J. Spectr. Theory (Online first)
DOI 10.4171/JST/522

Published by EMS Press

Counterexamples and weak (1,1) estimates of wave operators for fourth-order Schrödinger operators in dimension three

Haruya Mizutani, Zijun Wan, and Xiaohua Yao

Abstract. This paper is dedicated to investigating the L^p-bounds of wave operators $W_{\pm}(H,\Delta^2)$ associated with fourth-order Schrödinger operators $H=\Delta^2+V$ on $\mathbb{R}^3$ with real potentials satisfying $|V(x)|\lesssim\langle x\rangle^{-\mu}$ for some $\mu>0$. A recent work by Goldberg and Green (2021) has demonstrated that wave operators $W_{\pm}(H,\Delta^2)$ are bounded on $L^p(\mathbb{R}^3)$ for all $1<p<\infty$ under the condition that $\mu>9$ and zero is a regular point of H. In the paper, we aim to further establish endpoint estimates for $W_{\pm}(H,\Delta^2)$ in two significant ways. First, we provide counterexamples to illustrate the unboundedness of $W_{\pm}(H,\Delta^2)$ on the endpoint spaces $L^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ for non-zero compactly supported potentials V. Second, we establish weak $(1,1)$ estimates for the wave operators $W_{\pm}(H,\Delta^2)$ and their dual operators $W_{\pm}(H,\Delta^2)^*$ in the case where zero is a regular point and $\mu>11$. These estimates depend critically on the singular integral theory of Calderón–Zygmund on a homogeneous space $(X,d\omega)$ with a doubling measure $d\omega$.

1. Introduction

1.1. The main results

Let $H=\Delta^2+V(x)$ be the fourth-order Schrödinger operator on $\mathbb{R}^3$, where $V(x)$ is a real-valued potential satisfying $|V(x)|\lesssim\langle x\rangle^{-\mu}$, $x\in\mathbb{R}^3$ with some $\mu>0$ specified later and $\langle x\rangle=\sqrt{1+|x|^2}$. As $\mu>1$, it was well known (see, e.g., [1,23,26,27]) that the *wave operators*

$$W_{\pm}=W_{\pm}(H,\Delta^2):=\operatorname*{s-lim}_{t\to\pm\infty}e^{itH}e^{-it\Delta^2} \tag{1.1}$$

exist as partial isometries on $L^2(\mathbb{R}^3)$ and are asymptotically complete.

Note that $W_{\pm}$ are clearly bounded on $L^2(\mathbb{R}^3)$. Hence, it would be interesting to establish the following L^p-bounds of $W_{\pm}$ for $p\neq 2$:

$$\|W_{\pm}\varphi\|_{L^p(\mathbb{R}^3)}\lesssim\|\varphi\|_{L^p(\mathbb{R}^3)}. \tag{1.2}$$

Mathematics Subject Classification 2020: 47A40 (primary); 47A55 (secondary).
Keywords: Counterexample, weak L^1-boundedness, wave operator, fourth-order Schrödinger operators.

To explain the importance of these bounds, recall that $W_\pm$ satisfy the identities

$$W_\pm W_\pm^* = P_{\rm ac}(H), \quad W_\pm^* W_\pm = I,$$

and *the intertwining property* $f(H)W_\pm = W_\pm f(\Delta^2)$, where f is a Borel measurable function on $\mathbb{R}$. These formulas especially imply

$$f(H)P_{\rm ac}(H) = W_\pm f(\Delta^2) W_\pm^*. \tag{1.3}$$

By virtue of (1.3), the L^p-boundedness of $W_\pm, W_\pm^*$ can immediately be used to reduce the L^p-L^q estimates for the perturbed operator $f(H)$ to the same estimates for the free operator $f(\Delta^2)$ as follows:

$$\|f(H)P_{\rm ac}(H)\|_{L^p\to L^q} \le \|W_\pm\|_{L^q\to L^q}\ \|f(\Delta^2)\|_{L^p\to L^q}\ \|W_\pm^*\|_{L^p\to L^p}. \tag{1.4}$$

For many cases, under suitable conditions on f, it is feasible to establish the L^p-L^q bounds of $f(\Delta^2)$ by Fourier multiplier methods. Thus, in order to obtain the inequality (1.4), it is a key problem to prove the L^p-bounds (1.2) of $W_\pm$ and $W_\pm^*$.

Recently, in the regular case (i.e., zero is neither an eigenvalue nor a resonance of H), Goldberg and Green [16] have demonstrated that the wave operators $W_\pm$ are bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$ if $|V(x)| \lesssim \langle x\rangle^{-\mu}$ for some $\mu > 9$ and there are no embedded positive eigenvalues in the spectrum of $H = \Delta^2 + V$. Therefore, it is natural to consider whether the boundedness of $W_\pm$ holds for the endpoint cases, namely, when $p = 1$ and $p = \infty$.

The following theorem provides a negative answer, showing that the wave operators $W_\pm$ are unbounded on $L^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$ assuming that V is compactly supported on $\mathbb{R}^3$. Furthermore, weak $(1,1)$ estimates for $W_\pm$ can be established in the regular case, provided that $\mu > 11$.

In order to state our results, we denote by $\mathbb{B}(X,Y)$ the space of bounded operators from X to Y, $\mathbb{B}(X) = \mathbb{B}(X,X)$, and by $L^{1,\infty}(\mathbb{R}^3)$ the weak $L^1(\mathbb{R}^3)$. Moreover, we say that *zero is a regular point of* $H = \Delta^2 + V$ *if there only exists zero solution to* $H\psi = 0$ *in the weighted space* $L^2_{-s}(\mathbb{R}^3)$ *for all* $s > \frac{3}{2}$, where $L^2_{-s}(\mathbb{R}^3) = \langle\cdot\rangle^s L^2(\mathbb{R}^3)$.

Theorem 1.1. *Let* $H = \Delta^2 + V(x)$. *Suppose that* V *is compactly supported and* $V \not\equiv 0$ *such that zero is a regular point of* H *and* H *has no embedded eigenvalue in* $(0,\infty)$. *Then* $W_\pm, W_\pm^* \notin \mathbb{B}(L^1(\mathbb{R}^3)) \cup \mathbb{B}(L^\infty(\mathbb{R}^3))$.

Theorem 1.2. *Let* V *satisfy* $|V(x)| \lesssim \langle x\rangle^{-\mu}$ *for some* $\mu > 11$. *Assume also* H *has no embedded eigenvalue in* $(0,\infty)$ *and zero is a regular point of* H. *Then* $W_\pm, W_\pm^* \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$, *that is,*

$$|\{x\in\mathbb{R}^3 : |W_\pm f(x)| \ge \lambda\}| \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^3} |f(x)|\,dx, \quad \lambda > 0,$$

with the analogous estimate for $W_\pm^*$.

Remark 1.3. By the interpolation and the duality, Theorem 1.2 also implies $W_\pm \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$, while this is already known due to Goldberg and Green [16].

Finally, we would like to emphasize that the condition of the absence of embedded positive eigenvalues is a fundamental assumption when studying dispersive estimates and L^p-bounds of wave operators for higher-order Schrödinger operators. In fact, for any dimension $d \geq 1$, it is relatively straightforward to construct a potential function $V \in C_0^\infty(\mathbb{R}^d)$ such that $H = \Delta^2 + V$ has some positive eigenvalues, as demonstrated, for instance, in [10, Section 7.1].

On the other hand, it is worth noting that Feng et al. in [10] have proven that $H = \Delta^2 + V$ does not have any positive eigenvalues under the assumption that the potential V is bounded and satisfies the repulsive condition, meaning that $(x \cdot \nabla)V \leq 0$. Additionally, it is well established, as demonstrated by Kato in [22], that the Schödinger operator $-\Delta + V$ has no positive eigenvalues when the potential is bounded and satisfies the condition $V(x) = o(|x|^{-1})$ as $|x| \to \infty$. Consequently, these studies indicate that establishing the absence of positive eigenvalues for fourth-order Schrödinger operators is a more intricate task compared to second-order cases when dealing with bounded potential perturbations.

1.2. Further backgrounds

For the classical Schrödinger operator $H = -\Delta + V(x)$, since the seminal work [30] of Yajima, there exists a great number of interesting works on the L^p-boundedness for the wave operators $W_\pm$. More specifically, in the space dimension $d = 1$, the wave operators $W_\pm$ are bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ for both regular and zero resonance cases but in general unbounded on $L^p(\mathbb{R})$ for $p = 1, \infty$ (see, e.g., [2,4,29]). In the regular case, for dimension $d = 2$ the wave operators $W_\pm$ are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ but the result of endpoint is unknown (see [20, 32]). For dimensions $d \geq 3$, the wave operators $W_\pm$ are bounded on $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ in the regular case (see, for example, [3, 30, 31]). However, the existence of threshold resonances shrink the range of p, which depends on dimension d and the decay properties of zero energy eigenfunctions (see [5, 12, 14, 15, 21, 33–37]).

More recently, there exist several works for the L^p-boundedness of the wave operators $W_\pm$ for higher order Schrödinger operators $H = (-\Delta)^m + V(x)$ especially for $m = 2$. First of all, Goldberg and Green in [16] proved that for dimension $d = 3$ and $m = 2$, the wave operators $W_\pm$ extend to bounded operators on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ when zero is a regular point (the endpoint case is not mentioned in [16]). Then Erdoğan and Green in [7, 8] further showed that as $m > 1$ and $d > 2m$, $W_\pm$ are bounded on $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ for certain smooth potentials $V(x)$ in the regular case. Moreover, Erdoğan, Goldberg, and Green in [6] also obtained that for dimension $d > 4m-1$ and $\frac{2d}{d-4m+1} < p \le \infty$, the L^p boundedness of the wave operators may fail for compactly supported continuous potentials if the potential is not sufficiently smooth. In our previous work [25], we studied the case $d = 1$ and $m = 2$ and obtained that whatever zero is a regular point or a resonance of H, the wave operators $W_\pm$ are bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover, if in addition V is compactly supported, then $W_\pm$ are also bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$. On the other hand, $W_\pm$ are shown to be unbounded on both $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ at least for the regular case. More recently, Galtbayar and Yajima [13] have established the L^p-estimates of wave operator $W_\pm$ with zero resonances for the case $m = 2$ and $d = 4$.

In a forthcoming paper [24], the authors consider all the zero resonance cases for $H = \Delta^2 + V$ on $\mathbb{R}^3$ and show that $W_\pm \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$ in the first kind resonance case. For the second and third kind resonance cases, it is shown that $W_\pm \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < 3$ but $W_\pm \notin \mathbb{B}(L^p(\mathbb{R}^3))$ for any $3 \le p \le \infty$.

1.3. The ideas of the proof

Let us explain briefly the idea of the proof. We begin with the stationary representation of W_-:

$$W_- = I - \frac{2}{\pi i}\int_0^\infty \lambda^3 R_V^+(\lambda^4)V(R_0^+(\lambda^4) - R_0^-(\lambda^4))\,d\lambda,$$

where $R_0^\pm(\lambda) = (\Delta^2 - \lambda \mp i0)^{-1}$ and $R_V^\pm(\lambda) = (H - \lambda \mp i0)^{-1}$ are the free and perturbed limiting resolvents, respectively. Since the high energy part is already known to be bounded on L^p for all $1 \le p \le \infty$ by [16], it is enough to deal with the low energy part

$$W_-^L := \int_0^\infty \lambda^3\chi(\lambda)R_V^+(\lambda^4)V(R_0^+(\lambda^4) - R_0^-(\lambda^4))\,d\lambda,$$

with $\operatorname{supp}\chi \subset [-\lambda_0, \lambda_0]$ and $\lambda_0 \ll 1$. To regard W_-^L as an (singular) integral operator, we then use the asymptotic expansion of $R_V^+(\lambda^4)V$ near $\lambda = 0$. Note that the integral kernel of $R_0^\pm(\lambda^4)$ is explicit (see (3.2)). In [16], Goldberg and Green used the expansion

$$R_V^+(\lambda^4)V = R_0^+(\lambda^4)v\{QA_0Q + \lambda A_1 + \Gamma_2(\lambda)\}v, \quad v = |V|^{1/2}, \tag{1.5}$$

where $Q = I - P$, $P = \|V\|_{L^1}^{-1}\langle\cdot, v\rangle v$, $A_0, A_1 \in \mathbb{B}(L^2)$, and $\Gamma_k(\lambda)$ denotes a λ-dependent absolutely bounded operator on L^2 such that

$$\sum_{\ell=0}^{k} \||\lambda^\ell\partial_\lambda^\ell\Gamma_k(\lambda)|\|_{L^2\to L^2} \lesssim \lambda^k, \quad 0 < \lambda \le \lambda_0.$$

This formula was enough for $1<p<\infty$, while this is not the case for $p=1,\infty$ not only for the unboundedness, but also for the weak $(1,1)$ estimate. Hence, we compute the right-hand side of (1.5) more precisely to obtain

$$R_V^+(\lambda^4)V = R_0^+(\lambda^4)v\{QA_0Q + \lambda(QA_{1,0} + A_{0,1}Q) + \lambda\widetilde{P} + \lambda^2 A_2 + \Gamma_3(\lambda)\}v, \tag{1.6}$$

where $A_{1,0}, A_{0,1}, A_2 \in \mathbb{B}(L^2)$ and $\widetilde{P} = cP$ with some constant c. To ensure this expansion make sense, we need the condition $|V(x)| \lesssim \langle x\rangle^{-\mu}$ with $\mu > 11$.

By (1.6), W_-^L can be written as a sum of associated six integral operators. Moreover, using the explicit formula (3.2) of $R_0^\pm(\lambda^4)$ and the cancellation property

$$\int Qv(x)\,dx = 0,$$

we can categorize such six operators into three classes (I)–(III), where (I) is associated with $QA_0Q, \lambda(QA_{1,0} + A_{0,1}Q)$ and $\lambda^2 A_2$, (II) with $\lambda\widetilde{P}$, and (III) with $\Gamma_3(\lambda)$, respectively.

The operators in the class (I) can be shown to be bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$. Indeed, thanks to the translation invariance of L^p-norms and Minkowski's integral inequality (see e.g., (4.6)), the proof can be reduced to deal with an integral operator with the kernel bounded by

$$\min\{\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| \pm |y|\rangle^{-2}, \quad \langle |x| \pm |y|\rangle^{-4}\}.$$

Although classical Schur's test cannot be applied to this case, separating it into three regions $|x| \sim |y|$, $|x| \gg |y|$ and $|x| \ll |y|$, we can show it is bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$. For the class (III), we can apply Schur's test directly to obtain the L^p-boundedness for all $1 \le p \le \infty$. We would emphasize that the strong L^1 and L^∞ boundedness for the classes (I) and (III) are necessary to achieve the unboundedness of the full operator W_-^L on L^1 and L^∞.

For the class (II), we show that the operator associated with $\lambda\widetilde{P}$ and its adjoint are bounded from $L^1(\mathbb{R}^3)$ to $L^{1,\infty}(\mathbb{R}^3)$. To explain the main idea of this result, let us consider the following model kernel

$$\begin{aligned} K &= \frac{|x|}{|x|^4 - |y|^4} \\ &= \frac{1}{2|x|(|x|^2 + |y|^2)} + \frac{1}{4|x|^2(|x| + |y|)} + \frac{1}{4|x|^2(|x| - |y|)} =: \sum_{j=1}^{3} K_j, \end{aligned}$$

restricted on the region $\{(x,y) : ||x| - |y|| \ge 1\}$. Note that $T_{K_1}, T_{K_2} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$, since K_1, K_2 are dominated by $|x|^{-3} \in L^{1,\infty}(\mathbb{R}^3)$. To deal with T_{K_3}, we use the polar coordinate to rewrite $T_{K_3} f(x)$ as the following weighted 1D singular integral:

$$T_{K_3} f(x) = \int_0^\infty \frac{g(r)}{4|x|^2(|x|-r)} \chi_{\{||x|-r|\ge 1\}} r^2\, dr, \quad g(r) = \int_{S^2} f(r\omega)\, d\omega.$$

We then use the theory of general C-Z singular integrals on the homogeneous space to obtain that $T_{K_3} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$.

