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The non-degeneracy invariant of Brandhorst and Shimada's families of Enriques surfaces

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Abstract. Brandhorst and Shimada described a large class of Enriques surfaces, called $(\tau, \overline{\tau})$-generic, for which they gave generators for the automorphism groups and calculated the elliptic fibrations and the smooth rational curves up to automorphisms. In the present paper, we give lower bounds for the non-degeneracy invariant of such Enriques surfaces, we show that in most cases the invariant has generic value 10, and we present the first known example of complex Enriques surface with infinite automorphism group and non-degeneracy invariant not equal to 10.

1. Introduction

A fundamental feature of an Enriques surface Y is that it always has an elliptic pencil, and the understanding of these gives information about the geometry of Y. More specifically, in the current paper we are interested in studying the so-called *non-degeneracy invariant* $\mathrm{nd}(Y)$. This was introduced in [7] and it is defined as follows. Every elliptic pencil can be written as $|2F|$, where $F \in \mathrm{Pic}(Y)$ is called a *half-fiber*. Then, the non-degeneracy invariant $\mathrm{nd}(Y)$ is the maximum m such that there exist half-fibers $F_1, \dots, F_m$ such that $F_i \cdot F_j = 1 - \delta_{ij}$. Back to geometry, if $\mathrm{nd}(Y) = 10$, then Y can be realized as a degree 10 surface in $\mathbb{P}^5$ given by the intersection of 10 cubics (see the discussion in [11, Section 2.3]). In a way, $\mathrm{nd}(Y)$ can be thought of as a way of measuring how far we are from such a projective realization.

In characteristic different from 2, it is known that $4 \leq \mathrm{nd}(Y) \leq 10$. While the upper bound simply follows from the fact that $\mathrm{Num}(Y)$, the group of divisors on Y modulo numerical equivalence, has rank 10, the lower bound is a recent result [14]. The non-degeneracy invariant is known to be 10 for Enriques surfaces without smooth rational curves [6, Theorem 3.2] and for generic Enriques surfaces containing smooth rational curves (see [11, Section 4.2] and [5, Lemma 3.2.1]). The non-degeneracy invariant was also computed for Enriques surfaces with finite automorphism group [10, Section 8.9], and for specific examples of special Enriques surfaces with smooth rational curves and infinite automorphism group (see [9, Sections 4.1–4.3] and [16, Section 5]). Otherwise, computing $\mathrm{nd}(Y)$ is in general a hard problem as it is difficult to understand all the elliptic

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fibrations on Y, and the knowledge of their orbits under automorphisms is not sufficient to determine it.

In the present paper, we work over $\mathbb{C}$ (see Remark 3.2) and turn to a specific class of Enriques surfaces which were introduced by Brandhorst and Shimada in [4]. These are called $(\tau, \overline{\tau})$-*generic Enriques surfaces*, where τ is the ADE-type of a lattice spanned by a set of smooth rational curves on Y, and $\overline{\tau}$ is the ADE-type of its primitive closure in $\mathrm{Num}(Y)$. The 184 lattice-theoretic possibilities for all pairs $(\tau, \overline{\tau})$ were classified in [20]. Of these, 155 are obtained by *realizable* families of $(\tau, \overline{\tau})$-generic Enriques surfaces (see Definition 3.4). For these families, Brandhorst and Shimada study $\mathrm{Aut}(Y)$ and the sets of smooth rational curves and elliptic fibrations on Y up to automorphisms. Combining this information with [16, 17], we prove the following result. In the statement we do not include the values of the non-degeneracy invariants for the families $1, 172, 184$, since they are already known, see Remark 4.1.

Theorem 1.1 (Theorems 4.2 and 5.10). *For an integer* $i \in \{1, \dots, 184\}$, *let* Y_i *be the* i*-th realizable* $(\tau, \overline{\tau})$*-generic Enriques surface in* [4, Table 1] *with* $i \neq 1, 172, 184$. *Then,*

(1) $\mathrm{nd}(Y_{145}) = 4$.

(2) *We have the lower bounds*

$$\begin{aligned} &\mathrm{nd}(Y_{84}) \geq 9, && \mathrm{nd}(Y_{85}) \geq 7, && \mathrm{nd}(Y_{121}) \geq 9,\\ &\mathrm{nd}(Y_{122}) \geq 7, && \mathrm{nd}(Y_{123}) \geq 7, && \mathrm{nd}(Y_{143}) \geq 8,\\ &\mathrm{nd}(Y_{144}) \geq 8, && \mathrm{nd}(Y_{158}) \geq 9, && \mathrm{nd}(Y_{159}) \geq 7,\\ &\mathrm{nd}(Y_{171}) \geq 8, && \mathrm{nd}(Y_{176}) \geq 7. \end{aligned}$$

(3) *Finally, in the remaining* 140 *cases,* $\mathrm{nd}(Y_i) = 10$.

For each surface Y_i in the statement of Theorem 1.1, we provide an explicit sequence of half-fibers on Y_i realizing the claimed lower bound for $\mathrm{nd}(Y_i)$, see Section 6 for a reference. For the Enriques surface Y_{145}, using the work of Brandhorst and Shimada, the automorphism group of Y_{145} acts on cohomology as the infinite dihedral group. We give a full combinatorial description of its action on the set of smooth rational curves. We then assemble this information to compute the value of the non-degeneracy invariant. The details of the proof are spelled out in Section 5. To our knowledge, Y_{145} is the first example of Enriques surface with infinite automorphism group and $\mathrm{nd}(Y) < 10$. Different aspects of the Enriques surfaces Y_{145} have been studied in the literature: they appear in [2] as examples with "small" infinite automorphism group, their K3 covers have zero entropy [3, Remark 5.5], and some non-extendable isotropic sequences on them were computed in [13, Section 4].

The strategy we used to show that $\mathrm{nd}(Y_{145}) = 4$ applies in principle to the remaining 11 cases in Theorem 1.1 (2). However, the argument is harder to replicate: both the automorphism groups and the sets of orbit representatives of smooth rational curves are more complex. This will be object of future study.

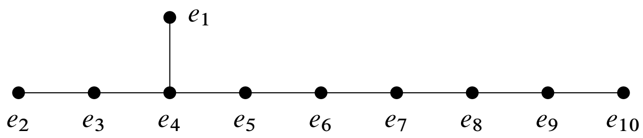

Figure 1. The E_{10} lattice.

2. Preliminaries on Enriques surfaces and the non-degeneracy invariant

2.1. Enriques surfaces and the E_{10} lattice

An *Enriques surface* Y is a smooth minimal projective connected algebraic surface of Kodaira dimension zero such that

$$h^0(Y, \omega_Y) = h^1(Y, \mathcal{O}_Y) = 0.$$

For an Enriques surface, the canonical class K_Y is the unique torsion element of $\mathrm{Pic}(Y)$ (more specifically, $2K_Y \sim 0$), and after quotienting by it we obtain $S_Y := \mathrm{Num}(Y)$, i.e. the group of divisors on Y modulo numerical equivalence. The intersection product among curves endows S_Y with the structure of a *lattice*: a finitely generated free abelian group L of finite rank together with a non-degenerate symmetric bilinear form $b: L \times L \to \mathbb{Z}$. As a lattice, S_Y is isometric to $U \oplus E_8$, where U is the hyperbolic lattice $\left(\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ and E_8 is the negative definite root lattice associated to the corresponding Dynkin diagram. We review root lattices more in detail in Section 3.1, but first we recall an alternative realization of the lattice $U \oplus E_8$.

Definition 2.1. The E_{10} *lattice* is defined to be $\mathbb{Z}^{10}$ together with the intersection form associated to the canonical basis $e_1, \dots, e_{10}$ as represented in Figure 1: $e_i^2 = -2$ and $e_i \cdot e_j = 1$ if the corresponding vertices are joined by an edge, and zero otherwise. A direct check shows that E_{10} is even, unimodular, and of signature $(1, 9)$ (see [19, Chapter V] for this standard terminology). It follows by [15, Theorem 1] that E_{10} is isometric to $U \oplus E_8$. Therefore, for any Enriques surface Y, S_Y is isometric to E_{10}.

In the current paper, it will be crucial to work with $\mathbb{Z}$-bases of S_Y in the following form.

Definition 2.2. Let Y be an Enriques surface and let $\mathcal{B} = \{B_1, \dots, B_{10}\}$ be a basis of S_Y. We call $\mathcal{B}$ an E_{10}*-basis* if the map $B_i \mapsto e_i$ extends to an isometry between S_Y and E_{10}.

2.2. The non-degeneracy invariant of an Enriques surface

Given an elliptic fibration $f: Y \to \mathbb{P}^1$ on an Enriques surface Y, then f has exactly two multiple fibers $2F$ and $2F'$ [1]. The curves F, F' are called the *half-fibers* of the elliptic fibration. The half-fibers on an Enriques surface can be used to define the following invariant of Y.

Definition 2.3. Let Y be an Enriques surface and let m be the maximum for which there exist half-fibers $F_1, \dots, F_m$ on Y such that $F_i \cdot F_j = 1 - \delta_{ij}$. Then m is called the *non-degeneracy invariant* of Y and it is denoted by $\mathrm{nd}(Y)$.

As we briefly reviewed in Section 1, if the Enriques surface Y is unnodal, or general nodal, then $\mathrm{nd}(Y) = 10$. Otherwise, it is in general a hard problem to compute $\mathrm{nd}(Y)$, which is known only in few cases (for instance, the Enriques surfaces with finite automorphism group). Motivated by this, in [16] we introduced a combinatorial version of the non-degeneracy invariant which we now review. First we recall that an *elliptic configuration* on Y is a curve C appearing in Kodaira's classification of singular fibers in elliptic fibrations [1, Chapter V, Table 3], but not a multiple of such a fiber (note that C does not have to be primitive in S_Y). In this case, we have that either $|C|$ is an elliptic pencil or $|2C|$ is an elliptic pencil of which C is a half-fiber [1, Chapter VIII, Lemma 17.3].

