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Local ε-conjecture and p-adic differential equations

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Abstract. Laurent Berger attached a p-adic differential equation $\mathbf{N}_{\mathrm{rig}}(M)$ with a Frobenius structure to an arbitrary de Rham (φ, Γ)-module M over a Robba ring. In this article, we compare the local epsilon conjecture for the cyclotomic deformation of M with that of $\mathbf{N}_{\mathrm{rig}}(M)$. We first define an isomorphism between the fundamental lines of their cyclotomic deformations using the second author's results on the big exponential map. As a main result of the article, we show that this isomorphism enables us to reduce the local epsilon conjecture for the cyclotomic deformation of M to that of $\mathbf{N}_{\mathrm{rig}}(M)$. The result can be regarded as a refined interpolation formula of the big exponential map.

1. Introduction and notations

In [18], Kato formulated a conjecture called the generalized Iwasawa main conjecture, which is a vast generalization of the Iwasawa main conjecture and Bloch–Kato conjecture. It claims the existence of so-called zeta isomorphisms for any family of p-adic Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, interpolating the zeta elements of geometric p-adic Galois representations. Note that a similar conjecture was formulated by Fontaine and Perrin-Riou in [13]. Since the zeta elements are conjectural bases in (the determinants of) the Galois cohomologies and closely related to the L-functions, it is natural to regard the zeta isomorphisms as algebraic counterparts of the L-functions. In [14, 17], Kato's local and global ε-conjectures are formulated as algebraic analogue of the functional equations of L-functions; the local ε-conjecture claims the existence of the local ε-isomorphisms, the algebraic analogue of local ε-factors for families of p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_l/\mathbf{Q}_l)$, and the global ε-conjecture states that the zeta isomorphisms satisfies the functional equations whose local factors are the local ε-isomorphisms.

The local ε-conjecture for $l \neq p$ is proved [16, 31]. But for the case $l = p$, which we treat in this paper, the existence of the local ε-isomorphisms are proved for limited families and the conjecture is still open. In particular, by generalizing the conjecture for (φ, Γ)-modules over relative Robba rings, the second author proves the existence of ε-isomorphisms for trianguline representations. The conjecture has turned out to be closely related to the Coleman isomorphisms [17, 30], the Perrin-Riou maps [3, 23], and also the p-adic local Langlands correspondence [15, 26].

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Our main theorem compares the local ε-isomorphisms of the following different objects. Let M be an arbitrary de Rham (φ, Γ)-module over the Robba ring. The first object is the cyclotomic deformation of M. The second one is the cyclotomic deformation of $\mathbf{N}_{\mathrm{rig}}(M)$, where $\mathbf{N}_{\mathrm{rig}}(M)$ is the p-adic differential equation attached to M by Laurent Berger. We remark that the existences of their local ε-isomorphisms are still conjectural. The main theorem claims that the difference of their local ε-isomorphisms is written as the generalized Perrin-Riou map defined by the second author in [24].

To make the statement of the main theorem more precise, we recall (φ, Γ)-modules over Robba rings and the local ε-conjecture for them.

A (φ, Γ)-module M is a module equipped with a suitable endomorphism $\varphi : M \to M$ and a continuous group action of $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$, where μ_{p^∞} is the group of p-power roots of unity in $\bar{\mathbf{Q}}_p$. There are several specific rings over which (φ, Γ)-modules are useful to study p-adic representations. An important case is the Robba rings $\mathcal{R}_L$ with their coefficients in local fields L; by results of Fontaine [12], Cherbonnier and Colmez [8] and Kedlaya [19], the category of p-adic representations over L can be embedded fully and faithfully into the one of (φ, Γ)-modules over $\mathcal{R}_L$. A lot of important notions of p-adic Hodge theory can be generalized to (φ, Γ)-modules over $\mathcal{R}_L$, such as the functors $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{D}_{\mathrm{dR}}$ [4], or Bloch–Kato's exponential maps [5, 24]. Another important feature is that, when a (φ, Γ)-module M is de Rham, Berger attached to M a p-adic differential equation $\mathbf{N}_{\mathrm{rig}}(M)$ with Frobenius structure; as its application, one can prove the p-adic monodromy theorem for p-adic representations by reducing it to that for p-adic differential equations, or Colmez–Fontaine's theorem [4, 6].

In [25], the second author formulated the local ε-conjecture for (φ, Γ)-modules over relative Robba rings, generalizing the Kato's conjecture for p-adic representations. We recall only the conjecture for the cyclotomic deformations of de Rham (φ, Γ)-modules, since it is the case we treat in this paper. Let L be a finite extension of $\mathbf{Q}_p$, and M be a (φ, Γ)-module over the Robba ring $\mathcal{R}_L$ with coefficients in L. Then, one can attach to M a (graded) invertible module $\Delta_L(M)$ over L and $\Delta_L^{\mathrm{Iw}}(M)$ over $\mathcal{R}_L^+(\Gamma)$ for any (φ, Γ)-module M over $\mathcal{R}_L$, where $\mathcal{R}_L^+(\Gamma) = \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}})$ is the ring of global sections of the Berthelot generic fiber $\mathcal{W} = \mathrm{Spf}(\mathcal{O}_L[\![\Gamma]\!]^{\mathrm{an}})$ associated to the Iwasawa algebra $\mathcal{O}_L[\![\Gamma]\!]$. When M is de Rham, the second author constructed a canonical trivialization isomorphism

$$\varepsilon_L^{\mathrm{dR}}(M) : L \xrightarrow{\sim} \Delta_L(M).$$

Its definition involves a lot of notions of p-adic Hodge theory, such as the theory of local constants (ε-constants and L-constants), Bloch–Kato's exponential and dual exponential maps, Hodge–Tate weights. Then the local ε-conjecture in this situation claims that, there exists a unique isomorphism

$$\varepsilon_L^{\mathrm{Iw}}(M) : \mathcal{R}_L^+(\Gamma) \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M)$$

interpolating $\varepsilon_L^{\mathrm{dR}}(M(\delta))$ for any de Rham character $\delta : \Gamma \to L^\times$, i.e. any character of the form $\delta = \chi^k \tilde{\delta}$ for $k \in \mathbf{Z}$ and a finite character $\tilde{\delta}$, where χ is the cyclotomic character. More precisely, $\varepsilon_L^{\mathrm{Iw}}(M)$ is required to make the following diagram

$$\begin{tikzcd} \Delta_L^{\mathrm{Iw}}(M) \otimes_{f_\delta} L \arrow[r, "\mathrm{ev}_\delta"] & \Delta_L\big(M(\delta)\big) \\ \mathcal{R}_L^+(\Gamma) \otimes_{f_\delta} L \arrow[u, "\varepsilon_L^{\mathrm{Iw}}(M)\otimes\mathrm{id}"] \arrow[r, "\mathrm{can}"'] & L \arrow[u, "\varepsilon_L^{\mathrm{dR}}(M(\delta))"'] \end{tikzcd}$$

commute for any de Rham character δ of Γ, where $f_\delta : \mathcal{R}_L^+(\Gamma) \to L$ is a continuous homomorphism of L-algebras given by $[g] \mapsto \delta(g)^{-1}$ and ev_δ is a canonical isomorphism induced by the specialization at f_δ. In the original article of Kato [17], he predicts the conjectural base $\varepsilon_{\mathcal{O}_L}^{\mathrm{Iw}}(T)$ of an invertible $\mathcal{O}_L[\![\Gamma]\!]$-module $\Delta_{\mathcal{O}_L}^{\mathrm{Iw}}(T)$ similarly defined for any $\mathcal{O}_L$-representation T of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. In [25, Conjecture 3.8 (v)], the second author predicts the equality

$$\varepsilon_{\mathcal{O}_L}^{\mathrm{Iw}}(T) \otimes_{\mathcal{O}_L[\![\Gamma]\!]} \mathrm{id}_{\mathcal{R}_L^+(\Gamma)} = \varepsilon_L^{\mathrm{Iw}}\big(\mathbf{D}_{\mathrm{rig}}^\dagger\big(T[1/p]\big)\big),$$

that is, the right hand side has an integral structure in the étale case.

The following is the main theorem of this paper, which can be regarded as an extension of the studies in [24, 25]. It roughly states that, for a general de Rham (φ, Γ)-module M over $\mathcal{R}_L$ and the p-adic differential equation $\mathbf{N}_{\mathrm{rig}}(M)$ attached to M, the differences of $\varepsilon_L(M(\delta))$ and $\varepsilon_L(\mathbf{N}_{\mathrm{rig}}(M)(\delta))$ for the de Rham characters δ of Γ are interpolated by the generalized Perrin-Riou map in [24].

Theorem. *There exists an $\mathcal{R}_L^+(\Gamma)$-linear isomorphism*

$$\mathrm{Exp}(M) : \Delta_L^{\mathrm{Iw}}\big(\mathbf{N}_{\mathrm{rig}}(M)\big) \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M)$$

whose specialization at any de Rham character δ of Γ makes the following diagram

$$\begin{tikzcd} & 1_L \arrow[dl, "\varepsilon_L^{\mathrm{dR}}(\mathbf{N}_{\mathrm{rig}}(M)(\delta))"'] \arrow[dr, "\varepsilon_L^{\mathrm{dR}}(M(\delta))"] & \\ \Delta_L\big(\mathbf{N}_{\mathrm{rig}}(M)(\delta)\big) \arrow[rr, "\mathrm{Exp}(M)_\delta"'] & & \Delta_L\big(M(\delta)\big) \end{tikzcd}$$

commute, where the isomorphism $\mathrm{Exp}(M)_\delta$ is defined by the following commutative diagram

$$\begin{tikzcd} \Delta_L\big(\mathbf{N}_{\mathrm{rig}}(M)(\delta)\big) \arrow[r, "\mathrm{Exp}(M)_\delta"] & \Delta_L\big(M(\delta)\big) \\ \Delta_L^{\mathrm{Iw}}\big(\mathbf{N}_{\mathrm{rig}}(M)\big) \otimes_{f_\delta} L \arrow[u, "\mathrm{ev}_\delta"] \arrow[r, "\mathrm{Exp}(M)\otimes\mathrm{id}"'] & \Delta_L^{\mathrm{Iw}}(M) \otimes_{f_\delta} L. \arrow[u, "\mathrm{ev}_\delta"'] \end{tikzcd}$$

In particular, if $\varepsilon_L^{\mathrm{Iw}}(\mathbf{N}_{\mathrm{rig}}(M))$ exists, then $\varepsilon_L^{\mathrm{Iw}}(M)$ also exists and is written as

$$\varepsilon_L^{\mathrm{Iw}}(M) = \mathrm{Exp}(M) \circ \varepsilon_L^{\mathrm{Iw}}\big(\mathbf{N}_{\mathrm{rig}}(M)\big).$$

We remark that our theorem can be regarded as a refined interpolation formula of the big exponential map for the Bloch–Kato's morphisms. The isomorphism $\mathrm{Exp}(M)$ is obtained by the generalized Perrin-Riou's big exponential map

$$\mathrm{Exp}_{M,h} : \mathrm{H}^1_{\psi}\big(\mathbf{N}_{\mathrm{rig}}(M)\big) \to \mathrm{H}^1_{\psi}(M)$$

defined in [24, Definition 3.7] for de Rham (φ, Γ)-modules M, in conjunction with [24, Theorem 3.14] on the determinant of $\mathrm{Exp}_{M,h}$, which we called the $\delta(M)$ theorem because it is a generalization of Perrin-Riou's famous conjecture $\delta(V)$ for crystalline representations V. The big exponential maps are first introduced by Perrin-Riou [27] for crystalline representations and used essentially in her study of p-adic L-functions, and then generalized to de Rham representations [9] and to de Rham (φ, Γ)-modules [24]. Their key feature is that they interpolate the Bloch–Kato's exponentials

$$\exp_{M(\chi^k\tilde{\delta})} : \mathrm{D}_{\mathrm{dR}}\big(M(\chi^k\tilde{\delta})\big)/\mathrm{Fil}^0\mathrm{D}_{\mathrm{dR}}\big(M(\chi^k\tilde{\delta})\big) \to \mathrm{H}^1_{\varphi,\gamma}\big(M(\chi^k\tilde{\delta})\big)$$

and it's dual, i.e. the dual exponentials $\exp^*_{M(\chi^k\tilde{\delta})^*}$ for suitable $k \in \mathbf{Z}$ (see, for example, [24, Theorem 3.10]). Our theorem can be seen as a refinement of such interpolation formulae. First, our big exponential map $\mathrm{Exp}(M)$ satisfies an interpolation formula interpolates for arbitrary twits $\delta = \chi^k\tilde{\delta}$, i.e. for all(!) $k \in \mathbf{Z}$. Second, the map $\mathrm{Exp}(M)$ interpolates not only the maps $\exp_{M(\delta)}$ and $\exp^*_{M(\delta)^*}$ but also another exponential map

$$\exp_{f,M(\delta)} : \mathbf{D}_{\mathrm{cris}}\big(M(\delta)\big) \to \mathrm{H}^1_{\varphi,\gamma}\big(M(\delta)\big)$$

defined in [7] for Galois representations and [25, Section 2] for (φ, Γ)-modules, which is closely related with the exceptional zeros for p-adic L-functions.

We also remark a relation of our theorem to the local ε-conjecture itself. The local ε-conjecture for the cyclotomic deformation of a general de Rham (φ, Γ)-module is not proved yet, and only the following special cases are proved.

- The case of rank 1 Galois representations (i.e. rank 1 étale (φ, Γ)-modules) is proved by Kato in [17] (proofs taking account of signs is given in [14] briefly and in [30] in detail).
- The case of crystalline representations is proved by Benois and Berger in [3] (the cyclotomic deformations of crystalline representations), which is generalized by Loeffler, Venjakob, and Zerbes in [23] (the abelian twists of crystalline representations), and by Bellovin and Venjakob in [1] (the crystalline families).
- The case of trianguline (φ, Γ)-modules over relative Robba rings, including the crystalline families and also the rigid analytic families of representations attached to eigenvarieties, is proved by the second author in [25].
- The case of rank 2 Galois representations is proved by the second author in [26] in almost all cases and completed by Rodrigues Jacinto [15], by showing its close relation to the p-adic local Langlands conjecture for $\mathrm{GL}_2(\mathbf{Q}_p)$.

By the last assertion of the theorem, we can reduce the local ε-conjecture for the cyclotomic deformation of arbitrary de Rham (φ, Γ)-module M to that of $\mathbf{N}_{\mathrm{rig}}(M)$. This reduction seems a useful approach, since $\mathbf{N}_{\mathrm{rig}}(M)$ is relatively simple (all of its Hodge–Tate weights are zero) and also has an additional structure of a p-adic differential equation with a Frobenius structure so that we can use the theory of p-adic differential equations. We also note that such a reduction is implicitly used to prove the trianguline case, and our theorem is stated as a conjecture [25, Remark 4.15]; see also Remark 4.2.2.

