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Boundary states of the Robin magnetic Laplacian

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Abstract. This article tackles the spectral analysis of the Robin Laplacian on a smooth bounded two-dimensional domain in the presence of a constant magnetic field. In the semiclassical limit, a uniform description of the spectrum located between the Landau levels is obtained. The corresponding eigenfunctions, called edge states, are exponentially localized near the boundary. By means of a microlocal dimensional reduction, our unifying approach allows on the one hand to derive a very precise Weyl law and a proof of quantum magnetic oscillations for excited states, and on the other hand to refine simultaneously old results about the low-lying eigenvalues in the Robin case and recent ones about edge states in the Dirichlet case.

1. Motivations and results

1.1. About the magnetic Robin Laplacian

We want to describe the spectrum of the semiclassical magnetic Laplacian

$$\mathcal{L}_h = (-ih\nabla - \mathbf{A})^2$$

on a smooth, bounded, and simply connected open Euclidean domain $\Omega \subset \mathbf{R}^2$, with boundary conditions of Robin type. The vector potential $\mathbf{A} : \overline{\Omega} \to \mathbf{R}^2$ is supposed to be smooth and generating a constant magnetic field of intensity 1:

$$\partial_1 A_2 - \partial_2 A_1 = 1.$$

The magnetic Robin boundary conditions are enforced by defining the operator $\mathcal{L}_h = \mathcal{L}_{h,A,\gamma}$ to be the selfadjoint operator associated with the quadratic form defined for all $\psi \in H^1(\Omega)$ by:

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) = \int_\Omega \left|(-ih\nabla - \mathbf{A})\psi\right|^2 \mathrm{d}x + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 \mathrm{d}s, \tag{1.1}$$

where $\gamma \in \mathbf{R} \cup \{+\infty\}$, and $\mathrm{d}s$ is the length measure of the boundary induced by the Euclidean metric. By convention, $\gamma = +\infty$ corresponds to the Dirichlet boundary condition $\psi \in H^1_0(\Omega)$. In the whole paper, our estimates will be uniform when $\gamma \in [-\gamma_0, +\infty]$ for an arbitrary fixed $\gamma_0 > 0$. When $\gamma \in \mathbf{R}$, the domain of $\mathcal{L}_h$ is given by

$$\begin{aligned}\mathrm{Dom}(\mathcal{L}_h) = \big\{\psi \in H^1(\Omega) : (-ih\nabla - \mathbf{A})^2\psi \in L^2(\Omega), \\ -ih\mathbf{n} \cdot (-ih\nabla - \mathbf{A})\psi = \gamma h^{\frac{3}{2}}\psi \text{ on } \partial\Omega\big\},\end{aligned}$$

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where $\mathbf{n}$ is the outward pointing normal to the boundary. Note that a change of gauge can be used to ensure that $\mathbf{A} \cdot \mathbf{n} = 0$. In this case, the magnetic Robin condition becomes a usual Robin condition:

$$-\mathbf{n} \cdot \nabla \psi = \gamma h^{-\frac{1}{2}} \psi. \tag{1.2}$$

We would like to establish accurate spectral asymptotics for $\mathcal{L}_h$ in regimes where the magnetic field plays a major role, competing with the Robin condition (this is the origin, as we will see, of the factor $h^{\frac{3}{2}}$ in the Robin condition). Until now, very accurate results are available only for the Neumann magnetic Laplacian (when $\gamma = 0$). In this case, the lowest eigenvalues have been analyzed in detail in [11] and uniform estimates have been recently established in [3] where a purely magnetic tunnelling effect formula has been proved. Let us also note that, in the special case of disks and annuli, the magnetic Laplacian with Robin condition has been the objet of physics papers (see, for instance, [35, Figure 8] where the real part of the eigenvalues appear as a function of B). They also consider the case of a complex Robin parameter, which could motivate interesting extensions of the present work to a non-selfadjoint setting.

When $\gamma \neq 0$, the only known semiclassical results go back to the works by Kachmar, see [25], where only the smallest eigenvalue has been estimated. In all these situations (except when $\gamma = +\infty$, where the first eigenvalue is asymptotic to h times the magnetic intensity—here, 1), one can show that the first eigenvalue becomes smaller than h as soon as h is small enough. This energy bound is usually associated with a localization behavior near the boundary of the eigenfunctions, which can be quantified by semiclassical Agmon estimates.

By a simple scaling, the semiclassical limit $h \to 0$ translates into a quantum regime where the intensity of the magnetic field tends to infinity. In the physics literature of thin conductors or electron gases (approximated by 2D domains) subject to a strong external magnetic field, it is well known that the presence of a boundary (or, more generally, of an abruptly changing magnetic field along a curve) generates a current along the boundary due to the presence of “bouncing modes” classically localized at a distance $\sqrt{E}/B$ to the boundary (E is the kinetic energy and B is the magnetic intensity: in this work $B = 1$), see for instance [17]. These so-called “edge states” or “boundary states” exist as soon as the Fermi level of the conductor lies strictly in between two consecutive Landau levels, and produce ballistic dynamics along the boundary. If the boundary $\partial\Omega$ is compact, this dynamics is quantized and produces new discrete energy levels. These are precisely the eigenvalues that we wish to describe in this work.

Heuristically, the localization near $\partial\Omega$ is often explained by the classical bouncing modes alluded to above, but it is also easy to understand from a quantum perspective. Indeed, if we forget the boundary condition, $\mathcal{L}_h$ acts as the magnetic Laplacian with constant magnetic field, on the Euclidean plane, $\mathcal{L}_{h,\mathbf{A}}^{\mathbf{R}^2}$. The spectrum of this so-called “bulk” operator is well-known and made of the famous Landau levels $\{(2n-1)h,\ n \geqslant 1\}$, which are infinitely degenerate eigenvalues. This suggests that, if one considers potential eigenvalues of $\mathcal{L}_h$ in a window of the form $I_h = [ha, hb]$ with $2n-1 < a < b < 2n+1$ for some integer $n \geqslant 0$ (for $n = 0$, we take $a = -\infty$), they cannot correspond to any bulk state, and hence the corresponding eigenfunctions should be localized near the boundary. This phenomenon has interesting physical applications; a famous one is the quantum Hall effect, when the domain is not simply connected, which expresses the collective effects of several boundaries on the total net current. Another application is the confinement of particles in small domains, or "quantum dots" (sometimes called "anti-dots" because one takes $B = 0$ inside the domain, and $B = 1$ outside), see [31, 37].

On the mathematics side, the existence of edge currents in a half-plane with Dirichlet boundary condition was shown in [7]. In a compact setting, the eigenfunction localization at the boundary has been observed (again in the Dirichlet case $\gamma = +\infty$, which is usually chosen in physics) in [16], which was one of our motivations for this work. The methods of [16] lead to a description of the spectrum in a thin spectral window, see [16, Corollary 2.7]. However the *exponential* decay away from the boundary was not established. In fact, as we will see, this decay does not follow from the usual Agmon estimates, but from a strategy *à la* Combes–Thomas (see the original article [5] or the review [24]). Such a strategy has been used recently in the context of the Bochner–Schrödinger operator, see [29, Section 3] and [30].

In this article we treat the general case $\gamma \in \mathbf{R} \cup \{+\infty\}$. This corresponds, physically, to a domain Ω coated with a very thin layer of a different material (see for instance [2]). Since Ω is bounded, the spectrum in I_h is always discrete and a first rough estimate shows that the number of eigenvalues lying in I_h, denoted by $N(\mathcal{L}_h, I_h)$, satisfies

$$N(\mathcal{L}_h, I_h) \leqslant C h^{-2}, \tag{1.3}$$

for some $C > 0$ and all $h > 0$ small enough (see Appendix A where we recall the origin of this estimate). Our goal is to obtain a very precise description, in the semiclassical regime, of the spectral elements corresponding to the interval I_h, much more accurate than (1.3). This includes the localization behavior near $\partial\Omega$ of the corresponding eigenfunctions. For instance, when $\gamma \in \mathbf{R}$, a consequence of our main result Theorem 1.7 is the appearance of a quite interesting phenomenon: for a given (low) energy, one can have boundary quasimodes corresponding to classical currents flowing in opposite directions, leading to magnetic oscillations of eigenvalues, see Theorem 1.12.

This work is also an opportunity to revisit the Neumann case analyzed in [11, 19] (see also [3]) by establishing more uniform asymptotic expansions, with slightly more general boundary conditions.

1.2. De Gennes operator with Robin condition

Our results will be expressed in terms of the eigenvalues of the de Gennes operator with Robin boundary condition. This operator, which appears naturally in the study of boundary induced magnetic effects [12, 39], is a differential operator of order two depending on the real parameters γ and σ and acting as

$$H[\gamma, \sigma] = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t - \sigma)^2,$$

on the domain

$$\mathrm{Dom}\left(H[\gamma,\sigma]\right) = \left\{u \in B^1(\mathbf{R}_+) : \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (t-\sigma)^2\right)u \in L^2(\mathbf{R}_+),\ u'(0) = \gamma\, u(0)\right\},$$

where

$$B^1(\mathbf{R}_+) = \left\{u \in H^1(\mathbf{R}_+) : \left[t \mapsto tu(t)\right] \in L^2(\mathbf{R}_+)\right\}.$$

It is well known that $H[\gamma,\sigma]$ is a self-adjoint elliptic operator with compact resolvent. Its spectrum can be written as a non-decreasing sequence of eigenvalues $(\mu_n(\gamma,\sigma))_{n\geqslant 1}$ (which are all simple due to the Cauchy–Lipschitz theorem). We denote by $u_n^{[\gamma,\sigma]}$ the normalized sequence of the corresponding eigenfunctions (with $u_n^{[\gamma,\sigma]}(0) > 0$). We let

$$\Theta^{[n-1]}(\gamma) := \inf_{\sigma\in\mathbf{R}} \mu_n(\gamma,\sigma).$$

The index $n-1$ is compatible with the notation used in the case of the de Gennes operator (case when $\gamma = 0$), see [12, Section 3.2]. The family $(H[\gamma,\sigma])_{(\gamma,\sigma)\in\mathbf{R}^2}$ is analytic of type (B) (in the sense of Kato, see [27, Chapter VII, Section 4]), i.e., the form domain does not depend on the parameters and the sesquilinear form is analytic as a function of γ or σ. By convention, we denote by $H[+\infty,\sigma]$ (i.e., we let $\gamma = +\infty$) the corresponding operator with Dirichlet boundary condition $u(0) = 0$.

The following proposition gathers the main properties of the functions $\mu_n(\gamma,\cdot)$ (which are usually called *dispersion curves*) that will be used in this article. Most of them have been established in [25] (see also [26], and [7] in the Dirichlet case).

Proposition 1.1. *Let us fix $n \geqslant 1$. When $\gamma \in \mathbf{R}$, the function $\mu_n(\gamma,\cdot)$ is analytic and*

$$\lim_{\sigma\to-\infty} \mu_n(\gamma,\sigma) = +\infty, \qquad \lim_{\sigma\to+\infty} \mu_n(\gamma,\sigma) = 2n-1. \tag{1.4}$$

Moreover, $\mu_n(\gamma,\cdot)$ has a unique minimum attained at $\sigma = \xi_{n-1}(\gamma)$, but not attained at infinity. This minimum is non-degenerate. The function $\mu_n(\gamma,\cdot)$ is decreasing on $(-\infty,\xi_{n-1}(\gamma))$ and increasing on $(\xi_{n-1}(\gamma),+\infty)$. In addition, we have, for all $n \geqslant 2$,

$$2n-3 < \Theta^{[n-1]}(\gamma) < 2n-1. \tag{1.5}$$

When $\gamma = +\infty$, that is when the Robin condition is replaced by the Dirichlet condition, $\mu_n(+\infty,\cdot)$ is still smooth, but now decreasing from $+\infty$ to $2n-1$.

The non-degeneracy of the minimum of $\mu_n(\gamma,\cdot)$ for $\gamma \in \mathbf{R}$ is obtained by adapting the Dauge–Helffer formula, see [25] for the case $n = 1$, which gives:

$$\partial_\sigma^2 \mu_n(\gamma,\sigma)_{|\sigma=\xi_{n-1}(\gamma)} = 2\xi_{n-1}(\gamma)\left|u_n^{[\gamma,\sigma]}(0)\right|^2. \tag{1.6}$$

The lower bound in (1.5) will be established in Appendix B. This proposition has the following elementary but important consequences for our analysis, which are illustrated in Figure 1.

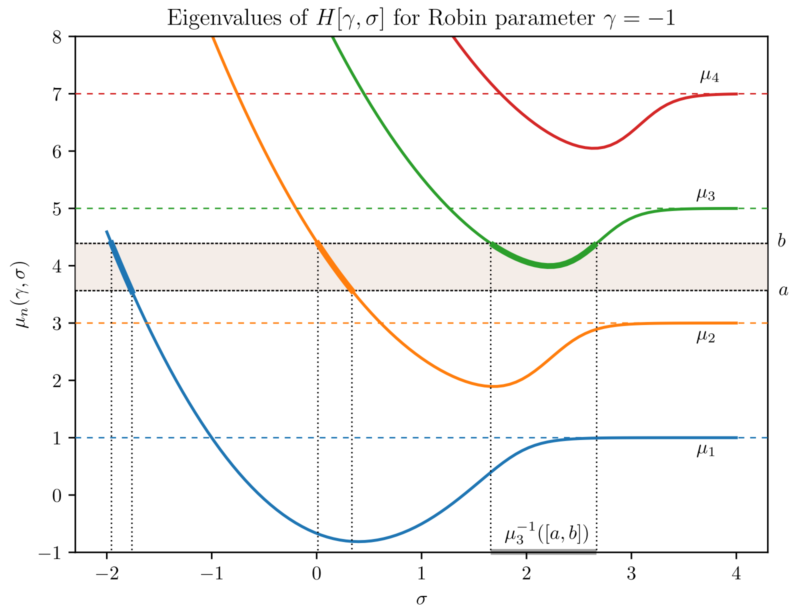

Figure 1. This figure shows the dispersion curves $\mu_k(\gamma,\cdot)$, for $\gamma = -1$. We visualize the preimage of a given interval $[a,b] \subset (2n-3, 2n-1)$ with $n = 3$. The curves are obtained by a standard finite difference numerical scheme.

Corollary 1.2. *Let $\gamma \in \mathbf{R} \cup \{+\infty\}$ be fixed. Let Θ be the set of all critical values of the functions μ_n: we have*

$$\Theta = \left\{\Theta^{[n-1]}(\gamma),\ n \geqslant 1\right\}.$$

Let Λ be the set of limit points of the functions μ_n at infinity:

$$\Lambda := \{2n-1,\ n \geqslant 1\}.$$

Let $[a,b] \subset \mathbf{R}$ be an interval disjoint from Λ. Let either $n = 1$ if $a < 1$ or $n \geqslant 2$ be such that $[a,b] \subset (2n-3, 2n-1)$. (In the case $n = 1$ we allow $a = -\infty$.) It follows from (1.4) *that for any integer $k \geqslant 1$, $\mu_k^{-1}([a,b])$ is compact.*

Let $p(k)$ be the number of connected components of $\mu_k^{-1}([a,b])$: we have

$$\begin{cases} p(k) = 1 & \text{if } 1 \leqslant k < n \\ p(n) = 0 & \text{if } \gamma = +\infty \\ p(n) = 1 & \text{if } \gamma \in \mathbf{R} \text{ and } \Theta^{[n-1]} \in [a,b] \\ p(n) = 2 & \text{if } \gamma \in \mathbf{R} \text{ and } \Theta^{[n-1]} < a \\ p(n) = 0 & \text{if } \gamma \in \mathbf{R} \text{ and } b < \Theta^{[n-1]} \\ p(k) = 0 & \text{if } k > n. \end{cases}$$

Therefore, when $\gamma \in \mathbf{R}$*,*

$$N(\gamma, a, b) := \#\{k \geqslant 1 : \mu_k(\gamma, \cdot)^{-1}([a, b]) \neq \emptyset\} = \begin{cases} n & \text{if } b \geqslant \Theta^{[n-1]}(\gamma) \\ n-1 & \text{otherwise}, \end{cases} \tag{1.7}$$

and if $\gamma = +\infty$ *(Dirichlet case) then* $\mu_1(+\infty, \cdot)$ *does not take any value in* $(-\infty, 1)$*, and* $N(\gamma, a, b) = n - 1$.

From now on, we denote by $N(\gamma, a, b) = N$ this cardinal.

Assumption 1.3. In the following, a and b are allowed to depend on h, as soon as they stay in an h-independent compact interval inside $(2n-3, 2n-1)$.

With this picture in mind, for each $k \geqslant 1$, we may construct a smooth function $\mathring{\mu}_k$, bounded with all its derivatives, which coincide with μ_k in a neighborhood of $\mu_k^{-1}([a, b])$. Indeed, let $\Xi_0 : \mathbf{R} \to \mathbf{R}$ be a smooth, bounded with all its derivatives, and increasing function such that for all $k \in \{1, \ldots, N\}$, $\mu_k(\gamma, \Xi_0(\sigma)) = \mu_k(\gamma, \sigma)$ in a neighborhood of $\mu_k^{-1}([a, b])$ and $\mu_k \circ \Xi_0$ takes its values in $(-\infty, a) \cup (b, +\infty)$ away from it. We consider

$$\mathring{\mu}_k := \mu_k(\gamma, \Xi_0(\cdot)), \tag{1.8}$$

where we omit the reference to the parameter γ to lighten the notation. In the following, we will more generally denote by $\mathring{\varphi}$, the function φ after Ξ_0.

1.3. Results

Let us now describe the main results of our article, which will be expressed in terms of pseudo-differential operators in one dimension.

1.3.1. A pseudo-differential framework. The bounded functions $\mathring{\mu}_k$ will be convenient to state our main theorem, which involves $h^{\frac{1}{2}}$-pseudo-differential operators with symbols in the usual class $S_{\mathbf{R}^2}(1)$ given by

$$S_{\mathbf{R}^2}(1) = \{a \in \mathcal{C}^\infty(\mathbf{R}^2_{s,\sigma}) : \forall \alpha \in \mathbf{N}^2,\ \exists C_\alpha > 0 : |\partial^\alpha a| \leqslant C_\alpha\}.$$

As we said before, the eigenfunctions of $\mathcal{L}_h$ will be localized near the boundary of Ω, which is a closed smooth curve with length $2L$. Our main result describes their distribution with the help of an $h^{\frac{1}{2}}$-pseudo-differential operator on the boundary (see for instance [13, Section 4.1] where similar considerations have been done in the context of discontinuous magnetic fields). Let us denote $\hbar = h^{\frac{1}{2}}$. We recall that the Weyl quantization of a symbol p is given by the formula:

$$(\mathrm{Op}_\hbar^{\mathrm{W}} p)\psi(x) = \frac{1}{2\pi\hbar} \int_{\mathbf{R}^2} e^{i(x-y)\eta/\hbar} p\Big(\frac{x+y}{2}, \eta\Big)\psi(y)\mathrm{d}y\mathrm{d}\eta, \quad \forall \psi \in \mathcal{S}(\mathbf{R}), \tag{1.9}$$

and that this formula defines a bounded operator form $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ if $p \in S_{\mathbf{R}^2}(1)$, by the Calderón–Vaillancourt theorem. To shorten the notation, we will sometimes write p^{W} instead of $\mathrm{Op}_\hbar^{\mathrm{W}} p$.

