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Gluing complexes of sheaves

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Abstract. We prove two variations of the classical gluing result of Beilinson–Bernstein–Deligne. We recast the problem of gluing in terms of filtered complexes in the total topos of a D-topos, in the sense of SGA 4, and prove our results using the filtered derived category.

1. Statements of results

1.1. The work in this paper grew out of investigating problems of gluing in two different contexts:

(a) The results of this article are used in [5] where the following problem is studied. Let $f : X \to S$ and $g : Y \to S$ be two morphisms of schemes over a field k and let $K \in D(X \times_k Y)$ be an object in the derived category of coherent sheaves. Motivated by the study of derived equivalences, we would like to understand reasonable conditions on K that ensure that K is the pushforward of a complex on the fiber product $X \times_S Y$. A variant of this was considered in the context of ∞-categories in [2], and the transition from that work to ordinary derived categories can be viewed as requiring a variant of the classical BBD gluing lemma [3, Théorème 3.2.4] for *cosimplicial* schemes. See in particular the proof of [5, Theorem 1.1].

(b) Given a finite diagram of schemes one can consider compatible collections of ℓ-adic complexes of sheaves on this diagram and ask about the relationship between this category and the derived category of ordinary ℓ-adic sheaves on the diagram. This problem arose in the study of sheaves on stacks, and its resolution again requires a variant of the BBD gluing lemma for diagrams of ℓ-adic sheaves.

We present here a general approach to these kinds of problems proving a generalization of the BBD gluing lemma for very general diagrams of sheaves.

1.2. Let D be a category and let T be a D-topos in the sense of [1, $\mathrm{V}^{\mathrm{bis}}$, Définition 1.2.1]. For an object $d \in D$ we write T_d for the fiber of T over d (so T_d is a topos), and for each morphism $\delta : d \to e$ in D we write $f_\delta : T_e \to T_d$ for the corresponding morphism of topoi (see loc. cit.).

We write $\mathrm{Sh}(T)$ for the category of sheaves in T. The category $\mathrm{Sh}(T)$ can be described as the category of systems $(\{F_d\}_{d \in D}, \{\sigma_\delta\}_{\delta \in \mathrm{Mor}(D)})$, consisting of an object $F_d \in T_d$

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for each $d \in D$ and for every morphism $\delta : d \to e$ in D a morphism $\sigma_\delta : f_\delta^* F_d \to F_e$ satisfying a natural compatibility with composition.

Let Λ be a sheaf of rings in T, and let Λ_d be its component in T_d. Let $\mathrm{Sh}(T, \Lambda)$ be the category of Λ-modules in $\mathrm{Sh}(T)$. For each $d \in D$ there is a restriction functor

$$e_d : \mathrm{Sh}(T) \to \mathrm{Sh}(T_d), \quad \big(\{F_d\}, \{\sigma_\delta\}\big) \mapsto F_d$$

which induces a functor (which we again denote by e_d)

$$\mathrm{Sh}(T, \Lambda) \to \mathrm{Sh}(T_d, \Lambda_d).$$

For an object $M \in \mathrm{Sh}(T, \Lambda)$ we often write M_d for $e_d M \in \mathrm{Sh}(T_d, \Lambda_d)$. For $* \in \{\emptyset, b, +, -\}$ we have the corresponding derived category $D^*(T, \Lambda)$ of $\mathrm{Sh}(T, \Lambda)$. We can also consider the triangulated subcategory $D^{(*)}(T, \Lambda) \subset D(T, \Lambda)$ consisting of complexes $M \in D(T, \Lambda)$ for which $M_d \in D^*(T_d, \Lambda_d)$ for all d.

Remark 1.3. As we discuss at the end of the introduction, the classical BBD gluing lemma [3, Théorème 3.2.4] can be formulated in terms of complexes in a Δ-topos, where Δ is the standard simplicial category. It is also interesting to consider the theory for $D = \Delta^{\mathrm{op}}$ (cosimplicial topoi), which arise naturally for example in [2, Theorem 4.7].

1.4. Given a category D and D-topos T, define Γ to be the category of systems $(\{M_d\}_{d\in D}, \{\varphi_\delta\})$, where

(i) $M_d \in D^b(T_d, \Lambda_d)$ is an object for each $d \in D$

(ii) for each morphism $\delta : c \to d$ we are given a morphism

$$\varphi_\delta : M_c \to Rf_{\delta *} M_d$$

compatible with compositions in D in the sense that for a triple $c \xrightarrow{\delta} d \xrightarrow{\varepsilon} e$ the diagram

$$\begin{array}{ccccc} M_c & \xrightarrow{\varphi_\delta} & Rf_{\delta *} M_d & \xrightarrow{\varphi_\varepsilon} & Rf_{\delta *} Rf_{\varepsilon *} M_e \\ & \searrow^{\varphi_{\varepsilon\delta}} & & & \downarrow \simeq \\ & & & & Rf_{\varepsilon\delta *} M_e \end{array}$$

commutes.

There is a functor

$$D^b(T, \Lambda) \to \Gamma, \quad M \mapsto \big(\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}\big) \tag{1.4.1}$$

sending a complex to its restrictions with the natural transition maps. The two basic problems we consider here are the following:

(i) Given $M, M' \in D^b(T, \Lambda)$, formulate conditions under which the map

$$\mathrm{Hom}_{D^b(T,\Lambda)}(M, M') \to \mathrm{Hom}_\Gamma\Big(\big(\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}\big), \big(\{M'_d\}, \{\varphi_\delta^{\mathrm{can}}\}\big)\Big) \tag{1.4.2}$$

is an isomorphism.

(ii) Given a system $(\{M_d\}, \{\varphi_\delta^{\mathrm{can}}\}) \in \Gamma$, formulate conditions on the system that imply that it is in the essential image of (1.4.1).

The main results are the following.

Theorem 1.5. *Let $M, M' \in D^b(T, \Lambda)$ be two objects such that for every morphism $\delta : c \to d$ in D we have*

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(M_c, Rf_{\delta *}M_d) = 0$$

for $i < 0$. Then the map (1.4.2) is an isomorphism.

Remark 1.6. Note that the condition in Theorem 1.5 could also be formulated as

$$\mathrm{Ext}^i_{D(T_d,\Lambda_d)}(\mathrm{L}f_\delta^* M_c, M_d) = 0$$

for $i < 0$.