The structure of the paper is as follows. In Section 2, we recall definitions about (φ, Γ)-modules over the Robba ring and prove a key theorem (Theorem 2.2.5) on a relation of Bloch–Kato's morphisms and some differential operators induced by the Γ-action. In Section 3, we recall (a special case of) the local ε-conjecture for (φ, Γ)-modules studied in [25], and construct our big exponential map $\mathrm{Exp}(M) : \Delta_L^{\mathrm{Iw}}(\mathbf{N}_{\mathrm{rig}}(M)) \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M)$ for a de Rham (φ, Γ)-module M; it is induced by the multiplication of some differential operators, and the construction depends heavily on [24], in particular on theorem $\delta(D)$. In Section 4, we state our main theorem and prove it, by introducing a notion which we call genericity, deducing the proof of the general case to the generic case by using the explicit construction of the $\varepsilon^{\mathrm{Iw}}$-isomorphisms for (φ, Γ)-modules of rank 1 in [25], and proving the generic case by applying the key theorem.

Notation. Let p be a prime number. We fix the algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic number field $\mathbf{Q}_p$. Let L be a finite extension of $\mathbf{Q}_p$. Let μ_{p^∞} be the group of p-power roots of unity in $\overline{\mathbf{Q}}_p$. We fix primitive p^n-th roots of unity $\zeta_{p^n} \in \mu_{p^\infty}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for any $n \in \mathbf{Z}_{\geq 1}$. We set $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$. The cyclotomic character on Γ is denoted by $\chi : \Gamma \xrightarrow{\sim} \mathbf{Z}_p^\times$, which is characterized by $\gamma'(\zeta) = \zeta^{\chi(\gamma')}$ for all $\zeta \in \mu_{p^\infty}$ and $\gamma' \in \Gamma$. Let $\Delta \subseteq \Gamma$ be the torsion subgroup of Γ and put $p_\Delta = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \sigma \in \mathbf{Q}[\Gamma]$. We fix a topological generator γ of the torsion free subgroup Γ_{free} of Γ. For a ring R, the objects of the category of graded invertible R-modules are written as the pairs $(\mathcal{L}, r)$ of an invertible R-module $\mathcal{L}$ and a continuous function $r : \mathrm{Spec}(R) \to \mathbf{Z}$, and the product $\boxtimes$ is defined by $(\mathcal{L}_1, r_1) \boxtimes (\mathcal{L}_2, r_2) := (\mathcal{L}_1 \otimes_R \mathcal{L}_2, r_1 + r_2)$. We put $1_R := (R, 0)$.

2. Review of the theory of (φ, Γ)-modules over the Robba ring

In this section, we first recall the definition of (φ, Γ)-modules over the Robba ring, their cohomologies, and some notions of p-adic Hodge theory following [6,20]. Then, we study several kinds of morphisms defined by differential operators induced by the Γ-action. Theorem 2.2.5 is the key result, which describes a relation between such morphisms and Bloch–Kato's exponentials defined for (φ, Γ)-modules over Robba rings in [24].

2.1. (φ, Γ)-modules over Robba rings

For each integer $n \in \mathbf{Z}_{\geq 1}$, put

$$\mathcal{R}_L^{(n)} = \Big\{ \sum_{i \in \mathbf{Z}} a_i T^i : a_i \in L,\ \sum_{i \in \mathbf{Z}} a_i T^i \text{ is convergent on } |\zeta_{p^n} - 1| \leq |T| < 1 \Big\},$$

which is naturally an L-Banach algebra. We put $\mathcal{R}_L = \cup_{n\geq 1}\mathcal{R}_L^{(n)}$. We equip it with the locally convex inductive limit topology and we call it the Robba ring over L. We also define a subring $\mathcal{R}_L^+$ of $\mathcal{R}_L$ by

$$\mathcal{R}_L^+ = \left\{\sum_{i\geq 0} a_i T^i : a_i \in L, \sum_{i\geq 0} a_i T^i \text{ is convergent on } 0 \leq |T| < 1\right\}.$$

Put $t = \log(1+T) \in \mathcal{R}_L^+$. There are an operator $\varphi : \mathcal{R}_L \to \mathcal{R}_L$ and a group action $\Gamma \times \mathcal{R}_L \to \mathcal{R}_L$ of Γ on $\mathcal{R}_L$, both of which are continuous and linear over L satisfying

$$\varphi(T) = (1+T)^p - 1, \quad \gamma'(T) = (1+T)^{\chi(\gamma')} - 1$$

for any $\gamma' \in \Gamma$. The tuple $((1+T)^i)_{i=0,\ldots,p-1}$ is a basis of $\mathcal{R}_L$ over $\varphi(\mathcal{R}_L)$, and we can define a left inverse $\psi : \mathcal{R}_L \to \mathcal{R}_L$ of φ by

$$\psi\left(\sum_{i=0}^{p-1} \varphi(f_i)(1+T)^i\right) = f_0$$

for $f_i \in \mathcal{R}_L$. The map ψ turns out to be continuous and commutes with Γ.

For each $n \in \mathbf{Z}_{\geq 1}$, set $L_n = \mathbf{Q}_p(\zeta_{p^n}) \otimes_{\mathbf{Q}_p} L$. Then one has a continuous Γ-equivariant homomorphism

$$\iota_n : \mathcal{R}_L^{(n)} \to L_n[\![t]\!]$$

of L-algebras defined in [6, Section I.2] such that

$$\iota_n(T) = \zeta_{p^n} \exp\left(\frac{t}{p^n}\right) - 1,$$

which satisfies the following commutative diagram

$$\begin{tikzcd}
\mathcal{R}_L^{(n)} \arrow[r, "\iota_n"] \arrow[d, "\varphi"'] & L_n[\![t]\!] \arrow[d, "\mathrm{incl}"] \\
\mathcal{R}_L^{(n+1)} \arrow[r, "\iota_{n+1}"'] & L_{n+1}[\![t]\!].
\end{tikzcd}$$

Definition 2.1.1. A (φ,Γ)-module over $\mathcal{R}_L$ is a free $\mathcal{R}_L$-module M of finite rank equipped with a semilinear endomorphism $\varphi : M \to M$ over $\mathcal{R}_L$ satisfying $\varphi^* M = M$ and a continuous Γ-action commuting to φ.

The following lemma is [6, Theorem 1.3.3].

Lemma 2.1.2. *Let M be a (φ,Γ)-module over $\mathcal{R}_L$. Then, there exists an integer $n \geq 1$ such that there exists a unique Γ-stable $\mathcal{R}_L^{(m)}$-submodule $M^{(m)} \subseteq M$ for each $m \geq n$ such that for any $m \geq n$ we have $\mathcal{R}_L^{(m+1)} \otimes_{\mathcal{R}_L^{(m)},\varphi} M^{(m)} = M^{(m+1)}$ and $\mathcal{R}_L \otimes_{\mathcal{R}_L^{(m)}} M^{(m)} = M$.*

The smallest integer n satisfying the property in Lemma 2.1.2 is denoted as $n(M)$.

For a (φ, Γ)-module M over $\mathcal{R}_L$, one can define ψ-operator on M by $\psi(\varphi(x) \otimes f) = x \otimes \psi(f)$ for $x \in M$ and $f \in \mathcal{R}_L$, which turns out to be well-defined, continuous and L-linear.

For each $n \geq n(M)$, define

$$\mathbf{D}^+_{\mathrm{dif},n}(M) = M^{(n)} \otimes_{\iota_n, \mathcal{R}_L^{(n)}} L_n[\![t]\!].$$

We put $\iota_n : M^{(n)} \to \mathbf{D}^+_{\mathrm{dif},n}(M) : x \mapsto x \otimes 1$ and

$$\mathrm{can}_n : \mathbf{D}^+_{\mathrm{dif},n}(M) \to \mathbf{D}^+_{\mathrm{dif},n+1}(M) : f(t) \otimes x \mapsto f(t) \cdot \iota_{n+1}\big(\varphi(x)\big)$$

for $f(t) \in L_n[\![t]\!]$ and $x \in M^{(n)}$. We set

$$\mathbf{D}_{\mathrm{dif},n}(M) = \mathbf{D}^+_{\mathrm{dif},n}(M)[1/t], \quad \mathbf{D}^{(+)}_{\mathrm{dif}}(M) = \varinjlim_n \mathbf{D}^{(+)}_{\mathrm{dif},n}(M),$$

where the injective limit is taken over $(\mathrm{can}_n)_{n \geq n(M)}$. On these modules, we can define Γ-actions induced by the diagonal action on $M^{(n)}$ and $L_n[\![t]\!]$.

We next introduce briefly notions for (φ, Γ)-modules related to p-adic Hodge theory. We recall that, as in Notation, Δ is the torsion subgroup of Γ and γ is a fixed topological generator of the torsion free subgroup Γ_{free} of Γ. Let R be a topological ring, X a topological module over R. If X is equipped with a continuous R-linear Γ-action, we denote its Γ_0-fixed part by X^{Γ_0} for any subgroup Γ_0 of Γ. We define a complex of R-modules

$$C^\bullet_\gamma(X) := \big[X^\Delta \xrightarrow{\gamma - 1} X^\Delta\big]$$

concentrated in degree $[0, 1]$. If X is furthermore equipped with a continuous R-linear action of φ or ψ commuting with the Γ-action, then we put, again as a complex of R-modules,

$$C^\bullet_{*,\gamma}(X) := \big[X^\Delta \xrightarrow{(\gamma-1, *-1)} X^\Delta \oplus X^\Delta \xrightarrow{(*-1)\oplus(1-\gamma)} X^\Delta\big]$$

concentrated in degree $[0, 2]$ for $* = \varphi, \psi$, and

$$C^\bullet_\psi(X) := \big[X^\Delta \xrightarrow{\psi - 1} X^\Delta\big]$$

concentrated in degree $[1, 2]$. For each complex $C^\bullet_\square(X)$ above, its i-th cohomology group is denoted as $\mathrm{H}^i_\square$. For a (φ, Γ)-module M over $\mathcal{R}_L$ of rank r, we use the following special notations

$$\mathbf{D}_{\mathrm{cris}}(M) := \mathrm{H}^0_\gamma\big(M[1/t]\big) = M[1/t]^\Gamma, \quad \mathbf{D}_{\mathrm{dR}}(M) := \mathrm{H}^0_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) = \mathbf{D}_{\mathrm{dif}}(M)^\Gamma.$$

These spaces are of dimension $\leq r$ over L, and we say M is crystalline (resp. de Rham) if $\dim_L(\mathbf{D}_{\mathrm{cris}}(M)) = r$ (resp. $\dim_L(\mathbf{D}_{\mathrm{dR}}(M)) = r$). For $i \in \mathbf{Z}$, we also define $\mathbf{D}^i_{\mathrm{dR}}(M) = \mathbf{D}_{\mathrm{dR}}(M) \cap t^i \mathbf{D}^+_{\mathrm{dif}}(M)$ and $t(M) = \mathbf{D}_{\mathrm{dR}}(M)/\mathbf{D}^0_{\mathrm{dR}}(M)$. When M is de Rham, then we say that $h \in \mathbf{Z}$ is a Hodge–Tate weight of M if $\mathbf{D}^{-h}_{\mathrm{dR}}(M)/\mathbf{D}^{-h+1}_{\mathrm{dR}}(M) \neq 0$, and refer to its dimension as the multiplicity of h. We put h_M as the sum of the Hodge–Tate weights of M with multiplicity.

2.2. Morphisms induced by differential operators

Let $\mathcal{W}$ (resp. $\mathcal{W}_1$) be the Berthelot generic fiber of the formal scheme $\mathrm{Spf}(\mathcal{O}_L[\![\Gamma]\!])$ (resp. $\mathrm{Spf}(\mathcal{O}_L[\![\Gamma_{\mathrm{free}}]\!])$) associated to the Iwasawa algebra $\mathcal{O}_L[\![\Gamma]\!]$ (resp. $\mathcal{O}_L[\![\Gamma_{\mathrm{free}}]\!]$). The ring of its global sections are denoted by $\mathcal{R}_L^+(\Gamma) = \Gamma(\mathcal{W}, \mathcal{O})$ (resp. $\mathcal{R}_L^+(\Gamma_{\mathrm{free}}) = \Gamma(\mathcal{W}_1, \mathcal{O})$). They are equipped with a natural topological L-algebra structure. We remark that there is a natural isomorphism $\mathcal{R}_L^+(\Gamma) \xrightarrow{\sim} \mathcal{R}_L^+(\Gamma_{\mathrm{free}}) \otimes_L L[\Delta]$, and an isomorphism $\mathcal{R}_L^+(\Gamma_{\mathrm{free}}) \xrightarrow{\sim} \mathcal{R}_L^+$ which sends $[\gamma] - 1$ to T (recall that $\gamma \in \Gamma_{\mathrm{free}}$ is a fixed topological generator). Let $\mathcal{D}(\Gamma, L)$ be the distribution algebra of Γ with values in L, i.e. the convolution algebra of the continuous L-linear duals of the topological L-vector space $\mathrm{LA}(\Gamma, L)$ of locally analytic functions on Γ with values in L. Then, it is known that the L-algebra map $L[\Gamma] \to \mathcal{D}(\Gamma, L)$ sending $[\gamma_0]$ ($\gamma_0 \in \Gamma$) to the Dirac measure δ_{γ_0} at γ_0 uniquely extends to a topological L-algebra isomorphism $\mathcal{R}_L^+(\Gamma) \xrightarrow{\sim} \mathcal{D}(\Gamma, L)$. One has a canonical inclusion $\mathrm{Lie}(\Gamma)_L \to \mathcal{D}(\Gamma, L)$. Explicitly, the sub L-vector space $\mathrm{Lie}(\Gamma)_L$ is generated by the derivation

$$\left[\mathrm{LA}(\Gamma, L) \ni f(\gamma_0) \mapsto \lim_{\gamma_0 \to e_\Gamma} \frac{f(\gamma_0) - f(e_\Gamma)}{\chi(\gamma_0) - 1} \in L\right] \in \mathcal{D}(\Gamma, L).$$

We remark that, by the isomorphism $\mathrm{R}_L^+(\Gamma) \xrightarrow{\sim} \mathcal{D}(\Gamma, L)$, this element corresponds to the element

$$\nabla_0 := \lim_{\gamma_0 \to e_\Gamma} \frac{[\gamma_0] - 1}{\chi(\gamma_0) - 1} = \frac{\log\left([\gamma']\right)}{\log\left(\chi(\gamma')\right)} \in \mathcal{R}_L^+(\Gamma),$$

where $\gamma' \in \Gamma$ is any non-torsion element (see, e.g. [28] for more details) and whose action on (φ, Γ)-modules is crucial in the article.

