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# Overgroups of exterior powers of an elementary group. Normalizers

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**Abstract.** We establish two characterizations of an algebraic group scheme $\bigwedge^m \mathrm{GL}_n$ over $\mathbb{Z}$. Geometrically, the scheme $\bigwedge^m \mathrm{GL}_n$ is a stabilizer of an explicitly given invariant form or, generally, an invariant ideal of forms. Algebraically, $\bigwedge^m \mathrm{GL}_n$ is isomorphic (as a scheme over $\mathbb{Z}$) to a normalizer of the elementary subgroup functor $\bigwedge^m \mathrm{E}_n$ and a normalizer of the subscheme $\bigwedge^m \mathrm{SL}_n$.

Our immediate goal is to apply both descriptions in the "sandwich classification" of overgroups of the elementary subgroup. Additionally, the results can be seen as a solution of the linear preserver problem for algebraic group schemes over $\mathbb{Z}$, providing a more functorial description that goes beyond geometry of the classical case over fields.

*In memory of Nikolai Aleksandrovich Vavilov,*
*our teacher, a brilliant mathematician, and a generous colleague.*

## 1. Introduction

The present work is a sequel of [18] where we have started the description of overgroups of exterior powers of an elementary group. In this paper, we carry out the second key step of the description: an explicit calculation of the normalizer of elementary groups in the corresponding general linear group.

In the case when $n$ is a multiple of $m$, we construct an $R$-linear form $f\colon V \times \cdots \times V \to R$ in $k$ variables, where $V = V(\varpi_m)$ and $R$ is an arbitrary commutative ring. We prove that $\bigwedge^m \mathrm{SL}_n$ coincides with the algebraic group $G_f$ of linear transformations preserving this form $f$:

$$G_f(R) := \big\{ g \in \mathrm{GL}_{\binom{n}{m}}(R) \mid f(gx^1, \ldots, gx^k) = f(x^1, \ldots, x^k) \big\}.$$

We deliver analogous description for $\bigwedge^m \mathrm{GL}_n$ in terms of the form $f$. Namely, this group scheme is equal to the stabilizer $\overline{G}_f$ of the ideal generated by the form $f$:

$$\overline{G}_f(R) := \big\{ g \in \mathrm{GL}_{\binom{n}{m}}(R) \mid g \text{ preserves the ideal } \langle f \rangle \big\}.$$

**Theorem 1.** *If $n/m$ is an integer greater than 2, then there are isomorphisms $\bigwedge^m \mathrm{SL}_n \cong G_f$, $\bigwedge^m \mathrm{GL}_n \cong \overline{G}_f$ of affine group schemes over $\mathbb{Z}$.*

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The theorem follows the traditional description of a Chevalley group as a stabilizer of a form and the corresponding extended Chevalley group as the stabilizer up to a scalar multiplier, see [6].

In the case when $n$ is not divisible by $m$, we construct an ideal $F$, a direct generalization of $\langle f \rangle$, such that $\bigwedge^m \mathrm{GL}_n$ coincides with a stabilizer of this ideal:

$$\overline{G}_F(R) := \left\{g \in \mathrm{GL}_{\binom{n}{m}}(R) \mid g \text{ preserves the ideal } F\right\}.$$

**Theorem 2.** *Using prior notation, $\bigwedge^m \mathrm{GL}_n$ and $\overline{G}_F$ are isomorphic as affine group schemes over $\mathbb{Z}$.*

Analogous description for the general case of $n$ and $m$ can be found in [29], as we discuss in Section 3. Indeed, the group scheme $\bigwedge^m \mathrm{GL}_n$ is a stabilizer of the Plücker ideal Plu generated by Plücker *quadratic* forms. However, our description goes further then just taking a subideal of Plu: ideal $F$ from Theorem 2 is a proper subideal of the radical $\sqrt{\mathrm{Plu}}$ with some nice properties.

In the theory of linear preserver problem and, more generally, in geometric invariant theory there exists a classic geometric interpretation of a normalizer $N_{\mathrm{GL}(V)}(G)$ of a group $G$ acting irreducibly on a vector space $V$: $N_{\mathrm{GL}(V)}(G)$ is equal to $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathcal{O})$, where $\mathcal{O}$ is a closed $G$-orbit in $\mathbb{P}(V)$ (we invite reader to consult [3, Theorem 3.2] and references there). Theorems 1 and 2 can be seen as an example of a *scheme-theoretic* incarnation of the statement. Authors hope to pursue this direction for wider class of groups in a future publication.

Now let $C, D$ be two subgroups of an abstract group $G$. Recall that the *transporter* of $C$ to $D$ is the set:

$$\mathrm{Tran}_G(C, D) = \{g \in G \mid C^g \le D\}.$$

We need a scheme-theoretic analogue [22, Section V.6]: *scheme-theoretic transporter* of $X$ to $Y$ inside an algebraic group $G$ is the functor $\mathrm{Tran}_G(X, Y)$ such that

$$\mathrm{Tran}_G(X, Y)(R) = \left\{g \in G(R) \mid z^g \in Y(\widetilde{R}) \text{ for all } R\text{-algebras } \widetilde{R} \text{ and } z \in X(\widetilde{R})\right\}.$$

The scheme-theoretic normalizer $N_G(X)$ is defined as a scheme-theoretic transporter $\mathrm{Tran}_G(X, X)$.

We denote the elementary subgroup of $\mathrm{GL}_n(R)$ by $\mathrm{E}_n(R)$ and the corresponding $m$-th exterior power of the elementary group by $\bigwedge^m \mathrm{E}_n(R)$. The following is our second result.

**Theorem 3.** *If $n \geqslant 4$ and $n/m$ is an integer greater than 2, then there are isomorphisms of the affine algebraic group schemes over $\mathbb{Z}$:*

$$N\left(\textstyle\bigwedge^m \mathrm{E}_n\right) \cong N\left(\textstyle\bigwedge^m \mathrm{SL}_n\right) \cong \mathrm{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{SL}_n\right) \cong \mathrm{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{GL}_n\right) \cong \textstyle\bigwedge^m \mathrm{GL}_n,$$

*where all scheme-theoretic normalizers and transporters are taken inside* $\mathrm{GL}_{\binom{n}{m}}$.

According to the results of [20] and forthcoming [19], we can replace the normalizers and transporters with their group-theoretic analogues for some classes of rings $R$. For example, $\mathrm{Tran}(\bigwedge^m \mathrm{E}_n(R), \bigwedge^m \mathrm{SL}_n(R))$ coincides with $\mathrm{Tran}(\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{SL}_n)(R)$ for algebras $R$ over infinite fields, see [20, Proposition 4.3]. In other words, the classic version of Theorem 3 with abstract transporters holds as well over these rings, see [28, 31, 32, Theorem 3], [27, Theorem 2], [1, Theorem 4] for analogues in other cases.

The paper is organized as follows. In Section 2 we present the basic notation. We recall the well-known description using the Plücker polynomials in Section 3, we construct an invariant form for $\bigwedge^m \mathrm{GL}_n$ for the case $n/m \in \mathbb{N}$ in Section 4, and, in Section 6, we generalize the latter description to an invariant system of forms for any $n, m$. Section 5 gives a geometric description of the quotient $\bigwedge^m \mathrm{GL}_n(R)$ by $\bigwedge^m(\mathrm{GL}_n(R))$. Finally, in Section 7 we discuss different notions of normalizers and transporters and prove Theorem 3.

## 2. Exterior powers of elementary groups

In this section, we introduce exterior powers of an elementary group and define the related concepts.

We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. *If there is no confusion, we denote the binomial coefficient* $\binom{n}{m}$ *by* $N$. Elements of $\bigwedge^m[n]$, the $m$-th exterior power of the set $[n]$, are ordered subsets $I \subseteq [n]$ of cardinality $m$ without repeating entries:

$$\textstyle\bigwedge^m[n] = \big\{(i_1, \ldots, i_m) \mid 1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n\big\}.$$

Let $R$ be a commutative ring and let $R^n$ be the right free $R$-module with the standard basis $\{e_1, \ldots, e_n\}$. $\bigwedge^m R^n$ is a free module of rank $N = \binom{n}{m}$ with the basis $e_{i_1} \wedge \cdots \wedge e_{i_m}$ with $(i_1, \ldots, i_m) \in \bigwedge^m[n]$. The products $e_{i_1} \wedge \cdots \wedge e_{i_m}$ are defined for an arbitrary set $\{i_1, \ldots, i_m\}$ via $e_{\sigma(i_1)} \wedge \cdots \wedge e_{\sigma(i_m)} = \mathrm{sgn}(\sigma)\, e_{i_1} \wedge \cdots \wedge e_{i_m}$ for $\sigma \in S_m$ a permutation of $[m]$. We can assume that $n \geqslant 2m$ due to the isomorphism $\bigwedge^m V^* \cong (\bigwedge^{\dim(V)-m} V)^*$ for an arbitrary free $R$-module $V$.

For every $m \leqslant n$, we have the Cauchy–Binet homomorphism $\bigwedge^m\colon \mathrm{GL}_n(R) \to \mathrm{GL}_N(R)$ defined via the diagonal action:

$$\textstyle\bigwedge^m(g)(e_{i_1} \wedge \cdots \wedge e_{i_m}) := (ge_{i_1}) \wedge \cdots \wedge (ge_{i_m}) \quad \text{for } e_{i_1}, \ldots, e_{i_m} \in R^n.$$

Thus $\bigwedge^m$ is a representation of the group $\mathrm{GL}_n(R)$. It is called the $m$-th *vector representation* or the $m$-th *fundamental representation*. The image group $\bigwedge^m(\mathrm{GL}_n(R))$ is called the $m$-th exterior power of the general linear group.

By $a_{i,j}$ we denote an entry of a matrix $a \in \mathrm{GL}_n(R)$ at the position $(i, j)$, where $1 \leqslant i, j \leqslant n$. Further, $e$ denotes the identity matrix and $e_{i,j}$ denotes the standard matrix unit, i.e., the matrix that has 1 at the position $(i, j)$ and zeros elsewhere. For entries of the inverse matrix we use the standard notation $a'_{i,j} := (a^{-1})_{i,j}$. The *[absolute] elementary group* $\mathrm{E}_n(R)$ is a subgroup of $\mathrm{GL}_n(R)$ generated by all elementary transvections $t_{i,j}(\xi) = e + \xi e_{i,j}$, where $1 \leqslant i \neq j \leqslant n$, $\xi \in R$. The set $\mathrm{E}^l(n, R)$ is a subset of $\mathrm{E}_n(R)$ consisting of products of at most $l$ elementary transvections. The exterior power of the elementary group $\bigwedge^m \mathrm{E}_n(R)$ is defined as the $\bigwedge^m$–image of the elementary group $\mathrm{E}_n(R)$.

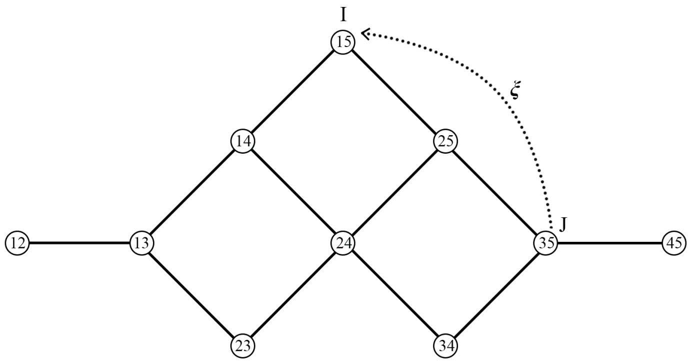



**Figure 1.** Weight diagram $(A_4, \varpi_2)$ and action of $t_{I,J}(\xi)$.

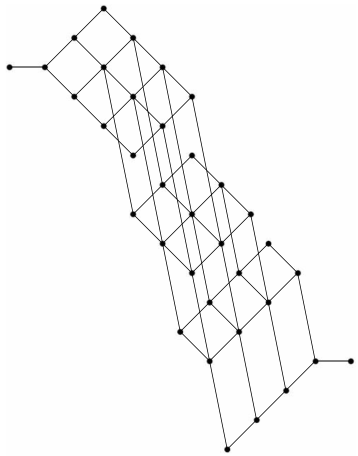


**Figure 2.** Weight diagram $(A_6, \varpi_3)$.

In the sequel, we use weight diagrams to illustrate internal combinatorics of equations. We refer the reader to [25] where the authors describe all the rules to construct weight diagrams. The exterior power of the elementary group $\bigwedge^m \mathrm{E}_n(R)$ corresponds to the representation of the Chevalley group of type $\Phi = A_{n-1}$ with the highest weight $\varpi_m$.

