

Environmental averaging

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Abstract. Many classical examples of models of self-organized dynamics, including the Cucker–Smale, Motsch–Tadmor, multi-species, and several others, include an alignment force that is based upon density-weighted averaging protocol. Those protocols can be viewed as special cases of “environmental averaging”. In this paper we formalize this concept and introduce a unified framework for systematic analysis of alignment models.

A series of studies are presented including the mean-field limit in deterministic and stochastic settings, hydrodynamic limits in the monokinetic and Maxwellian regimes, hypocoercivity and global relaxation for dissipative kinetic models, several general alignment results based on chain connectivity and spectral gap analysis. These studies cover many of the known results and reveal new ones, which include asymptotic alignment criteria based on connectivity conditions, new estimates on the spectral gap of the alignment force that do not rely on the upper bound of the macroscopic density, uniform gain of positivity for solutions of the Fokker–Planck-alignment model based on smooth environmental averaging. As a consequence, we establish unconditional relaxation result for global solutions to the Fokker–Planck-alignment model, which presents a substantial improvement over previously known perturbative results.

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1. Introduction

Many mathematical models of swarming behavior reflect the tendency of every agent to align its velocity to an averaged direction of motion of the crowd around. Although the rules that describe the average may not be given explicitly, most adhere to a few basic principles. First, agents react more to the closest neighbors, and second, the density of the swarm plays constructive role in defining a particular communication protocol. Such rules, in a broad sense, give rise to what is called *environmental averaging*.

Early computer simulations that incorporated an alignment mechanism along with other interaction forces produced first realistic visualizations of flocks and schools, see [3, 77]. A wide variety of applications ranging from swarming behavior of animals to technological implementations, see these sources [1, 6, 8, 26, 33, 51, 70, 72, 82, 92] and references therein, has ignited mathematical inquiries into theoretical foundation of alignment dynamics.

A prototypical example of a static averaging model arises in opinion dynamics, where each agent labeled by index $i \in [1, N]$ has a set of other agents \mathcal{N}_i to which it is connected. The opinion vector \mathbf{p}_i aligns to the opinions of connected agents via

$$\dot{\mathbf{p}}_i = \lambda \sum_{j \in \mathcal{N}_i} a_{ij}(t)(\mathbf{p}_j - \mathbf{p}_i) + \mathbf{F}_i, \quad \sum_j a_{ij}(t) = 1.$$

Here, \mathbf{F}_i incorporate all other forces such as adherence to convictions or random noise. If the graph of players is connected then the system naturally reaches the total consensus $\mathbf{p}_i \rightarrow \bar{\mathbf{p}}$. Forces may lead to non-trivial limiting states, such as Nash equilibria, see [27, 58, 70, 74].

In swarming dynamics the pioneering work of Vicsek et al. [96] introduced a discrete model of self-propelled particles with local interactions

$$\begin{cases} \mathbf{v}_i(k + 1) = v_0 \frac{\sum_{j:|x_j-x_i|<r_0} \mathbf{v}_j}{|\sum_{j:|x_j-x_i|<r_0} \mathbf{v}_j|} + \sigma \xi_n, \\ \mathbf{x}_i(k + 1) = \mathbf{x}_i(k) + \mathbf{v}_i(k + 1), \end{cases}$$

where ξ_n are random variables and $\sigma > 0$ is the noise intensity. The Vicsek averaging is spatially local and includes normalization to reflect the tendency of agents to adhere to a fixed characteristic speed. The model produces a number of emergent phenomena developing into global patterns such as mills or periodically rotating chains. Solutions undergo phase transitions from ordered to disordered states depending on the noise level, see [26] for a discussion.

A growing number of studies on flocking behavior is based on the Cucker–Smale system introduced in [24, 25],

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \sum_{j=1}^N m_j \phi(x_i - x_j)(v_j - v_i), \end{cases} \quad (x_i, v_i) \in \mathbb{R}^n \times \mathbb{R}^n, \quad i = 1, \dots, N. \quad (1.1)$$

Here, ϕ is a smooth radially symmetric and decreasing kernel, originally $\phi(r) = \frac{\lambda}{(1+r^2)^{\beta/2}}$, where $\lambda, \beta > 0$. The model provides a well-defined mathematical framework which admits justifiable kinetic and macroscopic descriptions, see [9, 16, 35, 41, 43, 82, 94]. It appeared, in part, in response to the need for a model whose long time behavior is not associated with perpetual connectivity assumptions on the flock as in prior studies. In fact, a simple criterion for alignment can be stated solely based on rate of decay of the kernel.

Theorem 1.1 ([24, 25]). *If $\beta \leq 1$, all solutions to (1.1) align exponentially fast to the mean velocity $\bar{v} = (1/\sum_{j=1}^N m_j) \sum_{j=1}^N m_j v_j$, while flock remains bounded*

$$\max_{i=1,\dots,N} |v_i - \bar{v}| \leq C e^{-\delta t}, \quad \max_{i,j=1,\dots,N} |x_i - x_j| \leq \bar{D},$$

where C, δ, \bar{D} depend only on the initial condition and parameters of the kernel. If $\beta > 1$, there are solutions that do not align.

Since its inception the Cucker–Smale system has seen numerous applications. A remarkable implementation to satellite navigation was proposed in [75], where a value of $\beta = 0.4$ was found to be most optimal for the purposes of the mission. Adaptations to control problems are addressed in [10, 12, 20]. Interacting agents immersed in an incompressible fluid lead to hybrid systems with Cucker–Smale component modeling the alignment force [40]. Multi-scale and multi-species flocks have been studied in [45, 88]. An important modification of the system with thermodynamic features was proposed in [42], see also [1]. Flocking analysis can be extended to non-linear alignment protocols as well [39, 50, 65, 93]. A comprehensive review of various other features of the Cucker–Smale dynamics based on hierarchy, angle of vision, and emergence of leaders can be found in [17]. In the context of alignment dynamics which includes potential attraction/repulsion or Rayleigh frictions forces, the emergent behavior has not yet been fully understood, although it is clear from these studies [22, 58, 80, 81, 88], that the effect of such forces on collective outcomes could be dramatic. In particular, the quadratic confinement potential drives the system to an aggregated harmonic oscillator, [80]. Some general N -dependent results in this direction can be achieved for the 3Zone model of Reynolds [77] with the use of the corrector method introduced in [29], see [82]. Lastly, we mention that the alignment criterion itself stated in Theorem 1.1 does not require the kernel to have any explicit form and has seen numerous extensions to include general fat-tail kernels and kernels with degenerate communication in short range, see [29, 41] and Section 4.1 below.