In this subsection, we consider several morphisms induced by an element λ of $\mathcal{R}_L^+(\Gamma)$, and then prove a theorem about relationships between such morphisms and Bloch–Kato's morphisms. We will apply the theorem to the case $\lambda = \nabla_0 - i$ for $i \in \mathbf{Z}$ later to prove our main theorem.

We recall natural $\mathcal{R}_L^+(\Gamma)$-actions on several objects related to a (φ, Γ)-module M over $\mathcal{R}_L$. We can equip $M^{(n)}$, $M^{(n)}[1/t]$ with natural $\mathcal{R}_L^+(\Gamma)$-actions for each $n \geq n(M)$, which extend to M and $M[1/t]$. Also, for each $n \geq n(M)$, we can equip $\mathbf{D}^+_{\mathrm{dif},n}(M)$ with a natural $\mathcal{R}_L^+(\Gamma)$-action. In fact, for any $n \geq 1$ and a finitely generated $L_n[\![t]\!]$-module X with a semilinear and continuous Γ-action with respect to the canonical Frechét topology, we can equip X with a natural $\mathcal{R}_L^+(\Gamma)$-action as follows. Since one has $X = \varprojlim_n X/t^n X$ with the quotient $X/t^n X$ is a finite dimensional L-vector space with L-linear continuous Γ-action, it suffices to define a natural $\mathcal{R}_L^+(\Gamma)$-action on arbitrary finite dimensional L-vector space M with an L-linear continuous Γ-action. First, it is easy to see that Γ-action on M naturally extends to a continuous $\mathcal{O}_L[\![\Gamma]\!]$-action. Since M is finite dimensional L-vector space, the action of $\mathcal{O}_L[\![\Gamma]\!][1/p]$-factors through a quotient R_0 of $\mathcal{O}_L[\![\Gamma]\!][1/p]$ of finite length. Since the maximal ideals of $\mathcal{O}_L[\![\Gamma]\!][1/p]$ bijectively correspond to closed maximal ideals of $\mathcal{R}_L^+(\Gamma)$, R_0 is also a quotient of $\mathcal{R}_L^+(\Gamma)$, i.e. the natural quotient map $\mathcal{O}_L[\![\Gamma]\!][1/p] \to R_0$ factors through the inclusion $\mathcal{O}_L[\![\Gamma]\!][1/p] \hookrightarrow \mathcal{R}_L^+(\Gamma)$.

From now until the end of this section, we consider the following situation. Let M, M' be (φ, Γ)-modules over $\mathcal{R}_L$ such that $M[1/t] = M'[1/t]$. Note that, in particular, we have $\mathbf{D}_{\mathrm{dif},m}(M) = \mathbf{D}_{\mathrm{dif},m}(M')$, $\mathbf{D}_{\mathrm{dR}}(M) = \mathbf{D}_{\mathrm{dR}}(M')$ and $\mathbf{D}_{\mathrm{cris}}(M) = \mathbf{D}_{\mathrm{cris}}(M')$. Let $\lambda \in \mathcal{R}_L^+(\Gamma)$ be any element. We assume that, there exists some $n \geq \max\{n(M), n(M')\}$ such that we have

$$\lambda\big(\mathbf{D}^+_{\mathrm{dif},m}(M)\big) \subseteq \mathbf{D}^+_{\mathrm{dif},m}(M')$$

in $\mathbf{D}_{\mathrm{dif},m}(M) = \mathbf{D}_{\mathrm{dif},m}(M')$ for all $m \geq n$.

Proposition 2.2.1. *For any $m \geq n$, we have $\lambda(M^{(m)}) \subseteq (M')^{(m)}$. In particular, we have $\lambda(M) \subseteq M'$.*

Proof. By [6, Section II.1], the submodule $(M')^{(m)} \subseteq (M')^{(m)}[1/t] = M^{(m)}[1/t]$ can be written as

$$(M')^{(m)} = \big\{x \in t^{-h} M^{(m)} : \iota_{m'}(x) \in \mathbf{D}^+_{\mathrm{dif},m'}(M') \text{ for all } m' \geq m\big\},$$

where $h \in \mathbf{Z}_{>0}$ is a sufficient large integer. Since $M^{(m)}$ is an $\mathcal{R}_L^+(\Gamma)$-module and ι_m commutes with $\mathcal{R}_L^+(\Gamma)$-action for any $m \geq n(M)$, one has

$$\iota_{m'}(\lambda x) = \lambda \iota_{m'}(x) \in \mathbf{D}^+_{\mathrm{dif},m'}(M')$$

for each $x \in M^{(m)}$ and $m' \geqq m$ by our assumption $\lambda(\mathbf{D}^+_{\mathrm{dif},m}(M)) \subseteq \mathbf{D}^+_{\mathrm{dif},m}(M')$ for all $m \geq n$, which shows that $\lambda x \in (M')^{(m)}$. ■

The following corollary is fundamental.

Corollary 2.2.2. *Multiplying by λ induces morphisms of complexes*

$$C^\bullet_{\varphi,\gamma}(M) \to C^\bullet_{\varphi,\gamma}(M'), \quad C^\bullet_{\psi,\gamma}(M) \to C^\bullet_{\psi,\gamma}(M')$$

of L-vector spaces,

$$C^\bullet_{\psi}(M) \to C^\bullet_{\psi}(M')$$

of $\mathcal{R}_L^+(\Gamma)$-modules, and

$$C^\bullet_{\gamma}\big(\mathbf{D}^{(+)}_{\mathrm{dif},m}(M)\big) \to C^\bullet_{\gamma}\big(\mathbf{D}^{(+)}_{\mathrm{dif},m}(M')\big)$$

of $L_m[\![t]\!]$-modules for each $m \geq n$.

Proof. Since the operators φ, ψ are continuous so that they commute with the $\mathcal{R}_L^+(\Gamma)$-action, Proposition 2.2.1 gives our assertion. ■

By abuse of notation, we use the same expression $\times\lambda$ for the morphisms defined in Proposition 2.2.1, the ones in Corollary 2.2.2, and the induced ones between their cohomologies, which will cause no confusion. We remark that the action $\times\lambda$ on $\mathrm{H}^i_\gamma(\mathbf{D}_{\mathrm{dif}}(M)) = \mathrm{H}^i_\gamma(\mathbf{D}_{\mathrm{dif}}(M'))$, $\mathbf{D}_{\mathrm{dR}}(M) = \mathbf{D}_{\mathrm{dR}}(M')$ and $\mathbf{D}_{\mathrm{cris}}(M) = \mathbf{D}_{\mathrm{cris}}(M')$ is just the multiplication by $\lambda(\mathbf{1}) \in L$. Here, for any $\lambda \in \mathcal{R}_L^+(\Gamma)$, we denote by $\lambda(\mathbf{1}) \in L$ the image of λ by the augmentation map $f_{\mathbf{1}} : \mathcal{R}_L^+(\Gamma) \to L : [\gamma'] \mapsto 1$ $(\gamma' \in \Gamma)$.

Recall the following morphisms defined in [24]:

$$\begin{aligned} \mathrm{can} : \mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) : &\quad [x, y] \mapsto \big[\iota_n(x)\big], \\ g_M : \mathbf{D}_{\mathrm{dR}}(M) = \mathrm{H}^0_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) \to \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) : &\quad \alpha \mapsto \big[\log\chi(\gamma)\alpha\big]. \end{aligned}$$

Since they commute with $\mathcal{R}^+_L(\Gamma)$-action, we immediately obtain the following lemma.

Lemma 2.2.3. *The action $\times\lambda$ induces the following commutative diagrams:*

$$\begin{tikzcd} \mathrm{H}^1_{\varphi,\gamma}(M) \arrow[r, "\times\lambda"] \arrow[d, "\mathrm{can}"'] & \mathrm{H}^1_{\varphi,\gamma}(M') \arrow[d, "\mathrm{can}"] \\ \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) \arrow[r, "\times\lambda(\mathbf{1})"'] & \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M')\big), \end{tikzcd} \quad \begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(M) \arrow[r, "\times\lambda(\mathbf{1})"] \arrow[d, "g_M"'] & \mathbf{D}_{\mathrm{dR}}(M') \arrow[d, "g_{M'}"] \\ \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) \arrow[r, "\times\lambda(\mathbf{1})"'] & \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M')\big). \end{tikzcd}$$

We next introduce a morphism of L-vector spaces

$$\exp_M : t(M) \to \mathrm{H}^1_{\varphi,\gamma}(M)$$

called the Bloch–Kato's exponential map, and if M is de Rham, then we have another one

$$\exp^*_{M^*} : \mathrm{H}^1_{\varphi,\gamma}(M) \to \mathbf{D}^0_{\mathrm{dR}}(M)$$

called the Bloch–Kato's dual exponential map that is the Tate dual of $\exp_{M^*}$. They are characterized by the following explicit formulae.

Theorem 2.2.4. *Let M be a (φ, Γ)-module over $\mathcal{R}_L$.*

(1) *For $x \in \mathbf{D}_{\mathrm{dR}}(M)$, there exists $n \geq n(M)$ and $\tilde{x} \in M^{(n)}[1/t]^\Delta$ such that for any $m \geq n$ we have*

$$\iota_m(\tilde{x}) - x \in \mathbf{D}^+_{\mathrm{dif},m}(M).$$

Using such an element $\tilde{x}$, we can calculate the value $\exp_M(x)$ as

$$\exp_M(x) = \big[(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x}\big].$$

(2) *We assume that M is de Rham. Then g_M is an isomorphism and $\exp^*_{M^*}$ is characterized by the following commutative diagram*

$$\begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(M) \arrow[r, "g_M"] & \mathrm{H}^1_\gamma\big(\mathbf{D}_{\mathrm{dif}}(M)\big) \\ \mathrm{H}^1_{\varphi,\gamma}(M) \arrow[r, "="'] \arrow[u, "\exp^*_{M^*}"] & \mathrm{H}^1_{\varphi,\gamma}(M). \arrow[u, "\mathrm{can}"'] \end{tikzcd}$$

Proof. See [24, Sections 2.3 and 2.4] or [25, Section 2B] for the definition of $\exp_M$, $\exp^*_{M^*}$ and the proofs of the above formulae. ∎

To state a relation between $\times\lambda$ and Bloch–Kato's morphisms, we need some preparation.

For any $\gamma_0 \in \Gamma_{\mathrm{free}} \setminus \{1\}$, we set

$$\omega := \frac{1}{\log\bigl(\chi(\gamma_0)\bigr)} \frac{d[\gamma_0]}{[\gamma_0]} \in \Omega^{1,\mathrm{an}}_{\mathcal{R}_L^+(\Gamma)/L} := \Gamma(\mathcal{W}, \Omega^1_{\mathcal{W}/L}).$$

This is independent of the choice of γ_0 since one has

$$\frac{1}{\log\bigl(\chi(\gamma_0^a)\bigr)} \frac{d[\gamma_0]^a}{[\gamma_0]^a} = \frac{1}{a\log\bigl(\chi(\gamma_0)\bigr)} a\frac{d[\gamma_0]}{[\gamma_0]} = \frac{1}{\log\bigl(\chi(\gamma_0)\bigr)} \frac{d[\gamma_0]}{[\gamma_0]}$$

for any non zero $a \in \mathbb{Z}_p$. Because one has $d[\gamma_0] = d([\gamma_0]-1)$ and $[\gamma_0]-1$ is a parameter of

$$\mathcal{R}_L^+(\Gamma_{\mathrm{free}}) \xrightarrow{\sim} \mathcal{R}_L^+$$

if γ_0 is a topological generator, one has

$$\Omega^{1,\mathrm{an}}_{\mathcal{R}_L^+(\Gamma)/L} = \mathcal{R}_L^+(\Gamma)\omega = L[\Delta] \otimes_L \mathcal{R}_L^+(\Gamma_{\mathrm{free}})\omega,$$

which is a free $\mathcal{R}_L^+(\Gamma)$-module of rank one. For each $\lambda \in \mathcal{R}_L^+(\Gamma)$, we define $\frac{d\lambda}{\omega} \in \mathcal{R}_L^+(\Gamma)$ by $d\lambda = \frac{d\lambda}{\omega} \cdot \omega$. Explicitly, if $\gamma \in \Gamma_{\mathrm{free}}$ is the fixed topological generator and λ is of the form $\lambda = y \otimes f([\gamma]-1)$ with $y \in L[\Delta]$ and $f(T) \in \mathcal{R}_L^+$, then one has

$$\frac{d\lambda}{\omega} = \log\bigl(\chi(\gamma)\bigr) y \otimes \frac{df}{dT}\bigl([\gamma]-1\bigr).$$

In the following theorem, we shall compare the Bloch–Kato's morphisms of M and M' using λ. It is the key theorem to prove our main theorem.

Theorem 2.2.5. *Let M, M' be (φ, Γ)-modules over $\mathcal{R}_L$ such that $M[1/t] = M'[1/t]$.*

(1) *The diagram*

$$\begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(M) \arrow[r, "\times\lambda(\mathbf{1})"] \arrow[d, "\exp_M"'] & \mathbf{D}_{\mathrm{dR}}(M') \arrow[d, "\exp_{M'}"] \\ \mathrm{H}^1_{\varphi,\gamma}(M) \arrow[r, "\times\lambda"'] & \mathrm{H}^1_{\varphi,\gamma}(M') \end{tikzcd}$$

commutes.

(2) *Assume that M or M' is (thus both are) de Rham. Then the diagram*

$$\begin{tikzcd} \mathbf{D}_{\mathrm{dR}}(M) \arrow[r, "\times\lambda(\mathbf{1})"] & \mathbf{D}_{\mathrm{dR}}(M') \\ \mathrm{H}^1_{\varphi,\gamma}(M) \arrow[r, "\times\lambda"'] \arrow[u, "\exp^*_{M^*}"] & \mathrm{H}^1_{\varphi,\gamma}(M'), \arrow[u, "\exp^*_{M'^*}"'] \end{tikzcd}$$

commutes.

(3) *Assume further that* $\lambda(\mathbf{1}) = 0$. *Then the diagram*

$$\begin{tikzcd}
\mathbf{D}_{\mathrm{dR}}(M) \arrow[r, "\times\frac{d\lambda}{\omega}(\mathbf{1})"] & \mathbf{D}_{\mathrm{dR}}(M') \arrow[d, "\exp_{M'}"] \\
\mathrm{H}^1_{\varphi,\gamma}(M) \arrow[u, "\exp^*_{M^*}"] \arrow[r, "\times\lambda"'] & \mathrm{H}^1_{\varphi,\gamma}(M')
\end{tikzcd}$$

commutes.