Theorem 1.7. *Let $(\{M_d\}, \{\varphi_\delta\})$ be an object of Γ such that there exists $a \leq b$ for which $M_d \in D^{[a,b]}(T_d, \Lambda_d)$ (the subcategory of complexes whose nonzero cohomology all lies between a and b) for all $d \in D$. Suppose one of the following conditions hold:*

(i) *for every diagram*

$$\begin{array}{ccc} d & & \\ \uparrow{\scriptstyle \delta} & & \\ c & \xrightarrow{\gamma} & e \end{array}$$

we have

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(Rf_{\delta *}M_d, Rf_{\gamma *}M_e) = 0 \tag{1.7.1}$$

for $i < 0$; or

(ii) *for every diagram*

$$\begin{array}{ccc} d & & \\ \downarrow{\scriptstyle \delta} & & \\ c & \xleftarrow{\gamma} & e \end{array}$$

we have

$$\mathrm{Ext}^i_{D(T_c,\Lambda_c)}(\mathrm{L}f_\delta^* M_d, \mathrm{L}f_\gamma^* M_e) = 0 \tag{1.7.2}$$

for $i < 0$

Then $(\{M_d\}, \{\varphi_\delta\})$ is in the essential image of (1.4.1).

Remark 1.8. Jacob Lurie explained in private correspondence the following alternate, though closely related, approach proving the above in a more general setting of ∞-categories. Contemplation of his argument led, in particular, to removal of certain unnecessary assumptions in earlier drafts.

Though we will not use ∞-categorical perspective in this article, for the reader familiar with this language we can rephrase the above as follows. For a ringed topos (T, Λ) let $\mathscr{D}(T, \Lambda)$ denote the ∞-category of complexes of Λ-modules in T. The association $(T, \Lambda) \mapsto \mathscr{D}(T, \Lambda)$ can be upgraded to an ∞-functor

$$(\text{ringed topoi}) \to (\text{stable } \infty\text{-categories}).$$

This is explained in a very general context in [6, Notation 2.4.4 and following paragraph].

For a ringed D-topos (T, Λ) we therefore get a functor

$$D \to (\text{stable } \infty\text{-categories}), \quad d \mapsto \mathscr{D}(T_d, \Lambda_d).$$

Using straightening/unstraightening (∞-categorical version of the Grothendieck construction) this gives rise to a cocartesian fibration of ∞-categories $U \to N(D)$ whose fiber over $d \in D$ is $\mathscr{D}(T_d, \Lambda_d)$ and for which the infinity category of sections $N(D) \to U$ is the derived category $\mathscr{D}(T, \Lambda)$ of the total topos. This allows for the problems considered in this article to be considered in a very general context: Let $U \to D$ be a cocartesian fibration of ∞-categories (where we switch notation replacing $N(D)$ by an arbitrary ∞-category as the base). Given a partial section defined on a subcomplex of D one can then ask for conditions under which this partially defined section extends uniquely to a section on all D. The negative vanishing conditions in the above results can be viewed as vanishing of the relevant obstruction spaces as one extends the partially defined section one simplex at a time.

Example 1.9. The classical BBD gluing theorem [3, Théorème 3.2.4] can be viewed as a special case of Theorem 1.7.

Let (T, Λ) be a topos and let $U \in T$ be an object covering the final object of T. Let $M_U \in D^b(T|_U, \Lambda_U)$ be an object equipped with an isomorphism

$$\varepsilon : \mathrm{pr}_1^* M_U \to \mathrm{pr}_2^* M_U$$

in $D(T|_{U\times U}, \Lambda_{U\times U})$ satisfying the cocycle condition over $U \times U \times U$. Suppose that

$$\mathscr{E}xt^i_{T|_U}(M_U, M_U) = 0 \tag{1.9.1}$$

for $i < 0$. Then we claim that (M, ε) is induced by a unique object of $D^b(T, \Lambda)$.

For this we apply Theorem 1.7 with $D = \Delta$ and consider the simplicial topos given by $T|_{U_\bullet}$, where $U_\bullet$ is the coskeleton of $U \to *$. So the fiber of $T|_{U_\bullet}$ over $[n] \in \Delta$ is the topos $T|_{U^{n+1}}$. For each $[n] \in \Delta$ let $M_n \in D(T|_{U_n}, \Lambda_{U_n})$ be the pullback of M_U along the first projection $U_n \to U$. The isomorphism ε defines maps φ_δ (using the cocycle condition) so we get an object

$$\big(\{M_n\}, \{\varphi_\delta\}\big) \in \Gamma. \tag{1.9.2}$$

The vanishing condition (1.9.1) implies that condition (ii) in Theorem 1.7 holds. Indeed the vanishing of the local Ext-groups implies that $\mathrm{RHom}(M, M) \in D^{\geq 0}(T, \Lambda)$ and therefore for all $[n] \in \Delta$ the complex $\mathrm{RHom}_{T|_{U_n}}(M_n, M_n)$ is also concentrated in degrees ≥ 0. Combined with the observation that for every morphism $\delta : [n] \to [m]$ in Δ the induced map $\mathrm{L} f_\delta^* M_n \to M_m$ is an isomorphism we get condition (ii). Therefore the system (1.9.2) is induced by a unique object $M_\bullet \in D(T|_{U_\bullet}, \Lambda_{U_\bullet})$. The object $M_\bullet$ is cartesian by construction. Therefore, by [7, Tag 0D8I] the object $M_\bullet$ is induced by a unique object $M \in D(T, \Lambda)$.

Remark 1.10. Note that the cartesian condition only enters in at the very end of the argument and the principal issue is to construct the object $M_\bullet$.

2. Preliminaries on the filtered derived category

2.1. Let (T, Λ) be a ringed topos and let $\mathrm{DF}(T, \Lambda)$ denote the filtered derived category of Λ-modules [4, V] (see also [7, Tag 05RX]). Objects of $\mathrm{DF}(T, \Lambda)$ are complexes K equipped with a finite decreasing filtration, indexed by $\mathbf{Z}$. Note that this differs slightly from the definition in [7, Tag 05RX], but this distinction will not be important here. We also consider the subcategory $D^bF(T, \Lambda)$ (resp. $D^+F(T, \Lambda)$, $D^-F(T, \Lambda)$) of $\mathrm{DF}(T, \Lambda)$ consisting of objects K such that for all i the complex $\mathrm{gr}^i_F K$ is a bounded (resp. bounded below, bounded above) complex.

Lemma 2.2. *Let $H \in \mathrm{DF}(\mathcal{A})$ be an object in the filtered derived category of an abelian category $\mathcal{A}$ such that $\mathrm{gr}^n H \in D^{\geq n}(\mathcal{A})$ for all $n \geq 0$. Then for all $s \geq 0$ we have $F^s H \in D^{\geq s}(\mathcal{A})$ and the sequence*

$$0 \to H^s(F^s H) \to H^s(\mathrm{gr}^s H) \to H^{s+1}(\mathrm{gr}^{s+1} H),$$

obtained from the projection map $H^s(F^s H) \to H^s(F^s H/F^{s+2} H)$ and the distinguished triangle

$$\mathrm{gr}^{s+1} H \to F^s H/F^{s+2} H \to \mathrm{gr}^s H \to \mathrm{gr}^{s+1} H[1],$$

is exact.