Let us emphasize that K is just a model kernel and the integral kernel K_P associated with $\lambda \widetilde{P}$ is in fact much more complicated. Indeed, we will show the following two different expressions:

$$\begin{aligned} &K_P(x,y) \\ &= -\frac{1+i}{4\pi} G(x)\Big(\frac{|x|\chi_{\{||x|-|y||\ge 1\}}}{|x|^4-|y|^4}\Big) G(y) + O\Big(\frac{1}{\langle x\rangle\langle y\rangle\langle |x|-|y|\rangle^2}\Big) \quad (1.7) \\ &= \frac{1}{8\pi(1+i)\|V\|_{L^1}^2} \int_{\mathbb{R}^6} v^2(u_1)v^2(u_2)\widetilde{K}_P(x-u_1, y-u_2)\, du_1\, du_2, \quad (1.8) \end{aligned}$$

where

$$G(x) = \frac{|x|}{\|V\|_{L^1}} \Big(\int_{\mathbb{R}^3} \frac{|V|(u)}{|x-u|}\, du \Big),$$

$$\widetilde{K}_P(z,w) = \frac{-4i|z|\chi_{\{||z|-|w||\ge 1\}}}{|z|^4-|w|^4} + \Psi(z,w),$$

and $T_\Psi \in \mathbb{B}(L^p)$ for all $1 \le p \le \infty$. The former equality (1.7) is used for proving the weak $(1,1)$ estimate and the latter one (1.8) for the unboundedness on L^1 and L^∞. In particular, for the unboundedness, we utilize the assumption that $\operatorname{supp} V \subset \{|x| \le R_0\}$ with some R_0 and take characteristic functions $f_1(y) = \chi_{\{|y|\le 1\}}$ and $f_R(y) = \chi_{\{|y|\le R\}}$ with $R \gg R_0$ to somehow estimate $\int_{\mathbb{R}^3} |T_{K_P} f_1| dx$ and $|(T_{K_P} f_R)(x)|$, respectively, then we show that $T_{K_P} f_1 \notin L^1(\mathbb{R}^3)$ and $\|T_{K_P} f_R\|_{L^\infty(\mathbb{R}^3)} \to \infty$ as $R \to \infty$, which implies the desired unboundedness of $W_\pm$ on $L^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$.

1.4. Some notations

Some notations used in the paper are listed as follows.

- $A \lesssim B$ (resp. $A \gtrsim B$) means $A \le CB$ (resp. $A \ge CB$) with some constant $C > 0$.
- $L^p = L^p(\mathbb{R}^n)$, $L^{1,\infty} = L^{1,\infty}(\mathbb{R}^n)$ denote the Lebesgue and weak L^1 spaces, respectively.

- For $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ positive almost everywhere and $1 \le p < \infty$,

$$L^p(w) = L^p(\mathbb{R}^n, w dx)$$

denotes the weighted L^p-space with the norm

$$\|f\|_{L^p(w)} = \left(\int |f(x)|^p w(x)\, dx\right)^{1/p}.$$

Set

$$w(E) := \int_E w(x)\, dx, \quad \text{for each Borel subset } E \subset \mathbb{R}^n.$$

Denote $L^{1,\infty}(w)$ as the weighted weak L^1 space with the quasi-norm

$$\|f\|_{L^{1,\infty}(w)} = \sup_{\lambda>0} \lambda w(\{x : |f(x)| > \lambda\}).$$

- Let $\{\varphi_N\}_{N\in\mathbb{Z}}$ be a homogeneous dyadic partition of unity on $(0, \infty)$, that is $\varphi_0 \in C_0^\infty(\mathbb{R}_+)$, $0 \le \varphi \le 1$, $\mathrm{supp}\,\varphi \subset \left[\frac{1}{4}, 1\right]$, $\varphi_N(\lambda) = \varphi_0(2^{-N}\lambda)$, $\mathrm{supp}\,\varphi_N \subset [2^{N-2}, 2^N]$ and

$$\sum_{N\in\mathbb{Z}} \varphi_N(\lambda) = 1, \quad \lambda > 0.$$

2. Some integrals related with wave operators

In this section, we prepare some basic criterions to the boundedness of integral operators related with the wave operators $W_\pm$. Throughout the paper, we always use T_K to denote the integral operator defined by the kernel $K(x, y)$:

$$T_K f(x) = \int_{\mathbb{R}^3} K(x, y) f(y)\, dy.$$

Moreover, we say that the kernel $K(x, y)$ of an operator T_K is *admissible* if it satisfies

$$\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |K(x, y)|\, dy + \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} |K(x, y)|\, dx < \infty.$$

Let us first recall of the classical Schur test lemma.

Lemma 2.1. *$T_K \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \le p \le \infty$ if its kernel $K(x, y)$ is admissible.*

Next, the following proposition is crucial to the L^p-boundedness of wave operators $W_\pm$.

Proposition 2.2. *Let the kernel $K(x, y)$ satisfy the following condition:*

$$|K(x, y)| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2}, \quad (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{2.1}$$

Then $T_K \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3))$ for $1 < p < \infty$. That is

$$\|T_K f\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}, \qquad 1 < p < \infty, \tag{2.2}$$

$$|\{x \in \mathbb{R}^3 : |(T_K f)(x)| \geq \lambda\}| \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^3} |f(x)|\, dx, \quad \lambda > 0. \tag{2.3}$$

Moreover, if there exists $\delta > 0$ such that $K(x, y)$ further satisfies one of the following two conditions:

$$|K(x, y)| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2-\delta}, \tag{2.4}$$

$$|K(x, y)| \lesssim \min\{\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2}, \langle |x| - |y|\rangle^{-3-\delta}\}, \tag{2.5}$$

then $T_K \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \leq p \leq \infty$.

Proof. Firstly, we decompose $K(x, y)$ as

$$\begin{aligned} K(x, y) &= K(x, y)(\chi_{\{\frac{1}{2}|x| \leq |y| \leq 2|x|\}} + \chi_{\{|y| < \frac{1}{2}|x|\}} + \chi_{\{|y| > 2|x|\}}) \\ &=: K_1(x, y) + K_2(x, y) + K_3(x, y), \end{aligned}$$

and denote T_{K_i} as the integral operators associated with the kernels $K_i(x, y)$ for $i = 1, 2, 3$. Using (2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^3} |K_1(x, y)| dy &\lesssim \frac{1}{\langle x\rangle^2} \int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} \langle |x| - |y|\rangle^{-2}\, dy \\ &\lesssim \frac{|x|^2}{\langle x\rangle^2} \int_{\frac{1}{2}|x|}^{2|x|} \langle |x| - r\rangle^{-2}\, dr \lesssim \int_{-\infty}^{+\infty} \langle r\rangle^{-2}\, dr \lesssim 1, \end{aligned}$$

uniformly in $x \in \mathbb{R}^3$. Similarly, we also have

$$\int_{\mathbb{R}^3} |K_1(x, y)|\, dx \lesssim \frac{1}{\langle y\rangle^2} \int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} \langle |x| - |y|\rangle^{-2}\, dx \lesssim 1,$$

uniformly in $y \in \mathbb{R}^3$. Hence, by Schur's test, we conclude that $T_{K_1} \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \leq p \leq \infty$.

Now, consider the integral operator T_{K_2}. Note that

$$|K_2(x, y)| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2}\, \chi_{\{|y| \leq \frac{1}{2}|x|\}}.$$

Then, for $f \in L^\infty(\mathbb{R}^3)$, we have

$$|T_{K_2} f(x)| \lesssim \Big(\int\limits_{|y| \le \frac{1}{2}|x|} \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|-|y|\rangle^{-2}\, dy \Big) \|f\|_{L^\infty(\mathbb{R}^3)}$$
$$\lesssim \frac{1}{\langle x\rangle^3} \Big(\int\limits_{|y| \le \frac{1}{2}|x|} \langle y\rangle^{-1}\, dy \Big) \|f\|_{L^\infty} \lesssim \|f\|_{L^\infty(\mathbb{R}^3)},$$

which yields $T_{K_2} \in \mathbb{B}(L^\infty(\mathbb{R}^3))$. On the other hand, if $f \in L^1(\mathbb{R}^3)$, then

$$|T_{K_2} f(x)| \lesssim \langle x\rangle^{-3} \Big(\int\limits_{|y| \le \frac{1}{2}|x|} \langle y\rangle^{-1} |f(y)|\, dy \Big) \le \langle x\rangle^{-3} \|f\|_{L^1(\mathbb{R}^3)}, \tag{2.6}$$

which leads to $T_{K_2} \in \mathbb{B}(L^1, L^{1,\infty})$ due to $\langle x\rangle^{-3} \in L^{1,\infty}(\mathbb{R}^3)$. By the Marcinkiewicz interpolation (see, e.g., Grafakos [18, p. 34]), we obtain that

$$T_{K_2} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3)) \quad \text{for all } 1 < p \le \infty.$$

Next, we deal with the third integral operator T_{K_3}. Clearly, $T_{K_3}^* = T_{K_3^*}$ with

$$|K_3^*(x, y)| = |\overline{K_3(y, x)}| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|-|y|\rangle^{-2}\, \chi_{\{|y| \le \frac{1}{2}|x|\}}.$$

By the same argument as in T_{K_2}, one has $T_{K_3^*} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p \le \infty$. Hence, $T_{K_3} \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \le p < \infty$ by the duality. Combining with the boundedness of T_{K_j} for $j = 1, 2, 3$, we conclude that $T_K \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$ as desired.

Finally, we shall show $T_K \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \le p \le \infty$ under the conditions (2.4) or (2.5). By the above argument, it suffices to show $T_{K_2} \in \mathbb{B}(L^1(\mathbb{R}^3))$. If (2.4) holds, then for any $f \in L^1(\mathbb{R}^3)$,

$$\int\limits_{\mathbb{R}^3} |T_{K_2} f(x)|\, dx \lesssim \int\limits_{\mathbb{R}^3} \Big(\int\limits_{|y| \le \frac{1}{2}|x|} \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|-|y|\rangle^{-2-\delta} |f(y)|\, dy \Big)\, dx$$
$$\lesssim \Big(\int\limits_{\mathbb{R}^3} \langle x\rangle^{-3-\delta}\, dx \Big) \|f\|_{L^1} \lesssim \|f\|_{L^1(\mathbb{R}^3)}.$$

That is, $T_{K_2} \in \mathbb{B}(L^1(\mathbb{R}^3))$. If (2.5) holds, then

$$|K_2(x, y)| \lesssim \langle |x|-|y|\rangle^{-3-\delta}\, \chi_{\{|y| < \frac{1}{2}|x|\}},$$

for some $\delta > 0$. Hence, again, we can obtain from the (2.6) that

$$\int\limits_{\mathbb{R}^3} |T_{K_2} f(x)|\, dx \lesssim \Big(\int\limits_{\mathbb{R}^3} \langle x\rangle^{-3-\delta}\, dx \Big) \Big(\int\limits_{|y| < \frac{1}{2}|x|} |f(y)|\, dy \Big) \lesssim \|f\|_{L^1(\mathbb{R}^3)}.$$

Thus, the whole proof of Proposition 2.2 has been finished. ∎

Remark 2.3. In Proposition 2.2, under condition (2.1), the strong estimates (2.2) of T_K have been obtained by Goldberg and Green [16, Lemma 2.1] using a different argument from one above. We also remark that the weak estimate (2.3) of T_K seems to be new.

As is seen in Section 4 below, Proposition 2.2 is not enough to prove Theorem 1.2 and we need to study some integral operators T_K with kernels like $K(x,y) = \frac{|x|}{|x|^4-|y|^4}$. To establish the L^p boundedness of such an operator T_K, we will make use of the theory of Caldeŕon–Zygmund on the A_p-weighted spaces and on homogeneous spaces with doubling measures. Although the proof of the following proposition is reduced to the Calderón–Zygmund theory of singular integrals, the kernel $\frac{|x|}{|x|^4-|y|^4}$ is not a standard Calderón–Zygmund kernel of $\mathbb{R}^3$, e.g., see Grafakos [18, p. 359].

Proposition 2.4. *Let T_K be the integral operator with the following truncated kernel*

$$K(x,y) := \frac{|x|\,\chi_{\{||x|-|y||\ge 1\}}}{|x|^4-|y|^4}, \quad (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Then the operator $T_K, T_K^* \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$.

Proof. It should be pointed out that [16, Lemma 3.3] implies $T_K \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$. Hence, in the sequel we mainly show the weak estimate for the endpoint case $p = 1$ with only a sketch of the proof for $1 < p < \infty$. Following a similar method as of [16], we reduce the integral in three space dimensions to the one-dimensional integral by the spherical coordinate transform. Let $g(s) := \int_{S^2} f(s\omega)d\omega$ for $s > 0$, where S^2 is the unite sphere of $\mathbb{R}^3$. Then

$$\begin{aligned} T_K f(x) &= \int\limits_{||x|-|y||\ge 1} \frac{|x|}{|x|^4-|y|^4} f(y)\,dy \\ &= \frac{|x|}{4} \int\limits_{||x|-\sqrt[4]{r}|\ge 1} \frac{r^{-\frac14} g(\sqrt[4]{r})}{|x|^4-r} dr := \frac{|x|}{4} G(|x|^4), \end{aligned}$$

where

$$G(s) = \int\limits_{|\sqrt[4]{s}-\sqrt[4]{r}|\ge 1} \frac{r^{-\frac14} g(\sqrt[4]{r})}{s-r}\,dr.$$

Note that in [16, Lemma 3.3] it was shown that the function $G(s)$ can be dominated by the maximal truncated Hilbert transform $\mathbb{H}^*(\tilde{g})(s)$ and Littlewood–Hardy maximal function $\mathbb{M}(\tilde{g})(s)$, where the function $\tilde{g}(r) := r^{-\frac14} g(\sqrt[4]{r})$. That is,

$$|G(s)| \lesssim \mathbb{H}^*(\tilde{g})(s) + \mathbb{M}(\tilde{g})(s), \quad s > 0.$$

Since

$$\int_{\mathbb{R}^3} |T_K f(x)|^p \, dx = \frac{\pi}{4^{p+1}} \int_0^\infty |G(s)|^p s^{\frac{p-1}{4}} \, ds,$$

and $|s|^{\frac{p-1}{4}}$ is A_p-weights for all $1 < p < \infty$, by using the boundedness of $\mathbb{H}^*$ and $\mathbb{M}$ on $L^p(\mathbb{R}, |s|^{\frac{p-1}{4}} ds)$ (see, e.g., Grafakos [18, Chapter 7]), then it immediately follows that

$$\begin{aligned}\int_{\mathbb{R}^3} |T_K f(x)|^p \, dx &\lesssim \int_0^\infty |\mathbb{H}^*(\tilde{g})(s)|^p s^{\frac{p-1}{4}} \, ds + \int_0^\infty |\mathbb{M}(\tilde{g})(s)|^p s^{\frac{p-1}{4}} \, ds \\ &\lesssim \int_0^\infty |\tilde{g}(r)|^p r^{\frac{p-1}{4}} \, dr \lesssim \|f\|^p_{L^p(\mathbb{R}^3)},\end{aligned}$$

which gives the integral operator $T_K \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 < p < \infty$.

We remark that the arguments above depend on the strong estimates of Hilbert transforms $\mathbb{H}^*(\tilde{g})$ and the Littlewood–Hardy maximal function $\mathbb{M}(\tilde{g})$ on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$, which do not directly work for $p = 1$ or ∞ due to the failure of strong estimates of $\mathbb{H}^*$ and $\mathbb{M}$ on these limiting spaces. Hence, in the following, we will use another argument to prove $T_K \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$.