It is insightful to rewrite the Cucker–Smale system as follows

$$\begin{aligned} \dot{x}_i &= v_i, & x_i &\in \Omega, \\ \dot{v}_i &= s_i([v]_i - v_i), & v_i &\in \mathbb{R}^n, \quad i = 1, \dots, N, \end{aligned} \tag{1.2}$$

where Ω is an environment (for most of our discussion either \mathbb{T}^n or \mathbb{R}^n), $[v]_i$ is an averaging protocol of the i th agent, $v = (v_1, \dots, v_N)$, and s_i is a specific communication

strength. Here,

$$s_i = \sum_{j=1}^N m_j \phi(x_i - x_j), \quad [v]_i = \frac{\sum_{j=1}^N m_j \phi(x_i - x_j) v_j}{\sum_{j=1}^N m_j \phi(x_i - x_j)}. \tag{1.3}$$

This form highlights two separate structural components of an alignment model – the averaging and communication strength. Varying these components allows to adapt the system to a particular modeling scenario. For example, it is argued in [69, 70] that if a flock consists of clusters with unbalanced sizes it is more realistic to incorporate a static strength parameter $s_i = \lambda > 0$, leading to what is called the Motsch–Tadmor model

$$\dot{v}_i = \lambda([v]_i - v_i).$$

Analysis of this model presents many challenges related to the lack of symmetry and momentum conservation. However, the analogue of Theorem 1.1 still holds, [70]. A modification that restores the symmetry was proposed in [82],

$$s_i = 1, \quad [v]_i = \int_{\mathbb{R}^n} \phi(x_i - \xi) \frac{\sum_{j=1}^N m_j \phi(\xi - x_j) v_j}{\sum_{j=1}^N m_j \phi(\xi - x_j)} d\xi. \tag{1.4}$$

This particular averaging appears instrumental in several other studies of flocking such as hydrodynamic limits [82], relaxation and hypocoercivity in kinetic dynamics [83], see also Sections 8 and 9. Its continuous variant emerged in the analysis of non-homogeneous turbulence in [61].

Another interesting example of a non-Galilean invariant environmental averaging is given by a class of segregation models. Let $\{g_l\}_{l=1}^L$ be a smooth partition of unity

$$\sum_{l=1}^L g_l = 1$$

subordinated to an open cover $\bigcup_{l=1}^L \Omega_l = \Omega$, where Ω is a compact environment. Let

$$s_i = 1, \quad [v]_i = \sum_{l=1}^L g_l(x_i) \frac{\sum_{j=1}^N m_j v_j g_l(x_j)}{\sum_{j=1}^N m_j g_l(x_j)}. \tag{1.5}$$

Here, the agents communicate predominantly in their own communities and exchange of information is facilitated through the borders. Consensus can be reached provided the border is sufficiently populated at all times, see Section 4 for a rigorous formulation. Many more examples are discussed in Section 2.

In the large crowd limit as $N \rightarrow \infty$ the components $s_i, [v]_i$ take macroscopic forms, which makes them suitable for statistical description of the alignment systems. For example, denoting $f_\phi = f * \phi$ for a distribution f , we can see that the Cucker–Smale model corresponds to

$$s_\rho = \rho\phi, \quad [u]_\rho = \frac{(u\rho)\phi}{\rho\phi}.$$

This averaging rule is also known as the Favre filtration, [34], which was introduced in the context of numerical simulations of turbulent flow. In the same manner, the averaging of (1.4) is given by the over-mollification of the Favre filtration

$$[u]_\rho = \left(\frac{(u\rho)\phi}{\rho\phi} \right)_\phi,$$

and the averaging of (1.5) becomes

$$[u]_\rho(x) = \sum_{l=1}^L g_l(x) \frac{\int_\Omega u g_l \rho \, dy}{\int_\Omega g_l \rho \, dy}.$$

All the operations above make mathematical sense for any probability measure $\rho \in \mathcal{P}(\Omega)$ and any bounded field $u \in L^\infty(\rho)$. In particular, we can go back to the discrete analogues by applying averaging on empirical pairs

$$\begin{aligned} \rho^N &= \sum_{i=1}^N m_i \delta_{x_i}, & u^N &= \sum_{i=1}^N v_i \mathbb{1}_{x_i}, \\ [v]_i &:= [u^N]_{\rho^N}(x_i), & s_i &:= s_{\rho^N}(x_i). \end{aligned} \tag{1.6}$$

It is therefore more inclusive to define averaging rules via macroscopic formulas.

Physical features of the system (1.2) are intimately connected to analytical properties of the pair $(s_\rho, [\cdot]_\rho)$. In most situations those properties are more naturally expressed in terms of the strength measure given by $d\kappa_\rho = s_\rho \, d\rho$. Thus, the preservation of κ -momentum

$$\int_\Omega [u]_\rho \, d\kappa_\rho = \int_\Omega u \, d\kappa_\rho,$$

implies conservation of the physical hydrodynamic momentum, $\frac{d}{dt} \int_\Omega u \, d\rho = 0$. The symmetry

$$\int_\Omega v \cdot [u]_\rho \, d\kappa_\rho = \int_\Omega [v]_\rho \cdot u \, d\kappa_\rho$$

implies a natural energy dissipation law

$$\frac{d}{dt} \int_\Omega |u|^2 \, d\rho(x) = -\frac{1}{2} \int_{\Omega \times \Omega} \phi_\rho(x, y) |u(x) - u(y)|^2 \, d\rho(x) \, d\rho(y), \tag{1.7}$$

where ϕ_ρ is a communication kernel representing a given averaging, see Section 3.2. The long time behavior analysis becomes connected to coercivity and positive-definiteness of the averaging, see Section 4.

In order to get more insight into such connections, it is useful to disassociate the averaging/strength pair $(\kappa_\rho, [\cdot]_\rho)$ from any particular differential law they are involved in, and take a “birds eye” look on its kinematic properties. For this purpose, we will delegate the concept of an *environmental averaging model* to a family of pairs

$$\mathcal{M} = \{(\kappa_\rho, [\cdot]_\rho) : \rho \in \mathcal{P}(\Omega)\},$$

parametrized by probability measures $\rho \in \mathcal{P}(\Omega)$, and satisfying a list of continuity assumptions stated below in Section 2. Through the study of such models it appears possible to build a unifying framework for many flocking and regularity results that have appeared scattered before, and to find substantially new ones that, otherwise, are obscured by specificity of a particular model. This will be the main objective of the present work. So, let us give a brief overview of the studies undertaken here.

(I) First, we develop basic functional analysis of the averaging models. Here we focus primarily on those properties that have direct physical interpretation in terms of dynamics of a particular system they are involved in. Those include representability (existence of a communication kernel), conservation, symmetry, and most importantly a quantitative version of positive definiteness – ball positivity, see Section 3. We also describe regularity conditions on the pairs $s_\rho, [u]_\rho$ necessary for developing a meaningful well-posedness theory for kinetic models done in Sections 5 and 7.