In the majority of existing constructions, $\bigwedge^m \mathrm{GL}_n(R)$ arises together with an action on the Weyl module $V(\varpi_m) = R^N$. We denote the weight set of the module $V(\varpi_m)$ by $\Lambda(\varpi_m)$. Then $\Lambda(\varpi_m) = \bigwedge^m [n]$.

Fix an admissible base $v^\lambda, \lambda \in \Lambda$ of the module $V = V(\varpi_m)$. We regard a vector $a \in V, a = \sum v^\lambda a_\lambda$, as a column of coordinates $a = (a_\lambda), \lambda \in \Lambda$.

In Figures 1 and 2 we reproduce the weight diagrams of the groups $\bigwedge^2 \mathrm{E}_5(R)$ and $\bigwedge^3 \mathrm{E}_7(R)$, which correspond to representations $(A_4, \varpi_2)$ and $(A_6, \varpi_3)$, respectively. We follow the convention of naturally ascending numbering of weights. On the diagrams, the highest weight is the leftmost one. Recall that in a weight diagram two weights are joined by an edge if their difference is a fundamental root.

The algebraic group scheme $\bigwedge^m \mathrm{GL}_n$ is, by definition, the *categorical* image of the group scheme $\mathrm{GL}_n$ under the Cauchy–Binet homomorphism. The group $\bigwedge^m \mathrm{GL}_n(R)$ is defined as $R$-points of the functor $\bigwedge^m \mathrm{GL}_n = \bigwedge^m \mathrm{GL}_n(\_)$. The [abstract] groups $\bigwedge^m(\mathrm{GL}_n(R))$ and $\bigwedge^m \mathrm{GL}_n(R)$ are different for a general ring $R$. We have a canonical inclusion

$$\bigwedge^m\big(\mathrm{GL}_n(R)\big) \leqslant \bigwedge^m \mathrm{GL}_n(R);$$

the quotient set is computed in Section 5.

Abstractly, elements of $\bigwedge^m \mathrm{GL}_n(R)$ are images of matrices under $\bigwedge^m$ with entries belonging to some extension of $R$. In other words, arbitrary element $\tilde{g} \in \bigwedge^m \mathrm{GL}_n(R)$ has the form $\tilde{g} = \bigwedge^m g$ where $g \in \mathrm{GL}_n(S)$ for some extension ring $S$ of $R$.

Below we show that the group $\bigwedge^m \mathrm{SL}_n(R)$ is the standard Chevalley group $G(\Phi, R)$, $\bigwedge^m \mathrm{GL}_n(R)$ is the extended Chevalley group $\overline{G}(\Phi, R)$, and $\bigwedge^m \mathrm{E}_n(R)$ coincides with the [absolute] elementary subgroup of $G(\Phi, R)$.

Recall that $\bigwedge^m \mathrm{E}_n(R)$ is normal not only in the image of the general linear group but in the bigger group $\bigwedge^m \mathrm{GL}_n(R)$. This fact follows from [24, Theorem 1].

**Theorem 4.** *Let $R$ be a commutative ring, $n \geqslant 3$, then*

$$\bigwedge^m \mathrm{E}_n(R) \trianglelefteq \bigwedge^m \mathrm{GL}_n(R).$$

We recall the explicit form of the exterior power of an elementary transvection [18] which we utilize later.

**Proposition 5.** *Let $t_{i,j}(\xi)$ be an elementary transvection in $\mathrm{E}_n(R)$, $n \geqslant 3$. Then*

$$\bigwedge^m t_{i,j}(\xi) = \prod_{L \in \bigwedge^{m-1}([n] \smallsetminus \{i,j\})} t_{L\cup i, L\cup j}\big(\mathrm{sgn}(L,i)\,\mathrm{sgn}(L,j)\xi\big) \qquad (2.1)$$

*for any $1 \leqslant i \neq j \leqslant n$.*

Similarly, one can get an explicit form of the torus elements $h_{\varpi_m}(\xi)$ of the group $\bigwedge^m \mathrm{GL}_n(R)$.

**Proposition 6.** *Let $d_i(\xi) = e + (\xi - 1)e_{i,i}$ be a torus generator, $1 \leqslant i \leqslant n$. Then the exterior power of $d_i(\xi)$ equals the diagonal matrix with diagonal entries 1 everywhere except in $\binom{n-1}{m-1}$ positions:*

$$\bigwedge^m\big(d_i(\xi)\big)_{I,I} = \begin{cases} \xi, & \text{if } i \in I, \\ 1, & \text{otherwise.} \end{cases}$$

As an example, consider $\bigwedge^3 t_{1,3}(\xi) = t_{124,234}(-\xi)t_{125,235}(-\xi)t_{145,345}(\xi) \in \bigwedge^3 \mathrm{E}_5(R)$ and $\bigwedge^4 d_2(\xi) = \mathrm{diag}(\xi, \xi, \xi, 1, \xi) \in \bigwedge^4 \mathrm{E}_5(R)$. It follows from the propositions that $\bigwedge^m t_{i,j}(\xi)$ belongs to $\mathrm{E}^{\binom{n-2}{m-1}}(N, R)$. In other words, the residue[1] of an exterior transvection

$$\mathrm{res}\,\big(\bigwedge^m t_{i,j}(\xi)\big)$$

equals the binomial coefficient $\binom{n-2}{m-1}$.

[1]The *residue* $\mathrm{res}(g)$ of a transformation $g$ is, by definition, the rank of $g - e$.

Let $I, J$ be two elements of $\bigwedge^m[n]$. We define a *distance* between $I$ and $J$ as the cardinality of the intersection $I \cap J$:

$$d(I, J) = |I \cap J|.$$

This combinatorial characteristic plays an analogous role of the distance function $d(\lambda, \mu)$ for roots $\lambda$ and $\mu$ on the weight diagram of a root system.

## 3. Stabilizer of the Plücker ideal

First we recall the well-known description of polyvector representations of the general linear group. In [29] the authors proved that $\bigwedge^m \mathrm{GL}_n(R)$ coincides with the stabilizer of the Plücker ideal.

Plücker polynomials are homogeneous quadratic polynomials

$$f_{I,J} \in \mathbb{Z}\big[x_H, H \in \textstyle\bigwedge^m[n]\big]$$

of Grassmann coordinates $x_H$. In general, Plücker polynomials can be represented in the form:

$$f_{I,J} = \sum_{j \in J \setminus I} \pm x_{I \cup \{j\}} x_{J \setminus \{j\}},$$

where $I \in \bigwedge^{m-1}[n]$ and $J \in \bigwedge^{m+1}[n]$. To clarify the sign of the factors, we extend the definition of the Grassmann coordinates as follows. If there are coinciding elements in the set $\{i_1, \ldots, i_m\}$, then $x_{i_1 \cdots i_m} = 0$; otherwise $x_{i_1 \cdots i_m} = \mathrm{sgn}(i_1, \ldots, i_m) x_{\{i_1 \cdots i_m\}}$. Thus the Plücker polynomials have the form:

$$f_{I,J} = \sum_{h=1}^{m+1} (-1)^h x_{i_1 \cdots i_{m-1} j_h} x_{j_1 \cdots \hat{j}_h \cdots j_{m+1}}.$$

A *Plücker ideal* $\mathrm{Plu} := \mathrm{Plu}_{n,m} \trianglelefteq R[x_I : I \in \bigwedge^m[n]]$ is generated by all Plücker relations $f_{I,J}$ with $I \in \bigwedge^{m-1}[n]$ and $J \in \bigwedge^{m+1}[n]$.

**Lemma 7.** *Let $R$ be an arbitrary commutative ring. The group $\bigwedge^m \mathrm{E}_n(R)$ preserves the Plücker ideal* Plu.

Following notation of the paper [29], we put $G_{nm}(R) := \mathrm{Fix}_R(\mathrm{Plu})$ for any commutative ring $R$, where $\mathrm{Fix}_R(\mathrm{Plu})$ is the set of $R$-linear transformations preserving the ideal Plu:

$$G_{nm}(R) := \big\{g \in \mathrm{GL}_N(R) \mid f(gx) \in \mathrm{Plu} \text{ for all } f \in \mathrm{Plu}\big\}.$$

**Lemma 8.** *For any $n, m$ the functor $R \mapsto \mathrm{Fix}_R(\mathrm{Plu})$ is an affine group scheme defined over $\mathbb{Z}$.*

Next results are classical known, see [6] and [35, Theorem 4]. Note that representation $\bigwedge^m$ is minuscule. Therefore it is irreducible and tensor indecomposable.

**Lemma 9.** *Let $K$ be an algebraically closed field. For any $n,m$ with $1\leqslant m\leqslant n-1$, the kernel of $\bigwedge^m$ for $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ equals $\mu_m$ and $\mu_d$ where $d = \gcd(n,m)$, respectively.*

**Lemma 10.** *As a subgroup of $\mathrm{GL}_N(K)$, the algebraic group $\bigwedge^m(\mathrm{GL}_n(K))$ is irreducible and tensor indecomposable. Moreover, except the case $n=2m\geqslant 4$, the group $\bigwedge^m(\mathrm{GL}_n(K))$ coincides with its normalizer. In the exceptional case, the group has index 2 in its normalizer.*

*The analogous result holds for $\bigwedge^m(\mathrm{SL}_n(K))$ as a subgroup of $\mathrm{SL}_N(K)$.*

Using the classification of maximal subgroups in classical groups by Gary Seitz [26, Table 1] (see also the survey [5] with corrections), it is easy to prove that $\bigwedge^m \mathrm{SL}_n(K)$ is maximal for an algebraically closed field $K$. The following statement is [29, Lemma 7].

**Lemma 11.** *Let $K$ be an algebraically closed field. For any $n, m$, $1 \leqslant m \leqslant n-1$ the groups $\bigwedge^m \mathrm{GL}_n(K)$ and $\bigwedge^m \mathrm{SL}_n(K)$ are maximal among connected closed subgroups in one of the following groups:*

| $\bigwedge^m \mathrm{GL}_n(K)$: | | $\bigwedge^m \mathrm{SL}_n(K)$: | |
|---|---|---|---|
| • *in* $\mathrm{GL}_N(K)$, | *if* $n \neq 2m$; | • *in* $\mathrm{SL}_N(K)$, | *if* $n \neq 2m$; |
| • *in* $\mathrm{GSp}_N(K)$, | *if* $n = 2m$ & *odd* $m$; | • *in* $\mathrm{Sp}_N(K)$, | *if* $n = 2m$ & *odd* $m$; |
| • *in* $\mathrm{GO}_N^0(K)$, | *if* $n = 2m$ & *even* $m$. | • *in* $\mathrm{SO}_N(K)$, | *if* $n = 2m$ & *even* $m$. |

*Besides, in the exceptional cases these classical groups are unique proper connected overgroups of $\bigwedge^m \mathrm{GL}_n(K)$ and $\bigwedge^m \mathrm{SL}_n(K)$, respectively.*

**Corollary 12.** *Suppose $K$ is an algebraically closed field; then $\bigwedge^m \mathrm{GL}_n(K) = G_{nm}^0(K)$.*

Finally, for the coincidence of the group schemes, we must prove that $G_{nm}$ is smooth or, what is essentially the same, to calculate the dimension of the Lie algebra $\mathrm{Lie}(G_{nm})$.

