\rho$ depends continuously on $\rho$. Therefore, the dependance on $\rho$ of

$$\mathbb{A}(\Lambda_\rho) = \max_{\Gamma\backslash\partial_\infty\Gamma^{(3)}} A_\rho$$

is continuous. ■

This gives examples of slim circles having low regularity. We will come back to these examples in Section 5.

**3.5.5. Deformation of the Farey triangulation and (non)-slimness.** We explore here another family $(\Lambda_\alpha)_{-\pi/2\leq\alpha\leq\pi/2}$ of limit sets of surface groups in a geometrically finite setting. We prove that the supremum $\mathbb{A}(\Lambda_\alpha)$ is always $\pi/2$ except for $\alpha = 0$ where it vanishes. This proves that, in general, the semicontinuity of $E \mapsto \mathbb{A}(E)$ is the best we can hope for. It also makes clear that in the previous section, the cocompactness assumption was crucial.

Let us describe the explicit construction that appeared in the works on PU(2, 1)-representations of the modular group PSL(2, $\mathbb{Z}$) by Falbel–Koseleff [19], Gusevskii–Parker [26] and Falbel–Parker [20] (see also [41, Section 8] for a survey).

Let $\Gamma$ be the group $(\mathbb{Z}_2)^{*3} = \langle \iota_1, \iota_2, \iota_3 | \iota_k^2 = 1\rangle$. We fix $(T_\alpha)_{\alpha\in[-\pi/2,\pi/2]}$, a continuous family of ideal triangles such that $\mathbb{A}(T_\alpha) = \alpha$. We denote by $p_1, p_2, p_3 \in \mathbb{S}^3$ the ideal vertices of $T_\alpha$ and by $q_k$ the projection of $p_k$ onto the geodesic $(p_{k+1}p_{k-1})$ (indices are taken mod. 3). Consider the half-turns $R_k$ about $q_k$. The $R_k$'s are conjugate to the transformation given by $(z_1, z_2) \mapsto (-z_1, -z_2)$ in ball model coordinates. Then, one defines a representation $\rho_\alpha : \Gamma \to \mathrm{PU}(2, 1)$ by setting

$$\rho_A(\iota_k) = R_k, \quad k = 1, 2, 3.$$

This gives rise to a continuous 1-parameter family of representations of $\Gamma$ in PU(2, 1). The groups $\rho_0(\Gamma)$ and $\rho_{\pm\pi/2}(\Gamma)$ are respectively $\mathbb{R}$ and $\mathbb{C}$-Fuchsian: they are discrete, isomorphic to $\Gamma$ and preserve totally geodesic disc, which are real if $\alpha = 0$ and complex if $\alpha = \pm\pi/2$. The orbits of the geodesics connecting the $p_k$'s generate the Farey tessellation in the $\mathbb{R}$ or $\mathbb{C}$-Fuchsian case. The striking result of the aforementioned works is as follows.

**Theorem** (Falbel–Koseleff, Gusevskii–Parker). *For any value of $\alpha$, the representation $\rho_\alpha$ is discrete and faithful. Moreover, the type of elements remains the same all along the deformation.*

The limit set $\Lambda_\alpha$ of $\rho_\alpha(\Gamma)$ is a $\mathbb{C}$-circle when $\alpha = \pm\pi/2$, an $\mathbb{R}$-circle if $\alpha = 0$, and it remains a topological circle when $0 < |\alpha| < \pi/2$. It is not slim unless $\alpha = 0$.

**Proposition 3.30.** *We have $\mathbb{A}(\Lambda_\alpha) = \frac{\pi}{2}$ unless $\alpha = 0$, in which case $\mathbb{A}(\Lambda_\alpha) = 0$.*

*Proof.* The group $\rho_\alpha(\Gamma)$ contains a primitive class of parabolic elements, unique up to conjugation in $\Gamma$, which is the one of $R_1R_2R_3$. It follows from the above works that this parabolic element is screw-parabolic for any value $\alpha \neq 0$, and 2-step unipotent if $\alpha = 0$. By Proposition 3.23, this proves that $\Lambda_\alpha$ is not slim unless $\alpha = 0$, in which case it is an $\mathbb{R}$-circle, so 0-slim. ∎

## 4. Deforming the foliation by arcs of $\mathbb{C}$-circles

We now come back to the foliation described in Corollary 2.16. Recall that it expresses that the complement of an $\mathbb{R}$-circle $R$ is foliated by arcs of $\mathbb{C}$-circles with endpoints on $R$. We study in this section how this picture deforms when $R$ is deformed among slim curves.

One important tool to understand this is to realise any slim circle as the boundary of a Möbius band in $\mathbf{H}^{1,1}_{\mathbb{C}}$.

### 4.1. The foliation and the unit tangent bundle over $\mathbf{H}^2_{\mathbb{R}}$

Before actually deforming slim circles, we explain another way to look at the foliation described in Section 2.6, which will be more adapted to the study of deformations. Along the way, we will come back to the natural isomorphism between the foliation and the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ over $\mathbf{H}^2_{\mathbb{R}}$. This last point will be useful for studying limit sets of surface groups.

**4.1.1. Reinterpretation of the foliation property.** Consider a subset $E$ in the sphere $\mathbb{S}^3$. We first define a notation for its complement and the subset of the sphere swept by arcs of $\mathbb{C}$-circles with endpoints in $E$.

**Definition 4.1.** For any subset $E$ of the sphere, we define the sets

$$\Omega_E = \mathbb{S}^3 \setminus E \quad \text{and} \quad M_E = \left\{(x, y, p) \in (\mathbb{S}^3)^3 \text{ such that } x \neq y \in E,\ p \in x \curvearrowright y\right\}.$$

Moreover, let $\mathcal{F}_E : M_E \to \mathbb{S}^3$ be the forgetful map $(x, y, p) \mapsto p$.

When the context makes things clear, we may drop the notation of the dependence in $E$, considering the sets $\Omega$, $M$ and the map $\mathcal{F}$. Corollary 2.16 may be rephrased as the following equivalent statement.

**Corollary 4.2.** *If $R$ is an $\mathbb{R}$-circle, the map $\mathcal{F}_R$ realises a homeomorphism $M_R \xrightarrow{\sim} \Omega_R$.*

The previous corollary splits in fact into three substatements, which we will study for slim circles:

- The map $\mathcal{F}_R$ takes values inside $\Omega_R$,
- it is actually surjective on $\Omega_R$,
- it is injective with continuous inverse.

The first point generalises readily in the context of slim subsets.

**Lemma 4.3.** *Let $E$ be a slim subset of $\mathbb{S}^3$. Then, the map*

$$\mathcal{F}_E : M_E \to \mathbb{S}^3$$

*takes values inside $\Omega_E$.*

*Proof.* Let $(x, y, p)$ be an element of $M_E$. Then, $p$ is a point of the $\mathbb{C}$-circle through $x$ and $y$, distinct from $x$ and $y$. This $\mathbb{C}$-circle meets $E$ at $x$ and $y$. As $E$ is slim, it cannot meet $E$ also in $p$. So $p$ belongs to the complement $\Omega_E$ of $E$. ∎

The goal of this whole Section 4 is to understand what happens with the last two points when $E$ is a slim circle more general than an $\mathbb{R}$-circle. We will prove the following theorem.

**Theorem 4.4.** *Let $E$ be a slim circle and consider the map $\mathcal{F}_E : M_E \to \Omega_E$. We have:*

- *[**Surjectivity**] If there is a Hausdorff-continuous family of slim circles $(E_t)$, for $0 \leq t \leq 1$, with $E = E_1$ and $E_0$ a $\mathbb{R}$-circle, then $\mathcal{F}_E$ is surjective.*
- *[**Non-Injectivity**] If $E$ is invariant by a non-real loxodromic transformation, then $\mathcal{F}_E$ is not injective.*

We will prove the two parts of this theorem independently and with arguments of very distinct flavor. The surjectivity property will be proven in Section 4.3; see Proposition 4.13. The non-injectivity statement will be proven in Section 4.5. This theorem raises in particular the following question: is $\mathcal{F}_E$ always surjective onto $\Omega_E$ for any slim circle $E$? Before going further, we continue to review the case of Corollary 2.16 and its link with the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$.