Firstly, we decompose $K(x, y)$ as follows:

$$\begin{aligned}&K(x, y) \\ &= \frac{\chi_{\{||x|-|y||\geq 1\}}}{2|x|(|x|^2 + |y|^2)} + \frac{\chi_{\{||x|-|y||\geq 1\}}}{4|x|^2(|x| + |y|)} + \frac{\chi_{\{||x|-|y||\geq 1\}}}{4|x|^2(|x| - |y|)} =: \sum_{j=1}^3 K_j(x, y),\end{aligned}$$

and write the integral operator T_K into the sum $\sum_{j=1}^3 T_{K_j}$, respectively. Let $f \in L^1(\mathbb{R}^3)$. Then for each $x \in \mathbb{R}^3$ we easily obtain that

$$|T_{K_1} f(x)| + |T_{K_2} f(x)| \lesssim |x|^{-3} \|f\|_{L^1(\mathbb{R}^3)}.$$

Since $|x|^{-3} \in L^{1,\infty}(\mathbb{R}^3)$, so it follows immediately that $T_{K_j} \in B(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$ for $j = 1, 2$.

Next, it remains to show $T_{K_3} \in B(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$. By the polar coordinate transform,

$$T_{K_3} f(x) = \int_0^\infty \frac{g(r)}{4|x|^2(|x| - r)} \chi_{\{||x|-r|\geq 1\}} \, r^2 \, dr = \mathbb{W}(g_0)(|x|),$$

where $g(r) = \int_{S^2} f(r\omega)\,d\omega$ and

$$\mathbb{W}(g_0)(s) := \int_{\mathbb{R}} \frac{\chi_{\{|s-r|\geq 1\}}}{4s^2(s-r)} g_0(r) r^2\,dr, \quad g_0(s) = \chi_{(0,\infty)}(s) g(s).$$

Let $d\mu(r) = r^2 dr$ be a Borel measure on the real line $\mathbb{R}$. Then $d\mu(r)$ is a doubling measure on $\mathbb{R}$ (see, e.g., Stein [28, p. 12]). In the following, we will regard the integral $\mathbb{W}(g_0)$ as a singular integral on $L^1(\mathbb{R}, d\mu)$ in order to establish the weak estimate of $T_{K_3} f$ on $L^1(\mathbb{R}^3)$.

In fact, in view of the following facts:

$$\begin{aligned}|\{x \in \mathbb{R}^3 : |T_{K_3} f(x)| > \lambda\}| &= |\{x \in \mathbb{R}^3 : |\mathbb{W}(g_0)(|x|)| > \lambda\}| \\ &= 4\pi \int_0^\infty \chi_{\{s\in\mathbb{R}:|\mathbb{W}(g_0)(s)|>\lambda\}} s^2\,ds \\ &= 4\pi \mu\{s \in \mathbb{R}^+ : |\mathbb{W}(g_0)(s)| > \lambda\},\end{aligned}$$

and

$$\int_{\mathbb{R}} |g_0(s)|\,d\mu(s) \leq \int_0^\infty \int_{S^2} |f(r\omega)| r^2\,d\omega\,dr = \|f\|_{L^1(\mathbb{R}^3)},$$

we can immediately conclude that the operator $T_{K_3} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$, if one has

$$\lambda \mu\{s \in \mathbb{R} : |\mathbb{W}(g_0)(s)| > \lambda\} \lesssim \int_{\mathbb{R}} |g_0(s)|\,d\mu(s), \quad \lambda > 0. \tag{2.7}$$

To obtain the weak estimate (2.7), we will make use of the theory of general C–Z singular integral on the homogeneous space $(X, d\mu)$ with a doubling measure μ. Indeed, in view of conclusions in Stein [28, p. 19, Theorem 1.3], it suffices to show that the integral $\mathbb{W}(f)$ on the homogeneous space $(\mathbb{R}, r^2 dr)$ satisfies the following two conditions:

(i) there exist some $q > 1$ and $A > 0$ such that

$$\|\mathbb{W}(f)\|_{L^q(\mathbb{R},d\mu)} \leq A\|f\|_{L^q(\mathbb{R},d\mu)}, \quad d\mu = r^2 dr;$$

(ii) the kernel $\mathcal{K}(s,r) = \frac{\chi_{\{|s-r|\geq 1\}}}{4s^2(s-r)}$ of the integral operator $\mathbb{W}(f)$, satisfies that

$$\int_{|s-r|\geq 2\delta} |\mathcal{K}(s,r) - \mathcal{K}(s,\bar{r})|\,d\mu(s) \leq A < \infty,$$

whenever $|r - \bar{r}| < \delta$ and $\delta > 0$.

Firstly, let us check the condition (i). Indeed, let $1 < q < \frac{3}{2}$, then

$$\int_{\mathbb{R}} |\mathbb{W}(f)(s)|^q \, d\mu(s) = 4^{-q} \int_{\mathbb{R}} \Big| \int_{|s-r|\geq 1} \frac{f(r)r^2}{s-r} \, dr \Big| s^{2-2q} \, ds$$
$$\lesssim \int_{\mathbb{R}} |f(r)r^2|^q r^{2-2q} \, dr = \|f\|^q_{L^q(\mathbb{R},d\mu)},$$

where in the second inequality above, we have used the weighted L^q estimates of the truncated Hilbert transform on $L^q(\mathbb{R}, w(r)dr)$ with a A_q-weight $w(r) = |r|^{2-2q}$ due to the fact $-1 < 2-2q < q-1$ as $1 < q < \frac{3}{2}$.

Next, we come to prove the condition (ii). Let $\delta > 0$ and $|r - \bar{r}| < \delta$. Then

$$\begin{aligned}
&\int_{|s-r|\geq 2\delta} |\mathcal{K}(s,r) - \mathcal{K}(s,\bar{r})| \, d\mu(s) \\
&= \frac{1}{4} \int_{|s-r|\geq 2\delta} \Big|\frac{\chi_{\{|s-r|\geq 1\}}}{s-r} - \frac{\chi_{\{|s-\bar{r}|\geq 1\}}}{s-\bar{r}}\Big| \, ds \\
&\lesssim \int_{|s-r|\geq 2\delta} \Big|\frac{\chi_{\{|s-r|\geq 1\}}}{s-r} - \frac{\chi_{\{|s-r|\geq 1\}}}{s-\bar{r}}\Big| \, ds + \int_{|s-r|\geq 2\delta} \Big|\frac{\chi_{\{|s-r|\geq 1\}} - \chi_{\{|s-\bar{r}|\geq 1\}}}{s-\bar{r}}\Big| \, ds \\
&:= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Note that $|r - \bar{r}| < \delta$ and $|s - r| \geq 2\delta$, which imply that $|s - \bar{r}| \geq \frac{1}{2}|s - r|$. Then

$$\mathrm{I} \leq \int_{|s-r|\geq 2\delta} \frac{2|r-\bar{r}|}{|(s-r)(s-\bar{r})|} \, ds \leq 2\delta \int_{|s-r|\geq 2\delta} \frac{ds}{|s-r|^2} = 4,$$

and

$$\begin{aligned}
\mathrm{II} &\leq \int_{|s-r|\geq 2\delta} \frac{(\chi_{\{|s-\bar{r}|\geq 1/2\}} - \chi_{\{|s-\bar{r}|\geq 1\}})}{|s-\bar{r}|} \, ds + \int_{1>|s-r|\geq 2\delta} \frac{\chi_{\{|s-\bar{r}|\geq 1\}}}{|s-\bar{r}|} ds \\
&\leq \int_{\frac{1}{2}\leq|s-\bar{r}|<1} \frac{ds}{|s-\bar{r}|} + \int_{|s-r|<1} ds \leq 2.
\end{aligned}$$

Thus, condition (ii) holds. Hence, by summarizing above all arguments we can conclude the desired estimate (2.7), and then $T_K \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$.

Finally, we observe that the kernel of T_K^* is given by

$$\overline{K(y,x)} = \chi_{\{||x|-|y||\geq 1\}} \Big(\frac{|y|}{2|x|^2(|x|^2 + |y|^2)} - \frac{1}{4|x|^2(|x|+|y|)} + \frac{1}{4|x|^2(|x|-|y|)} \Big).$$

The last two terms are equal to exactly K_2 and K_3, respectively. The first term is dominated by $\frac{|x|^{-3}}{4}$. Hence, the same argument as above shows $T_K^* \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3))$. ■

3. Stationary formula and resolvent expansion at zero

3.1. The stationary formulas of wave operators

First of all, we observe that it suffices to deal with W_- since (1.1) implies $W_+ f = \overline{W_- \overline{f}}$. The starting point is the following well-known stationary representation of W_- (see, e.g., Kuroda [23]):

$$W_- = I - \frac{2}{\pi i} \int_0^\infty \lambda^3 R_V^+(\lambda^4) V \big(R_0^+(\lambda^4) - R_0^-(\lambda^4)\big)\, d\lambda. \tag{3.1}$$

To explain the formula (3.1), we need to introduce some notations. Let

$$R_0(z) = (\Delta^2 - z)^{-1}, \quad R_V(z) = (H - z)^{-1}, \quad z \in \mathbb{C} \setminus [0, \infty),$$

be the resolvents of Δ^2 and $H = \Delta^2 + V(x)$, respectively. We denote by $R_0^\pm(\lambda)$, $R_V^\pm(\lambda)$ their boundary values (limiting resolvents) on $(0, \infty)$, namely

$$R_0^\pm(\lambda) = \lim_{\varepsilon \searrow 0} R_0(\lambda \pm i\varepsilon), \quad R_V^\pm(\lambda) = \lim_{\varepsilon \searrow 0} R_V(\lambda \pm i\varepsilon), \quad \lambda > 0.$$

The existence of $R_0^\pm(\lambda)$ as bounded operators from $L^2_s(\mathbb{R}^3)$ to $L^2_{-s}(\mathbb{R}^3)$ with $s > \frac{1}{2}$ follows from the limiting absorption principle for the resolvent $(-\Delta - z)^{-1}$ of the free Schrödinger operator $-\Delta$ (see, e.g., Agmon [1]) and the following equality:

$$R_0(z) = \frac{1}{2\sqrt{z}}\big((-\Delta - \sqrt{z})^{-1} - (-\Delta + \sqrt{z})^{-1}\big), \quad z \in \mathbb{C} \setminus [0, \infty), \operatorname{Im} \sqrt{z} > 0.$$

This formula above also gives the explicit expressions of the kernels of $R_0^\pm(\lambda^4)$:

$$R_0^\pm(\lambda^4, x, y) = \frac{1}{8\pi\lambda^2 |x - y|}(e^{\pm i\lambda|x-y|} - e^{-\lambda|x-y|}) = \frac{F_\pm(\lambda|x - y|)}{8\pi\lambda}, \tag{3.2}$$

where $x, y \in \mathbb{R}^3$ and $F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s})$. The existence of $R_V^\pm(\lambda)$ for $\lambda > 0$ under our assumption of Theorem 1.2 has been also already shown (see, e.g., [1, 23]).

3.2. Resolvent asymptotic expansions near zero

This section is mainly devoted to the study of asymptotic behaviors of the resolvent $R_V^+(\lambda^4)$ at low energy $\lambda \to +0$. We also prepare some elementary lemmas needed in the proof of our main theorems.

We begin with recalling the symmetric resolvent formula for $R_V^\pm(\lambda^4)$. Let $v(x) = |V(x)|^{1/2}$ and $U(x) = \operatorname{sgn} V(x)$, that is $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$. Let $M^\pm(\lambda) = U + vR_0^\pm(\lambda^4)v$ and $(M^\pm)^{-1}(\lambda) := (M^\pm(\lambda))^{-1}$.

Lemma 3.1. *For $\lambda > 0$, $M^\pm(\lambda)$ is invertible on $L^2(\mathbb{R}^3)$ and $R_V^\pm(\lambda^4)V$ has the form*

$$R_V^\pm(\lambda^4)V = R_0^\pm(\lambda^4)v(M^\pm)^{-1}(\lambda)v. \tag{3.3}$$

Proof. Due to the absence of embedded positive eigenvalue of H, it was well known that $M^\pm(\lambda)$ is invertible on $L^2(\mathbb{R}^3)$ for all $\lambda > 0$ (see, e.g., Agmon [1] and Kuroda [23]). Since $V = vUv$ and $1 = U^2$, we have

$$\begin{aligned} R_V^\pm(\lambda^4)v &= R_0^\pm(\lambda^4)v - R_V^\pm(\lambda^4)vUvR_0^\pm(\lambda^4)v \\ &= R_0^\pm(\lambda^4)v(1 + UvR_0^\pm(\lambda^4)v)^{-1} \\ &= R_0^\pm(\lambda^4)v(U + vR_0^\pm(\lambda^4)v)^{-1}U^{-1}. \end{aligned}$$

Multiplying Uv from the right, we obtain the desired formula for $R_V^\pm(\lambda^4)V$. ∎

Throughout the paper, we only use $M^+(\lambda)$, so we write $M(\lambda) = M^+(\lambda)$ for simplicity. In order to obtain the asymptotic behaviors of $R_V^+(\lambda^4)$ near $\lambda = 0$, we need to establish the asymptotic expansion of $M^{-1}(\lambda)$, which plays a crucial role in the paper. To this end, we introduce some notations. We say that an integral operator $T_K \in \mathbb{B}(L^2(\mathbb{R}^3))$ with the kernel K is *absolutely bounded* if $T_{|K|} \in \mathbb{B}(L^2(\mathbb{R}^3))$. Let

$$P := \frac{\langle\cdot, v\rangle v}{\|V\|_{L^1}}, \quad \widetilde{P} = \frac{8\pi}{(1+i)\|V\|_{L^1}}P = \frac{8\pi}{(1+i)\|V\|_{L^1}^2}\langle\cdot, v\rangle v, \quad Q := I - P. \tag{3.4}$$

Note that P is the orthogonal projection onto the span of v in $L^2(\mathbb{R}^3)$, i.e., $PL^2 = \operatorname{span}\{v\}$, and $Q(v) = 0$.

Lemma 3.2. *Let $H = \Delta^2 + V(x)$ with $|V(x)| \lesssim \langle x\rangle^{-\mu}$ for $x \in \mathbb{R}^3$. If 0 is a regular point of H and $\mu > 11$, then there exists $\lambda_0 > 0$ such that $M^{-1}(\lambda)$ satisfies the following asymptotic expansions on $L^2(\mathbb{R}^3)$ for $0 < \lambda \leq \lambda_0$:*

$$M^{-1}(\lambda) = QA_0Q + \lambda(QA_{1,0} + A_{0,1}Q) + \lambda\widetilde{P} + \lambda^2 A_2 + \Gamma_3(\lambda), \tag{3.5}$$

where A_0, $A_{1,0}$, $A_{0,1}$ and A_2 are λ-independent bounded operators on L^2 and $\Gamma_3(\lambda)$ are λ-dependent bounded operators on L^2 such that all the operators in the right *sides of* (3.5) *are absolutely bounded. Moreover,* $\Gamma_3(\lambda)$ *satisfy that for* $\ell = 0, 1, 2, 3$,

$$\| |\partial_\lambda^\ell \Gamma_3(\lambda)| \|_{L^2 \to L^2} \le C_\ell \lambda^{3-\ell}, \quad 0 < \lambda \le \lambda_0. \tag{3.6}$$

We remark that, in the regular case (i.e., zero is neither an eigenvalue nor a resonance of H), the expansion of $M^{-1}(\lambda)$ at zero has been obtained with different error terms in [9, 11, 16]. In Lemma 3.2 above, the expansion (3.5) contains more specific and higher order terms at the cost of fast decay of V in order to study the endpoint estimates of wave operators $W_\pm$ here. For reader's convenience, we give its simple proof in Appendix A. Moreover, it should be pointed out that asymptotic expansions of $M^{-1}(\lambda)$ were also established in the presence of zero resonance or eigenvalue in [9].