Proof. We proceed by descending induction on s. By the definition of $\mathrm{DF}(\mathcal{A})$ the result holds for s sufficiently big, so it suffices to show that if the result holds for $s + 1$ then it also holds for s. For this note first that by the distinguished triangle

$$F^{s+1} H \to F^s H \to \mathrm{gr}^s H \to F^{s+1} H[1]$$

and the inductive hypothesis, which implies that $H^j(F^{s+1}) = 0$ for $j < s + 1$, we get that $F^s H \in D^{\geq s}(\mathcal{A})$ and an exact sequence

$$0 \to H^s(F^s H) \to H^s(\mathrm{gr}^s H) \to H^{s+1}(F^{s+1} H).$$

Since the map

$$H^{s+1}(F^{s+1} H) \to H^{s+1}(\mathrm{gr}^{s+1} H)$$

is injective, by the inductive hypothesis, we then get the result. ∎

Lemma 2.3. *Let $E \in D^-F(T, \Lambda)$ and $E' \in D^+F(T, \Lambda)$ be objects such that*

$$\mathrm{Ext}^s(\mathrm{gr}^i E, \mathrm{gr}^j E') = 0$$

for all $i \leq j$ and $s < j - i$. Then the sequence

$$0 \to \mathrm{Hom}_{\mathrm{DF}(T,\Lambda)}(E, E') \to \bigoplus_i \mathrm{Hom}_{D(T,\Lambda)}(\mathrm{gr}^i E, \mathrm{gr}^i E') \to \bigoplus_i \mathrm{Hom}_{D(T,\Lambda)}\big(\mathrm{gr}^i E, \mathrm{gr}^{i+1} E'[1]\big)$$

is exact and $\mathrm{RHom}_{\mathrm{DF}(T,\Lambda)}(E, E') \in D^{\geq 0}(\mathrm{Ab})$.

Remark 2.4. The map

$$\bigoplus_i \mathrm{Hom}_{D(X)}(\mathrm{gr}^i E, \mathrm{gr}^i E') \to \bigoplus_i \mathrm{Hom}_{D(X)}\big(\mathrm{gr}^i E, \mathrm{gr}^{i+1} E'[1]\big) \tag{2.4.1}$$

is obtained as follows. Note that for all i there is a distinguished triangle

$$\mathrm{gr}^{i+1} E \to F^i/F^{i+2} \to \mathrm{gr}^i E \xrightarrow{\partial_i} \mathrm{gr}^{i+1} E[1],$$

and similarly for E'. The map (2.4.1) is obtained by sending a collection of maps (φ_i) to the differences of the two ways of going around the squares

$$\begin{array}{ccc} \mathrm{gr}^i E & \xrightarrow{\partial_i} & \mathrm{gr}^{i+1} E[1] \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i+1}[1]} \\ \mathrm{gr}^i E' & \xrightarrow{\partial_i} & \mathrm{gr}^{i+1} E'[1]. \end{array}$$

Proof of Lemma 2.3. We can reformulate our assumption on Ext-groups as follows. Recall [4, V, Proposition 1.4.9] that we have

$$H^s\big(\mathrm{gr}^n \mathrm{RHom}(E, E')\big) \simeq \bigoplus_i \mathrm{Ext}^s(\mathrm{gr}^i E, \mathrm{gr}^{i+n} E') \tag{2.4.2}$$

for all n and s. Using this, our assumption on Ext-groups can then be reformulated as saying that

$$H^s\big(\mathrm{gr}^n \mathrm{RHom}(E, E')\big) = 0$$

for $n \geq 0$ and $s < n$. Applying 2.2 with $H = \mathrm{RHom}(E, E')$ we conclude that the sequence

$$0 \to H^0\big(F^0 \mathrm{RHom}(E, E')\big) \to H^0\big(\mathrm{gr}^0 \mathrm{RHom}(E, E')\big) \to H^1\big(\mathrm{gr}^1 \mathrm{RHom}(E, E')\big)$$

is exact, which gives the result when combined with (2.4.2) and [4, V, Corollaire 1.4.6]. ∎

2.5. For an object $E \in D^b F(T, \Lambda)$ we get a bounded complex

$$\cdots \to P^s \to P^{s+1} \to \cdots$$

in $D(T, \Lambda)$ by setting $P^s := \mathrm{gr}^s E[s]$ and the maps $d_s : P^s \to P^{s+1}$ given by the ∂_i. Note that $d_{s+1} \circ d_s = 0$. This follows from the fact that $\partial_i : \mathrm{gr}^s E \to \mathrm{gr}^{s+1} E[1]$ factors through a map

$$\mathrm{gr}^s E \to F^{s+1} E / F^{s+3} E[1];$$

namely, the boundary map arising from the distinguished triangle

$$F^{s+1} E / F^{s+3} E \to F^s E / F^{s+3} E \to \mathrm{gr}^s E \to F^{s+1} E / F^{s+3} E[1].$$

Proposition 2.6. *Let $(P^\bullet, d_\bullet)$ be a bounded complex in $D(T, \Lambda)$ such that for $s \in \mathbf{Z}$ and $r \geq 0$ we have*

$$\mathrm{Ext}^i_{D(T,\Lambda)}(P^s, P^{s+r}) = 0 \tag{2.6.1}$$

for $i < 0$. Then there exists an object $E \in D^bF(T,\Lambda)$, unique up to unique isomorphism, inducing $(P^\bullet, d_\bullet)$ by the construction of Section 2.5.

Proof. For $(P^\bullet, d_\bullet)$ obtained from $E \in D^bF(T,\Lambda)$ the vanishing condition (2.6.1) is equivalent to

$$\mathrm{Ext}^s_{D(T,\Lambda)}(\mathrm{gr}^i E, \mathrm{gr}^j E) = 0$$

for $j \geq i$ and $s < j - i$. The uniqueness of E therefore follows from Lemma 2.3.