(II) Next we address the classical flocking result of Cucker and Smale for general environmental averaging models. We choose the kinetic description in the context of measure-valued solutions:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (s_\rho(v - [u]_\rho)f). \tag{1.8}$$

Here ρ and $u\rho$ are the macroscopic density and momentum, respectively. It is the most inclusive framework since it encapsulates the microscopic system (1.2) if applied to empirical measures

$$f = \sum_{i=1}^N m_i \delta_{x_i} \otimes \delta_{v_i},$$

and the pressureless hydrodynamic system if applied to mono-kinetic solutions

$$f = \rho(x, t) \delta_0(v - u(x, t)),$$

see (4.7). For global communication kernels the analogue of the original Cucker–Smale alignment criterion is stated in Theorem 4.2, see also Carrillo et al. [16] for the first result of this kind in kinetic formulation.

In the case of local communication, which is our primary focus, all alignment criteria can be sorted into two types – ones that rely on a chain-connectivity of the flock, and ones that make use of the spectral gap condition. The former approach is dynamic. It is based on the idea that connected misaligned components of the flock lose energy through the law (1.7) until full alignment is achieved. For the classical Cucker–Smale and topological singular models this was addressed in [68, 87]. Here we present a new result stated in Theorem 4.5 which gives a sufficient condition of ball-thickness, see (3.15): as long as the flock is connected at a local communication scale r of the kernel, and $\bar{\rho}_r(\text{supp } \rho) \gtrsim 1/t^{1/4}$ in the open space or $\bar{\rho}_r(\Omega) \gtrsim 1/t^{1/2}$ on the torus, the flock aligns. No control on the upper bound of the density is necessary.

The spectral gap approach is kinematic in nature. It relies on finding efficient bounds on the spectral gap of the averaging operator set in a proper function space. In fact, spectral gaps are relevant to flocking behavior in several contexts including relaxation problem for the Fokker–Planck-alignment model. So, it will be our primary focus in Section 4.4. A criterion proved in [92] states that a symmetric model aligns provided $\int_0^\infty \lambda(t) dt = \infty$, where

$$\lambda = \inf_{u \in L^2_\delta(\rho)} \frac{(u, \mathcal{L}_\rho u)_\rho}{(u, u)_\rho}, \quad \mathcal{L}_\rho u = s_\rho(u - [u]_\rho),$$

and $(u, v)_\rho = \int_\Omega u \cdot v d\rho$. In Proposition 4.9 we present an extension of this result to the non-symmetric case. For the Cucker–Smale model the bound $\lambda \gtrsim \rho_-^2/\rho_+$ was proved in the same work [92], see also Remark 4.11. This result is consistent with the chain-connectivity criterion stated above provided ρ_+ remains bounded. For systems with a singular kernel a similar result was established in [87]. With a view towards the relaxation problem, where reliance on ρ_+ is prohibitive, it is imperative to find bounds on the spectral gap independent of ρ_+ .

To this end we propose a somewhat different methodology – one that focuses directly on the averaging $[\cdot]_\rho$ in the framework of κ_ρ -weighted spaces:

$$(u, [u]_\rho)_{\kappa_\rho} \leq (1 - \varepsilon) \|u\|_{L^2(\kappa_\rho)}^2.$$

We introduce the *low energy method* tailored to finding estimates on ε solely in terms of ρ_- . The method applies to a special, but quite broad class of so-called *ball-positive models*, see Proposition 4.16. These include the segregation (1.5), the overmollified Motsch–Tadmor variant (1.4), and most notably the classical Cucker–Smale model (1.3) provided the defining kernel is Bochner-positive: $\phi = \psi * \psi$ for some $\psi \geq 0$. In particular, if applied to the Cucker–Smale model the method gives the following bound:

$$\varepsilon \gtrsim \bar{\rho}_r^3(\Omega). \tag{1.9}$$

(III) The next study is dedicated to justifying the kinetic description through a mean-field limit in both deterministic and stochastic contexts. As the number of agents grows $N \rightarrow \infty$, the microscopic system settles in the weak sense to a solution to the kinetic *Vlasov-alignment* equation (1.8):

$$\mu^N = \sum_{i=1}^N m_i \delta_{x_i} \otimes \delta_{v_i} \rightarrow f.$$

So far the limit has been rigorously justified for the Cucker–Smale and (1.4)-models [41, 43, 82]. In Section 5 we establish a much broader result which covers models with certain uniform regularity properties, see Definition 3.17, and is insensitive to the symmetry of a model. It applies in particular, to the Motsch–Tadmor and other similar models.

When system (1.2) is supplemented with density-weighted stochastic forces

$$\begin{aligned} dx_i &= v_i dt, \\ dv_i &= s_i([v]_i - v_i) dt + \sqrt{2\sigma s_i} dB_i, \quad i = 1, \dots, N, \end{aligned} \tag{1.10}$$

where the B_i 's are independent Brownian motions in \mathbb{R}^n , the limit “in law” settles to a solution of the *Fokker–Planck-alignment* equation

$$\partial_t f + v \cdot \nabla_x f = \sigma s_\rho \Delta_v f + \nabla_v \cdot (s_\rho (v - [u]_\rho) f). \tag{1.11}$$

For the additive noise and general convolution-type models the result was proved in [9], while the present case is treated in Section 6. The non-homogeneous diffusion requires a separate consideration, and is introduced for two reasons. First, it makes physical sense to put stochasticity where communication actually happens and is proportional to its strength. Random deviations get stronger with more active communication, so s_i acts as a thermalization parameter. Second, it ensures that the kinetic model (1.11) has a natural Maxwellian equilibrium. This will be instrumental in the study of relaxation.

(IV) Reading off the evolution of macroscopic quantities from (1.8), we obtain the hydrodynamic *Euler-alignment* system (EAS)

$$\begin{aligned} \partial_t \rho + \nabla \cdot (u\rho) &= 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \cdot \mathcal{R} &= ([u]_\rho - u) d\kappa_\rho, \end{aligned} \tag{1.12}$$

where \mathcal{R} is the Reynolds stress given by

$$\mathcal{R} = \int_{\mathbb{R}^n} (v - u) \otimes (v - u) f \, dv.$$

Here, we encounter the classical closure problem. One can achieve a specific form of \mathcal{R} by introducing various scaling regimes. This has been addressed in two situations. The monokinetic regime $f \rightarrow \rho(x, t)\delta_0(v - u(x, t))$ results in the pressureless EAS, $\mathcal{R} = 0$, and the analysis of this limit for the classical Cucker–Smale model goes back to [35, 52, 66], see also [82]. The convergence was established quantitatively in Wasserstein-1 metric. In Section 9.1 we produce a general result and upgrade the convergence to Wasserstein-2 under mild continuity assumptions on \mathcal{M} . It applies, in particular, to all the models listed here, including non-symmetric ones such as \mathcal{M}_{MT} .