In the following we give some elementary but useful lemmas.

Lemma 3.3. *Let* $\lambda > 0$ *and* $x, y \in \mathbb{R}^3$. *If* $F \in C^1(\mathbb{R}_+)$, *then*

$$F(\lambda|x-y|) = F(\lambda|x|) - \lambda \int_0^1 \langle y, w(x-\theta y)\rangle F'(\lambda|x-\theta y|)\, d\theta,$$

where $F'(s)$ *is the first order derivative of* $F(s)$, $\langle \cdot, \cdot \rangle$ *denotes the inner product of* $\mathbb{R}^3$, *and* $w(x) = \frac{x}{|x|}$ *for* $x \neq 0$ *and* $w(x) = 0$ *for* $x = 0$.

Proof. Let $G_\varepsilon(y) = F(\lambda\sqrt{\varepsilon^2 + |x-y|^2})$, $\varepsilon \neq 0$. Then $G_\varepsilon(y) \in C^1(\mathbb{R}^3)$ for $\varepsilon \neq 0$ and $F(\lambda|x-y|) = \lim_{\varepsilon \to 0} G_\varepsilon(y)$. By Taylor's expansions, we have

$$G_\varepsilon(y) = G_\varepsilon(0) + \int_0^1 \sum_{|\alpha|=1} (\partial^\alpha G_\varepsilon)(\theta y) y^\alpha\, d\theta. \tag{3.7}$$

Observe that

$$\partial_{y_j} G_\varepsilon(y) = \frac{-\lambda(x_j - y_j)}{(\varepsilon^2 + |x-y|^2)^{\frac{1}{2}}} F'(\lambda\sqrt{\varepsilon^2 + |x-y|^2}), \; j = 1, 2, 3.$$

Since there exists a constants $C = C(\lambda, x, y)$ such that $|(\partial_{y_i} G_\varepsilon)(\theta y)| \le C (i = 1, 2, 3)$ for $0 \le \theta \le 1$ and $0 < \varepsilon \le 1$, then by the Lebesgue dominated convergence theorem, we have for $x - \theta y \neq 0$,

$$\lim_{\varepsilon \to 0} \int_0^1 (\partial_{y_i} G_\varepsilon)(\theta y)\, d\theta = \int_0^1 \frac{-\lambda(x_j - \theta y_j)}{|x - \theta y|} F'(\lambda|x-\theta y|)\, d\theta, \quad j = 1, 2, 3,$$

and

$$\lim_{\varepsilon \to 0} \int_0^1 (\partial_{y_i} G_\varepsilon)(\theta y)\, d\theta = 0 \quad (j = 1, 2, 3)$$

for $x - \theta y = 0$. From Taylor expansions (3.7), we obtain that

$$F(\lambda|x-y|) = F(\lambda|x|) - \lambda \int_0^1 F'(\lambda|x-\theta y|)\langle y, w(x-\theta y)\rangle \, d\theta. \qquad \blacksquare$$

Below we apply Lemma 3.3 for the specific functions $F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s})$ to establish the following formulas used later.

Lemma 3.4. *Let Q be the orthogonal projection defined in* (3.4), $\lambda > 0$ *and* $F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s})$. *Then*

$$\begin{aligned}&\big(QvR_0^\pm(\lambda^4)f\big)(x)\\ &\quad = -\frac{1}{8\pi} Q\Big(v(x)\int_{\mathbb{R}^3}\Big(\int_0^1 \langle x, w(y-\theta x)\rangle \; F_\pm^{(1)}(\lambda|y-\theta x|)d\theta\Big) f(y)\,dy\Big)\end{aligned}$$

and

$$\begin{aligned}&\big(R_0^\pm(\lambda^4)vQf\big)(x)\\ &\quad = -\frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(\int_0^1 F_\pm^{(1)}(\lambda|x-\theta y|)\langle y, w(x-\theta y)\rangle\, d\theta\Big) v(y)(Qf)(y)\,dy,\end{aligned}$$

where $F_\pm^{(1)}(s) = s^{-2}((\pm is - 1)e^{\pm is} + (s+1)e^{-s})$ *denotes the first order derivative of* $F_\pm(s)$.

Remark 3.5. The above formulas for $QvR_0^\pm(\lambda^4)f$ and $R_0^\pm(\lambda^4)vQf$ can be written respectively as

$$\begin{aligned} QvR_0^\pm(\lambda^4)f &= \frac{1}{8\pi}Q\Big(\int_{\mathbb{R}^3} h_\ell(\lambda,x,y)f(y)\,dy\Big),\\ R_0^\pm(\lambda^4)vQf &= \frac{1}{8\pi}\int_{\mathbb{R}^3} h_r(\lambda,x,y)(Qf)(y)\,dy,\end{aligned}$$

where

$$h_\ell(\lambda,x,y) = -v(x)\int_0^1 \langle x, w(y-\theta x)\rangle \; F_\pm^{(1)}(\lambda|y-\theta x|)\,d\theta,$$

$$h_r(\lambda,x,y) = -v(y)\int_0^1 \langle y, w(x-\theta y)\rangle \; F_\pm^{(1)}(\lambda|x-\theta y|)\,d\theta.$$

Moreover, we also notice that

$$h_\ell(\lambda, x, y),\ h_r(\lambda, x, y) = O_{x,y}(1), \quad \lambda \to +0.$$

Here, we use $h(\lambda, x, y) = O_{x,y}(\lambda^k)$ to denote that $|h(\lambda, x, y)| \lesssim \lambda^k$ for fixed x, y. Compared with the free resolvent $|R_0^\pm(\lambda^4)(x, y)| \lesssim \lambda^{-1}$, such a gain of one order power of λ will be crucial to establish stronger point-wise estimates of integral kernels related to $W_\pm$ later.

Proof of Lemma 3.4. By (3.2) and applying Lemma 3.3 to $F_\pm$, we obtain

$$\begin{aligned} R_0^\pm(\lambda^4, x, y) &= \frac{F_\pm(\lambda|y-x|)}{8\pi\lambda} \\ &= \frac{F_\pm(\lambda|y|)}{8\pi\lambda} - \frac{1}{8\pi}\int_0^1 \langle x, w(y-\theta x)\rangle F_\pm'(\lambda|y-\theta x|)\, d\theta. \end{aligned}$$

Since $Q(v) = 0$, then it follows that

$$\begin{aligned} &\big(QvR_0^\pm(\lambda^4)f\big)(x) \\ &\quad = \frac{1}{8\pi\lambda} Q(v) \int_{\mathbb{R}^3} F_\pm(\lambda|y|) f(y) dy \\ &\qquad - \frac{1}{8\pi} Q\bigg(v \int_{\mathbb{R}^3} \bigg(\int_0^1 \langle x, w(y-\theta x)\rangle F_\pm'(\lambda|y-\theta x|)\, d\theta\bigg) f(y)\, dy\bigg) \\ &\quad = -\frac{1}{8\pi} Q\bigg(v \int_{\mathbb{R}^3} \bigg(\int_0^1 \langle x, w(y-\theta x)\rangle F_\pm'(\lambda|y-\theta x|)\, d\theta\bigg) f(y)\, dy\bigg). \end{aligned}$$

For $R_0^+(\lambda^4)vQf$, by taking

$$R_0^\pm(\lambda^4, x, y) = \frac{F_\pm(\lambda|x|)}{8\pi\lambda} - \frac{1}{8\pi}\int_0^1 \langle y, w(x-\theta y)\rangle F_\pm'(\lambda|x-\theta y|)\, d\theta,$$

the proof is analogous. ■

Moreover, we also need to frequently use the following lemmas later.

Lemma 3.6. *Let*

$$F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s}),$$
$$A_\pm(s) = e^{\mp is} F_\pm'(s) \quad \textit{and} \quad B_\pm(s) = e^{\mp is} F_\pm(s).$$

Then for any $\ell \in \mathbb{N}$, the following estimates hold:

$$|F_{\pm}^{(\ell)}(s)| \lesssim \langle s\rangle^{-1}, \qquad s > 0,$$
$$|A_{\pm}^{(\ell)}(s)| + |B_{\pm}^{(\ell)}(s)| \lesssim \langle s\rangle^{-\ell-1}, \quad s > 0,$$

where $F_{\pm}^{(\ell)}(s)$, $A_{\pm}^{(\ell)}(s)$ denote the ℓ^{th} order derivative of $F_{\pm}^{(\ell)}(s)$, $A_{\pm}^{(\ell)}(s)$, respectively.

Proof. We only prove the estimates of $A_{\pm}(s)$ due to similarity. Firstly, we calculate that

$$A_{\pm}(s) = s^{-2}((\pm is - 1) + (s+1)e^{(-1\mp i)s}).$$

For each $\ell \in \mathbb{N}$, it follows by Leibniz's rule that

$$|A_{\pm}^{(\ell)}(s)| \lesssim s^{-\ell-2}\Big((s+1) + \sum_{k=0}^{\ell} s^k e^{-s}\Big),$$

which gives

$$|A_{\pm}^{(\ell)}(s)| \lesssim s^{-\ell-1} \quad \text{for } s \geq 1.$$

Additionally, by Taylor's expansion of $e^{(-1\mp)s}$, we obtain

$$A_{\pm}(s) = \sum_{k=0}^{\infty}(k+1-i)(-1\mp i)^{k+1}\frac{s^k}{(k+1)!},$$

which gives $A_{\pm}(s) \in C^{\infty}(\mathbb{R})$. Hence, $|A_{\pm}^{(\ell)}(s)| \lesssim s^{-\ell-1}$ for $s > 0$ and $\ell \in \mathbb{N}$. ■

Finally, we record the following well-known lemma, e.g., see [17, Lemma 3.8].

Lemma 3.7. *Let α and β satisfy $0 < \alpha < n < \beta$. Then*

$$\int_{\mathbb{R}^n} \frac{1}{\langle y\rangle^{\beta}|x-y|^{\alpha}}dy \lesssim \langle x\rangle^{-\alpha}.$$

4. The proof of Theorem 1.2

In this section we consider the proof of Theorem 1.2. The stationary formula (3.1) of W_- is decomposed into the low and high energy parts as follows: fixed $\lambda_0 > 0$ small enough, let $\chi \in C_0^{\infty}(\mathbb{R})$ be such that $\chi \equiv 1$ on $\left(-\frac{\lambda_0}{2}, \frac{\lambda_0}{2}\right)$ and $\operatorname{supp}\chi \subset [-\lambda_0, \lambda_0]$. We define

$$W_-^L = \int_0^{\infty} \lambda^3\chi(\lambda)R_V^+(\lambda^4)V(R_0^+(\lambda^4) - R_0^-(\lambda^4))\,d\lambda, \tag{4.1}$$

$$W_-^H = \int_0^{\infty} \lambda^3(1-\chi(\lambda))R_V^+(\lambda^4)V(R_0^+(\lambda^4) - R_0^-(\lambda^4))\,d\lambda.$$

Then $W_- = I - \frac{2}{\pi i}(W_-^L + W_-^H)$. In view of the decomposition, it suffices to estimate W_-^H and W_-^L, separately. Indeed, in the work [16, Proposition 4.1], it has been proved that high energy part W_-^H is bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$ with the decay rate $\mu > 5$ for $V(x)$. Hence, it only remains to deal with the low energy part W_-^L.

Now, we will prove the following conclusion.

Theorem 4.1. *Under the assumption in Theorem 1.2, the low energy part W_-^L defined by (4.1) satisfies the same statement as that in Theorem 1.2.*

Throughout this section, we thus always assume that $|V(x)| \lesssim \langle x \rangle^{-\mu}$ with $\mu > 11$ and zero is a regular point of H. Substituting the expansion (3.5) into (3.3), if $0 < \lambda \le \lambda_0$, then we have

$$R^+(\lambda^4)V = R_0^+(\lambda^4)v\{QA_0Q + \lambda(QA_{1,0} + A_{0,1}Q) + \lambda\widetilde{P} + \lambda^2 A_2 + \Gamma_3(\lambda)\}v.$$

Hence, W_-^L can be written as follows:

$$W_-^L = T_{K_0} + T_{K_{1,0}} + T_{K_{0,1}} + T_{K_P} + T_{K_2} + T_{K_3}, \tag{4.2}$$

where the kernels of six operators in the right side of the (4.2) are given by the following integrals:

$$K_0(x,y) = \int_0^\infty \lambda^3\chi(\lambda)\big(R_0^+(\lambda^4)vQA_0Qv(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda, \tag{4.3a}$$

$$K_{1,0}(x,y) = \int_0^\infty \lambda^4\chi(\lambda)\big(R_0^+(\lambda^4)vQA_{1,0}v(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda, \tag{4.3b}$$

$$K_{0,1}(x,y) = \int_0^\infty \lambda^4\chi(\lambda)\big(R_0^+(\lambda^4)vA_{0,1}Qv(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda, \tag{4.3c}$$

$$K_P(x,y) = \int_0^\infty \lambda^4\chi(\lambda)\big(R_0^+(\lambda^4)v\widetilde{P}v(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda, \tag{4.3d}$$

$$K_2(x,y) = \int_0^\infty \lambda^5\chi(\lambda)\big(R_0^+(\lambda^4)vA_2v(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda, \tag{4.3e}$$

$$K_3(x,y) = \int_0^\infty \lambda^3\chi(\lambda)\big(R_0^+(\lambda^4)v\Gamma_3(\lambda)v(R_0^+ - R_0^-)(\lambda^4)\big)(x,y)\,d\lambda. \tag{4.3f}$$

In view of this formula (4.2) for W_-^L, Theorem 4.1 follows from the corresponding boundedness of these six integral operators. By virtue of Lemma 3.4 and Remark 3.5, the six operators $T_{K_j}, T_{K_P}, T_{K_{ij}}$ are classified into the following three cases.

Class I. $T_{K_0}, T_{K_{1,0}}, T_{K_{0,1}}, T_{K_2}$, where all integrands can be dominated by $C\lambda^3$ for fixed x, y in their corresponding kernel integrals (4.3). (For short, we may set $O_{x,y}(\lambda^3)$ below).

Class II. T_{K_P} with $O_{x,y}(\lambda^2)$.

Class III. T_{K_3} with $O_{x,y}(\lambda^4)$.

In particular, all the six operators above are in fact well-defined integral operators. Note that, since $|v(x)| \lesssim \langle x\rangle^{-\mu/2}$ with $\mu > 11$, we have

$$\begin{aligned}\|\langle x\rangle^k vBv\langle x\rangle^k f\|_{L^1(\mathbb{R}^3)} &\le \|\langle x\rangle^k v\|_{L^2}^2 \|B\|_{L^2\to L^2}\|f\|_{L^\infty}\\ &\lesssim \|\langle x\rangle^{2k} V\|_{L^1(\mathbb{R}^3)}\|f\|_{L^\infty(\mathbb{R}^3)},\end{aligned}$$

for all $B = QA_0Q,\ QA_{1,0},\ A_{0,1}Q,\ \widetilde{P},\ A_2,\ \Gamma_3(\lambda)$, and $k < \frac{\mu-3}{2}$. Hence, in all cases, $\langle x\rangle^k vBv\langle x\rangle^k$ is an absolutely bounded integral operator for any $k \le 3$ at least, satisfying

$$\int_{\mathbb{R}^6} \langle x\rangle^k |(vBv)(x,y)|\langle y\rangle^k \, dx\, dy \lesssim \|\langle x\rangle^{2k} V\|_{L^1(\mathbb{R}^3)} < \infty, \tag{4.4}$$

where we use the notation $(vBv)(x,y) = v(x)B(x,y)v(y)$.