Let $\mathbf{T}_{2L} = \mathbf{R}/2L\mathbf{Z}$, and $L^2(\mathbf{T}_{2L})$ be the subset of $L^2_{\mathrm{loc}}(\mathbf{R})$ of $2L$-periodic functions, equipped with the usual L^2 norm on $[0, 2L]$. When $p \in S_{\mathbf{T}_{2L}\times\mathbf{R}}(1)$, i.e., $p \in S_{\mathbf{R}^2}(1)$ and is $2L$-periodic in its first variable s, then for any $\theta \in \mathbf{R}$, the operator given in (1.9) induces a bounded operator from $e^{i\theta\cdot}L^2(\mathbf{T}_{2L})$ to $e^{i\theta\cdot}L^2(\mathbf{T}_{2L})$—we denote by $e^{i\theta\cdot}$ the function $x \mapsto e^{i\theta x}$.

Remark 1.4. The space $e^{i\theta\cdot}L^2(\mathbf{T}_{2L})$ only depends on the class of θ modulo $\frac{\pi}{L}$; it is equal to the subspace of functions in $L^2_{\mathrm{loc}}(\mathbf{R})$ equipped with the Floquet boundary condition $\psi(x + 2L) = e^{i2L\theta}\psi(x)$. The operator $\mathrm{Op}^{\mathrm{W}}_{\hbar} p$ acting on $e^{i\theta\cdot}L^2(\mathbf{T}_{2L})$ is unitarily equivalent to $\mathrm{Op}^{\mathrm{W}}_{\hbar} p(x, \eta + \hbar\theta)$ acting on $L^2(\mathbf{T}_{2L})$.

1.3.2. Main theorem. Since our main result describes the spectrum of $\mathcal{L}_h$ "modulo $\mathcal{O}(h^2)$", we need to make this notion precise.

Definition 1.5. In this article, we will say that the spectra of two self-adjoint operators T_1 and T_2 depending on h coincide in I_h modulo $\mathcal{O}(h^\alpha)$, $\alpha \in \mathbf{R} \sqcup \{+\infty\}$, when there exists $C, h_0 > 0$ such that, for all $h \in (0, h_0)$,

- T_1 and T_2 have discrete spectrum in $I_h + [-Ch^\alpha, Ch^\alpha]$,
- for all interval $J_h \subset I_h$ we can find an interval K_h such that $J_h \subset K_h$ with $\mathrm{d}_H(K_h, J_h) \leqslant Ch^\alpha$ and

$$\operatorname{rank} \mathbb{1}_{J_h}(T_1) \leqslant \operatorname{rank} \mathbb{1}_{K_h}(T_2), \quad \operatorname{rank} \mathbb{1}_{J_h}(T_2) \leqslant \operatorname{rank} \mathbb{1}_{K_h}(T_1),$$

  where d_H denotes the Hausdorff distance:

$$\mathrm{d}_H(A, B) = \sup_{(a,b)\in A\times B} \max\big(d(a, B), d(b, A)\big).$$

This definition translates to discrete subsets of $\mathbf{R}$ as follows: for each discrete subset $S \subset \mathbf{R}$, we associate the sum of Dirac masses $\delta_S := \sum_{s\in S} \delta_s$, and consider the corresponding self-adjoint operator whose spectral measure is δ_S. Then we say that two discrete subsets A_1 and A_2 coincide modulo $\mathcal{O}(h^\alpha)$ when the spectra of the corresponding operators coincide modulo $\mathcal{O}(h^\alpha)$ in the above sense. In order to deal with multiplicities, we will, by convention, associate with the *disjoint union* $S \sqcup S'$ the operator corresponding to the spectral measure $\delta_S + \delta_{S'}$.

Remark 1.6. Let us make some comments about Definition 1.5.

- The relation "the spectra of T_1 and T_2 coincide in I_h modulo $\mathcal{O}(h^\alpha)$" is an equivalence relation. It is obviously symmetric and reflexive (taking $K_h = J_h$). The transitivity follows from the triangle inequality for d_H.
- If the spectra of T_1 and T_2 coincide in I_h modulo $\mathcal{O}(h^\alpha)$, then, for all $\widetilde{I}_h \subset I_h$, the spectra of T_1 and T_2 coincide in $\widetilde{I}_h$ modulo $\mathcal{O}(h^\alpha)$.
- If the spectra of T_1 and T_2 coincide in I_h modulo $\mathcal{O}(h^\alpha)$, we have

$$\mathrm{d}_H\big(\mathrm{sp}(T_1) \cap I_h, \mathrm{sp}(T_2) \cap I_h\big) = \mathcal{O}(h^\alpha).$$

- If the endpoints of the interval I_h stay away from an h^β-neighborhood of the spectrum, with $\beta < \alpha$, then for h small enough T_1 and T_2 have exactly the same number of eigenvalues inside I_h, counted with multiplicities.
- The notion described in Definition 1.5 already appears under various forms in the literature (see, for instance, the view point in [22, Section 1] and [18, Section 4]).

We can now state our main result, where we use, among others, the eigenvalues $\mu_k(\gamma, \sigma)$ and eigenfunctions $u_k^{[\gamma,\sigma]}$ of the de Gennes operator (Section 1.2), the integer N defined in (1.7), and the notation introduced in (1.8).

Theorem 1.7. *Under Assumption 1.3, the spectrum of $\mathcal{L}_h$ in $[ha, hb]$ coincides with that of $h\mathfrak{M}_h$ modulo $\mathcal{O}(h^2)$, where*

$$\mathfrak{M}_h := \begin{bmatrix} m_1^{\mathrm{W}} & 0 & \cdots & 0 \\ 0 & m_2^{\mathrm{W}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & m_N^{\mathrm{W}} \end{bmatrix}$$

is a bounded operator acting diagonally on $e^{i\theta(h)\cdot} L^2(\mathbf{T}_{2L})^N$. Here

$$\theta(h) = \frac{|\Omega|}{|\partial\Omega| h},$$

and each m_k^{W} is an $h^{\frac{1}{2}}$-pseudo-differential operator with symbol in $S_{\mathbf{T}_{2L}\times\mathbf{R}}(1)$. Let us denote by (s, σ) the (canonical) variables in $\mathbf{T}_{2L} \times \mathbf{R}$. Then we have:

- *the principal symbol of m_k^{W} is $\mathring{\mu}_k(\sigma)$;*
- *its subprincipal symbol is $-\kappa(s)\mathring{C}_k(\sigma)$ with*

$$C_k(\sigma) = \left\langle \left((\tau - \sigma)\tau^2 - \partial_\tau - 2\tau(\sigma - \tau)^2\right) u_k^{[\gamma,\sigma]}(\tau),\ u_k^{[\gamma,\sigma]}(\tau) \right\rangle_{L^2(\mathbf{R}_+)}, \tag{1.10}$$

and $\kappa(s)$ is the curvature of the boundary at the point of curvilinear abscissa s.

Remark 1.8. One can check that, for all $k \geqslant 1$, $C_k(\xi_{k-1}(\gamma))$ has the same sign as $\gamma_0^{[k-1]} - \gamma$, see Proposition B.5 where the threshold $\gamma_0^{[k-1]}$ is discussed. Proposition B.5 also corrects a mistake in [25, Lemma II.3 and (2.24)], where it is stated that $C_1(\xi_0(\gamma))$ is always positive.

It is important to notice that Theorem 1.7 is actually a diagonalization result since it reduces the spectral analysis of $\mathcal{L}_h$ to that of a family of pseudo-differential operators in one dimension: the spectrum of $\mathfrak{M}_h$ is the superposition (counting multiplicities) of the spectra of the m_k^{W}, $k = 1, \ldots, N$. As it turns out, the spectrum of each of these pseudo-differential operators can be completely described using (refinements of) old and new results in the literature. Indeed, notice that the principal symbols $\mathring{\mu}_k$ have a special feature:

they depend only on the frequency variable σ, and, as functions of σ, they have at most a unique critical point, which is a nondegenerate minimum (Proposition 1.1). Hence, from a microlocal viewpoint, only two situations must be considered. Let $E \in [a,b]$, either E is a regular value of $\mathring{\mu}_k$ (or $\mu_k(\gamma,\cdot)$, equivalently), and then the well-known Bohr–Sommerfeld rules apply, or E is a critical value of $\mathring{\mu}_k$, in which case the Hamiltonian $(s,\sigma) \mapsto \mathring{\mu}_k(\sigma)$ admits a transversally non-degenerate minimum on a circle, and the recent study [8] of folded quantum action variables applies.

1.3.3. Eigenvalues in a regular spectral window. Our first application concerns the case where the interval $[a,b]$ consists of regular values of all μ_k. We will use the following well-known spectral result, an extension to all orders of the Bohr–Sommerfeld rules (see, for instance, [9, 20, 21, 38]), which we prove in Section 5.2.1.

Proposition 1.9. *Consider an $\hbar$-pseudo-differential operator $\mathsf{P}_\hbar \in \mathrm{Op}_\hbar^{\mathrm{W}}(S_{\mathbf{R}^2}(1))$ with symbol $2L$-periodic with respect to s and with principal symbol $(s,\sigma) \mapsto \mu(\sigma)$ and subprincipal symbol $(s,\sigma) \mapsto -\kappa(s)C(\sigma)$. We consider its realization on $e^{is\theta(\hbar^2)}L^2(\mathbf{T}_{2L})$. Let E be a regular value of μ for which $\mu^{-1}(E)$ is a finite set of points $\sigma_1^E, \dots, \sigma_p^E$.*

Then, there exists $\varepsilon > 0$ such that $[E-\varepsilon, E+\varepsilon]$ is a set of regular values of μ, and $\mu^{-1}([E-\varepsilon, E+\varepsilon])$ is the disjoint union $\Sigma_1 \sqcup \dots \sqcup \Sigma_p$ where each $\Sigma_q \subset \mathbf{R}$ is a compact interval containing σ_q^E in its interior. Let $\varepsilon > 0$ be any value satisfying the above conditions. For each $q = 1, \dots, p$, let $\widetilde{\Sigma}_q$ be an open interval containing Σ_q such that the $\widetilde{\Sigma}_q$'s are pairwise disjoint. Then the following holds.

For each $q = 1, \dots, p$, there exists a smooth map $\widetilde{\Sigma}_q \ni \sigma \mapsto f_q(\sigma,\hbar) \in \mathbf{R}$ with an asymptotic expansion, in the smooth topology,

$$f_q(\sigma,\hbar) \sim f_{q,0}(\sigma) + \hbar f_{q,1}(\sigma) + \hbar^2 f_{q,2}(\sigma) + \cdots$$

depending only on the symbol of $\mathsf{P}_\hbar$ in the cylinder $\mathbf{T}_{2L} \times \Sigma_q$, such that the spectrum of $\mathsf{P}_\hbar$ inside $[E-\varepsilon, E+\varepsilon]$ coincides, modulo $\mathcal{O}(\hbar^\infty)$, with the disjoint union

$$\left(\bigsqcup_{q=1}^{p} \left\{ f_q(\sigma,\hbar),\ \sigma \in \hbar\left(\tfrac{\pi}{L}\mathbf{Z} + \theta(\hbar^2)\right) \cap \widetilde{\Sigma}_q \right\} \right) \cap [E-\varepsilon, E+\varepsilon],$$

see Definition 1.5. Moreover, we have

$$f_{q,0}(\sigma) = \mu(\sigma)_{|\Sigma_q}, \tag{1.11}$$

$$f_{q,1}(\sigma) = \frac{-C(\sigma)_{|\Sigma_q}}{2L} \int_0^{2L} \kappa(s)\mathrm{d}s. \tag{1.12}$$

Combining Proposition 1.9 and Theorem 1.7, we get the following result, where we use the notation of Corollary 1.2 and Theorem 1.7.

Corollary 1.10 (Spectrum of $\mathcal{L}_h$ at regular values). *Let $[a,b]$ be an interval disjoint from Θ and Λ. For each $k = 1, \dots, N$, for each $q = 1, \dots, p(k)$, let $\Sigma_{k,q} \subset \mathbf{R}$ be an interval* *such that $\mu_k(\gamma,\cdot)$ is a diffeomorphism from $\Sigma_{k,q}$ to a neighborhood of $[a,b]$, in such a way that all $\Sigma_{k,q}$ are pairwise disjoint and $\bigcup_{q=1}^{p(k)}\Sigma_{k,q}$ contains $\mu_k(\gamma,\cdot)^{-1}([a,b])$. Then there exists a smooth map $\Sigma_{k,q}\ni\sigma\mapsto f_{k,q}(\sigma,\hbar)\in\mathbf{R}$ with an asymptotic expansion (in the smooth topology)*

$$f_{k,q}(\sigma,\hbar)\sim f_{k,q,0}(\sigma)+\hbar f_{k,q,1}(\sigma)+\hbar^2 f_{k,q,2}(\sigma)+\cdots$$

such that the spectrum of $\mathcal{L}_h$ in $[ha,hb]$ coincides, modulo $\mathcal{O}(h^2)$, with the disjoint union

$$\Big(\bigsqcup_{k=1}^{N}\bigsqcup_{q=1}^{p(k)}\{hf_{k,q}(\sigma,h^{\frac12}),\ \sigma\in h^{\frac12}\big(\tfrac{\pi}{L}\mathbf{Z}+\theta(h)\big)\cap\Sigma_{k,q}\}\Big)\cap[ha,hb].$$

Moreover, we have,

when $\sigma\in\Sigma_{k,q}$,

$$f_{k,q,0}(\sigma)=\mu_k(\gamma,\sigma), \tag{1.13}$$

$$f_{k,q,1}(\sigma)=-\langle\kappa\rangle C_k(\sigma), \tag{1.14}$$

where $\langle\kappa\rangle$ is the average curvature:

$$\langle\kappa\rangle=\frac{1}{2L}\int_0^{2L}\kappa(s)\mathrm{d}s=\frac{\pi}{L}.$$

Since the leading terms (1.13) and (1.14) do not depend on q (apart from the domain of definition $\Sigma_{k,q}$) we obtain that the spectrum of $\mathcal{L}_h$ in $[ha,hb]$ coincides, modulo $\mathcal{O}(h^2)$, with the disjoint union

$$\bigsqcup_{k=1}^{N}\{h\mu_k(\gamma,\sigma)-h^{\frac32}\langle\kappa\rangle\mathring{C}_k(\sigma),\ \sigma\in h^{\frac12}\big(\tfrac{\pi}{L}\mathbf{Z}+\theta(h)\big)\}\cap[ha,hb]. \tag{1.15}$$

As a first application of this corollary, we obtain a very accurate formula for the number of eigenvalues of $\mathcal{L}_h$ in $[ha,hb]$, this number being much smaller than what the crude estimate (1.3) says:

Theorem 1.11 (Precise Weyl formula). *Let $I_h=[ha,hb]$ where $[a,b]$ is an interval disjoint from Θ and Λ. Then the number of eigenvalues of $\mathcal{L}_h$ in I_h is*

$$N(\mathcal{L}_h,I_h)=\Big\lfloor\frac{L}{\pi h^{1/2}}\sum_{k,q}\delta_{k,q}^{[0]}+\frac{L\langle\kappa\rangle}{\pi}\sum_{k,q}\delta_{k,q}^{[1]}+\mathcal{O}(h^{1/2})\Big\rfloor,$$

where we use the notation $\sum_{k,q}:=\sum_{k=1}^{N}\sum_{q=1}^{p(k)}$, and

$$\delta_{k,q}^{[0]}:=|\alpha_{k,q}-\beta_{k,q}|,\quad \delta_{k,q}^{[1]}:=\frac{C_k(\beta_{k,q})}{|\mu_k'(\beta_{k,q})|}-\frac{C_k(\alpha_{k,q})}{|\mu_k'(\alpha_{k,q})|},$$

with $\alpha_{k,q}:=\mu_{k,q}^{-1}(a)$, $\beta_{k,q}:=\mu_{k,q}^{-1}(b)$.

In this statement we have denoted $\mu_{k,q} := \mu_k(\gamma, \cdot)_{|\Sigma_{k,q}}$. Notice that, since the remainder term $\mathcal{O}(h^{1/2})$ tends to 0, we obtain that, when h is small enough, $N(\mathcal{L}_h, I_h)$ is equal to the integer part of $\frac{L}{\pi h^{1/2}} \sum_{k,q} \delta^{[0]}_{k,q} + \frac{L\langle\kappa\rangle}{\pi} \sum_{k,q} \delta^{[1]}_{k,q}$, or this plus or minus 1. Note that the one term asymptotics in Theorem 1.11 is already known when $\gamma = +\infty$ and that it can be seen as a consequence of [6, Corollary 1.3]. It is also related to [15, Theorem 1.1] dealing with positive variable magnetic fields and $\gamma = 0$. When B is constant, Theorem 1.11 is refinement of [15, Theorem 1.1] since it exhibits the second order term. The strategy of our proof could also likely be used to recover [15, Theorem 1.2], see Section 1.3.4.

In a second application, we focus on the regular eigenvalues of $\mathcal{L}_h$ below the first Landau level, and investigate how the eigenvalues move when h varies (by the scaling mentioned in the introduction, this corresponds to the variation of the quantum energies when the external magnetic field is modified). This variation of eigenvalues is mainly due to the strong flux term $\theta(h) = \frac{|\Omega|}{|\partial\Omega| h}$, see (1.15). When $\gamma \in \mathbf{R}$, the eigenvalues below the first Landau level are described by only two intervals $\Sigma_{1,1}$ and $\Sigma_{1,2}$, for which the sense of variation of the approximate eigenvalues with respect to h are opposite. Hence, we obtain a strongly oscillating behavior for these eigenvalues, which is a generalization to excited states of the Little–Parks effect, see [14].

Theorem 1.12 (Magnetic quantum oscillations). *Let $\gamma \in \mathbf{R}$. Let $I_h = [ha, hb]$ with $a > \Theta^0(\gamma)$ and $b < 1$. There exists $h_0 > 0$, $C > 0$ and $M > 0$ such that the following holds. Let $h < h_0$, and let $j \in \mathbf{N}$ be such that the j-th eigenvalue $\lambda_j(\gamma, h)$ of $\mathcal{L}_h$ belongs to I_h. Then there exists $C_i = C_i(j, h)$, $i = 1, 2, 3$, with $0 < C_1 < C_2 < C_3 \leqslant M$ such that, letting $h_i := h + C_i h^2$, we have*

- $\lambda_j(h_2) \geqslant \lambda_j(h_1) + C h^{3/2}$,
- $\lambda_j(h_2) \geqslant \lambda_j(h_3) + C h^{3/2}$.

Moreover, the gap between consecutive eigenvalues is—(roughly) periodically with period $\mathcal{O}(h^2)$—smaller than $\mathcal{O}(h^2)$, precisely: there exists h' such that $|h - h'| = \mathcal{O}(h^2)$ and $\lambda_j(h') - \lambda_{j+1}(h') = \mathcal{O}(h^2)$, and there exists h'' such that $|h - h''| = \mathcal{O}(h^2)$ and $\lambda_{j+1}(h'') - \lambda_j(h'') \geqslant C h^{3/2}$.