Definition 2.4. Let $\mathcal{R}$ be a finite collection of smooth rational curves on an Enriques surface Y. Let $\mathsf{HF}(Y, \mathcal{R})$ be the set of numerical equivalence classes $h \in S_Y$ in the following form:

- $h = [C]$, where C is an elliptic configurations with irreducible components in $\mathcal{R}$ and C is a half-fiber of an elliptic pencil on Y;
- $h = \frac{1}{2}[C]$, where C is an elliptic configurations with irreducible components in $\mathcal{R}$ and C is a fiber of an elliptic pencil on Y.

Then we define the *combinatorial non-degeneracy invariant of Y relative to $\mathcal{R}$*, $\mathrm{cnd}(Y, \mathcal{R})$, to be the maximum m such that there exist $f_1, \dots, f_m \in \mathsf{HF}(S, \mathcal{R})$ such that $f_i \cdot f_j = 1 - \delta_{ij}$.

As $\mathrm{cnd}(Y, \mathcal{R})$ only considers classes of half-fibers supported in $\mathcal{R}$, it gives a lower bound for $\mathrm{nd}(Y)$. In [17] we implemented a code that computes $\mathrm{cnd}(Y, \mathcal{R})$ by listing all elliptic configurations supported in $\mathcal{R}$ and then, with a recursive procedure, finds all the maximal sequences $f_1, \dots, f_m \in \mathsf{HF}(S, \mathcal{R})$ satisfying $f_i \cdot f_j = 1 - \delta_{ij}$. The procedure terminates as $\mathcal{R}$ is a finite set of smooth rational curves.

2.3. The combinatorial non-degeneracy invariant cnd(Y)

We now introduce an additional version of non-degeneracy invariant which explores the behavior of $\mathrm{cnd}(Y, \mathcal{R})$ as $\mathcal{R}$ varies.

Definition 2.5. Let Y be an Enriques surface. We define the *combinatorial non-degeneracy invariant of Y* as follows:

$$\mathrm{cnd}(Y) = \max\left\{\mathrm{cnd}(Y, \mathcal{R}) \mid \mathcal{R} \subseteq \mathcal{R}(Y) \text{ is finite}\right\}.$$

Remark 2.6. It is an immediate consequence of the definition that we have the following order relations among the different invariants we introduced above:

$$\mathrm{cnd}(Y, \mathcal{R}) \le \mathrm{cnd}(Y) \le \mathrm{nd}(Y).$$

The next proposition pinpoints a (nonempty) class of Enriques surfaces where the invariants $\mathrm{cnd}(Y)$ and $\mathrm{nd}(Y)$ are equal.

Proposition 2.7. *Let Y be an Enriques surface such that every elliptic fibration admits a fiber or a half-fiber supported on the union of smooth rational curves. Then we have that* $\mathrm{nd}(Y) = \mathrm{cnd}(Y)$.

Proof. Let $m = \mathrm{nd}(Y)$ and let $f_1 = [F_1], \ldots, f_m = [F_m]$ be numerical equivalence classes of half-fibers on Y such that $f_i \cdot f_j = 1 - \delta_{ij}$. By hypothesis, for all $i \in \{1, \ldots, m\}$, $|2F_i| = |nC_i|$ where $\mathrm{Supp}(C_i) = \bigcup_j R_j^{(i)}$ for some smooth rational curves $R_j^{(i)}$ and $n = 1$ or 2 if C_i is a fiber or a half-fiber respectively. Define $\mathcal{R} = \{R_j^{(i)} \mid i, j\}$. Then

$$\mathrm{nd}(Y) = m \leq \mathrm{cnd}(Y, \mathcal{R}) \leq \mathrm{cnd}(Y) \leq \mathrm{nd}(Y) \implies \mathrm{nd}(Y) = \mathrm{cnd}(Y). \qquad \blacksquare$$

We now give an example where $\mathrm{nd}(Y)$ does not coincide with $\mathrm{cnd}(Y)$.

Proposition 2.8. *Let Y be a general nodal Enriques surface. Then* $\mathrm{cnd}(Y) < 10$.

Proof. Suppose $(f_1, \ldots, f_{10})$ is an isotropic sequence realizing $\mathrm{cnd}(Y) = 10$. By the definition of $\mathrm{cnd}(Y)$ and [10, Theorem 6.5.5 (ii)], it is necessary that every pencil $|2F_i|$ has a reducible fiber of type $\widetilde{A}_1$. So, let us write $2f_i = R_i + S_i$ with R_i and S_i smooth rational curves.

The first step is to show that for all $i, j \in \{1, \ldots, 10\}$, $i \neq j$, the fibers $R_i + S_i$ and $R_j + S_j$ have a component in common. By contradiction, assume otherwise. Then, we have that

$$f_i \cdot f_j = 1 \implies 4 = (R_i + S_i) \cdot (R_j + S_j) = R_i \cdot R_j + R_i \cdot S_j + S_i \cdot R_j + S_i \cdot S_j.$$

Since no two (-2)-curves on Y are disjoint by [10, Corollary 6.5.2], then the intersection products $R_i \cdot R_j, R_i \cdot S_j, S_i \cdot R_j, S_i \cdot S_j$ equal 1, which contradicts [10, Lemma 6.5.1].

In the second step of the proof, we show that there exists a smooth rational curve R_0 which is an irreducible component of $R_i + S_i$ for all i. Let us start by considering $R_1 + S_1$ and $R_2 + S_2$. As they have a common irreducible component by the first step, we can assume up to relabeling that $R_0 := S_1 = S_2$. Let us consider $R_i + S_i$, $i \in \{3, \ldots, 10\}$, and assume by contradiction that $R_0 \neq R_i, S_i$. Then, by the first step, we must have that

$$R_i + S_i = R_1 + R_2.$$

Up to relabeling, we may assume that $R_i = R_1$ and $S_i = R_2$. Now, consider $R_j + S_j$, $j \in \{3, \ldots, 10\} \setminus \{i\}$. As $R_j + S_j$ has a component in common with $R_i + S_i = R_1 + R_2$, then we can assume up to relabeling that $R_j = R_1$. At the same time, $R_j + S_j$ has a component in common with $R_2 + R_0$, which is impossible because $R_j + S_j$ would equal $R_1 + R_2$ or $R_1 + R_0$, contradicting the fact that the 10 fibers of type $\widetilde{A}_1$ that we fixed are distinct. So, R_0 is a component of $R_i + S_i$. Summarizing, we can write for all i that

$$R_i + S_i = R_i + R_0.$$

Finally, we study the intersection matrix $\mathfrak{R} = (R_i \cdot R_j)_{0 \le i,j \le 10}$. For $i = 1, \dots, 10$, we have that $R_0 \cdot R_i = 2$ as $R_0 + R_i$ is an elliptic configuration of type $\widetilde{A}_1$. Lastly, for $i, j \neq 0, i \neq j$, we have that

$$4 = (R_0 + R_i) \cdot (R_0 + R_j) = 2 + R_i \cdot R_j \implies R_i \cdot R_j = 2.$$

It can be checked directly that the matrix $\mathfrak{R}$ has rank 11, which implies that $R_0, \dots, R_{10}$ generate a sublattice of S_Y of rank 11, which cannot be. ∎

3. Smooth rational curves and automorphisms of Enriques surfaces

From the discussion so far, it emerged that the more we know about smooth rational curves on an Enriques surface Y, the more we understand $\mathrm{nd}(Y)$. In [4] Brandhorst and Shimada studied the distribution of smooth rational curves on Y in relation to the automorphism group $\mathrm{Aut}(Y)$. Below we recall some aspects of their work.

3.1. Root lattices and the $(\tau, \overline{\tau})$-generic Enriques surfaces

We follow the exposition in [4, Section 1.1]. An *ADE-lattice* is an even, negative definite lattice R generated by *roots*, i.e. vectors $v \in R$ such that $v^2 = -2$. It is well known that an ADE-lattice R has a basis consisting of roots whose associated dual graph is the disjoint union of some of the Dynkin diagrams A_n $(n \ge 1)$, D_n $(n \ge 4)$, and E_6, E_7, E_8. This *ADE-type* for the lattice R is denoted by $\tau(R)$.

In [20] Shimada classified the ADE-sublattices of E_{10} up to the action of $O^{\mathcal{P}}(E_{10})$, which is the group of isometries of E_{10} which preserve a positive half-cone $\mathcal{P}$, that is one of the two connected components of $\{v \in E_{10} \mid v \cdot v > 0\}$.

Theorem 3.1 ([20]). *The following hold:*

(1) *Let $R_1, R_2 \subseteq E_{10}$ be two ADE-lattices. Denote by $\overline{R}_1, \overline{R}_2$ their respective primitive closures in E_{10}. Then also $\overline{R}_1, \overline{R}_2$ are ADE-lattices and $(\tau(R_1), \tau(\overline{R}_1)) = (\tau(R_2), \tau(\overline{R}_2))$ if and only if R_1 and R_2 are in the same $O^{\mathcal{P}}(E_{10})$-orbit.*

(2) *Let $R \subseteq E_{10}$ be an ADE-sublattice. Then there are* 184 *possibilities for the pairs $(\tau(R), \tau(\overline{R}))$. These are listed in* [20, Table 1] *(see also* [4, Table 1]*).*

Remark 3.2. Given an ADE-sublattice $R \subseteq E_{10}$, it is natural to ask whether there exists an Enriques surface Y together with a configuration of smooth rational curves $C_1, \dots, C_n \subseteq Y$ whose dual graph equals $\tau(R)$. By [20, Corollary 1.8], we have that $\tau(R)$ is realized in this way on a complex Enriques surface if and only if the fourth column of [20, Table 1] does not have the symbol "–", which occurs in 175 cases out of the 184 possibilities for $(\tau(R), \tau(\overline{R}))$. As our goal is to study these Enriques surfaces, we work over $\mathbb{C}$.