To construct $E \in D^bF(T,\Lambda)$ inducing a given $(P^\bullet, d_\bullet)$ we proceed by induction of the number of terms in $P^\bullet$. For an integer s let $\sigma_{\leq s}P^\bullet$ be the complex in $D(T,\Lambda)$ with $(\sigma_{\leq s}P^\bullet)^i = P^i$ if $i \leq s$ and 0 if $i > s$. We then have a term-wise split exact sequence [7, Tag 014I] of complexes in $D(T,\Lambda)$

$$0 \to P^s[-s] \to \sigma_{\leq s}P^\bullet \to \sigma_{\leq s-1}P^\bullet \to 0.$$

This defines a distinguished triangle in $K(D(T,\Lambda))$; in particular, a map

$$\delta_s : \sigma_{\leq s-1}P^\bullet \to P^s[-s+1].$$

Concretely this is simply the map induced by $d_{s-1} : P^{s-1} \to P^s$. Note that the assumptions on $(P^\bullet, d_\bullet)$ are also satisfied by $(\sigma_{\leq s}P^\bullet, d_\bullet)$ and therefore by induction it suffices to show that if $(\sigma_{\leq s-1}P^\bullet, d_\bullet)$ is obtained from an object $E_{s-1} \in D^bF(T,\Lambda)$ then so is $(\sigma_{\leq s}P^\bullet, d_\bullet)$. For this it suffices, in turn, to show that δ_s is induced by a morphism in the filtered derived category

$$\tilde{\delta}_s : E_{s-1} \to \left(P^s[-s+1], G_s\right),$$

where for an integer q we write G_q for the filtration on $P^s[-s+1]$ for which $G^i_qP^s[-s+1]$ equals $P^s[-s+1]$ if $i \leq q$ and 0 otherwise. For this note that we have

$$\mathrm{Ext}^r\left(\mathrm{gr}^i(\sigma_{\leq s-1}E), \mathrm{gr}^j\left(P^s[-s+1], G_{s-1}\right)\right) = 0$$

if $j \neq s-1$ or $i \geq s$, since in this case one of the factors is 0, and for $i \leq s-1$ we have

$$\mathrm{Ext}^r\left(\mathrm{gr}^i(\sigma_{\leq s-1}E), \mathrm{gr}^{s-1}\left(P^s[-s+1], G_{s-1}\right)\right) = \mathrm{Ext}^{r-s+1+i}(P^i, P^s).$$

In particular, this vanishes if $r - s + 1 + i < 0$, or equivalently, $r < (s-1) - i$. Therefore by Lemma 2.3 the map δ_s lifts to a map

$$\delta'_s : E_{s-1} \to \left(P^s[-s+1], G_{s-1}\right)$$

in the filtered derived category. Let

$$\tilde{\delta}_s : E_{s-1} \to \left(P^s[-s+1], G_s\right)$$

be the composition of δ'_s with the natural map

$$\left(P^s[-s+1], G_{s-1}\right) \to \left(P^s[-s+1], G_s\right). \qquad \blacksquare$$

3. Filtered complexes in a D-topos

3.1. The functor e_d discussed in 1.2 has both a right and left adjoint. The right adjoint

$$\lambda_d : \mathrm{Sh}(T_d, \Lambda_d) \to \mathrm{Sh}(T, \Lambda)$$

sends an object $N \in \mathrm{Sh}(T_d, \Lambda_d)$ to an object of $\mathrm{Sh}(T, \Lambda)$ whose e-component is given by

$$\prod_{\delta : e \to d} f_{\delta *} N.$$

The left adjoint

$$s_d : \mathrm{Sh}(T_d, \Lambda_d) \to \mathrm{Sh}(T, \Lambda)$$

sends an object $N \in \mathrm{Sh}(T_d, \Lambda_d)$ to an object of $\mathrm{Sh}(T, \Lambda)$ whose e-component is given by

$$\oplus_{\delta : d \to e} f_\delta^* N.$$

Since e_d is exact the functor λ_d takes injectives to injectives. For an object $M \in \mathrm{Sh}(T, M)$ the natural map $M \to \prod_d \lambda_d M_d$ is injective. Therefore if we choose for each d an inclusion $M_d \hookrightarrow J_d$ of M_d into an injective Λ_d-module J_d then we get an inclusion into an injective $M \hookrightarrow \prod_d \lambda_d J_d$. In particular, every injective object in $\mathrm{Sh}(T, \Lambda)$ is a direct summand of a sheaf of the form $\prod_d \lambda_d J_d$ with each J_d injective in $\mathrm{Sh}(T_d, \Lambda_d)$.

Notation 3.2. Let ND denote the nerve of D (a simplicial set). In degree k the elements of ND_k are the diagrams in D

$$\underline{d} : d_0 \to d_1 \to \cdots \to d_k$$

consisting of k composable morphisms. For $a \in D$ let $a \backslash ND_k$ denote the set of pairs $(\underline{d}, \rho)$ consisting of an object $\underline{d} \in ND_k$ and a morphism $\rho : a \to d_0$. Similarly let ND_k/b denote the set of pairs $(\underline{d}, \gamma)$ consisting of an object $\underline{d} \in ND_k$ together with a morphism $\gamma : d_k \to b$, and let $a \backslash ND_k/b$ denote the collection of triples $(\underline{d}, \rho, \gamma)$ consisting of an object $\underline{d} \in ND_k$ and morphisms $\rho : a \to d_0$ and $\gamma : d_k \to b$. Note that

$$a \backslash ND_k/b = \coprod_{\sigma : a \to b} (a \backslash ND_k/b)_\sigma,$$

where $(a \backslash ND_k/b)_\sigma$ denotes the subset of triples for which the induced morphism $a \to b$ is σ.

For $\underline{d} \in ND_k$ let $f_{\underline{d}} : T_{d_k} \to T_{d_0}$ be the morphism induced by the composition $d_0 \to d_k$.

3.3. For a sheaf $M \in \mathrm{Sh}(T, \Lambda)$ we can associate an augmented cosimplicial object of $\mathrm{Sh}(T, \Lambda)$

$$M \to \widetilde{C}(M),$$

as well as an augmented simplicial object $\mathrm{Sh}(T, \Lambda)$

$$\widetilde{L}(M) \to M$$

as follows.

The object $\widetilde{C}(M)$ is defined by

$$[k] \mapsto \prod_{\underline{d} \in ND_k} \lambda_{d_0}(f_{\underline{d}*} M_{d_k}).$$

So the restriction of $\widetilde{C}(M)$ to T_a is the cosimplicial object of $\mathrm{Sh}(T_a, \Lambda_a)$ given by

$$[k] \mapsto \prod_{\underline{d} \in a \backslash ND_k} \rho_* f_{\underline{d}*} M_{d_k}.$$

The transition maps are induced by the simplicial structure on ND. Note that

$$\widetilde{C}(M)_0 = \prod_{d \in D} \lambda_d M_d.$$

The adjunction maps $M \to \lambda_d M_d$ induce the augmentation $M \to \widetilde{C}(M)$.

The simplicial object $\widetilde{L}(M)$ is defined similarly by the formula

$$[k] \mapsto \oplus_{\underline{d} \in ND_k} s_{d_k}(f_{\underline{d}}^* M_{d_0}).$$

So for $b \in D$ the restriction of $\widetilde{L}(M)$ to T_b is given by

$$[k] \mapsto \oplus_{\underline{d} \in ND_k / b} \gamma^* f_{\underline{d}}^* M_{d_0}.$$

We have $\widetilde{L}(M)_0 = \bigoplus_{d \in D} s_d M_d$, and the augmentation $\widetilde{L}(M) \to M$ is induced by the adjunction maps $s_d M_d \to M$.