Proof. First we prove (1). Let α be an element of $\mathbf{D}_{\mathrm{dR}}(M)$. By Theorem 2.2.4 (1), there exist an integer $n \geq \max\{n(M), n(M')\}$ and an element $x \in M^{(n)}[1/t]^{\Delta}$ such that

$$\iota_m(x) - \alpha \in \mathbf{D}^+_{\mathrm{dif},m}(M)$$

for any $m \geq n$. Then one has

$$\exp_M(\alpha) = \big[(\gamma - 1)x, (\varphi - 1)x\big] \in \mathrm{H}^1_{\varphi,\gamma}(M).$$

Thus, its image under the map $\times\lambda : \mathrm{H}^1_{\varphi,\gamma}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M')$ is equal to

$$\big[\lambda\big((\gamma - 1)x\big), \lambda\big((\varphi - 1)x\big)\big] = \big[(\gamma - 1)(\lambda x), (\varphi - 1)(\lambda x)\big] \in \mathrm{H}^1_{\varphi,\gamma}(M').$$

This is nothing but $\exp_{M'}(\lambda\alpha)$ because $\lambda x \in M'^{(n)}[1/t]^{\Delta}$ satisfies

$$\iota_m(\lambda x) - \lambda(\alpha) = \lambda(\iota_m(x) - \alpha) \in \mathbf{D}^+_{\mathrm{dif},m}(M').$$

for any $m \geq n$ by Proposition 2.2.1.

(2) follows immediately by Theorem 2.2.4 (2) and Lemma 2.2.3.

We shall prove (3). Assume that M and M' are de Rham and $\lambda(\mathbf{1}) = 0$. We remark that the latter implies that one can write

$$\lambda p_{\Delta} = (\gamma - 1)\lambda_0$$

for some $\lambda_0 \in \mathcal{R}^+_L(\Gamma_{\mathrm{free}})$ and $p_{\Delta} := \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}[\sigma] \in L[\Delta]$. Let $[x, y] \in \mathrm{H}^1_{\varphi,\gamma}(M)$ and put $\alpha = \exp^*_{M^*}([x, y]) \in \mathbf{D}^0_{\mathrm{dR}}(M)$. By replacing n by a larger n' if necessary, we may assume that $x \in (M^{(n)})^{\Delta}$. Take $m \geq n$ arbitrary. By Theorem 2.2.4 (2), one has

$$\big[\iota_m(x)\big] = \big[\log\chi(\gamma)\alpha\big] \in \mathrm{H}^1_{\gamma}\big(\mathbf{D}^+_{\mathrm{dif},m}(M)\big),$$

and hence one obtains

$$\iota_m(x) - \log\chi(\gamma)\alpha \in (\gamma - 1)\mathbf{D}^+_{\mathrm{dif},m}(M)^{\Delta}.$$

Applying $\lambda p_{\Delta}/(\gamma - 1) = \lambda_0 \in \mathcal{R}^+_L(\Gamma_{\mathrm{free}})$ on the both sides gives

$$\iota_m\left(\frac{\lambda p_{\Delta}}{\gamma - 1}(x)\right) - \log\big(\chi(\gamma)\big)\frac{\lambda p_{\Delta}}{\gamma - 1}\alpha \in \lambda p_{\Delta}\big(\mathbf{D}^+_{\mathrm{dif},m}(M)^{\Delta}\big) \subseteq \mathbf{D}^+_{\mathrm{dif},m}(M')^{\Delta}.$$

Since one has

$$\log\big(\chi(\gamma)\big)\frac{\lambda p_\Delta}{\gamma-1}\alpha = \log\big(\chi(\gamma)\big)\frac{\lambda p_\Delta}{\gamma-1}(\mathbf{1})\alpha = \frac{d\lambda}{\omega}(\mathbf{1})\alpha,$$

we obtain

$$\iota_m\left(\frac{\lambda p_\Delta}{\gamma-1}(x)\right) - \frac{d\lambda}{\omega}(\mathbf{1})\alpha \in \mathbf{D}^+_{\mathrm{dif},m}(M')^\Delta.$$

Since $\frac{\lambda p_\Delta}{\gamma-1}(x) \in (M^{(n)})^\Delta \subseteq (M^{(n)}[1/t])^\Delta = (M'^{(n)}[1/t])^\Delta$ and we have taken $m \geq n$ arbitrary, the explicit formula for $\exp_{D'}$ gives that

$$\begin{aligned}\exp_{M'}\left(\frac{d\lambda}{\omega}(\mathbf{1})\alpha\right) &= \left[(\gamma-1)\frac{\lambda p_\Delta}{\gamma-1}(x), (\varphi-1)\frac{\lambda p_\Delta}{\gamma-1}(x)\right]\\ &= \left[\lambda x, \frac{\lambda p_\Delta}{\gamma-1}(\varphi-1)(x)\right]\\ &= \left[\lambda x, \frac{\lambda p_\Delta}{\gamma-1}(\gamma-1)(y)\right]\\ &= \lambda[x,y],\end{aligned}$$

which proves (3). ∎

3. Big exponential maps in the local ε-conjecture for (φ, Γ)-modules

In this section, we first recall briefly the definition of the de Rham ε-isomorphisms for (φ, Γ)-modules, and state the local ε-conjecture for the cyclotomic deformation. Then, we define the big exponential maps and study their several properties.

3.1. de Rham ε-isomorphisms for (φ, Γ)-modules

We recall de Rham ε-isomorphisms over Robba rings following [25].

First, for each local field $L/\mathbf{Q}_p$ and each (φ, Γ)-module M over $\mathcal{R}_L$, we define a graded line $\Delta_L(M)$ over L called the fundamental line attached to M as follows.

By [22], the complex $C^\bullet_{\varphi,\gamma}(M)$ is a perfect complex of L-vector spaces, and we put

$$\Delta_{L,1}(M) = \mathrm{Det}_L\big(C^\bullet_{\varphi,\gamma}(M)\big),$$

which is a graded line over L. Here, Det_L is the determinant functor over L defined by Knudsen–Mumford [21]. We define another graded L-vector space $\Delta_{L,2}(M)$ as follows. By the classification of rank 1 (φ, Γ)-modules over $\mathcal{R}_L$ [10, Proposition 3.1], there exists a unique continuous homomorphism

$$\delta_{\det_{\mathcal{R}_L}(M)} : \mathbf{Q}_p{}^\times \to L^\times$$

such that there exists an isomorphism

$$\det_{\mathcal{R}_L}(M) \cong \mathcal{R}_L(\delta_{\det_{\mathcal{R}_L}(M)}),$$

and we define

$$\mathcal{L}_L(M)=\{x\in\det_{\mathcal{R}_L}(M)\mid\varphi(x)=\delta_{\det_{\mathcal{R}_L}(M)}(p)x,\ \gamma(x)=\delta_{\det_{\mathcal{R}_L}(M)}\big(\chi(\gamma')\big)x\ (\gamma'\in\Gamma)\},$$

which turns out to be an L-vector space of dimension 1. We then define a graded line over L

$$\Delta_{L,2}(M)=\big(\mathcal{L}_L(M),r_M\big),$$

where we put $r_M=\mathrm{rank}_{\mathcal{R}_L}(M)$. Finally, we define a graded line $\Delta_L(M)$ over L called its fundamental line by

$$\Delta_L(M)=\Delta_{L,1}(M)\boxtimes\Delta_{L,2}(M).$$

We also define the fundamental line $\Delta_L^{\mathrm{Iw}}(M)$ for the cyclotomic deformation of a general (φ,Γ)-module M over $\mathcal{R}_L$. By [20, Theorem 4.4.1], the complex $C^\bullet_\psi(M)$ is a perfect complex over $\mathcal{R}_L^+(\Gamma)$, thus we may define

$$\Delta_{L,1}^{\mathrm{Iw}}(M):=\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(C^\bullet_\psi(M)\big).$$

We also define

$$\Delta_{L,2}^{\mathrm{Iw}}(M):=\Delta_{L,2}(M)\otimes_L\mathcal{R}_L^+(\Gamma),$$

and define the fundamental line for the cyclotomic deformation

$$\Delta_L^{\mathrm{Iw}}(M):=\Delta_{L,1}^{\mathrm{Iw}}(M)\boxtimes\Delta_{L,2}^{\mathrm{Iw}}(M).$$

Recall that for any continuous character $\delta:\Gamma\to L^\times$, we can consider a (φ,Γ)-module $M(\delta)=Me_\delta$ with a formal element e_δ on which we have

$$\varphi(xe_\delta)=\varphi(x)e_\delta,\quad\gamma'(xe_\delta)=\delta(\gamma')\gamma'(x)e_\delta$$

for any $x\in M$ and $\gamma'\in\Gamma$. In particular, we put $M^*=\mathrm{Hom}_{\mathcal{R}_L}(M,\mathcal{R}_L)(\chi)$.

As studied in [25, Section 4A], one has canonical isomorphisms

$$\begin{aligned}\mathrm{ev}_{\delta,j}&:\Delta_{L,j}^{\mathrm{Iw}}(M)\otimes_{f_\delta}L\xrightarrow{\sim}\Delta_{L,j}\big(M(\delta)\big),\\ \mathrm{can}_{\delta,j}&:\Delta_{L,j}^{\mathrm{Iw}}(M)\otimes_{g_\delta}\mathcal{R}_L^+(\Gamma)\xrightarrow{\sim}\Delta_{L,j}^{\mathrm{Iw}}\big(M(\delta)\big)\end{aligned}$$

for $j=1,2,\emptyset$, where $f_\delta:\mathcal{R}_L^+(\Gamma)\to L$ (resp. $g_\delta:\mathcal{R}_L^+(\Gamma)\to\mathcal{R}_L^+(\Gamma)$) is the continuous homomorphism of L-algebra extending $\gamma'\mapsto\delta^{-1}(\gamma')$ (resp. $\gamma'\mapsto\delta^{-1}(\gamma')\gamma'$).

The local ε-conjecture concerns canonical bases of the fundamental lines for de Rham (φ,Γ)-modules, which we recall briefly as follows: see [25, Section 3C] for the precise definition. Let M be a de Rham (φ,Γ)-module over $\mathcal{R}_L$. Define $\mathbf{1}_L:=(L,0)$ as the trivial line. We define the following two isomorphisms

$$\begin{aligned}\theta_{\mathrm{dR}}(M)&:\mathbf{1}_L\xrightarrow{\sim}\Delta_{L,1}(M)\boxtimes\mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M)\big),\\ f_M&:\Delta_{L,2}(M)\xrightarrow{\sim}\mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M)\big).\end{aligned}$$

To define the isomorphism $\theta_{\mathrm{dR}}(M)$, we first recall that there exists an exact sequence of L-vector spaces

$$C^\bullet(M): 0 \to \mathrm{H}^0_{\varphi,\gamma}(M) \to \mathbf{D}_{\mathrm{cris}}(M) \xrightarrow{x\mapsto((1-\varphi)x,\bar{x})} \mathbf{D}_{\mathrm{cris}}(M) \oplus t(M) \xrightarrow{\exp_{f,M}\oplus\exp_M} \mathrm{H}^1_{\varphi,\gamma}(M)_f \to 0$$

called the Bloch–Kato's fundamental sequence for M. Here, the first map is a canonical inclusion, $\exp_{f,M}: \mathbf{D}_{\mathrm{cris}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M)$ is another exponential map, and

$$\mathrm{H}^1_{\varphi,\gamma}(M)_f := \mathrm{Im}\big(\exp_{f,M} \oplus \exp_M : \mathbf{D}_{\mathrm{cris}}(M) \oplus t(M) \to \mathrm{H}^1_{\varphi,\gamma}(M)\big);$$

see [25, Section 2B] for details. Taking the dual and using Tate duality and the de Rham duality for $C^\bullet(M^*)$, we obtain

$$\widetilde{C}^\bullet(M^*): 0 \to \mathrm{H}^1_{\varphi,\gamma}(M)_{/f} \to \mathbf{D}_{\mathrm{cris}}(M^*)^\vee \oplus \mathbf{D}^0_{\mathrm{dR}}(M) \to \mathbf{D}_{\mathrm{cris}}(M^*)^\vee \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0.$$

We define the canonical isomorphism $\theta_{\mathrm{dR}}(M)$ as the inverse of the isomorphism

$$\theta_{\mathrm{dR}}(M): \Delta_{L,1}(M) \boxtimes \mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M)\big) \xrightarrow{(\sharp)} \mathrm{Det}_L\big(C^\bullet(M)\big)^{-1} \boxtimes \mathrm{Det}_L\big(\widetilde{C}^\bullet(M^*)\big) \xrightarrow{(\flat)} 1_L \boxtimes 1_L \xrightarrow{\mathrm{can}} 1_L,$$

where the isomorphism ($\sharp$) is defined by cancelation $X \boxtimes X^{-1} \xrightarrow{\sim} 1_L : a \otimes f \mapsto f(a)$ for graded invertible modules $X = \mathbf{D}_{\mathrm{cris}}(M), \mathbf{D}_{\mathrm{cris}}(M^*)^\vee$, and the one ($\flat$) is by the isomorphisms

$$\mathrm{Det}_L\big(C^\bullet(M)\big) \to 1_L, \quad \mathrm{Det}_L\big(\widetilde{C}^\bullet(M^*)\big) \to 1_L$$

corresponding to the zero morphisms

$$C^\bullet(M) \to [\cdots \to 0 \to \cdots], \quad \widetilde{C}^\bullet(M^*) \to [\cdots \to 0 \to \cdots]$$

under the determinant functor respectively.

Next we define the isomorphism f_M. Since M is de Rham, we have $\mathbf{D}_{\mathrm{dif}}(M) = \mathbf{D}_{\mathrm{dR}}(M) \otimes_L L_\infty((t))$ where $L_\infty((t)) = \cup_{n\geq 1} L_n((t))$. By [25, Lemma 3.4], a map

$$\mathcal{L}_L(M) \to \mathbf{D}_{\mathrm{dif},n}\big(\det_{\mathcal{R}_L}(M)\big) : x \mapsto \frac{1}{\varepsilon_L\big(W(M)\big)} \frac{1}{t^{h_M}} x$$

for sufficient large n induces an isomorphism $f_M : \Delta_{L,2}(M) \xrightarrow{\sim} \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(M))$. Here, the constant $\varepsilon_L(W(M)) \in L_\infty$ is defined by using the Weil–Deligne representation $W(M)$ attached to M and the fixed basis $(\zeta_{p^n})_n \in \mathbf{Z}_p(1)$, via the theory of ε-constants of Deligne–Langlands [11], and Fontaine–Perrin-Riou [13].