We will be interested in understanding the geometry of nodal Enriques surfaces Y whose universal K3 cover satisfies specific conditions with respect to a configuration of smooth rational curves on Y generating a sublattice of S_Y of fixed ADE-type. Recall that for a lattice (L, b) and a positive integer m, $L(m)$ denotes the lattice (L, mb).

Definition 3.3. Let Y be an Enriques surface with universal K3 cover $X \to Y$. Let $(\tau, \overline{\tau})$ one of the pairs in [4, Table 1]. Then Y is called $(\tau, \overline{\tau})$-*generic* provided the following hold:

(1) Consider $H^{2,0}(X) \subseteq T_X \otimes \mathbb{C}$, where T_X denotes the transcendental lattice of the K3 surface X. Then the group of isometries of T_X preserving $H^{2,0}(X)$ is equal to $\{\pm \mathrm{id}_{T_X}\}$.

(2) Let $R \subseteq E_{10}$ be an ADE-sublattice such that $(\tau(R), \tau(\overline{R})) = (\tau, \overline{\tau})$. Define M_R to be the sublattice of $E_{10}(2) \oplus R(2)$ given by

$$\langle (v, 0), (w, \pm w)/2 \mid v \in E_{10},\ w \in R \rangle.$$

Then there exist isometries $M_R \cong S_X$ and $E_{10} \cong S_Y$ such that the following diagram commutes:

$$\begin{array}{ccc} E_{10}(2) & \hookrightarrow & M_R \\ \cong \downarrow & & \downarrow \cong \\ S_Y(2) & \hookrightarrow & S_X. \end{array}$$

Definition 3.4. Among the 175 cases in Remark 3.2 of $(\tau, \overline{\tau})$ which can be realized geometrically by smooth rational curves on an Enriques surface, 155 of them are $(\tau, \overline{\tau})$-generic. These are the cases not marked with "×" in the fifth column of [4, Table 1]. We will focus on these Enriques surfaces, which we will refer to as *realizable* $(\tau, \overline{\tau})$-generic Enriques surfaces.

3.2. Automorphisms and smooth rational curves on $(\tau, \overline{\tau})$-generic Enriques surfaces

Let Y be a $(\tau, \overline{\tau})$-generic Enriques surface. Consider the natural homomorphism

$$\mathrm{Aut}(Y) \to O^{\mathcal{P}}(S_Y)$$

and denote by $\mathrm{aut}(Y)$ its image. One of the main results in [4] is the computation of a finite generating set for $\mathrm{aut}(Y)$ and the study of its action on $\mathcal{R}(Y)$, which denotes the set of smooth rational curves on Y. The construction is quite technical [4, Section 6.1], but for our purposes it suffices to focus on two of its main ingredients: the sets $\mathcal{R}_{\text{temp}}$ and $\mathcal{H}$. The former set $\mathcal{R}_{\text{temp}}$ is a specific subset of $\mathcal{R}(Y)$ with the property that the composition

$$\mathcal{R}_{\text{temp}} \hookrightarrow \mathcal{R}(Y) \twoheadrightarrow \mathcal{R}(Y)/\,\mathrm{aut}(Y)$$

is surjective, while $\mathcal{H}$ is a subset of $\mathrm{aut}(Y)$ that acts in a specific way on a chamber decompositions of the nef cone of Y. We focus on $\mathcal{R}_{\text{temp}}$ because it contains explicit examples of smooth rational curves on Y that we can use to study $\mathrm{nd}(Y)$, and we can apply the automorphisms in $\mathcal{H}$ to $\mathcal{R}_{\text{temp}}$ to obtain more smooth rational curves if needed.

Moreover, the elements in $\mathcal{R}_{\text{temp}}$ and $\mathcal{H}$ are conveniently described in [21] in terms of a fixed E_{10}-basis of S_Y. So, a curve in $\mathcal{R}_{\text{temp}}$ is an integral vector of dimension 10 and an automorphism in $\mathcal{H}$ is a 10×10 matrix with entries in $\mathbb{Z}$. In the next example we explain how to extract this information from [21] (see also [22]).

Example 3.5. Consider the $(\tau, \overline{\tau})$-generic Enriques surface number 145. To find the data corresponding to $\mathcal{R}_{\text{temp}}$ and $\mathcal{H}$ in the file [21, Enrs.txt], we can first search for `no := 145`. In the corresponding record named `Rats`, which is before the label 145, we can find `Ratstemp`, and then different records named `rat`. For each one of these, we consider `ratY`, which give the following vectors:

$$
\begin{aligned}
R_0 &:= (4,2,4,6,5,4,3,2,1,0), & R_1 &:= (2,2,3,4,3,2,1,0,0,0),\\
R_2 &:= (0,0,0,0,0,0,0,0,0,1), & R_3 &:= (0,0,0,0,0,0,0,0,1,0),\\
R_4 &:= (0,0,0,0,0,0,0,1,0,0), & R_5 &:= (0,0,0,0,0,0,1,0,0,0),\\
R_6 &:= (0,0,0,0,0,1,0,0,0,0), & R_7 &:= (0,0,0,0,1,0,0,0,0,0),\\
R_8 &:= (0,0,1,0,0,0,0,0,0,0), & R_9 &:= (0,0,0,1,0,0,0,0,0,0).
\end{aligned}
$$

This means that, for a fixed E_{10}-basis $\{B_1, \ldots, B_{10}\}$ of S_Y, the vectors

$$
\begin{aligned}
&4B_1 + 2B_2 + 4B_3 + 6B_4 + 5B_5 + 4B_6 + 3B_7 + 2B_8 + B_9,\\
&2B_1 + 2B_2 + 3B_3 + 4B_4 + 3B_5 + 2B_6 + B_7, B_{10}, B_9, B_8, B_7, B_6, B_5, B_3, B_4
\end{aligned}
$$

are numerically equivalent to smooth rational curve on Y, giving the curves in $\mathcal{R}_{\text{temp}}$. The record `HHH` encodes the automorphisms in $\mathcal{H}$, whose action on S_Y is described by the different `gY`. For the Enriques surface 145 there are two such `gY`, and these have to be thought of as 10×10 matrices in the E_{10}-basis $\{B_1, \ldots, B_{10}\}$.

4. The non-degeneracy invariant for $(\tau, \overline{\tau})$-generic Enriques surfaces

4.1. The main result

Using [16, 17] we compute a lower bound, and in the majority of the cases the exact value, for the non-degeneracy invariant of the realizable $(\tau, \overline{\tau})$-generic Enriques surfaces in [4, Table 1].

Remark 4.1. Let Y_i be the i-th realizable $(\tau, \overline{\tau})$-generic Enriques surface in [4, Table 1]. By [4, Section 7.4], the Enriques surfaces Y_{172} and Y_{184} have finite automorphism group. These Enriques surfaces were classified in [12], and more precisely we have that Y_{172} and Y_{184} are respectively of type I and II. For these Enriques surfaces, the non-degeneracy invariants were computed in [10, Propositions 8.9.6 and 8.9.9] (see also [16, Table 3]). The Enriques surface Y_1 is a general nodal Enriques surfaces by combining [4, Section 6.5] and [10, Theorem 6.5.5 (ii)], and thus we have $\mathrm{nd}(Y_1) = 10$. This leaves us with 152 cases of $\mathrm{nd}(Y_i)$ to compute.

Theorem 4.2. *Let Y_i be the i-th realizable $(\tau, \overline{\tau})$-generic Enriques surface in* [4, Table 1] *with $i \neq 1, 172, 184$. Then,*

(1) $\mathrm{nd}(Y_{145}) = 4$.

(2) *We have that*

$$\begin{aligned} \mathrm{nd}(Y_{84}) &\geq 9, & \mathrm{nd}(Y_{85}) &\geq 7, & \mathrm{nd}(Y_{121}) &\geq 9, \\ \mathrm{nd}(Y_{122}) &\geq 7, & \mathrm{nd}(Y_{123}) &\geq 7, & \mathrm{nd}(Y_{143}) &\geq 8, \\ \mathrm{nd}(Y_{144}) &\geq 8, & \mathrm{nd}(Y_{158}) &\geq 9, & \mathrm{nd}(Y_{159}) &\geq 7, \\ \mathrm{nd}(Y_{171}) &\geq 8, & \mathrm{nd}(Y_{176}) &\geq 7. \end{aligned}$$

Moreover, in the above 11 *cases,* $\mathrm{nd}(Y_i) = \mathrm{cnd}(Y_i)$.

(3) *Finally, in the remaining* 140 *cases we have that* $\mathrm{nd}(Y_i) = 10$.

Proof. The value $\mathrm{nd}(Y_{145}) = 4$ is computed in Section 5 (in particular, see Theorem 5.10). In the remaining cases where $\mathrm{nd}(Y_i) \geq m$, we provide an explicit non-degenerate isotropic sequence of length m (if $m = 10$, then we obtain the 141 cases in (3)). To find such sequences we use the code [17] with input the smooth rational curves in $\mathcal{R}_{\mathrm{temp}}$ provided by [21], and in some case $\mathcal{R}_{\mathrm{temp}}$ together with some curves in $\mathcal{R}_{\mathrm{temp}} \cdot \mathrm{aut}(Y_i)$ (see Section 4.2 for more detail). The output, which can be tested directly without using again [17] (see Section 4.3 for more detail), is discussed in Section 6. Finally, the claim in (2) that $\mathrm{nd}(Y_i) = \mathrm{cnd}(Y_i)$ for the eleven examples mentioned follows by Proposition 2.7 combined with the fact that every elliptic fibration on Y_i admits a fiber or a half-fiber supported on the union of smooth rational curves, as it can be argued by [23]. ∎

Remark 4.3. It follows immediately from Theorem 4.2 that there are no $(\tau, \overline{\tau})$-generic Enriques surfaces with non-degeneracy invariant equal to 5 or 6. To our knowledge, examples of Enriques surfaces with non-degeneracy invariant 5, 6, or 9 are not known.