By incorporating a strong penalization force of Fokker–Planck type, one can achieve another regime where f settles to a Maxwellian. This results in the Euler-alignment system with isothermal pressure tensor $\mathcal{R} = \rho \text{Id}$. The Cucker–Smale model was analyzed in [53–55], and (1.4) was analyzed in [83], see also [21] for a new development in the mildly singular case. Section 9.2 presents a general result.

We note that kinetic closure is not the only way to model flocking on the macroscopic level. A general class of systems with entropic pressure introduced in [93], which includes kinetic ones as a particular example, is amenable to flocking analysis as well.

(V) The most comprehensive study in this present work is related to well-posedness and relaxation of the Fokker–Planck-alignment model (1.11) on the periodic environment $\Omega = \mathbb{T}^n$. The motivation for this study is rooted in the original question of *emergence* – formation of collective outcome from purely local interactions. On the periodic domain, if the communication kernel ϕ has a short reach, $\text{supp } \phi \subset [0, r_0]$, then there exists a

family of unaligned solutions where agents rotate along parallel geodesics with various velocities (or even perpendicular geodesics with mutually rational velocities). These are called *locked states*. Such solutions form a measure-zero set in the ensemble of initial data $(x_1, \dots, x_N, v_1, \dots, v_N)$. No deterministic approach to establishing alignment based on generic data that avoids locked states has been explored yet, except for the one-dimensional case [29]. It is natural, however, to look into this problem in stochastic settings of (1.10), where locked states are being disrupted instantly. One can expect a collective outcome in two limiting steps: first $t \rightarrow \infty$, then $\sigma \rightarrow 0$. For large crowd distributions governed by (1.11) this can be viewed as a relaxation problem: in the first step, we obtain convergence to the Maxwellian

$$f \rightarrow \mu_{\sigma, \bar{u}} = \frac{1}{|\Omega|(2\pi\sigma)^{n/2}} e^{-(v-\bar{u})^2/2\sigma}, \tag{1.13}$$

which in turn aggregates on the monokinetic state $\delta_0(v - \bar{u}) \otimes dx$ as $\sigma \rightarrow 0$. The latter represents a perfectly aligned configuration.

This program has seen some success in the past. The relaxation itself for the linear problem is a classical and well-understood subject, see [97] and references therein. With the non-linear alignment force the works [18, 32] establish relaxation for perturbative solutions near equilibrium in the case of the Cucker–Smale and purely local models, respectively. The first global result was proved in [83] in the context of the (1.4)-model, where linear technique was adapted to the non-linear problem enabled by special cancellations in the alignment forcing.

In Section 8 we extend this technique further and prove a much more general result that pertains to a wide variety of models. Proposition 8.1 lists a set of functional requirement on a given solution to imply exponential relaxation. This applies in particular to perturbative solutions, but the main application manifests itself in global relaxation for ball-positive models. It comes in conjunction with the detailed well-posedness theory for the Fokker–Planck-alignment equations developed in Section 7. We prove that most models \mathcal{M} with good regularity properties facilitate the classical kinetic diffusion effect – spread of positivity of solutions expressed by the instant gain of Gaussian tails

$$f(t, x, v) \geq b e^{-a|v|^2}.$$

The spread of positivity is a well-known result observed in many kinetic equations, see [2, 13, 36, 46, 49, 71] and references therein. The novel additional aspect of our result stated in Proposition 7.3 is that the constants a, b depend only on the entropy and L^∞ -bound on the drift $s_\rho[u]_\rho$. For many models, including the Cucker–Smale, the latter two can be controlled by initial condition only. Consequently, for those models we obtain uniform control over the lower bound on the density, and hence, the spectral gap through (1.9). In such cases relaxation result is unconditional. Let us summarize the result specifically for the original Cucker–Smale model.

Theorem 1.2. *Any classical solution f to (1.11) based on the Cucker–Smale model with Bochner-positive kernel ϕ relaxes exponentially fast to the global Maxwellian (1.13).*

Theorem 8.8 contains the full list of models to which a similar result applies. We note again that previously this result was established only in perturbative regime by Duan et al. [32]. Concerning other models, in particular non-symmetric models such as Motsch–Tadmor, we obtain relaxation near equilibrium in Fisher information sense. The complete statement is given in Theorem 8.7.

Finally, let us comment on what is not included in our study and what would be highly desirable to address in the near future. First, we include no forces, focusing mainly on the core alignment mechanism. Potential forces, such as confinement, attraction/repulsion etc, have a great impact on collective outcomes and play major role in applications, [14, 17, 22, 80, 81]. Second, we treat only linear couplings in the alignment force. Several recent studies [39, 50, 65, 93] highlight the importance of non-linear couplings as well. In our general framework non-linearity Γ can be incorporated by considering the system

$$\dot{v}_i = s_i [\Gamma(v - v_i)]_i.$$

Developing regularity and relaxation theory, say, for the kinetic counterpart would be crucial to understanding more intricate non-linear phenomena of self-organization. Third, our framework does not presume communication to be singular, either mildly or strongly. Such models were introduced in [31, 76, 84–87] to analyze the effects of enhanced local communication and its role in emergent dynamics, see the survey [72]. Finally, we leave the analysis of hydrodynamic models in our general framework to future research as it shifts the focus far from the thread of this work, see [15, 44, 60, 70, 82, 94] and the literature therein. However, we will share a new prospective on modeling macroscopic alignment in Section 9.3.

2. Basic concept and examples

Let Ω denote an n -dimensional *environment*. We mostly focus on the cases when Ω is either the open space \mathbb{R}^n , periodic domain \mathbb{T}^n , a finite set of points, or Cartesian products of the above. Denote by $\mathcal{P}(\Omega)$ the set of probability measures on Ω . An *environmental averaging model* is a family of pairs

$$\mathcal{M} = \{(\kappa_\rho, [\cdot]_\rho) : \rho \in \mathcal{P}(\Omega)\}$$

satisfying the following functional requirements:

- (ev1) For every $\rho \in \mathcal{P}(\Omega)$, κ_ρ is a finite positive measure on Ω . We call it communication strength;
- (ev2) $[\cdot]_\rho$ is a linear bounded operator on the weighted space $L^2(\Omega, d\kappa_\rho) := L^2(\kappa_\rho)$;
- (ev3) $[\cdot]_\rho$ is a linear bounded operator on $L^\infty(\kappa_\rho)$, with the properties (κ_ρ -a.e.):

$$[u]_\rho \geq 0 \text{ for all } u \geq 0 \quad \text{and} \quad [\mathbb{1}_\Omega]_\rho = \mathbb{1}_\Omega. \tag{2.1}$$

Here and throughout $\mathbb{1}_A$ denotes the characteristic function of a set A . If $u = (u_1, \dots, u_m)$ is a vector field (where m may be unrelated to the dimension n) we assume that the operator $[u]_\rho$ is acting on each coordinate:

$$[u]_\rho = ([u_1]_\rho, \dots, [u_m]_\rho). \quad (2.2)$$

Although the averaging models are generally assumed to be defined over all densities $\rho \in \mathcal{P}(\Omega)$, to fulfill further regularity assumptions on the averaging operation it may be necessary to restrict the probabilities ρ to a narrower admissible class $\mathcal{D} \subset \mathcal{P}$. The most encountered examples include “thick” flocks, see Section 3.7.