**4.1.2. Back to the unit tangent bundle.** The relevance of the set $M_E$ is made clearer when we see how closely it is related to the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ over $\mathbf{H}^2_{\mathbb{R}}$. Recall that the latter is homeomorphic to the set of triples $(x, y, z)$ of distinct points in $\partial\mathbf{H}^2_{\mathbb{R}}$ that are cyclically positively oriented: a triple $(x, y, z)$ is associated with the unit tangent vector to the geodesic from $x$ to $y$ in $\mathbf{H}^2_{\mathbb{R}}$ with base point the orthogonal projection of $z$ on this geodesic.

Let $E$ be a slim circle. Using a parametrisation $\varphi : \partial\mathbf{H}^2_{\mathbb{R}} \xrightarrow{\sim} E$, we now define a natural map $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}} \to M_E$. A straightforward geometric lemma will prove useful.

**Lemma 4.5.** *Let $x$, $y$ and $z$ be distinct points in $\mathbb{S}^3$. Then, there exists a unique point $p \in x \curvearrowright y$ such that the projections of $p$ and $z$ on the (real) geodesic $xy$ coincide.*

This lemma is used in the following construction.

**Definition 4.6.** Let $E$ be a slim circle in $\mathbb{S}^3$ and $\varphi : \partial\mathbf{H}^2_{\mathbb{R}} \to E$ a homeomorphism. Then, we define the map $\Phi_{E,\varphi} : \mathrm{UT}\mathbf{H}^2_{\mathbb{R}} \to M_E$ by sending a point $(x, y, z)$ to the point $(\varphi(x), \varphi(y), p)$ in $M_E$ where $p \in x \curvearrowright y$ is the unique point whose projection on the real geodesic $\varphi(x), \varphi(y)$ coincides with the one of $\varphi(z)$.

We will often denote this map simply by $\Phi$. This map is natural: if $\varphi$ is equivariant for a representation $\rho$ of a group $\Gamma \subset \mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$ to $\mathrm{PU}(2, 1)$, then so is $\Phi$.

In the case where $R$ is an $\mathbb{R}$-circle, $R$ is the boundary of a real hyperbolic plane. So one can parametrise it by a map $\varphi : \partial\mathbf{H}^2_{\mathbb{R}} \to R$ which is $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$-equivariant. Denote for simplicity $M = M_R$, $\Omega = \mathbb{S}^3 \setminus R$, $\mathcal{F} = \mathcal{F}_R$ the forgetful map and $\Phi = \Phi_{R,\varphi}$ the map we just defined. The following proposition then expresses the foliation described in Section 2.6 using this map $\Phi$.

**Proposition 4.7.** *Let $R$ be an $\mathbb{R}$-circle. Then, the map $\Phi : \mathrm{UT}\mathbf{H}^2_{\mathbb{R}} \to M_R$ is an $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$-equivariant homeomorphism.*

*Moreover, the composition $\mathcal{F}_R \circ \Phi$ induces an $\mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$-equivariant homeomorphism $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}} \simeq \Omega_R$. This homeomorphism sends orbits of the geodesic flow to arcs $a \curvearrowright b$, $a \neq b \in R$.*

Note that for an $\mathbb{R}$-Fuchsian representation $\rho$ of a surface group $\Gamma \subset \mathrm{Isom}(\mathbf{H}^2_{\mathbb{R}})$, the map $\mathcal{F} \circ \Phi$ descends to a CR-spherical uniformisation $U(\Gamma\backslash\mathbf{H}^2_{\mathbb{R}}) \simeq \rho(\Gamma)\backslash\Omega$.

With the foliation reinterpreted, we can move on and see how each of its aspects vary when deforming the $\mathbb{R}$-circle into a slim circle $E$. The first tool is the construction of a surface $\mathrm{RP}(E)$ in $\mathbb{CP}^2$, homeomorphic to $\mathbb{RP}^2$ and that extends $E$ outside the sphere.

### 4.2. Extensions of slim circles

In this section, we will use extensively the identification between a line $\mathcal{L}(x, y)$ and its polar point $x \boxtimes y$. Recall that for $p \in \mathbb{S}^3$, the polar point to $\mathcal{L}(p, p) = p^\perp$ is $p$ itself.

Consider the simplest example of a slim circle, i.e. an $\mathbb{R}$-circle $R$. By definition, it is the intersection of a copy of a real projective plane $\mathbb{RP}^2 \subset \mathbb{CP}^2$ with the sphere $\mathbb{S}^3$. The part outside of the ball $\mathbf{H}^2_{\mathbb{C}}$ is the projective plane minus an open disc: it is a Möbius band. Moreover, we have a natural parametrisation of this closed Möbius band by

$$(R \times R)/[(x, y) \sim (y, x)],$$

given by the map $(x, y) \to \mathcal{L}(x, y)$ from $R \times R$ to $\mathbb{CP}^2$.

The goal of this section is to extend this construction to any slim circle $E$ and define an extension $\mathrm{RP}(E)$ to $\mathbb{CP}^2$ similar to the case of $\mathbb{R}$-circles. Our construction of the extension outside the ball $\mathbf{H}^2_{\mathbb{C}}$ is canonical, whereas inside we make some arbitrary choices. We will mainly focus on what happens outside later on, so this will not be a problem. Recall that a subset of the sphere is hyperconvex if no three points are contained in a $\mathbb{C}$-circle; see Definition 3.8.

**Proposition 4.8.** *Let $E$ be a horizontal and hyperconvex circle. Then, the map $\mathcal{L}$ from $E \times E$ to $\mathbb{CP}^2$ defines an embedding of the Möbius band $E \times E/[(x, y) \sim (y, x)]$ into $\mathbf{H}^{1,1}_{\mathbb{C}}$ whose intersection with the sphere $\mathbb{S}^3$ is $E$.*

*Proof.* As $E$ is horizontal, $\mathcal{L}$ is continuous on $E \times E$. Moreover, for any $x \neq y$ in $E$, we have $\mathcal{L}(x, y) = x \boxtimes y$, which is a point outside the closed ball, whereas $\mathcal{L}(x, x) = x \in E$. So $\mathcal{L}$ descends into a continuous map of $E \times E/[(x, y) \sim (y, x)]$ into $\mathbb{CP}^2 \setminus \mathbf{H}^2_{\mathbb{C}}$ whose intersection with the sphere $\mathbb{S}^3$ is $E$.

As $E \times E/(x, y) \sim (y, x)$ is compact, the last point to prove is the injectivity of our map. The proof is similar to Corollary 2.16: we have to check that $x \boxtimes y = x' \boxtimes y'$ implies $\{x, y\} = \{x', y'\}$. Suppose $x \boxtimes y = x' \boxtimes y'$. If this point belongs to $\mathbb{S}^3$, then $x = y = x' = y'$ and $\{x, y\} = \{x', y'\}$. If not, then $x \neq y$ and $x' \neq y'$ and $x, y, x', y'$ all lie in $E$ and in the $\mathbb{C}$-circle $(x \boxtimes y)^\perp \cap \mathbb{S}^3$. As $E$ is hyperconvex, there are at most two intersection points, which are $\{x, y\} = \{x', y'\}$. ∎

We also want to extend $E$ inside the complex hyperbolic space. We do not have a natural construction as above, so we will arbitrarily choose a good enough extension. From now on, we choose an arbitrary origin $o$ in $\mathbf{H}^2_{\mathbb{C}}$. We denote by $D(E)$ the union of all (real) geodesics from $o$ to a point $x \in E$. As two distinct geodesics from $o$ cannot meet again in $\mathbf{H}^2_{\mathbb{C}}$, the set $D(E)$ is a disc inside the ball $\mathbf{H}^2_{\mathbb{C}}$, whose closure meets the sphere $\mathbb{S}^3$ exactly at $E$.

**Definition 4.9.** For a slim circle $E$, we denote by $\mathrm{RP}(E)$ the union of $D(E)$ and $\mathcal{L}(E, E)$.

The Möbius band $\mathcal{L}(E, E)$ is invariant by any $\mathrm{PU}(2,1)$-transformation leaving $E$ invariant, by construction. Moreover, we will see that it varies continuously as $E$ varies among slim circles. But the disc $D(E)$ does not enjoy the first property. This raises the question: Is it possible to construct a natural disc $D(E)$ bounded by $E$, i.e. such that it is invariant by any $\mathrm{PU}(2,1)$-transformation leaving $E$ invariant and it varies continuously with $E$?