See also Figure 2. The proof of this theorem is given in Section 5.2.4. We believe that this is the first mathematical treatment of quantum magnetic oscillations for excited states in the first Landau band. In principle, similar oscillations for eigenvalues between higher Landau levels could be obtained in the same vein. However, the growing number of connected components $\Sigma_{k,q}$ involved would make the analysis (and statement) quite complicated.

Remark 1.13. These applications illustrate the fact that Corollary 1.10 gives a very accurate description of the spectrum of $\mathcal{L}_h$ by providing us with explicit approximations of the eigenvalues in $[ha, hb]$ modulo $\mathcal{O}(h^2)$. When $\gamma = +\infty$ (i.e., in the Dirichlet case), it also improves the description given in [16, Corollary 2.7] concerned with a thin spectral window containing a regular value. Moreover, although our results are formulated in terms of approximation of the eigenvalues, the strategy, based on microlocal projections, leading to Theorem 1.7 can also be used to describe the eigenspaces of $\mathcal{L}_h$ in terms of those of $\mathfrak{M}_h$.

1.3.4. Critical values. Our main theorem also applies to the case when the spectral window contains a critical value, i.e., an element of Θ, see Corollary 1.2 (such a critical value is the unique non-degenerate global minimum of a unique dispersion curve, see Proposition 1.1). To illustrate this, let us focus on the low-lying eigenvalues. The following corollary improves [25, Theorem I.5, $\alpha = \frac{1}{2}$] by establishing the spectral asymptotics of the lowest eigenvalues and by exhibiting spectral gaps of order $h^{\frac{7}{4}}$ instead of $h^{\frac{3}{2}}$ in the case of regular values for each given dispersion curve. It also extends to any Robin parameter the result obtained by Fournais and Helffer in [11] when $\gamma = 0$.

Once Theorem 1.7 is applied and reduces the analysis to a single $\hbar$-pseudo-differential operator, this corollary becomes essentially an application of [8, Proposition 6.8], see details in Section 5.2.2.

Corollary 1.14. *Consider $\gamma \neq \gamma_0^{[0]}$ with $\gamma_0^{[0]}$ defined in Remark 1.8, and let*

$$\epsilon = \operatorname{sign}(\gamma_0^{[0]} - \gamma) = \operatorname{sign}\big(C_1\big(\xi_0(\gamma)\big)\big).$$

Assume that $\epsilon\kappa$ admits a unique maximum at $s_{\max}$, which is non-degenerate. Then, for all $j \geqslant 1$, uniformly when $jh^{\frac{1}{4}} = o(1)$,

$$\lambda_j(\gamma, h) = \Theta^{[0]}(\gamma)h - \kappa(s_{\max})C_1\big(\xi_0(\gamma)\big)h^{\frac{3}{2}} + \frac{h^{\frac{7}{4}}(2j-1)}{2}\sqrt{k_2 C_1\big(\xi_0(\gamma)\big)\mu_1''\big(\gamma, \xi_0(\gamma)\big)} + o(h^{\frac{7}{4}}),$$

with $k_2 = -\kappa''(s_{\max})$, and where we recall that $\xi_0(\gamma)$ is given in Proposition 1.1.

Remark 1.15. Let us end the description of our results with a few comments about consequences and extensions following from our approach.

(i) Corollary 1.14 describes the low-lying eigenvalues with some uniformity in j (which was not the case in [11]), in an interval of the form $(-\infty, \Theta^{[0]}(\gamma)h + Ch^{3/2}]$. On the other hand, Corollary 1.10 gives the spectrum in any interval of the form $[ha, hb]$ with $a > \Theta^0(\gamma)$ and $b < 1$. Hence we have a spectral interval between these two regimes which we do not describe here. But actually, by using refined spectral results for 1D pseudo-differential operators, and in particular the strategy of [8] in the case where κ is a Morse function, it should also be possible to close this gap. However, this would require an analysis of the hyperbolic singularities arising from the minima of $\epsilon\kappa$, where we expect both a concentration of the eigenfunctions and a higher density of eigenvalues.

(ii) When $\gamma > \gamma_0^{[0]}$, the proof of Corollary 1.14 shows that the eigenfunctions (associated with the low-lying eigenvalues) are concentrated near the points of minimal curvature. This contrasts with the Neumann case when the points of maximal curvature play the role of attractive wells. This phenomenon was not observed before, see Remark 1.8.

(iii) The case $\gamma = \gamma_0^{[0]}$ is critical since $C_1(\xi_0(\gamma)) = 0$. However, our analysis can still be used by computing additional subprincipal terms in our effective operator method. A similar phenomenon has recently been observed in the study of the magnetic Dirac operator [1, Section 8] and also in the analysis of the magnetic Schrödinger operator with discontinuous magnetic fields [13]. In this case, we have, for all $j \geqslant 1$,

$$\lambda_j(\gamma,h) = \Theta^{[0]}(\gamma)h + h^2\lambda_j(\mathcal{A}_h) + o(h^2),$$

where $\mathcal{A}_h = \frac{\partial_\sigma^2\mu(\gamma,\xi_0(\gamma))}{2}(D_s + \theta(h) - h^{-\frac{1}{2}}\xi_0(\gamma))^2 + C_\gamma\kappa^2(s)$, for some $C_\gamma \in \mathbf{R}$. In this transition regime, the effective operator is not semiclassical.

(iv) When the curvature κ is constant, in the case $\gamma \in \mathbf{R}$, we are in a degenerate situation rather similar to the case when $\gamma = \gamma_0^{[0]}$. Concerning the operators m_k^W of Theorem 1.7, this case corresponds to [8, Proposition 6.4]. We can prove an expansion in the form

$$\lambda_j(\gamma,h) = \Theta^{[0]}(\gamma)h - \kappa C_1\big(\xi_0(\gamma)\big)h^{\frac{3}{2}} + h^2\lambda_j(\mathcal{A}_h) + o(h^2).$$

Here, the eigenvalues of $\mathcal{A}_h$ will generate magnetic oscillations, see [8, Theorem 2.2; $k = 0$]. When $\gamma = 0$ and $j = 1$, a similar estimate is described in [11, Theorem 5.3.1].

1.4. Organization of the article

In Section 2, we prove that the eigenfunctions associated with eigenvalues of $\mathcal{L}_h$ in $[ha, hb]$ are exponentially localized near the boundary of Ω, see Proposition 2.1. Note that the strategy used to derive this localization deviates from the usual variational method (see, for instance, [18] or [36, Proposition 4.7]), which fails since we want to consider eigenvalues between two consecutive Landau levels. To overcome this issue, our strategy, which eventually generalizes the variational method, is based on establishing the bijectivity of the magnetic Laplacian between exponentially weighted L^2 spaces. In Section 3, by means of tubular coordinates (s,t) near the boundary and a rescaling $t = h^{\frac{1}{2}}\tau$, we introduce a model operator $\mathcal{N}_\hbar$ depending on the effective semiclassical parameter $\hbar = h^{\frac{1}{2}}$, acting on $2L$-periodic functions and involving a flux term $\mathfrak{f}_0$, see (3.3) and (3.2). We also show that the eigenfunctions of $\mathcal{L}_h$ are roughly microlocalized in a compact set of the phase space attached to the boundary, see Proposition 3.1. This allows to prove that the spectrum of $\mathcal{L}_h$ (between the Landau levels) is located near that of $\mathcal{N}_\hbar$, see Proposition 3.2. However, one will see that Proposition 3.2 is not directly useful to establish our main theorems. It is rather a pretext to motivate the introduction of $\mathcal{N}_\hbar$ and to describe the spectral estimates required to prove that spectra coincide modulo $\mathcal{O}(h^\infty)$. Actually, one will compare directly the spectrum of $\mathcal{L}_h$ to that of an effective operator on the boundary of Ω. For that purpose, in Section 4, we construct a Grushin problem in order to invert the pseudo-differential operator $\mathfrak{N}_\hbar$ (which acts as $\mathcal{N}_\hbar$ with $\mathfrak{f}_0$ replaced by 0). This method is inspired by the works of Martinez and Sjöstrand, see, for instance, the presentation in [32,41]. It has been adapted to magnetic operators by Keraval in [28] and it has recently shown its efficiency to describe the low-lying eigenvalues of various magnetic operators (see, for instance, [3,13] and also [33] in a non-selfadjoint context). The novelty in the present paper is to use it to tackle the description of larger eigenvalues for magnetic Schrödinger operators with boundaries, when several dispersion curves are involved (see Figure 1), and not only the first one as in [3] or [13]. In order to use this method, we write a semiclassical expansion of $\mathcal{N}_\hbar$, see Proposition 4.2. The principal operator symbol is the de Gennes operator (with Robin condition), which can be inverted in the spectral window $[a,b]$ up to considering an augmented matrix involving the eigenfunctions of the de Gennes operator, see Lemma 4.3. This allows to build an approximate inverse of an augmented version of $\mathfrak{N}_\hbar$ denoted by $\mathrm{Op}_\hbar^{\mathrm{W}}\mathcal{P}_\hbar$, see (4.3) (and the left and right quasi inverses (4.5) and (4.6)). Thanks to these quasi inverses, the bijectivity of $\mathcal{L}_h - z$ is reduced to that of a pseudo-differential operator on $\mathbf{T}_{2L}$ whose matrix symbol is $M_\hbar$, modulo some remainders, see Proposition 4.4 where the eigenfunctions of $\mathcal{L}_h$ are directly used as quasimodes for $M_\hbar^{\mathrm{W}}$. In Section 5, we perform the spectral analysis of $M_\hbar^{\mathrm{W}}$ by using that the principal matrix symbol M_0 is diagonal with uniform gaps between the diagonal entries. We deduce Proposition 1.9 and Corollary 1.14. In Appendix A, we recall the origin of the estimate (1.3). Appendix B is devoted to the de Gennes operator with Robin conditions: a couple of known results are recalled and useful new ones are established.

2. Exponential localization near the boundary and consequences

Let us consider a smooth function $\Phi_0 : \overline{\Omega} \to \mathbf{R}_+$ that coincides with $\mathrm{dist}(x,\partial\Omega)$ near $\partial\Omega$, and which vanishes only on $\partial\Omega$. Such a function can be constructed as follows. Let $\epsilon > 0$ be such that the ϵ-neighborhood of $\partial\Omega$, which we call Ω_1, admits a trivialization by the geodesic exponential: in other words $\Omega_1 \simeq \mathbf{T} \times [-\epsilon,\epsilon]$ with coordinates (s,t), and for any $x(s,t) \in \Omega_1$, we have $\mathrm{dist}(x,\partial\Omega) = |t|$, and $t > 0$ if $x \in \Omega$. We denote by $t : \Omega_1 \to \mathbf{R}$ the corresponding (smooth) map $x \mapsto t$. Let $\Omega_0 \subset \Omega$ be the complementary set of the $\epsilon/2$-neighborhood of $\partial\Omega$. Thus, $\Omega_0 \cup \Omega_1$ is an open neighborhood of $\overline{\Omega}$. Let (χ_0,χ_1), be an associated partition of unity. The function $\Phi_0 := \chi_0 + t\chi_1$ meets our requirements.

Next, we extend Φ_0 to a smooth function on $\mathbf{R}^2$ that also belongs to $W^{2,\infty}(\mathbf{R}^2)$.

The following proposition states that the eigenfunctions of $\mathcal{L}_h$ associated with eigenvalues in I_h are localized near the boundary of Ω. The estimates look like Agmon's estimates, but they are not obtained via variational means as it is the case in many magnetic settings. Here, they follow from resolvent estimates using the distance to the Landau levels.

Proposition 2.1. *There exist $\alpha > 0, C > 0, h_0 > 0$ such that for all $h \in (0,h_0)$ and all eigenfunctions ψ associated with an eigenvalue in I_h, we have*

$$\int_\Omega e^{2\alpha\Phi_0(x)/h^{1/2}}|\psi(x)|^2 \mathrm{d}x \leq C\|\psi\|^2, \tag{2.1}$$

$$\int_\Omega e^{2\alpha\Phi_0(x)/h^{1/2}}\big|(-ih\nabla-\mathbf{A})\psi\big|^2\mathrm{d}x \leqslant Ch\|\psi\|^2. \tag{2.2}$$

2.1. Preliminaries

In the following, $\mathcal{L}_h^{\mathbf{R}^2}$ denotes the operator $(-ih\nabla-\mathbf{A})^2$ acting on the Hilbert space $L^2(\mathbf{R}^2)$. By using the gauge invariance, we assume in the whole section that $\mathbf{A}=\frac{1}{2}(-x_2,x_1)$. Due to our choice of eigenvalue λ, we deduce that $\mathcal{L}_h^{\mathbf{R}^2}-\lambda$ is bijective and that there exists $C>0$ such that, for all $h>0$,

$$\big\|(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)^{-1}\big\| \leqslant Ch^{-1}.$$

More precisely, we can take $C=\min(|2n-3-a|,|2n-1-b|)^{-1}$. We let $\Phi=\alpha\Phi_0$, with $\alpha>0$ to be determined, and consider the conjugated operator

$$\begin{aligned}\mathcal{L}_h^{\Phi} &:= e^{\Phi/h^{1/2}}\mathcal{L}_h^{\mathbf{R}^2}e^{-\Phi/h^{1/2}}\\ &= (-ih\nabla-\mathbf{A}+ih^{\frac12}\nabla\Phi)^2\\ &= \mathcal{L}_h^{\mathbf{R}^2}+2ih^{\frac12}\nabla\Phi\cdot(-ih\nabla-\mathbf{A})-h|\nabla\Phi|^2-ih^{\frac32}\Delta\Phi.\end{aligned} \tag{2.3}$$

The following lemma tells us that the invertibility is preserved for $\mathcal{L}_h^{\Phi}-\lambda$ if α is small enough.

Lemma 2.2. *There exists $C>0$ such that for all $h>0$ and all $\alpha>0$,*

$$h^{\frac12}\big\|\nabla\Phi\cdot(-ih\nabla-\mathbf{A})(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)^{-1}\big\| \leqslant C\alpha. \tag{2.4}$$

In particular, $\mathcal{L}_h^{\Phi}-\lambda$ is bijective as soon as $\alpha\leqslant\alpha_0$ and α_0 is chosen small enough. With such a choice of α_0, there exists $C>0$ such that, for all $h>0$, and all $\alpha\leqslant\alpha_0$,

$$\big\|(\mathcal{L}_h^{\Phi}-\lambda)^{-1}\big\| \leqslant \frac{C}{h}. \tag{2.5}$$

Proof. Consider $v\in L^2(\mathbf{R}^2)$ and let $u=(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)^{-1}v$. We have

$$(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)u=v,$$

so that, by taking the scalar product with u and using that $\lambda\leqslant Ch$,

$$\big\|(-ih\nabla-\mathbf{A})u\big\|^2 \leqslant Ch\|u\|^2+\|u\|\|v\|.$$

Therefore, since $\nabla\Phi_0\in L^\infty$, there is a new constant $C'>0$ such that

$$h^{\frac12}\big\|\nabla\Phi\cdot(-ih\nabla-\mathbf{A})u\big\| \leqslant C'\alpha h\|u\|+C'h^{\frac12}\alpha\|u\|^{\frac12}\|v\|^{\frac12}.$$

Since $\|u\|=\|(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)^{-1}v\|\leqslant Ch^{-1}\|v\|$, we see that

$$h^{\frac12}\big\|\nabla\Phi\cdot(-ih\nabla-\mathbf{A})(\mathcal{L}_h^{\mathbf{R}^2}-\lambda)^{-1}v\big\| \leqslant \widetilde{C}\alpha\|v\|,$$

which gives (2.4).

Let us now deal with the bijectivity. We have

$$\mathcal{L}_h^{\Phi} - \lambda = \mathcal{L}_h^{\mathbf{R}^2} - \lambda + B$$

with

$$B := 2ih^{\frac{1}{2}}\nabla\Phi \cdot (-ih\nabla - \mathbf{A}) - h|\nabla\Phi|^2 - ih^{\frac{3}{2}}\Delta\Phi.$$

Since $\nabla\Phi_0$ and $\Delta\Phi_0$ are bounded, we deduce from (2.4) that

$$\left\| B(\mathcal{L}_h^{\mathbf{R}^2} - \lambda)^{-1} \right\| \leqslant C\alpha + C_1\alpha + h^{1/2}C_2\alpha^2 \leqslant \widetilde{C}\alpha,$$

when α is small enough. On the other hand,

$$\mathcal{L}_h^{\Phi} - \lambda = \big(\mathrm{Id} + B(\mathcal{L}_h^{\mathbf{R}^2} - \lambda)^{-1}\big)(\mathcal{L}_h^{\mathbf{R}^2} - \lambda);$$

For α small enough, we deduce that $\mathrm{Id} + B(\mathcal{L}_h^{\mathbf{R}^2} - \lambda)^{-1}$ is invertible, and thus so is $\mathcal{L}_h^{\Phi} - \lambda$. ∎

In order to prove Proposition 2.1, we need to localize on an $h^{1/2}$-neighborhood of $\partial\Omega$. For this purpose, we introduce two functions $\chi_h \in \mathcal{C}_0^\infty(\Omega)$ and $\widetilde{\chi}_h \in \mathcal{C}^\infty(\overline{\Omega})$ as follows.

$$\chi_h \colon \begin{cases} \Omega \to [0,1], \\ x \mapsto g\big(\Phi_0(x)/h^{\frac{1}{2}}\big) \end{cases} \quad \text{and} \quad \widetilde{\chi}_h \colon \begin{cases} \overline{\Omega} \to [0,1], \\ x \mapsto 1 - g\big(\Phi_0(x)/2h^{\frac{1}{2}}\big), \end{cases}$$

where g is a smooth non-decreasing function on $\mathbf{R}$, valued in $[0,1]$, equal to 0 on $(-\infty,1)$ and to 1 on $(2,+\infty)$. In particular,

$$\mathrm{supp}(\chi_h) \cap \overline{\Omega} \subset \big\{x \in \Omega,\ h^{-\frac{1}{2}}\Phi_0(x) \geqslant 1\big\}, \tag{2.6}$$

and

$$\mathrm{supp}(\nabla\chi_h) \cap \overline{\Omega} \subset \big\{x \in \Omega : h^{-\frac{1}{2}}\Phi_0(x) \in [1,2]\big\} \subset \big\{x \in \Omega : \widetilde{\chi}_h(x) = 1\big\}. \tag{2.7}$$

Note that the following properties hold:

- $\chi_h = 1$ away from an $h^{1/2}$-neighborhood of $\partial\Omega$,
- $\nabla\chi_h$ is supported in an $h^{1/2}$-neighborhood of $\partial\Omega$,
- $\mathbb{1}_{\mathrm{supp}\,\nabla\chi_h} \leqslant \widetilde{\chi}_h$,
- $\widetilde{\chi}_h = 0$ away from an $h^{1/2}$-neighborhood of $\partial\Omega$.