Now, let us finish the proof of Theorem 4.1 in the following three propositions corresponding to the three classes I–III above.

Proposition 4.2. *Let $K \in \{K_0, K_{1,0}, K_{0,1}, K_2\}$. Then $T_K \in \mathbb{B}(L^p)$ for all $1 \le p \le \infty$.*

Proof. All the kernels $K_0, K_{1,0}, K_{0,1}$, and K_2 can be written as the difference of the following two kernels

$$K^{\pm}_{\alpha\beta}(x,y) := \int_0^\infty \lambda^{5-\alpha-\beta}\chi(\lambda)(R_0^+(\lambda^4)vQ_\alpha BQ_\beta vR_0^\pm(\lambda^4))(x,y)\, d\lambda,$$

with some $B \in \mathbb{B}(L^2)$ so that $Q_\alpha BQ_\beta$ is absolutely bounded, where we set $Q_1 = Q$, $Q_0 = I$ (the identity) and

$$(\alpha,\beta) = \begin{cases} (1,1) & \text{for } K = K_0,\\ (1,0) & \text{for } K = K_{1,0},\\ (0,1) & \text{for } K = K_{0,1},\\ (0,0) & \text{for } K = K_2.\end{cases}$$

Then we shall show $T_{K^\pm_{\alpha\beta}}$ satisfies the desired assertion for all pairs (α,β) above. To this end, we consider two cases (i) $\alpha = \beta = 1$, (ii) $\beta = 0$ or $\alpha = 0$.

Case (i). By Lemma 3.4 and Remark 3.5, we can rewrite $K_{11}^{\pm}$ as follows:

$$K_{11}^{\pm}(x,y) = \frac{1}{64\pi^2}\int_0^\infty \lambda^3\chi(\lambda)\Big(\int_{\mathbb{R}^6} \mathfrak{I}_1 (vQA_0Qv)(u_1,u_2)\mathfrak{I}_2\, du_1\, du_2\Big)\, d\lambda, \quad (4.5)$$

where

$$\mathfrak{I}_1 := \int_0^1 \langle u_1, w(x-\theta_1 u_1)\rangle F_+^{(1)}(\lambda|x-\theta_1 u_1|)\, d\theta_1,$$

$$\mathfrak{I}_2 := \int_0^1 \langle u_2, w(y-\theta_2 u_2)\rangle F_\pm^{(1)}(\lambda|y-\theta_2 u_2|)\, d\theta_2,$$

and $F_\pm^{(1)}(s)$ is the first order derivative of $F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s})$, $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^3$, and $w(x) = \frac{x}{|x|}$ for $x \neq 0$ and $w(x) = 0$ for $x = 0$.

By changing the order of integrals in (4.5), then it follows that

$$\begin{aligned}|K_{11}^{\pm}(x,y)| \le \frac{1}{64\pi^2} &\int_{\mathbb{R}^6\times[0,1]^2} \big(|u_1|v(u_1)|(QA_0Q)(u_1,u_2)||u_2|v(u_2)\big)\\ &\times\left|\int_0^\infty \lambda^3\chi(\lambda)F_+^{(1)}(\lambda|x-\theta_1u_1|)F_\pm^{(1)}(\lambda|y-\theta_2u_2|)\, d\lambda\right| du\, d\theta,\end{aligned}$$

where $(u,\theta) = (u_1,u_2,\theta_1,\theta_2) \in \mathbb{R}^6\times[0,1]^2$.

Let

$$G_{11}^{\pm}(X,Y) = \int_0^\infty \lambda^3\chi(\lambda)F_+^{(1)}(\lambda|X|)F_\pm^{(1)}(\lambda|Y|)d\lambda, \quad X,Y\in\mathbb{R}^3.$$

Then

$$\begin{aligned}&|K_{11}^{\pm}(x,y)|\\ &\quad\lesssim \int_{\mathbb{R}^6\times[0,1]^2} (|u_1||(vQA_0Qv)(u_1,u_2)||u_2|)|G_{11}^{\pm}(x-\theta_1u_1, y-\theta_2u_2)|\, du\, d\theta.\end{aligned} \quad (4.6)$$

Denote by $T_{G_{11}^{\pm}}$ the integral operator associated with $G_{11}^{\pm}(x,y)$. Then, by (4.4) and (4.6), Minkowski's inequality, and the translation invariance of L^p-norm, we can reduce the L^p-boundedness of $T_{K_{11}^{\pm}}$ to the L^p-boundedness of $T_{G_{11}^{\pm}}$ based on the following inequality:

$$\|T_{K_{11}^{\pm}}\|_{L^p\to L^p} \lesssim \||x|^2V\|_{L^1}\|QA_0Q\|_{L^2\to L^2}\|T_{G_{11}^{\pm}}\|_{L^p\to L^p}, \quad 1\le p\le\infty.$$

Indeed, to establish the L^p-boundedness of $T_{G_{11}^{\pm}}$ for all $1\le p\le\infty$, by Proposition 2.2 it suffices to prove that $G_{11}^{\pm}(x,y)$ satisfies the following point-wise estimate:

$$|G_{11}^{\pm}(x,y)| \lesssim \min\{\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|\pm|y|\rangle^{-2}, \langle |x|\pm|y|\rangle^{-4}\}, \quad x,y\in\mathbb{R}^3. \quad (4.7)$$

Now, we rewrite $G_{11}^{\pm}(x,y)$ as an oscillatory integral,

$$G_{11}^{\pm}(x,y) = \int_0^\infty \lambda^3\chi(\lambda)e^{i\lambda(|x|\pm|y|)}A_+(\lambda|x|)A_\pm(\lambda|y|)d\lambda, \quad x,y\in\mathbb{R}^3, \quad (4.8)$$

where

$$A_\pm(s) := e^{\mp is}F_\pm^{(1)}(s) = s^{-2}((\pm is-1)+(s+1)e^{(-1\mp i)s}),$$

which by Lemma 3.6, satisfies the following estimates:

$$|A_\pm^{(\ell)}(s)| \lesssim \langle s\rangle^{-\ell-1}, \quad s>0,\ \ell\in\mathbb{N}_0. \quad (4.9)$$

To estimate the integral (4.8), we decompose χ by using the dyadic partition of unity $\{\varphi_N\}$ defined in Section 1.4, as

$$\chi(\lambda) = \sum_{N=-\infty}^{N_0}\tilde\chi_N(\lambda), \quad \tilde\chi_N(\lambda) := \chi(\lambda)\varphi_N(\lambda), \quad \lambda>0,$$

where $N_0 \lesssim \log\lambda_0 \lesssim -1$ since $\operatorname{supp}\chi\subset[-\lambda_0,\lambda_0]$. Then we decompose

$$G_{11}^{\pm}(x,y) = \sum_{N=-\infty}^{N_0}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\Psi_N(\lambda,x,y)d\lambda := \sum_{N=-\infty}^{N_0}E_N^{\pm}(x,y),$$

where

$$\Psi_N(\lambda,x,y) := \lambda^3\tilde\chi_N(\lambda)A_+(\lambda|x|)A_\pm(\lambda|y|).$$

Note that $\operatorname{supp}\tilde\chi_N\subset[2^{N-2},2^N]$ and

$$|\partial_\lambda^\ell\tilde\chi_N(\lambda)| \lesssim 2^{-N\ell}, \quad \ell\in\mathbb{N}_0. \quad (4.10)$$

Hence, by Leibniz's formula, (4.9), and (4.10), we have

$$|\partial_\lambda^k\Psi_N(\lambda,x,y)| \lesssim 2^{(3-k)N}\langle 2^N|x|\rangle^{-1}\langle 2^N|y|\rangle^{-1}, \quad k\in\mathbb{N}_0.$$

Thus, by k-times integration by parts for $E_N^{\pm}(x,y)$, it follows that

$$|E_N^{\pm}(x,y)| \lesssim 2^{(4-k)N}||x|\pm|y||^{-k}\langle 2^N|x|\rangle^{-1}\langle 2^N|y|\rangle^{-1}, \quad k\in\mathbb{N}_0, \tag{4.11}$$

which leads to the following estimates for $N \le N_0$:

$$|E_N^{\pm}(x,y)| \lesssim \begin{cases} \dfrac{2^{2N}}{\langle x\rangle\langle y\rangle} & \text{by } k=0 \text{ of (4.11);} \\ \dfrac{2^N}{1+2^{2N}(|x|\pm|y|)^2}\dfrac{1}{\langle x\rangle\langle y\rangle||x|\pm|y||} & \text{by } k=1,3 \text{ of (4.11);} \\ \dfrac{2^N}{1+2^{2N}(|x|\pm|y|)^2}\dfrac{1}{||x|\pm|y||^3} & \text{by } k=3,5 \text{ of (4.11).} \end{cases}$$

So, we get that

$$|G_{11}^{\pm}(x,y)| \le \sum_{N=-\infty}^{N_0}|E_N^{\pm}(x,y)| \lesssim \begin{cases} \dfrac{1}{\langle x\rangle\langle y\rangle}; \\ \dfrac{1}{\langle x\rangle\langle y\rangle(|x|\pm|y|)^2}; \\ \dfrac{1}{(|x|\pm|y|)^4}. \end{cases} \tag{4.12}$$

Therefore, we have

$$|G_{11}^{\pm}(x,y)| \lesssim \frac{1}{\langle x\rangle\langle y\rangle} \lesssim \min\Big\{\frac{1}{\langle x\rangle\langle y\rangle\langle |x|\pm|y|\rangle^2}, \frac{1}{\langle |x|\pm|y|\rangle^4}\Big\},$$

if $||x|\pm|y||\le 1$. On the other hand, if $||x|\pm|y||\ge 1$, then it is clear from (4.12) again that

$$|G_{11}^{\pm}(x,y)| \lesssim \min\Big\{\frac{1}{\langle x\rangle\langle y\rangle\langle |x|\pm|y|\rangle^2}, \frac{1}{\langle |x|\pm|y|\rangle^4}\Big\}.$$

Thus, we obtain the desired estimate (4.7).

Case (ii). Let $\alpha=0$ or $\beta=0$. As in case (i), it similarly follows from (3.2) and Lemma 3.4 that

$$|K_{\alpha\beta}^{\pm}(x,y)| \lesssim \int_{\mathbb{R}^6\times[0,1]^2} \big(|u_1|^{\alpha}|(vQ_{\alpha}BQ_{\beta}v)(u_1,u_2)||u_2|^{\beta}\big) \times |G_{\alpha\beta}^{\pm}(x-\theta_1u_1, y-\theta_2u_2)|\, du\, d\theta,$$

where $(\alpha,\beta)=(1,0),(0,1),(0,0)$, and

$$G_{\alpha\beta}^{\pm}(X,Y)=\int_0^{\infty}\lambda^{5-\alpha-\beta}\chi(\lambda)F_{+}^{(\alpha)}(\lambda|X|)F_{\pm}^{(\beta)}(\lambda|Y|)\,d\lambda, \quad X,Y\in\mathbb{R}^3.$$

Then, by using the same arguments as above, we can obtain the same estimate (4.7) as for $G^{\pm}_{\alpha\beta}$ and then the same L^p-boundedness of $T_{K^{\pm}_{\alpha\beta}}$ for all $1 \le p \le \infty$. Hence, this completes the proof of Proposition 4.2. ∎

Next, we consider the operator T_{K_3} in the class (III).

Proposition 4.3. *The operator T_{K_3} satisfies the same statement as that in Proposition 4.2.*

Proof. We show that $|K_3(x, y)| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| - |y|\rangle^{-2-\delta}$ for some $\delta > 0$, which, together with Lemmas 2.1 and Proposition 2.2, implies the desired assertion. The proof is more involved than in the previous case since $\Gamma_3(\lambda)$ depends on λ.

As before, based on the free resolvent formula 3.2, we can write that

$$\begin{aligned}
K_3(x, y) &= \int_0^\infty \lambda^3 \chi(\lambda)\big(R_0^+(\lambda^4) v\Gamma_3(\lambda)v(R_0^+ - R_0^-)(\lambda^4)\big)(x, y)\, d\lambda,\\
&= \int_0^\infty \lambda^4 \chi(\lambda)\Big(\int_{\mathbb{R}^6} F_+(\lambda|x - u_1|)\widetilde{\Gamma}(\lambda, u_1, u_2)\\
&\qquad\qquad \times (F_+ - F_-)(\lambda|y - u_2|)\, du_1\, du_2\Big)\, d\lambda,\\
&:= (K_3^+(x, y) - K_3^-(x, y)),
\end{aligned}$$

where we set

$$\widetilde{\Gamma}(\lambda, u_1, u_2) = \frac{1}{64\pi^2\lambda^3}(v\Gamma_3(\lambda)v)(u_1, u_2) \quad \text{for } \lambda > 0.$$

Let

$$\Phi^{\pm}(x, y, u_1, u_2) = (|x - u_1| - |x|) \pm (|y - u_2| - |y|).$$

Then

$$K_3^{\pm}(x, y) = \int_0^\infty e^{i\lambda(|x|\pm|y|)}\lambda^4\chi(\lambda)b^{\pm}(\lambda, x, y)d\lambda,$$

where

$$\begin{aligned}
&b^{\pm}(\lambda, x, y)\\
&\quad= \int_{\mathbb{R}^6} e^{i\lambda\Phi^{\pm}(x,y,u_1,u_2)} B_+(\lambda|x - u_1|)\widetilde{\Gamma}(\lambda, u_1, u_2)B_{\pm}(\lambda|y - u_2|)du_1du_2,
\end{aligned}$$

and

$$B_{\pm}(s) = e^{\mp is}F_{\pm}(s) = s^{-1}(1 - e^{(-1\mp i)s}).$$

Firstly, using Leibniz formula, (3.6), Lemma 3.6, and Lemma 3.7, it follows that

$$\begin{aligned}
&|\partial_\lambda^\ell b^\pm(\lambda,x,y)| \\
&\lesssim \lambda^{-\ell-2}\Big(\int_{\mathbb{R}^3}\frac{\langle u_1\rangle^{2\ell}|V|(u_1)}{|x-u_1|^2}\,du_1\Big)^{\frac12}\Big(\int_{\mathbb{R}^3}\frac{\langle u_2\rangle^{2\ell}|V|(u_2)}{|y-u_2|^2}\,du_2\Big)^{\frac12} \\
&\lesssim \lambda^{-\ell-2}\,\langle x\rangle^{-1}\langle y\rangle^{-1},
\end{aligned}\tag{4.13}$$

for $0<\lambda\lesssim 1$, $x,y\in\mathbb{R}^3$ and $\ell=0,1,2,3$. Next, to deal with $K_3^\pm$, we decompose χ, by using the dyadic partition of unity $\{\varphi_N\}$ defined in Section 1.4, as

$$\chi(\lambda)=\sum_{N=-\infty}^{N_0}\tilde\chi_N(\lambda),\quad \tilde\chi_N(\lambda):=\chi(\lambda)\varphi_N(\lambda),\quad \lambda>0,$$

where $N_0\lesssim\log\lambda_0\lesssim -1$, $\operatorname{supp}\tilde\chi_N\subset[2^{N-2},2^N]$, and $|\partial_\lambda^\ell\tilde\chi_N(\lambda)|\le C_\ell 2^{-N\ell}$ for all $\ell\in\mathbb{N}_0$. Let $K_{3,N}^\pm$ be given by $K_3^\pm$ with χ replaced by $\tilde\chi_N$ and decompose $K_3^\pm$ as

$$K_3^\pm=\sum_{N\le N_0}K_{3,N}^\pm.$$

Since $\lambda\sim 2^N$ on $\operatorname{supp}\tilde\chi_N$, we know by (4.13) that

$$|K_{3,N}^\pm(x,y)|\lesssim 2^{2N}\langle x\rangle^{-1}\langle y\rangle^{-1}\int_{\operatorname{supp}\tilde\chi_N}d\lambda\lesssim 2^{3N}\langle x\rangle^{-1}\langle y\rangle^{-1},\quad x,y\in\mathbb{R}^3.$$