4.2. Computer-based construction of the isotropic sequences

Fix a $(\tau, \overline{\tau})$-generic Enriques surface Y_i. To compute a lower bound for $\mathrm{nd}(Y_i)$, the program [17] needs as input a basis for S_{Y_i} and a finite collection $\mathcal{R}$ of smooth rational curves on Y_i (for the explanation of the algorithm see [16, Section 4]). Following [4], we take an E_{10}-basis for S_{Y_i} and $\mathcal{R} = \mathcal{R}_{\mathrm{temp}}$ if $i \neq 43, 78, 84, 121, 158$ (recall the smooth rational curves in $\mathcal{R}_{\mathrm{temp}}$ are already expressed in the E_{10} basis in [21]). For $i = 43, 78, 84, 121, 158$, we obtain the claimed lower bound on $\mathrm{nd}(Y_i)$ by taking $\mathcal{R}$ equal to $\mathcal{R}_{\mathrm{temp}}$ union a finite subset of $\mathcal{R}_{\mathrm{temp}} \cdot \mathrm{aut}(Y_i)$. More precisely, we consider the action of the automorphisms in $\mathcal{H}$ (see Example 3.5).

Remark 4.4. We collect some subtleties behind the above calculations.

(1) In the cases where $\mathrm{nd}(Y_i) = 10$, it is sometimes enough to consider a proper subset of curves in $\mathcal{R}_{\mathrm{temp}}$ to construct a non-degenerate isotropic sequence of length 10.

(2) In Theorem 4.2, for the cases where $\mathrm{nd}(Y_i) \geq m$ with $m \neq 10$ and $i \neq 43, 78, 84$, we used all the curves in $\mathcal{R}_{\mathrm{temp}}$. We also used some of the automorphisms in $\mathcal{H}$ to produce new smooth rational curves not in $\mathcal{R}_{\mathrm{temp}}$, but this did not improve the lower bound m.

(3) The code [17] needs that the chosen smooth rational curves in $\mathcal{R}$ generate S_{Y_i} over $\mathbb{Q}$. We have that $\mathcal{R}_{\mathrm{temp}}$ has this property, except for $i = 84$.

(4) For $i = 43, 78, 84, 121, 158$, by only using $\mathcal{R} = \mathcal{R}_{\mathrm{temp}}$ we obtain the lower bounds $\mathrm{nd}(Y_{43}) \geq 6$, $\mathrm{nd}(Y_{78}) \geq 7$, $\mathrm{nd}(Y_{84}) \geq 4$, $\mathrm{nd}(Y_{121}) \geq 6$, $\mathrm{nd}(Y_{158}) \geq 8$, instead of the better 10, 10, 9, 9, 9, respectively.

4.3. How to test the correctness of the output

One way of proving parts (2) and (3) of Theorem 4.2 is by directly checking the output discussed in Section 6. More precisely, let $\mathrm{nd}(Y_k) \geq m$ be one of the inequalities claimed in (2) or (3) of Theorem 4.2 and let $F_1, \dots, F_m$ be the corresponding sequence of half-fibers we want to check. Using the data in [21], we can find the coordinates of each smooth rational curve in the support of $F_1, \dots, F_m$ with respect to the fixed E_{10} basis, and consequently express each half-fiber $F_1, \dots, F_m$ in this basis. (Sometimes, smooth rational curves are obtained by acting on some $R_i \in \mathcal{R}_{\mathrm{temp}}$ with automorphisms in $\mathcal{H} = \{H_0, H_1, \dots\}$). We point out that the obtained vectors have integral entries, which guarantees that $F_1, \dots, F_m \in S_{Y_k}$, and that the entries are not simultaneously divisible by 2, which guarantees that there are no fibers among $F_1, \dots, F_m$. By organizing the 10-dimensional vectors as rows of a matrix F, we can check the following equality by simple matrix multiplication:

$$F M_{E_{10}} F^{\mathsf{T}} = \mathbf{1}_m - \mathbf{I}_m.$$

Here, $\mathbf{1}_m$ is the $m \times m$ matrix with entries equal to 1 and $\mathbf{I}_m$ is the $m \times m$ identity matrix. This means that $F_i \cdot F_j = 1 - \delta_{ij}$ for all i, j. The last thing to verify is that $F_1, \dots, F_m$ are actually half-fibers. For this, let C_i be the elliptic configuration associated to F_i. That is, if F_i has a coefficient $1/2$, then $C_i = 2F_i$. Otherwise, $C_i = F_i$. Then we have to check that the dual graph of the curves R_i in the support of C_i form an extended Dynkin diagram and that the coefficients of the smooth rational curves R_i in C_i match the multiplicities of the irreducible components of the singular fibers in Kodaira's classification.

Example 4.5. For $Y_k = Y_{158}$, the claimed sequence of half-fibers is the following:

$$F_1 = \frac{1}{2}(R_0 + R_2), \quad F_2 = \frac{1}{2}(R_2 + R_{16}), \quad F_3 = \frac{1}{2}(R_2 + R_2 \cdot H_2),$$
$$F_4 = \frac{1}{2}(R_3 + R_4), \quad F_5 = \frac{1}{2}(R_3 + R_{12}), \quad F_6 = \frac{1}{2}(R_2 + R_{14}),$$
$$F_7 = (R_2 + R_8), \quad F_8 = \frac{1}{2}(R_1 + R_6 + R_8 + R_{15} + 2R_7),$$
$$F_9 = \frac{1}{2}(R_5 + R_6 + R_8 + R_9 + 2R_{11}).$$

Using the data in [21] we can write $R_0, R_1, R_2, R_3, \ldots, R_9, R_{11}, R_{12}, R_{14}, R_{15}, R_{16}$, $R_2 \cdot H_2$ in the E_{10}-basis as

$$
\begin{aligned}
&R_0 = (2,1,2,3,2,1,0,0,0,0), && R_1 = (2,1,2,3,2,2,2,1,0,0),\\
&R_2 = (8,5,10,15,14,13,10,6,4,2), && R_3 = (8,5,10,15,14,13,10,7,4,1),\\
&R_4 = (4,1,4,7,6,5,4,3,2,1), && R_5 = (0,1,0,0,0,0,0,0,0,0),\\
&R_6 = (0,0,0,0,0,0,0,0,0,1), && R_7 = (0,0,0,0,0,0,0,0,1,0),\\
&R_8 = (0,0,0,0,0,0,0,1,0,0), && R_9 = (2,1,2,4,4,4,4,3,2,1),\\
&R_{11} = (4,2,5,8,7,6,4,2,1,0), && R_{12} = (4,3,6,9,8,7,4,3,2,1),\\
&R_{14} = (4,3,6,9,8,7,6,4,2,0), && R_{15} = (4,3,6,9,8,8,6,4,2,1),\\
&R_{16} = (6,3,8,13,12,11,8,6,4,2), && R_2 \cdot H_2 = (12,7,14,23,20,19,14,10,6,2).
\end{aligned}
\tag{4.1}
$$

So, the coordinates of $F_1, \ldots, F_9$ in the same basis are

$$
\begin{aligned}
F_1 &= (5,3,6,9,8,7,5,3,2,1), & F_2 &= (7,4,9,14,13,12,9,6,4,2),\\
F_3 &= (10,6,12,19,17,16,12,8,5,2), & F_4 &= (6,3,7,11,10,9,7,5,3,1),\\
F_5 &= (6,4,8,12,11,10,7,5,3,1), & F_6 &= (6,4,8,12,11,10,8,5,3,1),\\
F_7 &= (8,5,10,15,14,13,10,7,4,2), & F_8 &= (3,2,4,6,5,5,4,3,2,1),\\
F_9 &= (5,3,6,10,9,8,6,4,2,1).
\end{aligned}
$$

The matrix equality $F M_{E_{10}} F^{\intercal} = \mathbf{1}_9 - \mathbf{I}_9$ holds true. Below we write the intersection matrix of the smooth rational curves in (4.1):

$$
\begin{pmatrix}
-2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 2\\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0\\
2 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 2 & 2\\
2 & 0 & 0 & -2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0\\
0 & 0 & 2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 2\\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0\\
0 & 0 & 0 & 2 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2\\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 2 & 0 & 0 & 2\\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 2 & 2\\
2 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0\\
2 & 2 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 2\\
2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & -2
\end{pmatrix}.
$$

From this matrix we can extract that the extended Dynkin diagrams corresponding to the elliptic configuration associated to $F_1, \ldots, F_9$, which are respectively:

$$6 \times \widetilde{A}_1^{\mathrm{F}},\ \widetilde{A}_1^{\mathrm{HF}},\ 2 \times \widetilde{D}_4^{\mathrm{F}}.$$

The coefficients of the smooth rational curves in (4.1) indeed match the multiplicities from Kodaira's classification. This completes the check for $i = 158$, confirming that

$$\mathrm{nd}(Y_{158}) \geq 9.$$

To assist with the above checking for the 151 cases in Theorem 4.2 (2) and (3), we provide appropriate scripts in [18].