2.2. Proof of Proposition 2.1

Let us consider $\lambda \in [ha, hb] \cap \mathrm{sp}(\mathcal{L}_h)$ and an associated eigenfunction $\psi \in \mathrm{Dom}(\mathcal{L}_h)$. We have

$$\big((-ih\nabla - \mathbf{A})^2 - \lambda\big)\psi = 0.$$

Let $\varphi = e^{\Phi/h^{1/2}}\psi$. Using (2.3) in Ω, the equation becomes

$$(\mathcal{L}_h^{\Phi} - \lambda)\varphi = 0. \tag{2.8}$$

Then, we have

$$\begin{aligned}(\mathcal{L}_h^{\Phi} - \lambda)(\chi_h\varphi) &= [\mathcal{L}_h^{\Phi}, \chi_h]\varphi \\ &= e^{\Phi/h^{1/2}}[\mathcal{L}_h, \chi_h]e^{-\Phi/h^{1/2}}\varphi \\ &= e^{\Phi/h^{1/2}}\big(-h^2\Delta\chi_h - 2ih\nabla\chi_h\cdot(-ih\nabla - \mathbf{A})\big)e^{-\Phi/h^{1/2}}\varphi \\ &= \big(-h^2\Delta\chi_h - 2ih\nabla\chi_h\cdot(-ih\nabla - \mathbf{A} + ih^{\frac12}\nabla\Phi)\big)\varphi. \end{aligned}\tag{2.9}$$

We have $\|h^2(\Delta\chi_h)\varphi\| \leqslant Ch\|\widetilde{\chi}_h\varphi\|$. (Here and in the rest of the paper, C denotes a constant that is independent on h but that can vary from line to line.) Let us explain how to deal with the last term. We have

$$h\big\|\nabla\chi_h\cdot(-ih\nabla - \mathbf{A} + ih^{\frac12}\nabla\Phi)\varphi\big\| \leqslant h\big\|\nabla\chi_h\cdot(-ih\nabla - \mathbf{A})\varphi\big\| + Ch\|\widetilde{\chi}_h\varphi\|.$$

Let us temporarily admit that, for α small enough,

$$h\big\|\nabla\chi_h\cdot(-ih\nabla - \mathbf{A})\varphi\big\| \leqslant Ch\|\widetilde{\chi}_h\varphi\|. \tag{2.10}$$

We then immediately deduce from (2.9) that

$$\big\|(\mathcal{L}_h^{\Phi} - \lambda)(\chi_h\varphi)\big\| \leqslant \widetilde{C}h\|\widetilde{\chi}_h\varphi\|. \tag{2.11}$$

Since $\chi_h\varphi \in \mathrm{Dom}(\mathcal{L}_h^{\mathbf{R}^2})$ we obtain from (2.5) that

$$\|\chi_h\varphi\| \leqslant C\,\|\widetilde{\chi}_h\varphi\|,$$

which implies that

$$\|\varphi\| \leqslant \widetilde{C}\big(\|\widetilde{\chi}_h\varphi\| + \big\|(1-\chi_h)\varphi\big\|\big),$$

showing that φ is localized near $\partial\Omega$. More precisely, recalling that $\varphi = e^{\Phi/h^{\frac12}}\psi$, using that $\Phi_0(x) = \mathrm{dist}(x, \partial\Omega)$ near the boundary, and the fact that the supports of $\widetilde{\chi}_h$ and $1 - \chi_h$ lie in neighborhood of the boundary of size $h^{\frac12}$, we deduce (2.1).

Let us now deal with (2.2). We have the Agmon identity

$$\mathrm{Re}\,\langle\mathcal{L}_h\psi, e^{2\Phi/h^{1/2}}\psi\rangle = \mathfrak{Q}_{h,\mathbf{A}}(e^{\Phi/h^{1/2}}\psi) - h\|e^{\Phi/h^{1/2}}\psi\nabla\Phi\|^2,$$

which follows from (2.3) where we see that $\mathrm{Re}\,\mathcal{L}_h^{\Phi} = \mathcal{L}_h - h|\nabla\Phi|^2$ and we notice that

$$\mathrm{Re}\big\langle\mathcal{L}_h\psi, e^{2\Phi/h^{1/2}}\psi\big\rangle = \big\langle(\mathrm{Re}\,\mathcal{L}_h^{\Phi})e^{\Phi/h^{1/2}}\psi, e^{\Phi/h^{1/2}}\psi\big\rangle.$$

Recall also that, when $u \in \mathrm{Dom}(\mathcal{L}_h)$, then $\langle\mathcal{L}_h u, u\rangle = \mathfrak{Q}_{h,\mathbf{A}}(u)$, see (1.1).

Then, by using that ψ is an eigenfunction, we get

$$\int_\Omega \left|(-ih\nabla - \mathbf{A})(e^{\Phi/h^{1/2}}\psi)\right|^2 \mathrm{d}x + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |e^{\Phi/h^{1/2}}\psi|^2 \mathrm{d}s - h\|e^{\Phi/h^{1/2}}\psi\nabla\Phi\|^2$$
$$= \lambda\|e^{\Phi/h^{1/2}}\psi\|^2.$$

With (2.1), we find

$$\int_\Omega \left|(-ih\nabla - \mathbf{A})(e^{\Phi/h^{1/2}}\psi)\right|^2 \mathrm{d}x + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |e^{\Phi/h^{1/2}}\psi|^2 \mathrm{d}s \leqslant Ch\|\psi\|^2.$$

From a classical trace theorem (see for instance [10, Section 5.5]), there exists $C > 0$ such that for all $\varepsilon > 0$, we have

$$\int_{\partial\Omega} |\varphi|^2 \mathrm{d}s \leqslant C\left(\varepsilon^{-1}\|\varphi\|^2 + \varepsilon\big\|\nabla|\varphi|\big\|^2\right).$$

With the diamagnetic inequality (see for instance [12, Theorem 2.1.1]), we deduce that

$$h^2 \int_{\partial\Omega} |\varphi|^2 \mathrm{d}s \leqslant C\left(h^2\varepsilon^{-1}\|\varphi\|^2 + \varepsilon\big\|(-ih\nabla - \mathbf{A})\varphi\big\|^2\right),$$

and then

$$h^{\frac{3}{2}} \int_{\partial\Omega} |\varphi|^2 \mathrm{d}s \leqslant C\left(h^{\frac{3}{2}}\varepsilon^{-1}\|\varphi\|^2 + \varepsilon h^{-\frac{1}{2}}\big\|(-ih\nabla - \mathbf{A})\varphi\big\|^2\right).$$

Taking $\varepsilon = \frac{h^{\frac{1}{2}}}{2|c|C}$ implies that

$$\int_\Omega \left|(-ih\nabla - \mathbf{A})(e^{\Phi/h^{1/2}}\psi)\right|^2 \mathrm{d}x \leqslant \tilde{C}h\|\psi\|^2.$$

Computing a commutator gives (2.2).

It remains to explain why (2.10) holds. From (2.3) we can write

$$\mathcal{L}_h^\Phi = L_1 + ih^{\frac{1}{2}}L_2, \tag{2.12}$$

with

$$L_1 = (-ih\nabla - \mathbf{A})^2 - h|\nabla\Phi|^2,$$
$$L_2 = 2\nabla\Phi\cdot(-ih\nabla - \mathbf{A}) - ih\Delta\Phi.$$

From (2.12) and (2.8), we get

$$(L_1 - \lambda + ih^{\frac{1}{2}}L_2)\varphi = 0.$$

For $j = 1, 2$, we have

$$\mathrm{Re}\left\langle (L_1 - \lambda)\varphi, (\partial_j\chi_h)^2\varphi\right\rangle - h^{\frac{1}{2}}\,\mathrm{Im}\left\langle L_2\varphi, (\partial_j\chi_h)^2\varphi\right\rangle = 0.$$

Thanks to the classical localization formula (see, for instance, [36, Proposition 4.2]), we have

$$\operatorname{Re}\big\langle (L_1-\lambda)\varphi, (\partial_j\chi_h)^2\varphi\big\rangle = \big\|(-ih\nabla-\mathbf{A})\big[(\partial_j\chi_h)\varphi\big]\big\|^2 - h\int_\Omega |\nabla\Phi|^2\big|(\partial_j\chi_h)\varphi\big|^2 \mathrm{d}x \\ -\lambda\|\partial_j\chi_h\varphi\|^2 - h^2\big\|\nabla(\partial_j\chi_h)\varphi\big\|^2.$$

Moreover,

$$\begin{aligned}\big|\operatorname{Im}\big\langle L_2\varphi,(\partial_j\chi_h)^2\varphi\big\rangle\big| &= \big|\operatorname{Im}\big\langle(\partial_j\chi_h)L_2\varphi,(\partial_j\chi_h)\varphi\big\rangle\big| \\ &\leqslant \big|\operatorname{Im}\big\langle L_2\big((\partial_j\chi_h)\varphi\big),(\partial_j\chi_h)\varphi\big\rangle\big| + \big|\big\langle[L_2,\partial_j\chi_h]\varphi,(\partial_j\chi_h)\varphi\big\rangle\big| \\ &\leqslant Ch\big\|(\partial_j\chi_h)\varphi\big\|^2 + C\alpha\big\|(-ih\nabla-\mathbf{A})(\partial_j\chi_h)\varphi\big\|\|\partial_j\chi_h\varphi\| \\ &\quad + Ch\big\|\nabla(\partial_j\chi_h)\varphi\big\|\|\partial_j\chi_h\varphi\|.\end{aligned}$$

Due to the properties of χ_h, we have

$$\big|\operatorname{Im}\big\langle L_2\varphi,(\partial_j\chi_h)^2\varphi\big\rangle\big| \leqslant C\|\tilde\chi_h\varphi\| + Ch^{-\frac12}\alpha\big\|(-ih\nabla-\mathbf{A})(\partial_j\chi_h)\varphi\big\|^2 \\ + Ch^{-\frac12}\alpha\|\tilde\chi_h\varphi\|^2 + Ch^{-\frac12}\|\tilde\chi_h\varphi\|^2.$$

Therefore,

$$\big\|(-ih\nabla-\mathbf{A})[(\partial_j\chi_h)\varphi]\big\|^2 \leqslant C\|\tilde\chi_h\varphi\|^2 + C\alpha\big\|(-ih\nabla-\mathbf{A})(\partial_j\chi_h)\varphi\big\|^2.$$

Taking α small enough, we get

$$\big\|(-ih\nabla-\mathbf{A})\big[(\partial_j\chi_h)\varphi\big]\big\|^2 \leqslant C\|\tilde\chi_h\varphi\|^2.$$

Computing a commutator, we get (2.10).

3. An operator on a semi-cylinder

3.1. A model operator

The exponential localization near the boundary at a scale of order $h^{\frac12}$ given by Proposition 2.1 invites us to use the classical tubular coordinates (s,t) near the boundary. We recall that these coordinates are defined thanks to the map

$$\Gamma : \mathbf{T}_{2L}\times(0,t_0)\ni(s,t)\mapsto\Gamma(s)-t\mathbf{n}(s), \quad \mathbf{T}_{2L} := \mathbf{R}/2L\mathbf{Z},$$

which is injective if t_0 is small enough. Its Jacobian is $a(s,t)=1-t\kappa(s)$, where κ is the curvature of the boundary at the point $\Gamma(s)$. Here Γ is a counterclockwise parametrization by the curvilinear abscissa. Thus, Γ induces a smooth diffeomorphism between $\mathbf{T}_{2L}\times(0,t_0)$ and $\Omega_{t_0} := \Gamma(\mathbf{T}_{2L}\times(0,t_0))$.

By using [12, Appendix F], we can check that the magnetic Laplacian acts locally near the boundary in these coordinates as

$$\widetilde{\mathcal{L}}_h = a(s,t)^{-1}\Big(-ih\partial_s - t + \mathfrak{f}_0 + \kappa(s)\frac{t^2}{2}\Big)a(s,t)^{-1}\Big(-ih\partial_s - t + \mathfrak{f}_0 + \kappa(s)\frac{t^2}{2}\Big) - h^2 a(s,t)^{-1}\partial_t a(s,t)\partial_t,$$

in the ambient Hilbert space $L^2(a\mathrm{d}s\mathrm{d}t)$. Here $\mathfrak{f}_0 = \frac{|\Omega|}{|\partial\Omega|}$. The boundary condition (1.2) becomes

$$\partial_t\psi(s,0) = \gamma h^{-\frac{1}{2}}\psi(s,0).$$

Of course the operator $\widetilde{\mathcal{L}}_h$ is only defined near $t = 0$. We would like to consider a global operator. This can be done by inserting cutoff functions with respect to t. We let $\check{t} = t\zeta(h^{-\frac{1}{2}+\eta}t)$ with $\eta \in (0,\frac{1}{2})$ and ζ a smooth cutoff function equal to 1 near 0.

Let us consider the differential operator acting as

$$\widetilde{\mathcal{L}}_h = a(s,\check{t})^{-1}\Big(-ih\partial_s - t + \mathfrak{f}_0 + \kappa(s)\frac{\check{t}^2}{2}\Big)a(s,\check{t})^{-1}\Big(-ih\partial_s - t + \mathfrak{f}_0 + \kappa(s)\frac{\check{t}^2}{2}\Big) - h^2 a(s,\check{t})^{-1}\partial_t a(s,\check{t})\partial_t,$$

on the domain

$$\mathrm{Dom}(\widetilde{\mathcal{L}}_h) = \big\{u \in L^2(\mathbf{T}_{2L}\times\mathbf{R}_+) : -\partial_t^2 u \in L^2(\mathbf{T}_{2L}\times\mathbf{R}_+), (-ih\partial_s - t + \mathfrak{f}_0)^2 u \in L^2(\mathbf{T}_{2L}\times\mathbf{R}_+),\ \partial_t u(\cdot,0) = \gamma h^{-\frac{1}{2}}u(\cdot,0)\big\}.$$

The ambient Hilbert space is $L^2(a(s,\check{t})\mathrm{d}s\mathrm{d}t) = L^2(\mathrm{d}s\mathrm{d}t)$, with $2L$-periodic condition with respect to s.

The exponential localization of the original eigenfunctions at the scale $h^{\frac{1}{2}}$ near the boundary suggests to consider the partial rescaling

$$(s,t) = (s,\hbar\tau),$$

where $\hbar = h^{\frac{1}{2}}$. We consider the new operator, acting in the ambient Hilbert space $L^2(\hat{a}_\hbar\mathrm{d}s\mathrm{d}\tau) = L^2(\mathrm{d}s\mathrm{d}\tau)$,

$$\widehat{\mathcal{L}}_h = \hat{a}_\hbar(s,\tau)^{-1}p_{s,\hbar}\hat{a}_\hbar(s,\tau)^{-1}p_{s,\hbar} - \hat{a}_\hbar(s,\tau)^{-1}\partial_\tau\hat{a}_\hbar(s,\tau)\partial_\tau, \tag{3.1}$$

with

$$p_{s,\hbar} = -i\hbar\partial_s - \tau + \hbar^{-1}\mathfrak{f}_0 + \hbar\kappa(s)\frac{\widehat{\tau}^2}{2}, \tag{3.2}$$

and where $\hat{a}_\hbar(s,\tau) = 1 - \hbar\widehat{\tau}\kappa$ with $\widehat{\tau} = \zeta(\hbar^{2\eta}\tau)\tau$.

The boundary condition becomes

$$\partial_\tau\psi(s,0) = \gamma\psi(s,0).$$

The domain is given by

$$\mathrm{Dom}(\widehat{\mathcal{L}}_h) = \big\{u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+) : -\partial_\tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+),\\ (-i\hbar\partial_s - \tau + \hbar^{-1}\mathfrak{f}_0)^2 u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+),\ \partial_\tau u(\cdot, 0) = \gamma u(\cdot, 0)\big\}.$$

In fact, it will even be more convenient to deal with the following operator

$$\mathcal{N}_\hbar = \hat{a}_\hbar(s,\tau)^{-1} p^{\Xi_0}_{s,\hbar} \hat{a}_\hbar(s,\tau)^{-1} p^{\Xi_0}_{s,\hbar} - \hat{a}_\hbar(s,\tau)^{-1}\partial_\tau \hat{a}_\hbar(s,\tau)\partial_\tau, \tag{3.3}$$

where we recall that Ξ_0 was defined in (1.8), and

$$p^{\Xi_0}_{s,\hbar} := \Xi_0(\cdot + \hbar^{-1}\mathfrak{f}_0)^{\mathrm{W}} - \tau + \hbar\kappa(s)\frac{\widehat{\tau}^2}{2},$$
$$\mathrm{Dom}(\mathcal{N}_\hbar) = \big\{u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+) : -\partial_\tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+),\\ \tau^2 u \in L^2(\mathbf{T}_{2L} \times \mathbf{R}_+),\ \partial_\tau u(\cdot, 0) = \gamma u(\cdot, 0)\big\}.$$

3.2. Microlocalization of the eigenfunctions of $\mathcal{L}_h$

In fact, we can prove that the eigenfunctions of $\mathcal{L}_h$ associated with eigenvalues in $[ha, hb]$ are roughly microlocalized with respect to $\sigma + \hbar^{-1}\mathfrak{f}_0$, the (shifted) dual variable of s. In order to quantify this, we consider the compact set

$$K = \bigcup_{j\geqslant 1} \{\sigma \in \mathbf{R} : \mu_j(\sigma) \in [a,b]\} \subset [\sigma_{\min}, \sigma_{\max}] =: \widetilde{K}. \tag{3.4}$$

Note that K is indeed compact due to the properties of the μ_j (tending to $+\infty$ in $-\infty$) and to the choice of $[a,b]$, which does not contain Landau levels (the limits of the μ_j in $+\infty$).

The following result establishes a rather rough microlocalization result (with respect to σ) for the eigenfunctions: it tells us that the eigenfunctions are microlocalized in the compact set $\widetilde{K}$. To quantify this, we consider a smooth function Ξ with values in $[0,1]$ such that $\Xi = 0$ near $\widetilde{K}$ and 1 away from $\widetilde{K}$.

We let $\widehat{\lambda} = h^{-1}\lambda$.

Proposition 3.1. *Let us consider the eigenvalue equation $\mathcal{L}_h\psi = \lambda\psi$ for $\lambda \in [ha, hb]$. Then,*

$$\widehat{\mathcal{L}}_h\varphi = \widehat{\lambda}\varphi + \mathcal{O}(h^\infty)\|\psi\|. \tag{3.5}$$

with $\varphi = \widehat{\chi}_\hbar\widehat{\psi}$, where $\widehat{\psi} = \psi \circ \Gamma(s, \hbar\tau)$ and $\widehat{\chi}_\hbar(\tau) = \chi(\hbar^\eta\tau)$ for a smooth cutoff function χ equal to 0 away from $\tau = 0$.

Moreover,

$$\mathrm{Op}^{\mathrm{W}}_\hbar\big(\Xi(\sigma + \hbar^{-1}\mathfrak{f}_0)\big)\varphi = \mathcal{O}(\hbar^\infty)\|\psi\|. \tag{3.6}$$

Proof. The estimate (3.5) follows from the localization near the boundary (see Proposition 2.1).

Then, let us only prove that (3.6) holds when Ξ is 0 near $(-\infty, \sigma_{\max} + \frac{\epsilon}{2})$ and 1 on $(\sigma_{\max} + \epsilon, +\infty)$, the estimate following from similar arguments on $(-\infty, \sigma_{\min} - \epsilon)$.