Using $\theta_{\mathrm{dR}}(M)$ and f_M, we define

$$\varepsilon^{\mathrm{dR}}_L(M) = (\mathrm{id} \boxtimes f_M^{-1}) \circ \big(\Gamma(M)\theta_{\mathrm{dR}}(M)\big) : 1_L \xrightarrow{\sim} \Delta_L(M)$$

and call it the de Rham ε-isomorphism for M. Here, the Γ-constant $\Gamma(M)$ for M is defined by $\Gamma(M) = \prod_{1\le i\le r}\Gamma^*(h_i)^{-1}$, where for $k \in \mathbf{Z}$ we put

$$\Gamma^*(k) = \begin{cases} (k-1)! & (k \ge 1) \\ \frac{(-1)^k}{(-k)!} & (k \le 0). \end{cases}$$

Now we can state the local ε-conjecture for cyclotomic deformation for (φ, Γ)-modules.

Conjecture 3.1.1. *For each finite extension $L/\mathbf{Q}_p$ and each de Rham (φ, Γ)-module M over $\mathcal{R}_L$, there exists an isomorphism*

$$\varepsilon_L^{\mathrm{Iw}}(M) : 1_{\mathcal{R}_L^+(\Gamma)} \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M)$$

satisfying the following commutative diagram

$$\begin{tikzcd} \Delta_L^{\mathrm{Iw}}(M) \otimes_{f_\delta} L \arrow[r, "\mathrm{ev}_\delta"] & \Delta_L(M(\delta)) \\ 1_{\mathcal{R}_L^+(\Gamma)} \otimes_{f_\delta} L \arrow[u, "\varepsilon_L^{\mathrm{Iw}}(M)\otimes \mathrm{id}"] \arrow[r, "\mathrm{can}"'] & 1_L . \arrow[u, "\varepsilon_L^{\mathrm{dR}}(M(\delta))"'] \end{tikzcd}$$

for arbitrary de Rham characters $\delta : \Gamma \to L^\times$.

Since the set of all the de Rham characters is Zariski dense in the weight space $\mathcal{W}$, the isomorphism $\varepsilon_L^{\mathrm{Iw}}(M)$ is uniquely determined (if it exists).

3.2. Big exponential maps

Throughout this section, let L be a finite extension of $\mathbf{Q}_p$, M a de Rham (φ, Γ)-modules of rank r over $\mathcal{R}_L$, and $N = \mathbf{N}_{\mathrm{rig}}(M)$ its associated p-adic differential equation define by Berger in [6]. Note that N is characterized as the (φ, Γ)-module in $M[1/t]$ satisfying $N[1/t] = M[1/t]$ and

$$\mathbf{D}^+_{\mathrm{dif},n}(N) = L_n[\![t]\!] \otimes_L \mathbf{D}_{\mathrm{dR}}(M)$$

for a sufficient large n.

In this subsection, we construct the big exponential map of M

$$\mathrm{Exp}(M) : \Delta_L^{\mathrm{Iw}}(N) \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M),$$

and prove its properties. Its construction involves the theory of big exponential map, especially the $\delta(D)$-theorem studied in [24], which generalizes the original $\delta(V)$-conjecture in [27].

First, we shall construct

$$\mathrm{Exp}_1 : \Delta_{L,1}^{\mathrm{Iw}}(N) \xrightarrow{\sim} \Delta_{L,1}^{\mathrm{Iw}}(M)$$

as follows.

We have a decomposition $\mathcal{R}_L^+(\Gamma) = \oplus_\eta \mathcal{R}_L^+(\Gamma)e_\eta$, where η runs over the characters on Δ and e_η is the idempotent in $L[\Delta]$ corresponding to η. Because $\mathcal{R}_L^+(\Gamma)e_\eta$ is a domain (in fact, it is non-canonically isomorphic to $\mathcal{R}_L^+$), we define $P_{\mathrm{tors}} = \oplus_\eta (Pe_\eta)_{\mathrm{tors}}$ for an $\mathcal{R}_L^+(\Gamma)$-module P, where each module $(Pe_\eta)_{\mathrm{tors}}$ is the torsion sub-$\mathcal{R}_L^+(\Gamma)e_\eta$-module of Pe_η. By [20, Proposition 4.3.6], all the complexes $\mathrm{H}^1_\psi(M)_{\mathrm{tors}}[0]$, $\mathrm{H}^1_\psi(M)[0]$ and $\mathrm{H}^2_\psi(M)[0]$ over $\mathcal{R}_L^+(\Gamma)$ are perfect, thus we have canonical isomorphisms

$$\begin{aligned}
&\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(C^\bullet_\psi(M)\big)\\
&\cong \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(M)[0]\big)^{-1} \boxtimes \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(M)[0]\big)\\
&\cong \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(M)_{\mathrm{free}}[0]\big)^{-1} \boxtimes \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(M)_{\mathrm{tors}}[0]\big)^{-1} \boxtimes \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(M)[0]\big),
\end{aligned}$$

where we set $\mathrm{H}^1_\psi(M)_{\mathrm{free}} = \mathrm{H}^1_\psi(M)/\mathrm{H}^1_\psi(M)_{\mathrm{tors}}$. Extending the coefficients to the total fraction ring $Q(\mathcal{R}_L^+(\Gamma))$ of $\mathcal{R}_L^+(\Gamma)$, we have

$$\begin{aligned}
&\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(C^\bullet_\psi(M)\big) \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big)\\
&\quad\cong \mathrm{Det}_{Q(\mathcal{R}_L^+(\Gamma))}\Big(\mathrm{H}^1_\psi(M)_{\mathrm{free}} \otimes Q\big(\mathcal{R}_L^+(\Gamma)\big)[0]\Big)^{-1} \boxtimes \big(1_{Q(\mathcal{R}_L^+(\Gamma))}\big)^{-1} \boxtimes 1_{Q(\mathcal{R}_L^+(\Gamma))}\\
&\quad= \Big(\bigwedge^r \Big(\mathrm{H}^1_\psi(M)_{\mathrm{free}} \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big)\Big)^{-1}, -r\Big).
\end{aligned}$$

We can calculate the image of $\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}(C^\bullet_\psi(M))$ under the last isomorphism using the characteristic ideals defined in [24], which we recall in the following. The ring $\mathcal{R}_L^+(\Gamma)$ can be written as

$$\mathcal{R}_L^+(\Gamma) \cong \varprojlim \mathcal{R}_{L,n}^+(\Gamma),$$

where $\mathcal{R}_{L,n}^+(\Gamma) = \mathcal{O}_L[\![\Gamma]\!][\mathfrak{m}^n/p]^\wedge[1/p]$ with $\mathfrak{m}$ the Jacobson radical of $\mathcal{O}_L[\![\Gamma]\!]$. It turns out that $\mathcal{R}_{L,n}^+(\Gamma)e_\eta$ is a PID for each η. For $P = \mathrm{H}^1_\psi(M)_{\mathrm{tors}}, \mathrm{H}^2_\psi(M)$, by [20, Lemma 4.3.4] we know that $P \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_{L,n}^+(\Gamma)e_\eta$ is a finitely generated torsion module over $\mathcal{R}_{L,n}^+(\Gamma)e_\eta$, and we define $\mathrm{char}_{\mathcal{R}_{L,n}^+(\Gamma)}(P \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_{L,n}^+(\Gamma))$ as the principal ideal of $\mathcal{R}_{L,n}^+(\Gamma)$ whose η-component is $\mathrm{char}_{\mathcal{R}_{L,n}^+(\Gamma)e_\eta}(P \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_{L,n}^+(\Gamma)e_\eta)$. We then define the characteristic ideal of P by

$$\mathrm{char}_{\mathcal{R}_L^+(\Gamma)}(P) := \varprojlim_n \mathrm{char}_{\mathcal{R}_{L,n}^+(\Gamma)}\big(P \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_{L,n}^+(\Gamma)\big) \subseteq \mathcal{R}_L^+(\Gamma).$$

Note that the equality $\mathrm{char}_{\mathcal{R}_L^+(\Gamma)}(P) = \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}(P)^{-1}$ holds under the identification

$$\begin{aligned}
\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}(P)^{-1} &\subseteq \mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}(P)^{-1} \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big)\\
&= \mathrm{Det}_{Q(\mathcal{R}_L^+(\Gamma))}(0)^{-1} = \big(Q\big(\mathcal{R}_L^+(\Gamma)\big), 0\big).
\end{aligned}$$

In fact, P is coadmissible by [20, Lemma 4.3.4] and thus $P \cong \varprojlim_n (P \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_{L,n}^+(\Gamma))$ by the theory of coadmissible modules [29, Corollary 3.3]. Since the determinant functor commutes with base change, it suffices therefore to show the equality after base change to $\mathcal{R}_L^+(\Gamma) \to \mathcal{R}_{L,n}^+(\Gamma)$ holds for each n, which is straightforward. Therefore, under the last isomorphism

$$\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}\big(C_\psi^\bullet(M)\big) \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big) \\ \xrightarrow{\sim} \Big(\bigwedge^r \Big(\mathrm{H}_\psi^1(M)_{\mathrm{free}} \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big)\Big)^{-1}, -r\Big).$$

in the previous paragraph, the image of $\mathrm{Det}_{\mathcal{R}_L^+(\Gamma)}(C_\psi^\bullet(M))$ is calculated as

$$\bigwedge^r \mathrm{H}_\psi^1(M)_{\mathrm{free}}^{-1} \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}_\psi^1(M)_{\mathrm{tors}}\big) \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}_\psi^2(M)\big)^{-1}.$$

On the other hand, let $h > 0$ be a sufficient large integer satisfying $\mathbf{D}_{\mathrm{dR}}^{-h}(M) = \mathbf{D}_{\mathrm{dR}}(M)$. We put

$$\nabla_0 := \frac{\log \gamma'}{\log \chi(\gamma')} = -\frac{1}{\log \chi(\gamma')} \sum_{1 \le n} \frac{(1-\gamma')^n}{n} \in \mathcal{R}_L^+(\Gamma)$$

and $\nabla_i := \nabla_0 - i$ for each $i \in \mathbf{Z}$, where $\gamma' \in \Gamma - \{1\}$ is close enough to 1 (it is independent of the choice of γ'.) Then by [24, Lemma 3.6] we have

$$\left(\prod_{i=0}^{h-1} \nabla_i\right)\big(\mathbf{D}_{\mathrm{dif},m}^+(N)\big) \subseteq \mathbf{D}_{\mathrm{dif},m}^+(M)$$

for any $m \ge n(M)$, and we define a morphism $\mathrm{Exp}_{(h)}(M) : \mathrm{H}_\psi^1(N) \to \mathrm{H}_\psi^1(M)$ as the induced one by Corollary 2.2.2;

$$\mathrm{Exp}_{(h)}(M) := \times \prod_{i=0}^{h-1} \nabla_i : \mathrm{H}_\psi^1(N) \to \mathrm{H}_\psi^1(M).$$

It induces a map $\overline{\mathrm{Exp}_{(h)}}(M) : \mathrm{H}_\psi^1(N)_{\mathrm{free}} \to \mathrm{H}_\psi^1(M)_{\mathrm{free}}$, which turns out to be injective by [24, Lemma 3.13]. Since $\nabla_i \in \mathcal{R}_L^+(\Gamma)$ is a non-zero-divisor for any integer i, we can define a modified map

$$\overline{\mathrm{Exp}}(M) : \wedge^r \mathrm{H}_\psi^1(N)_{\mathrm{free}} \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big) \to \wedge^r \mathrm{H}_\psi^1(M)_{\mathrm{free}} \otimes_{\mathcal{R}_L^+(\Gamma)} Q\big(\mathcal{R}_L^+(\Gamma)\big)$$

by

$$\overline{\mathrm{Exp}}(M) = \bigwedge^r \overline{\mathrm{Exp}_{(h)}}(M) \otimes \frac{1}{\prod_{i=1}^r \prod_{j_i=1}^{h-h_i-1} \nabla_{h_i+j_i}} \cdot \mathrm{id}_{Q(\mathcal{R}_L^+(\Gamma))},$$

where $h_1, \ldots, h_r$ are the Hodge–Tate weights of M with multiplicity. Note that the right hand side doesn't depend on h, which justifies our notation $\overline{\mathrm{Exp}}(M)$.

To define $\mathrm{Exp}_1(M)$, the main part of $\mathrm{Exp}(M)$, the following theorem is essential. It is nothing but theorem $\delta(D)$ in the context of the local ε-conjecture.

Theorem 3.2.1. *$\overline{\mathrm{Exp}}(M)$ is an isomorphism of $Q(\mathcal{R}_L^+(\Gamma))$-modules. Moreover, by restriction, it induces an isomorphism of $\mathcal{R}_L^+(\Gamma)$-modules*

$$\bigwedge^r \mathrm{H}^1_\psi(N)_{\mathrm{free}} \otimes \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(N)_{\mathrm{tors}}\big)^{-1} \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(N)\big)$$
$$\to \bigwedge^r \mathrm{H}^1_\psi(M)_{\mathrm{free}} \otimes \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(M)_{\mathrm{tors}}\big)^{-1} \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(M)\big).$$

Proof. Since $\overline{\mathrm{Exp}}(M)$ is a multiplication of a product of $\nabla_i^{\pm} \in Q(\mathcal{R}_L^+(\Gamma))$ with $i \in \mathbf{Z}$, it is an isomorphism as $Q(\mathcal{R}_L^+(\Gamma))$-modules.