From the previous discussion, we deduce that $\mathrm{RP}(E)$ is topologically a projective plane $\mathbb{RP}^2$.

**Proposition 4.10.** *For any slim circle $E$, the set* $\mathrm{RP}(E)$ *is homeomorphic to* $\mathbb{RP}^2$.

*Proof.* The disc $D(E)$ is a disc whose boundary is $E$. $\mathcal{L}(E, E)$ is a Möbius band whose boundary is also $E$. Their union is thus homeomorphic to the gluing of a disc and a Möbius band along their boundary: it is a real projective plane. ■

Now, we want to understand how these surfaces $\mathrm{RP}(E)$ deform when deforming $E$ inside the set of slim circles.

### 4.3. Deformations of slim subsets and surjectivity

We investigate in this section $\mathcal{C}^0$-deformations of horizontal sets, i.e. continuous deformations of a parametrisation of horizontal sets. It is easily seen that $\mathcal{C}^0$-deformations do not preserve horizontality. For example, fix a loxodromic one-parameter subgroup $L$, and take a continuous family $p_t$ of points in the sphere such that only $p_0$ belongs to the surface singled out by Proposition 3.7. Then, the family of sets $(\overline{L \cdot p_t})_t$ is $\mathcal{C}^0$-continuous but only $\overline{L \cdot p_0}$ is horizontal.

But the additional quantitative information given by slimness guarantees that deformations remain horizontal and the projective lines given by the line map $\mathcal{L}$ vary continuously. We will consider a deformation as a map $f_t : E \to \mathbb{S}^3$ where $f_0$ is the identity map and $t \in (-\varepsilon, \varepsilon)$. The continuity of the deformation is, by definition, the continuity of $f_t$ as a function on $E \times (-\varepsilon, \varepsilon)$.

**Proposition 4.11.** *Let $E \subset \mathbb{S}^3$ be a horizontal subset and $\varepsilon > 0$. For $t \in (-\varepsilon, \varepsilon)$, let $E_t = f_t(E)$ be a continuous deformation of $E$. We assume that there is $0 \leq \alpha < \frac{\pi}{2}$ such that all the sets $E_t$ are $\alpha$-slim.*

*Then, the sets $E_t$ are all horizontal and the map $(t, p, q) \mapsto \mathcal{L}(f_t(p), f_t(q))$ is continuous on $(-\varepsilon, \varepsilon) \times E^2$.*

*Proof.* The horizontality of $E_t$ is granted by the assumption that they are slim. What we really want to control is the second point.

We argue by contradiction. Suppose there exist converging sequences $p_n \to p$, $q_n \to q$ in $E$ and $t_n \to t$ in $(-\varepsilon, \varepsilon)$ such that the sequence of lines $l_n = \mathcal{L}(f_{t_n}(p_n), f_{t_n}(q_n))$ does not converge to $l = \mathcal{L}(f_t(p), f_t(q))$.

We first note that this cannot happen if $p \neq q$, by the assumption that $E_t$ is a continuous deformation: we have $f_{t_n}(p_n) \to f_t(p)$ and $f_{t_n}(q_n) \to f_t(q)$. $f_t$ is still a homeomorphism, so $f_t(p) \neq f_t(q)$ and the lines $l_n$ converge to $l$ by continuity of $\mathcal{L}$ outside the diagonal.

Assume now $p = q$. Fix a sequence $r_n$ of points in $E$ such that $f_{t_n}(r_n) \to r \neq p$. Suppose for the sake of contradiction that $l_n \to l_\infty \neq l = \mathcal{L}(p, p)$. Then, by Lemma 3.6, $\mathbb{A}(f_{t_n}(p_n), f_{t_n}(q_n), f_{t_n}(r_n))$ goes to $\pm\pi/2$. This is impossible, as all the $f_s(E)$ are $\alpha$-slim. This concludes the proof. ∎

A corollary is that the surfaces $\mathrm{RP}(E_t)$ vary continuously.

**Corollary 4.12.** *Under the hypothesis of the previous proposition, the map $(t, p) \mapsto f_t(p)$ can be extended to a homotopy $(t, p) \mapsto F_t(p)$ between $\mathrm{RP}(E)$ and $\mathrm{RP}(E_s)$.*

*Proof.* Fix $t \in (-\varepsilon, \varepsilon)$ and let us define the map $F_t : \mathrm{RP}(E) \to \mathrm{RP}(E_t)$. Let $p$ be a point in $\mathrm{RP}(E)$. We shall consider three cases:

(1) If $p$ is in $E$, then we set $F_t(p) = f_t(p)$.

(2) If $p$ is in $D(E)$, then it is on a geodesic $ox$ for a unique $x \in E$, at distance $d \geq 0$ from $o$. We set $F_t(p)$ to be the point at distance $d$ from $o$ in the geodesic $of_t(x)$.

(3) If $p$ is outside the ball, then the $\mathbb{C}$-circle polar to $p$ meets $E$ in two points $\{x, y\}$, so that $p = \mathcal{L}(x, y)$. We define $F_t(p)$ to be the point $\mathcal{L}(f_t(x), f_t(y))$.

From the previous section and proposition, we see that $t, p \to F_t(p)$ is continuous, that $F_0$ is the identity map and that for each $t$, $F_t$ realises a homeomorphism from $E$ to $E_t$. ∎

This corollary is the crucial point to prove that for slim deformations $E$ of an $\mathbb{R}$-circle, the map $\mathcal{F}_E$ is still surjective or, equivalently, they still are maximal slim subsets of $\mathbb{S}^3$. The following proposition rephrases the Surjectivity item of Theorem 4.4.

**Proposition 4.13.** *Fix $0 < \alpha < \frac{\pi}{2}$. For $t \in [0, 1]$, let $\varphi_t : \mathbb{S}^1 \to \mathbb{S}^3$ be a continuous deformation such that, for each $t$, the set $E_t = \varphi_t(\mathbb{S}^1)$ is $\alpha$-slim, and moreover $E_0$ is an $\mathbb{R}$-circle.*

*Then, for all $t$, the map $\mathcal{F}_{E_t} : M_{E_t} \to \Omega_{E_t}$ is surjective. Equivalently, the set $E_t$ is a maximal slim circle of $\mathbb{S}^3$: any slim set containing $E_t$ is $E_t$ itself.*

The proof uses that the surfaces $\mathrm{RP}(E_t)$ intersect any (complex) line in $\mathbb{CP}^2$, as shown in the following lemma.

**Lemma 4.14.** *Under the assumption of the theorem, for any line $l \subset \mathbb{CP}^2$, and any $0 \leq t \leq 1$, the intersection between $l$ and $\mathrm{RP}(E_t)$ is non-empty.*

*Proof.* Any complex line $l$ meets the usual $\mathbb{RP}^2$ at any point inside $l \cap \bar{l}$. The intersection is moreover transverse unless $l$ is a real line.

Now, we work in the homology group $H_2(\mathbb{CP}^2, \frac{\mathbb{Z}}{2\mathbb{Z}})$, with its intersection form denoted by $i$. The previous remark translates in this setting into the property

$$i\big([l], [\mathbb{RP}^2]\big) = 1 \in \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

Note that working with $\frac{\mathbb{Z}}{2\mathbb{Z}}$-coefficients avoids problems related to the non-orientability of $\mathbb{RP}^2$.

The previous Corollary 4.12 proves that, for any $t$, the surface $\mathrm{RP}(E_t)$ is a continuous deformation of $\mathrm{RP}(E_0)$. By assumption, $E_0$ is an $\mathbb{R}$-circle so that $\mathrm{RP}(E_0)$ is a copy of $\mathbb{RP}^2$. So we obtain $[\mathrm{RP}(E_t)] = [\mathbb{RP}^2]$. This in turn translates into the intersection property $i([l], [\mathrm{RP}(E_t)]) = 1$.

We conclude that any line $l$ intersects $\mathrm{RP}_t$. Moreover, if the intersection is transverse, it is an odd number of points. ∎

We can now conclude the proof of the theorem.

*Proof.* Fix $t \in [0, 1]$, and $p \in \mathbb{S}^3 \setminus E_t$. We want to prove that $p$ belongs to an arc $x \curvearrowright y$, with $x \neq y \in E_t$. This implies that $p$ belongs to the image of $\mathcal{F}_{E_t}$.