In particular, if $||x|\pm|y||\le 1$, then

$$|K_{3,N}^\pm(x,y)|\lesssim 2^{3N}\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|\pm|y|\rangle^{-3}.$$

On the other hand, if $||x|\pm|y||>1$, then we obtain by integrating by parts that

$$K_{3,N}^\pm(x,y)=\frac{i}{(|x|\pm|y|)^3}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\partial_\lambda^3\big(\lambda^4\tilde\chi_N(\lambda)b^\pm(\lambda,x,y)\big)\,d\lambda.$$

Then (4.10), (4.13), and the support property of $\tilde\chi_N$ imply

$$\begin{aligned}
|K_{3,N}^\pm(x,y)|&\lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|\pm|y|\rangle^{-3}2^{-N}\int_{2^{N-2}}^{2^N}d\lambda\\
&\lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|\pm|y|\rangle^{-3},
\end{aligned}$$

as $||x|\pm|y||>1$. Therefore, $K_{3,N}^\pm(x,y)$ satisfies

$$\begin{aligned}
|K_{3,N}^\pm(x,y)|&\lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\min\{2^{3N},\langle |x|\pm|y|\rangle^{-3}\}\\
&\lesssim 2^{3N(1-\theta)}\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|\pm|y|\rangle^{-3\theta},\quad \theta\in[0,1].
\end{aligned}$$

In particular, for instance, taking $\theta = \frac{5}{6}$, then we obtain

$$|K_3^{\pm}(x,y)| \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| \pm |y|\rangle^{-5/2} \sum_{N \le N_0} 2^{N/2} \lesssim \langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x| \pm |y|\rangle^{-5/2}.$$

Therefore, the desired result follows by Lemma 2.1 and Proposition 2.2. ∎

Finally, we deal with the class (II), namely the operator T_{K_P}. First recall that

$$\widetilde{P} = \frac{8\pi}{(1+i)\|V\|_{L^1}} P, \quad P = \frac{1}{\|V\|_{L^1}} \langle \cdot, v\rangle v.$$

Proposition 4.4. *Let* $1 < p < \infty$. *Then* $T_{K_P} \in \mathbb{B}(L^1(\mathbb{R}^3), L^{1,\infty}(\mathbb{R}^3)) \cap \mathbb{B}(L^p(\mathbb{R}^3))$.

Remark 4.5. It will be proved in Section 5 that $T_{K_P} \notin \mathbb{B}(L^\infty(\mathbb{R}^3)) \cup B(L^1(\mathbb{R}^3))$.

Proof of Proposition 4.4. By using (3.2) we first calculate that

$$\begin{aligned} K_P(x,y) &= \frac{8\pi}{(1+i)\|V\|_{L^1}} \int_0^\infty \lambda^4 \chi(\lambda)(R_0^+(\lambda^4) v P v (R_0^+ - R_0^-)(\lambda^4))(x,y)\, d\lambda \\ &= \frac{1}{8\pi(1+i)\|V\|_{L^1}} \int_0^\infty \lambda^2 \chi(\lambda) \\ &\quad \times \Big(\int_{\mathbb{R}^6} F_+(\lambda|x-u_1|)(vPv)(u_1,u_2) \\ &\qquad \times (F_+ - F_-)(\lambda|y-u_2|)\, du_1\, du_2 \Big) d\lambda, \end{aligned}$$

where $F_\pm(s) = s^{-1}(e^{\pm is} - e^{-s})$ and $(F_+ - F_-)(s) = s^{-1}(e^{is} - e^{-is})$.

Note that

$$(vPv)(u_1,u_2) = \frac{v^2(u_1)v^2(u_2)}{\|V\|_{L^1}}, \quad (u_1,u_2) \in \mathbb{R}^6.$$

Hence, we can rewrite $K_P(x,y)$ as

$$\begin{aligned} &K_P(x,y) \\ &= \frac{1}{8\pi(1+i)\|V\|_{L^1}^2} \int_0^\infty \chi(\lambda) \Big(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} (e^{i\lambda|x-u_1|} - e^{-\lambda|x-u_1|}) \\ &\qquad \times (e^{i\lambda|y-u_2|} - e^{-i\lambda|y-u_2|})\, du_1\, du_2 \Big) d\lambda. \end{aligned} \tag{4.14}$$

Let $z = x - u_1$ and $w = y - u_2$. Then

$$
\begin{aligned}
&(e^{i\lambda|z|} - e^{-\lambda|z|})(e^{i\lambda|w|} - e^{-i\lambda|w|}) \\
&\quad = e^{i\lambda(|z|+|w|)} - e^{i\lambda(|z|-|w|)} - e^{-\lambda(|z|-i|w|)} + e^{-\lambda(|z|+i|w|)}.
\end{aligned}
$$

So we can decompose $K_P(x,y)$ as follows:

$$
K_P(x,y) = \frac{1}{8\pi(1+i)\|V\|_{L^1}^2}\big(K_P^1(x,y) - K_P^2(x,y) - K_P^3(x,y) + K_P^4(x,y)\big), \tag{4.15}
$$

where

$$
K_P^1(x,y) = \int_0^\infty \chi(\lambda)\bigg(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} e^{i\lambda(|x-u_1|+|y-u_2|)}\, du_1\, du_2\bigg)\, d\lambda,
$$

$$
K_P^2(x,y) = \int_0^\infty \chi(\lambda)\bigg(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} e^{i\lambda(|x-u_1|-|y-u_2|)}\, du_1\, du_2\bigg)\, d\lambda,
$$

$$
K_P^3(x,y) = \int_0^\infty \chi(\lambda)\bigg(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} e^{-\lambda(|x-u_1|-i|y-u_2|)}\, du_1\, du_2\bigg)\, d\lambda,
$$

$$
K_P^4(x,y) = \int_0^\infty \chi(\lambda)\bigg(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} e^{-\lambda(|x-u_1|+i|y-u_2|)}\, du_1\, du_2\bigg)\, d\lambda.
$$

In the following, we will estimate these kernels $K_P^j(x,y)$ ($j = 1,2,3,4$) case by case. We only deal with the $K_P^1(x,y)$ due to similarity. For this end, let

$$
\psi_1(\lambda,x,y) := \int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|} e^{i\lambda((|x-u_1|-|x|)+(|y-u_2|-|y|))} du_1 du_2. \tag{4.16}
$$

Then, we obtain

$$
K_P^1(x,y) = \int_0^\infty e^{i\lambda(|x|+|y|)}\chi(\lambda)\psi_1(\lambda,x,y)d\lambda.
$$

By integration by parts, it follows that

$$
\begin{aligned}
&K_P^1(x,y)\\
&= \frac{1}{i(|x|+|y|)}\Big(-\psi_1(0,x,y) - \int_0^\infty e^{i\lambda(|x|+|y|)}\partial_\lambda(\chi\psi_1)\Big)\,d\lambda\\
&= -\frac{\psi_1(0,x,y)}{i(|x|+|y|)} - \frac{\partial_\lambda\psi_1(0,x,y)}{(|x|+|y|)^2} - \frac{1}{(|x|+|y|)^2}\int_0^\infty e^{i\lambda(|x|+|y|)}\partial_\lambda^2(\chi\psi_1)d\lambda.
\end{aligned}
\tag{4.17}
$$

By using (4.16), Lemma 3.7 and the decay condition of potential V, we have

$$
\begin{aligned}
&|\psi_1(\lambda,x,y)| + |\partial_\lambda\psi_1(\lambda,x,y)| + \Big|\int_0^\infty e^{i\lambda(|x|+|y|)}\partial_\lambda^2(\chi\psi_1)\,d\lambda\Big|\\
&\lesssim \Big(\int_{\mathbb{R}^3}\frac{\langle u_1\rangle^2 v^2(u_1)}{|x-u_1|}\,du_1\Big)\Big(\int_{\mathbb{R}^3}\frac{\langle u_2\rangle^2 v^2(u_2)}{|y-u_2|}\,du_2\Big) \lesssim \frac{1}{\langle x\rangle\langle y\rangle}.
\end{aligned}
$$

Therefore, (4.17) implies that

$$
K_P^1(x,y) = \frac{i}{(|x|+|y|)}\Big(\int_{\mathbb{R}^6}\frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}du_1du_2\Big) + O\Big(\frac{1}{\langle x\rangle\langle y\rangle(|x|+|y|)^2}\Big),
$$

where we use $h(x,y) = O(g(x,y))$ to denote $|h(x,y)| \lesssim |g(x,y)|$. Similarly, we obtain that

$$
\begin{aligned}
K_P^2(x,y) &= \frac{i}{(|x|-|y|)}\Big(\int_{\mathbb{R}^6}\frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}\,du_1\,du_2\Big)\\
&\quad + O\Big(\frac{1}{\langle x\rangle\langle y\rangle(|x|-|y|)^2}\Big),\\
K_P^3(x,y) &= \frac{1}{|x|-i|y|}\Big(\int_{\mathbb{R}^6}\frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}\,du_1\,du_2\Big)\\
&\quad + O\Big(\frac{1}{\langle x\rangle\langle y\rangle(|x|+|y|)^2}\Big),\\
K_P^4(x,y) &= \frac{1}{|x|+i|y|}\Big(\int_{\mathbb{R}^6}\frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}\,du_1\,du_2\Big)\\
&\quad + O\Big(\frac{1}{\langle x\rangle\langle y\rangle(|x|+|y|)^2}\Big).
\end{aligned}
$$

Therefore, by (4.15) it follows that

$$K_P(x,y) = -\frac{1+i}{4\pi\|V\|_{L^1}^2}\Big(\int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}\,du_1\,du_2\Big)\frac{|x|^2|y|}{|x|^4-|y|^4} + O\Big(\frac{1}{\langle x\rangle\langle y\rangle(|x|-|y|)^2}\Big).$$

By (4.14) and Lemma 3.7, we also have

$$|K_P(x,y)| \lesssim \int_{\mathbb{R}^6} \frac{v^2(u_1)v^2(u_2)}{|x-u_1||y-u_2|}\,du_1\,du_2 \lesssim \frac{1}{\langle x\rangle\langle y\rangle} \quad \text{for all } (x,y)\in\mathbb{R}^3\times\mathbb{R}^3.$$

Hence, we can finally write $K_P(x,y)$ into the following form:

$$K_P(x,y) = -\frac{(1+i)}{4\pi}G(x)\Big(\frac{|x|\chi_{\{||x|-|y||\ge 1\}}}{|x|^4-|y|^4}\Big)G(y) + O\Big(\frac{1}{\langle x\rangle\langle y\rangle\langle |x|-|y|\rangle^2}\Big), \tag{4.18}$$

where

$$G(x) = \frac{|x|}{\|V\|_{L^1}}\Big(\int_{\mathbb{R}^3}\frac{|V|(u)}{|x-u|}\,du\Big).$$

Note that $|G(x)| \lesssim |x|\langle x\rangle^{-1} < \infty$ by Lemma 3.7. Then Propositions 2.2 and 2.4 imply that $T_{K_P}, T_{K_P}^* \in \mathbb{B}(L^1, L^{1,\infty}) \cap \mathbb{B}(L^p)$ for all $1 < p < \infty$. ∎

In one word, putting Propositions 4.2–4.4 all together, we have finished the proof of Theorem 4.1.

Remark 4.6. Although the expression of $K_P(x,y)$ in (4.18) is suitable to show the weak L^1-boundedness (i.e., $T_{K_P} \in \mathbb{B}(L^1, L^{1,\infty})$), however it is ineffective to disprove the L^1-L^1 and L^∞-L^∞ boundedness of T_{K_P}. This is because the second part of (4.18) just represents a kernel form satisfying weak L^1-estimate but lacks specificity. In Section 5 we will employ alternative formula for $K_P(x,y)$ to show $T_{K_P} \notin \mathbb{B}(L^\infty(\mathbb{R}^3)) \cup \mathbb{B}(L^1(\mathbb{R}^3))$ assuming that V has compact support.

5. The proof of Theorem 1.1

This section is devoted to showing Theorem 1.1. Throughout the section, we assume that $V \not\equiv 0$, $\operatorname{supp} V \subset B(0,R_0)$ for some $R_0 > 0$, zero is a regular point of H and H has no embedded eigenvalues in $(0,\infty)$, where $B(0,R) = \{x \in \mathbb{R}^3\ |x| \le R\}$.

Recall that $W_- = I - \frac{2}{\pi i}(W_-^L + W_-^H)$. Except for T_{K_P}, all the other terms in W_-^L in the right side of (4.2) and the high-energy part W_-^H are bounded on $L^p(\mathbb{R}^3)$ for all

$1 \le p \le \infty$ by Propositions 4.2 and 4.3, and [16, Proposition 4.1]. Theorem 1.1 thus follows from the following proposition.

Proposition 5.1. *Let $f_R = \chi_{B(0,R)}$. Then $\|T_{K_P} f_R\|_{L^\infty(\mathbb{R}^3)} \to \infty$ as $R \to \infty$, and $T_{K_P} f_1 \notin L^1(\mathbb{R}^3)$. As a consequence, T_{K_P} is neither bounded on $L^\infty(\mathbb{R}^3)$ nor on $L^1(\mathbb{R}^3)$.*

To prove Proposition 5.1, we begin with the following lemma which gives another expression of $K_P(x,y)$.

Lemma 5.2. *Let $K_P(x,y)$ be the kernel of the operator T_{K_p} defined in (4.2). Then*

$$K_P(x,y) = \mathbb{G}(x,y) + \mathbb{F}(x,y),$$

where

$$\mathbb{G}(x,y) = \frac{-1-i}{4\pi\|V\|_{L^1}^2} \int\limits_{\mathbb{R}^6} |V(u_1)V(u_2)| \frac{|x-u_1|\chi_{\{||x-u_1|-|y-u_2||\ge 1\}}}{|x-u_1|^4 - |y-u_2|^4}\, du_1\, du_2,$$

$$\mathbb{F}(x,y) = \frac{1}{8\pi(1+i)\|V\|_{L^1}^2} \int\limits_{\mathbb{R}^6} |V(u_1)V(u_2)| \Psi(x-u_1, y-u_2)\, du_1\, du_2,$$

and $\Psi(z,w)$ is an admissible kernel on $\mathbb{R}^3 \times \mathbb{R}^3$ such that T_Ψ is bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$. As a consequence, $T_{\mathbb{F}} \in \mathbb{B}(L^p(\mathbb{R}^3))$ for each $1 \le p \le \infty$.