In order to lighten the notation, we will use a slight abuse of notation by writing

$$\Xi^{\mathrm{W}} := \mathrm{Op}_{\hbar}^{\mathrm{W}}\big(\Xi(\sigma + \hbar^{-1}\mathfrak{f}_0)\big). \tag{3.7}$$

Then, we write

$$(\widehat{\mathcal{L}}_h - \widehat{\lambda})\Xi^{\mathrm{W}}\varphi = [\widehat{\mathcal{L}}_h, \Xi^{\mathrm{W}}]\varphi + \mathcal{O}(\hbar^\infty)\|\psi\|.$$

Thanks to the explicit expression (3.1), we get

$$\big\|[\widehat{\mathcal{L}}_h, \Xi^{\mathrm{W}}]\varphi\big\| \leqslant C\hbar\|\underline{\Xi}^{\mathrm{W}}\varphi\| + C\hbar\|\underline{\Xi}^{\mathrm{W}}\partial_\tau\varphi\| + \mathcal{O}(\hbar^\infty)\|\psi\|, \tag{3.8}$$

and we can write, by using the support of $\chi(h^{-\frac{1}{2}+\eta}t)$,

$$\widehat{\mathcal{L}}_h = \widehat{\mathcal{L}}_0 + \mathcal{R}_\hbar, \quad \widehat{\mathcal{L}}_0 = -\partial_\tau^2 + p_{s,\hbar,0}^2, \; p_{s,\hbar,0} = -i\hbar\partial_s + \hbar^{-\frac{1}{2}}\mathfrak{f}_0 - \tau, \tag{3.9}$$

where the remainder $\mathcal{R}_\hbar$ can be written as

$$\mathcal{R}_\hbar = \hbar^{1-2\eta}R_{\hbar,2}(s,\tau)p_{s,\hbar,0}^2 + \hbar^{1-4\eta}R_{\hbar,1}(s,\tau)p_{s,\hbar,0} + \hbar^{2-8\eta}R_{\hbar,3} + \hbar R_{\hbar,4}\partial_\tau, \tag{3.10}$$

the $R_{\hbar,j}$ being smooth functions, uniformly bounded in $\hbar$.

Then, we consider an increasing function $\sigma \mapsto \widetilde{\Xi}(\sigma) \in (\sigma_{\max} + \frac{\epsilon}{4}, +\infty)$ that coincides with Id on $(\sigma_{\max} + \frac{\epsilon}{2}, +\infty)$. We let

$$\widehat{\mathcal{L}}_0^{\mathrm{cut}} = \mathrm{Op}_\hbar^{\mathrm{W}}\big(-\partial_\tau^2 + \big(\widetilde{\Xi}(\sigma + \hbar^{-1}\mathfrak{f}_0) - \tau\big)^2\big),$$

acting on $L^2(\mathbf{T}_{2L} \times \mathbf{R}_+)$, where the superscript "cut" refers to the replacement of $-i\hbar\partial_s + \hbar^{-1}\mathfrak{f}_0$ by $\widetilde{\Xi}^{\mathrm{W}}$ (with the same abuse of notation as in (3.7)). We notice that $\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda}$ is bijective (with an inverse uniformly bounded in $\hbar$) due to the choice of $\widetilde{\Xi}$ and the definition of $\sigma_{\max}$. Moreover, we have $\{\Xi \neq 0\} \subset \{\widetilde{\Xi} = \mathrm{Id}\}$ so that, with (3.9),

$$(\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda} + \mathcal{R}_\hbar^{\mathrm{cut}})\Xi^{\mathrm{W}}\varphi = [\widehat{\mathcal{L}}_h, \Xi^{\mathrm{W}}]\varphi + \mathcal{O}(\hbar^\infty)\|\psi\|,$$

which can be written as

$$\big(\mathrm{Id} + \mathcal{R}_\hbar^{\mathrm{cut}}(\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda})^{-1}\big)(\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda})\Xi^{\mathrm{W}}\varphi = [\widehat{\mathcal{L}}_h, \Xi^{\mathrm{W}}]\varphi + \mathcal{O}(\hbar^\infty)\|\psi\|.$$

By using (3.10) and applying the Calderón–Vaillancourt theorem, we get that

$$\big\|\mathcal{R}_\hbar^{\mathrm{cut}}(\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda})^{-1}\big\| = \mathcal{O}(\hbar^{1-4\eta}).$$

Thus, the operator $\mathrm{Id} + \mathcal{R}_\hbar^{\mathrm{cut}}(\widehat{\mathcal{L}}_0^{\mathrm{cut}} - \widehat{\lambda})^{-1}$ is bijective as soon as $\hbar$ is small enough.

With (3.8), this provides us first with

$$\|\Xi^{\mathrm{W}}\varphi\|^2 \leqslant C\hbar\|\underline{\Xi}^{\mathrm{W}}\varphi\|^2 + C\hbar\|\underline{\Xi}^{\mathrm{W}}\partial_\tau\varphi\|^2 + \mathcal{O}(\hbar^\infty)\|\psi\|^2,$$

and then

$$\|\Xi^{\mathrm{W}}\partial_\tau\varphi\|^2 + \|\Xi^{\mathrm{W}}\varphi\|^2 \leqslant C\hbar\|\underline{\Xi}^{\mathrm{W}}\varphi\|^2 + C\hbar\|\underline{\Xi}^{\mathrm{W}}\partial_\tau\varphi\|^2 + \mathcal{O}(\hbar^\infty)\|\psi\|^2.$$

The estimate (3.6) follows by induction on the size of the support of Ξ. ∎

3.3. First spectral estimates

The aim of the following proposition is to establish that the spectrum of $\mathcal{L}_h$ in I_h is close to that of $h\mathcal{N}_\hbar$ and thus that $\mathcal{N}_\hbar$ is a nice auxiliary operator to describe the spectrum of $\mathcal{L}_h$. In fact, we will see that this proposition is not necessary to prove our spectral estimates, but its proof is instructive.

Proposition 3.2. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$ the following holds. Let us consider an interval $J_h \subset I_h$. Then, there exists an interval $\widehat{J}_h$ such that $J_h \subset \widehat{J}_h \subset I_h$ with $\mathrm{d_H}(J_h, \widehat{J}_h) = \mathcal{O}(h^\infty)$ and*

$$\operatorname{rank} \mathbb{1}_{J_h}(\mathcal{L}_h) \leqslant \operatorname{rank} \mathbb{1}_{\widehat{J}_h}(h\mathcal{N}_\hbar). \tag{3.11}$$

Moreover, for all $\lambda \in I_h \cap \operatorname{sp}(\mathcal{L}_h)$,

$$\operatorname{dist}\big(\lambda, h \operatorname{sp}(\mathcal{N}_\hbar)\big) = \mathcal{O}(h^\infty). \tag{3.12}$$

Proof. Let us start by proving (3.12). Let us consider an eigenvalue $\lambda \in I_h$ of $\mathcal{L}_h$. We write the eigenvalue equation $\mathcal{L}_h\psi = \lambda\psi$.

With Proposition 3.1, we can write (3.5). Then, with (3.6), we deduce that

$$h\mathcal{N}_\hbar\varphi = \lambda\varphi + \mathcal{O}(\hbar^\infty)\|\psi\|.$$

Thus, (3.12) follows from the spectral theorem.

Let us now consider (3.11), which deals with multiplicities. Let us write

$$\operatorname{sp}(\mathcal{L}_h) \cap J_h = \{\lambda_1, \dots, \lambda_p\}$$

(where the λ_j are distinct) and underline that these eigenvalues depend on h as well as p. Consider the associated eigenspaces $(E_j)_{1\leqslant j\leqslant p}$ and note that

$$\dim \bigoplus_{j=1}^{p} E_j = \mathcal{O}(h^{-2})$$

thanks to the Weyl estimate (1.3). With the same notation as above, we consider the spaces of quasimodes $(\hat{\chi}_\hbar \widehat{E}_j)_{1\leqslant j\leqslant p}$. Thanks to Proposition 3.1 (and the rough Weyl estimate), $\dim(\hat{\chi}_\hbar \widehat{E}_j) = \dim E_j$, as soon as h is small enough. Moreover, we have

$$\Big\|\Big(\bigoplus_{j=1}^{p} h\mathcal{N}_\hbar - \lambda\Big)\varphi\Big\| \leqslant \varepsilon_h\|\varphi\|, \quad \varepsilon_h = \mathcal{O}(h^\infty),$$

for all $\varphi = (\varphi_1, \dots, \varphi_p) \in \bigoplus_{j=1}^{p} \hat{\chi}_\hbar \widehat{E}_j$ and where $\lambda = (\lambda_1, \dots, \lambda_p)$.

We set $J_h = [a_h, b_h]$ and $\widehat{J}_h = [a_h - \varepsilon_h, b_h + \varepsilon_h]$. If $\operatorname{rank} \mathbb{1}_{\widehat{J}_h}(h\mathcal{N}_\hbar) < \operatorname{rank} \mathbb{1}_{J_h}(\mathcal{L}_h)$, then the projection $\Pi : \bigoplus_{j=1}^{p} \hat{\chi}_\hbar \widehat{E}_j \to \operatorname{ran} \mathbb{1}_{\widehat{J}_h}(h\mathcal{N}_\hbar)$ could not be injective. Considering a non-zero φ in its kernel, the spectral theorem would give $\|(\bigoplus_{j=1}^{p} h\mathcal{N}_\hbar - \lambda)\varphi\| > \varepsilon_h\|\varphi\|$, which is a contradiction when $\varphi \neq 0$. Therefore, (3.11) follows. ∎

4. A Grushin problem

4.1. A pseudo-differential operator with operator-valued symbol

Recalling Remark 1.4, we notice that the operator $\mathcal{N}_\hbar$ can be seen as a pseudo-differential operator acting as

$$\mathfrak{N}_\hbar = \hat{a}_\hbar(s,\tau)^{-1}\mathcal{T}_\hbar \hat{a}_\hbar(s,\tau)^{-1}\mathcal{T}_\hbar - \hat{a}_\hbar(s,\tau)^{-1}\partial_\tau \hat{a}_\hbar(s,\tau)\partial_\tau,$$

on functions of the form $e^{is\mathfrak{f}_0/h}L^2(\mathbf{T}_{2L}\times\mathbf{R}_+)$ and where

$$\mathcal{T}_\hbar = \Xi_0^{\mathrm{W}} - \tau + \hbar\frac{\kappa}{2}\hat{\tau}^2.$$

In fact, it will be convenient to see $\mathfrak{N}_\hbar$ as a pseudo-differential operator with operator-valued symbol. At a formal level, the principal symbol of $\mathfrak{N}_\hbar$ is

$$n_0(s,\sigma) = -\partial_\tau^2 + (\Xi_0(\sigma)-\tau)^2$$

equipped with the domain

$$\mathrm{Dom}(n_0) = \left\{\psi \in B^2(\mathbf{R}_+) : \psi'(0) = c\psi(0)\right\}.$$

The vector space $B^2(\mathbf{R}_+)$ is equipped with the (s,σ)-independent norm

$$\|\psi\|^2_{B^2(\mathbf{R}_+)} = \|\psi''\|^2 + \|\psi'\|^2 + \left\|\langle t\rangle^2\psi\right\|^2.$$

With this convention, we may write that $n_0 \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}_+), L^2(\mathbf{R}_+)))$.

We say that $\Psi \in S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}_+), L^2(\mathbf{R}_+)))$ when, for all $\alpha\in\mathbf{N}^2$, there exists $C_\alpha > 0$ such that for all $(s,\sigma)\in\mathbf{R}^2$,

$$\|\partial^\alpha\Psi\|_{\mathcal{L}(B^2(\mathbf{R}_+),L^2(\mathbf{R}_+))} \leqslant C_\alpha.$$

Such symbols might also depend on $\hbar$; in this case, the constant C_α is uniform in $\hbar$.

Lemma 4.1. *The operator $\mathfrak{N}_\hbar$ can be written as the Weyl quantization of a symbol in $S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}_+), L^2(\mathbf{R}_+)))$.*

Proof. We can write

$$\mathcal{T}_\hbar = \mathrm{Op}_\hbar^{\mathrm{W}}\left(\Xi_0(\sigma) - \tau + \hbar\frac{\kappa}{2}\hat{\tau}^2\right),$$

the symbol ($2L$-periodic with respect to s) belonging to the class

$$S\big(\mathbf{R}^2, \mathcal{L}\big(B^1(\mathbf{R}_+), L^2(\mathbf{R}_+)\big)\big) \cap S\big(\mathbf{R}^2, \mathcal{L}\big(B^2(\mathbf{R}_+), B^1(\mathbf{R}_+)\big)\big).$$

The functions $a_\hbar(s,\tau)$ and $a_\hbar(s,\tau)^{-1}$ are bounded uniformly with respect to $\hbar$ (and so are all their derivatives). Then, the conclusion follows from the composition theorem for pseudo-differential operators, see [28, Theorem 2.1.12]. ∎

In the following, we let $\mu = \hbar^{2\eta}$ and $\zeta_\mu(\tau) = \zeta(\mu\tau)$. This is convenient when expanding the operator in powers of $\hbar$ (μ will be considered a parameter). This expansion allows to describe rather accurately the symbol of $\mathfrak{N}_\hbar$ by expanding it in powers of $\hbar$. An analogous description for a very similar operator can be found in great detail in [13, Section 4.2].

Proposition 4.2. *The operator $\mathfrak{N}_\hbar$ can be written as follows:*

$$\mathfrak{N}_\hbar = \mathfrak{n}_0 + \hbar\mathfrak{n}_1 + \hbar^2\mathcal{R}_\hbar^{(2)} + \hbar w_\hbar \partial_\tau, \tag{4.1}$$

where, for some $N \in \mathbb{N}$, $C, \hbar_0 > 0$, we have, for all $\hbar \in (0, \hbar_0)$,

(i) *$w_\hbar$ is a smooth function supported in $\{(s,\tau) : C^{-1}\hbar^{-2\eta} \leqslant \langle\tau\rangle \leqslant C\hbar^{-2\eta}\}$ and such that $w_\hbar = \mathcal{O}(\langle\tau\rangle)$,*

(ii) *$\mathcal{R}_\hbar^{(2)}$ is a pseudo-differential operator whose symbol belongs to a bounded set in the space of symbols $S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}_+), L^2(\mathbf{R}_+, \langle\tau\rangle^{-N}\mathrm{d}\tau)))$.*

Moreover, the $\mathfrak{n}_j$ are given by $\mathfrak{n}_j = \mathrm{Op}_\hbar^{\mathrm{W}} n_j$ with

$$n_0 = -\partial_\tau^2 + \big(\Xi_0(\sigma) - \tau\big)^2,$$

$$n_1 = \kappa(s)\big[\big(\Xi_0(\sigma) - \tau\big)\zeta_\mu^2\tau^2 + \zeta_\mu\partial_\tau + 2\zeta_\mu\tau\big(\Xi_0(\sigma) - \tau\big)^2\big].$$

In particular, we can write $\mathfrak{N}_\hbar = \mathrm{Op}_\hbar^{\mathrm{W}}(n_\hbar)$ with a symbol $n_\hbar$ satisfying

$$n_\hbar = n_0 + \hbar n_1 + \hbar^2 r_\hbar^{(2)} + \hbar w_\hbar \partial_\tau,$$

where $r_\hbar^{(2)}$ belongs to the class of operator symbols $S(\mathbf{R}^2, \mathcal{L}(B^2(\mathbf{R}_+), L^2(\mathbf{R}_+, \langle\tau\rangle^{-N}\mathrm{d}\tau)))$ uniformly in $\hbar$.

4.2. Dimensional reduction

The aim of this section is to analyse the spectrum of $\mathcal{N}_\hbar$. This can be done thanks to a Grushin reduction. The principal symbol of $\mathfrak{N}_\hbar$ is the “de Gennes operator” with Robin boundary conditions. Explicitly, we have

$$n_0(s,\sigma) = -\partial_\tau^2 + \big(\Xi_0(\sigma) - \tau\big)^2.$$

The increasing sequence of its (simple) eigenvalues is $(\mu_k(\Xi_0(\sigma)))_{k\geqslant 1}$. We recall that the functions μ_k are described in Proposition 1.1.

Now, consider the window

$$[a,b] \subset (2n-3, 2n-1).$$

For simplicity, let us denote

$$\overset{\circ}{u}_k := \overset{\circ}{u}_k^{[\gamma,\sigma]},$$

see Section 1.2 and (1.8). Let N be defined as in (1.7).

Lemma 4.3. *For all $z \in [a,b]$, let us consider the matrix operator*

$$\mathcal{P}_0(z) = \begin{pmatrix} n_0(s,\sigma) - z & \Pi^* \\ \Pi & 0 \end{pmatrix} : B^2(\mathbf{R}_+) \times \mathbf{C}^N \to L^2(\mathbf{R}_+) \times \mathbf{C}^N,$$

where $\Pi^*(\alpha) = \sum_{j=1}^N \alpha_j \mathring{u}_j$ and $\Pi\psi = (\langle \psi, \mathring{u}_j\rangle)_{1\leqslant k\leqslant N}$.

Then, $\mathcal{P}_0(z)$ is bijective with inverse

$$\mathcal{Q}_0(z) = \begin{pmatrix} q_0 & \Pi^* \\ \Pi & z - M_0(\sigma) \end{pmatrix}, \quad q_0 = \big(n_0(s,\sigma) - z\big)^{-1}(\Pi^*\Pi)^{\perp},$$

where $M_0(\sigma)$ is the diagonal $N \times N$ matrix whose diagonal is $(\mathring{\mu}_1, \dots, \mathring{\mu}_N)$.

Proof. Let $g \in L^2(\mathbf{R}^+)$ and $\beta \in \mathbf{C}^N$. Let us look for $f \in \mathrm{Dom}(n_0)$ and $\alpha \in \mathbf{C}^N$ such that

$$\mathcal{P}_0(z)(f \oplus \alpha) = g \oplus \beta.$$

In other words,

$$\big(n_0(s,\sigma) - z\big) f + \Pi^*\alpha = g, \quad \Pi f = \beta.$$

Let $E = \mathrm{span}(\mathring{u}_1, \dots, \mathring{u}_N)$, and $F = E^{\perp}$. We can write $f = f_E + f_F$ where

$$f_E = \sum_{j=1}^N \langle f, \mathring{u}_j\rangle \mathring{u}_j = \Pi^*\Pi f, \quad f_F = (\Pi^*\Pi)^{\perp} f.$$

We have

$$\begin{aligned} \big(n_0(s,\sigma) - z\big) f_F &= -\big(n_0(s,\sigma) - z\big) f_E - \Pi^*\alpha + g \\ &= -\big(n_0(s,\sigma) - z\big) \sum_{j=1}^N \beta_j \mathring{u}_j - \Pi^*\alpha + g, \end{aligned}$$

so that

$$\big(n_0(s,\sigma) - z\big) f_F = -\sum_{j=1}^N \beta_j (\mathring{\mu}_j - z)\mathring{u}_j - \Pi^*\alpha + g. \tag{4.2}$$

The space F is stable by $n_0(s,\sigma) - z$.