Consider the line $p^\perp$ in $\mathbb{CP}^2$. By the previous lemma, it intersects $\mathrm{RP}(E_t)$. This intersection can only happen in $\mathbf{H}^{1,1}_{\mathbb{C}}$, as $p$ is not in $E_t = \mathrm{RP}(E_t) \cap \mathbb{S}^3$. This means, by construction, that $p$ is a point $x \boxtimes y$ for some $x \neq y \in E_t$, which in turn implies that one of the arcs $x \curvearrowright y$ or $y \curvearrowright x$ contains $p$. ∎

**Remark 4.15.** The proof of Proposition 4.13 is valid as soon as $E_0$ satisfies that

$$i\big(\big[RP(E_0)\big], [l]\big) = 1.$$

Before moving on and proving the last point of Theorem 4.4, we exhibit in the next subsection a simple example where the foliation does indeed deform as a new foliation.

### 4.4. A one-parameter deformation of the foliation by arcs of $\mathbb{C}$-circles

Let us come back to the example we have studied in Section 3.5.1. For any angle $0 < \theta < 2\pi$, we consider the subset of $\mathbb{S}^3$ defined in Heisenberg coordinates by

$$E_\theta = \big\{[x, 0],\ x \in \mathbb{R}_+\big\} \cup \big\{[ye^{i\theta}, 0],\ y \in \mathbb{R}_+\big\}.$$

Note that when $\theta = \pi$, the curve $E_\theta$ is the boundary of $\mathbf{H}^2_{\mathbb{R}}$ in $\mathbb{S}^3$.

**Theorem 4.16.** *For any $\theta \in [\pi/2, 3\pi/2]$, the set of arcs of $\mathbb{C}$-circles with endpoints in $E_\theta$ defines a foliation of $\mathbb{S}^3 \setminus E_\theta$.*

*Proof.* To prove Theorem 4.16, we need to

(1) prove that any point $p \in \mathbb{S}^3$ outside $E_\theta$ belongs to some arc of $\mathbb{C}$-circle hitting $E_\theta$ twice;

(2) prove that any two arcs of $\mathbb{C}$-circle both hitting $E_\theta$ twice never meet unless they share at least one endpoint in $E_\theta$.

The first point follows directly from Proposition 4.13 and Proposition 3.19: $E_\theta$ is $|\pi - \theta|/2$-slim, and it is obtained from an $\mathbb{R}$-circle by a homotopy which is given by the bending.

To prove the second point, we use a computational criterion to determine when two $\mathbb{C}$-circles are disjoint. From Lemma 2.12, we know that, given four points $a, b, c, d$ in $\mathbb{S}^3$ such that $a \neq b$ and $c \neq d$, the $\mathbb{C}$-circles $C_{ab}$ and $C_{cd}$ spanned by $(a, b)$ and $(c, d)$ respectively are disjoint if and only if

$$\big\langle (\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{c} \boxtimes \boldsymbol{d}), (\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{c} \boxtimes \boldsymbol{d}) \big\rangle \neq 0. \tag{28}$$

So what we need to do is to take $a, b, c, d$ as above in $E_\theta$ and prove the left-hand side of (28) does not vanish when $|\theta - \pi| \leqslant \pi/2$ unless one of $a = c, a = d, b = c, b = d$ happens. We denote by $\Delta_1 = \{[x, 0],\ x \in \mathbb{R}_+\}$ and $\Delta_2 = \{[ye^{i\theta}, 0],\ y \in \mathbb{R}_+\}$ the two half-lines whose union is $E_\theta$.

Considering the possible relative positions of the two arcs of $\mathbb{C}$-circles, we are left with the following four cases.

(1) The four endpoints all belong to one of the $\Delta_i$'s,

(2) Three of the endpoints lie in $\Delta_1$ and one lies in $\Delta_2$,

(3) One of the two arcs has its endpoints in $\Delta_1$ and the other in $\Delta_2$,

(4) Both arcs have one endpoint in $\Delta_1$ and one in $\Delta_2$.

The first case follows directly from Corollary 2.16. We will not give details for each of the other three cases, but let us consider the fourth one, which is the most intricate. Assume that $a$ and $c$ are in $\Delta_1$ and that $b$ and $d$ are in $\Delta_2$. We may then choose lifts as in (8) so that there exist four non-negative real numbers $x$, $y$, $z$, $t$, such that ($x \neq 0$ or $z \neq 0$) and ($y \neq 0$ or $t \neq 0$), for which we have

$$\boldsymbol{a} = \begin{bmatrix} -x^2 \\ x \\ 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -y^2 \\ ye^{i\theta} \\ 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} -z^2 \\ z \\ 1 \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} -t^2 \\ te^{i\theta} \\ 1 \end{bmatrix}. \tag{29}$$

Plugging these values into the left-hand side of (28) and reorganising, the condition becomes

$$0 \neq (x - z)(t - y)(-\alpha \cos^2 \theta + \beta \cos \theta - \gamma) \tag{30}$$

where

$$\begin{aligned}
\alpha &= 16xyzt(x + z)(y + t), \\
\beta &= 4\big((xt + yz)^2 + (ty + xz)^2 + 2(tx + yz)(xy + tz)\big)(ty + xz), \\
\gamma &= 4\big((t^2 + 2ty + z^2)ty^2z + (t^2 + 2xz + z^2)tx^2z, \\
&\qquad + (x^2 + 2ty + y^2)t^2xy + (x^2 + y^2 + 2xz)xyz^2\big).
\end{aligned}$$

The conditions on $x$, $y$, $z$ and $t$ impose that $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\gamma > 0$. Since $\cos\theta \leqslant 0$ the right-hand side of (30) can only vanish if $x = z$ or $t = y$, which is the expected result in this case.

The remaining two cases are treated in the same way, only simpler since the analogue of (30) has degree 1 in $\cos\theta$. ∎

### 4.5. Obstruction to the foliation property and rigidity

We prove in this section the second part of Theorem 4.4, namely, the non-injectivity. It shows that the deformation of the previous subsection is very specific and we cannot hope to deform the foliation in general. First of all, we introduce a bit of vocabulary for loxodromic transformations. Let $\gamma \in \mathrm{PU}(2,1)$ be loxodromic. Let $\lambda = re^{i\theta}$ be its eigenvalue of greatest modulus, with $\theta \in (-\pi, \pi]$ and $r > 1$. $\lambda$ is well-defined only up to multiplication by $e^{\frac{2i\pi}{3}}$, so we normalise by choosing $-\pi/3 < \theta \leq \pi/3$. Up to conjugation, we choose the representative of $\gamma$ in $\mathrm{SU}(2,1)$ to be the diagonal matrix with diagonal entries $(\lambda, \overline{\lambda}/\lambda, 1/\overline{\lambda})$. The trace of this matrix is real if and only if $\lambda$ is real. Equivalently, the trace of $\gamma^3$ is well-defined for $\gamma \in \mathrm{PU}(2,1)$ and it is real if and only if the trace of the chosen lift is real.

**Definition 4.17.** A loxodromic element $\gamma$ is *non-real* if its trace is non-real.

Its *rotation factor* is the angle $3\theta \in (-\pi, \pi]$ where $\theta$ is the normalised argument of its eigenvalue of greatest modulus.

Note that $3\theta = \pi$ corresponds to a real loxodromic element even if its rotation factor is not 0. With this definition, we can state the following slightly more general version of the non-injectivity property. Note in particular that the slimness assumption is not fully needed and the hyperconvexity property is enough.

**Theorem 4.18.** *Let $E$ be a hyperconvex circle which is invariant under the action of a non-real loxodromic map $\gamma \in \mathrm{PU}(2,1)$ with fixed repelling and attracting fixed points $p^-, p^+ \in E$. Then, there exists an infinite family of $\mathbb{C}$-circles $C_n$ such that*

(1) *for all $n$, $C_n$ meets $E$ twice,*

(2) *for all $n$, $C_n$ meets the $\mathbb{C}$-circle through $p^-$ and $p^+$ outside $\{p^-, p^+\}$.*

Let us make a series of reductions before actually proving the theorem. First, note that $E \setminus \{p^-, p^+\}$ has two connected components. So, up to passing to the action of $\gamma^2$, we may assume that $\gamma$ preserves each of these components. Note that if $\gamma$ is non-real, $\gamma^2$ has a rotation factor different from 0. Moreover, up to conjugation, one can suppose that $\gamma$ is diagonal with fixed points 0 and $\infty$ in the Heisenberg model. The action of $\gamma$ then forces $E$ to spiral around 0. This can be stated, in Heisenberg coordinates.