Proof. Recall that $v = \sqrt{|V|}$. By (4.14), we can write

$$K_P(x,y) = \frac{1}{8\pi(1+i)\|V\|_{L^1}^2} \int\limits_{\mathbb{R}^6} |V(u_1)V(u_2)| \widetilde{K}_P(x-u_1, y-u_2)\, du_1\, du_2,$$

where

$$\widetilde{K}_P(z,w) = \int_0^\infty \chi(\lambda) \Big(\frac{e^{i\lambda|z|} - e^{-\lambda|z|}}{|z|}\Big)\Big(\frac{e^{i\lambda|w|} - e^{-i\lambda|w|}}{|w|}\Big)\, d\lambda. \tag{5.1}$$

We set

$$\Psi(z,w) := \widetilde{K}_P(z,w) + \frac{4i|z|\chi_{\{||z|-|w||\ge 1\}}}{|z|^4 - |w|^4},$$

so that $K_P(x,y) = \mathbb{G}(x,y) + \mathbb{F}(x,y)$ as expressed above. If $T_\Psi \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \le p \le \infty$, then Minkowski's integral inequality and the invariance of L^p-norm under the translation yield

$$\|T_{\mathbb{F}}\|_{L^p \to L^p} \le \frac{1}{8\sqrt{2}\pi} \|T_\Psi\|_{L^p \to L^p}.$$

By virtue of Schur's test, it thus suffices to show that Ψ is an admissible kernel on $\mathbb{R}^3 \times \mathbb{R}^3$, that is,

$$\sup_{z\in\mathbb{R}^3} \int_{\mathbb{R}^3} |\Psi(z,w)|\, dw + \sup_{w\in\mathbb{R}^3} \int_{\mathbb{R}^3} |\Psi(z,w)|\, dz < \infty. \tag{5.2}$$

To this end, we write $\Psi = \Psi_1 + \Psi_2$, where

$$\begin{aligned}
\Psi_1(z,w) &= \widetilde{K}_P(z,w)\chi_{\{||z|-|w||<1\}}, \\
\Psi_2(z,w) &= \Big(\widetilde{K}_P(z,w) + \frac{4i|z|}{|z|^4 - |w|^4}\Big)\chi_{\{||z|-|w||\ge 1\}}.
\end{aligned}$$

We first deal with Ψ_1. Since

$$|F_\pm(s)| = \Big|\frac{e^{\pm is} - e^{-s}}{s}\Big| \lesssim \min\Big\{1, \frac{1}{s}\Big\},$$

it follows from (5.1) that

$$\begin{aligned}
|\widetilde{K}_P(z,w)| &\lesssim \int_0^\infty \lambda^2 \chi(\lambda)|F_+(\lambda|z|)||(F_+ - F_-)(\lambda|w|)|\, d\lambda \\
&\lesssim \min\Big\{1, \frac{1}{|z|}, \frac{1}{|w|}, \frac{1}{|z||w|}\Big\}.
\end{aligned}$$

Using the bound $|\widetilde{K}_P(z,w)| \lesssim 1$, we obtain

$$\sup_{|z|\le 1} \int_{\mathbb{R}^3} |\Psi_1(z,w)|\, dw \lesssim \sup_{|z|\le 1} \int_{||z|-|w||<1} dw < \infty.$$

When $|z| \ge 1$, using the bound $|\widetilde{K}_P(z,w)| \lesssim |z|^{-1}|w|^{-1}$, we have

$$\begin{aligned}
\sup_{|z|\ge 1} \int_{\mathbb{R}^3} |\Psi_1(z,w)|\, dw &\lesssim \sup_{|z|\ge 1} \Big(\frac{1}{|z|} \int_{||z|-|w||<1} \frac{1}{|w|}\, dw\Big) \\
&\lesssim \sup_{|z|\ge 1} \Big(\frac{1}{|z|} \int_{|z|-1}^{|z|+1} r\, dr\Big) < \infty.
\end{aligned}$$

Thus,

$$\sup_{z\in\mathbb{R}^3} \int_{\mathbb{R}^3} |\Psi_1(z,w)|\, dw < \infty. \tag{5.3}$$

The same argument also shows

$$\sup_{w\in\mathbb{R}^3}\int_{\mathbb{R}^3}|\Psi_1(z,w)|\,dz<\infty. \tag{5.4}$$

To deal with Ψ_2, integrating by parts in (5.1) yields

$$\begin{aligned}\widetilde{K}_P(z,w)&=\frac{1}{|z||w|}\int_0^\infty(e^{i\lambda|z|}-e^{-\lambda|z|})(e^{i\lambda|w|}-e^{-i\lambda|w|})\chi(\lambda)d\lambda\\&=\frac{1}{|z||w|}\Big(-\frac{1}{i(|z|+|w|)}+\frac{1}{i(|z|-|w|)}+\frac{1}{|z|+i|w|}-\frac{1}{|z|-i|w|}\Big)\\&\quad+\frac{1}{|z||w|}\int_0^\infty\Big(-\frac{e^{i\lambda(|z|+|w|)}}{i(|z|+|w|)}+\frac{e^{i\lambda(|z|-|w|)}}{i(|z|-|w|)}\\&\qquad+\frac{e^{-\lambda(|z|+i|w|)}}{|z|+i|w|}-\frac{e^{-\lambda(|z|-i|w|)}}{|z|-i|w|}\Big)\chi'(\lambda)d\lambda.\end{aligned}$$

Since

$$\frac{1}{|z||w|}\Big(-\frac{1}{i(|z|+|w|)}+\frac{1}{i(|z|-|w|)}+\frac{1}{|z|+i|w|}-\frac{1}{|z|-i|w|}\Big)=\frac{-4i|z|}{|z|^4-|w|^4},$$

we find

$$\begin{aligned}\Psi_2(z,w)=\frac{\chi_{\{||z|-|w||\geq 1\}}}{|z||w|}\int_0^\infty\Big(-&\frac{e^{i\lambda(|z|+|w|)}}{i(|z|+|w|)}+\frac{e^{i\lambda(|z|-|w|)}}{i(|z|-|w|)}\\&+\frac{e^{-\lambda(|z|+i|w|)}}{|z|+i|w|}-\frac{e^{-\lambda(|z|-i|w|)}}{|z|-i|w|}\Big)\chi'(\lambda)\,d\lambda.\end{aligned} \tag{5.5}$$

Using this expression, we shall show that

$$|\Psi_2(z,w)|\leq C_N\begin{cases}|z|^{-1}|w|^{-1}\langle|z|-|w|\rangle^{-N} & \text{for all } (z,w)\in\operatorname{supp}\Psi_2,\\ \langle z\rangle^{-N} & \text{if } |w|\leq\frac{1}{2},\\ \langle w\rangle^{-N} & \text{if } |z|\leq\frac{1}{2},\end{cases} \tag{5.6}$$

which implies

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^3}\int_{\mathbb{R}^3} |\Psi_2(z,w)|\,dw \\
&\quad\le \sup_{|z|\le 1/2}\int_{\mathbb{R}^3} |\Psi_2(z,w)|dw + \sup_{|z|>1/2}\Big(\int_{|w|\le 1/2} + \int_{|w|>1/2}\Big)|\Psi_2(z,w)|\,dw \\
&\quad\lesssim \sup_{|z|\le 1/2}\int_{\mathbb{R}^3} \langle w\rangle^{-N} dw \\
&\qquad + \sup_{|z|>1/2}\Big(\int_{|w|\le 1/2} 1dw + \int_{\substack{|w|>1/2,\\ ||z|-|w||\ge 1}} |z|^{-1}|w|^{-1}\langle |z|-|w|\rangle^{-N}\,dw\Big) < \infty
\end{aligned}$$

and similarly

$$\sup_{w\in\mathbb{R}^3}\int_{\mathbb{R}^3} |\Psi_2(z,w)|dz < \infty.$$

These two bounds, (5.3), and (5.4) imply (5.2).

It remains to show (5.6). To prove the first estimate in (5.6), we observe that, since χ' is compactly supported in $(0,\infty)$, if we integrate by parts the integral in (5.5), then the boundary terms at $\lambda = 0,\infty$ vanish identically. Taking into account this fact and the bounds

$$||z|\pm|w|| \ge ||z|-|w|| \ge 1, \quad ||z|\pm i|w|| \ge ||z|-|w|| \ge 1, \quad \text{on supp}\,\Psi_2,$$

we make use of integration by parts N times to obtain

$$|\Psi_2(z,w)| \le C_N|z|^{-1}|w|^{-1}||z|-|w||^{-N} \le C_N|z|^{-1}|w|^{-1}\langle |z|-|w|\rangle^{-N}.$$

For the second estimate in (5.6), using the formula

$$\frac{e^{\lambda(a+b)}}{a+b} - \frac{e^{\lambda(a-b)}}{a-b} = be^{\lambda a}\Big(\frac{\lambda}{a+b}\frac{e^{\lambda b}-1}{\lambda b} - \frac{\lambda}{a-b}\frac{e^{-\lambda b}-1}{\lambda b} - \frac{2}{a^2-b^2}\Big)$$

with $(a,b) = (i|z|,-i|w|)$ or $(-|z|,i|w|)$, we rewrite the integrand of Ψ_2 as

$$\begin{aligned}
&\frac{\chi'(\lambda)}{|z||w|}\Big(\frac{e^{\lambda(i|z|-i|w|)}}{i|z|-i|w|} - \frac{e^{\lambda(i|z|+i|w|)}}{i|z|+i|w|} + \frac{e^{\lambda(-|z|+i|w|)}}{-|z|+i|w|} - \frac{e^{\lambda(-|z|-i|w|)}}{-|z|-i|w|}\Big) \\
&= \frac{e^{i\lambda|z|}}{|z|}\Big(\frac{\lambda\chi'(\lambda)}{i|z|-i|w|}\frac{e^{-i\lambda|w|}-1}{\lambda|w|} - \frac{\lambda\chi'(\lambda)}{i|z|+i|w|}\frac{e^{i\lambda|w|}-1}{\lambda|w|} - \frac{2i\chi'(\lambda)}{|z|^2-|w|^2}\Big) \\
&\quad + \frac{e^{-\lambda|z|}}{|z|}\Big(\frac{\lambda\chi'(\lambda)}{-|z|+i|w|}\frac{e^{i\lambda|w|}-1}{\lambda|w|} + \frac{\lambda\chi'(\lambda)}{|z|+i|w|}\frac{e^{-i\lambda|w|}-1}{\lambda|w|} - \frac{2i\chi'(\lambda)}{|z|^2+|w|^2}\Big).
\end{aligned}$$

Since for any $\ell \ge 0$ there exists C_ℓ such that for any $\lambda > 0$ and w with $|w| \le \frac{1}{2}$,

$$\Big|\partial_\lambda^\ell\Big(\frac{e^{\pm i\lambda|w|}-1}{\lambda|w|}\Big)\Big| \le C_\ell, \quad \ell = 0, 1, 2, \dots,$$

with the bound $|z| \ge \frac{1}{2}$ under the restrictions $||z| - |w|| \ge 1$ and $|w| \le \frac{1}{2}$ at hand, we obtain the second estimate in (5.6) by integrating by parts N times in (5.5). Changing the role of z and w, we also obtain the third estimate in (5.6) by the same argument. ■

Proof of Proposition 5.1. By Lemma 5.2, $T_{\mathbb{F}} \in \mathbb{B}(L^p(\mathbb{R}^3))$ for all $1 \le p \le \infty$. To disprove the L^1- and L^∞- boundedness of T_{K_P}, it thus is enough to prove $T_{\mathbb{G}} \notin \mathbb{B}(L^1(\mathbb{R}^3)) \cup \mathbb{B}(L^\infty(\mathbb{R}^3))$. Let

$$\Phi(u_1, u_2, x) = \int\limits_{|y| \le R} \frac{|x-u_1|\ \chi_{\{||x-u_1|-|y-u_2|| \ge 1\}}}{|x-u_1|^4 - |y-u_2|^4} dy$$

be such that

$$T_{\mathbb{G}} f_R(x) = \frac{-1-i}{4\pi\|V\|_{L^1}^2} \int\limits_{\mathbb{R}^6} |V(u_1)V(u_2)|\Phi(u_1, u_2, x) du_1 du_2. \tag{5.7}$$

(i) *The unboundedness of* $T_{\mathbb{G}}$ *on* L^∞. Suppose $R \ge 1$ and $R + 2R_0 + 1 \le |x| \le R + 2R_0 + 2$. We shall claim that

$$\Phi(u_1, u_2, x) \ge \frac{\pi}{2} \ln\Big(1 + \frac{R - R_0}{2R_0 + 1}\Big), \tag{5.8}$$

uniformly for $u_1, u_2 \in B(0, R_0)$ if R is large enough. If (5.8) holds, then by (5.7) we obtain

$$\begin{aligned} |T_{\mathbb{G}} f_R(x)| &= \frac{1}{2\sqrt{2}\pi\|V\|_{L^1}^2} \int |V(u_1)V(u_2)|\Phi(u_1, u_2, x)\, du_1\, du_2 \\ &\ge \frac{1}{4\sqrt{2}} \ln\Big(1 + \frac{R - R_0}{2R_0 + 1}\Big). \end{aligned}$$

This implies that $\|T_{\mathbb{G}} f_R\|_{L^\infty} \to \infty$ as $R \to \infty$ and thus $T_{\mathbb{G}} \notin \mathbb{B}(L^\infty(\mathbb{R}^3))$ since $\|f_R\|_{L^\infty} = 1$.

To prove (5.8), we let $|u_1| \le R_0$, $|u_2| \le R_0$ and set $z = y - u_2$ so that

$$\Phi(u_1, u_2, x) = \int\limits_{|z+u_2| \le R} \frac{|x-u_1|\ \chi_{\{||x-u_1|-|z|| \ge 1\}}}{|x-u_1|^4 - |z|^4}\, dz.$$

Since $|x-u_1| \geq |x|-|u_1| \geq R+R_0+1 \geq |z|+1$ if $|z+u_2| \leq R$, we have $\chi_{\{||x-u_1|-|z|| \geq 1\}}=1$ and

$$\begin{aligned}
\Phi(u_1,u_2,x) &= \int\limits_{|z+u_2|\leq R} \frac{|x-u_1|}{|x-u_1|^4-|z|^4}\,dz \\
&\geq \int\limits_{|z|\leq R-R_0} \frac{|x-u_1|}{|x-u_1|^4-|z|^4}\,dz \\
&= 4\pi \int_0^{R-R_0} \frac{|x-u_1|r^2}{|x-u_1|^4-r^4}\,dr \\
&= 2\pi \int_0^{R-R_0} |x-u_1|\Big(\frac{1}{|x-u_1|^2-r^2}-\frac{1}{|x-u_1|^2+r^2}\Big)\,dr \\
&\geq 2\pi\Big(\int_0^{\frac{R-R_0}{|x-u_1|}} \frac{1}{1-s^2}\,ds\Big) - 2\pi\Big(\int_{\mathbb{R}} \frac{1}{1+s^2}\,ds\Big) \\
&= \pi \ln\Big(1+\frac{2(R-R_0)}{|x-u_1|-R+R_0}\Big) - 2\pi^2.
\end{aligned} \tag{5.9}$$

Since $|x-u_1|-R+R_0 \leq 4R_0+2$, by the monotonicity of $\ln\big(1+\frac{1}{x}\big)$, (5.9) hence implies (5.8) for sufficiently large R.

(ii) *The unboundedness of* $T_{\mathbb{G}}$ *on* L^1. Let $|x| \geq 3R_0+2$, $|u_1| \leq R_0$, $|u_2| \leq R_0$ and $|y| \leq 1$. Then

$$|x-u_1| \geq |x|-|u_1| \geq 2R_0+2 \geq 2|y-u_2|,$$

which implies

$$|x-u_1|-|y-u_2| \geq \frac{1}{2}|x-u_1| \geq R_0+1 \geq 1.$$

Hence, when $|x| \geq 3R_0+2$,

$$\begin{aligned}
|T_{\mathbb{G}} f_1(x)| = \frac{1}{2\sqrt{2}\pi\|V\|_{L^1}^2} \int\limits_{\mathbb{R}^6} |V(u_1)V(u_2)| \\
\times \Big(\int\limits_{|y|\leq 1} \frac{|x-u_1|}{|x-u_1|^4-|y-u_2|^4}\,dy\Big)\,du_1\,du_2
\end{aligned}$$

Since $|x-u_1|^4 - |y-u_2|^4 \le |x-u_1|^4$ and $|x-u_1| \le |x| + |u_1| \le \frac{4}{3}|x|$, this shows

$$\begin{aligned}
&\int_{|x|\le R} |T_{\mathbb{G}} f_1|\,dx \\
&\quad \gtrsim \frac{1}{\|V\|_{L^1}^2} \int_{\mathbb{R}^6} |V(u_1)V(u_2)| \\
&\qquad\qquad \times \Big(\int_{3R_0+2\le|x|\le R} \int_{|y|\le 1} \frac{|x-u_1|}{|x-u_1|^4 - |y-u_2|^4}\,dy\,dx \Big)\,du_1\,du_2 \\
&\quad \gtrsim \frac{1}{\|V\|_{L^1}^2} \int_{\mathbb{R}^6} \int_{3R_0+2\le|x|\le R} \frac{|V(u_1)V(u_2)|}{|x-u_1|^3}\,dx\,du_1\,du_2 \\
&\quad \gtrsim \int_{3R_0+2\le|x|\le R} \frac{1}{|x|^3} dx \gtrsim \ln\Big(\frac{R}{3R_0+2}\Big) \to \infty
\end{aligned}$$

as $R \to \infty$. Therefore, $T_{\mathbb{G}} f_1 \notin L^1(\mathbb{R}^3)$ and $T_{\mathbb{G}} \notin \mathbb{B}(L^1(\mathbb{R}^3))$ since $f_1 \in L^1(\mathbb{R}^3)$. ∎

A. Proof of Lemma 3.2

We prove Lemma 3.2 on the expansion of $M^{-1}(\lambda)$ near $\lambda = 0$ in regular case. Before the proof, we list the following lemma used in the proof.