Most natural models are *material* – a property of adherence to the support of the flock. Namely, we say that the model \mathcal{M} is material if

(ev4) there exists bounded family of non-negative functions $s_\rho \in L^{\infty}_+(\Omega)$ with

$$\sup_{\rho \in \mathcal{P}(\Omega)} \|s_\rho\|_{L^\infty(\Omega)} \leq \bar{S}$$

such that $\kappa_\rho = \rho s_\rho$. We also call s_ρ a (specific) strength function;

(ev5) $[u]_\rho = 0$, provided $u = 0$ ρ -a.e.

On the microscopic level one considers discretely distributed density and velocity fields associated to a set of N agents $\{x_i\}_{i=1}^N$:

$$\rho^N = \sum_{i=1}^N m_i \delta_{x_i}, \quad u^N = \sum_{i=1}^N v_i \mathbb{1}_{x_i}.$$

Assuming that the model is material we can unambiguously compute the values of the average and strength at the agents’ locations

$$[v]_i := [u^N]_{\rho^N}(x_i), \quad s_i := s_{\rho^N}(x_i).$$

The agent-based system (1.2) is stated precisely in terms of these discrete components.

2.1. Examples

Let us list several classical examples, and some new ones, and show how they fit into the definition of environmental averaging.

Example 2.1. The most obvious example is the global averaging

$$s_\rho = 1, \quad [u]_\rho = \int_{\Omega} u \rho \, dx. \quad (\mathcal{M}_{\text{glob}})$$

and the system (1.2), in this case, expresses alignment with all-to-all communication

$$\dot{v}_i = \sum_{j=1}^N m_j (v_j - v_i).$$

The extreme opposite is the pure identity model

$$s_\rho = 1, \quad [u]_\rho = u \mathbb{1}_{\text{supp } \rho}. \tag{\mathcal{M}_I}$$

The agent-based version obviously leads to a stalled system. However, the utility of this model in the kinetic formulation will present itself in the study of hydrodynamic limits, see Section 9.

Example 2.2. The classical Cucker–Smale system has been discussed in detail in the introduction. Let us recall that in this case the pair is given by

$$s_\rho = \rho_\phi, \quad [u]_\rho = \frac{(u\rho)_\phi}{\rho_\phi}. \tag{\mathcal{M}_{CS}}$$

Here and throughout we denote for short $f_\phi = f * \phi$, and ϕ is assumed to be infinitely smooth. In this case the averaging $[u]_\rho = u_F$ is also known as the Favre filtration used in large eddy simulations of compressible turbulence, [34]. Its remarkable property comes from the fact that if ρ satisfies the continuity equation

$$\partial_t \rho + \nabla \cdot (u\rho) = 0,$$

then the filtered density ρ_ϕ satisfies the continuity equation relative to the Favre-filtered velocity field

$$\partial_t \rho_\phi + \nabla \cdot (u_F \rho_\phi) = 0.$$

An important implication of this equation will be discussed in Section 9.3.

Properties (ev1) and (ev3) are obvious here. To verify (ev2) we notice that $\kappa_\rho = \rho \rho_\phi$. So using that for any $\rho \in \mathcal{P}(\Omega)$,

$$|(u\rho)_\phi|^2 \leq (|u|^2 \rho)_\phi \rho_\phi, \tag{2.3}$$

we obtain

$$\int_\Omega |[u]_\rho|^2 d\kappa_\rho = \int_\Omega |(u\rho)_\phi|^2 \frac{d\rho}{\rho_\phi} \leq \int_\Omega (|u|^2 \rho)_\phi d\rho = \int_\Omega |u|^2 \rho_\phi d\rho = \|u\|_{L^2(\kappa_\rho)}^2.$$

We can see that the \mathcal{M}_{CS} -model is contractive. The contractivity generally holds even in L^p -spaces for any conservative model, see Lemma 3.9.

Example 2.3. If we set $s_\rho = 1$, the example above turns into another well-known model, the so-called Motsch–Tadmor model [69, 70]:

$$s_\rho = 1, \quad [u]_\rho = \frac{(u\rho)_\phi}{\rho_\phi}. \tag{\mathcal{M}_{MT}}$$

The model was introduced to mediate some issues arising in application of the Cucker–Smale averaging to multi-scale flocks, where a large and distant sub-flock overpowers the dynamics of a smaller sub-flock, see also [82, 88] for more discussion.

The only non-trivial property (ev2) holds for the admissible class of thick densities $\mathcal{D} = \{\rho \in \mathcal{P} : \inf \rho_\phi > 0\}$ under no assumption on the kernel ϕ . However, if the kernel ϕ is local and compactly supported, i.e.

$$c \mathbb{1}_{|x| < r_0} \leq \phi(x) \leq C \mathbb{1}_{|x| < R_0}, \quad R_0 > r_0, \tag{2.4}$$

(the latter holds automatically on compact Ω), then the L^2 -boundedness holds for any $\rho \in \mathcal{P}(\Omega)$ uniformly over $\mathcal{P}(\Omega)$. Indeed, using (2.3),

$$\int_{\Omega} |[u]_\rho|^2 \, d\rho \leq \int_{\Omega} (|u|^2 \rho)_\phi \frac{d\rho}{\rho_\phi} = \int_{\Omega} |u|^2 \left(\frac{\rho}{\rho_\phi}\right)_\phi \, d\rho.$$

According to [53, Lemma 5.2], and see also the appendix, under the condition (2.4) we have

$$\left(\frac{\rho}{\rho_\phi}\right)_\phi \leq C, \tag{2.5}$$

where C depends only on the constants the appear in (2.4) and dimension. This implies the desired result.

Example 2.4. We can interpolate between \mathcal{M}_{CS} and \mathcal{M}_{MT} and consider a general power law for the specific strength function:

$$s_\rho = \rho_\phi^\beta, \quad [u]_\rho = \frac{(u\rho)_\phi}{\rho_\phi}, \quad \beta \geq 0. \tag{\mathcal{M}_\beta}$$

All these models satisfy the requirements (ev1) and (ev3) obviously, and (ev2) follows as above provided we have the following generalization of (2.5), which is proved in the appendix: under (2.4),

$$\left(\frac{\rho}{\rho_\phi^{1-\beta}}\right)_\phi \leq C \rho_\phi^\beta, \quad \forall 0 \leq \beta \leq 1, \tag{2.6}$$

where C depends only on the constants the appear in (2.4), β , and the dimension.