**Lemma 13.** *For any field $K$ the dimension of the Lie algebra $\mathrm{Lie}(G_{nm,K})$ does not exceed $n^2$.*

Using [36, Theorem 1.6.1], we get the following result.

**Theorem 14.** *For any $n, m$, $1 \leqslant m \leqslant n-1$ there is an isomorphism of affine groups schemes over $\mathbb{Z}$:*

$$G_{nm} \cong \begin{cases} \mathrm{GL}_n/\mu_m, & \text{if } n \neq 2m, \\ \mathrm{GL}_n/\mu_m \leftthreetimes \mathbb{Z}/2\mathbb{Z}, & \text{if } n = 2m. \end{cases}$$

## 4. Exterior powers as the stabilizer of invariant forms I

Next we present an alternative description of $\bigwedge^m \mathrm{GL}_n(R)$ as a stabilizer of a form. Analogous forms are well known for classical and exceptional groups in the standard representation over an arbitrary ring, see [27, 28, 30–32]. Conveniently for the reader, a general approach was developed by Skip Garibaldi and Robert Guralnick [12, 13]. We also refer to [2, Section 4.4] where the author constructed cubic invariant forms for $\bigwedge^m \mathrm{SL}_n$.

The following theorem is classically known and can be found in [9, Chapter 2, Sections 5–7] for characteristic 0 and can be deduced from [7, 33] as all primes are *almost very good* in type $A_n$ or, nicely summarized, [23, Theorem 1 (4)] for fields of positive characteristic.

**Proposition 15.** *Let $K$ be an algebraically closed field. Then $\bigwedge^m \mathrm{GL}_n(K)$ is a group of similarities of a form only in the case $n/m \in \mathbb{N}$ and $n/m \geqslant 3$. Moreover, this form is unique in the space of $n/m$-tensors and it is equal to*

- $q^m_{[n]}(x) = \sum \mathrm{sgn}(I_1, \dots, I_{\frac{n}{m}})\, x_{I_1} \cdots x_{I_{\frac{n}{m}}}$ *for even $m$;*
- $q^m_{[n]}(x) = \sum \mathrm{sgn}(I_1, \dots, I_{\frac{n}{m}})\, x_{I_1} \wedge \cdots \wedge x_{I_{\frac{n}{m}}}$ *for odd $m$,*

*where the sums in the both cases range over all unordered partitions of the set $[n]$ into $m$-element subsets $I_1, \dots, I_{\frac{n}{m}}$.*

Henceforth, we use the uniform notation $q(x)$ for these forms and we assume that $m$ is even (unless otherwise specified); the case of odd $m$ can be addressed analogously.

So in the case of an algebraically closed field $K$, the abstract group $\bigwedge^m \mathrm{GL}_n(K)$ consists of matrices $g \in \mathrm{GL}_N(K)$ for which there is a multiplier function $\lambda = \lambda(g) \in K^*$ such that $q(gx) = \lambda(g)q(x)$ for all $x \in K^N$. The calculation of $\lambda$ on a generic diagonal matrix $d_i(\xi) \in \mathrm{GL}_n(K)$ shows that $\lambda(g) = \det(g)$. Since the coefficients of these forms equal $\pm 1$, the forms are defined over $\mathbb{Z}$. The same calculation confirms the answer over an arbitrary ring:

$$q(\textstyle\bigwedge^m g \cdot x) = \det(g) \cdot q(x) \quad \text{for } g \in \mathrm{GL}_n(R).$$

To get a direct analog of Proposition 15 over arbitrary rings, we change our focus from forms of high degree to the corresponding multilinear forms. Concretely, let

$$k := \frac{n}{m} \in \mathbb{N},$$

then a [full] *polarization* for the forms $q(x) = q^m_{[n]}(x)$ is a $k$-linear form $f^m_{[n]}$:

$$f(x) = f^m_{[n]}(x^1, \dots, x^k) = \sum \mathrm{sgn}(I_1, \dots, I_k)\, x^1_{I_1} \cdots x^k_{I_k},$$

where the sum ranges over all *ordered* partitions of the set $[n]$ into $m$-element subsets.

**Proposition 16.** *Let $R$ be an arbitrary commutative ring and $n/m \in \mathbb{N}$. The form $f$ is invariant under the action of $\bigwedge^m \mathrm{E}_n(R)$ and it is multiplied by $\xi$ under the action of a weight element $\bigwedge^m d_i(\xi)$.*

*Proof.* As we noted previously, the multiplier $\lambda(g)$ is equal to the determinant. Indeed, $\lambda(g)$ is a one-dimensional representation, i.e., is a homomorphism $\mathrm{GL}_n(R) \to \mathrm{GL}_1(R)$. Moreover, $\lambda(g)$ is a polynomial map that equals the determinant of $g$ over $\mathbb{C}$. Thus $\lambda(g) = \det(g)$ for an arbitrary ring $R$. And then the statement is obvious. But below we prove the proposition by direct calculation.

We show that

$$f(gx^1, \ldots, gx^k) = \xi f(x^1, \ldots, x^k),$$

where $g = \bigwedge^m d_i(\xi)$. Since $I_1, \ldots, I_k$ is an ordered partition of $[n]$, the number $i$ belongs to the index of only one variable $x^l_{I_l}$ in every monomial $x^1_{I_1} \cdots x^k_{I_k}$ of the form $f$. Thus every monomial of $f(gx^1, \ldots, gx^k)$ has the form $\pm x^1_{I_1} \cdots x^{l-1}_{I_{l-1}} \xi x^l_{I_l} x^{l+1}_{I_{l+1}} \cdots x^k_{I_k}$.

Now let $g = \bigwedge^m t_{i,j}(\xi)$. By (2.1) the matrix $g$ is equal to the product of transvections $t_{iL,jL}(\operatorname{sgn}(i, L)\operatorname{sgn}(j, L)\xi)$ with $L \in \bigwedge^{m-1}([n] \setminus \{i, j\})$. Therefore exactly $\binom{n-2}{m-1}$ coordinates change in the vector $gx$, $x \in R^N$: $(gx)_{iL} = x_{iL} + \operatorname{sgn}(i, L)\operatorname{sgn}(j, L)\xi x_{jL}$. Then in the form $f(gx^1, \ldots, gx^k) - f(x^1, \ldots, x^k)$ all monomials have the form:

$$\pm x^1_{I_1} \cdots x^{l-1}_{I_{l-1}} \big(\operatorname{sgn}(i, L)\operatorname{sgn}(j, L)\xi x^l_{jL}\big) x^{l+1}_{I_{l+1}} \cdots x^k_{I_k},$$

where $I_l = iL$, $L \in \bigwedge^{m-1}([n] \setminus \{i, j\})$. Let $I_1, \ldots, I_k$ be a partition of $[n]$ where $I_l = iL_1$, $I_p = jL_2$, $L_1, L_2 \in \bigwedge^{m-1}([n] \setminus \{i, j\})$. Then the indices $\widetilde{I}_1, \ldots, \widetilde{I}_k$, where $\widetilde{I}_l = jL_1$, $\widetilde{I}_p = iL_2$, form a partition of $[n]$ as well. Therefore the sum of the corresponding monomials equals

$$\begin{aligned}&\operatorname{sgn}(I_1, \ldots, I_k) x^1_{I_1} \cdots x^p_{jL_2} \cdots x^{l-1}_{I_{l-1}} \big(\operatorname{sgn}(i, L_1)\operatorname{sgn}(j, L_1)\xi x^l_{jL_1}\big) x^{l+1}_{I_{l+1}} \cdots x^k_{I_k}\\ &\quad + \operatorname{sgn}(\widetilde{I}_1, \ldots, \widetilde{I}_k) x^1_{\widetilde{I}_1} \cdots x^l_{jL_1} \cdots x^{l-1}_{\widetilde{I}_{l-1}} \big(\operatorname{sgn}(i, L_2)\operatorname{sgn}(j, L_2)\xi x^p_{jL_2}\big) x^{l+1}_{\widetilde{I}_{l+1}} \cdots x^k_{\widetilde{I}_k}.\end{aligned}$$

It remains to check that the corresponding signs are opposite:

$$\operatorname{sgn}(I_1, \ldots, I_k)\operatorname{sgn}(i, L_1)\operatorname{sgn}(j, L_1) = -\operatorname{sgn}(\widetilde{I}_1, \ldots, \widetilde{I}_k)\operatorname{sgn}(i, L_2)\operatorname{sgn}(j, L_2).$$

Multiplying this equality by $\operatorname{sgn}(j, L_2)\operatorname{sgn}(j, L_1)$, we obtain

$$\operatorname{sgn}(I_1, \ldots, I_k)\operatorname{sgn}(i, L_1)\operatorname{sgn}(j, L_2) = -\operatorname{sgn}(\widetilde{I}_1, \ldots, \widetilde{I}_k)\operatorname{sgn}(i, L_2)\operatorname{sgn}(j, L_1).$$

And this is equivalent to

$$\operatorname{sgn}(I_1, \ldots, I_k) = -\operatorname{sgn}(\widetilde{I}_1, \ldots, \widetilde{I}_k),$$

where the indices $I_p, \widetilde{I}_p$ and $I_l, \widetilde{I}_l$ are unordered.

If $m$ is even, then this equality is equivalent to $\operatorname{sgn}(iL_1, jL_2) = -\operatorname{sgn}(jL_1, iL_2)$. Since $iL_1, jL_2$ and $jL_1, iL_2$ differ by an odd number of transpositions, the signs are opposite. Similarly, $I_1, \ldots, I_k$ and $\widetilde{I}_1, \ldots, \widetilde{I}_k$ differ by an odd number of transpositions for odd $m$. ∎

We denote the ring of all polynomials in (families of) variables

$$x^1 = \{x^1_I\}_{I \in \bigwedge^m [n]}, \ldots, x^k = \{x^k_I\}_{I \in \bigwedge^m [n]}$$

with $R$-coefficients by $R[x^1, \ldots, x^k]$. We consider a $\mathbb{Z}^k$-grading on this ring given by sums of degrees in each of the families $x^1, \ldots, x^k$, e.g., the form

$$f = f^m_{[n]}(x^1, \ldots, x^k)$$

has grading $(1, \dots, 1)$ as exactly one of variables from each families appears in each monomial of $f$. The submodule of all forms with grading $(1, \dots, 1)$ we denote by

$$R[x^1, \dots, x^k]_{(1,\dots,1)}.$$

Applying the calculations similar to the previous proof, we get the uniqueness result for $\bigwedge^m \mathrm{E}_n(R)$–semi-invariant forms.

**Proposition 17.** *Let $R$ be an arbitrary ring and suppose $n/m \in \mathbb{N}$. Then every $\bigwedge^m \mathrm{E}_n(R)$–semi-invariant form in the space of multilinear forms $R[x^1, \dots, x^k]_{(1,\dots,1)}$ is a multiple of $f = f^m_{[n]}(x^1, \dots, x^k)$.*

*Proof.* Consider arbitrary $F(x^1, \dots, x^k) = \sum a_{I_1,\dots,I_k} x^1_{I_1} \cdots x^k_{I_k} \in R[x^1, \dots, x^k]_{(1,\dots,1)}$.