3.4. Let $C(M)$ (resp. $L(M)$) denote the normalized complex associated to $\widetilde{C}(M)$ (resp. $\widetilde{L}(M)$) so we have maps of complexes

$$M \to C(M), \quad L(M) \to M. \tag{3.4.1}$$

We will prove that these maps are quasi-isomorphisms (see 3.8 below). For later purposes, however, we will show this using some slightly more general considerations.

3.5. Let $\mathscr{A}$ be an additive category with infinite products and let $F : E \to \mathscr{A}$ be a functor (we will apply this below with $E = a \backslash D$ or $E = D/a$ for $a \in D$ and $\mathscr{A} = \mathrm{Sh}(T_a, \Lambda_a)$ – hence the change in notation). For $e \in E$ write F_e for the value of F on e. We can then repeat the construction of $\widetilde{C}(M)$ above to get a cosimplicial object $\widetilde{C}(F)$ in $\mathscr{A}$ given by

$$[k] \mapsto \prod_{\underline{e} \in NE_k} F_{e_k}.$$

3.6. Suppose now that E has an initial object $b \in E$. For $k \geq 1$ define

$$h_k : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k-1}$$

to be the map whose component in the factor corresponding to $d_0 \to \cdots \to d_{k-1}$ is the projection

$$\prod_{e_0 \to \cdots \to e_k} F_{e_k} \to F_{d_{k-1}}$$

onto the component given by

$$b \to d_0 \to \cdots \to d_{k-1}.$$

Define

$$h_0 : \widetilde{C}(F)_0 = \prod_{e \in E} F_e \to F_b$$

to be the projection onto the b-th factor, and let $d_{-1} : F_b \to C(F)$ to be the product of the maps $\sigma_e : F_b \to F_e$ given by the fact that b is the initial object in E.

Lemma 3.7. *For every $k \geq 0$ we have*

$$\mathrm{id}_{\widetilde{C}(F)_k} = d_{k-1}h_k + h_{k+1}d_k,$$

where $d_k : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k+1}$ is given by the alternating sum of the maps $\delta_i : \widetilde{C}(F)_k \to \widetilde{C}(F)_{k+1}$ provided by the cosimplicial structure.

Proof. Fix

$$\underline{d} = (d_0 \to d_1 \to \cdots \to d_k) \in NE_k$$

and let us calculate the composition of the maps $\widetilde{C}(F)_k \to \widetilde{C}(F)_k$ in question with the projection onto the $\underline{d}$-th factor of $\widetilde{C}(F)_k$. For $\underline{e} \in ND_k$ write $F_{\underline{e}}$ for the factor of $\widetilde{C}(F)_k$ corresponding to $\underline{e}$ (so $F_{\underline{e}} = F_{e_k}$, but this notation reflects also which factor in the product we are considering).

Let $0 \leq i_0 \leq k$ be the smallest integer for which $d_i \neq b$. For both of the maps $d_{k-1}h_k$ and $h_{k+1}d_k$ the compositions in question factor through the projection from $\prod_{\underline{e} \in NE_k} F_{e_k}$ to the product of $F_{\underline{d}}$ with the factors of the form

$$F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k}$$

for $i \geq i_0$ (note here that there may be several different choices of i corresponding to the same factor). Thus it suffices to calculate the individual factors

$$F_{\underline{d}} \to F_{\underline{d}}, \quad F_{b \to d_0 \to \cdots \hat{d}_i \cdots \to d_k} \to F_{\underline{d}}$$

of our maps.

On the factor $F_{\underline{d}}$ the map $d_{k-1}h_k$ is given by (with the convention that if $i_0 = 0$ then the sum is 0)

$$\left(\sum_{j=0}^{i_0-1}(-1)^j\right) \cdot \mathrm{id}_{F_{\underline{d}}}$$

and the map $h_{k+1}d_k$ is given by

$$\left(\sum_{j=0}^{i_0}(-1)^j\right)\cdot \mathrm{id}_{F_{\underline{d}}},$$

so their sum is $\mathrm{id}_{F_{\underline{d}}}$.

To calculate the maps on a factor $F_{b\to d_0\to\cdots\hat{d}_i\cdots\to d_k}$ let J denote the set of those elements j for which

$$(d_0 \to \cdots \hat{d}_j \cdots \to d_k) = (d_0 \to \cdots \hat{d}_i \cdots \to d_k).$$

On a factor $F_{b\to d_0\to\cdots\hat{d}_i\cdots\to d_k}$ the map $d_{k-1}h_k$ is given by $\sum_{j\in J}(-1)^j$ times the natural map

$$F_{b\to d_0\to\cdots\hat{d}_i\cdots\to d_k} \simeq F_{d_0\to\cdots\hat{d}_i\cdots\to d_k} \to F_{\underline{d}},$$

whereas the map $h_{k+1}d_k$ is given by $\sum_{j\in J}(-1)^{j+1}$ times this map. The two maps therefore cancel on the factor of $F_{b\to d_0\to\cdots\hat{d}_i\cdots\to d_k}$. ■

Corollary 3.8. *The maps* (3.4.1) *are quasi-isomorphisms.*

Proof. It suffices to verify that the maps restrict to quasi-isomorphisms over each T_a $(a \in D)$.

To prove that $M_a \to C(M)_a$ is a quasi-isomorphism apply Lemma 3.7 with

$$F : a\backslash D \to \mathrm{Sh}(T_a, \Lambda_a)$$

sending $\delta : a \to d$ to $f_{\delta *}M_d$.

To get that $L(M)_a \to M_a$ is a quasi-isomorphism apply Lemma 3.7 to the functor

$$F : (D/a)^{\mathrm{op}} \to \mathrm{Sh}(T_a, \Lambda_a)^{\mathrm{op}}$$

sending

$$(\delta : d \to a) \mapsto f_\delta^* M_d.$$ ■

3.9. We view $C(M)$ and $L(M)$ as filtered objects using the "stupid filtration", so for $k \geq 0$ we have

$$\mathrm{gr}^k C(M) = \prod_{\underline{d}\in ND_k} \lambda_{d_0}(f_{\underline{d}*}M_{d_k})[-k],$$

$$\mathrm{gr}^{-k} L(M) = \oplus_{\underline{d}\in ND_k} s_{d_k}(f_{\underline{d}}^* M_{d_0})[k].$$

Since the filtrations involved are infinite we cannot directly apply our results on the filtered derived category. To get around this, note that for all $n \in \mathbf{Z}$ the objects $C(M)/\mathrm{Fil}^n$, (resp. $\mathrm{Fil}^n L(M)$) define projective (resp. inductive) systems in $\mathrm{DF}(T, \Lambda)$ and we have

$$C(M) \simeq \mathrm{holim}_n\, C(M)/\mathrm{Fil}^n, \quad L(M) \simeq \mathrm{hocolim}_{n\to-\infty}\big(\mathrm{Fil}^n(M)\big)$$

in $D(T, \Lambda)$. Indeed this follows from noting that for any index p the map

$$\mathscr{H}^p\big(C(M)/\mathrm{Fil}^{n+1}\big) \to \mathscr{H}^p\big(C(M)/\mathrm{Fil}^n\big) \ \big(\text{resp. } \mathscr{H}^p\big(\mathrm{Fil}^{n+1}(M)\big) \to H^p\big(\mathrm{Fil}^{n+1}(M)\big)\big)$$

is an isomorphism for n sufficiently large (resp. sufficiently negative); see also [7, Tag 0CQE].