**Lemma 4.19.** *Let $\gamma$ be a loxodromic map fixing 0 and $\infty$ with rotation factor $\beta = 3\theta \neq 0$. Let $c : [0, +\infty]$ be a path $c(s) = [z(s), t(s)]$ that is a homeomorphism onto its image. Assume that this image is $\gamma$-invariant and hyperconvex, and moreover $c(0) = 0$ and $c(+\infty) = \infty$.*

*Then, any continuous lift $s \mapsto \tilde{\arg}(z(s))$ of the argument of $z$ over $0 < s < +\infty$ is onto $\mathbb{R}$ and proper.*

*Proof.* Since $c$ is hyperconvex, the vertical projection $s \mapsto z(s) = \frac{1}{2}\overline{\Pi_\infty(c(s))}$ is injective; see Remark 3.17. As $c(s)$ is never 0 for $0 < s < +\infty$, the quantity $z(s)$ never vanishes, and the lift of argument is well-defined once the lift of the argument of $c(1)$ is chosen.

Applying the loxodromic element in coordinates, we compute that

$$\gamma \cdot c(s) = \left[re^{-3i\theta} z(s), r^2 t(s)\right] = c(s').$$

The lift $\widetilde{\arg}(z(s'))$ of the argument of $z(s')$ is of the form $\widetilde{\arg}(z(s)) - 3\theta + 2k\pi$ for some $k \in \mathbb{Z}$. Note that, by our assumption on the rotation factor, we have $-3\theta + 2k\pi \neq 0$. So the image of $\widetilde{\arg}(z(s))$ is invariant by translation by $-3\theta + 2k\pi \neq 0$. This proves it is onto $\mathbb{R}$ and proper. ■

*Proof of Theorem* 4.18. Following the discussions above, we can assume that $p^- = 0$, $p^+ = \infty$ with their usual lifts, and $\gamma$ preserves both connected components of $E \setminus \{p^-, p^+\}$.

Let $a$ be close to $p^-$ and $b$ to $p^+$. Using the local parametrisation given by Lemma 3.5, we can write for lifts:

$$\boldsymbol{a} = \boldsymbol{p}^- + x\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+ + o(x) \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{p}^+ + y\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+ + o(y).$$

These coordinates are directly linked to the Heisenberg coordinates:

$$\boldsymbol{a} = \left[z(a), t(a)\right] \quad \text{and} \quad \boldsymbol{b} = \left[z(b), t(b)\right],$$

where

$$z(a) = -\frac{x}{2} + o(x) \quad \text{and} \quad \frac{2}{\bar{z}(b)} = y + o(y).$$

In particular, Lemma 4.19 implies that the arguments mod. $2\pi$ of $x$ and $y$ oscillate infinitely as $a$ goes to $p^-$ and $b$ to $p^+$.

By Lemma 2.12, the fact that the $\mathbb{C}$-circle through $a, b$ intersects the one through $p^-, p^+$ is equivalent to the equality $(\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+) = 0$. We can compute directly, once noticing that $\boldsymbol{p}^- \boxtimes (\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+) = \frac{1}{2}\boldsymbol{p}^-$ and $(\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+) \boxtimes \boldsymbol{p}^+ = \frac{1}{2}\boldsymbol{p}^+$. Indeed, we have

$$\boldsymbol{a} \boxtimes \boldsymbol{b} = \boldsymbol{p}^- \boxtimes \boldsymbol{p}^+ + \frac{x\boldsymbol{p}^+ + y\boldsymbol{p}^-}{2} + o\left(\sqrt{|x|^2 + |y|^2}\right).$$

Computing the box-product with $\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+$ leads to

$$(\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+) = \frac{y\boldsymbol{p}^- - x\boldsymbol{p}^+}{2} + +o\left(\sqrt{|x|^2 + |y|^2}\right).$$

One can now compute the square of the Hermitian norm of this last vector, getting

$$\left\langle(\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+), (\boldsymbol{a} \boxtimes \boldsymbol{b}) \boxtimes (\boldsymbol{p}^- \boxtimes \boldsymbol{p}^+)\right\rangle = -\mathrm{Re}(x\bar{y}) + o(x^2 + y^2).$$

Since the arguments of $x$ and $y$ oscillate, this last value has an infinite number of change of signs as $a$ goes to $p^-$ and $b$ to $p^+$. Hence, it vanishes infinitely many times: this gives an infinite number of $\mathbb{C}$-circles hitting the one through $p^-, p^+$. ■

This concludes the proof of Theorem 4.4. It is the main ingredient in the following rigidity theorem. Recall that a representation $\rho$ is a convex-cocompact and slim deformation of an $\mathbb{R}$-Fuchsian representation $\rho_0$ if there exists a continuous path of convex-cocompact and slim representations connecting $\rho$ to $\rho_0$. The following result characterises those deformations of an $\mathbb{R}$-Fuchsian representation remain $\mathbb{R}$-Fuchsian among convex-cocompact slim deformations, in terms of the persistence of the foliation by arcs of $\mathbb{C}$-circles.

**Theorem 4.20.** *Let $\Sigma$ be a closed hyperbolic surface. Denote by $\Gamma$ the fundamental group of $\Sigma$. Consider $\rho_0 : \Gamma \to \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ an $\mathbb{R}$-Fuchsian representation. Let $\rho$ be a convex-cocompact and slim deformation of $\rho_0$, whose limit set is denoted by $\Lambda_\rho$. We have the following:*

(1) *The limit set $\Lambda_\rho$ is a maximal slim circle.*

(2) *The following two conditions are equivalent:*

  (a) *The arcs $x \curvearrowright y$, for $x \neq y \in \Lambda_\rho$, are pairwise disjoint.*

  (b) *The representation $\rho$ is $\mathbb{R}$-Fuchsian.*

*Proof.* The first item follows from Proposition 4.13: the map $\mathcal{F}$ is surjective on $\Omega_\rho$, which means that any point in $\Omega_\rho := \mathbb{S}^3 \setminus \Lambda_\rho$ belongs to an arc with endpoints in $\Lambda_\rho$. In particular, any superset of $\Lambda_\rho$ has 3 points on a $\mathbb{C}$-circle, so it is not slim.

The second point is a corollary of Theorem 4.18. Indeed, from [2] the fact that all loxodromic elements in the $\rho(\Gamma)$ have real trace implies that $\rho(\Gamma)$ preserves a totally geodesic real plane and therefore $\rho$ is $\mathbb{R}$-Fuchsian. So, let $\rho$ be a non-$\mathbb{R}$-Fuchsian deformation of $\rho_0$. Then, some element $\rho(\gamma)$ is a non-real loxodromic transformation. The limit set $\Lambda_\rho$ is invariant under this element. Then, Theorem 4.18 implies that some arcs $x \curvearrowright y$ intersect. The other implication is direct. ∎

## 5. Crown-type spherical CR uniformisation of 3-manifolds

We now look at the geometric meaning of slimness and the deformed foliation in the equivariant case, i.e. assuming that the slim circle is the limit set of a convex-cocompact group. We will see that it gives CR-spherical uniformisations on unit tangent bundles and drilled unit tangent bundles.

So let $\Gamma$ be a lattice in $\mathrm{PO}(2,1)$ and denote by $\rho_0$ the $\mathbb{R}$-Fuchsian representation given by $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$. Let $\Lambda_0$ be the $\mathbb{R}$-circles which is the limit set of $\rho_0(\Gamma)$ and $\Omega_0$ its complement. We have seen in Proposition 2.17 that we have a natural identification $\rho_0(\Gamma)\backslash\Omega_0 \simeq \mathrm{UT}(\Gamma\backslash\mathbf{H}^2_{\mathbb{R}})$. This is a CR-uniformisation of the unit tangent bundle to the surface.