Moreover, thanks to the self-adjointness of n_0, the min-max principle and the fact that $\min \mathring{\mu}_{N+1} \geqslant 2N + 1 > z$, there exists $c > 0$ such that, for all $u \in \mathrm{Dom}(n_0) \cap F$,

$$\big\langle (n_0 - z)u, u\big\rangle = \langle n_0 u, u\rangle - z\|u\|^2 \geqslant (\mathring{\mu}_{N+1} - z)\|u\|^2 \geqslant c\|u\|^2.$$

Thus, the operator $(n_0 - z)_{|F}$ is injective with closed range and, by considering the adjoint, we deduce that it is bijective. We also notice that

$$\big\|(n_0 - z)_{|F}^{-1}\big\| \leqslant (\mathring{\mu}_{N+1} - z)^{-1} \leqslant c^{-1}.$$

Then (4.2) has a solution if and only if the right-hand-side belongs to F, that is

$$-\sum_{j=1}^{N} \beta_j(\mathring{\mu}_j - z)\mathring{u}_j - \Pi^*\alpha + g \in F$$

which means that, for all $k \in \{1, \dots, N\}$,

$$-\beta_k(\mathring{\mu}_k - z) - \alpha_k + \langle g, \mathring{u}_k\rangle = 0.$$

We deduce that

$$\alpha = \Pi g + \big(z - M_0(\sigma)\big)\beta.$$

This unique solution is given by

$$f_F = (\Pi^*\Pi)^\perp\big(n_0(s,\sigma) - z\big)^{-1} g.$$

Therefore,

$$f = f_E + f_F = \Pi\beta + (\Pi^*\Pi)^\perp\big(n_0(s,\sigma) - z\big)^{-1} g.$$ ■

Let us now consider the full symbol

$$\mathcal{P}_\hbar(z) := \begin{pmatrix} n_\hbar - z & \Pi^* \\ \Pi & 0 \end{pmatrix},$$

which may be expanded in powers of $\hbar$ as

$$\mathcal{P}_\hbar(z) = \mathcal{P}_0(z) + \hbar\mathcal{P}_1(z) + \mathcal{R}_\hbar,$$

with

$$\mathcal{P}_0(z) = \begin{pmatrix} n_0(s,\sigma) - z & \Pi^* \\ \Pi & 0 \end{pmatrix}, \quad \mathcal{P}_1(z) = \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{R}_\hbar(z) = \begin{pmatrix} r_\hbar & 0 \\ 0 & 0 \end{pmatrix},$$

where $r_\hbar = \hbar^2 r_\hbar^{(2)} + \hbar\tilde{w}_{\hbar,1} + \hbar w_{\hbar,2}\partial_\tau$, see Proposition 4.2.

We notice that

$$\mathcal{P}_\hbar^{\mathrm{W}} = \begin{pmatrix} \mathfrak{N}_\hbar - z & \mathfrak{P}^* \\ \mathfrak{P} & 0 \end{pmatrix}, \quad \mathfrak{P} = \Pi^{\mathrm{W}}. \tag{4.3}$$

Since the principal symbol of $\mathcal{P}_\hbar^{\mathrm{W}}$ is bijective, it is natural to try to construct an approximate inverse in the semiclassical limit. Let us look for an approximate inverse whose symbol is in the form

$$\mathcal{Q}_{\hbar,1} = \mathcal{Q}_0(z) + \hbar\mathcal{Q}_1(z). \tag{4.4}$$

As in [28], we are led to choose

$$\mathcal{Q}_1 = -\mathcal{Q}_0\mathcal{P}_1\mathcal{Q}_0 = -\begin{pmatrix} q_0 n_1 q_0 & q_0 n_1 \Pi^* \\ \Pi n_1 q_0 & \Pi n_1 \Pi^* \end{pmatrix}.$$

This choice is convenient since the composition theorem for pseudo-differential operators (see [28]) implies that

$$\mathcal{Q}_{\hbar,1}^{\mathrm{W}}\big(\mathcal{P}_0(z)+\hbar\mathcal{P}_1\big)^{\mathrm{W}} = \mathrm{Id} + \hbar\left(\frac{1}{i}\{\mathcal{Q}_0,\mathcal{P}_0\} + \mathcal{Q}_0\mathcal{P}_1 + \mathcal{Q}_1\mathcal{P}_0\right)^{\mathrm{W}} + \mathcal{O}_{L^2(\mathbf{T}_{2L}\times\mathbf{R}_+,\langle\tau\rangle^N\mathrm{d}s\mathrm{d}\tau)\times L^2(\mathbf{T}_{2L})\to L^2(\mathbf{T}_{2L}\times\mathbf{R}_+)\times L^2(\mathbf{T}_{2L})}(\hbar^2),$$

where the remainder is estimated thanks to the Calderón–Vaillancourt theorem (see [28, Theorem 2.1.16]) and the resolvent estimate in Lemma B.6 (applied with an appropriate $\alpha > 0$). The $\hbar$-term vanishes due the choice of $\mathcal{Q}_1$ and that fact that the Poisson bracket is actually 0 since the principal symbol does not depend on s. With this choice, the bottom right coefficient, denoted by $\mathcal{Q}_{\hbar,1}^{\pm}$, of the matrix $\mathcal{Q}_{\hbar,1}$ is

$$\mathcal{Q}_{\hbar,1}^{\pm} = z - M_0(\sigma) - \hbar\Pi n_1\Pi^*.$$

This invites to consider the effective matrix pseudo-differential operator whose symbol is

$$M_\hbar = M_0(\sigma) + \hbar M_1(s,\sigma),$$

with

$$M_1(s,\sigma) = \kappa(s)\Pi\mathcal{C}\big(\tau,\Xi_0(\sigma)\big)\Pi^*,$$
$$\mathcal{C}(\tau,\xi) = (\xi-\tau)\zeta_\mu^2\tau^2 + \zeta_\mu\partial_\tau + 2\zeta_\mu\tau(\xi-\tau)^2.$$

Using again the composition theorem to deal with the remainder $\mathcal{R}_\hbar$, we get

$$\mathcal{Q}_{\hbar,1}^{\mathrm{W}}\mathcal{P}_\hbar^{\mathrm{W}} = \mathrm{Id} + \mathcal{O}_{L^2(\mathbf{T}_{2L}\times\mathbf{R}_+,\langle\tau\rangle^N\mathrm{d}s\mathrm{d}\tau)\times L^2(\mathbf{T}_{2L})\to L^2(\mathbf{T}_{2L}\times\mathbf{R}_+)\times L^2(\mathbf{T}_{2L})}(\hbar^2) + \mathcal{Q}_{\hbar,1}^{\mathrm{W}}\begin{pmatrix}\hbar w_\hbar\partial_\tau & 0\\ 0 & 0\end{pmatrix}^{\mathrm{W}}. \tag{4.5}$$

Moreover, similar arguments show that $\mathcal{Q}_{\hbar,1}^{\mathrm{W}}$ is also an approximate right inverse of $\mathcal{P}_\hbar^{\mathrm{W}}$ in the sense that

$$\mathcal{P}_\hbar^{\mathrm{W}}\mathcal{Q}_{\hbar,1}^{\mathrm{W}} = \mathrm{Id} + \mathcal{O}_{L^2(\mathbf{T}_{2L}\times\mathbf{R}_+,\langle\tau\rangle^N\mathrm{d}s\mathrm{d}\tau)\times L^2(\mathbf{T}_{2L})\to L^2(\mathbf{T}_{2L}\times\mathbf{R}_+)\times L^2(\mathbf{T}_{2L})}(\hbar^2) + \begin{pmatrix}\hbar w_\hbar\partial_\tau & 0\\ 0 & 0\end{pmatrix}^{\mathrm{W}}\mathcal{Q}_{\hbar,1}^{\mathrm{W}}. \tag{4.6}$$

Proposition 4.4. *The spectrum of $\mathcal{L}_h$ in $[ha,hb]$ coincides (with multiplicity) with that of $h\mathrm{Op}_\hbar^{\mathrm{W}} M_\hbar$ modulo $\mathcal{O}(h^2)$.*

Proof. First, we consider ψ an eigenfunction of $\mathcal{L}_h$ associated with $\lambda \in [ha,hb]$. We use (4.5) with $z = h^{-1}\lambda$ to get that

$$\mathcal{Q}_{\hbar,1}^{\mathrm{W}}\mathcal{P}_\hbar^{\mathrm{W}}\begin{pmatrix}\varphi\\0\end{pmatrix} = \begin{pmatrix}\varphi\\0\end{pmatrix} + \mathcal{O}(\hbar^2)\|\varphi\|,$$

where φ denotes the function ψ after multiplication by a cutoff function in t and rescaling as in Proposition 3.1. Note that we used the exponential decay in τ of our quasimode φ (which comes from that of ψ) to control the remainder term in (4.5). We infer that

$$\mathfrak{P}^*\mathfrak{P}\varphi = \varphi + \mathcal{O}(\hbar)\|\varphi\|, \quad (\widehat{\lambda} - \mathrm{Op}_\hbar^{\mathrm{W}} M_\hbar)\mathfrak{P}\varphi = \mathcal{O}(\hbar^2)\|\varphi\|, \tag{4.7}$$

where we used that the principal symbol of the top right coefficient of $\mathcal{Q}_{\hbar,1}$ is Π^*. Since $\mathfrak{P}^*$ is bounded uniformly in $\hbar$ (as the quantization of a bounded symbol), the first relation implies that

$$\|\varphi\| \leqslant C\|\mathfrak{P}\varphi\|.$$

Then, from the second relation and the spectral theorem, we deduce that

$$\mathrm{dist}\big(\widehat{\lambda}, \mathrm{sp}(\mathrm{Op}_\hbar^{\mathrm{W}} M_\hbar)\big) \leqslant C\hbar^2.$$

This means that the spectrum of $h^{-1}\mathcal{L}_h$ in the window $[ha, hb]$ is at a distance of order $\hbar^2$ to the spectrum of the effective operator $\mathrm{Op}_\hbar^{\mathrm{W}} M_\hbar$.

Let us now proceed as in the proof of Proposition 3.2 and keep the same notation. We have

$$\Big\|\Big(\bigoplus_{j=1}^{p} \mathcal{N}_\hbar - \widehat{\lambda}\Big)\varphi\Big\| \leqslant \varepsilon_h\|\varphi\|, \quad \varepsilon_h = \mathcal{O}(h^\infty),$$

for all $\varphi = (\varphi_1, \dots, \varphi_p) \in \bigoplus_{j=1}^{p} \widehat{\chi}_\hbar \widehat{E}_j$ and where $\widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_p)$.

Similarly as (4.7), we have

$$\|\varphi\| \leqslant C\|\mathfrak{P}\varphi\|, \tag{4.8}$$

and

$$\Big(\bigoplus_{j=1}^{p} \mathrm{Op}_\hbar^{\mathrm{W}} M_\hbar - \widehat{\lambda}\Big)\mathfrak{P}\varphi = \mathcal{O}(\hbar^2)\|\mathfrak{P}\varphi\|,$$

where $\mathfrak{P}\varphi = (\mathfrak{P}\varphi_1, \dots, \mathfrak{P}\varphi_p)$. Due to (4.8), the action of the map $\mathfrak{P}$ is injective on $\bigoplus_{j=1}^{p} \widehat{\chi}_\hbar \widehat{E}_j$. Therefore, as in the proof of Proposition 3.2, the spectral theorem provides us with

$$\mathrm{rank}\, \mathbb{1}_{J_h}(\mathcal{L}_h) \leqslant \mathrm{rank}\, \mathbb{1}_{K_h}(hM_\hbar^{\mathrm{W}}),$$

where $J_h \subset I_h$ and K_h is an interval such that $J_h \subset K_h$ and $\mathrm{d_H}(J_h, K_h) = \mathcal{O}(h^2)$.

Let us now prove the converse estimate. We use (4.6) with an eigenvalue $z = \widehat{\lambda}$ of $M_\hbar^{\mathrm{W}}$ and for f a corresponding eigenfunction. We have

$$\mathcal{P}_\hbar^{\mathrm{W}} \mathcal{Q}_{\hbar,1}^{\mathrm{W}} \begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} + \mathcal{O}(\hbar^2)\|f\|, \tag{4.9}$$

where the remainder term involving $w_\hbar$ has been controlled by using the exponential decay of the eigenfunctions of the de Gennes–Robin operator n_0.

Then, the first line in (4.9) gives

$$(\mathcal{N}_\hbar - \widehat{\lambda})\big(\mathcal{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f = \mathcal{O}(\hbar^2)\|f\|, \tag{4.10}$$

whereas the second line gives

$$\mathfrak{P}\big(\mathfrak{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f = \mathcal{O}(\hbar^2)\|f\|,$$

which leads to

$$\|f\| \leqslant C\big\|\big(\mathfrak{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f\big\|.$$

With (4.10), we get

$$(\mathcal{N}_\hbar - \widehat{\lambda})\big(\mathfrak{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f = \mathcal{O}(\hbar^2)\big\|\big(\mathfrak{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f\big\|.$$

Now, by using the exponential decay of $(\mathfrak{Q}_{\hbar,1}^{+})^{\mathrm{W}} f$ and the rough microlocalization of f in the support of Ξ_0 (since the principal symbol of the scalar pseudo-differential operator $M_\hbar$ is n_0), we get the quasimode estimate

$$(\mathcal{L}_h - \lambda)\psi^{\text{quasi}} = \mathcal{O}(h^2)\|\psi^{\text{quasi}}\|,$$

with $\psi^{\text{quasi}}(x) = \chi(t(x)/\hbar^{1-\gamma})\Psi^{\text{quasi}} \circ \Gamma^{-1}(x)$ where

$$\Psi^{\text{quasi}}(s,t) = \big(\mathfrak{Q}_{\hbar,1}^{+}\big)^{\mathrm{W}} f(s, \hbar^{-1}t),$$

and χ is a smooth cutoff function equal to 1 near 0 and 0 away from a neighborhood of $t = 0$ and $\gamma \in (0,1)$ is chosen small enough so that

$$t\zeta(h^{-\frac{1}{2}+\eta}t) = t$$

on the support of $\chi(t/\hbar^{1-\gamma})$. Note that Ψ^{quasi} satisfies the Robin condition at $t = 0$ since $\mathfrak{Q}_{\hbar,1}^{+}$ (as well as $\mathfrak{Q}_0$, see (4.4)) takes values in a space of functions satisfying the Robin condition. In particular, ψ^{quasi} belongs to the domain of $\mathcal{L}_h$.

The spectral theorem shows that λ is close to the spectrum of $\mathcal{L}_h$ at a distance or order at most $\mathcal{O}(h^2)$. The argument concerning the multiplicities can again be used (as above) by exchanging the roles of $\mathcal{N}_\hbar$ and $M_\hbar^{\mathrm{W}}$. The conclusion follows. ■

Remark 4.5. In Proposition 3.2 we only proved one inclusion of spectra. In contrast, Proposition 4.4 is stronger, since it provides an equality modulo $\mathcal{O}(h^2)$, in the sense of Definition 1.5. Indeed, in the proof of Proposition 4.4, we only have to use quasimodes for $\mathcal{N}_\hbar$ and not necessarily the true eigenfunctions of $\mathcal{N}_\hbar$ (whose existence in the spectral window of interest is not obvious). Our presentation avoids the spectral analysis of $\mathcal{N}_\hbar$ (existence of the discrete spectrum, Agmon estimates, etc.) by comparing directly the spectra of $\mathcal{L}_h$ and of the effective operator.

5. Analysis of the effective operator

This section is devoted to the spectral study of $M_\hbar^{\mathrm{W}}$ in $[a,b]$. Let us diagonalize this operator, up to a remainder of order $\mathcal{O}(\hbar^2)$. Note that, by using the exponential decay of the eigenfunctions of n_0, we may (and so do we) replace ζ_μ by 1.

5.1. Asymptotic diagonalization and end of the proof of Theorem 1.7

The end of the proof follows from classical arguments (see, for instance, [23, Section 3.1] where such arguments are used). We notice that the spectrum of

$$\mathcal{T}_\hbar = \exp(\hbar A^{\mathrm{W}}) M_\hbar^{\mathrm{W}} \exp(-\hbar A^{\mathrm{W}})$$

is the same as the one of $M_\hbar^{\mathrm{W}}$, as soon as A belongs to $S(1)$ and is $2L$-periodic with respect to s. In this case, we recall that A^{W} is bounded from $L^2(\mathbf{T}_{2L})$ to $L^2(\mathbf{R}_{2L})$ (and thus its exponential is well-defined as an element of $\mathcal{L}(L^2(\mathbf{T}_{2L}))$ thanks to the classical power series). Let us explain how to choose A. By expanding the exponential, we have

$$\mathcal{T}_\hbar = (\mathrm{Id} + \hbar A^{\mathrm{W}}) M_\hbar^{\mathrm{W}} (\mathrm{Id} - \hbar A^{\mathrm{W}}) + \mathcal{O}(\hbar^2),$$

and thus

$$\mathcal{T}_\hbar = M_\hbar^{\mathrm{W}} + \hbar [A^{\mathrm{W}}, M_\hbar^{\mathrm{W}}] + \mathcal{O}(\hbar^2),$$

so that

$$\mathcal{T}_\hbar = M_0^{\mathrm{W}} + \hbar \big(M_1 + [A, M_0]\big)^{\mathrm{W}} + \mathcal{O}(\hbar^2).$$

Therefore, A should be chosen so that $M_1 - [M_0, A]$ is diagonal. The map $\mathrm{Skew}_N(\mathbf{R}) \ni A \mapsto [M_0, A] \in \mathrm{Sym}_N^0(\mathbf{R})$ is well-defined and an isomorphism since M_0 is diagonal with distinct real entries, where $\mathrm{Skew}_N(\mathbf{R})$ is the vector space of skew-symmetric matrices and $\mathrm{Sym}_N^0(\mathbf{R})$ the space of symmetric matrices with null diagonal. It is actually easy to compute its inverse. Consider M a symmetric matrix with null diagonal. We want to find $A \in \mathrm{Skew}_N(\mathbf{R})$ such that $[M_0, A] = M$. For all $j \in \{1, \dots, N\}$, we have

$$(M_0 - \mathring{\mu}_j) A e_j = M e_j,$$

and then, for all $k \in \{1, \dots, N\}$,

$$(\mathring{\mu}_k - \mathring{\mu}_j) \langle A e_j, e_k \rangle = \langle M e_j, e_k \rangle.$$

Thus, for all $k \neq j$,

$$\langle A e_j, e_k \rangle = (\mathring{\mu}_k - \mathring{\mu}_j)^{-1} \langle M e_j, e_k \rangle,$$

which determines a unique $A \in \mathrm{Skew}_N(\mathbf{R})$.

Since there is a uniform gap between the $\mathring{\mu}_j$ (with respect to σ), we get the existence of a skew-symmetric A in $S(1)$ such that

$$\underbrace{M_1 - \mathrm{diag}(M_1)}_{\in \mathrm{Sym}_N^0(\mathbf{R})} + [A, M_0] = 0.$$

With this choice, we get

$$\mathcal{T}_\hbar = M_0^{\mathrm{W}} + \hbar\, \mathrm{diag}(M_1^{\mathrm{W}}) + \mathcal{O}(\hbar^2).$$

Note that, for all $j \in \{1, \dots, N\}$, we have

$$\langle M_1 e_j, e_j \rangle = \kappa(s) \big\langle \mathcal{C}\big(\tau, \Xi_0(\sigma)\big) u_j^{[\Xi_0(\sigma),\gamma]}, u_j^{[\Xi_0(\sigma),\gamma]} \big\rangle.$$

By the spectral theorem, we deduce that the spectra of $M_\hbar^{\mathrm{W}}$ and $M_0^{\mathrm{W}} + \hbar \operatorname{diag}(M_1^{\mathrm{W}})$ coincide modulo $\mathcal{O}(\hbar^2)$. This procedure can be continued at any order.