Lemma A.1 ([19, Lemma 2.1]). *Let A be a closed operator and S be a projection. Suppose $A+S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$a := S - S(A+S)^{-1}S$$

has a bounded inverse in SH, and in this case

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1} S a^{-1} S (A+S)^{-1}.$$

Proof of Lemma 3.2. Firstly, we expand $M(\lambda)$ as follows for small λ by Taylor expanding the exponentials in $F_+(\lambda|x-y|) = (\lambda|x-y|)^{-1}(e^{i(\lambda|x-y|)} - e^{-(\lambda|x-y|)})$:

$$M(\lambda) = U + vR_0^+(\lambda^4)v = \frac{a}{\lambda}P + T + a_1\lambda v G_1 v + O(\lambda^3 v(x)|x-y|^4 v(y)),$$

where

$$T = U + vG_0 v, \qquad G_0 = -\frac{|x-y|}{8\pi}, \quad G_1(x,y) = |x-y|^2,$$
$$a = \frac{1+i}{8\pi}\|V\|_{L^1}, \quad a_1 = \frac{1-i}{8\pi\cdot 3!}, \qquad v = \sqrt{|V|},$$

and where $O(\lambda^3 v(x)|x-y|^4 v(y))$ denotes a λ-dependent absolutely bounded operator whose kernel is dominated by $C\lambda^3 v(x)|x-y|^4 v(y)$ for some $C > 0$. Next, we are devoted to obtaining (3.5). Write

$$M(\lambda) = \frac{a}{\lambda}\Big(P + \frac{\lambda}{a}T + \frac{a_1}{a}\lambda^2 vG_1v + O\big(\lambda^4 v(x)|x-y|^4 v(y)\big)\Big) := \frac{a}{\lambda}\widetilde{M}(\lambda).$$

Clearly, it suffices to establish the inverse of $\widetilde{M}(\lambda)$ in order to obtain $M^{-1}(\lambda)$ for small λ. For convenience, in the following, we also use $O(\lambda^k)$ as an absolutely bounded operator on $L^2(\mathbb{R}^3)$, whose bound is dominated by $C\lambda^k$.

Note that by Neumann series expansion, the operator $\widetilde{M}(\lambda) + Q$ is inverse for λ sufficiently small, and its inverse operator is given by

$$(\widetilde{M}(\lambda) + Q)^{-1} = I - \sum_{k=1}^{3} \lambda^k B_k + O(\lambda^4),$$

where $B_k (1 \le k \le 3)$ are absolutely bounded operators in $L^2(\mathbb{R}^3)$ as follows:

$$B_1 = \frac{1}{a}T, \quad B_2 = \frac{a_1}{a}vG_1v - \frac{1}{a^2}T^2, \quad B_3 = -\frac{a_1}{a^2}(TvG_1v + vG_1vT) + \frac{1}{a^3}T^3.$$

Let

$$\begin{aligned} M_1(\lambda) &:= Q - Q(\widetilde{M}(\lambda) + Q)^{-1}Q \\ &= \frac{\lambda}{a}\big(QTQ + a\lambda QB_2Q + a\lambda^2 QB_3Q + O(\lambda^3)\big) := \frac{\lambda}{a}\widetilde{M}_1(\lambda). \end{aligned}$$

Since zero is a regular point of H, i.e., QTQ is invertible on $QL^2(\mathbb{R}^3)$, then $\widetilde{M}_1(\lambda)$ is invertible on $QL^2(\mathbb{R}^3)$. By Neumann series expansion, as λ sufficiently small, one has on $QL^2(\mathbb{R}^3)$:

$$\begin{aligned} M_1^{-1}(\lambda) &= \frac{a}{\lambda}\widetilde{M}_1^{-1}(\lambda) \\ &= \frac{a}{\lambda}D_0 - a^2 D_0B_2D_0 + \lambda\big(a^3 D_0(B_2D_0)^2 - a^2 D_0B_3D_0\big) + O(\lambda^2), \end{aligned}$$

where $D_0 := (QTQ)^{-1}$. Thus, according to Lemma A.1, the inverse operator $\widetilde{M}^{-1}(\lambda)$ exists for sufficiently small λ, and

$$\widetilde{M}^{-1}(\lambda) = (\widetilde{M}(\lambda) + Q)^{-1} + (\widetilde{M}(\lambda) + Q)^{-1} QM_1^{-1}(\lambda)Q(\widetilde{M}(\lambda) + Q)^{-1}.$$

Hence, we finally obtain as sufficiently small λ,

$$
\begin{aligned}
M^{-1}(\lambda) = \frac{\lambda}{a}\widetilde{M}^{-1}(\lambda) &= D_0 \\
&+ \lambda\Big(\frac{1}{a}Q - \frac{1}{a}D_0T - \frac{1}{a}TD_0 + \frac{1}{a}D_0T^2D_0 - a_1D_0vG_1vD_0\Big) \\
&+ \frac{1}{a}\lambda P + \lambda^2 A_2 + O(\lambda^3) \\
&:= QA_0Q + \lambda(QA_{1,0} + A_{0,1}Q) + \lambda\tilde{P} + \lambda^2 A_2 + \Gamma_3(\lambda),
\end{aligned}
$$

where A_0, $A_{1,0}$, $A_{0,1}$, and A_2 are absolutely bounded operators on $L^2(\mathbb{R}^3)$ independent of λ, and the error term $\Gamma_3(\lambda)$ satisfies the desired bounds (3.6). So we complete the whole proof. ∎

Acknowledgments. The authors would like to express their thanks to Professor Avy Soffer for his interests and insightful discussions about topics on higher-order operators. We also thank the anonymous referee for insightful and helpful comments which greatly improve the present version.

Funding. H. Mizutani is partially supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) #JP21K03325. Z. Wan and X. Yao are partially supported by NSFC grants No.11771165 and 12171182.

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **2** (1975), no. 2, 151–218 Zbl 0315.47007 MR 0397194

[2] G. Artbazar and K. Yajima, The L^p-continuity of wave operators for one dimensional Schrödinger operators. *J. Math. Sci. Univ. Tokyo* **7** (2000), no. 2, 221–240 Zbl 0976.34071 MR 1768465

[3] M. Beceanu and W. Schlag, Structure formulas for wave operators. *Amer. J. Math.* **142** (2020), no. 3, 751–807 Zbl 1445.35234 MR 4101331

[4] P. D'Ancona and L. Fanelli, L^p-boundedness of the wave operator for the one-dimensional Schrödinger operator. *Comm. Math. Phys.* **268** (2006), no. 2, 415–438 Zbl 1127.35053 MR 2259201

[5] M. B. Erdoğan, M. Goldberg, and W. R. Green, On the L^p boundedness of wave operators for two-dimensional Schrödinger operators with threshold obstructions. *J. Funct. Anal.* **274** (2018), no. 7, 2139–2161 Zbl 1516.35132 MR 3762098

[6] M. B. Erdoğan, M. Goldberg, and W. R. Green, Counterexamples to L^p boundedness of wave operators for classical and higher order Schrödinger operators. *J. Funct. Anal.* **285** (2023), no. 5, article no. 110008 Zbl 07694897 MR 4593124

[7] M. B. Erdoğan and W. R. Green, The L^p-continuity of wave operators for higher order Schrödinger operators. *Adv. Math.* **404** (2022), no. part B, article no. 108450 Zbl 07537713 MR 4418888

[8] M. B. Erdoğan and W. R. Green, A note on endpoint L^p-continuity of wave operators for classical and higher order Schrödinger operators. *J. Differential Equations* **355** (2023), 144–161 Zbl 1525.35084 MR 4542551

[9] M. B. Erdoğan, W. R. Green, and E. Toprak, On the fourth order Schrödinger equation in three dimensions: dispersive estimates and zero energy resonances. *J. Differential Equations* **271** (2021), 152–185 Zbl 1455.35212 MR 4151179

[10] H. Feng, A. Soffer, Z. Wu, and X. Yao, Decay estimates for higher-order elliptic operators. *Trans. Amer. Math. Soc.* **373** (2020), no. 4, 2805–2859 Zbl 1440.35054 MR 4069234

[11] H. Feng, A. Soffer, and X. Yao, Decay estimates and Strichartz estimates of fourth-order Schrödinger operator. *J. Funct. Anal.* **274** (2018), no. 2, 605–658 Zbl 1379.58013 MR 3724151

[12] D. Finco and K. Yajima, The L^p boundedness of wave operators for Schrödinger operators with threshold singularities. II. Even dimensional case. *J. Math. Sci. Univ. Tokyo* **13** (2006), no. 3, 277–346 Zbl 1142.35060 MR 2284406

[13] A. Galtbayar and K. Yajima, The L^p-boundedness of wave operators for fourth order Schrödinger operators on $\mathbb{R}^4$. *J. Spectr. Theory* **14** (2024), no. 1, 271–354 Zbl 07861442 MR 4741063

[14] M. Goldberg and W. R. Green, The L^p boundedness of wave operators for Schrödinger operators with threshold singularities. *Adv. Math.* **303** (2016), 360–389 Zbl 1351.35029 MR 3552529

[15] M. Goldberg and W. R. Green, On the L^p boundedness of wave operators for four-dimensional Schrödinger operators with a threshold eigenvalue. *Ann. Henri Poincaré* **18** (2017), no. 4, 1269–1288 Zbl 1364.81223 MR 3626303

[16] M. Goldberg and W. R. Green, On the L^p boundedness of the wave operators for fourth order Schrödinger operators. *Trans. Amer. Math. Soc.* **374** (2021), no. 6, 4075–4092 Zbl 07344659 MR 4251223

[17] M. Goldberg and M. Visan, A counterexample to dispersive estimates for Schrödinger operators in higher dimensions. *Comm. Math. Phys.* **266** (2006), no. 1, 211–238 Zbl 1110.35073 MR 2231971

[18] L. Grafakos, *Classical Fourier analysis*. Third edn., Grad. Texts in Math. 249, Springer, New York, 2014 Zbl 1304.42001 MR 3243734

[19] A. Jensen and G. Nenciu, A unified approach to resolvent expansions at thresholds. *Rev. Math. Phys.* **13** (2001), no. 6, 717–754 Zbl 1029.81067 MR 1841744

[20] A. Jensen and K. Yajima, A remark on L^p-boundedness of wave operators for two-dimensional Schrödinger operators. *Comm. Math. Phys.* **225** (2002), no. 3, 633–637 Zbl 1057.47011 MR 1888876

[21] A. Jensen and K. Yajima, On L^p boundedness of wave operators for 4-dimensional Schrödinger operators with threshold singularities. *Proc. Lond. Math. Soc. (3)* **96** (2008), no. 1, 136–162 Zbl 1182.35089 MR 2392318

[22] T. Kato, Growth properties of solutions of the reduced wave equation with a variable coefficient. *Comm. Pure Appl. Math.* **12** (1959), 403–425 Zbl 0091.09502 MR 0108633

[23] S. T. Kuroda, Scattering theory for differential operators. I. Operator theory. *J. Math. Soc. Japan* **25** (1973), 75–104 Zbl 0245.47006 MR 0326435

[24] H. Mizutani, Z. Wan, and X. Yao, L^p-boundedness of wave operators for fourth order Schrödinger operators with zero resonances on $\mathbb{R}^3$. [v1] 2023, [v2] 2024, arXiv:2311.06763

[25] H. Mizutani, Z. Wan, and X. Yao, L^p-boundedness of wave operators for bi-Schrödinger operators on the line. *Adv. Math.* **451** (2024), article no. 109806 MR 4761956

[26] M. Reed and B. Simon, *Methods of modern mathematical physics.* II. Fourier analysis, self-adjointness. Academic Press, New York and London, 1972 Zbl 0308.47002 MR 0493420

[27] M. Reed and B. Simon, *Methods of modern mathematical physics.* IV. Analysis of operators. Academic Press, New York and London, 1978 Zbl 0401.47001 MR 0493421

[28] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. Princeton Math. Ser. 43, Princeton University Press, Princeton, NJ, 1993 Zbl 0821.42001 MR 1232192

[29] R. Weder, The $W_{k,p}$-continuity of the Schrödinger wave operators on the line. *Comm. Math. Phys.* **208** (1999), no. 2, 507–520 Zbl 0945.34070 MR 1729096

[30] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan* **47** (1995), no. 3, 551–581 Zbl 0837.35039 MR 1331331

[31] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators. III. Even-dimensional cases $m \geq 4$. *J. Math. Sci. Univ. Tokyo* **2** (1995), no. 2, 311–346 Zbl 0841.47009 MR 1366561

[32] K. Yajima, L^p-boundedness of wave operators for two-dimensional Schrödinger operators. *Comm. Math. Phys.* **208** (1999), no. 1, 125–152 Zbl 0961.47004 MR 1729881

[33] K. Yajima, The L^p boundedness of wave operators for Schrödinger operators with threshold singularities. I. The odd dimensional case. *J. Math. Sci. Univ. Tokyo* **13** (2006), no. 1, 43–93 Zbl 1115.35094 MR 2223681

[34] K. Yajima, Remarks on L^p-boundedness of wave operators for Schrödinger operators with threshold singularities. *Doc. Math.* **21** (2016), 391–443 Zbl 1339.35203 MR 3505130

[35] K. Yajima, L^1 and L^∞-boundedness of wave operators for three-dimensional Schrödinger operators with threshold singularities. *Tokyo J. Math.* **41** (2018), no. 2, 385–406 Zbl 1414.35022 MR 3908801

[36] K. Yajima, The L^p-boundedness of wave operators for four-dimensional Schrödinger operators. In *The physics and mathematics of Elliott Lieb—the 90th anniversary*. Vol. II, pp. 517–563, EMS Press, Berlin, 2022 Zbl 1500.81034 MR 4531373

[37] K. Yajima, The L^p-boundedness of wave operators for two dimensional Schrödinger operators with threshold singularities. *J. Math. Soc. Japan* **74** (2022), no. 4, 1169–1217 Zbl 1523.35084 MR 4499832

Received 7 January 2024; revised 20 July 2024.

Haruya Mizutani
Department of Mathematics, Graduate School of Science, Osaka University,
1-1 Machikaneyama-cho, Toyonaka-shi 560-0043, Japan; haruya@math.sci.osaka-u.ac.jp

Zijun Wan
Department of Mathematics, School of Mathematics and Statistics, Central China Normal University, 152 Luoyu Road, 430079 Wuhan, P. R. China; zijunwan@mails.ccnu.edu.cn

Xiaohua Yao (corresponding author)
School of Mathematics and Statistics, Key Laboratory of Nonlinear Analysis and Applications (Ministry of Education), Central China Normal University, 152 Luoyu Road, 430079 Wuhan, P. R. China; yaoxiaohua@ccnu.edu.cn