4.4. A conjecture about $\mathrm{nd}(Y_{158})$

By Theorem 4.2, we have that

$$\mathrm{nd}(Y_{84}), \mathrm{nd}(Y_{121}), \mathrm{nd}(Y_{158}) \geq 9.$$

As there are no known examples of Enriques surfaces with non-degeneracy invariant 9, it is worthwhile to explore whether at least one of these three lower bounds is attained. As $\mathrm{aut}(Y_{158})$ has the smallest number of generators, Y_{158} is a good candidate to study more in detail. We run the code [17] with large (finite) samples of smooth rational curves $\mathcal{R}$ in $\mathcal{R}(Y_{158}) = \mathcal{R}_{\mathrm{temp}} \cdot \mathrm{aut}(Y_{158})$ always obtaining that $\mathrm{cnd}(Y_{158}, \mathcal{R}) = 9$. This computational evidence motivates the following conjecture.

Conjecture 4.6. *The realizable $(\tau, \overline{\tau})$-generic Enriques surfaces* 158 *satisfies*

$$\mathrm{nd}(Y_{158}) = 9.$$

4.5. A question about $\mathrm{cnd}(Y_1)$

For the $(\tau, \overline{\tau})$-generic Enriques surface Y_1, we found that $\mathrm{cnd}(Y_1) \geq \mathrm{cnd}(Y_1, \mathcal{R}) = 7$ for several choices of $\mathcal{R}$ among the finite sets of smooth rational curves containing $\mathcal{R}_{\mathrm{temp}}$. Additionally, by Proposition 2.8 we have that $\mathrm{cnd}(Y_1) < 10$, so the possibilities are $\mathrm{cnd}(Y_1) \in \{7, 8, 9\}$. It would be interesting to know the value of $\mathrm{cnd}(Y_1)$.

5. The non-degeneracy invariant of the Enriques surface 145

In this section we fix $Y := Y_{145}$ to be the realizable $(\tau, \overline{\tau})$-generic Enriques surface 145 in [4, Table 1]. In this case, $(\tau, \overline{\tau}) = (E_8, E_8)$. Our goal is to prove that $\mathrm{nd}(Y) = 4$ in Theorem 5.10. In what follows, we fix an E_{10}-basis of S_Y compatible with the computational data in [21].

5.1. Automorphisms and smooth rational curves on Y

Next, we describe $\mathrm{aut}(Y)$. A description of the full automorphism group $\mathrm{Aut}(Y)$ appears already in [2, Theorem 4.12]. At the same time, it is necessary for our purposes to describe explicit generators compatibly with [4].

Start by first considering the 10 rational curves $R_0, \ldots, R_9$ in $\mathcal{R}_{\mathrm{temp}}$ on Y which were introduced in Example 3.5. The dual graph of the union of $R_0, \ldots, R_9$ appears in Figure 2.

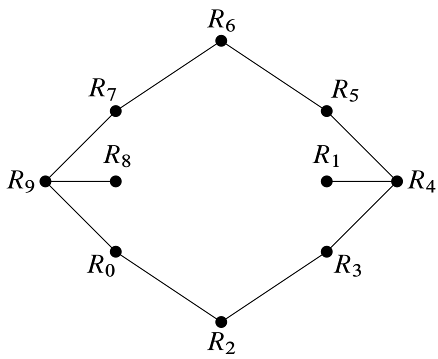

Figure 2. Intersection graph of the smooth rational curves $R_0, \dots, R_9$ on the Enriques surface Y_{145}.

Recall that a pair $(2F_1, 2F_2)$ of genus one pencils such that $F_1 \cdot F_2 = 1$ is called a *U-pair*. To a U-pair one can associate the *bielliptic map*, generically of degree 2, which is induced by the linear series $|2F_1 + 2F_2|$. Therefore, a U-pair gives rise to an involution on Y. We refer to [8, Section 3.3] for more properties of the linear series associated to the U-pairs.

Definition 5.1. Consider the following fibers of elliptic fibrations:

$$G_1 = 2(R_0 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_9),$$
$$G_2 = R_0 + R_1 + R_3 + R_8 + 2(R_4 + R_5 + R_6 + R_7 + R_9),$$
$$G_3 = 4R_9 + 3R_0 + 3R_7 + 2R_2 + 2R_6 + 2R_8 + R_3 + R_5.$$

Let $\boldsymbol{\varepsilon}$ be the involution associated to the U-pair (G_1, G_2). Another involution $\boldsymbol{\delta}$ is associated to the U-pair (G_1, G_3). Observe that $G_3 \cdot R_1 = 0$, so R_1 is a component of a singular fiber of $|G_3|$. By the classification in [23], such a fiber forms a $\widetilde{A}_1$ configuration: we denote by R_1' its other component. Finally, define $\boldsymbol{\gamma} := \boldsymbol{\delta} \circ \boldsymbol{\varepsilon}$.

Lemma 5.2. *The automorphisms $\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\gamma} \in \mathrm{Aut}(Y)$ constructed above satisfy the following properties:*

(a) *The involution $\boldsymbol{\varepsilon}$ fixes R_2 and R_6 and it acts as a transposition on the pairs of curves*
$$(R_0, R_3),\ (R_1, R_8),\ (R_4, R_9),\ (R_5, R_7).$$

(b) *The involution $\boldsymbol{\delta}$ fixes R_4, R_8, R_9, and it acts as a transposition on the pairs*
$$(R_0, R_7),\ (R_2, R_6),\ (R_3, R_5),\ (R_1, R_1').$$

(c) *The automorphism $\boldsymbol{\gamma}$ has infinite order and it acts as a transposition on the pairs*
$$(R_0, R_5),\ (R_2, R_6),\ (R_3, R_7),\ (R_4, R_9).$$

Moreover, let us define $R_{8,k} := \boldsymbol{\gamma}^k(R_8)$. Then the following hold:

(c.1) $R_{8,-1} = R_1$;

(c.2) $R_{8,m} \neq R_{8,n}$ *for all* $m, n \in \mathbb{Z}$, $m \neq n$;

(c.3) $R_{8,k} \neq R_0, \ldots, R_9$ *for all* $k \in \mathbb{Z} \setminus \{-1, 0\}$.

Proof. By [10, Lemma 8.7.5], $\boldsymbol{\varepsilon}$ preserves the elliptic fibrations $|G_1|$ and $|G_2|$. The automorphism $\boldsymbol{\varepsilon}$ is numerically non-trivial: this follows from [10, Lemma 8.2.5] and the fact that the singular fibers in $|G_1|$ and $|G_2|$ have only 7 common components. The common components $R_3, R_4, R_5, R_6, R_7, R_9, R_0$ form an A_7-configuration. Then, we claim that $\boldsymbol{\varepsilon}$ must act on the A_7-configuration as the non-trivial symmetry of the A_7 diagram. Indeed, otherwise it would fix the classes of all curves R_i, $i = 0, \ldots, 9$. Since these classes are independent in S_Y, $\boldsymbol{\varepsilon}$ would act trivially on a basis of $S_Y \otimes \mathbb{Q}$, and hence numerically trivially. This is a contradiction. The claimed action of $\boldsymbol{\varepsilon}$ on R_1, R_2, R_8 follows since $\boldsymbol{\varepsilon}$ preserves the intersection products. This shows part (a).

Consider now $\boldsymbol{\delta}$. Observe first that the $\widetilde{E}_7$-fiber G_3 must be preserved, and hence that R_8 must be fixed. Likewise, $\boldsymbol{\delta}$ maps R_4 to itself. Next, we claim that $\boldsymbol{\delta}(R_3) = R_3$ and $\boldsymbol{\delta}(R_5) = R_5$ if and only if $\boldsymbol{\delta}(R_1) = R_1$ and $\boldsymbol{\delta}(R_1') = R_1'$. In fact, the curve R_4 is a bisection of $|G_3|$ meeting G_3 in R_3 and R_5, and meeting the singular fiber $R_1 + R_1'$ once in each component. The restriction $\boldsymbol{\delta}|_{R_4}$ has at least two fixed points given by the intersection with the two half-fibers of $|G_3|$. Since an automorphism of $R_4 \cong \mathbb{P}^1$ with three fixed points is the identity, and since $\boldsymbol{\delta}$ preserves the fibration $|G_3|$, the map $\boldsymbol{\delta}|_{R_4}$ fixes $R_4 \cap R_3$ and $R_4 \cap R_5$ only if it is the identity, which establishes the claim. Arguing as for $\boldsymbol{\varepsilon}$, the automorphism $\boldsymbol{\delta}$ acts numerically non-trivially. Then, we claim that $\boldsymbol{\delta}$ must act on the A_7-configuration of common curves among G_1 and G_3 as a reflection. Suppose otherwise for the sake of contradiction. In particular, the classes R_3 and R_5 are preserved by $\boldsymbol{\delta}$. As explained above, this implies that $\boldsymbol{\delta}$ also preserves R_1, and hence that it is numerically trivial, arguing as for $\boldsymbol{\varepsilon}$. This is a contradiction, hence $\boldsymbol{\delta}$ acts as a reflection on the A_7-configuration and transposes R_1 and R_1'.