For the latter assertion, we first remark that in $\wedge^r \mathrm{H}^1_\psi(M)_{\mathrm{free}} \otimes_{\mathcal{R}_L^+(\Gamma)} Q(\mathcal{R}_L^+(\Gamma))$ we have

$$\begin{aligned}
&\overline{\mathrm{Exp}}(M)\Big(\bigwedge^r \mathrm{H}^1_\psi(N)_{\mathrm{free}}\Big)\\
&= \overline{\mathrm{Exp}_{(h)}}(M)\Big(\bigwedge^r \mathrm{H}^1_\psi(N)_{\mathrm{free}}\Big) \otimes \Big(\prod_{i=1}^r \prod_{j_i=1}^{h-h_i-1} \nabla_{h_i+j_i}\Big)^{-1}\\
&= \det_{\mathcal{R}_L^+(\Gamma)}\big(\overline{\mathrm{Exp}_{(h)}}(M)\big) \cdot \bigwedge^r \mathrm{H}^1_\psi(M)_{\mathrm{free}} \otimes \Big(\prod_{i=1}^r \prod_{j_i=1}^{h-h_i-1} \nabla_{h_i+j_i}\Big)^{-1},
\end{aligned}$$

where $\det_{\mathcal{R}_L^+(\Gamma)}(\overline{\mathrm{Exp}_{(h)}}(M)) \subseteq \mathcal{R}_L^+(\Gamma)$ is the determinant ideal of $\overline{\mathrm{Exp}_{(h)}}(M)$. Therefore, the claim is equivalent to the equality

$$\begin{aligned}
\Big(\prod_{i=1}^r \prod_{j_i=1}^{h-h_i-1} \nabla_{h_i+j_i}\Big)^{-1} &\det_{\mathcal{R}_L^+(\Gamma)}\big(\overline{\mathrm{Exp}_{(h)}}(M)\big) \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(N)_{\mathrm{tors}}\big)^{-1}\\
&\cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^1_\psi(M)_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(M)\big) \cdot \mathrm{char}_{\mathcal{R}_L^+(\Gamma)}\big(\mathrm{H}^2_\psi(N)\big)^{-1}
\end{aligned}$$

of fractional ideals in $Q(\mathcal{R}_L^+(\Gamma))$, which is proved as the theorem $\delta(D)$ [24, Theorem 3.14]. ∎

Definition 3.2.2. We define an isomorphism

$$\mathrm{Exp}_1(M) : \Delta^{\mathrm{Iw}}_{L,1}(N) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L,1}(M)$$

as the isomorphism corresponding to the one appearing in Theorem 3.2.1 under the covariant endofunctor $X \mapsto X^{-1}$ of the category of the graded invertible modules defined in [21].

Second, we shall define

$$\mathrm{Exp}_2(M) : \Delta^{\mathrm{Iw}}_{L,2}(N) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L,2}(M).$$

Lemma 3.2.3. *Under the canonical identification of* $\det_{\mathcal{R}_L}(M)[1/t]$ *and* $\det_{\mathcal{R}_L}(N)[1/t]$, *we have*

$$\det_{\mathcal{R}_L}(N) = \mathbf{N}_{\mathrm{rig}}\big(\det_{\mathcal{R}_L}(M)\big) = t^{-h_M} \det_{\mathcal{R}_L}(M).$$

Proof. The first equality follows from $\det_L(\mathbf{D}_{\mathrm{dR}}(M)) = \mathbf{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L}(M))$. The second one follows from the fact that for a general 1-dimensional (φ, Γ)-module M corresponding a continuous character δ, we have $\mathbf{N}_{\mathrm{rig}}(M) = t^{-h_M} M$. This shows the second equality. ■

Lemma 3.2.3 justifies the following definition.

Definition 3.2.4. We define the isomorphism

$$\mathrm{Exp}_2(M) : \Delta^{\mathrm{Iw}}_{L,2}(N) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L,2}(M)$$

as the scalar extension of the isomorphism

$$\mathcal{L}_2(N) \xrightarrow{\sim} \mathcal{L}_2(M) : \quad x \mapsto (-t)^{h_M} x.$$

We define the big exponential map of M as follows.

Definition 3.2.5. We define the isomorphism

$$\mathrm{Exp}(M) : \Delta^{\mathrm{Iw}}_{L}(N) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L}(M)$$

as the product $\mathrm{Exp}(M) := \mathrm{Exp}_1(M) \boxtimes \mathrm{Exp}_2(M)$ and call it the big exponential map of M.

We also define relative big exponential maps, which are useful to prove our main theorem.

Definition 3.2.6. Let M' be another (φ, Γ)-module such that $M[1/t] = M'[1/t]$. We define the isomorphism

$$\mathrm{Exp}_j(M, M') : \Delta^{\mathrm{Iw}}_{L,j}(M) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L,j}(M')$$

as the composition $\mathrm{Exp}_j(M') \circ \mathrm{Exp}_j(M)^{-1}$ for each $j = 1, 2, \emptyset$.

We note that the definition of $\mathrm{Exp}_j(M, M')$ is justified by the equality $\mathbf{N}_{\mathrm{rig}}(M) = \mathbf{N}_{\mathrm{rig}}(M')$.

The following proposition is used when we reduce the proof of our main theorem to the generic case.

Proposition 3.2.7. *Let $C^\bullet : 0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of de Rham (φ, Γ)-modules, and $\mathbf{N}_{\mathrm{rig}}(C^\bullet) : 0 \to N_1 \to N_2 \to N_3 \to 0$ the exact one corresponding $C^\bullet$ via the functor $\mathbf{N}_{\mathrm{rig}}$. Then we have*

$$\begin{CD}
\Delta^{\mathrm{Iw}}_{L,j}(M_1) \boxtimes \Delta^{\mathrm{Iw}}_{L,j}(M_3) @>>> \Delta^{\mathrm{Iw}}_{L,j}(M_2) \\
@V{\mathrm{Exp}_j(M_1)\boxtimes\mathrm{Exp}_j(M_3)}VV @VV{\mathrm{Exp}_j(M_2)}V \\
\Delta^{\mathrm{Iw}}_{L,j}(N_1) \boxtimes \Delta^{\mathrm{Iw}}_{L,j}(N_3) @>>> \Delta^{\mathrm{Iw}}_{L,j}(N_2),
\end{CD}$$

for $j = 1, 2, \emptyset$, where the horizontal isomorphisms are induced by $C^\bullet$ and $\mathbf{N}_{\mathrm{rig}}(C^\bullet)$ respectively.

Proof. The Hodge–Tate weights of M_2 is the same as the union of the ones of M_1 and M_3 with multiplicity, and thus we have $h_{M_2} = h_{M_1} + h_{M_3}$. This gives the commutativity for each $j = 1, 2$ by the definition of Exp_j and so for $j = \emptyset$. ■

Big exponential maps are compatible with twists by characters on Γ as follows.

Lemma 3.2.8. *Let $\delta : \Gamma \to L^\times$ be a de Rham character. Then the diagram*

$$\begin{tikzcd} \Delta^{\mathrm{Iw}}_{L,j}(M) \otimes_{g_\delta} \mathcal{R}^+_L(\Gamma) \arrow[r, "\mathrm{can}_{\delta,j}"] \arrow[d, "\mathrm{Exp}_j(M,M')\otimes\mathrm{id}"'] & \Delta^{\mathrm{Iw}}_{L,j}\big(M(\delta)\big) \arrow[d, "\mathrm{Exp}_j(M(\delta),M'(\delta))"] \\ \Delta^{\mathrm{Iw}}_{L,j}(M') \otimes_{g_\delta} \mathcal{R}^+_L(\Gamma) \arrow[r, "\mathrm{can}_{\delta,j}"'] & \Delta^{\mathrm{Iw}}_{L,j}\big(M'(\delta)\big), \end{tikzcd}$$

commutes for $j = 1, 2, \emptyset$.

Proof. The case $j = 2$ can be checked easily by definition.

We shall show the case $j = 1$. Let h_δ be the Hodge–Tate weight of $\mathcal{R}_L(\delta)$. Then, we can write $\delta = \chi^{h_\delta}\tilde{\delta}$, where $\tilde{\delta} : \Gamma \to L^\times$ is a finite character on Γ. Recall that the expression $\nabla_0 = \log\gamma'/\log\chi(\gamma')$ holds for any $\gamma' \in \Gamma - \{1\}$ close enough to 1. Thus, taking γ' such that $\tilde{\delta}(\gamma') = 1$, for any $i \in \mathbf{Z}$ we have

$$g_\delta(\nabla_i) = g_\delta\left(\frac{\log\gamma'}{\log\chi(\gamma')} - i\right) = \frac{\log\delta^{-1}(\gamma')\gamma'}{\log\chi(\gamma')} - i = \left(\frac{\log\gamma'}{\log\chi(\gamma')} - h_\delta\right) - i = \nabla_{i+h_\delta},$$

which gives our assertion, because the vertical arrows for $j = 1$ are induced by the multiplications of products of $\nabla_i^\pm$ for some $i \in \mathbf{Z}$ by definition. ■

4. Interpolation formula of $\mathrm{Exp}(M)$ for local ε-isomorphisms

In this section, we first state the main result and its corollary. Its proof will be divided into the next three subsections. We use an explicit construction of $\varepsilon^{\mathrm{Iw}}_L$-isomorphisms for (φ,Γ)-modules of rank 1, which is one of the main results in [25].

4.1. Statement of main result

Let L be a finite extension of $\mathbf{Q}_p$, M a de Rham (φ,Γ)-module over $\mathcal{R}_L$, and $N = \mathbf{N}_{\mathrm{rig}}(M)$ the p-adic differential equation corresponding to M. For any character $\delta : \Gamma \to L^\times$, we define an isomorphism $\mathrm{Exp}_j(M)_\delta : \Delta_{L,j}(N(\delta)) \xrightarrow{\sim} \Delta_{L,j}(M(\delta))$ as the one characterized by the commutative diagram

$$\begin{tikzcd} \Delta_{L,j}\big(N(\delta)\big) \arrow[r, "\mathrm{Exp}_j(M)_\delta"] & \Delta_{L,j}\big(M(\delta)\big) \\ \Delta^{\mathrm{Iw}}_{L,j}(N) \otimes_{f_\delta} L \arrow[r, "\mathrm{Exp}_j(M)\otimes\mathrm{id}"'] \arrow[u, "\mathrm{ev}_\delta"] & \Delta^{\mathrm{Iw}}_{L,j}(M) \otimes_{f_\delta} L \arrow[u, "\mathrm{ev}_\delta"'] \end{tikzcd}$$

for each $j = 1, 2, \emptyset$.

The following is the main theorem of this paper.

Theorem 4.1.1. *For any de Rham character $\delta : \Gamma \to L^\times$, the diagram*

$$\begin{tikzcd} & 1_L \arrow[dl, "\varepsilon_L^{\mathrm{dR}}(N(\delta))"'] \arrow[dr, "\varepsilon_L^{\mathrm{dR}}(M(\delta))"] & \\ \Delta_L\big(N(\delta)\big) \arrow[rr, "\mathrm{Exp}(M)_\delta"'] & & \Delta_L\big(M(\delta)\big) \end{tikzcd}$$

commutes.

Since de Rham ε-isomorphisms are compatible with base change, a similar statement for any de Rham character $\delta : \Gamma \to \overline{\mathbf{Q}}_p^\times$ is deduced from the above case by enlarging L if necessary.

Remark 4.1.2. The dual exponential map

$$\exp^*_{N^*} : \mathrm{H}^1_{\varphi,\gamma}(N) \to \mathbf{D}_{\mathrm{dR}}(N),$$

which $\varepsilon_L^{\mathrm{dR}}(N)$ consists mainly of, can be treated in a different way. Let $q = p$ if p is odd and $q = 4$ if $p = 2$. By Theorem 2.2.4, the composition of $\exp^*_{N^*}$ and the map

$$\mathrm{pr}_{\mathbf{Q}_p,N} : \mathrm{H}^1_\psi(N) \to \mathrm{H}^1_{\psi,\gamma}(N) : x \mapsto \left[\left(\frac{\log \chi(\gamma)}{q}x, 0\right)\right]$$

can be written as q^{-1} times the composition

$$T_{\mathbf{Q}_p} : \mathrm{H}^1_\psi(N) = (N^{(n)})^{\psi=1} \xrightarrow{\iota_n} \mathbf{D}^+_{\mathrm{dif}}(N) = L_n[\![t]\!] \otimes_L \mathbf{D}_{\mathrm{dR}}(N) \xrightarrow{t=0} \mathbf{D}_{\mathrm{dR}}(N).$$

We remark that the above maps $\mathrm{pr}_{\mathbf{Q}_p,N}$ and $T_{\mathbf{Q}_p}$ are used to express the interpolation formulae in [24].

Remark 4.1.3. Since $\varepsilon^{\mathrm{dR}}$-isomorphisms consist particularly of Bloch–Kato's exponential maps and dual exponential maps, Theorem 4.1.1 can be regarded as a generalized interpolation formula of big exponential maps in the context of the local ε-conjecture; our theorem treats general de Rham (φ, Γ)-modules and covers all of the twists by de Rham characters on Γ, that is, $\chi^k\tilde{\delta}$ for any $k \in \mathbf{Z}$ and any finite character $\tilde{\delta}$.

We remark that our theorem can be regarded as a refined interpolation formula, because $\varepsilon^{\mathrm{dR}}$-isomorphisms consist also of another exponential map

$$\exp_{f,M} : \mathbf{D}_{\mathrm{cris}}(M) \to \mathrm{H}^1_{\varphi,\gamma}(M),$$

which is important to study the exceptional zeros of p-adic L-functions (see [2, Proposition 2.2.4] for example).

The following corollary is an important consequence.

Corollary 4.1.4. *The existence of* $\varepsilon_L^{\mathrm{Iw}}(M)$ *is equivalent to that of* $\varepsilon_L^{\mathrm{Iw}}(N)$ *for* $N = \mathbf{N}_{\mathrm{rig}}(M)$. *More precisely, if one of them exists, then the other one also exists and we have the following commutative diagram:*

$$\begin{tikzcd} & 1_{\mathcal{R}_L^+(\Gamma)} \arrow[dl, "\varepsilon_L^{\mathrm{Iw}}(N)"'] \arrow[dr, "\varepsilon_L^{\mathrm{Iw}}(M)"] & \\ \Delta_L^{\mathrm{Iw}}(N) \arrow[rr, "\mathrm{Exp}(M)"'] & & \Delta_L^{\mathrm{Iw}}(M). \end{tikzcd}$$

Proof. If $\varepsilon_L^{\mathrm{Iw}}(N)$ (resp. $\varepsilon_L^{\mathrm{Iw}}(M)$) exists, then we define $\varepsilon_L^{\mathrm{Iw}}(M)$ (resp. $\varepsilon_L^{\mathrm{Iw}}(N)$) by

$$\varepsilon_L^{\mathrm{Iw}}(M) := \mathrm{Exp}(M) \circ \varepsilon_L^{\mathrm{Iw}}(N) \quad \big(\text{resp. } \varepsilon_L^{\mathrm{Iw}}(N) := \mathrm{Exp}(M)^{-1} \circ \varepsilon_L^{\mathrm{Iw}}(M)\big).$$

Since the isomorphism $\varepsilon_L^{\mathrm{Iw}}(N)$ (resp. $\varepsilon_L^{\mathrm{Iw}}(M)$) satisfies the commutative diagram in Conjecture 3.1.1 for arbitrary de Rham character δ by assumption, the isomorphism $\varepsilon_L^{\mathrm{Iw}}(M)$ (resp. $\varepsilon_L^{\mathrm{Iw}}(N)$) also satisfies the commutative diagram for arbitrary de Rham δ (in Conjecture) by Theorem 4.1.1, which shows that $\varepsilon_L^{\mathrm{Iw}}(M)$ (resp. $\varepsilon_L^{\mathrm{Iw}}(N)$) satisfies the conjecture. ∎

By this corollary, the conjecture for all the de Rham (φ, Γ)-modules is reduced to that for de Rham (φ, Γ)-modules with a structure of p-adic differential equation (equivalently, de Rham (φ, Γ)-modules with all Hodge–Tate weights 0). This equivalence was in fact effectively used in the second author's proof of the conjecture for rank 1 case [25, Propositions 4.11 and 4.16] (see also Remark 4.2.2.)