Example 2.5. More suitable for modeling local communication, a symmetric version of the Motsch–Tadmor model can be defined by applying extra convolution to the Favre filtration:

$$s_\rho = 1, \quad [u]_\rho = \left(\frac{(u\rho)_\phi}{\rho_\phi}\right)_\phi. \tag{\mathcal{M}_\phi}$$

This gives rise to the discrete averaging given by (1.4). Here we assume as always that $\phi \in C^\infty$ and it is a mollifier: $\phi \in L^1_+(\Omega)$ with $\int \phi \, dx = 1$.

The model was introduced in [82, 83] and played various roles. It was proved to define a globally hypocoercive kinetic dynamics, and was also used to extend Figalli and Kang’s hydrodynamic limit in the monokinetic regime [35] to flocks with compact support, see also Section 9.

More versions of \mathcal{M}_ϕ can be obtained by looking into different strengths by analogy with the \mathcal{M}_β -model, or by replacing ρ with a more general barotropic pressure:

$$\kappa_\rho = p(\rho), \quad [u]_\rho = \left(\frac{(up)_\phi}{p_\phi} \right)_\phi, \tag{\mathcal{M}_{\phi,p}}$$

where $p \geq 0$ is a function of ρ . Here, the support of the strength function may not coincide with ρ , or s_ρ may be unbounded, which makes it a non-material model. Also the class of admissible densities \mathcal{D} may be restricted depending on the pressure law $p(\rho)$. For example, in the ideal gas case $p = \rho^\gamma$ we naturally assume $\mathcal{D} = L^\gamma(\Omega)$.

One interesting case is obtained when $p = 1$, resulting in

$$\kappa_\rho = 1, \quad [u]_\rho = u_{\phi*\phi}. \tag{\mathcal{M}_{\phi\phi}}$$

In this case the average and the strength do not depend on the density at all, and consequently define a non-material model.

Example 2.6 (Topological models). A new way of modeling interactions which implement topological, rather than Euclidean measure of distance, has long been advocated by many empirical studies [7, 11, 73, 79]. The first symmetric topological model was introduced in [87], see also [59, 67, 78], although it incorporated singular communication. Its smooth variant fits within our framework of environmental averaging.

To define such a model let us consider a basic symmetric domain $\mathcal{O}_0 = \mathcal{O}(-\mathbf{e}_1, \mathbf{e}_1)$ connecting two points $-\mathbf{e}_1$ and \mathbf{e}_1 , and for any pair (x, y) , let $\mathcal{O}(x, y)$ be the domain connecting x and y obtained by rotation and dilation of \mathcal{O}_0 . Let $\chi_{\mathcal{O}(x,y)}$ be some mollification of the characteristic function $\mathbb{1}_{\mathcal{O}(x,y)}$. We introduce the topological “distance” given by

$$d_\rho(x, y) = \int_\Omega \chi_{\mathcal{O}(x,y)}(\zeta) d\rho(\zeta).$$

Now let $\phi(d, z): \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ be a smooth non-negative kernel, radial in z . We define

$$\phi_\rho(x, y) = \phi(d_\rho(x, y), x - y).$$

The kernel incorporates both metric and topological distances. Note that due to the symmetry of the domain $\mathcal{O}(x, y)$, the kernel is also symmetric.

Let us define

$$s_\rho(x) = \int_\Omega \phi_\rho(x, y) d\rho(y), \quad [u]_\rho = \frac{\int_\Omega \phi_\rho(x, y) u(y) d\rho(y)}{\int_\Omega \phi_\rho(x, y) d\rho(y)}. \tag{\mathcal{M}_{CS}^{\text{topo}}}$$

This is the full topological variant of \mathcal{M}_{CS} . As these models bear relevance to biological applications it makes most sense to assume inverse dependence on the topological distance. For example,

$$\phi(d, z) = \frac{\psi(z)}{(\varepsilon + d^2)^{\alpha/2}}, \quad \alpha \geq 0,$$

where ψ is a smooth kernel and $\varepsilon > 0$ is a parameter ($\varepsilon = 0$ would correspond to the fully singular case).

By analogy we can also define a topological version of \mathcal{M}_{MT} :

$$s_\rho(x) = 1, \quad [u]_\rho = \text{same}, \quad (\mathcal{M}_{MT}^{\text{topo}})$$

or the β -model

$$s_\rho(x) = \left(\int_\Omega \phi_\rho(x, y) \rho(y) \, dy \right)^\beta, \quad [u]_\rho = \text{same}. \quad (\mathcal{M}_\beta^{\text{topo}})$$

There is no reasonable topological counterpart of the mollified model \mathcal{M}_ϕ , since there is no way to guarantee that ϕ_ρ integrates to 1 at all times.

Example 2.7 (Models with strict segregation). A family of examples with segregated alignment protocol can be built by setting $s_\rho = 1$, fixing a σ -algebra \mathcal{F} of Borel subsets of Ω and considering the conditional expectation $\mathbb{E}_\rho(f|\mathcal{F})$ relative to $d\rho$. Define

$$[u]_{\rho, \mathcal{F}} = \mathbb{E}_\rho(u|\mathcal{F}). \quad (\mathcal{M}_{\text{cond}})$$

For a given filtration $\{\Omega, \emptyset\} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \rightarrow \mathcal{B}$, we can define a martingale chain of averages

$$[u]_{\rho, n} = \mathbb{E}_\rho(u|\mathcal{F}_n),$$

which naturally connects the global averaging model with the purely local one, since $[u]_{\rho, n} \rightarrow u$ in any $L^p(\rho)$, $1 \leq p < \infty$.

Such an averaging operation models strict segregation between disjoint subalgebras of \mathcal{F} , so-called “neighborhoods”. Let us consider one specific example. Suppose \mathcal{F} is the algebra spanned by a partitioning of Ω into subsets A_1, \dots, A_L . Then

$$[u]_{\rho, \mathcal{F}} = \sum_{l=1}^L \frac{\mathbb{1}_{A_l}}{\rho(A_l)} \int_{A_l} u \rho \, dx. \quad (\mathcal{M}_{\mathcal{F}})$$

If $u_0 = u_0^l$ within each cube A_l , and initial density ρ_0 stays away from the borders ∂A_l , then for a short period of time the solution satisfies a pure transport equation

$$\rho_t + u_0^l \cdot \nabla_x \rho = 0$$

on each A_l . So, the flock will travel with constant velocity within each neighborhood and will remain segregated until one piece reaches the boundary of its neighborhood and starts communicating with others.

Example 2.8 (Smooth segregation). Since in practice there is always a gradual transition between neighborhoods, it makes sense to consider a smooth version of the model above, which is also more amenable to analysis. Let us assume that Ω is compact, and consider any smooth partition of unity $g_l \in C^\infty(\Omega)$, $g_l \geq 0$, and $\sum_{l=1}^L g_l = 1$. Most naturally,

such a partition can be obtained by subordinating it to an open cover $\{\mathcal{O}_l\}_{l=1}^L$ of Ω , so that $\text{supp } g_l \subset \mathcal{O}_l$. We define the model by setting all $s_\rho = 1$, and

$$[u]_\rho(x) = \sum_{l=1}^L \frac{g_l(x)}{\rho(g_l)} \int_\Omega u g_l \rho \, dy, \quad \rho(g_l) = \int_\Omega g_l \rho \, dx. \tag{M_seg}$$

In this model the boundaries are not sharp as in the previous version, and there is some exchange of information that occurs across the adjacent neighborhoods.