Now, let us deform $\rho_0$ into a convex cocompact representation $\rho$. Denote by $\Lambda_\rho$ the limit set of $\rho(\Gamma)$ and by $\Omega_\rho$ its complement. In fact, $\rho(\Gamma)\backslash\Omega_\rho$ is still homeomorphic to $\mathrm{UT}(\Gamma\backslash\mathbf{H}^2_{\mathbb{R}})$ (see Proposition 5.9). We want to use arcs of $\mathbb{C}$-circles to construct natural CR-uniformisation on the unit tangent bundle drilled along a geodesic. For that, we need to define supersets of the limit set, which we call *crowns*.

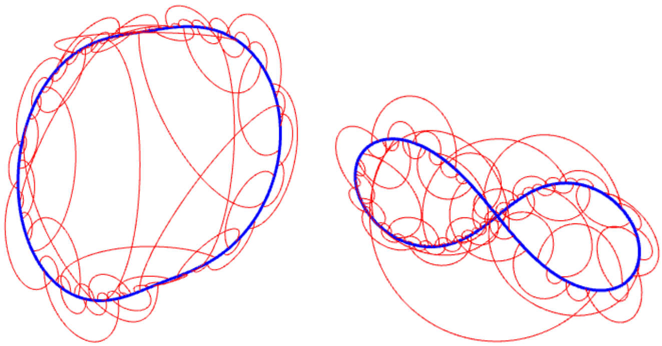

**Figure 9.** Two views of an approximation of the crown $\mathrm{Crown}_{\Gamma,\gamma}$, where $\Gamma$ is the $\mathbb{R}$-Fuchsian $(3,3,4)$-triangle group, and $\gamma$ is the word $\iota_3\iota_2\iota_1\iota_2$ (see Example 5.6). Here, the (thick) blue curve is the $\mathbb{R}$-circle which is the limit set of the group $\Gamma$. The red (thin) curves form the orbit of the axis at infinity of $\gamma$.

### 5.1. Crowns

Let $\gamma \in \mathrm{PO}(2,1)$ be a loxodromic element. The axis of $\gamma$ in $\mathbf{H}^2_{\mathbb{R}}$ is naturally oriented. We call again the axis of $\gamma$ and denote by $\mathrm{axis}(\gamma)$ the oriented lift of the $\mathbf{H}^2_{\mathbb{R}}$-axis of $\gamma$ to the unit tangent bundle $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$. The goal of this section is to explore the analogy between this notion of axis in $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ and arcs of $\mathbb{C}$-circles in $\mathbb{S}^3$. This analogy has already been noticed in the discussion following Corollary 2.16, and in Proposition 4.7.

For any loxodromic element $\delta \in \mathrm{PU}(2,1)$, we denote by $\alpha_\delta$ the arc of $\mathbb{C}$-circle $a^- \curvearrowright a^+$, where $a^+$ and $a^-$ are the attracting and repelling fixed points of $\delta$. Note that $\alpha_\delta$ is naturally oriented toward $a^+$. We call $\alpha_\delta$ the *axis at infinity* of $\delta$.

**Definition 5.1.** Let $\Delta$ be a convex-cocompact subgroup of $\mathrm{PU}(2,1)$ whose limit set $\Lambda_\Delta$ is a topological circle, and let $\delta \in \Delta$ be a loxodromic element. We call the *crown associated with* $\delta$ the subset of $\mathbb{S}^3$ defined as

$$\mathrm{Crown}_{\Delta,\delta} = \Lambda_\Delta \cup \Big( \bigcup_{g\in\Delta} g\cdot\alpha_\delta \Big). \tag{31}$$

We denote by $\Omega_{\Delta,\delta} \subset \Omega_\Delta$ the complement of $\mathrm{Crown}_{\Delta,\delta}$ in $\mathbb{S}^3$ (see Figure 9).

Note that by construction $\mathrm{Crown}_{\Delta,\delta}$ and $\Omega_{\Delta,\delta}$ are $\Delta$-invariant. These two objects only depend on the $\Delta$-conjugacy class of $\delta$. Moreover, the action of $\Delta$ on $\Omega_{\Delta,\delta}$ is properly discontinuous. In fact, the stabiliser in $\Delta$ of $\alpha_\delta$ is the cyclic group generated by $\delta$, so that the union may be rewritten:

$$\bigcup_{g\in\Delta} g\cdot\alpha_\delta = \bigcup_{[g]\in\Delta/<\delta>} [g]\cdot\alpha_\delta.$$

**Definition 5.2.** We say that $\mathrm{Crown}_{\Delta,\delta}$ is *embedded* whenever the arcs of $\mathbb{C}$-circles $g\cdot\alpha_\delta$ are pairwise disjoint.

For a cocompact $\mathbb{R}$-Fuchsian group $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$, the situation is clear.

**Proposition 5.3.** *Let $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ be a cocompact $\mathbb{R}$-Fuchsian group with limit set $\Lambda = \partial \mathbf{H}^2_{\mathbb{R}} \subset \mathbb{S}^3$. Denote by $\Sigma$ the surface $\Gamma \backslash \mathbf{H}^2_{\mathbb{R}}$. Then, for any loxodromic element $\gamma \in \Gamma$, we have the following:*

(1) $\mathrm{Crown}_{\Gamma,\gamma}$ *is embedded.*

(2) *The quotient $\Gamma \backslash \Omega_{\Gamma,\gamma}$ is homeomorphic to the 3-manifold obtained by drilling out from the unit tangent bundle* $\mathrm{UT}\Sigma$ *the orbit of the geodesic flow corresponding to $\alpha$.*

*Proof.* The first item follows directly from Corollary 2.16. The second item follows from Proposition 4.7. Using the notation therein, the map $\mathcal{F}_\Lambda \circ \Phi$ restricts as a $\Gamma$-equivariant homeomorphism from the complement in $\mathrm{UT}\mathbf{H}^2_{\mathbb{R}}$ of the union of all axes of elements conjugate to $\gamma$ in $\Gamma$ to $\Omega_{\Gamma,\gamma}$. The result is obtained by taking quotients. ∎

In a more general setting, we first see that the horizontality of the limit set implies that crowns are closed.

**Lemma 5.4.** *Let $\Delta$ be a convex-cocompact subgroup of* $\mathrm{PU}(2,1)$ *whose limit set $\Lambda_\Delta$ is a horizontal circle, and let $\delta \in \Delta$ be a loxodromic element. Then, the crown* $\mathrm{Crown}_{\Delta,\delta}$ *is closed and its complement $\Omega_{\Delta,\delta}$ open.*

The lemma is stated for convex-cocompact groups, but the proof below actually works for geometrically finite groups.

*Proof.* Using convex-cocompactness, we know that the projection of the axis in $\mathbf{H}^2_{\mathbb{C}}$ of $\delta$ is a closed geodesic in the convex-core of $\Delta \backslash \mathbf{H}^2_{\mathbb{C}}$. This means that the orbit of the pair $(\delta_-, \delta_+)$ of fixed points of $\delta$ in $\Lambda_\Delta$ is discrete in the set $\Lambda^{(2)}_\Delta$ of pairs of distinct points in $\Lambda_\Delta$.

So let $(x_n)$ be a sequence of points in the crown $\mathrm{Crown}_{\Delta,\delta}$ converging to a point $x$ in $\mathbb{S}^3$. We want to prove that $x$ belongs to the crown. As the limit set $\Lambda_\Delta$ is closed, one may assume up to extraction that each $x_n$ belongs to an arc $a_n \curvearrowright b_n$. Let, up to another extraction, $a_\infty, b_\infty$ be the limits in $\Lambda_\Delta$ of the sequences $a_n, b_n$.

If $a_\infty \neq b_\infty$, then by discreteness, the sequences $a_n$ and $b_n$ are stationary and for $n$ big enough $x_n \in a_\infty \curvearrowright b_\infty$. The closed arc $\overline{a_\infty \curvearrowright b_\infty}$ is included in the crown, which implies that $x$ belong to the crown as well.

If $a_\infty = b_\infty$, then by horizontality, the whole $\mathbb{C}$-circle through $a_n$ and $b_n$ converges to the point $a_\infty$. So $x = a_\infty$ belongs to the limit set and to the crown. ∎

Let $\Sigma$ be a closed hyperbolic surface and $\lambda$ a closed oriented geodesic of $\Sigma$. We say that $\lambda$ is *filling* whenever the complement of $\lambda$ in $\Sigma$ is a union of topological discs. We denote by $\mathrm{UT}\Sigma(\lambda)$ the unit tangent bundle drilled out along the natural lift of the oriented geodesic $\lambda$. A direct corollary of Proposition 5.3 reads as follows.