For part (c), the action of $\boldsymbol{\gamma}$ on R_i for $i \neq 8$ follows from parts (a) and (b) above. We now prove that $\boldsymbol{\gamma}$ has infinite order. Denote by $\boldsymbol{\gamma}_0$ the isometry of S_Y which is induced by $\boldsymbol{\gamma}$. First, observe that $\boldsymbol{\gamma}_0^2$ is the identity on the $\widetilde{A}_7$-configuration. Hence, by the proof of [2, Lemma 4.10], we have that $\boldsymbol{\gamma}_0^2$ is an element of a subgroup of $O(S_Y)$ isomorphic to $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$. In particular, as $\boldsymbol{\gamma}_0^2$ is not the identity ($\boldsymbol{\gamma}_0^2(R_1) = R_1'$), we have that $\boldsymbol{\gamma}_0^2$ has order 2 or infinite. So, if we show that $\boldsymbol{\gamma}_0^4$ is a nontrivial isometry, then we can conclude that $\boldsymbol{\gamma}_0^2$, and hence $\boldsymbol{\gamma}, \boldsymbol{\gamma}_0$, has infinite order. Consider the $\widetilde{E}_7$ elliptic configurations given by

$$M = 4R_4 + 3R_3 + 3R_5 + 2R_2 + 2R_6 + 2R_1 + R_0 + R_7,$$
$$M' = 4R_4 + 3R_3 + 3R_5 + 2R_2 + 2R_6 + 2R_1' + R_0 + R_7.$$

By [23], the elliptic fibrations $|M|, |M'|$ have a $\widetilde{A}_1$ fiber each, which both contain the curve R_8. Explicitly, we can write these two $\widetilde{A}_1$ fibers as $R_8 + R$ and $R_8 + S$ for some smooth rational curves R and S. As $\boldsymbol{\gamma}^2(M) = M'$, we have that $\boldsymbol{\gamma}^2(R_8 + R) = R_8 + S$. So, $\boldsymbol{\gamma}^2(R_8) = R_8$ or S. We show that the former is impossible by contradiction. If $\boldsymbol{\gamma}^2(R_8) = R_8$, then $\boldsymbol{\gamma}(R_1') = R_8$. By applying $\boldsymbol{\delta}$ to both sides, we obtain $\boldsymbol{\varepsilon}(R_1') = \boldsymbol{\delta}(R_8) = R_8$, hence $\boldsymbol{\varepsilon}(R_8) = R_1'$, which cannot be as $\boldsymbol{\varepsilon}(R_8) = R_1$ and $R_1 \neq R_1'$. We argued that $\boldsymbol{\gamma}^2(R_8) = S$, which forces $\boldsymbol{\gamma}^2(R) = R_8$, whence $\boldsymbol{\gamma}^4(R) = \boldsymbol{\gamma}^2(R_8) = S$. As it also holds $\boldsymbol{\gamma}_0^4(R) = S$, this proves that $\boldsymbol{\gamma}_0^4$ is not the trivial isometry.

We now can prove part (c.2). If false, then let k be a nonzero integer such that $\boldsymbol{\gamma}^k(R_8) = R_8$. Therefore, also $\boldsymbol{\gamma}^{2k}(R_8) = R_8$. Additionally, $\boldsymbol{\gamma}^{2k}$ fixes the $\widetilde{A}_7$-configuration. We have that

$$\begin{aligned}\boldsymbol{\gamma}^{2k}(R_8) = R_8 &\implies \boldsymbol{\gamma}^{-2k}(R_8) = R_8 \implies \boldsymbol{\gamma}^{-2k}\big(\boldsymbol{\varepsilon}(R_1)\big) = R_8 \\ &\implies \boldsymbol{\varepsilon}\big(\boldsymbol{\gamma}^{2k}(R_1)\big) = R_8 \implies \boldsymbol{\gamma}^{2k}(R_1) = \boldsymbol{\varepsilon}(R_8) = R_1.\end{aligned}$$

As $\boldsymbol{\gamma}^{2k}$ fixes each curve $R_0, \dots, R_9$, we have that $\boldsymbol{\gamma}_0^{2k}$ is the trivial isometry, contradicting the fact proved above that $\boldsymbol{\gamma}_0$ has infinite order. Finally, let us show (c.3). By the above argument, we already know that $R_{8,k} \neq R_1, R_8$. If $R_{8,k} = R_j$ for $j \neq 1, 8$, then $R_{8,k+2} = R_j$ (recall $\boldsymbol{\gamma}^2$ fixes the $\widetilde{A}_7$-configuration), which contradicts (c.2). ∎

The remaining part of this section relies on the computational data [21]. To help the reader reproduce the computational arguments, we introduce some notations and conventions, and reconcile our considerations so far with [4]. In particular, recall that [4] fixes a E_{10}-basis of S_Y. The coordinates of the curves $R_0, \dots, R_9$ with respect to this basis are written as row vectors in Example 3.5. By letting $\boldsymbol{\varepsilon}$ and $\boldsymbol{\gamma}$ act on S_Y, the associated matrices in this E_{10}-basis are written below. To denote them, we use the corresponding Greek letters, but not in boldface.

$$\varepsilon = \begin{pmatrix} -3 & -2 & -4 & -6 & -5 & -4 & -3 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} -9 & -4 & -10 & -16 & -13 & -10 & -7 & -6 & -4 & -2 \\ -8 & -3 & -8 & -14 & -12 & -10 & -8 & -6 & -4 & -2 \\ 10 & 4 & 11 & 18 & 15 & 12 & 9 & 6 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From now on, we see $\mathrm{aut}(Y)$ as a group of matrices.

Lemma 5.3. *The group* $\mathrm{aut}(Y)$ *is generated by* ε *and* γ, *which satisfy the relations*

$$\varepsilon^2 = \mathbf{I}_{10}, \quad \varepsilon\gamma = \gamma^{-1}\varepsilon.$$

In particular, $\mathrm{aut}(Y) \cong \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$.

Proof. The claimed relations can be checked directly (this can also be checked at level of the automorphisms $\boldsymbol{\varepsilon}, \boldsymbol{\gamma}$). The matrix γ matches an element of $\mathscr{H}$, extracted from [21, Enrs.txt] as explained in Example 3.5. The other matrix in the record is γ^{-1}, and these are the only two elements in $\mathscr{H}$. The matrix ε appears in the record `autcham` inside `V0`. In `autcham` there are two possible `gY`, and ε equals the record `gY` different from $\mathbf{I}_{10}$. So, by the discussion in [4, Section 6.1], we have that ε and γ generate $\mathrm{aut}(Y)$. Finally, the isomorphism of groups $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{aut}(Y)$ is explicitly given by $(n, i) \mapsto \gamma^n \varepsilon^i$. ∎

Recall the convention in [4] that the automorphisms act on the smooth rational curves by matrix multiplication on the right. We follow this convention, for consistency with the work of Brandhorst–Shimada. So, for instance, the action of $\boldsymbol{\gamma}$ on R_8 is given by the matrix multiplication

$$R_8 \cdot \gamma = (10, 4, 11, 18, 15, 12, 9, 6, 4, 2),$$

meaning that the coordinate vector of $\boldsymbol{\gamma}(R_8)$ is $R_8 \cdot \gamma$, where in the latter we identify as usual the smooth rational curve R_8 with its numerical class in S_Y and coordinate vector. Although both the intersection product among curves and the matrix action of $\mathrm{aut}(Y)$ are denoted by "·", the difference between the two will always be clear from the context.

As an illustration, we explicitly compute the coordinate vectors of $R_8 \cdot \gamma^k$ as the transpose of

$$\begin{pmatrix} 4k^2 + 4k + 1 - (-1)^k \\ 2k^2 + k + \frac{1}{2} - \frac{1}{2}(-1)^k \\ 4k^2 + 4k + 2 - (-1)^k \\ 7k^2 + 7k + 2 - 2(-1)^k \\ 6k^2 + 6k + \frac{3}{2} - \frac{3}{2}(-1)^k \\ 5k^2 + 5k + 1 - (-1)^k \\ 4k^2 + 4k + \frac{1}{2} - \frac{1}{2}(-1)^k \\ 3(k^2 + k) \\ 2(k^2 + k) \\ k^2 + k \end{pmatrix}. \tag{5.1}$$

This can be verified by using induction twice, one for the positive and one for the negative integers. Alternatively, the above claim can be checked directly with `SageMath`, as it can compute abstract k-th powers of matrices. We point out that (5.1) can be used to produce an alternative argument for Lemma 5.2 (c).

Corollary 5.4. *The only smooth rational curves on Y are $R_0, R_2, R_3, R_4, R_5, R_6, R_7, R_9$ and the infinitely many $R_{8,k}$ for $k \in \mathbb{Z}$, where in particular $R_8 = R_{8,0}$ and $R_1 = R_{8,-1}$.*

Proof. By the discussion in [4, Section 6.2], we have that the set $\mathcal{R}(Y)$ of smooth rational curves on Y can be obtained by acting with $\operatorname{aut}(Y)$ on $\mathcal{R}_{\text{temp}} = \{R_0, \dots, R_9\}$. At the same time, by Lemma 5.3, every element of $\operatorname{aut}(Y)$ can be written as $\boldsymbol{\varepsilon}^i \circ \boldsymbol{\gamma}^n$ for $n \in \mathbb{Z}$, $i \in \{0, 1\}$. So, we have that

$$\mathcal{R}(Y) = \left\{ R_j \cdot (\gamma^n \varepsilon^i) \mid n \in \mathbb{Z},\ i \in \{0,1\},\ j \in \{0, \dots, 9\} \right\}.$$

Then, combining the results in Lemma 5.2 with the equality (given by the right action)

$$R_j \cdot (\gamma^n \varepsilon) = R_j \cdot (\varepsilon \gamma^{-n}) = (R_j \cdot \varepsilon) \cdot \gamma^{-n},$$

we obtain that

$$\mathcal{R}(Y) = \mathcal{R}_{\text{temp}} \sqcup \left\{ R_{8,k} \mid k \in \mathbb{Z} \setminus \{0, -1\} \right\},$$

which gives the description of $\mathcal{R}(Y)$ in the statement. ∎

Remark 5.5 (The Barth–Peters example). It can be proved that the (E_8, E_8)-generic Enriques surface Y under analysis already appeared in [2, Section 4]. We briefly recall the construction of Barth and Peters. Consider $Q := \mathbb{P}^1 \times \mathbb{P}^1$ together with the involution given in an affine patch by

$$\mathbf{s}\colon (x, y) \mapsto (-x, -y).$$

For certain $(4,4)$-curves $B \subseteq Q$ (see [2, Section 4.1, Case 1]), the double cover $\overline{X} \to Q$ branched along B has two A_7 singularities. The minimal resolution X of $\overline{X}$ is a K3 surface, and $\mathbf{s}$ lifts to a fixed point free involution $\boldsymbol{\sigma}$ on X. Then, the very general Enriques surface $Y = X/\langle \boldsymbol{\sigma} \rangle$ is (E_8, E_8)-generic. We omit the proof of this as it is not needed in the computations that follow.