There are ways to combine several averaging models into one that describe evolution of a multi-flock. Here “multi” may mean several things – either multiple subflocks with their own communication rules combine into a mega-flock with some global communication between subflocks, or it could mean the use of several communication rules within and between subgroups, which we call “species”. Both of these variants were studied in [45, 88].

Example 2.9 (Multi-species). When a big flock contains groups of agents with distinct characteristics, communication between different groups may be facilitated by different rules, or communication kernels $\phi^{\alpha\beta}$. A model that accommodates such various communication rules was introduced in [45]:

$$\begin{aligned} \dot{x}_i^\alpha &= v_i^\alpha, \quad i = 1, \dots, N^\alpha, \quad \alpha = 1, \dots, A, \\ \dot{v}_i^\alpha &= \sum_{\beta=1}^A \sum_{j=1}^{N^\beta} m_j^\beta \phi^{\alpha\beta}(x_j^\beta - x_i^\alpha)(v_j^\beta - v_i^\alpha). \end{aligned} \tag{2.7}$$

Here, each communication protocol is of Cucker–Smale type.

Such multi-species models can be generalized and fit into the framework of environmental averaging we discuss here. To do that, suppose we have an array of A^2 material models $\mathcal{M}_{\alpha\beta}$, $\alpha, \beta = 1, \dots, A$ defined over the same environment Ω . We can combine them into a new multi-model on the product space $\Omega \times A$. To account for possible variations of masses of sub-flocks, we fix a set of masses $\{M^\alpha\}_\alpha$ with the total mass being $M = \sum_\alpha M^\alpha$, and encode them into the set of admissible densities \mathcal{D}^A over $\Omega \times A$. Namely, we say that $\rho \in \mathcal{D}^A$ is admissible if

$$\rho = \frac{1}{M} \sum_{\alpha=1}^A M^\alpha \rho^\alpha \otimes \delta_\alpha,$$

where $\rho^\alpha \in \mathcal{P}$. We define a cumulative strength function by

$$s_\rho(x, \alpha) = \sum_{\beta=1}^A M^\beta s_{\rho^\beta}^{\alpha\beta}(x).$$

The corresponding averaging of a function $u = \{u^\alpha\}_\alpha$ is given by

$$[u]_\rho(x, \alpha) = \frac{1}{s_\rho(x, \alpha)} \sum_{\beta=1}^A M^\beta s_{\rho^\beta}^{\alpha\beta}(x) [u^\beta]_{\rho^\beta}(x).$$

In terms of this average one can see directly, that the model (2.7) takes the canonical form

$$\dot{v} = s_\rho([v]_\rho - v).$$

Example 2.10 (Multi-flocks). Let us recall the multi-flock model introduced in [88]:

$$\begin{aligned} \dot{x}_i^\alpha &= v_i^\alpha, \\ \dot{v}_i^\alpha &= \sum_{j=1}^{N_\alpha} m_j^\alpha \phi^\alpha(x_i^\alpha - x_j^\alpha)(v_j^\alpha - v_i^\alpha) + \varepsilon \sum_{\beta=1, \beta \neq \alpha}^A M^\beta \psi(X^\alpha, X^\beta)(V^\beta - v_i^\alpha). \end{aligned} \quad (2.8)$$

The model represents A groups of agents evolving according to their own communication, Cucker–Smale type in this particular case, while communication between groups is facilitated through another protocol which involves a kernel ψ and alignment to macroscopic parameters of each subflock, namely their center of masses

$$X^\alpha = \frac{1}{M^\alpha} \sum_{i=1}^{N_\alpha} m_i^\alpha x_i^\alpha, \quad M^\alpha = \sum_{i=1}^{N_\alpha} m_i^\alpha,$$

and momenta

$$V^\alpha = \frac{1}{M^\alpha} \sum_{i=1}^{N_\alpha} m_i^\alpha v_i^\alpha.$$

This idea can be made more formal via an asymptotic analysis detailed in [88].

In general, let $\{\mathcal{M}^\alpha\}_{\alpha=1}^A$ be a family of material models defined over the same environment Ω . We define the admissible set of densities \mathcal{D}^A as in the previous example. For any $\rho = \{\rho^\alpha\}_\alpha \in \mathcal{D}^A$, we define the strength function by

$$s_\rho(x, \alpha) = M^\alpha s_{\rho^\alpha}^\alpha(x),$$

and for $u = \{u^\alpha\}_\alpha$ the average is given by

$$[u]_\rho(x, \alpha) = [u^\alpha]_{\rho^\alpha}^\alpha(x).$$

So far this model incorporates only internal flock communications. To combine these into an interactive multi-flock we assume that the communication between sub-flocks is facilitated through another averaging model $(s_\rho^{\text{ext}}, [u]_\rho^{\text{ext}})$. The multiflock model (2.8) can be written as a system over $\Omega \times A$:

$$\dot{v} = s_\rho([v]_\rho - v) + \varepsilon s_{\rho^A}^{\text{ext}}([V]_{\rho^A}^{\text{ext}} - v),$$

where $\rho^A = \sum_{\alpha=1}^A M^\alpha \delta_{X^\alpha}$ and $V = \sum_{\alpha=1}^A \mathbb{1}_{V^\alpha}$.

Example 2.11 (Models on finite sets). The last but not least example on our list is the family of models on finite environments $\Omega = \{x_1, \dots, x_N\}$. These will be an essential tool to prove results about continuous models, see Appendix B. Finite models illustrate a

situation when all the agents are planted in their places and simply play the role of labels. They do not give rise to any inertial systems of type (1.2). However, they do give rise to families of first order linear systems for $v_i = v(x_i) \in \mathbb{R}^m$,

$$\dot{v}_i = s_i([v]_i - v_i),$$

for each distribution of masses $\rho = (m_1, \dots, m_N)$. Since the averages act coordinate-wise (see (2.2)), the systems for each coordinate decouple and we can assume that v_i are scalars. In this case the properties of the model can be reduced to the properties of the corresponding reproducing matrix associated with the average:

$$A = (a_{ij})_{i,j=1}^N, \quad a_{ij} = [\mathbb{1}_{x_j}](x_i).$$

Property (ev3) implies that A has non-negative entries, and $A\mathbb{1} = \mathbb{1}$, that is, A is right-stochastic.

3. Classes of models and their properties

In this section we will systematize functional properties of environmental averaging models without association with any dynamical law. We introduce several important classes based on their operator-theoretical classification, which will be used extensively in subsequent studies.