**Corollary 5.5.** *For any hyperbolic surface $\Sigma$ and any closed oriented geodesic $\lambda$, the* 3-*manifold* $\mathrm{UT}\Sigma(\lambda)$ *admits a CR-spherical uniformisation with an* $\mathbb{R}$*-Fuchsian holonomy.*

*Proof.* Let $\Gamma \subset \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ be the fundamental group of $\Sigma$ and $\gamma \in \Gamma$ a primitive element whose oriented axis is $\lambda$. Then, Proposition 5.3 states that $\mathrm{UT}\Sigma(\lambda)$ is homeomorphic to $\Gamma\backslash\Omega_{\Gamma,\gamma}$. ∎

Note that one can construct a lot of cusped hyperbolic 3-manifolds in that way: Theorem 1.12 in [21] states that $\mathrm{UT}\Sigma(\lambda)$ is hyperbolic as soon as $\lambda$ is filling (see also Calegari's blog [12]).

**Example 5.6.** Let $2 \leq p \leq q \leq r$ be three integers, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Consider the group

$$\Gamma = \left\langle \iota_1, \iota_2, \iota_3 | \iota_k^2 = (\iota_1\iota_2)^p = (\iota_2\iota_3)^q = (\iota_3\iota_1)^r = 1 \right\rangle.$$

It is the $(p,q,r)$-triangle group, which is hyperbolic. It can be seen uniquely – up to conjugacy – as a subgroup of $\mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$. Each of the $\iota_k$'s is a complex reflection of order two that fixes pointwise a complex line of $\mathbf{H}^2_{\mathbb{C}}$ which intersects $\mathbf{H}^2_{\mathbb{R}}$ along the geodesic $\gamma_k$. Let $\Gamma_2$ be the even subgroup of $\Gamma$, and let $\gamma$ be the geodesic in $\mathbf{H}^2_{\mathbb{R}}$ which is the axis of a (hyperbolic) element $w \in \Gamma_2$. By Proposition 5.3, the quotient of $\mathbb{S}^3 \setminus \mathrm{Crown}_{\Gamma_2,\gamma}$ is homeomorphic to the complement of the axis of $\gamma$ in the unit tangent bundle of the orbisurface $\Gamma_2\backslash\mathbf{H}^2_{\mathbb{R}}$.

In the special case where $(p,q,r) = (3,3,4)$ and $w$ is the word $\iota_3\iota_2\iota_1\iota_2$, then the resulting 3-manifold is the figure eight knot complement. This fact is proved in [14].

### 5.2. Deformations

We now prove that, after a small deformation of $\rho_0$, the crown deforms and gives rise to new CR-spherical uniformisations of the drilled unit tangent bundle, with non-$\mathbb{R}$-Fuchsian holonomies. By the analysis of the previous Section 4, the arcs of $\mathbb{C}$-circles in the crown could intersect. We prove that it is not the case, at least locally.

**Theorem 5.7.** *Let $\Sigma$ be a closed hyperbolic surface and $\lambda$ an oriented closed geodesic. Denote by $\Gamma$ its fundamental group and by $\gamma$ a primitive element whose axis lifts $\lambda$. Consider $\rho_0 : \Gamma \to \mathrm{PO}(2,1) \subset \mathrm{PU}(2,1)$ an $\mathbb{R}$-Fuchsian representation.*

*Then, there exists a neighbourhood $U$ of $\rho_0$ (of convex-cocompact and slim deformations $\rho$ of $\rho_0$) such that for any $\rho$ in $U$, we have the following:*

(1) $\mathrm{Crown}_{\rho(\Gamma),\rho(\gamma)}$ *is embedded and homotopic in* $\mathbb{S}^3$ *to* $\mathrm{Crown}_{\rho_0(\Gamma),\rho_0(\gamma)}$.

(2) *The quotient* $\rho(\Gamma)\backslash\Omega_{\rho(\Gamma),\rho(\gamma)}$ *is homeomorphic to* $\mathrm{UT}\Sigma(\lambda)$.

In order to prove this theorem, we need two different arguments: first that the crowns $\mathrm{Crown}_{\rho(\Gamma),\rho(\gamma)}$ remain embedded along a small deformation of $\rho_0$ and second that the whole quotient $\rho(\Gamma)\backslash\Omega_\rho$ is always homeomorphic to the unit tangent bundle $\mathrm{UT}\Sigma$.

For the first argument, we prove in the following lemma that we have indeed only a finite number of arcs to watch to ensure that no intersections happen.

**Lemma 5.8.** *Let $\rho : \Gamma \to \mathrm{PU}(2,1)$ be a convex-cocompact and slim representation of a surface group, with limit set $\Lambda_\rho$. Fix a compact subset $K$ of $\Omega_\rho$.*

*Then, for all $\gamma \in \Gamma$ with $\gamma$ loxodromic, the following set is finite:*

$$\left\{[\delta] \in \Gamma/\langle\gamma\rangle \text{ such that } \rho(\delta)\cdot\alpha_{\rho(\gamma)} \cap K \neq \emptyset\right\}.$$

For the proof of this lemma, we use the polarity that identifies any $\mathbb{C}$-circle with a point in $\mathbf{H}^{1,1}_{\mathbb{C}}$. Recall from Section 4.2 that the Möbius band $\mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ in $\mathrm{RP}(\Lambda_\rho)$ is exactly the set of points in $\mathbf{H}^{1,1}_{\mathbb{C}}$ polar to $\mathbb{C}$-circles that hit $\Lambda_\rho$ twice.

*Proof.* Let $H \subset \mathbf{H}^{1,1}_{\mathbb{C}}$ be the set of points polar to $\mathbb{C}$-circles meeting $K$:

$$H := \{p \in \mathbf{H}^{1,1}_{\mathbb{C}} \mid p^\perp \cap K \neq \emptyset\}.$$

*Step* 1. Let us prove that the intersection $H \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ is compact.

Note that $H$ is closed in $\mathbf{H}^{1,1}_{\mathbb{C}}$, by compacity of $K$. Moreover, the intersection of $\overline{H}$ with $\mathbb{S}^3$ is exactly $K$: indeed, it consists of points $p$ in $\mathbb{S}^3$ whose polar line $p^\perp$ meets $K$. But the only point in $\mathbb{S}^3 \cap p^\perp$ is $p$ itself, so $p \in K$. Therefore, the closure of $H \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ in $\mathbb{S}^3 \cup \mathbf{H}^{1,1}_{\mathbb{C}}$ is $(K \cup H) \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho)$. But $K \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho) = K \cap \Lambda_\rho = \emptyset$ because $\mathcal{L}(\Lambda_\rho, \Lambda_\rho) \cap \mathbb{S}^3 = \Lambda_\rho$. So $H \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ is closed in the compact set $\mathbb{S}^3 \cup \mathbf{H}^{1,1}_{\mathbb{C}}$, hence compact.

*Step* 2. The set of points polar to the axes in the orbit of $\alpha_{\rho(\gamma)}$ is discrete in $\mathcal{L}(\Lambda_\rho, \Lambda_\rho)$.

Indeed, the orbit of the geodesic axis$(\gamma)$ in $\mathbf{H}^2_{\mathbb{R}}$ is discrete in the space of geodesic of $\mathbf{H}^2_{\mathbb{R}}$. Equivalently, using polarity in the real case, the orbit $\mathcal{O}_0$ of points polar to these geodesic axes in $\mathbb{RP}^2$ is discrete in the Möbius band

$$\mathbf{H}^{1,1}_{\mathbb{R}} = \partial_\infty\Gamma^2/(x,y) \sim (y,x).$$

Denote by $p_\gamma$ the polar point to the axis at infinity $\alpha_{\rho(\gamma)}$ of $\rho(\gamma)$. By construction, we have $p_\gamma = \rho(\gamma)_- \boxtimes \rho(\gamma)_+$. For any $\delta \in \Gamma$, the polar to $\rho(\delta)\cdot\alpha_{\rho(\gamma)}$ is the point $\rho(\delta)\cdot p_\gamma = (\rho(\delta)\cdot\rho(\gamma)_-) \boxtimes (\rho(\delta)\cdot\rho(\gamma)_+)$. One can express this in other, more adapted, terms. Recall from Section 3.5.4 that we have a boundary map $B_\rho : \partial_\infty \mathbf{H}^2_{\mathbb{R}} \to \Lambda_\rho$. This boundary map induces an embedding $(x,y) \to \mathcal{L}(B_\rho(x), B_\rho(y))$ of $\mathbf{H}^{1,1}_{\mathbb{R}}$ into $\mathbf{H}^{1,1}_{\mathbb{C}}$ whose image is the Möbius band $\mathcal{L}(\Lambda_\rho, \Lambda_\rho)$; see Proposition 4.8.