Finally, we connect the geometry of Y studied in Section 5.1 with the results and notations of Barth–Peters. Following [8, Proposition 3.3.15], the surface $W := Q/\langle \mathbf{s} \rangle$ is a 4-nodal quartic symmetroid del Pezzo surface in $\mathbb{P}^4$. The bielliptic map $Y \to W$ associated with the U-pair (G_1, G_3) in Definition 5.1 makes the following diagram commute:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Q & \longrightarrow & W. \end{array}$$

In particular, the covering involution of $Y \to W$, which we denoted by $\boldsymbol{\delta}$, coincides with the involution induced on Y by $\boldsymbol{\sigma}_3$ on X (see [2, Section 4.2]). While [2] does not describe an explicit infinite order generator of $\operatorname{aut}(Y)$, there is a precise description of the subgroup G of $\mathrm{O}(S_Y)$ which fixes $\{R_0, \dots, R_9\} \setminus \{R_1, R_8\}$: it is an infinite dihedral group, which is generated by an involution α_0 and an infinite order isometry α_1 [2, Lemma 4.10 (a)]. One can verify that $\boldsymbol{\gamma}^2$ induces on S_Y the isometry $\alpha_4 := \alpha_1^4$. This implies, together with the description of $\operatorname{aut}(Y)$ from [4], that $\operatorname{aut}(Y) \cap G$ is generated by α_4.

Type	Representatives of elliptic fibrations up to $\mathrm{aut}(Y)$
$\widetilde{A}_7^{\mathrm{HF}}$	$R_0 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_9$
$\widetilde{A}_1^{\mathrm{F}} + \widetilde{E}_7^{\mathrm{F}}$	$\big(A := \frac{1}{2}(R_8 + R_{8,-2}),\ \frac{1}{2}(3R_0 + 2R_2 + R_3 + 3R_5 + 2R_6 + R_7 + 4R_4 + 2R_1)\big)$
$\widetilde{D}_8^{\mathrm{F}}$	$D_a := \frac{1}{2}(R_0 + R_1 + R_3 + 2R_4 + 2R_5 + 2R_6 + 2R_7 + 2R_9 + R_8)$ $D_b := \frac{1}{2}(2R_0 + R_1 + 2R_2 + 2R_3 + 2R_4 + R_5 + R_7 + 2R_9 + R_8)$
$\widetilde{E}_8^{\mathrm{F}}$	$E_a := \frac{1}{2}(R_1 + 2R_4 + 3R_5 + 4R_6 + 5R_7 + 6R_9 + 4R_0 + 2R_2 + 3R_8)$ $E_b := \frac{1}{2}(2R_6 + 4R_7 + 6R_9 + 5R_0 + 4R_2 + 3R_3 + 2R_4 + R_1 + 3R_8)$

Table 1. Elliptic fibrations on the Enriques surface 145.

5.2. Elliptic fibrations on Y

The elliptic fibrations on Y up to $\mathrm{aut}(Y)$ are classified in [4, Theorem 1.21]. In the next lemma we give explicit representatives in term of the given smooth rational curves $\mathcal{R}(Y)$. First, it will be convenient to recall the following terminology and notation from [16].

Definition 5.6. The *type of an elliptic configuration* supported on $\mathbb{R}(Y)$ is the associated extended Dynkin diagram, together with the information of being a fiber or a half-fiber. The *type of an elliptic fibration* is the formal sum of the types of its singular fibers supported on $\mathcal{R}(Y)$. For instance, $(2\widetilde{A}_1^{\mathrm{HF}} + \widetilde{D}_6^{\mathrm{F}})$ refers to the fibrations whose singular fibers are three elliptic configurations, two of type $\widetilde{A}_1^{\mathrm{HF}}$ and one of type $\widetilde{D}_6^{\mathrm{F}}$.

Lemma 5.7. *Up to the action of* $\mathrm{aut}(Y)$*, the types of elliptic fibrations are as in Table 1.*

Proof. Let us introduce the notation A, D_a, D_b, E_a, E_b as in the table above. Using the graph in Figure 2, one can check directly that the given representatives have the corresponding type. What is left to prove is that the two representatives for $\widetilde{D}_8^{\mathrm{F}}$ and $\widetilde{E}_8^{\mathrm{F}}$ are not in the same $\mathrm{aut}(Y)$-orbit.

We start by discussing the case of $\widetilde{D}_8^{\mathrm{F}}$. We need to show that $D_a \neq D_b \cdot \sigma$ for all $\sigma \in \mathrm{aut}(Y)$. As $D_b \cdot \varepsilon = D_b$, the possible $D_b \cdot \sigma$ are given by $D_{b,k} := D_b \cdot \gamma^k$ for $k \in \mathbb{Z}$. Therefore, what we need to show is that D_a is different from $D_{b,k}$ for all $k \in \mathbb{Z}$. For this purpose, it is enough to compute the intersection between D_a and $D_{b,k}$ for every k, and show that this is never equal to 0. It can be checked inductively that

$$D_a \cdot D_{b,k} = k^2 + \frac{1}{2}(-1)^k + \frac{1}{2},$$

which is never equal to zero for $k \in \mathbb{Z}$.

We use similar ideas for $\widetilde{E}_8^{\mathrm{F}}$. In this case, $E_b \cdot \varepsilon \neq E_b$, so we consider

$$E_{b,\varepsilon} := E_b \cdot \varepsilon, \quad E_{b,k} := E_b \cdot \gamma^k, \quad E_{b,\varepsilon,k} := E_{b,\varepsilon} \cdot \gamma^k.$$

Notice that by Lemma 5.3 $E_b \cdot \gamma^k \cdot \varepsilon = E_b \cdot \varepsilon \cdot \gamma^{-k}$. Therefore, to prove that E_a is not in the same orbit as E_b, it is enough to prove that the intersections $E_a \cdot E_{b,k}$ and $E_a \cdot E_{b,\varepsilon,k}$

are nonzero for all $k \in \mathbb{Z}$. We have that

$$E_a \cdot E_{b,k} = 4k^2 + 3,$$
$$E_a \cdot E_{b,\varepsilon,k} = 4k^2 + 2(-1)^k - 4k + 3.$$

These are nonzero for all $k \in \mathbb{Z}$, proving what we needed. ∎

5.3. Computing nd(Y)

We start by computing the maximum number of elliptic fibration of the same type which can appear in the same isotropic sequence (this idea was also used in [16]).

Lemma 5.8. *On the Enriques surface Y, the maximum number of elliptic fibrations of the same type (see Lemma 5.7) that can appear in the same isotropic sequence is given as follows:*

Type	*Maximum*
$\widetilde{A}_7^{\mathrm{HF}}$	1
$\widetilde{A}_1^{\mathrm{F}}+\widetilde{E}_7^{\mathrm{F}}$	1
$\widetilde{D}_8^{\mathrm{F}}$	2
$\widetilde{E}_8^{\mathrm{F}}$	2

Proof. We analyze each case separately.

($\widetilde{A}_7^{\mathrm{HF}}$) The generators ε and γ of aut(Y) map $R_0 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_9$ to itself, so there is exactly one elliptic configuration on Y of type $\widetilde{A}_1^{\mathrm{HF}}$. Therefore, the maximum we are looking for is 1.

($\widetilde{A}_1^{\mathrm{F}} + \widetilde{E}_7^{\mathrm{F}}$) Consider the representative A of type $\widetilde{A}_1$. First, we observe that

$$A \cdot (\varepsilon\gamma^{-1}) = A.$$

This implies that, for all $\sigma \in \mathrm{aut}(Y)$, $A \cdot \sigma = A \cdot \gamma^k$ for some $k \in \mathbb{Z}$. This is because $\sigma = \gamma^n \varepsilon^i$ for some $n \in \mathbb{Z}$ and $i \in \{0, 1\}$, and either $i = 0$, or $i = 1$ and hence

$$A \cdot (\gamma^n \varepsilon) = A \cdot (\varepsilon\gamma^{-n}) = A \cdot (\varepsilon\gamma^{-1})\gamma^{-n+1} = A \cdot \gamma^{-n+1}.$$

Therefore, we define

$$A_k := A \cdot \gamma^k,$$

which give all the other fibrations of the same type $\widetilde{A}_1^{\mathrm{F}}$. Now, assume that two diagrams of type $\widetilde{A}_1^{\mathrm{F}}$ are in the same isotropic sequence. Then, up to the action of aut(Y), we can assume that one of them to be $A_0 = A$ while the other is A_k for some $k \in \mathbb{Z} \setminus \{0\}$. We compute that

$$A_0 \cdot A_k = k^2 - \frac{1}{2}(-1)^k + \frac{1}{2},$$

which is always even, and hence it never equals 1. In conclusion, there can be only one diagram of type $\widetilde{A}_1^{\mathrm{F}}$ in the same non-degenerate isotropic sequence.