Remark 4.1.5. When M is a crystalline (φ, Γ)-module of rank r, we can construct $\varepsilon_L^{\mathrm{Iw}}(M)$ by Corollary 4.1.4 since we have the following concrete construction of (a scalar extension of) $\varepsilon_L^{\mathrm{Iw}}(N)$ by using the relation of N and $\mathbf{D}_{\mathrm{cris}}(M)$. We recall an isomorphism

$$\mathcal{R}_L(\Gamma) \xrightarrow{\sim} (\mathcal{R}_L)^{\psi=0} : \lambda \mapsto \lambda(1+T)^{-1},$$

and, for any (φ, Γ)-module D, another one

$$D^{\psi=1} \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_L(\Gamma) \xrightarrow{\sim} D^{\psi=0}$$

induced by the map $1-\varphi : D^{\psi=1} \to D^{\psi=0}$ [20, Proposition 4.3.8]. Using these, we obtain the following isomorphism

$$\begin{aligned} F : \mathcal{R}_L(\Gamma) \otimes_L \mathbf{D}_{\mathrm{cris}}(N) &\cong \mathcal{R}_L^{\psi=0} \otimes_L \mathbf{D}_{\mathrm{cris}}(N) \cong \big(\mathcal{R}_L \otimes_L \mathbf{D}_{\mathrm{cris}}(N)\big)^{\psi=0} \\ &\cong N^{\psi=0} \cong N^{\psi=1} \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_L(\Gamma). \end{aligned}$$

Note that the third isomorphism comes from the equality $N = \mathcal{R}_L \otimes_L \mathbf{D}_{\mathrm{cris}}(M) = \mathcal{R}_L \otimes_L \mathbf{D}_{\mathrm{cris}}(N)$, which follows from the fact that the (φ, Γ)-module $\mathcal{R}_L \otimes_L \mathbf{D}_{\mathrm{cris}}(M) \subseteq M[1/t]$ satisfies

$$\mathbf{D}^+_{\mathrm{dif},n(M)}\big(\mathcal{R}_L \otimes_L \mathbf{D}_{\mathrm{cris}}(M)\big) = L_{n(M)}[\![t]\!] \otimes_L \mathbf{D}_{\mathrm{cris}}(M) = L_{n(M)}[\![t]\!] \otimes_L \mathbf{D}_{\mathrm{dR}}(M).$$

Since we have $\mathrm{H}^2_\psi(N)\otimes_{\mathcal{R}^+_L(\Gamma)}\mathcal{R}_L(\Gamma)=0$ because $\mathrm{H}^2_\psi(N)$ is finite over L,

$$\mathcal{L}_L(N)=\mathcal{L}_L\big(\mathcal{R}_L\otimes_L\mathbf{D}_{\mathrm{cris}}(N)\big)=\big(\det_L\big(\mathbf{D}_{\mathrm{cris}}(N)\big),r\big)$$

by the definition of $\mathcal{L}_L(N)$, and $\mathcal{R}_L(\Gamma)$ is flat over $\mathcal{R}^+_L(\Gamma)$ [19, Proposition 3.4.1], F gives an isomorphism

$$1_{\mathcal{R}_L(\Gamma)}\xrightarrow{\sim}\Delta^{\mathrm{Iw}}_L(N)\otimes_{\mathcal{R}^+_L(\Gamma)}\mathcal{R}_L(\Gamma),$$

which turns out to be equal to $\varepsilon^{\mathrm{Iw}}_L(N)\otimes_{\mathcal{R}^+_L(\Gamma)}\mathcal{R}_L(\Gamma)$ by comparing with the construction of $\varepsilon^{\mathrm{Iw}}$-isomorphisms in [25]. In fact, one can construct $\varepsilon^{\mathrm{Iw}}_L(N)$ itself by decomposing N via a triangulation with respect to eigenvectors of $\mathbf{D}_{\mathrm{cris}}(N)$ into unramified (φ,Γ)-modules of rank 1 and using the construction [25, Definition 4.1] of the local epsilon isomorphisms of their cyclotomic deformations, and can show that the above construction gives $\varepsilon^{\mathrm{Iw}}_L(N)\otimes_{\mathcal{R}^+_L(\Gamma)}\mathcal{R}_L(\Gamma)$ (and also descends to $\varepsilon^{\mathrm{Iw}}_L(N)$).

Before proving the main theorem, we shall state an equivalent relative version of Theorem 4.1.1. As before, let L be a finite extension of $\mathbf{Q}_p$ and M, M' de Rham (φ,Γ)-modules over $\mathcal{R}_L$ with $M[1/t]=M'[1/t]$. For any character $\delta:\Gamma\to L^\times$, we denote $\mathrm{Exp}_j(M,M')_\delta$ as the isomorphism characterized by the commutative diagram

$$\begin{tikzcd}
\Delta_{L,j}\big(M(\delta)\big) \arrow[r, "\mathrm{Exp}_j(M,M')_\delta"] & \Delta_{L,j}\big(M'(\delta)\big) \\
\Delta^{\mathrm{Iw}}_{L,j}(M)\otimes_{f_\delta}L \arrow[u, "\mathrm{ev}_\delta"] \arrow[r, "\mathrm{Exp}_j(M,M')\otimes\mathrm{id}"'] & \Delta^{\mathrm{Iw}}_{L,j}(M')\otimes_{f_\delta}L \arrow[u, "\mathrm{ev}_\delta"']
\end{tikzcd}$$

for each $j=1,2,\emptyset$.

Theorem 4.1.6. *For any de Rham character $\delta:\Gamma\to L^\times$, the diagram*

$$\begin{tikzcd}
& 1_L \arrow[dl, "\varepsilon^{\mathrm{dR}}_L(M(\delta))"'] \arrow[dr, "\varepsilon^{\mathrm{dR}}_L(M'(\delta))"] & \\
\Delta_L\big(M(\delta)\big) \arrow[rr, "\mathrm{Exp}(M,M')_\delta"'] & & \Delta_L\big(M'(\delta)\big)
\end{tikzcd}$$

commutes.

We shall prove Theorem 4.1.1 in the rest of the paper as follows. In the Section 4.2, we prove Theorem 4.1.1 for (φ,Γ)-modules of rank 1 by using the explicit construction of $\varepsilon^{\mathrm{Iw}}_L$-isomorphisms for (φ,Γ)-modules of rank 1 in [25]. In the Section 4.3, we introduce a special class of (φ,Γ)-modules called generic, and reduce the proof of 4.1.1 for general (φ,Γ)-modules to that for generic ones using the result for rank 1 case. In the final Section 4.4, we complete the proof of Theorem 4.1.1 by proving Theorem 4.1.6 for generic (φ,Γ)-modules by using the key result Theorem 2.2.5.

4.2. Proof for rank one case

We prove Theorem 4.1.1 when M is of rank 1. We use the explicit construction of $\varepsilon_L^{\mathrm{Iw}}(M)$ obtained in [25].

Theorem 4.2.1. *When M is of rank 1, the diagram of Theorem 4.1.1 commutes.*

Proof. By [25, Theorem 3.11], the isomorphisms $\varepsilon_L^{\mathrm{Iw}}(M)$ and $\varepsilon_L^{\mathrm{Iw}}(N)$ exist. Moreover, since we have $N = t^{-h_M}M$, it suffices to show that the diagram

$$\begin{tikzcd} & \mathbf{1}_{\mathcal{R}_L^+(\Gamma)} \arrow[dl, "\varepsilon_L^{\mathrm{Iw}}(M)"'] \arrow[dr, "\varepsilon_L^{\mathrm{Iw}}(tM)"] & \\ \Delta_L^{\mathrm{Iw}}(M) \arrow[rr, "\mathrm{Exp}(M{,}tM)"'] & & \Delta_L^{\mathrm{Iw}}(tM) \end{tikzcd}$$

commutes.

Since the objects in the diagram are free, we shall show that the diagram commutes after base change to the inclusion $\mathcal{R}_L^+(\Gamma) \hookrightarrow \mathcal{R}_L(\Gamma)$. By the explicit construction in [25, Section 4A], for a general (φ,Γ)-module M of rank 1, the isomorphism

$$\varepsilon_L^{\mathrm{Iw}}(M) \otimes_{\mathcal{R}_L^+(\Gamma)} \mathrm{id}_{\mathcal{R}_L(\Gamma)} : \mathbf{1}_{\mathcal{R}_L(\Gamma)} \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(M) \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_L(\Gamma)$$

is obtained by the isomorphisms

$$\theta_1 = 1-\varphi : \Delta_{L,1}^{\mathrm{Iw}}(M) \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_L(\Gamma) \xrightarrow{\sim} \big((\mathcal{R}_L e_{\delta_M})^{\psi=0}, 1\big)^{-1},$$
$$\theta_2 : \mathcal{R}_L(\Gamma) \otimes_L L e_{\delta_M} \xrightarrow{\sim} (\mathcal{R}_L e_{\delta_M})^{\psi=0}; \lambda \otimes e_{\delta_M} \mapsto \lambda\big((1+X)^{-1} e_{\delta_M}\big),$$

where we put $\delta_M : \mathbf{Q}_p^{\times} \to L^{\times}$ as the character corresponding to M. Since $\mathrm{Exp}_1(M, tM)$ is induced by multiplying ∇_{h_M}, and we can calculate

$$\begin{aligned} &\nabla_{h_M}\big(\lambda\big((1+X)^{-1}e_{\delta_M}\big)\big) \\ &\quad = \lambda\big(\big(\nabla_0\big((1+X)^{-1}e_{\delta_M}\big)\big) - h_M(1+X)^{-1}e_{\delta_M}\big) \\ &\quad = \lambda\big(-t(1+X)^{-1}e_{\delta_M} + (1+X)^{-1}(h_M e_{\delta_M}) - h_M(1+X)^{-1}e_{\delta_M}\big) \\ &\quad = -\lambda\big((1+X)^{-1}te_{\delta_M}\big), \end{aligned}$$

our assertion follows from the equality $\mathrm{Exp}_2(M,tM)(e_{\delta_M}) = -te_{\delta_M}$. ∎

Remark 4.2.2. Theorem 4.2.1 shows that our main theorem exactly generalizes the [25, Proposition 4.13], which is proved in a different way and used in the proof of the local ε-conjecture for (φ,Γ)-modules of rank 1.

4.3. Reduction to generic case

In this subsection, we define genericity of a (φ,Γ)-module and reduce the proof of our main theorem for the general case to that for the generic case.

Definition 4.3.1. A (φ, Γ)-module M over $\mathcal{R}_L$ is generic if for any continuous character $\delta : \Gamma \to \overline{\mathbf{Q}}_p^\times$ we have $\mathbf{D}_{\mathrm{cris}}(M(\delta)) = 0$ and $\mathbf{D}_{\mathrm{cris}}(M(\delta)^*) = 0$.

For general (φ, Γ)-modules M, M' with $M[1/t] = M'[1/t]$, M is generic if and only if M' is generic by definition of $\mathbf{D}_{\mathrm{cris}}$. In particular, if a generic (φ, Γ)-module M is de Rham, then the attached p-adic differential equation $N = \mathbf{N}_{\mathrm{rig}}(M)$ is also generic.

Lemma 4.3.2. *Let M be a generic (φ, Γ)-module over $\mathcal{R}_L$. Then we have* $\mathrm{H}^2_\psi(M) = \mathrm{H}^2_\psi(M^*) = 0$.

Proof. We shall show $\mathrm{H}^2_\psi(M) = 0$. For any continuous character $\delta : \Gamma \to \overline{\mathbf{Q}}_p^\times$, we have

$$\mathrm{H}^0_{\varphi,\gamma}\big(M(\delta)^*\big) \subseteq \big(M(\delta)^*\big)^\Gamma \subseteq \mathbf{D}_{\mathrm{cris}}\big(M(\delta)^*\big) = 0,$$

thus the Tate duality gives $\mathrm{H}^2_{\varphi,\gamma}(M(\delta)) = 0$. For any maximal ideal $\mathfrak{m} \subseteq \mathcal{R}_L^+(\Gamma)$, we obtain

$$\begin{aligned}\mathrm{H}^2_\psi(M)/\mathfrak{m}\mathrm{H}^2_\psi(M) &= \mathrm{H}^2_\psi(M) \otimes_{\mathcal{R}_L^+(\Gamma)} \big(\mathcal{R}_L^+(\Gamma)/\mathfrak{m}\big)\\ &\cong M^\Delta/(\psi - 1, \mathfrak{m}) \cong \mathrm{H}^2_{\varphi,\gamma}\big(M(\delta_\mathfrak{m}^{-1})\big) = 0\end{aligned}$$

by taking the corresponding character $\delta_\mathfrak{m} : \Gamma \to \overline{\mathbf{Q}}_p^\times$. Since $\mathrm{H}^2_\psi(M)$ is finitely generated as a module over $\mathcal{R}_L^+(\Gamma)$ (and even over L [20, Proposition 3.3.2]) and so is the localization $\mathrm{H}^2_\psi(M)_\mathfrak{m}$ over $\mathcal{R}_L^+(\Gamma)_\mathfrak{m}$, Nakayama's lemma gives $\mathrm{H}^2_\psi(M)_\mathfrak{m} = 0$. Since $\mathfrak{m}$ is arbitrary, we obtain our assertion. ■

The cohomologies of a (φ, Γ)-module whose second ψ-cohomology and that of its dual vanish are quite simple.

Lemma 4.3.3. *Let M be a (φ, Γ)-module over $\mathcal{R}_L$ of rank r such that* $\mathrm{H}^2_\psi(M) = \mathrm{H}^2_\psi(M^*) = 0$. *Then one has* $\mathrm{H}^i_{\varphi,\gamma}(M) = 0$ *for* $i = 0, 2$, *and* $\dim_L(\mathrm{H}^1_{\varphi,\gamma}(M)) = r$. *The first ψ-cohomology* $\mathrm{H}^1_\psi(M)$ *is free of rank r over* $R_L^+(\Gamma)$ *and for any continuous character* $\delta : \Gamma \to L^\times$, *one has a canonical isomorphism* $\mathrm{H}^1_\psi(M) \otimes_{f_\delta} L \cong \mathrm{H}^1_{\varphi,\gamma}(M(\delta))$.