We can extend $C(-)$ to $D^+(T, \Lambda)$ by applying the above construction termwise to bounded below injective complexes to get a functor

$$C(-) : D^+(T, \Lambda) \to \big(\text{projective systems in } D^+F(T, \Lambda)\big)$$

such that the composition of $C(-)$ with the forgetful functor followed by holim is the identity functor on $D^+(T, \Lambda) \to D^+(T, \Lambda)$.

Similarly we define

$$L(-) : D^-(T, \Lambda) \to \big(\text{inductive systems in } D^-F(T, \Lambda)\big)$$

by considering flat complexes.

4. Proof of Theorem 1.5

We proceed with the notation of the theorem.

To show surjectivity of the map (1.4.1) fix a compatible collection of maps

$$\sigma_d : M_d \to M_{d'} \tag{4.0.1}$$

defining a morphism in Γ. We show that these maps are induced by a morphism in $D(T, \Lambda)$ as follows.

View M as an object of $\mathrm{DF}(T, \Lambda)$ by setting $\mathrm{Fil}^i M = M$ for $i \leq 0$ and 0 otherwise. Consider the projective system $C(M')/\mathrm{Fil}^n$ of objects of $\mathrm{DF}(T, \Lambda)$. We have $\mathrm{gr}^i M = 0$ for $i \neq 0$ and

$$\begin{aligned}\mathrm{Ext}^s_{D(T,\Lambda)}\big(\mathrm{gr}^0 M, \mathrm{gr}^j C(M')\big) &= \mathrm{Ext}^{s-j}_{D(T,\Lambda)}\Big(M, \prod_{\underline{d} \in ND_j} \lambda_{d_0}(Rf_{\underline{d}*}M'_{d_k})\Big) \\ &= \prod_{\underline{d} \in ND_j} \mathrm{Ext}^{s-j}_{D(T_{d_0},\Lambda_{d_0})}(M_{d_0}, Rf_{\underline{d}*}M'_{d_k}).\end{aligned}$$

By assumption these groups vanish for $j \geq 0$ and $s < j$. Therefore the assumptions of Lemma 2.3 hold and it follows that the map

$$\sigma := \prod_{d \in D} \sigma_d : M \to \prod_{d \in D} \lambda_d M'_d$$

lifts uniquely to a compatible collection of morphisms $M \to C(M')/\mathrm{Fil}^n$ in the filtered derived category. Passing to the homotopy limit over n we then get a morphism $M \to C(M') \simeq M'$ in $D(T, \Lambda)$ inducing the σ_k proving the surjectivity.

For the injectivity, note that a morphism $\tau : M \to M'$ in $D(T,\Lambda)$ induces, by composition with the map $M' \to C(M')$, a compatible collection of maps in the filtered derived category $M \to C(M')/\mathrm{Fil}^n$ which recovers τ by passing to the homotopy limit. By the preceding discussion these maps in the filtered derived category are uniquely determined by the associated maps on graded pieces, which implies that τ is determined by its image in Γ. ∎

5. Proof of Theorem 1.7

5.1. Proof under assumption (i)

The uniqueness follows from Theorem 1.5.

For the existence set

$$P^k := \prod_{\underline{d}\in ND_k} \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k}) \in D(T,\Lambda),$$

and let $d_k : P^k \to P^{k+1}$ be the maps induced by the alternating sums of the φ_δ.

For a given

$$\underline{d} = (d_0 \to \cdots \to d_k) \in ND_k$$

and $s \leq k$ let $(ND_s)_{\underline{d}} \subset ND_s$ be the images of $\underline{d}$ under the various degeneracy maps $ND_k \to ND_s$. So $(ND_s)_{\underline{d}}$ is a finite set and there is a projection

$$\pi^s_{\underline{d}} : P^s \to P^s_{\underline{d}} := \prod_{\underline{e}\in(ND_s)_{\underline{d}}} \lambda_{e_0}(Rf_{\underline{e}*}M_{e_s}).$$

Note that for $s \leq k$ the composition

$$P^{s-1} \xrightarrow{d_{s-1}} P^s \xrightarrow{\pi^s_{\underline{d}}} P^s_{\underline{d}}$$

factors through $\pi^{s-1}_{\underline{d}}$. This implies that if $\sigma_{\leq s}P^\bullet$ is defined as in the proof of Proposition 2.6, then we get a map of complexes in $D(T,\Lambda)$

$$\pi_{\underline{d}} : \sigma_{\leq s}P^\bullet \to \sigma_{\leq s}P^\bullet_{\underline{d}}.$$

Since $(ND_s)_{\underline{d}}$ is finite, our vanishing assumption (1.7.1) implies that

$$\mathrm{Ext}^i_{D(T,\Lambda)}(P^s_{\underline{d}}, P^{s+r}) = 0$$

for all $i < 0$, $r \geq 0$ and $s \leq k$. In particular, by Proposition 2.6 there exists for every $\underline{d} \in ND_k$ and $s \leq k$ a unique filtered complex $K^{(\underline{d})}_s \in \mathrm{DF}(T,\Lambda)$ inducing the complex $\sigma_{\leq s}P^\bullet_{\underline{d}}$. Furthermore, for $\underline{d}' \in ND_{k'}$ for which $\underline{d} \in (ND_k)_{\underline{d}'}$ we have by Theorem 1.5 a unique morphism in $\mathrm{DF}(T,\Lambda)$

$$\tau^s_{\underline{d},\underline{d}'} : K^{(\underline{d}')}_s \to K^{(\underline{d})}_s$$

inducing the natural map $\sigma_{\leq s}P^\bullet_{\underline{d}'} \to \sigma_{\leq s}P^\bullet_{\underline{d}}$.