We first prove that for each nonzero $a_{I_1,\dots,I_k}$ the indices $I_1, \dots, I_k$ form a partition of $[n]$. Assume that there exists $j \in [n]$ such that $j \notin I_1 \cup \dots \cup I_k$ for some tuple $(I_1, \dots, I_k)$ with $a_{I_1,\dots,I_k} \neq 0$. Choose an arbitrary $i$ appearing in at least one $I_i$; without loss of generality, $I_1 = iL_1$. Action by $\bigwedge^m t_{ij}(\zeta)$ on the monomial $a_{I_1,\dots,I_k} x^1_{I_1} \cdots x^k_{I_k}$ contains the monomial $\pm\zeta \cdot a_{I_1,\dots,I_k} x^1_{jL_1} \cdots x^k_{I_k}$. This monomial appears only for

$$a_{I_1,\dots,I_k} x^1_{I_1} \cdots x^k_{I_k}$$

due to the conditions on $j$. We get a contradiction with the semi-invariancy of $F$, so each $j \in [n]$ appears in at least one $I_1, \dots, I_k$.

As each $I_i$ has cardinality $m$, the cardinality of their union is at most $m \cdot k = n$. Therefore $a_{I_1,\dots,I_k} \neq 0$ implies that $\{I_i\}$ forms a (non-intersecting) partition of $[n]$.

For $a_{I_1,\dots,I_k} \neq 0$, we take arbitrary $i \in I_1$ with $I_1 = iL_1$ and $j \in I_2$ with $I_2 = jL_2$. Then action of $\bigwedge^m t_{ij}(1)$ on $a_{I_1,\dots,I_k} x^1_{I_1} \cdots x^k_{I_k}$ has the form

$$a_{I_1,\dots,I_k} x^1_{I_1} \cdots x^k_{I_k} + \mathrm{sgn}(j, L_1) \cdot a_{I_1,\dots,I_k} x^1_{jL_1} x^2_{I_2} \cdots x^k_{I_k}.$$

The latter term does not appear in $F$ as $jL_1 \cap I_2 = j$, therefore we need to cancel it out to get the semi-invariancy. Then

$$\mathrm{sgn}(j, L_1) \cdot a_{I_1,\dots,I_k} x^1_{jL_1} x^2_{I_2} \cdots x^k_{I_k}$$

is forced to be equal to $-\mathrm{sgn}(j, L_2) \cdot a_{jL_1,iL_2,\dots,I_k} x^1_{jL_1} x^2_{jL_2} \cdots x^k_{I_k}$ coming from the action on the monomial $a_{jL_1,iL_2\dots,I_k} x^1_{jL_1} x^2_{iL_2} \cdots x^k_{I_k}$. In other words, for every $i \neq j$ from the disjoint partition $iL_1 \sqcup jL_2 \sqcup \dots \sqcup I_k = [n]$ we get the equation:

$$\mathrm{sgn}(j, L_1) \cdot a_{iL_1,jL_2,\dots,I_k} + \mathrm{sgn}(j, L_2) \cdot a_{jL_1,iL_2,\dots,I_k} = 0.$$

Thus the final step of Proposition 16 proof implies that every non-zero $a_{I_1,\dots,I_k}$ coincides with $\mathrm{sgn}(I_1, \dots, I_k) \cdot a$ for some shared $a \in R$. ■

Let us define a group $G_f(R)$ as the group of linear transformations preserving the form $f(x^1, \dots, x^k)$:

$$G_f(R) := \{g \in \mathrm{GL}_N(R) \mid f(gx^1, \dots, gx^k) = f(x^1, \dots, x^k)\}.$$

It is an analogue of the Chevalley group for the exterior powers. We define an analogue of the extended Chevalley group:

$$\overline{G}_f(R) := \big\{g \in \mathrm{GL}_N(R) \mid \text{ there exists } \lambda = \lambda(g) \in R^* \text{ such that } f(gx^1, \dots, gx^k) = \lambda(g) f(x^1, \dots, x^k)\big\}.$$

The functors $R \mapsto \overline{G}_f(R)$ and $R \mapsto G_f(R)$ define affine group schemes over $\mathbb{Z}$. Combining Proposition 16 and the reasonings before it for all rings $R$, we have the morphism of group schemes:

$$\iota\colon \textstyle\bigwedge^m \mathrm{GL}_n \longrightarrow \overline{G}_f \quad \text{ or, after Theorem 14,} \quad \iota\colon \mathrm{GL}_n / \mu_m \longrightarrow \overline{G}_f.$$

Ideally, we can expect the group $\bigwedge^m \mathrm{GL}_n(R)$ to coincide with $\overline{G}_f(R)$ (and $\bigwedge^m \mathrm{SL}_n(R)$ to coincide with $G_f(R)$) in the case $n/m \in \mathbb{N}$. Theorem 1 is a precise form of the expectation:

**Theorem 18.** *If $n/m$ is an integer greater than 2, then the group $\bigwedge^m \mathrm{GL}_n(R)$ coincides with $\overline{G}_f(R)$, and $\bigwedge^m \mathrm{SL}_n(R)$ coincides with $G_f(R)$ for an arbitrary ring $R$.*

**Remark 19.** If $n = 2m$ and 2 is not a zero-divisor, then $\overline{G}_f(R) = \mathrm{GO}_N(R)$ or $\mathrm{GSp}_N(R)$ depending on the parity of $m$. So in this case $\bigwedge^m \mathrm{GL}_n(R)$ is a subgroup of the orthogonal or the symplectic group, respectively. Moreover, if $(n, m) = (4, 2)$, then $\mathrm{GO}_6(R)$ equals $\bigwedge^2 \mathrm{GL}_4(R)$.

In general case, stabilizer of a quadratic form and its polarization do not coincide. Therefore, we only have the inclusion $\mathrm{GO}_N(R) \leqslant \overline{G}_f(R)$ or $\mathrm{GSp}_N(R) \leqslant \overline{G}_f(R)$.

The *proof* of the theorem follows the classic Waterhouse Lemma [36, Theorem 1.6.1]. This result essentially reduces the verification of an isomorphism of affine group schemes to the isomorphism of their groups of points over algebraically closed fields and the dual numbers[2] over such fields.

We note that an alternative proof based on [8, Exp. VI_B, Corollary 2.6] can be developed, but we do not pursue this direction here.

**Lemma 20.** *Let $G$ and $H$ be affine group schemes of finite type over $\mathbb{Z}$ where $G$ is flat, and let $\varphi\colon G \to H$ be a morphism of group schemes. Assume that the following conditions are satisfied for any algebraically closed field $K$:*

(1) $\dim(G_K) \geqslant \dim_K(\mathrm{Lie}(H_K))$,

(2) *$\varphi$ induces monomorphisms of the groups of points*

$$G(K) \longrightarrow H(K) \quad \text{and} \quad G\big(K[\delta]\big) \longrightarrow H\big(K[\delta]\big),$$

(3) *the normalizer $\varphi(G^0(K))$ in $H(K)$ is contained in $\varphi(G(K))$.*

[2] Recall that the algebra $K[\delta]$ of dual numbers over a field is isomorphic as a $K$-module to $K \oplus K\delta$ with multiplication given by $\delta^2 = 0$.

*Here $G^0$ denotes the connected component of the identity in $G$, $G_K$ denotes the extension of scalars of $G$, and* $\mathrm{Lie}(H_K)$ *denotes the Lie algebra of the scheme $H_K$.*

*Then $\varphi$ is an isomorphism of group schemes over $\mathbb{Z}$.*

In the case under consideration, the preliminary assumptions on the schemes are satisfied. Indeed, the schemes are of finite type being subschemes of appropriate $\mathrm{GL}_n$. The flatness condition follows from smoothness of the Chevalley–Demazure scheme $G$. All groups $G_K^0$ are smooth connected schemes of the same dimension. Moreover, we showed in the previous section that the normalizer of $\bigwedge^m \mathrm{GL}_n(K)$ in $\mathrm{GL}_N(K)$ coincides with $\bigwedge^m \mathrm{GL}_n(K)$. Thus condition (3) holds true.

As we mentioned above, Theorem 14 shows that instead of a morphism $\mathrm{GL}_n / \mu_m \to \overline{G}_f$ we can consider the morphism (which we call $\iota$ as well) $\bigwedge^m \mathrm{GL}_n \to \overline{G}_f$. Then Proposition 16 shows that $\bigwedge^m \mathrm{E}_n(R)$ is a subgroup of $\overline{G}_f(R)$ (as abstract groups) for any ring $R$. A standard argument shows that $\bigwedge^m \mathrm{E}_n(R)$ is dense in $\bigwedge^m \mathrm{GL}_n(R)$ for any local ring $R$. Therefore $\iota$ is a monomorphism for any local ring $R$. So condition (2) follows.

For $R = K$, an algebraically closed field, we can prove an even stronger statement:

**Proposition 21.** *Suppose $K$ is an algebraically closed field and $n \neq 2m$; then*

$$\textstyle\bigwedge^m \mathrm{GL}_n(K) = \overline{G}_f^0(K) \quad \text{and} \quad \bigwedge^m \mathrm{SL}_n(K) = G_f(K).$$

*Proof.* The group $\bigwedge^m \mathrm{GL}_n(K)$ preserves the invariant form $f(x^1, \dots, x^k)$ by Proposition 15, thus $\bigwedge^m \mathrm{GL}_n(K) \leqslant \overline{G}_f(K)$. Since $\bigwedge^m \mathrm{GL}_n(K)$ is connected, we have

$$\textstyle\bigwedge^m \mathrm{GL}_n(K) \leqslant \overline{G}_f^0(K).$$

Further, from Lemma 11 it follows that $\bigwedge^m \mathrm{GL}_n(K)$ is maximal among connected closed subgroups in $\mathrm{GL}_N(K)$. Since $\overline{G}_f(K)$ is a proper subgroup of $\mathrm{GL}_N(K)$, we obtain the reverse inclusion. For the group $\bigwedge^m \mathrm{SL}_n(K)$ the proof is similar. ∎

To deal with condition (1), it only remains to evaluate the dimension of the Lie algebras $\overline{G}_f$ and $G_f$. We follow the ideas of William Waterhouse [36, Lemmas 3.2, 5.3, and 6.3].

Let $K$ be an arbitrary field. Then Lie algebra $\mathrm{Lie}((G_f)_K)$ of an affine group scheme $(G_f)_K$ is most naturally interpreted as the kernel of homomorphism $G_f(K[\delta]) \to G_f(K)$ sending $\delta$ to 0, see [4, 15, 16, 34]. Practically, if $G$ is a subscheme of $\mathrm{GL}_n$, then $\mathrm{Lie}(G_K)$ consists of all matrices $e + z\delta$, $z \in \mathrm{M}_n(K)$, satisfying the equations defining $G(K)$. Formally, the statement takes the following form when $G$ is the stabilizer of a system of polynomials.

**Lemma 22.** *Let $\varphi_1, \dots, \varphi_s \in K[x_1, \dots, x_t]$. Then a matrix $e + z\delta$ with $z \in \mathrm{M}_t(K)$ belongs to* $\mathrm{Lie}(\mathrm{Fix}_K(\varphi_1, \dots, \varphi_s))$ *if and only if*

$$\sum_{1 \leqslant i,j \leqslant t} z_{ij} x_i \frac{\partial \varphi_h}{\partial x_j} = 0$$

*for all $h = 1, \dots, s$.*

To illustrate the argument that will be utilized for Theorem 23, we first provide an outline of the proof of Lemma 13 for scheme $G_{nm}$.