5.2. Spectral consequences

The aim of this last section is to prove Proposition 1.9 and Corollary 1.14.

5.2.1. Proof of Proposition 1.9. We could not find this particular statement in the literature, because (a) we have to deal with non-connected level sets of the principal symbol, and (b) we have Floquet periodic conditions, with $\hbar$-dependent Floquet exponent. The first issue is treated with usual microlocal arguments: each connected component carries with itself a Bohr–Sommerfeld asymptotic series, as in [21], and the initial spectrum is obtained, modulo $\mathcal{O}(\hbar^\infty)$, by the superposition (with multiplicities) of all these series. The second one is easily included in the general theory thanks to the "sheaf" approach of [42]. Indeed, near each point of the energy level curve $\sigma = \text{const}$, the operator $\mathsf{P}_\hbar$ is microlocally a usual $\hbar$-pseudo-differential operator, and the quantum Darboux–Carathéodory normal form holds. Therefore, the Bohr–Sommerfeld cocycle of [42, Proposition 5.6] holds; the difference being that the condition for a global section should include the Floquet exponent θ. This gives a Bohr–Sommerfeld rule for quantized energies E (for each connected component) of the form

$$\mathcal{A}(E) + \hbar m(E)\tfrac{\pi}{2} + \hbar \mathcal{K}(E) + \mathcal{O}(\hbar^2) = 2\pi\hbar\big(\ell + \tfrac{L}{\pi}\theta\big), \quad \ell \in \mathbf{Z}, \tag{5.1}$$

where $\mathcal{A}(E)$ is the action integral (here $\mathcal{A}(E) = 2L\sigma$ when $E = \mu(\sigma)$), $m(E)$ the Maslov index (which vanishes here, because the curves $\sigma = \text{const}$ project diffeomorphically on the s variable), and $\mathcal{K}(E)$ is the integral of the subprincipal form [42, Definition 3.2] along the energy level set. In order to compute $\mathcal{K}$, we notice that the Hamiltonian vector field of $\mu(\sigma)$ is $\mu'(\sigma)\frac{\partial}{\partial s}$ and hence the subprincipal form is $\frac{-r}{\mu'(\sigma)}\mathrm{d}s$, where r is the subprincipal symbol of $\mathsf{P}_\hbar$ (here $r = -C(\sigma)\kappa(s)$). Hence, for $E = \mu(\sigma)$, we have

$$\mathcal{K}(E) = \frac{C(\sigma)}{\mu'(\sigma)}_{|\Sigma_q} \int_0^{2L} \kappa(s)\mathrm{d}s.$$

Inverting the formal series (5.1), we get

$$E = \mu(\sigma) - \mathcal{K}\big(\mu(\sigma)\big)\tfrac{1}{2L}\mu'(\sigma) + \mathcal{O}(\hbar^2), \quad \sigma = \tfrac{\pi}{L}\hbar\big(\ell + \tfrac{L}{\pi}\theta\big)$$

which gives (1.11) and (1.12).

5.2.2. Proof of Corollary 1.14. Thanks to Proposition 4.4 and the considerations in Section 5.1, we know that the spectrum of $\mathcal{L}_h$ in $[ha, hb]$ coincides with that of $hM_\hbar^{\mathrm{W}}$ modulo $\mathcal{O}(h^2)$. In the present section, since we are interested in the low-lying eigenvalues, we take

$a = -\infty$ and $b = \Theta_0(\gamma) + \varepsilon < 1$ (for $\varepsilon > 0$ small enough). Therefore, we have $N = 1$ and the matrix symbol $M_\hbar$ reduces to a scalar symbol:

$$M_\hbar(s,\sigma) = \mu_1(\gamma,\sigma) + \hbar\kappa(s)\big\langle \mathcal{C}\big(\tau, \Xi_0(\sigma)\big) u_1^{[\Xi_0(\sigma),\gamma]}, u_1^{[\Xi_0(\sigma),\gamma]}\big\rangle.$$

We are interested in the spectrum of $M_\hbar^{\mathrm{W}}$ (when acting on $e^{is\mathfrak{f}_0/h}L^2(\mathbf{T}_{2L})$). Hence, Corollary 1.14 can be obtained by [8] (see in particular the Morse case, Section 6.3.1) followed by a standard Birkhoff normal form (here, the Floquet exponent $\mathfrak{f}_0/h$ plays no role because the analysis is local near a point in the boundary $\partial\Omega$). Here are the details.

Thanks to the Weyl asymptotic formula for pseudo-differential operators (see, for instance, [43, Theorem 14.11]), the counting function $\mathrm{N}(M_\hbar^{\mathrm{W}}, \Theta_0(\gamma) + \varepsilon)$ (giving the number of eigenvalues less than $\Theta_0(\gamma) + \varepsilon$) satisfies

$$\begin{aligned}\mathrm{N}\big(M_\hbar^{\mathrm{W}}, \Theta_0(\gamma) + \varepsilon\big) &= \frac{1}{2\pi\hbar}\int_{\{(s,\sigma):\mu_1(\sigma)\leqslant\Theta_0(\gamma)+\varepsilon\}} \mathrm{d}s\mathrm{d}\sigma + o(\hbar^{-1}) \\ &= \frac{L}{\pi\hbar}\big|\{\sigma : \mu_1(\sigma) \leqslant \Theta_0(\gamma) + \varepsilon\}\big|\big(1 + o(1)\big).\end{aligned}$$

Now, we take $\varepsilon = \hbar^\eta$, for some given $\eta > 0$.

Due to the non-degeneracy of the minimum of $\sigma \mapsto \mu_1(\gamma, \Xi_0(\sigma))$, the eigenfunctions associated with eigenvalues less than b are microlocalized in a neighborhood of $\xi_0(\gamma)$ of size $\hbar^{\eta/2}$ (and so are all the linear combinations of such eigenfunctions due to the Weyl estimate). This invites us to expand the symbol near $\xi_0(\gamma)$:

$$\begin{aligned}M_\hbar(s,\sigma) = \Theta_0(\gamma) &+ \frac{\partial_\sigma^2\mu\big(\gamma,\xi_0(\gamma)\big)}{2}\big(\sigma - \xi_0(\gamma)\big)^2 - \hbar\kappa(s)C_1\big(\xi_0(\gamma)\big) \\ &+ \mathcal{O}\big(\big|\sigma - \xi_0(\gamma)\big|^3 + \hbar\big|\sigma - \xi_0(\gamma)\big|\big).\end{aligned} \tag{5.2}$$

Therefore, $M_\hbar$ is relative perturbation of the symbol of a classical electric Schrödinger operator. The corresponding operator is

$$\mathcal{M}_\hbar = \Theta_0(\gamma) + \frac{\partial_\sigma^2\mu\big(\gamma,\xi_0(\gamma)\big)}{2}\big(\hbar D_s - \xi_0(\gamma)\big)^2 - \hbar\kappa(s)C_1\big(\xi_0(\gamma)\big).$$

Let us only consider the case when $\gamma < \gamma_0^{[0]}$ (i.e., $\epsilon = 1$). The assumption that κ has a unique maximum, which is non-degenerate, allows to use the harmonic approximation near the maximum of κ (and even a Birkhoff normal form, see, for instance, [36, Chapter 5] or the original references [4, 40]). The eigenvalues of $\mathcal{M}_\hbar$ satisfy

$$\begin{aligned}\lambda_j(\mathcal{M}_\hbar) = \Theta_0(\gamma) &- \kappa_{\max}C_1\big(\xi_0(\gamma)\big)\hbar \\ &+ \left((2j-1)\sqrt{\frac{k_2C_1\big(\xi_0(\gamma)\big)\mu_1''\big(\gamma,\xi_0(\gamma)\big)}{4}}\right)\hbar^{\frac{3}{2}} + o(\hbar^{\frac{3}{2}}),\end{aligned}$$

uniformly in $j \geqslant 1$ such that $j\hbar^{\frac{1}{2}} = o(1)$.

We recall that $\hbar = h^{\frac{1}{2}}$. We get Corollary 1.14 by noticing, thanks to a perturbation analysis using (5.2), that the spectra of $hM_\hbar^{\mathrm{W}}$ and $h\mathcal{M}_\hbar$ below $h(\Theta_0(\gamma) + \hbar^\eta)$ coincide modulo $o(h^{\frac{7}{4}})$.

5.2.3. Proof of Theorem 1.11. By Theorem 1.7, and Definition 1.5 we have, for $\epsilon = \mathcal{O}(h)$,

$$N\big(h\mathfrak{M}_h, \big[h(a+\epsilon), h(b-\epsilon)\big]\big) \\ \leqslant N\big(\mathcal{L}_h, [ha, hb]\big) \leqslant N\big(h\mathfrak{M}_h, \big[h(a-\epsilon), h(b+\epsilon)\big]\big). \tag{5.3}$$

From Corollary 1.10, for any interval $[a', b']$ disjoint from Θ and Λ, the number of eigenvalues of $\mathfrak{M}_h$ in $[ha', hb']$ is bounded by $C\frac{b'-a'}{h^{1/2}}$ for some constant $C > 0$. Applying this with (a', b') equal, respectively, to the four intervals $(a, a+\epsilon)$, $(b-\epsilon, b)$, $(a-\epsilon, a)$, and $(b, b+\epsilon)$, it follows from (5.3) that

$$\begin{aligned} N\big(\mathcal{L}_h, [ha, hb]\big) &= N\big(h\mathfrak{M}_h, [ha, hb]\big) + \mathcal{O}(\epsilon h^{-1/2}) \\ &= N\big(h\mathfrak{M}_h, [ha, hb]\big) + \mathcal{O}(h^{1/2}). \end{aligned}$$

Therefore, it is enough to estimate $N(h\mathfrak{M}_h, [ha, hb])$, for which we apply Corollary 1.10 (which is actually a description of the spectrum of $\mathfrak{M}_h$). This corollary says that the number of eigenvalues of $h\mathfrak{M}_h$ inside $[ha, hb]$, including multiplicities, is given, modulo $\mathcal{O}(h^2)$, by the number of integers $\ell \in \mathbf{Z}$ such that

$$h^{\frac{1}{2}}\big(\tfrac{\pi}{L}\ell + \theta(h)\big) \in f_{k,q,h}^{-1}\big([a, b]\big), \tag{5.4}$$

for some admissible (k, q), where $f_{k,q,h}(\sigma) := f_{k,q}(\sigma, h^{1/2})$.

To simplify notations, let us momentarily fix (k, q) and denote $f_h := f_{k,q,h} =: f_0 + h^{1/2} f_1 + \mathcal{O}(h)$, where f_0 and f_1 are defined in (1.13) and (1.14). By assumption, f_0 is monotonous on $\Sigma_{k,q}$, let us assume that it is increasing; the decreasing case is obtained by swapping (a, b). For h small enough, f_h is also increasing and hence $f_h^{-1}([a, b]) = [f_h^{-1}(a), f_h^{-1}(b)]$. Therefore, the solutions to (5.4) are exactly the integers belonging to the interval

$$\frac{Lh^{-1/2}}{\pi}\big[f_h^{-1}(a), f_h^{-1}(b)\big] - \frac{L}{\pi}\theta(h). \tag{5.5}$$

Let $\sigma = f_h^{-1}(a)$; of course σ depends on h, but since $\sigma \in \Sigma_{k,q}$, it is bounded and we have $\sigma = f_0^{-1}(a) + \mathcal{O}(h^{1/2})$. Therefore, $f_1(\sigma) = f_1(f_0^{-1}(a)) + \mathcal{O}(h^{1/2})$. According to the statement of Theorem 1.11, we denote $\alpha := f_0^{-1}(a)$. Writing $f_0(\sigma) = a - h^{1/2} f_1(\alpha) + \mathcal{O}(h)$ we get, by Taylor expansion,

$$\sigma = \alpha - h^{1/2}(f_0^{-1})'(a) f_1(\alpha) + \mathcal{O}(h) = \alpha + h^{1/2}\frac{\langle\kappa\rangle C_k(\alpha)}{\mu_k'(\alpha)} + \mathcal{O}(h).$$

Using the analogous formula for $f_h^{-1}(b)$, we may compute the difference $f_h^{-1}(b) - f_h^{-1}(a)$ and obtain the length of the interval (5.5):

$$\frac{Lh^{-1/2}}{\pi}\big(f_h^{-1}(b) - f_h^{-1}(a)\big) = \frac{Lh^{-1/2}}{\pi}(\beta - \alpha) + \frac{L\langle\kappa\rangle}{\pi}\left(\frac{C_k(\beta)}{\mu_k'(\beta)} - \frac{C_k(\alpha)}{\mu_k'(\alpha)}\right) + \mathcal{O}(h^{1/2})$$

which gives Theorem 1.11 by summing over admissible (k, q).

5.2.4. Proof of Theorem 1.12. We use the notation of Theorem 1.7. By Proposition 1.9, the self-adjoint operators m_k^W acting on $e^{i\theta(h)\cdot}L^2(\mathbf{T}_{2L})$ satisfy the Gårding inequality:

$$m_k^W \geqslant \min \mu_k - \mathcal{O}(h^{1/2}) = \Theta^{[k-1]} - \mathcal{O}(h^{1/2}) > b \quad \forall k = 2, \dots, N,\ \forall h < h_0$$

for h_0 small enough. Hence the spectrum of $\mathcal{L}_h$ in I_h coincides, modulo $\mathcal{O}(h^2)$, with the spectrum of hm_1^W in that interval. In other words, for this choice of interval I_h, the disjoint unions of Corollary 1.10 reduce to a union of the two components $(k = 1, q = 1)$ and $(k = 1, q = 2)$, and the spectrum in I_h coincides modulo $\mathcal{O}(h^2)$ with

$$\bigsqcup_{q=1,2} \left\{hf_{1,q}(\sigma, h^{\frac{1}{2}}),\ \sigma \in h^{\frac{1}{2}}\big(\tfrac{\pi}{L}\mathbf{Z} + \theta(h)\big) \cap \Sigma_{1,q}\right\} \cap [ha, hb].$$

So eigenvalues λ_j in I_h are associated with integers $\ell = \ell(h) \in \mathbf{Z}$ such that $h^{\frac{1}{2}}(\frac{\pi}{L}\ell + \theta(h)) \in \Sigma_{1,1} \cup \Sigma_{1,2}$; therefore there are constants α, β, independent on h, such that

$$\sigma_\ell(h) := h^{\frac{1}{2}}\big(\tfrac{\pi}{L}\ell + \theta(h)\big) \in [\alpha, \beta].$$

Hence,

$$\frac{\pi}{L}\ell \in \left[\frac{\alpha}{h^{1/2}} - \theta(h), \frac{\beta}{h^{1/2}} - \theta(h)\right].$$

Recalling that $\theta(h) = \frac{|\Omega|}{|\partial\Omega| h}$, we get that, for $h < h_0 := \sqrt{\frac{|\Omega|}{\beta|\partial\Omega|}}$, ℓ must be negative. Thus, for each fixed ℓ, $\sigma_\ell(h)$ increases when h decreases to zero. In other words, because of the non-zero flux term, the corresponding semiclassical eigenvalues $hf_{1,q}(\sigma_\ell(h), h^{\frac{1}{2}})$ "move to the right" (in the sense of Figure 1) towards the Landau level $\mu_1 = 1$.

Let us now describe the semiclassical branches, i.e., the curves

$$h \mapsto hf_{1,q}\big(\sigma_\ell(h), h^{\frac{1}{2}}\big), \quad q = 1, 2,\ \ell \in \mathbf{Z}_-. \tag{5.6}$$

We may assume that the intervals $\Sigma_{1,q}$ satisfy $\Sigma_{1,1} \leqslant \Sigma_{1,2}$. Recall from Proposition 1.1 (or Figure 1) that there exists $c > 0$ such that $\mu'_{1|\Sigma_{1,1}} \leqslant -c$ while $\mu'_{1|\Sigma_{1,2}} \geqslant c$. Hence, in view of the semiclassical expansion of $f_{k,q}$, and up to reducing c, we get

$$f'_{1,1} \leqslant -c \quad \text{and} \quad f'_{1,2} \geqslant c \tag{5.7}$$

uniformly for $h \leqslant h_0$ small enough. Thus, for each fixed admissible $\ell \in \mathbf{Z}_-$, the branch (5.6) generated by $\Sigma_{1,1}$ (i.e., corresponding to $q = 1$) is an increasing curve, while the branch corresponding to $q = 2$ is decreasing. Moreover, the semiclassical branches generated by $\Sigma_{1,1}$ and associated with different integers $\ell_1 \neq \ell_2$ will never cross as h varies, and their mutual vertical distance is bounded below as

$$\left|hf_{1,q}\big(\sigma_{\ell_1}(h), h^{\frac{1}{2}}\big) - hf_{1,q}\big(\sigma_{\ell_2}(h), h^{\frac{1}{2}}\big)\right| \geqslant h^{\frac{3}{2}}\frac{\pi c}{L}. \tag{5.8}$$

Hence, in view of (5.7), we see that the horizontal distance between these curves is $\mathcal{O}(h^2)$. Of course, the same holds for the branches associated with $\Sigma_{1,2}$, which thus form a collection of disjoint decreasing curves. Therefore, the superposition of all branches is a

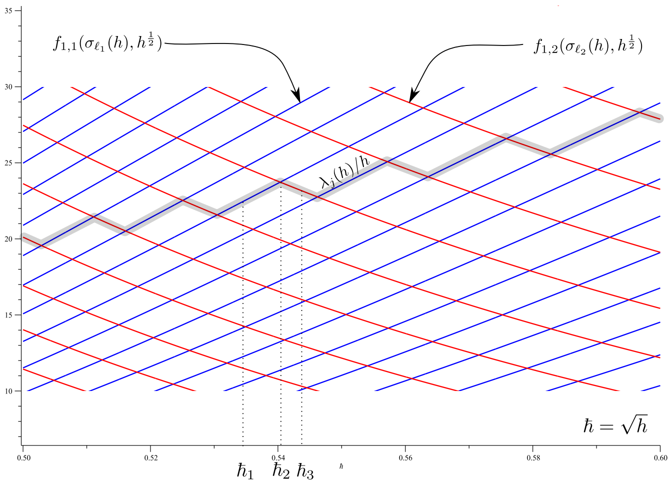

Figure 2. Illustration of the collection of semiclassical branches of eigenvalues. Here we plot the graphs of $f_{1,q}(\sigma_\ell(h), h)$ with respect to the variable $\hbar = \sqrt{h}$, for $q = 1$ (blue curves) and $q = 2$ (red curves). The continuous curve of $\lambda_j(h)/h$, for fixed j, where $\lambda_j(h)$ is the exact eigenvalue of $\mathcal{L}_h$, lies within the greyed stair-case like curve (of vertical width $\mathcal{O}(h)$).

deformed grid intersected with the window $(0, h_0] \times [ha, hb]$, see Figure 2. In particular, there are many crossing points, and the horizontal distance between consecutive crossing points along a fixed branch is $\mathcal{O}(h^2)$.