The above expression of $\rho(\delta)\cdot p_\gamma$ means that the orbit $\mathcal{O}_\rho$ of $p_\gamma$ in $\mathrm{RP}(\Lambda_\rho)$ is exactly the image of $\mathcal{O}_0$ by this embedding. This implies that this orbit $\mathcal{O}_\rho$ is discrete in the Möbius band $\mathcal{L}(\Lambda_\rho, \Lambda_\rho)$.

*Step* 3. The set of the statement is in natural bijection through polarity with the intersection of $\mathcal{O}_\rho$ and $H$. We have proven that $\mathcal{O}_\rho$ is discrete in $\mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ and that $H \cap \mathcal{L}(\Lambda_\rho, \Lambda_\rho)$ is compact. So the intersection between the orbit $\mathcal{O}_\rho$ and $H$ is finite. ∎

The second argument is classical in the world of geometric structure and follows from the Ehresmann–Thurston principle. The next proposition is a consequence of a theorem of Guichard and Wienhard [25, Theorem 9.12]. The language in which [25] states and proves this theorem is quite different from ours, so we provide a short proof.

**Proposition 5.9.** *Let $\rho$ be a convex cocompact deformation of $\rho_0$. Then, $\rho(\Gamma)\backslash\Omega_\rho$ is diffeomorphic to* $\mathrm{UT}\Sigma$.

*Proof.* For any convex-cocompact representation $\rho$, let $X_\rho$ denote the quotient $\rho(\Gamma)\backslash\Omega_\rho$. Note that $\Lambda_\rho$ is a circle in the sphere, so the quotient $X_\rho$ is connected. We prove, following Guichard–Wienhard, that the diffeomorphism class of $X_\rho$ is constant under small deformation.

We first note that $X_\rho$ is compact as follows from convex-cocompactness: in the geometrically finite setting [7, Definition F1], the orbifold with boundary

$$\rho(\Gamma)\backslash(\mathbf{H}^2_{\mathbb{C}} \cup \Omega_\rho) = \big(\rho(\Gamma)\backslash\mathbf{H}^2_{\mathbb{C}}\big) \cup X_\rho$$

has a finite number of ends, each associated with a class of maximal parabolic subgroups of $\rho(\Gamma)$. Under the convex-cocompact assumption, $\rho(\Gamma)$ has no parabolic subgroup, so there are no ends: $X_\rho$ is a closed subset of a compact set.

Let us now prove that after a small deformation $\rho'$ of $\rho$, the quotient $X_{\rho'}$ is homeomorphic to $X_\rho$. Indeed, by construction, $\rho$ is the holonomy representation of a spherical CR uniformisation of $X_\rho$. Let $\widehat{X} \simeq \Omega_\rho$ be the $\Gamma$-covering of $X_\rho$ and fix a compact fundamental domain $D \subset \widehat{X}$. The developing map $s_\rho : \widehat{X} \to \Omega_\rho \subset \mathbb{S}^3$ sends $D$ to a compact set disjoint from the compact set $\Lambda_\rho$. By the Ehresmann–Thurston principle [22], any small deformation $\rho'$ is also a holonomy representation of a spherical CR structure on $X_\rho$. The developing map $s_{\rho'} : \widehat{X} \to \mathbb{S}^3$ is close to $s_\rho$. As $\Lambda_{\rho'}$ varies continuously (Proposition 3.29), the two compact sets $s_{\rho'}(D)$ and $\Lambda_{\rho'}$ remain disjoint for small enough deformations. Hence, the image of $s_{\rho'}$ avoids $\Lambda'_\rho$ and $s_{\rho'}$ is a local diffeomorphism from $\widehat{X}$ to $\Omega_{\rho'}$. By $\Gamma$-equivariance, it descends to a local diffeomorphism $\varphi_{\rho'}$ from $X_\rho$ to $X_{\rho'}$. Both $X_\rho$ and $X_{\rho'}$ are compact and connected, so $\varphi_{\rho'}$ is a covering. As $\varphi_{\rho'}$ deforms to $\varphi_\rho = \mathrm{Id}_{X_\rho}$ when $\rho'$ deforms to $\rho$, $\varphi_{\rho'}$ is actually a homeomorphism isotopic to the identity.

We finally prove that for any $\rho$ in the connected component of $\rho_0$, the space $X_\rho$ is homeomorphic to $X_{\rho_0} \simeq \mathrm{UT}\Sigma$. This is a connectedness argument: the homeomorphy class of $X_\rho$ is fixed under small deformations. So, the set of representations $\rho$ in this connected component such that

$$X_\rho \simeq X_{\rho_0}$$

is a non-empty open subset of this connected component. But its complement is also open. So the complement is empty. This proves the proposition. ∎

With these two preliminary results at hand, we may proceed with the proof of Theorem 5.7.

*Proof of Theorem 5.7.* In order to prove the first point, we have to prove that the arcs of $\mathbb{C}$-circles $\rho(\delta)\cdot\alpha_{\rho(\gamma)}$ are pairwise disjoint for $\rho$ close to $\rho_0$.

We can choose a compact $K \subset \mathbb{S}^3$ such that for all small enough deformations $\rho$ of $\rho_0$, $K$ avoids $\Lambda_\rho$ and contains a fundamental domain for $\rho(\Gamma)$ acting on $\Omega_\rho$. So, if some intersection happens between two arcs $\rho(\delta)\cdot\alpha_{\rho(\gamma)}$ and $\rho(\delta')\cdot\alpha_{\rho(\gamma)}$, one such intersection also happens inside $K$. So we just have to control the behaviour of arcs that meet $K$.

If the deformations are small enough, $\Lambda_\rho$ is always slim (Proposition 3.28). This implies that the set of arcs $\rho(\delta) \cdot \alpha_{\rho(\gamma)}$ intersecting $K$ is finite, by the previous Lemma 5.8. Moreover, this set is locally constant. So there is an open neighbourhood $U$ of $\rho_0$, for which the following set is finite:

$$\{\delta \in \Gamma,\ \exists \rho \in U,\ \rho(\delta) \cdot \alpha_{\rho(\gamma)} \text{ intersects } K\}.$$

So we have to control a finite set of arcs of $\mathbb{C}$-circles. At $\rho_0$, from Proposition 5.3, we know that these finite number of arcs do not meet. As they vary continuously with $\rho$, it remains true in a small neighbourhood.

The second point follows: in the quotient $\rho(\Gamma)\backslash\Omega_{\rho(\Gamma)} \simeq \mathrm{UT}\Sigma$, the projection of the set of arcs $\delta \cdot \alpha_\gamma$ is a curve, which varies continuously with $\rho$ from the first point. For the $\mathbb{R}$-Fuchsian representation $\rho_0$, the axis at infinity $\alpha_\gamma$ identifies with the geodesic $\mathrm{axis}(\gamma) = \lambda$ in $\mathrm{UT}\Sigma$, so its projection remains homotopic to $\lambda$ throughout the deformation. As a consequence, the quotient $\rho(\Gamma)\backslash\Omega_{\rho(\Gamma),\rho(\gamma)}$ is homeomorphic to $\mathrm{UT}\Sigma(\lambda)$. ∎

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**Elisha Falbel**
Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR 7586), Sorbonne Université, 4, place Jussieu, 75252 Paris, France; elisha.falbel@imj-prg.fr

**Antonin Guilloux**
Institut de Mathématiques de Jussieu-Paris Rive Gauche (UMR 7586), Sorbonne Université, 4, place Jussieu, 75252 Paris; Institut Fourier (UMR 5582), Université Grenoble Alpes, CNRS, 38058 Grenoble, France; antonin.guilloux@imj-prg.fr

**Pierre Will**
Université Grenoble Alpes, CNRS, IF, 38000 Grenoble, France;
pierre.will@univ-grenoble-alpes.fr