*Proof of Lemma* 13. We apply Lemma 22 to the case of the stabilizer of Plücker polynomials $f_{K,L}(x)$, where $K \in \bigwedge^{m-1}[n]$, $L \in \bigwedge^{m+1}[n]$. There are three types of equations on entries $z_{I,J}$, see [29, proof of Proposition 3]:

- $d(I,J) \leqslant m-2$, so we are in the case $|I \cup J| \geqslant m+2$, and then $z_{I,J} = 0$;
- $d(I,J) = d(M,H) = m-1$ and $I - J = H - M$, then $z_{I,J} = \pm z_{H,M}$;
- $d(I,J) = d(M,H) = m-1$ and $I - H = J - M$, then $z_{I,I} \pm z_{H,H} = \pm z_{J,J} \pm z_{M,M}$,

where we conceive indices $I \in \bigwedge^m[n]$ as roots of the corresponding representation, see the proof of Theorem 23 and the example next to this theorem for a detailed description of such approach.

The first case does not contribute to dimension of the Lie algebra. Matrix entries $z_{I,J}$ from the second case give the contribution equal to $n(n-1)$. And the third case contributes no more than $n$ linearly independent variables. Summing up, we get the upper bound equal to $n^2$. ■

We consider the schemes $G_f(K)$ and $\overline{G}_f(K)$. The Lie algebra $\mathrm{Lie}(G_f(K))$ consists of all matrices $g = e + y\delta$, $y \in \mathrm{M}_N(K)$, satisfying the condition

$$f(gx^1, \ldots, gx^k) = f(x^1, \ldots, x^k)$$

for all $x^1, \ldots, x^k \in K^N$. Similarly, $\mathrm{Lie}(\overline{G}_f(K))$ consists of all matrices $g = e + y\delta$ with $y \in \mathrm{M}_N(K)$ satisfying the condition $f(gx^1, \ldots, gx^k) = \lambda(g) f(x^1, \ldots, x^k)$ for all $x^1, \ldots, x^k \in K^N$.

**Theorem 23.** *If $n \neq 2m$, then for any field $K$ the dimension of the Lie algebra* $\mathrm{Lie}(\overline{G}_f(K))$ *does not exceed $n^2$, whereas the dimension of the Lie algebra* $\mathrm{Lie}(G_f(K))$ *does not exceed* $n^2 - 1$.

*Proof.* First observe that the conditions on elements of the Lie algebra $\mathrm{Lie}(G_f(K))$ are obtained from the corresponding conditions for elements of $\mathrm{Lie}(\overline{G}_f(K))$ by substituting $\lambda(g) = 1$. Let $g$ be a matrix satisfying the above conditions for all $x^1, \ldots, x^k \in K^N$. Plugging in $g = e + y\delta$ and using that the form $f$ is $k$-linear, we get

$$\delta\big(f(yx^1, x^2, \ldots, x^k) + \cdots + f(x^1, \ldots, x^{k-1}, yx^k)\big) = \big(\lambda(g) - 1\big) f(x^1, \ldots, x^k).$$

Now we show that the entries of the matrix $y$ are subject to exactly the same linear dependencies, as in the case $G_{nm}$. By definition $f(e_{I_1}, \ldots, e_{I_k}) = 0$ for all indices $I_1, \ldots, I_k \in \bigwedge^m[n]$, except the cases where $\{I_j\}$ is a partition of the set $[n] = I_1 \sqcup \cdots \sqcup I_k$.

- If $d(I,J) \leqslant m-2$, then $y_{I,J} = 0$. Indeed, in this case then there is a set of pairwise disjoint indices $I_2, \ldots, I_k \in \bigwedge^m([n] \smallsetminus I)$ such that $d(J, I_2) \geqslant 1$, $d(J, I_3) \geqslant 1$ and $d(J, I_4) = \cdots = d(J, I_k) = 0$. Put $x^1 := e_J$, $x^l := e_{I_l}$, $2 \leqslant l \leqslant k$. Then

$f(x^1, yx^2, \dots, x^k) = \cdots = f(x^1, x^2, \dots, yx^k) = 0$. It follows that

$$f(yx^1, x^2, \dots, x^k) = \pm y_{I,J} = 0.$$

- If $d(I, J) = m - 1$ and $I - J = H - M$, then $y_{I,J} = \pm y_{H,M}$. Here there is a set of pairwise disjoint indices $M, I_3, \dots, I_k \in \bigwedge^m([n] \smallsetminus I)$ such that $d(J, M) = 1$ and $d(J, I_3) = \cdots = d(J, I_k) = 0$. Put $x^1 := e_J, x^2 := e_M, x^l := e_{I_l}, 3 \leqslant l \leqslant k$ and denote by $H$ the index $[n] \smallsetminus (J \cup I_2 \cup \cdots \cup I_k)$. Then $f(x^1, x^2, yx^3, \dots, x^k) = \cdots = f(x^1, x^2, \dots, yx^k) = 0$. It follows that

$$f(yx^1, x^2, \dots, x^k) + f(x^1, yx^2, x^3 \dots, x^k) = 0.$$

  But $f(yx^1, x^2, \dots, x^k) = \mathrm{sgn}(I, M, I_3, \dots, I_k) \cdot y_{I,J}$, and $f(x^1, yx^2, x^3, \dots, x^k) = \mathrm{sgn}(J, H, I_3, \dots, I_k) \cdot y_{H,M}$.
- Finally, if $d(I, M) = m - 1$ and $I - M = H - J$, then $y_{I,I} - y_{M,M} = y_{H,H} - y_{J,J}$. Indeed, there is a set of pairwise disjoint indices $I_3, \dots, I_k \in \bigwedge^m([n] \setminus (I \cup J))$. In other words, $I, J, I_3, \dots, I_k$ is a partition of the set $[n]$. Put $x^1 := e_I, x^2 := e_J, x^l := e_{I_l}$, where $3 \leqslant l \leqslant k$. Then

$$\big(\lambda(g) - 1\big) = \delta(y_{I,I} + y_{J,J} + y_{I_3,I_3} + \cdots + y_{I_k,I_k}).$$

  On the other hand, $H, M, I_3, \dots, I_k$ is a partition of $[n]$ too, where $I \cup J = H \cup M$. Substituting $x^1 := e_H, x^2 := e_M, x^l := e_{I_l}$ for all $3 \leqslant l \leqslant k$, we get

$$\big(\lambda(g) - 1\big) = \delta(y_{M,M} + y_{H,H} + y_{I_3,I_3} + \cdots + y_{I_k,I_k}).$$

  Combining the obtained equalities, we see $y_{I,I} + y_{J,J} = y_{M,M} + y_{H,H}$.

Therefore the obtained relations are the same as the relations in the previous lemma. The matrix entries $y_{I,J} = 0$ with $d(I, J) \leqslant m - 2$ do not contribute to the dimension of the Lie algebra. The entries $y_{I,J}$ with $d(I, J) = m - 1$ give the contribution equal to the number of roots of $\Phi$, namely, $(n^2 - n)$. Finally, the latter item allows us to express all entries $y_{I,I}$ as linear combinations of the entries $y_{K_j,K_j}, 1 \leqslant j \leqslant n$, where each fundamental root of $\Phi$ occurs among the pairwise differences of the weights $K_j$. For instance, one can use the weights $\{1, \dots, m-1, p\}, m \leqslant p \leqslant n$, and $\{1, \dots, \hat{i}, \dots, m+1\}, 1 \leqslant i < m$, see [29]. Figure 3 shows their location in the weight diagram $(A_5, \varpi_2)$. Therefore the dimension of the Lie algebra $\mathrm{Lie}(\overline{G}_f(K))$ does not exceed $n^2 - n + n = n^2$. The same argument is also applicable for the case of $\mathrm{Lie}(G_f(K))$. It suffices to set $\lambda(g) = 1$. Again, we conclude that the dimension of $\mathrm{Lie}(G_f(K))$ does not exceed $n^2$.

To conclude the proof of the theorem, we must reduce the dimension of $\mathrm{Lie}(G_f(K))$. For the sake of brevity, we conceive indices $I \in \bigwedge^m[n]$ as roots of the corresponding representation, and we write roots $\alpha = c_1\alpha_1 + \cdots + c_{n-1}\alpha_{n-1} \in A_{n-1}$ in the Dynkin form $c_1 \cdots c_{n-1}$, where $\alpha_j$ are the simple roots of $A_{n-1}$. For example, $\delta = 1 \cdots 1$ is the maximal root of $A_{n-1}$. Suppose $K_1$ is the highest weight of the representation, and $I_2, \dots, I_k$ is

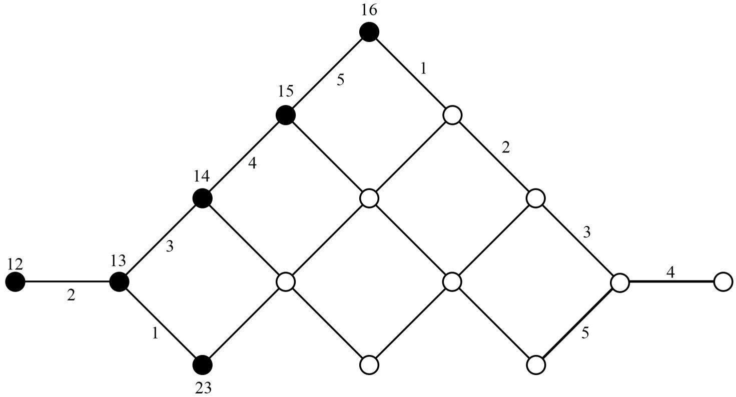



**Figure 3.** Diagonal weights in $(A_5, \varpi_2)$.

the standard partition of the set $[n] \setminus K_1$ into $m$-element subsets, i.e., $I_2 > I_3 > \cdots > I_k$. Substituting $x^1 := e_{K_1}, x^2 := e_{I_2}, \ldots, x^k := e_{K_k}$, we get

$$y_{K_1,K_1} + y_{I_2,I_2} + \cdots + y_{I_k,I_k} = 0.$$

Further, note that for every $j$: $K_1 - I_j = c_1^j \alpha_1 + \cdots + c_{n-1}^j \alpha_{n-1}$. Using already proven relations $y_{I,I} - y_{M,M} = y_{H,H} - y_{J,J}$ for $I - M = H - J$, express all diagonal entries $y_{I_j,I_j}$ as linear combinations of the entries $y_{K_j,K_j}$. Thus we find a non-trivial relation among $y_{K_j,K_j}$. Below we do this for arbitrary exterior power in detail.