Proof. For example, see [20, Section 5]. ■

We reduce the proof of our main theorem to the generic case via the next proposition.

Proposition 4.3.4. *Assume that Theorem 4.1.1 holds for any L and for all of the de Rham generic (φ, Γ)-modules over $\mathcal{R}_L$. Then, Theorem 4.1.1 holds unconditionally.*

Proof. Let M be a de Rham (φ, Γ)-module over $\mathcal{R}_L$ of rank r. We prove Theorem 4.1.1 for M and $N = \mathbf{N}_{\mathrm{rig}}(M)$ by induction on r.

The base case $r = 1$ has been proved as Theorem 4.2.1.

Suppose that $r \geq 2$. We assume that Theorem 4.1.1 holds for all the de Rham (φ, Γ)-modules over $\mathcal{R}_L$ of rank $\leqq r - 1$. If M is not generic, we have $\mathbf{D}_{\mathrm{cris}}(M(\delta)) \neq 0$ for some character $\delta : \Gamma \to \overline{\mathbf{Q}}_p^\times$. Extending L if necessary, we may assume that $\delta(\Gamma) \subseteq L^\times$ and that there is a nonzero φ-eigenvector $x \otimes e_\delta \in \mathbf{D}_{\mathrm{cris}}(M(\delta)) = (M(\delta)[1/t])^\Gamma$ with $x \in M[1/t]$. Then, the submodule $\mathcal{R}_L[1/t]x \subseteq M[1/t]$ is stable under (φ, Γ)-actions. Since $\mathcal{R}_L$ is a Bézout domain, it turns out that its saturation $M' := \mathcal{R}_L[1/t]x \cap M \subsetneq M(\delta)$ and the quotient $M(\delta)/M'$ are (φ, Γ)-modules. Therefore, by considering an exact sequence

$$0 \to M'(\delta^{-1}) \to M \to \big(M(\delta)/M'\big)(\delta^{-1}) \to 0$$

of de Rham (φ, Γ)-modules, Lemma 3.2.7 gives our assertion. ∎

4.4. Proof for generic case

In this subsection, we shall show Theorem 4.1.6 for de Rham generic (φ, Γ)-modules. This gives Theorem 4.1.6 in full generality in conjunction with Proposition 4.3.4.

In the following, let M, M' be de Rham generic (φ, Γ)-modules of rank r over $\mathcal{R}_L$ satisfying $M[1/t] = M'[1/t]$. For a technical reason, we introduce another (φ, Γ)-module M^+. Let $h_1 \leq \cdots \leq h_r$ be the Hodge–Tate weights of M. Let $\alpha_1, \ldots, \alpha_r \in \mathbf{D}_{\mathrm{dR}}(M)$ be a basis; taking along the filtration of $\mathbf{D}_{\mathrm{dR}}(M)$, we may assume that $t^{h_i}\alpha_i \in \mathbf{D}^+_{\mathrm{dif}}(M)$ for each i, and that $(t^{h_i}\alpha_i)_{1\leq i\leq r}$ is a basis of $\mathbf{D}^+_{\mathrm{dif}}(M)$. Then [6, Theorem II.1.2] gives that there exists a unique (φ, Γ)-module $M^+ \subseteq M$ such that

$$\mathbf{D}^+_{\mathrm{dif},n(M)}(M^+) = L_{n(M)}[\![t]\!] \cdot t^{h_1+1}\alpha_1 \oplus \Big(\bigoplus_{2\leq i\leq r} L_{n(M)}[\![t]\!] \cdot t^{h_i}\alpha_i\Big).$$

It is clear that M^+ is also de Rham and generic.

Note that, since the big exponential maps are transitive by definition, $t^h\mathbf{N}_{\mathrm{rig}}(M) \subseteq M, M'$ for a sufficient large $h \in \mathbf{Z}_{>0}$, and t^hN is obtained by the above procedure repeatedly starting from M, it suffices to prove the case $M' = M^+$. Moreover, by Lemma 3.2.8, we may assume that $\delta = \mathbf{1}$.

In summary, it is sufficient to prove that the diagram

$$\begin{tikzcd} & \mathbf{1}_L \arrow[dl, "\varepsilon^{\mathrm{dR}}_L(M)"'] \arrow[dr, "\varepsilon^{\mathrm{dR}}_L(M^+)"] & \\ \Delta_L(M) \arrow[rr, "\mathrm{Exp}(M,M^+)_\mathbf{1}"'] & & \Delta_L(M^+). \end{tikzcd}$$

commutes.

Lemma 4.4.1. *The diagram*

$$\begin{tikzcd} \mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M)\big) \arrow[r, "\times(-1)"] & \mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M^+)\big) \\ \Delta_{L,2}(M) \arrow[u, "f_M"] \arrow[r, "\mathrm{Exp}_2(M,M^+)_\mathbf{1}"'] & \Delta_{L,2}(M^+), \arrow[u, "f_{M^+}"'] \end{tikzcd}$$

commutes.

Proof. This follows from the direct calculation

$$\begin{aligned} f_{M^+}\big(\mathrm{Exp}_2(M,M^+)_{\mathbf{1}}(x)\big) &= f_{M^+}(-tx) = -\frac{1}{\varepsilon(M^+)}\frac{1}{t^{h_{M^+}}}\otimes\varphi^n(tx) \\ &= -\frac{1}{\varepsilon(M^+)}\frac{t}{t^{h_{M^+}}}\otimes\varphi^n(x) = -\frac{1}{\varepsilon(M^+)}\frac{1}{t^{h_M}}\otimes\varphi^n(x) \\ &= -f_M(x), \end{aligned}$$

where $x \in \mathcal{L}_L(M)$ is any element and $n \geq \max\{n(M), n(M^+)\}$. We note that the last equality follows from the fact that for two de Rham (φ,Γ)-modules D, D' with $D[1/t] = D'[1/t]$, the corresponding filtered $(\varphi, N, \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p))$-modules are the same, so are the attached ε-constants. ■

Thus, our claim is deduced from the following lemma.

Lemma 4.4.2. *The diagram*

$$\begin{tikzcd} & \mathbf{1}_L \arrow[dl, "\Gamma(M)\theta(M)"'] \arrow[dr, "\Gamma(M^+)\theta(M^+)"] & \\ \Delta_{L,1}(M)\boxtimes \mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M)\big) \arrow[rr, "-\mathrm{Exp}_1(M,M^+)_{\mathbf{1}}\otimes_L \mathrm{id}"'] & & \Delta_{L,1}(M^+)\boxtimes \mathrm{Det}_L\big(\mathbf{D}_{\mathrm{dR}}(M^+)\big) \end{tikzcd}$$

commutes.

Proof. We first give explicit descriptions of the isomorphisms appearing in the diagram. Let D be a general de Rham generic (φ,Γ)-module. By Lemma 4.3.3, we have canonical quasi-isomorphisms $C^\bullet_\psi(D) \cong \mathrm{H}^1_\psi(D)[-1]$ and $C^\bullet_{\varphi,\gamma}(D) \cong \mathrm{H}^1_{\varphi,\gamma}(D)[-1]$. Thus, the isomorphism

$$\mathrm{Exp}_1(M,M^+)_{\mathbf{1}} : \Delta_{L,1}(M) \xrightarrow{\sim} \Delta_{L,1}(M^+)$$

is induced by the isomorphism

$$\mathrm{Det}_L\big(\mathrm{H}^1_{\varphi,\gamma}(M)\big) \xrightarrow{\sim} \mathrm{Det}_L\big(\mathrm{H}^1_{\varphi,\gamma}(M^+)\big) : x_1\wedge\cdots\wedge x_r \mapsto \nabla_{h_1}(x_1\wedge\cdots\wedge x_r).$$

Next we consider $\theta(D)$. By the assumption of genericity, we have $\mathrm{H}^i_{\varphi,\gamma}(D) = 0$ for $i = 0, 2$ and $\mathbf{D}_{\mathrm{cris}}(D) = 0$ again by Lemma 4.3.3. Thus, the fundamental line for D is a complex

$$C^\bullet(D) : 0 \to t(D)_3 \xrightarrow{\exp_D} \mathrm{H}^1_{\varphi,\gamma}(D)_{f,4} \to 0,$$

and by taking dual as complex and then using Tate duality and the de Rham duality for the complex $C^\bullet(D^*)$, we obtain another complex

$$\widetilde{C}^\bullet(D^*) : 0 \to \mathrm{H}^1_{\varphi,\gamma}(D)_{/f,-4} \xrightarrow{-\exp^*_{D^*}} \mathbf{D}^0_{\mathrm{dR}}(D)_{-3} \to 0,$$

where the index appearing at each space expresses its degree in the complexes. Using the above forms of $C^\bullet(D)$ and $\widetilde{C}^\bullet(D^*)$, we obtain the following explicit expression of $\theta(D)$

by definition; if we put $d_0(D) = \dim_L(\mathbf{D}^0_{\mathrm{dR}}(D))$, then for any basis $(\beta_i)_{1\le i\le r}$ of $\mathbf{D}_{\mathrm{dR}}(D)$ such that $\beta_{r-d_0(D)+1}, \ldots, \beta_r$ span $\mathbf{D}^0_{\mathrm{dR}}(D)$, $\theta(D)^{-1}$ is characterized by the correspondence

$$\left[\exp_D(\overline{\beta_1}) \wedge \cdots \wedge \exp_D(\overline{\beta_{r-d_0(D)}}) \wedge \beta^{*D}_{r-d_0(D)+1} \wedge \cdots \wedge \beta^{*D}_r \mapsto 1\right] \otimes (\beta_1 \wedge \cdots \wedge \beta_r) \mapsto (-1)^{d_0(D)},$$

where β_i^{*D} are any lifts of β_i with respect to $\exp^*_{D^*}$.

Using the above descriptions, we can state the asserted commutativity in a more concrete form. We define the element X of $(\mathrm{Det}_L(\mathrm{H}^1_{\varphi,\gamma}(D(M))))^{-1} \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(M))$ as

$$\left[\exp_M(\overline{\alpha_r}) \wedge \cdots \wedge \exp_M(\overline{\alpha_{d_0(M)+1}}) \wedge \alpha_1^{*M} \wedge \cdots \wedge \alpha^{*M}_{d_0(M)} \mapsto 1\right] \otimes \big(\alpha_r \wedge \cdots \wedge \alpha_{d_0(M)+1} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{d_0(M)}\big).$$

In the following, we write $\mathcal{E}$ instead of the isomorphism $\mathrm{Exp}_1(M, M^+)_{\mathbf{1}} \otimes_L \mathrm{id}$ for simplicity of notation. Then, since X is a basis by the definition of $(\alpha_i)_{1\le i\le r}$, our claim reduces to show the commutativity at X, that is, the equality

$$\Gamma(M^+)^{-1}\theta(M^+)^{-1}\big(-\mathcal{E}(X)\big) = \Gamma(M)^{-1}\theta(M)^{-1}(X),$$

or, furthermore, by the description of $\theta(D)$ for $D = M$, the equality

$$-\Gamma(M^+)^{-1}\theta(M^+)^{-1}\big(\mathcal{E}(X)\big) = (-1)^{d_0(M)}\Gamma(M)^{-1}.$$

By our construction of M^+, we have $\nabla_{h_1}(\mathbf{D}^+_{\mathrm{dif},m}(M)) \subseteq \mathbf{D}^+_{\mathrm{dif},m}(M^+)$ for all $m \ge n(M)$. Therefore, we can verify the above equality essentially by Theorem 2.2.5 as follows.

For the case $h_1 < 0$, Theorem 2.2.5 (i) gives that $\nabla_{h_1}(\alpha_1^{*M}) = -h_1\alpha_1^{*M^+}$, so one obtains

$$\mathcal{E}(X) = (-h_1)^{-1}\left[\exp_{M^+}(\overline{\alpha_r}) \wedge \cdots \wedge \exp_{M^+}(\overline{\alpha_{d_0(M)+1}}) \wedge \alpha_1^{*M^+} \wedge \cdots \wedge \alpha^{*M^+}_{d_0(M)} \mapsto 1\right] \otimes \big(\alpha_r \wedge \cdots \wedge \alpha_{d_0(M)+1} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{d_0(M)}\big).$$

Since we have $\mathbf{D}^0_{\mathrm{dR}}(M^+) = \mathbf{D}^0_{\mathrm{dR}}(M)$, by the description of $\theta(D)$ for $D = M^+$ we obtain

$$\theta(M^+)^{-1}\big(\mathcal{E}(X)\big) = (-1)^{d_0(M^+)}(-h_1)^{-1} = (-1)^{d_0(M)+1}h_1^{-1}.$$

Thus, the desired equality is rewritten as

$$h_1^{-1}\Gamma(M^+)^{-1} = \Gamma(M)^{-1},$$

which holds by the relation $\Gamma^*(k+1) = k \cdot \Gamma^*(k)$ for any nonzero $k \in \mathbf{Z}$.

The case $h_1 > 0$ follows similarly to the previous case $h_1 < 0$ by using Theorem 2.2.5 (ii) instead of Theorem 2.2.5 (i).

For the last case $h_1 = 0$, canceling $\Gamma(M) = \Gamma(M^+)$ from the equality our assertion becomes the following one:

$$-\theta(M^+)^{-1}\big(\mathcal{E}(X)\big) = (-1)^{d_0(M)}.$$

Theorem 2.2.5 (iii) gives that $\nabla_0(\alpha_1^{*M}) = \exp_{M^+}(\overline{\alpha_1})$, thus we obtain

$$\begin{aligned}\mathcal{E}(X) = \big[\exp_{M^+}(\overline{\alpha_r}) \wedge \cdots \wedge \exp_{M^+}(\overline{\alpha_{d_0(M)+1}}) \wedge \exp_{M^+}(\overline{\alpha_1}) \wedge \alpha_2^{*M^+} \wedge \cdots \wedge \alpha_{d_0(M)}^{*M^+} \mapsto 1\big]\\ \otimes \big(\alpha_r \wedge \cdots \wedge \alpha_{d_0(M)+1} \wedge \alpha_1 \wedge \cdots \wedge \alpha_{d_0(M)}\big).\end{aligned}$$

In this case, the elements $\alpha_2, \ldots, \alpha_{d_0(M)}$ span $\mathbf{D}^0_{\mathrm{dR}}(M^+)$, therefore we can use again the previous explicit description of $\theta(D)$ for $D = M^+$ and obtain

$$\theta(M^+)^{-1}\big(\mathcal{E}(X)\big) = (-1)^{d_0(M^+)} = (-1)^{d_0(M)-1},$$

which completes all the cases and finishes the proof. ∎

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