We now construct for each integer $n \geq 0$ an object $K_n \in \mathrm{DF}(T, \Lambda)$ with maps

$$q_{n,\underline{d}} : K_n \to K_n^{(\underline{d})}$$

for $\underline{d} \in ND_k$ and $k \geq n$, and distinguished triangles

$$P^{n+1}\big[-(n+1)\big] \to K_{n+1} \xrightarrow{t_{n+1}} K_n \to P^{n+1}[-n] \tag{5.1.1}$$

such that the following hold:

(i) $\mathrm{gr}^i K_n = \begin{cases} P^i[-i] & \text{if } i \leq n \\ 0 & \text{otherwise,} \end{cases}$

(ii) the filtered structure on K_n induces the maps d_k (as in Section 2.5),

(iii) the map t_{n+1} induces an isomorphism $K_{n+1}/\mathrm{Fil}^{n+1} \simeq K_n$ in $\mathrm{DF}(T, \Lambda)$ compatible with the isomorphisms in (i).

(iv) For $k' \geq k$, $\underline{d}' \in ND_{k'}$ and $\underline{d} \in (ND_k)_{\underline{d}'}$ the diagram

$$\begin{array}{ccc} & K_n & \\ {\scriptstyle q_{n,\underline{d}}}\swarrow & & \searrow{\scriptstyle q_{n,\underline{d}'}} \\ K_n^{(\underline{d})} & \xrightarrow{\tau^n_{\underline{d},\underline{d}'}} & K_n^{(\underline{d}')} \end{array}$$

commutes.

For this we proceed by induction on n.

For $n = 0$ we take $K_0 = P^0$ with the $q_{0,\underline{d}}$ the projection maps.

To pass from n to $n+1$ note that K_{n+1} will be specified by a map

$$\alpha_n : K_n \to P^{n+1}[-n] = \prod_{\underline{d} \in ND_{n+1}} \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n]$$

in $\mathrm{DF}(T, \Lambda)$, where P^{n+1} is viewed as filtered with $\mathrm{Fil}^u P^{n+1} = P^{n+1}$ for $u \leq n+1$ and 0 for $u > n+1$. To give this map it suffices to specify for each $\underline{d} \in ND_{n+1}$ a map

$$\alpha_{n,\underline{d}} : K_n \to \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n],$$

and we take for this map the composition of $K_n \xrightarrow{q_{n,\underline{d}}} K_n^{(\underline{d})}$ and the map

$$K_n^{(\underline{d})} \to \lambda_{d_0}(Rf_{\underline{d}*}M_{d_k})[-n]$$

arising from $K_{n+1,\underline{d}}$. The above properties follow from the construction.

In particular, we get a tower of complexes

$$\cdots \to K_{n+1} \to K_n \to \cdots \to K_0,$$

with distinguished triangles as in (5.1.1). Let K denote the homotopy limit of the K_n.

Precisely, K is defined to be the cocone of the map

$$1 - t : \prod_n K_n \to \prod_n K_n.$$

We claim that K is the desired object of $D(T, \Lambda)$.

By our assumptions there exists an integer n_0 such that $M_d \in D^{\geq n_0}(T_d, \Lambda_d)$ for all d. Using the distinguished triangle (5.1.1) we obtain that for every s there exists an integer r such that the map

$$\mathscr{H}^s(K_m) \to \mathscr{H}^s(K_{m-1})$$

is an isomorphism for $m \geq r$. This in turn implies that for every s the map

$$\mathscr{H}^s(1-t) : \mathscr{H}^s\Big(\prod_n K_n\Big) \to \mathscr{H}^s\Big(\prod_n K_n\Big)$$

is surjective with kernel $\mathscr{H}^s(K)$. From this we conclude that the projection map $K \to K_n$ induces an isomorphism $\mathscr{H}^s(K) \to \mathscr{H}^s(K_n)$ for n sufficiently big.

Fix $a \in D$. The restriction of P^n to T_a is the complex

$$P_a^n = \prod_{\underline{d} \in N(a \backslash D_k)} R\rho_* Rf_{d_0 *} M_{d_k}$$

with the transition maps induced by the φ_δ. If we view M_a as a filtered object with

$$\mathrm{Fil}^r M_a = \begin{cases} M_a & \text{for } r \leq 0 \\ 0 & \text{for } r > 0 \end{cases}$$

then by the vanishing (1.7.1) we have

$$\mathrm{Ext}^s_{D(T_a,\Lambda_a)}(\mathrm{gr}^i M_a, \mathrm{gr}^j K_n) = \mathrm{Ext}^{s-j}_{D(T_a,\Lambda_a)}(\mathrm{gr}^i M_a, P_a^j) = 0$$

for $s < j$ and all i. It follows from this and Lemma 2.3 that the natural map

$$M_a \to \mathrm{gr}^0 K_n = \prod_{\delta : a \to d} Rf_{\delta *} M_d$$

lifts uniquely to a morphism $\beta_{a,n} : M_a \to K_{n,a}$ in $\mathrm{DF}(T_a, \Lambda_a)$. By the uniqueness, the diagram

$$\begin{array}{ccc} M_a & \xrightarrow{\beta_{a,n}} & K_{n,a} \\ & \searrow^{\beta_{a,n-1}} & \downarrow t_n \\ & & K_{n-1,a} \end{array}$$

commutes. The maps $\beta_{a,n}$ therefore induce a map $\beta_a : M_a \to K_a$. To see that this map is an isomorphism note that the spectral sequence of the filtered complex K_a (see for example [7, Tag 012K]) takes the form

$$E_1^{p,q} = \mathscr{H}^q(P^p) \implies \mathscr{H}^{p+q}(K_a),$$

and the differentials

$$\mathscr{H}^q(P^p) \to \mathscr{H}^q(P^{p+1})$$

are induced by our given maps $P^p \to P^{p+1}$. From this it follows that the map β induces an isomorphism

$$\mathscr{H}^q(M_a) \to E_2^{0,q} = \mathrm{Ker}\big(\mathscr{H}^q(P^0) \to \mathscr{H}^q(P^1)\big),$$

and that $E_2^{p,q} = 0$ for $p \neq 0$. Indeed consider the functor

$$F : a\backslash D \to D(T_a, \Lambda_a), \quad (\delta : a \to d) \mapsto Rf_{\delta *}M_d,$$

and form the associated cosimplicial object $\widetilde{C}(F)$ in $D(T_a, \Lambda_a)$. Then we see that the complex $E_1^{*,q}$ is equal to the complex obtained by taking q-th cohomology sheaves level-wise of $\widetilde{C}(F)$ and then taking total complex. By Lemma 3.7 it follows that the natural map

$$\mathscr{H}^q(M_a) \to E_1^{*,q}$$

is a quasi-isomorphism.

We therefore obtain the desired isomorphisms $\beta : M_a \simeq K_a$, and by the construction these isomorphisms are compatible with the transitions maps φ_δ.

5.2. Proof under assumption (ii)

Once again the uniqueness follows from Theorem 1.5.