In this notation,

$$K_1 - I_2 = 12\cdots m \cdots 210\cdots 0, \quad K_1 - I_3 = 12\cdots \underbrace{m\cdots m}_{m+1 \text{ times}} \cdots 210\cdots 0,$$

and in general

$$K_1 - I_j = 12\cdots \underbrace{m\cdots m}_{(j-2)\cdot m+1} \cdots 21\, \underbrace{0\cdots 0}_{n-mj}$$

for $2 \leqslant j \leqslant k$. Recall that our numbering of the roots $K_j$ is such that $\alpha_m = K_1 - K_2, \alpha_{m+1} = K_2 - K_3, \ldots, \alpha_{n-1} = K_{n-m} - K_{n-m+1}, \alpha_{m-1} = K_2 - K_{n-m+2}, \alpha_{m-2} = K_{n-m+2} - K_{n-m+3}, \ldots, \alpha_1 = K_{n-1} - K_n$ (for the exterior squares $\alpha_{m-1} = \alpha_1 = K_2 - K_{n-m+2}$). Then for $3 \leqslant j \leqslant k$, we have

$$\begin{aligned} &y_{K_1,K_1} - y_{I_j,I_j} \\ &\quad = (y_{K_{n-1},K_{n-1}} - y_{K_n,K_n}) + 2(y_{K_{n-2},K_{n-2}} - y_{K_{n-1},K_{n-1}}) \\ &\qquad + \cdots + (m-1)(y_{K_2,K_2} - y_{K_{n-m+2},K_{n-m+2}}) \\ &\qquad + m\big((y_{K_1,K_1} - y_{K_2,K_2}) + \cdots + (y_{K_{m(j-2)+1},K_{m(j-2)+1}} - y_{K_{m(j-2)+2},K_{m(j-2)+2}})\big) \\ &\qquad + (m-1)(y_{K_{m(j-2)+2},K_{m(j-2)+2}} - y_{K_{m(j-2)+3},K_{m(j-2)+3}}) \\ &\qquad + \cdots + 2(y_{K_{m(j-1)-1},K_{m(j-1)-1}} - y_{K_{m(j-1)},K_{m(j-1)}}) \\ &\qquad + (y_{K_{m(j-1)},K_{m(j-1)}} - y_{K_{m(j-1)+1},K_{m(j-1)+1}}) \end{aligned}$$

$$= my_{K_1,K_1} + (m-1)y_{K_2,K_2} - y_{K_{m(j-2)+2},K_{m(j-2)+2}}$$
$$- \cdots - y_{K_{m(j-1)+1},K_{m(j-1)+1}} - y_{K_{n-m+2},K_{n-m+2}} - \cdots - y_{K_n,K_n},$$

and for $j = 2$, we have

$$\begin{aligned} y_{K_1,K_1} - y_{I_2,I_2} &= (y_{K_{n-1},K_{n-1}} - y_{K_n,K_n}) + 2(y_{K_{n-2},K_{n-2}} - y_{K_{n-1},K_{n-1}}) \\ &\quad + \cdots + (m-1)(y_{K_2,K_2} - y_{K_{n-m+2},K_{n-m+2}}) \\ &\quad + m(y_{K_1,K_1} - y_{K_2,K_2}) + (m-1)(y_{K_2,K_2} - y_{K_3,K_3}) \\ &\quad + \cdots + 2(y_{K_{m-1},K_{m-1}} - y_{K_m,K_m}) + (y_{K_m,K_m} - y_{K_{m+1},K_{m+1}}) \\ &= my_{K_1,K_1} + (m-2)y_{K_2,K_2} - y_{K_3,K_3} - \cdots - y_{K_{m+1},K_{m+1}} \\ &\quad - y_{K_{n-m+2},K_{n-m+2}} - \cdots - y_{K_n,K_n}. \end{aligned}$$

It remains to add up all the obtained equalities with the equation $y_{K_1,K_1} + y_{I_2,I_2} + \cdots + y_{I_k,I_k} = 0$. Thus the final equation on diagonal entries is the following:

$$\big(m(k-1)-k\big)y_{K_1,K_1} + \big((m-1)(k-1)-1\big)y_{K_2,K_2} - y_{K_3,K_3} - \cdots - y_{K_{n-m+1},K_{n-m+1}}$$
$$- (k-1)y_{K_{n-m+2},K_{n-m+2}} - \cdots - (k-1)y_{K_n,K_n} = 0.$$

This is precisely the desired non-trivial linear relation among the entries $y_{K_j,K_j}$, which, over a field of any characteristic, shows that the dimension of our Lie algebra is 1 smaller than the above bound. Thus $\dim \mathrm{Lie}(G_f(K)) \leqslant n^2 - 1$, as claimed. ∎

Let us give an example of the proof calculations for the case of $\bigwedge^2 \mathrm{E}_6(R)$. Figure 3 shows the location of $K_j$ in the weight diagram. We have $y_{12,12} + y_{34,34} + y_{56,56} = 0$ as the form is preserved.

- Since $12 - 34 = \alpha_1 + 2\alpha_2 + \alpha_3$, it follow that

$$\begin{aligned} y_{12,12} - y_{34,34} &= (y_{13,13} - y_{23,23}) + 2(y_{12,12} - y_{13,13}) + (y_{13,13} - y_{14,14}) \\ &= 2y_{12,12} - y_{14,14} - y_{23,23}. \end{aligned}$$

- Since $12 - 56 = \alpha_1 + 2(\alpha_2 + \alpha_3 + \alpha_4) + \alpha_5$, we have

$$\begin{aligned} y_{12,12} - y_{56,56} &= (y_{13,13} - y_{23,23}) \\ &\quad + 2\big((y_{12,12} - y_{13,13}) + (y_{13,13} - y_{14,14}) + (y_{14,14} - y_{15,15})\big) \\ &\quad + (y_{15,15} - y_{16,16}) \\ &= 2y_{12,12} + y_{13,13} - y_{15,15} - y_{16,16} - y_{23,23}. \end{aligned}$$

Adding up these three equations, we get a non-trivial linear relation among the entries $y_{K_j,K_j}$:

$$y_{12,12} + y_{13,13} - y_{14,14} - y_{15,15} - y_{16,16} - 2y_{23,23} = 0.$$

Now we verified all the conditions from Lemma 20 and are ready to complete the proof of Theorem 18.

**Theorem 1.** *If $n/m$ is an integer greater than 2, then there are isomorphisms $\bigwedge^m \mathrm{SL}_n \cong G_f$, $\bigwedge^m \mathrm{GL}_n \cong \overline{G}_f$ of affine group schemes over $\mathbb{Z}$.*

*Proof.* Consider the Cauchy–Binet morphism $\bigwedge^m$ of algebraic groups:

$$\bigwedge^m : \mathrm{GL}_n \longrightarrow \mathrm{GL}_N .$$

From Lemma 9, it follows that the kernel of this morphism equals $\mu_m$. Proposition 16 implies that its image is contained in $\overline{G}_f$. Hence $\bigwedge^m$ induces a monomorphism of algebraic groups:

$$\iota : \mathrm{GL}_n / \mu_m \longrightarrow \overline{G}_f .$$

We wish to apply Lemma 20 to this morphism $\iota$. We know that $\dim(\bigwedge^m \mathrm{GL}_{n,K}) = n^2$ (as an image of $\mathrm{GL}_{n,K}$ under the Cauchet–Binet homomorphism with a finite kernel) for an algebraically closed field $K$. Theorem 23 implies that $\dim(\mathrm{Lie}(\overline{G}_{f,K})) \leqslant n^2$ with the same assumption on the field $K$. Therefore Condition (1) of Lemma 20 holds true. As we discussed after Lemma 20, Conditions (2) and (3) are also satisfied.

This means that we can apply Lemma 20 to conclude that $\iota$ is an isomorphism of affine group schemes over $\mathbb{Z}$.

The proof for the schemes $\bigwedge^m \mathrm{SL}_n$ and $G_f$ is similar and so it is omitted. ∎

## 5. Difference between two exterior powers

The isomorphism $\iota : \bigwedge^m \mathrm{GL}_n \to \overline{G}_f$ from the previous section shows that for arbitrary rings the class of transvections from $\bigwedge^m \mathrm{GL}_n(R)$ is strictly larger than the images $\bigwedge^m g$, $g \in \mathrm{GL}_n(R)$:

$$\bigwedge^m \big(\mathrm{GL}_n(R)\big) < \bigwedge^m \mathrm{GL}_n(R) \quad \text{for a general ring } R.$$

Indeed, suppose $n \neq 2m$ (otherwise, one has to consider the argument for the corresponding connected component of the group). Then the exact sequence of affine group schemes

$$1 \longrightarrow \mu_m \longrightarrow \mathrm{GL}_n \longrightarrow \mathrm{GL}_n / \mu_m \longrightarrow 1$$

gives an exact sequence of Galois cohomology

$$\begin{aligned} 1 \longrightarrow \mu_m(R) \longrightarrow \mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n / \mu_m(R) \\ \longrightarrow H^1(R, \mu_m) \longrightarrow H^1(R, \mathrm{GL}_n) \longrightarrow H^1(R, \mathrm{GL}_n / \mu_m). \end{aligned}$$

The values of all these cohomology sets are well known, see [17, Chapter III, §2], [29, §9], or in the case of exterior square [36]. $H^1(R, \mathrm{GL}_n)$ classifies projective $R$-modules $P$ of rank $n$. In particular, $H^1(R, \mathrm{GL}_1)$ classifies invertible $R$-modules, i.e., finitely generated projective $R$-modules of rank 1. The set $H^1(R, \mathrm{GL}_1)$ has a group structure induced by a tensor product. This group is called the Picard group $\mathrm{Pic}(R)$ of the ring $R$. Its elements are twisted forms of the free $R$-module $R$.

Let us consider the following exact sequence for description of $H^1(R, \mu_m)$:

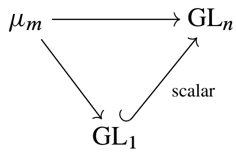
$$1 \longrightarrow \mu_m \longrightarrow \mathrm{GL}_1 \xrightarrow{(\_)^m} \mathrm{GL}_1 \longrightarrow 1,$$

where $(\_)^m$ is the $m^{th}$ power. Since $(\mathrm{GL}_1)^m(R) = R^{*m}$, we have

$$1 \longrightarrow R^*/R^{*m} \longrightarrow H^1(R, \mu_m) \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R),$$

where the rightmost arrow is induced by $(\_)^m$. Thus the cohomology group $H^1(R, \mu_m)$ classifies projective $R$-modules $P$ of rank 1 together with the isomorphism $P^{\otimes m} = R$.

To describe the group $\mathrm{GL}_n/\mu_m(R)$ it remains to calculate the kernel of $H^1(R, \mu_m) \to H^1(R, \mathrm{GL}_n)$. Observe that the morphism $\mu_m \to \mathrm{GL}_n$ passes through $\mathrm{GL}_1 = \mathbb{G}_m$:




Since $H^1(R, \mathrm{GL}_n)$ classifies projective $R$-modules of rank $n$ and the embedding $\mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$ sends $\lambda$ to $\lambda e$, the map $H^1(R, \mathrm{GL}_1) \to H^1(R, \mathrm{GL}_n)$ sends an invertible module $P$ to $\bigoplus_1^n P$. Therefore the kernel of $H^1(R, \mu_m) \to H^1(R, \mathrm{GL}_n)$ contains the whole group $R^*/R^{*m}$ and, in addition, elements $P$ of the Picard group $\mathrm{Pic}(R)$ such that $P^{\otimes m} \cong R$ and $\bigoplus_1^n P$ is free ($\cong R^n$).

Summarizing both arguments, we see that *the quotient of* $\bigwedge^m \mathrm{GL}_n(R)$ *by* $\bigwedge^m(\mathrm{GL}_n(R))$ *contains a copy of the group* $R^*/R^{*m}$. *The quotient by this group is isomorphic to a subgroup of the Picard group* $\mathrm{Pic}(R)$ *consisting of invertible modules* $P$ *over* $R$ *such that* $P^{\otimes m} \cong R$ *and* $\bigoplus_1^n P$ *is free.*

For the special linear group the argument is similar. The exact sequence of affine group schemes

$$1 \longrightarrow \mu_d \longrightarrow \mathrm{SL}_n \longrightarrow \mathrm{SL}_n/\mu_d \longrightarrow 1$$

gives the exact sequence of Galois cohomology

$$\begin{aligned} 1 &\longrightarrow \mu_d(R) \longrightarrow \mathrm{SL}_n(R) \longrightarrow \mathrm{SL}_n/\mu_d(R) \\ &\longrightarrow H^1(R, \mu_d) \longrightarrow H^1(R, \mathrm{SL}_n) \longrightarrow H^1(R, \mathrm{SL}_n/\mu_d), \end{aligned}$$

where $d = \gcd(n, m)$. The values of all these cohomology sets are also well known, for instance see [17, Chapter III, §2].

The determinant map $\det\colon \mathrm{GL}_n \to \mathrm{GL}_1$ induces a map of pointed sets

$$(\det)^1_*\colon H^1(R, \mathrm{GL}_n) \longrightarrow \mathrm{Pic}(R).$$

Suppose $[T] \in H^1(R, \mathrm{GL}_n)$ is a class represented by a projective module $T$ of rank $n$. For any automorphism $\alpha$ of $T$, the determinant $\det(\alpha) \in R$ is the induced automorphism of the $n$-th exterior power $\bigwedge^n T$. Thus $(\det)^1_*([T]) = [\bigwedge^n T]$.