Consider now the exact eigenvalues $\lambda_j \in [ha, hb]$. By Corollary 1.10, each λ_j must be $\mathcal{O}(h^2)$-close to one of the semiclassical branches. For fixed $\ell \in \mathbf{Z}_-$, modifying the value of h by an amount of order $\mathcal{O}(h^2)$ amounts to shifting the abscissa $\sigma_\ell(h)$ by an amount proportional to $h^{\frac{1}{2}}$. Therefore, by suitably choosing C_1 and setting $h_1 := h + C_1 h^2$ we may assume that $\lambda_j(h_1)$ corresponds to a unique increasing branch (parameterized by $\Sigma_{1,1}$): when h varies in an interval of size $\mathcal{O}(\varepsilon h^2)$ around h_1, with $\varepsilon > 0$ small enough, there is a unique and fixed $\ell_1 \in \mathbf{Z}$ such that

$$\left|\lambda_j(h) - h f_{1,1}\left(\sigma_{\ell_1}(h), h^{\frac{1}{2}}\right)\right| = \mathcal{O}(h^2).$$

Next, we choose $C_2 > C_1$ so that with $h_2 := h + C_2 h^2$, $\sigma_{\ell_1}(h_2)$ is $\mathcal{O}(h^3)$ close to the first crossing on the right-hand side of $\sigma_{\ell_1}(h_1)$. The exponent 3 is not important, any exponent $N \geqslant 3$ will work as well. We have

$$\left|\lambda_j(h_2) - h_2 f_{1,1}\left(\sigma_{\ell_1}(h_2), h_2^{\frac{1}{2}}\right)\right| = \mathcal{O}(h^2).$$

and since $|f'_{1,1}| \geqslant c$, we obtain a constant $C > 0$ such that

$$\lambda_j(h_2) \geqslant \lambda_j(h_1) + C h^{3/2}.$$

On the right hand side of the crossing, the integer ℓ_1, and the increasing branch, do not longer correspond to the eigenvalue λ_j (this branch will now correspond to λ_{j+1}). Instead, we have to select the branch parameterized by $\Sigma_{1,2}$, labeled by some $\ell_2 \in \mathbf{Z}_-$;

then, as before, with a suitable $C_3 > C_2$, we have, with $h_3 := h + C_3 h^2$

$$\big|\lambda_j(h_3) - h_3 f_{1,2}\big(\sigma_{\ell_2}(h_3), h_3^{\frac{1}{2}}\big)\big| = \mathcal{O}(h^2),$$

and hence, since the new branch is now decreasing,

$$\lambda_j(h_3) \leqslant \lambda_j(h_2) - C h^{3/2}.$$

Note that, in the above analysis, the constants C_j depend on h, but in a uniform way: they belong to a fixed compact interval contained in $(0, +\infty)$. The above estimates are then uniform for $h \leqslant h_0$ if h_0 is chosen small enough.

We now turn to the last statement of the theorem. We choose h_2 as before, but with more precision: we can always select the exact crossing point h' between the semiclassical branches, i.e.,

$$h' f_{1,1}\big(\sigma_{\ell_1}(h'), h'^{\frac{1}{2}}\big) = h' f_{1,2}\big(\sigma_{\ell_2}(h'), h'^{\frac{1}{2}}\big).$$

This gives

$$\lambda_j(h') - \lambda_{j+1}(h') = \mathcal{O}(h^2).$$

Finally, for any value of h sufficiently far from the crossing, for instance $h'' = h_1$ or h_3, the vertical estimate (5.8) ensures that

$$\lambda_{j+1}(h'') - \lambda_j(h'') \geqslant C h^{3/2},$$

for some $C > 0$, which finishes the proof of the theorem.

A. A rough Weyl estimate

The aim of this section is recall why (1.3) holds. Thanks to the Young inequality, we have, for all $\psi \in H^1(\Omega)$,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geqslant \frac{h^2}{2}\|\nabla\psi\|^2 - 2\|A\|_\infty^2\|\psi\|^2 + \gamma h^{\frac{3}{2}} \int_{\partial\Omega} |\psi|^2 \mathrm{d}s.$$

When $\gamma \geqslant 0$, we get that

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geqslant \frac{h^2}{2}\|\nabla\psi\|^2 - 2\|A\|_\infty^2\|\psi\|^2.$$

When $\gamma < 0$, we use a classical trace theorem: there exists $C > 0$ such that, for all $\varepsilon > 0$,

$$\int_{\partial\Omega} |\psi|^2 \mathrm{d}s \leqslant \varepsilon\|\nabla\psi\|^2 + C\varepsilon^{-1}\|\psi\|^2.$$

By choosing $\varepsilon = -\frac{\sqrt{h}}{4\gamma} > 0$, we deduce that

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geqslant \frac{h^2}{4}\|\nabla\psi\|^2 - 2\|A\|_\infty^2\|\psi\|^2 - 4\gamma^2 C h\|\psi\|^2.$$

In both cases, there exists $\widetilde{C} > 0$ such that, for all $h \in (0,1)$ and all $\psi \in H^1(\Omega)$,

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geqslant \frac{h^2}{4}\|\nabla\psi\|^2 - \widetilde{C}\|\psi\|^2.$$

With the min-max principle, this shows that, for all λ,

$$N(\mathcal{L}_h,\lambda) \leqslant N\Big(-\Delta^{\mathrm{Neu}}, 4\frac{\lambda+\widetilde{C}}{h^2}\Big).$$

The conclusion follows from the Weyl asymptotics for the Neumann Laplacian, which is the same at the main order as in the Dirichlet case, see, for instance, [34, Introduction].

B. Spectral analysis of De Gennes operator

Lemma B.1. *For each $\gamma \in \mathbf{R}$, $n \geqslant 2$, we have*

$$\mu_n(\gamma,\sigma) > 2n-3.$$

In particular, we have

$$\Theta^{[n-1]}(\gamma) > 2n-3.$$

Proof. From the Sturm–Liouville theory, $u_n^{[\gamma,\sigma]}$ admits $n-1$ zeros on $\mathbf{R}_+$. We denote by $z_{n,1}(\gamma,\sigma)$ its first zero. We consider the function

$$v_n^{[\gamma,\sigma]}(t) = u_n^{[\gamma,\sigma]}\big(t + z_{n,1}(\gamma,\sigma)\big),$$

which satisfies the Dirichlet condition at 0 and

$$H^{\mathrm{Dir}}\big[\sigma - z_{n,1}(\gamma,\sigma)\big]v_n^{[\gamma,\sigma]} = \mu_n(\gamma,\sigma)v_n^{[\gamma,\sigma]},$$

where $H^{\mathrm{Dir}}[\sigma]$ is the Dirichlet realization of $-\partial_\tau^2 + (\sigma-\tau)^2$ on $L^2(\mathbf{R}_+)$. The function $v_n^{[\gamma,\sigma]}$ has exactly $n-2$ zeros on $\mathbf{R}_+$. By the Sturm's oscillation theorem, $v_n^{[\gamma,\sigma]}$ is the $(n-1)$-th eigenfunction of $H^{\mathrm{Dir}}[\sigma - z_{n,1}(\gamma,\sigma)]$. Therefore we have

$$\mu_n(\gamma,\sigma) = \mu_{n-1}^{\mathrm{Dir}}\big(\sigma - z_{n,1}(\gamma,\sigma)\big).$$

Moreover, by monotonicity of the Dirichlet problem, for all $\sigma \in \mathbf{R}$,

$$\mu_{n-1}^{\mathrm{Dir}}(\sigma) > 2n-3. \qquad \blacksquare$$

The following proposition is obtained by adapting the proof of [25, Theorem II.2].

Proposition B.2. *Let $n \geqslant 1$. If σ is a critical point of $\mu_n(\gamma,\cdot)$, we have*

$$\mu_n(\gamma,\sigma) = \sigma^2 - \gamma^2.$$

Lemma B.3. *When $\gamma \in \mathbf{R}$, we have the following relations*

$$\int_0^{+\infty} \big(t - \xi_{n-1}(\gamma)\big)\big|u_n^{[\gamma,\xi_{n-1}(\gamma)]}(t)\big|^2 \mathrm{d}t = 0, \tag{B.1}$$

$$\int_0^{+\infty} \big(t - \xi_{n-1}(\gamma)\big)^3\big|u_n^{[\gamma,\xi_{n-1}(\gamma)]}(t)\big|^2 \mathrm{d}t = \frac{1}{6}\big[1 + 2\gamma\xi_{n-1}(\gamma)\big]u_n^{[\gamma,\xi_{n-1}(\gamma)]}(0). \tag{B.2}$$

Proof. We let

$$u_n^{[\gamma]} = u_n^{[\gamma,\xi_{n-1}(\gamma)]}.$$

Let us consider the differential operator:

$$L = -\partial_t^2 + \big(t - \xi_{n-1}(\gamma)\big)^2 - \Theta^{[n-1]}(\gamma).$$

Note that for any polynomial p, we have:

$$Lv = \Big(p^{(3)} - 4\big[\big(t - \xi_{n-1}(\gamma)\big)^2 - \Theta(\gamma)^{[n-1]}\big]p' - 4\big(t - \xi_{n-1}(\gamma)\big)p\Big)u_n^{[\gamma]}, \tag{B.3}$$

and

$$\int_0^{+\infty} u_n^{[\gamma]}(t)(Lv)(t)\mathrm{d}t = \int_0^{+\infty} Lu_n^{[\gamma]}(t)v(t)\mathrm{d}t + \big(v'(0) - \gamma v(0)\big)u_n^{[\gamma]}(0), \tag{B.4}$$

for $v = 2p[u_n^{[\gamma]}]' - p'u_n^{[\gamma]}$. Taking $p = 1$, we get

$$-4\int_0^{+\infty} \big(t - \xi_{n-1}(\gamma)\big)\big|u_n^{[\gamma]}(t)\big|^2 \mathrm{d}t = 2\big(\xi_{n-1}(\gamma)^2 - \gamma^2 - \Theta^{[n-1]}(\gamma)\big)\big|u_n^{[\gamma]}(0)\big|^2.$$

Recalling Proposition B.2, the above formula proves (B.1). To prove (B.2), we take $p = (t - \xi_{n-1}(\gamma))^2$. Then, we have

$$v'(0) - \gamma v(0) = -2\big(2\gamma\xi_{n-1}(\gamma) + 1\big)u_n^{[\gamma]}(0).$$

We get now from (B.3) and (B.4)

$$-12\int_0^{+\infty} \big(t - \xi_{n-1}(\gamma)\big)^3\big|u_n^{[\gamma]}(t)\big|^2 \,\mathrm{d}t = -2\big(2\gamma\xi_{n-1}(\gamma) + 1\big)\big|u_n^{[\gamma]}(0)\big|^2. \qquad \blacksquare$$

Lemma B.4. *We have*

$$\begin{aligned} C_j\big(\xi_{j-1}(\gamma)\big) &= \frac{1}{2}\big(u_j^{[\gamma]}(0)\big)^2 - \int_0^{+\infty} \big(t - \xi_{j-1}(\gamma)\big)^3\big(u_j^{[\gamma]}(t)\big)^2 \mathrm{d}t \\ &= \frac{1}{3}\big(1 - \gamma\xi_{j-1}(\gamma)\big)\big(u_j^{[\gamma]}(0)\big)^2, \end{aligned}$$

where C_j is defined in (1.10).

Proof. We write

$$(\sigma - t)t^2 + 2t(\sigma - t)^2 = (t - \sigma)^3 - \sigma^2(t - \sigma).$$

We take $\sigma = \xi_{j-1}(\gamma)$. The conclusion follows from Lemma B.3. ∎

Proposition B.5. *Let us fix $j \geqslant 1$. When $\gamma \in \mathbf{R}$, there exists $\gamma_0^{[j-1]} > 0$, such that, $C_j(\xi_{j-1}(\gamma))$ is positive if $\gamma < \gamma_0^{[j-1]}$ and negative if $\gamma > \gamma_0^{[j-1]}$.*

Proof. We notice that, for $\gamma \leqslant 0$, we get $C_j(\xi_{j-1}(\gamma)) < 0$.

Now, for $\gamma > 0$. From Proposition B.2, we can rewrite $C_j(\xi_{j-1}(\gamma))$ as

$$C_j\big(\xi_{j-1}(\gamma)\big) = \frac{1}{3}\left(1 - \gamma\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}\right)\big(u_j^{[\gamma]}(0)\big)^2,$$

Since $(u_j^{[\gamma]}(0))^2 > 0$, then, to study the sign of $C_j(\xi_{j-1}(\gamma))$ it is sufficient to study the sign of the function f defined by $f(\gamma) = 1 - \gamma\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}$. We have

$$f'(\gamma) = -\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)} - \frac{\gamma}{2}\frac{\big(\Theta^{[j-1]}(\gamma)\big)' + 2\gamma}{\sqrt{\gamma^2 + \Theta^{[j-1]}(\gamma)}}.$$

We can use [25, Section B] (which can be adapted to $j \geqslant 1$) to deduce that $f'(\gamma) < 0$. Therefore, f is increasing on $[0, +\infty[$.

Let us notice now that $f(0) = 1$ and $\lim_{\gamma \to +\infty} f(\gamma) = -\infty$. This establishes the existence of a unique zero of $f(\gamma)$, denoted by $\gamma_0^{[j-1]}$. ∎

Lemma B.6. *Let $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{N}$. Consider the interval $[a, b]$. We consider $\Pi\psi = (\langle\psi, u_j^{[\gamma,\sigma]}\rangle)_{1\leqslant j\leqslant n}$, where n is the number of dispersion curves $\mu_j(\gamma, \sigma)$ taking values in $[a, b]$ (see the discussion at the beginning of Section 3.2). We consider $\widehat{K}$ a neighborhood of $\widetilde{K}$.*

There exists $C_{\alpha,\beta} > 0$ such that for all $z \in [a, b]$ and all $\sigma \in \widehat{K}$, the following holds. For all $v \in L^2(\mathbf{R}_+)$ such that $\langle t\rangle^\alpha v \in L^2(\mathbf{R}_+)$, we have

$$\big\|\langle t\rangle^{-\alpha}\partial_\sigma^\beta\big(H[\gamma, \sigma] - z\big)^{-1}(\Pi^*\Pi)^\perp\big(\langle t\rangle^\alpha v\big)\big\| \leqslant C_{\alpha,\beta}\|v\|.$$

Proof. Let us only prove this estimate for $\beta = 0$. We consider $z \in [a, b]$. Let us consider $v \in \mathcal{S}(\overline{\mathbf{R}}_+)$ and let u be the unique solution to the equation

$$\big(H[\gamma, \sigma] - z\big)u = (\Pi^*\Pi)^\perp\big(\langle\epsilon t\rangle_k^\alpha v\big) \tag{B.5}$$

that is orthogonal to $(u_j^{[\gamma,\sigma]})_{1\leqslant j\leqslant n}$, with

$$\langle t\rangle_k = \big(1 + t^2\chi_k^2\big)^{\frac{1}{2}},$$

where χ_k is a smooth non-negative function equal to 0 on $[0, 1]$ and to 1 on $[2k, +\infty)$ and such that $|\chi_k'| \leqslant k^{-1}$. In particular, the weight is 1 near 0. Here $\epsilon > 0$ is a parameter to be chosen small enough.

We have seen in Lemma 2.2 that

$$\begin{pmatrix} H[\gamma,\sigma]-z & \Pi^* \\ \Pi & 0 \end{pmatrix} : B^2(\mathbf{R}_+)\times\mathbf{C}^n \to L^2(\mathbf{R}_+)\times\mathbf{C}^n$$

is bijective. Thus, equation (B.5) is equivalent to

$$\begin{pmatrix} H[\gamma,\sigma]-z & \Pi^* \\ \Pi & 0 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} (\Pi^*\Pi)^{\perp}\big(\langle\epsilon t\rangle_k^{\alpha} v\big) \\ 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} \langle\epsilon t\rangle_k^{-\alpha} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} H[\gamma,\sigma]-z & \Pi^* \\ \Pi & 0 \end{pmatrix}\begin{pmatrix} \langle\epsilon t\rangle_k^{\alpha} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H[\gamma,\sigma]+R_{\epsilon,k}-z & \langle\epsilon t\rangle_k^{-\alpha}\Pi^* \\ \Pi\big(\,\cdot\,\langle\epsilon t\rangle_k^{\alpha}\big) & 0 \end{pmatrix},$$

where

$$R_{\epsilon,k} = -\alpha\epsilon^2(t^2\chi_k^2)'\frac{1}{1+\epsilon^2t^2\chi_k^2}\partial_t - \left(\frac{\epsilon^2\alpha}{2}\frac{(t^2\chi_k^2)''}{1+\epsilon^2t^2\chi_k^2} + \left(\frac{\alpha}{2}-1\right)\frac{\epsilon^4\alpha}{2}\frac{(t^2\chi_k^2)'^2}{(1+\epsilon^2t^2\chi_k^2)^2}\right).$$

With

$$u = \langle\epsilon t\rangle_k^{\alpha}\tilde{u},$$

we get

$$\mathcal{H}[\gamma,\sigma,\epsilon,k]\begin{pmatrix} \tilde{u} \\ 0 \end{pmatrix} = \begin{pmatrix} \langle\epsilon t\rangle_k^{-\alpha}(\Pi^*\Pi)^{\perp}\big(\langle\epsilon t\rangle_k^{\alpha} v\big) \\ 0 \end{pmatrix},$$

with

$$\mathcal{H}[\gamma,\sigma,\epsilon,k] = \begin{pmatrix} H[\gamma,\sigma]+R_{\epsilon,k}-z & \langle\epsilon t\rangle_k^{-\alpha}\Pi^* \\ \Pi\big(\,\cdot\,\langle\epsilon t\rangle_k^{\alpha}\big) & 0 \end{pmatrix}.$$

Thanks to the exponential decay of the $u_j^{[\gamma,\sigma]}$ (which is uniform for $\sigma\in\widehat{K}$), we notice that $\mathcal{H}[\gamma,\sigma,\epsilon,k]$ is bijective as soon as ϵ is small enough and k large enough. Moreover,

$$\big\|\mathcal{H}[\gamma,\sigma,\epsilon,k]^{-1}\big\| \leqslant C.$$

This implies that

$$\|\tilde{u}\| \leqslant C\big\|\langle\epsilon t\rangle_k^{-\alpha}(\Pi^*\Pi)^{\perp}\big(\langle\epsilon t\rangle_k^{\alpha} v\big)\big\|,$$

and then (by using again the exponential decay of the eigenfunctions)

$$\big\|\langle\epsilon t\rangle_k^{-\alpha}u\big\| \leqslant C\|v\|.$$

Taking the limit $k\to+\infty$, the Fatou lemma gives

$$\big\|\langle\epsilon t\rangle^{-\alpha}u\big\| \leqslant C\|v\|.$$

This provides us with the desired estimate since $\langle\epsilon t\rangle\langle t\rangle^{-1}\in[\epsilon,1]$. ■

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