For existence define for $k \geq 0$

$$P^{-s} := \oplus_{\underline{d} \in ND_s} s_{d_k}(\mathbf{L}f_{\underline{d}}^* M_{d_0})$$

and let $d_{-s} : P^{-s} \to P^{-s+1}$ be the map obtained by taking the alternating sum of the maps given by the simplicial structure. We then get a complex, concentrated in degrees $(-\infty, 0]$, in $D(T, \Lambda)$.

For $\underline{d} \in ND_k$ we can also consider for $s \leq k$ the complex

$$P_{\underline{d}}^{-s} := \oplus_{\underline{e} \in (ND_s)_{\underline{d}}} s_{e_s}(\mathbf{L}f_{\underline{e}}^* M_{e_0}) \subset P^{-s}.$$

This defines a subcomplex

$$\sigma_{\geq -s} P_{\underline{d}}^{\bullet} \subset \sigma_{\geq -s} P^{\bullet}$$

in $D(T, \Lambda)$.

Our assumptions imply that for all $s, r \in \mathbf{Z}$ and $i < 0$, $\mathrm{Ext}^i(P^r, P_{\underline{d}}^s) = 0$. By Proposition 2.6 the complex $P_{\underline{d}}^{-s}$ is induced by a unique filtered complex $F_{-s}^{(d)} \in \mathrm{DF}(T, \Lambda)$. Moreover, for $k \leq k'$ and $\underline{d}' \in ND_{k'}$ for which $\underline{d} \in (ND_k)_{\underline{d}'}$ we have unique maps

$$\tau_{\underline{d},\underline{d}'}^{-s} : F_{-s}^{(d)} \to F_{-s}^{(d')}$$

in $\mathrm{DF}(T, \Lambda)$ inducing the natural maps on complexes in $D(T, \Lambda)$.

We now construct for each integer $n \geq 0$ an object $F_{-n} \in \mathrm{DF}(T, \Lambda)$ with maps

$$q_{n,\underline{d}} : F_{-n}^{(\underline{d})} \to F_{-n}$$

for $\underline{d} \in ND_k$ and $k \geq n$, and distinguished triangles

$$F_{-n} \xrightarrow{t_{n+1}} F_{-n-1} \to P^{n+1}[n+1] \to F_{-n}[1]$$

such that the following hold:

(i) $\quad \mathrm{gr}^s F_{-k} = \begin{cases} P^s & \text{if } 0 \geq s \geq -k \\ 0 & \text{otherwise,} \end{cases}$

(ii) the filtered structure on F_{-n} induces the maps d_k (as in Section 2.5),

(iii) the map t_{n+1} induces an isomorphism $F_{-n} \simeq \mathrm{Fil}^{-n} F_{-n-1}$ in $\mathrm{DF}(T, \Lambda)$ compatible with the isomorphisms in (i).

(iv) For $k' \geq k$, $\underline{d}' \in ND_{k'}$ and $\underline{d} \in (ND_k)_{\underline{d}'}$ the diagram

$$\begin{array}{ccc} & F_{-n} & \\ {\scriptstyle q_{n,\underline{d}}} \nearrow & & \nwarrow {\scriptstyle q_{n,\underline{d}'}} \\ F_{-n}^{(\underline{d})} & \xrightarrow{\tau^{-s}_{\underline{d},\underline{d}'}} & F_{-n}^{(\underline{d}')} \end{array}$$

commutes.

The objects F_{-n} are constructed by induction on n. For set $F_0 := P^0$.

To obtain F_{-n-1} given F_{-n}, note that F_{-n-1} is determined by a morphism

$$\alpha : P^{n+1}[n+1] \to F_{-n}[1],$$

or equivalently for each $\underline{d} \in ND_{n+1}$ a map

$$\alpha_{\underline{d}} : s_{d_k}(\mathbf{L} f_{\underline{d}}^* M_{d_0})[n+1] \to F_{-n}[1].$$

We take for $\alpha_{\underline{d}}$ the map induced by the natural map

$$F_{-n-1}^{(\underline{d})} \to F_{-n}^{(\underline{d})}[1] \to F_{-n}[1].$$

The above properties follow immediately from the construction.

We therefore get a sequence of objects in $D(T, \Lambda)$

$$F_0 \to F_{-1} \to F_{-2} \to \cdots .$$

We let K denote the homotopy colimit of this sequence.

We have

$$P^0 = \bigoplus_{d \in D} s_d M_d,$$

and therefore for $a \in D$

$$P_a^0 = \bigoplus_{\delta : d \to a} \mathrm{L} f_\delta^* M_d .$$

The maps φ_δ therefore define a map $e_a : P_a^0 \to M_a$. By Lemma 2.3 these maps are given by unique maps $e_{k,a} : F_{k,a} \to M_a$ in $\mathrm{DF}(T_a, \Lambda_a)$, where M_a is viewed as being filtered with $\mathrm{Fil}^i M_a = 0$ for $i > 0$ and $\mathrm{Fil}^0 M_a = M_a$. The uniqueness of the maps imply that $e_{k,a}$ restricts to $e_{k+1,a}$ on $F_{k+1,a}$ and we consequently get a map $e_a : K_a \to M_a$. By the same argument as in the proof under assumption (i), using the spectral sequence of a filtered complex and looking at cohomology sheaves, we find that e_a is an isomorphism compatible with the maps φ_δ.

This completes the proof of Theorem 1.7. ∎

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References

[1] M. Artin, A. Grothendieck, J. L. Verdier, P. Deligne, and B. Saint-Donat (eds.), *Théorie des topos et cohomologie étale des schémas. Tome 2. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4)*. Lecture Notes in Math. 270, Springer, Berlin, 1972 Zbl 0237.00012 MR 354653

[2] D. Ben-Zvi, J. Francis, and D. Nadler, Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Amer. Math. Soc.* **23** (2010), no. 4, 909–966 Zbl 1202.14015 MR 2669705

[3] A. A. Beĭlinson, J. Bernstein, and P. Deligne, Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, pp. 5–171, Astérisque 100, Soc. Math. France, Paris, 1982 Zbl 0536.14011 MR 0751966

[4] L. Illusie, *Complexe cotangent et déformations. I*. Lecture Notes in Math. 239, Springer, Berlin, 1971 Zbl 0224.13014 MR 0491680

[5] M. Lieblich and M. Olsson, Derived equivalences over base schemes and support of complexes. 2022, arXiv:2208.14378v1

[6] J. Lurie, Derived algebraic geometry VIII: Quasi-coherent sheaves and tannaka duality theorems. Preprint, https://www.math.ias.edu/~lurie/papers/DAG-VIII.pdf visited on 6 April 2024

[7] The Stacks project, https://stacks.math.columbia.edu visited on 6 April 2024

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