Consider another exact sequence of groups:

$$1 \longrightarrow \mathrm{SL}_n(R) \longrightarrow \mathrm{GL}_n(R) \xrightarrow{\det} \mathrm{GL}_1(R) \longrightarrow 1.$$

We describe the cohomology set $H^1(R, \mathrm{SL}_n)$. Let $M$ be a projective $R$-module of rank $n$ such that $\bigwedge^n M \cong R$. And let $\delta_M\colon \bigwedge^n M \to R$ be a fixed isomorphism. An isomorphism $\psi\colon M \to N$ is called an isomorhism of pairs $(M, \delta_M) \cong (N, \delta_N)$ if $\delta_N \circ \bigwedge^n \psi = \delta_M$. By $[M, \delta_M]$ denote the class of isomorphisms $(M, \delta_M)$. Then for any automorphism $\psi$ of $(M, \delta_M)$, we have $\delta_M \circ \bigwedge^n \psi = \delta_M$. This yields that $\det(\psi) = 1$. Therefore the set $H^1(R, \mathrm{SL}_n)$ is determined by the classes $[M, \delta_M]$, i.e., by projective modules $M$ of rank $n$ together with the fixed isomorphism $\bigwedge^n M \cong R$. And the map

$$H^1(R, \mathrm{SL}_n) \to H^1(R, \mathrm{GL}_n)$$

corresponds to $[M, \delta_M] \mapsto [M]$.

As before, we use the description of $H^1(R, \mu_d)$ in terms of $R^*/R^{*d}$ and projective modules $P$ of rank 1 such that $P^{\otimes d} \cong R$. The map $H^1(R, \mu_d) \to H^1(R, \mathrm{SL}_n)$ sends a module $P$ to the pair $[\bigoplus_1^n P, \delta_P^{\mathrm{can}}]$ where $\delta_P^{\mathrm{can}}$ is the canonical isomorphism induced by the multiplication $\delta^{\mathrm{can}}\colon R^n \to R$.

Summing up, we see that *the quotient of* $\bigwedge^m \mathrm{SL}_n(R)$ *by* $\bigwedge^m(\mathrm{SL}_n(R))$ *contains a copy of the group* $R^*/R^{*d}$. *The quotient by this group consists of pairs* $(P, \alpha)$ *where* $P$ *is an element of the Picard group* $\mathrm{Pic}(R)$ *such that*

$$P^{\otimes m} \cong R \quad \textit{and} \quad \alpha\colon \bigoplus_1^n P \longrightarrow R^n$$

*is an isomorphism such that* $\delta_P^{\mathrm{can}} = \delta^{\mathrm{can}} \circ \alpha$.

## 6. Exterior powers as the stabilizer of invariant forms II

In the previous sections, we completely analyzed the case of one invariant form. However if $n/m \notin \mathbb{N}$, as we assume for this section, then the group $\bigwedge^m \mathrm{GL}_n(R)$ has only an *ideal* of invariant forms.

Let us extend the definition of $q(x)$ from Section 4. Previously considered form

$$q(x) = q^m_{[n]}(x)$$

is associated to the set $[n] = \{1, \dots, n\}$. In this section, we use forms associated to an arbitrary subsets of $[n]$ with fixed cardinality. Namely, we define $q^m_V(x)$ for an arbitrary $n_1$-subset $V \subseteq [n]$, where $n_1/m \in \mathbb{N}$:

- $q^m_V(x) = \sum \mathrm{sgn}(I_1, \dots, I_{\frac{n_1}{m}})\, x_{I_1} \cdots x_{I_{\frac{n_1}{m}}}$ for even $m$;
- $q^m_V(x) = \sum \mathrm{sgn}(I_1, \dots, I_{\frac{n_1}{m}})\, x_{I_1} \wedge \dots \wedge x_{I_{\frac{n_1}{m}}}$ for odd $m$,

where the sums in the both cases range over all unordered partitions of the set $V$ into $m$-element subsets $I_1, \dots, I_{\frac{n_1}{m}}$.

As usual, $f^m_V(x^1, \dots, x^k)$ denotes the [*full*] *polarization* of $q^m_V(x)$, where $k := \frac{n_1}{m}$. We ignore the power $m$ in the notation $f^m_V(x^1, \dots, x^k)$ and $q^m_V(x)$ if it is clear from context.

Let $n = lm + r$ where $l, r \in \mathbb{N}$ and $l$ is the maximal such. Consider the ideal $F = F_{n,m}$ of the ring $\mathbb{Z}[x_I]$ generated by the forms $f_V(x^1, \dots, x^k)$ for all possible $ml$-element subsets $V \subsetneq [n]$. We define the extended Chevalley group $\overline{G}_F(R)$ as the group of linear transformations preserving the ideal $F$:

$$\overline{G}_F(R) := \Big\{ g \in \mathrm{GL}_N(R) \mid \text{ there exist } \lambda_{V_1}, \dots, \lambda_{V_p} \in R^*, c(V_k, V_l) \in R \text{ such that}$$
$$f_{V_j}(gx^1, \dots, gx^k) = \lambda_{V_j}(g) f_{V_j}(x^1, \dots, x^k) + \sum_{l \neq j} c(V_j, V_l) \cdot f_{V_l}(x^1, \dots, x^k)$$
$$\text{for all } j \text{ satisfying } 1 \leqslant j \leqslant p \Big\}.$$

First we must show that $\overline{G}_F$ is a group scheme. We use the following standard argument.

Let $f_1, \dots, f_s$ be arbitrary polynomials in $t$ variables with coefficients in a commutative ring $R$. We are interested in the linear changes of variables $g \in \mathrm{GL}_t(R)$ that preserve the condition that all these polynomials simultaneously vanish. In other words, we consider all $g \in \mathrm{GL}_t(R)$ preserving the ideal $A$ of the ring $R[x_1, \dots, x_t]$ generated by $f_1, \dots, f_s$. It is well known (see, e.g., [10, Lemma 1] or [36, Proposition 1.4.1]) that the set $G_A(R) = \mathrm{Fix}_R(A) = \mathrm{Fix}_R(f_1, \dots, f_s)$ of all such linear variable changes $g$ forms a group. For any $R$-algebra $S$ with 1, we can consider $f_1, \dots, f_s$ as polynomials with coefficients in $S$. Thus the group $G(S)$ is defined for all $R$-algebras. It is clear that $G(S)$ depends functorially on $S$. It is easy to provide examples showing that $S \mapsto G(S)$ may fail to be an affine group scheme over $R$. This is due to the fact that $G_A(R)$ is defined by congruences, rather than equations, in its matrix entries. However in [36, Theorem 1.4.3] a simple sufficient condition was found, that guarantees that $S \mapsto G(S)$ is an affine group scheme. Denote by $R[x_1, \dots, x_t]_r$ the submodule of polynomials of degree at most $r$. The following lemma is [36, Corollary 1.4.6].

**Lemma 24.** *Let $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_t]$ be polynomials of degree at most $r$ and let $A$ be the ideal they generate. Then for the functor $S \mapsto \mathrm{Fix}_S(f_1, \dots, f_s)$ to be an affine group scheme, it suffices that the rank of the intersection $A \cap R[x_1, \dots, x_t]_r$ does not change under reduction modulo any prime $p \in \mathbb{Z}$. This is true in particular if all generators of $A$ remain independent modulo $p$ for all prime $p$.*

We apply this lemma for the ideal $F$ in $\mathbb{Z}[x_I]$.

**Lemma 25.** *Let $n = ml + r$, where $m, l \in \mathbb{N}$. Then the functor $R \mapsto \overline{G}_F(R)$ is an affine group scheme over $\mathbb{Z}$.*

*Proof.* Let us show that for any prime $p$ the polynomials $f_{V_j}$ are linear independent modulo $p$. Indeed, specializing $x_I$ appropriately, we can guarantee that one of these polynomials takes value $\pm 1$, while all other vanish. Let $I_1 \sqcup \dots \sqcup I_l = V_j$ be a partition of some $ml$-element subset $V_j \subset [n]$. Set $x_{I_j} := 1$ for $i = 1, \dots, l$ and $x_I := 0$ otherwise. The monomial $x_{I_1} \cdots x_{I_l}$ occurs only in one form corresponding to the partition $V_j = I_1 \sqcup \dots \sqcup I_l$. Thus the value of the polynomial $f_{V_j}$ is $\mathrm{sgn}(I_1, \dots, I_l) = \pm 1$. ∎

Our immediate goal is to prove the coincidence of $\overline{G}_F$ and $\bigwedge^m \mathrm{GL}_n$. Lemma 20 is useful for this again. Using the results of the previous two sections, we only must verify coincidence of $\bigwedge^m \mathrm{GL}_n(K)$ and $\overline{G}_F^0(K)$ for algebraically closed fields and smoothness of $\overline{G}_F$.

The proof of the following proposition is completely analogous to the proof of Proposition 21.

**Proposition 26.** *Suppose $K$ is an algebraically closed field. Then*

$$\textstyle\bigwedge^m \mathrm{GL}_n(K) = \overline{G}_F^0(K).$$

To verify that the scheme $\overline{G}_F$ is smooth one needs to evaluate the dimension of the Lie algebra. As above, it is possible to identify the Lie algebra $\mathrm{Lie}(\overline{G}_F(K))$ with a homomorphism kernel sending $\delta$ to 0 in $K[\delta]$. Thus $\mathrm{Lie}(\overline{G}_F(K))$ consists of the matrices $g = e + y\delta$ where $y \in \mathrm{M}_N(K)$ satisfying the following conditions

$$f_{V_j}(gx^1, \dots, gx^k) = \lambda_{V_j}(g) f_{V_j}(x^1, \dots, x^k) + \sum_{l \neq j} c(V_j, V_l) f_{V_l}(x^1, \dots, x^k)$$

for $1 \leqslant j \leqslant p$ and $x^1, \dots, x^k \in K^N$.

**Theorem 27.** *For any field $K$ the dimension of the Lie algebra* $\mathrm{Lie}(\overline{G}_F(K))$ *does not exceed $n^2$.*

*Proof.* Let $g$ be a matrix satisfying the above conditions for all $1 \leqslant j \leqslant p$ and $x^1, \dots, x^k \in K^N$. Plugging in $g = e + y\delta$ and using that the form $f_{V_j}$ is $k$-linear, we get

$$\begin{aligned} &\delta\big(f_{V_j}(yx^1, x^2, \dots, x^k) + \dots + f_{V_j}(x^1, \dots, x^{k-1}, yx^k)\big) \\ &\quad = (\lambda_{V_j}(g) - 1) f_{V_j}(x^1, \dots, x^k) + \sum_{l \neq j} c(V_j, V_l) f_{V_l}(x^1, \dots, x^k) \end{aligned}$$

for all $1 \leqslant j \leqslant p$.

Now we show that the entries of the matrix $y$ are subject to the same linear dependences, as in Theorem 23. By the very definition of the forms, $f_{V_j}(e_{I_1}, \dots, e_{I_k}) = 0$ except the cases when $\{I_l\}$ is a partition of the set $V_j = I_1 \sqcup \dots \sqcup I_k$.