($\widetilde{D}_8^{\mathrm{F}}$) Recall the curves $D_a, D_b, D_{b,k}$ from the proof of Lemma 5.7. Also, note that $D_a \cdot \varepsilon = D_a$ and introduce $D_{a,k} := D_a \cdot \gamma^k$. These give all the fibers of type $\widetilde{D}_8$ on Y. Now, assume two diagrams of type $\widetilde{D}_8$ are in the same isotropic sequence. Up to $\mathrm{aut}(Y)$ we have the following possibilities:

$$(D_a, D_{a,k}),\ (D_b, D_{b,k}),\ (D_a, D_{b,k}).$$

We can compute the pairwise intersections as

$$D_a \cdot D_{a,k} = k^2 - \frac{1}{2}(-1)^k + \frac{1}{2},$$
$$D_b \cdot D_{b,k} = k^2 - \frac{1}{2}(-1)^k + \frac{1}{2},$$
$$D_a \cdot D_{b,k} = k^2 + \frac{1}{2}(-1)^k + \frac{1}{2}.$$

The first two are different from 1 for every k, while the third is equal to 1 if and only if $k = 0, \pm 1$. Finally, we observe that $(D_a, D_{b,k})$ cannot be further extended, as adding $D_{a,h}$ or $D_{b,h}$ for some $h \in \mathbb{Z}$ will result in two elements of the sequence having intersection not equal to 1.

($\widetilde{E}_8^{\mathrm{F}}$) Recall the curves $E_a, E_b, E_{b,\varepsilon}, E_{b,k}, E_{b,\varepsilon,k}$ from the proof of Lemma 5.7. Also, introduce $E_{a,\varepsilon} := E_a \cdot \varepsilon$, $E_{a,k} := E_a \cdot \gamma^k$, and $E_{a,\varepsilon,k} := E_{a,\varepsilon} \cdot \gamma^k$. These give all the fibers of type $\widetilde{E}_8$ on Y. Now, assume that two diagrams of type $\widetilde{E}_8^{\mathrm{F}}$ are in the same isotropic sequence. Up to $\mathrm{aut}(Y)$ we have the following possibilities:

$$(E_a, E_{a,k}),\ (E_a, E_{a,\varepsilon,k}),\ (E_b, E_{b,k}),\ (E_b, E_{b,\varepsilon,k}),\ (E_a, E_{b,k}),\ (E_a, E_{b,\varepsilon,k}).$$

We can compute the pairwise intersections as

(1) $E_a \cdot E_{a,k} = 4k^2 - 2(-1)^k + 2$;
(2) $E_a \cdot E_{a,\varepsilon,k} = 4k^2 - 4k + 4$;
(3) $E_b \cdot E_{b,k} = 4k^2 - 2(-1)^k + 2$;
(4) $E_b \cdot E_{b,\varepsilon,k} = 4k^2 - 4k + 4$;
(5) $E_a \cdot E_{b,k} = 4k^2 + 3$;
(6) $E_a \cdot E_{b,\varepsilon,k} = 4k^2 + 2(-1)^k - 4k + 3$.

We observe that in cases (1), (2), (3), (4), (5) the intersection product is not equal to 1 for any k, while in case (6) $E_a \cdot E_{b,\varepsilon,k} = 1$ if and only if $k = 1$. From the above calculations also follows that the isotropic sequence $(E_a, E_{b,\varepsilon,k})$ cannot be further extended because by adding a curve $E_{a,h}$, $E_{a,\varepsilon,h}$, $E_{b,h}$, or $E_{b,\varepsilon,h}$ for some $h \in \mathbb{Z}$ we will have a pairwise intersection not equal to 1. ■

Proposition 5.9. *Let Y be the realizable* $(\tau,\overline{\tau})$*-generic Enriques surface* 145 *in* [4, Table 1]. *Then,* $\mathrm{cnd}(S) = 4$.

Proof. An explicit example of non-degenerate isotropic sequence of length 4, also described in [13, Remark 4.3], is given by the numerical equivalence classes of

$$\begin{aligned} R_0 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7 + R_9 \quad & (\widetilde{A}_7^{\mathrm{HF}}) \\ \frac{1}{2}(R_0 + R_8 + R_1 + R_3 + 2R_9 + 2R_7 + 2R_6 + 2R_5 + 2R_4) \quad & (\widetilde{D}_8^{\mathrm{F}}) \\ \frac{1}{2}(R_1 + R_5 + R_7 + R_8 + 2R_4 + 2R_3 + 2R_2 + 2R_0 + 2R_9) \quad & (\widetilde{D}_8^{\mathrm{F}}) \\ \frac{1}{2}(2R_2 + 3R_0 + 4R_9 + 3R_7 + 2R_6 + R_5 + 2R_8 + R_3) \quad & (\widetilde{E}_7^{\mathrm{F}}). \end{aligned}$$

This shows $\mathrm{cnd}(Y) \geq 4$. To prove that equality holds, we assume by contradiction that there exists an isotropic sequence of five elements in $\mathsf{HF}(Y, \mathcal{R}(Y))$ (see Definition 2.4). By Lemma 5.8, the types of the corresponding five elliptic fibrations can only be one of the following combinations:

- $(\widetilde{E}_7^{\mathrm{F}} + \widetilde{A}_1^{\mathrm{F}}), 2 \times (\widetilde{D}_8^{\mathrm{F}}), 2 \times (\widetilde{E}_8^{\mathrm{F}})$;
- $(\widetilde{A}_7^{\mathrm{HF}}), 2 \times (\widetilde{D}_8^{\mathrm{F}}), 2 \times (\widetilde{E}_8^{\mathrm{F}})$;
- $(\widetilde{A}_7^{\mathrm{HF}}), (\widetilde{E}_7^{\mathrm{F}} + \widetilde{A}_1^{\mathrm{F}}), (\widetilde{D}_8^{\mathrm{F}}), 2 \times (\widetilde{E}_8^{\mathrm{F}})$;
- $(\widetilde{A}_7^{\mathrm{HF}}), (\widetilde{E}_7^{F} + \widetilde{A}_1^{\mathrm{F}}), 2 \times (\widetilde{D}_8^{\mathrm{F}}), (\widetilde{E}_8^{\mathrm{F}})$.

Recall from the beginning of the proof of Lemma 5.8 that the half-fiber of type $\widetilde{A}_7$ is invariant under $\mathrm{aut}(Y)$. Moreover, for all $k \in \mathbb{Z}$, we have that

$$\widetilde{A}_7 \cdot E_{a,k} = \widetilde{A}_7 \cdot E_{a,\varepsilon,k} = \widetilde{A}_7 \cdot E_{b,k} = \widetilde{A}_7 \cdot E_{b,\varepsilon,k} = 2.$$

This implies that no isotropic sequence on Y contains an elliptic fibration of type $\widetilde{A}_7^{\mathrm{HF}}$ and another of type $\widetilde{E}_8^{\mathrm{F}}$. So, the only combination of types allowed in the isotropic sequence of length 5 is

$$(\widetilde{A}_1^{\mathrm{F}} + \widetilde{E}_7^{\mathrm{F}}),\ 2 \times (\widetilde{D}_8^{\mathrm{F}}),\ 2 \times (\widetilde{E}_8^{\mathrm{F}}). \tag{5.2}$$

Now consider one of the two half-fibers of type $\widetilde{E}_8$. Up to $\mathrm{aut}(Y)$, this is either E_a or E_b. Assume we have E_a (an analogous argument holds for E_b). We now show that there exists exactly one half-fiber of type $\widetilde{D}_8$ which has intersection 1 with E_a, giving a contradiction as we have two of such half-fibers from (5.2). We consider then the following intersection numbers:

- $E_a \cdot D_{a,k} = 2k^2 - \frac{1}{2}(-1)^k - k + \frac{3}{2} = 1$ if and only if $k = 0$;
- $E_a \cdot D_{b,k} = 2k^2 + \frac{1}{2}(-1)^k - k + \frac{3}{2} \neq 1$ for all $k \in \mathbb{Z}$.

So, $D_{a,0} = D_a$ is the only half-fiber of type $\widetilde{D}_8$ such that $E_a \cdot D_a = 1$. In conclusion, we cannot have on Y an isotropic sequence of elements in $\mathsf{HF}(Y, \mathcal{R}(Y))$ of length 5, hence $\mathrm{cnd}(S) = 4$. ∎

Theorem 5.10. *Let Y be the $(\tau, \overline{\tau})$-generic Enriques surface* 145 *in* [4, Table 1]. *Then, we have that* $\mathrm{nd}(Y) = 4$.

Proof. By the classification of elliptic fibrations modulo automorphisms [4, Theorem 1.21] and the complete table of these, which can be found in [23], we obtain that every elliptic fibration on Y admits a fiber or a half-fiber supported on the union of smooth rational curves. So, we can apply Proposition 2.7 to argue that $\mathrm{nd}(Y) = \mathrm{cnd}(Y)$. Finally, $\mathrm{cnd}(Y)$ was shown to equal 4 in Proposition 5.9. ∎

6. List of isotropic sequences

We computed explicit non-degenerate isotropic sequences realizing the lower bounds in Theorem 4.2. A digital version of these data can be accessed by following the article's DOI. Recall that for each case, the curves R_i and the automorphisms H_i are taken from the data `Rats-Ratstemp` and `Autrec-HHH` of [21] respectively, and they are numbered sequentially starting from zero. We only list the realizable $(\tau, \overline{\tau})$-generic Enriques surfaces, so we skip $Y_{26}, Y_{48}, Y_{49}, \ldots$ (we refer to [4, Table 1] for the complete list of non-existent cases). Recall that the non-degeneracy invariants for Y_1, Y_{172}, Y_{184} are known: see Remark 4.1 and the discussion in Section 4.5 about $\mathrm{cnd}(Y_1)$. These data are also available in [18].

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