- If $d(I, J) \leqslant m - 2$ ($|I \cup J| \geqslant m + 2$), then $y_{I,J} = 0$. Indeed, then there is a set of pairwise disjoint indices $I_2, \dots, I_k \in \bigwedge^m(V_j \smallsetminus I)$ such that $d(J, I_2) \geqslant 1$, $d(J, I_3) \geqslant 1$ and $d(J, I_4) = \dots = d(J, I_k) = 0$. Set $x^1 := e_J$, $x^l := e_{I_l}$, $2 \leqslant l \leqslant k$. Then $f_{V_j}(x^1, yx^2, \dots, x^k) = \dots = f_{V_j}(x^1, x^2, \dots, yx^k) = 0$. It follows that

$$f_{V_j}(yx^1, x^2, \dots, x^k) = \pm y_{I,J} = 0.$$

- If $d(I, J) = d(M, H) = m - 1$, then $y_{I,J} = \pm y_{H,M}$. Here there is a set of pairwise disjoint indices $M, I_3, \dots, I_k \in \bigwedge^m(V_j \smallsetminus I)$ such that $d(J, M) = 1$ and $d(J, I_3) = \dots = d(J, I_k) = 0$. Set $x^1 := e_J$, $x^2 := e_M$, $x^l := e_{I_l}$, $3 \leqslant l \leqslant k$ and denote by $H$

the index $V_j \setminus (J \cup I_2 \cup \cdots \cup I_k)$. Then

$$f_{V_j}(x^1, x^2, yx^3, \ldots, x^k) = \cdots = f_{V_j}(x^1, x^2, \ldots, yx^k) = 0.$$

It follows that

$$f_{V_j}(yx^1, x^2, \ldots, x^k) + f_{V_j}(x^1, yx^2, x^3 \ldots, x^k) = 0.$$

But

$$f_{V_j}(yx^1, x^2, \ldots, x^k) = \operatorname{sgn}(I, M, I_3, \ldots, I_k) \cdot y_{I,J},$$
$$f_{V_j}(x^1, yx^2, x^3, \ldots, x^k) = \operatorname{sgn}(J, H, I_3, \ldots, I_k) \cdot y_{H,M}.$$

- Finally, for diagonal entries the following condition holds

$$y_{I,I} - y_{M,M} = y_{H,H} - y_{J,J},$$

where $d(I,J) = d(H,M) = 0$ and $I \cup J = H \cup M$. In this case there is a set of pairwise disjoint indices $I_3, \ldots, I_k \in \bigwedge^m(V_j \setminus (I \cup J))$. In other words, $I, J, I_3, \ldots, I_k$ is a partition of the set $V_j$. Put $x^1 := e_I$, $x^2 := e_J$, $x^l := e_{I_l}$ where $3 \leqslant l \leqslant k$. Since $f_{V_l}(x^1, \ldots, x^k) = 0$ for all $l \neq j$, we get

$$\big(\lambda_{B_j}(g) - 1\big) = \delta(y_{I,I} + y_{J,J} + y_{I_3,I_3} + \cdots + y_{I_k,I_k}).$$

On the other hand, $H, M, I_3, \ldots, I_k$ is partition of the set $V_j$ too, where $I \cup J = H \cup M$. Substituting $x^1 := e_H$, $x^2 := e_M$, $x^l := e_{I_l}$ for all $3 \leqslant l \leqslant k$, we have

$$\big(\lambda_{B_j}(g) - 1\big) = \delta(y_{M,M} + y_{H,H} + y_{I_3,I_3} + \cdots + y_{I_k,I_k}).$$

Combining the obtained qualities, we see that $y_{I,I} + y_{J,J} = y_{M,M} + y_{H,H}$.

Thus, as in the proof of Theorem 23, it turns out that the dimension of the Lie algebra $\mathrm{Lie}(\overline{G}_F(K))$ does not exceed $n^2$: the entries $y_{I,J}$ do not contribute to the dimension when $d(I,J) \leqslant m-2$, they make a contribution $n(n-1)$ when $d(I,J) = m-1$ and, finally, they make a contribution $n$ for $d(I,J) = m$. ∎

Consequently we verified all the condition from Lemma 20 and can conclude that $\bigwedge^m \mathrm{GL}_n$ equals the stabilizer of $F$. The proof is similar to the proof of Theorem 1.

**Theorem 2.** *Using prior notation, $\bigwedge^m \mathrm{GL}_n$ and $\overline{G}_F$ are isomorphic as affine group schemes over $\mathbb{Z}$.*

## 7. Normalizer theorem

We modify our approach in proving Theorem 3 by contrasting it with Theorems 1 and 2. Specifically, in Theorem 28, we establish that the functors of $R$-points coincide for the group schemes under consideration, for an arbitrary ring $R$.

**Theorem 28.** *If $n \geqslant 4$ and ${}^{n}/_{m}$ is an integer greater than 2, then for any commutative ring $R$, we have*

$$N\left(\textstyle\bigwedge^m \mathrm{E}_n\right)(R) = N\left(\textstyle\bigwedge^m \mathrm{SL}_n\right)(R) = \operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n, \textstyle\bigwedge^m \mathrm{SL}_n\right)(R)$$
$$= \operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n, \textstyle\bigwedge^m \mathrm{GL}_n\right)(R) = \textstyle\bigwedge^m \mathrm{GL}_n(R),$$

*where all normalizers and transporters are taken inside the group scheme $\mathrm{GL}_{\binom{n}{m}}$.*

Before proving the theorem, we address the issue of group-theoretic vs. scheme-theoretic objects appearing in the theorem. Classically, the theorem is formulated with normalizers and transporters as abstract groups. For example, the (group of $R$-points of the) transporter

$$\operatorname{Tran}(\textstyle\bigwedge^m \mathrm{E}_n, \textstyle\bigwedge^m \mathrm{SL}_n)(R)$$
$$:= \left\{g \in \mathrm{GL}_{\binom{n}{m}}(R) \mid z^g \in \textstyle\bigwedge^m \mathrm{SL}_n(\widetilde{R}) \text{ for all } R\text{-algebras } \widetilde{R} \text{ and } z \in \textstyle\bigwedge^m \mathrm{E}_n(\widetilde{R})\right\}$$

should be replaced with the transporter (as an abstract group)

$$\operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n(R), \textstyle\bigwedge^m \mathrm{SL}_n(R)\right)$$
$$:= \left\{g \in \mathrm{GL}_{\binom{n}{m}}(R) \mid z^g \in \textstyle\bigwedge^m \mathrm{SL}_n(R) \text{ for all } z \in \textstyle\bigwedge^m \mathrm{E}_n(R)\right\}.$$

In this presentation, we immediately see the inclusion

$$\operatorname{Tran}(\textstyle\bigwedge^m \mathrm{E}_n, \textstyle\bigwedge^m \mathrm{SL}_n)(R) \leqslant \operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n(R), \textstyle\bigwedge^m \mathrm{SL}_n(R)\right).$$

The next proposition [20, Lemma 4.1, Proposition 4.3] presents other more nontrivial inclusions between different version of the normalizers and transporters.

**Proposition 29.** *In the assumptions of Theorem 3 and 28, the following inclusions hold:*

$$N\left(\textstyle\bigwedge^m \mathrm{E}_n(R)\right) = \operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n(R), \textstyle\bigwedge^m \mathrm{SL}_n(R)\right) \geqslant N\left(\textstyle\bigwedge^m \mathrm{SL}_n(R)\right),$$
$$N\left(\textstyle\bigwedge^m \mathrm{E}_n\right)(R) = \operatorname{Tran}\left(\textstyle\bigwedge^m \mathrm{E}_n, \textstyle\bigwedge^m \mathrm{SL}_n\right)(R) = N\left(\textstyle\bigwedge^m \mathrm{SL}_n\right)(R).$$

The question of when all these groups coincide is quite tricky. For example, [20, Proposition 4.5] proves it in a general situation for algebras over infinite fields; and [19] proves it for our case for an arbitrary $R$ with $\operatorname{char}(R) \neq 2$.

*Proof of Theorem 28 (and Theorem 3).* First, the equality of the first three sets follows from Proposition 29. Moreover, a standard Lie-theoretic argument [11, Chapter 4, Corollary 3.9] shows that $N(\bigwedge^m \mathrm{SL}_n)(R)$ is a group scheme, so all three of them are.

Second, we prove the inclusion $\bigwedge^m \mathrm{GL}_n(R) \leqslant N(\bigwedge^m \mathrm{SL}_n)(R)$ via Theorem 18. Indeed, $g \in \bigwedge^m \mathrm{GL}_n(R)$ implies that $g$ stabilizes the form $f$ up to a scalar $\lambda(g)$. Then, for an arbitrary $R$-algebra $\widetilde{R}$, the element $gbg^{-1}$ stabilizes $f$ as $\lambda(g)\lambda(g^{-1}) = 1$ and $b \in \bigwedge^m \mathrm{SL}_n(\widetilde{R})$ stabilizes $f$.

Third, we show the inclusion $\mathrm{Tran}(\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{SL}_n)(R) \leqslant \bigwedge^m \mathrm{GL}_n(R)$. We pick an element $g \in \mathrm{Tran}(\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{SL}_n)(R)$ and an element $h \in \bigwedge^m \mathrm{E}_n(\widetilde{R})$. Then $a := ghg^{-1}$ belongs to $\bigwedge^m \mathrm{SL}_n(\widetilde{R})$, and thus

$$f(ax^1, \dots, ax^k) = f(x^1, \dots, x^k).$$

Substituting $(gx^1, \dots, gx^k)$ for $(x^1, \dots, x^k)$, we get

$$f(ghx^1, \dots, ghx^k) = f(gx^1, \dots, gx^k).$$

Consider the form $D\colon R^N \times \cdots \times R^N \to R$ defined by the rule

$$D(x^1, \dots, x^k) := f(gx^1, \dots, gx^k).$$

By our assumption, one has

$$D(hx^1, \dots, hx^k) = D(x^1, \dots, x^k)$$

for all $h \in \bigwedge^m \mathrm{E}_n(\widetilde{R})$. Hence the form $D$ is invariant under the action of $\bigwedge^m \mathrm{E}_n(\widetilde{R})$. Thus Proposition 17 shows us

$$D(x^1, \dots, x^k) = \lambda \cdot f(x^1, \dots, x^k) \quad \text{for some } \lambda \in \widetilde{R}.$$

As the transporter is a group, we can plug in $g^{-1}$ instead of $g$. Thereby we conclude that $\lambda$ is invertible. This shows that $g$ belongs to the group $\overline{G}_f(\widetilde{R})$. But initially $g \in \mathrm{GL}_N(R)$, so $g$ belongs to $\overline{G}_f(R)$ which by Theorem 1 coincides with $\bigwedge^m \mathrm{GL}_n(R)$.

Finally, the equality $\mathrm{Tran}(\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{SL}_n)(R) = \mathrm{Tran}(\bigwedge^m \mathrm{E}_n, \bigwedge^m \mathrm{GL}_n)(R)$ follows from Proposition 17 and Theorem 1. Indeed, if $z^g$ (with $z$ and $g$ are from $\widetilde{R}$-points of the group schemes) belongs to $\bigwedge^m \mathrm{GL}_n \cong \overline{G}_f$, then the scalar of semi-invariancy is $\det(z^g) = \det(z) = 1$. Therefore $z^g$ belongs to $G_f \cong \bigwedge^m \mathrm{SL}_n$. ∎

**Remark 30.** We turn to the structure theory of Lie groups for proving that $N(\bigwedge^m \mathrm{SL}_n)$ is a group. Alternatively, we can employ the proved isomorphism $N(\bigwedge^m \mathrm{SL}_n) \cong G_f$ to deduct explicit equations, as in [21], for the functor $N(\bigwedge^m \mathrm{SL}_n)$ and, using Jacobi's complementary formula, verify that they cut out a group scheme.

**Remark 31.** The equivalence $\mathrm{Tran}(E(\Phi, -), G(\Phi, -)) \cong \mathrm{Tran}(E(\Phi, -), \overline{G}(\Phi, -))$ holds in a general situation. It is enough to use the argument of [20, Lemma 4.1] and immediate generalization of the main theorem of [14] to the extended Chevalley group $\overline{G}(\Phi, -)$.

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