3.1. Mapping properties. Jensen inequality

Let us discuss functional basics of environmental averages, and direct consequences of mapping properties stated in (ev2) and (ev3).

First of all, order preserving maps (2.1) obey the maximum principle

$$\min f \leq [f]_\rho \leq \max f, \tag{3.1}$$

and consequently are contractive on $L^\infty(\kappa_\rho)$:

$$\|[f]_\rho\|_\infty \leq \|f\|_\infty. \tag{3.2}$$

Next, let us look into L^∞ -adjoint operator $[\cdot]_\rho^*$. Technically, it maps $(L^\infty)^* \rightarrow (L^\infty)^*$ and if restricted to L^1 it still lands into $(L^\infty)^*$ from this general prospective. However, the extra structure of the averaging allows us to conclude more.

Lemma 3.1. *The operator $[\cdot]_\rho^*$ has the following properties:*

- (1) $[\cdot]_\rho^*: L^1(\kappa_\rho) \rightarrow L^1(\kappa_\rho)$, and hence, $[\cdot]_\rho$ is weak*-continuous on $L^\infty(\kappa_\rho)$;
- (2) $[\cdot]_\rho^*$ is order preserving;
- (3) $[\cdot]_\rho^*: L^1_+(\kappa_\rho) \rightarrow L^1_+(\kappa_\rho)$ is an isometry.

Proof. Let us fix $f \in L^1(\kappa_\rho)$ and for every measurable set A define

$$\nu_f(A) = \int_{\Omega} f[\mathbb{1}_A]_\rho \, d\kappa_\rho.$$

This defines finite σ -additive measure. Indeed, if $A = \bigcup_{i=1}^\infty A_i$, a disjoint union, then

$$\mathbb{1}_{\bigcup_{i=1}^N A_i} \rightarrow \mathbb{1}_{\bigcup_{i=1}^\infty A_i}$$

in $L^2(\kappa_\rho)$. By (ev2), we then also have $[\mathbb{1}_{\bigcup_{i=1}^N A_i}]_\rho \rightarrow [\mathbb{1}_{\bigcup_{i=1}^\infty A_i}]_\rho$ in $L^2(\kappa_\rho)$. Then up to a subsequence, the same convergence holds κ_ρ -a.e. By the Lebesgue dominated convergence theorem we obtain

$$\nu_f(\mathbb{1}_{\bigcup_{i=1}^N A_i}) \rightarrow \nu_f(\mathbb{1}_{\bigcup_{i=1}^\infty A_i}).$$

Furthermore, if $\kappa_\rho(A) = 0$, then by (ev2) $[\mathbb{1}_A]_\rho = 0$ a.e., and hence $\nu_f(A) = 0$. This implies that ν_f is absolutely continuous with respect to κ_ρ . Hence, there exists a function $g \in L^1(\kappa_\rho)$ such that

$$\int_{\Omega} f[\mathbb{1}_A]_\rho \, d\kappa_\rho = \int_{\Omega} g\mathbb{1}_A \, d\kappa_\rho.$$

By approximation and continuity (3.2) we obtain the same relation

$$\int_{\Omega} f[h]_\rho \, d\kappa_\rho = \int_{\Omega} gh \, d\kappa_\rho,$$

for any $h \in L^\infty(\kappa_\rho)$. This means that $[f]_\rho^* = g \in L^1(\kappa_\rho)$. We have proved (1).

Preservation of order (2) follows directly from (ev3) since if $f \in L^1_+(\kappa_\rho)$, then

$$\int [f]_\rho^* g \, d\kappa_\rho = \int f[g]_\rho \, d\kappa_\rho \geq 0$$

for all $g \in L^\infty_+(\kappa_\rho)$. Hence, $[f]_\rho^* \geq 0$. Moreover,

$$\int [f]_\rho^* \, d\kappa_\rho = \int f[\mathbb{1}]_\rho \, d\kappa_\rho = \int f \, d\kappa_\rho,$$

which proves (3). ■

As a consequence, we obtain the following point-wise Jensen inequality for averages.

Lemma 3.2. *For any $u \in L^\infty(\kappa_\rho)$, the following Jensen inequality holds κ_ρ -a.e.:*

$$\psi([u]_\rho(x)) \leq [\psi(u)]_\rho(x), \tag{3.3}$$

where ψ is a continuous convex even and monotonically increasing on \mathbb{R}^+ function.

Proof. By Lemma 3.1, for every $A \subset \Omega$, there exists $f_A \in L^1_+(\kappa_\rho)$, $\|f_A\|_1 = 1$, such that

$$\frac{1}{\kappa_\rho(A)} \int_A [u]_\rho \, d\kappa_\rho = \int_\Omega u f_A \, d\kappa_\rho.$$

Then by the classical Jensen inequality we have

$$\begin{aligned} \psi\left(\frac{1}{\kappa_\rho(A)} \int_A [u]_\rho \, d\kappa_\rho\right) &= \psi\left(\left|\int_\Omega u f_A \, d\kappa_\rho\right|\right) \leq \psi\left(\int_\Omega |u| f_A \, d\kappa_\rho\right) \\ &\leq \int_\Omega \psi(|u|) f_A \, d\kappa_\rho = \int_\Omega \psi(u) f_A \, d\kappa_\rho = \frac{1}{\kappa_\rho(A)} \int_A [\psi(u)]_\rho \, d\kappa_\rho. \end{aligned}$$

Since this holds for any A , by the Lebesgue differentiation theorem and continuity of ψ , as $A \rightarrow \{x\}$ for a.e. x we obtain (3.3), as desired. ■

One of the useful consequences of Jensen’s inequality is extrapolation to L^p -spaces for $p < 2$ and a bound on the L^p -norms.

Lemma 3.3. *Suppose $[1]_\rho^* \in L^\infty(\kappa_\rho)$. Then $[\cdot]_\rho: L^p(\kappa_\rho) \rightarrow L^p(\kappa_\rho)$ for all $1 \leq p \leq \infty$, and*

$$\|[\cdot]_\rho\|_{L^p(\kappa_\rho) \rightarrow L^p(\kappa_\rho)} \leq \|[1]_\rho^*\|_\infty^{1/p}.$$

Proof. For $p = \infty$, the result is simply the axiom (ev3). For $p < \infty$, we use Jensen’s inequality

$$\int_\Omega |[u]_\rho|^p \, d\kappa_\rho \leq \int_\Omega [|u|^p]_\rho \, d\kappa_\rho = \int_\Omega |u|^p [1]_\rho^* \, d\kappa_\rho \leq \|u\|_p^p \|[1]_\rho^*\|_\infty,$$

and the result follows. ■

In some of our studies we will encounter the need to quantify boundedness of the weighted averages $s_\rho[\cdot]_\rho$ on $L^2(\rho)$. This is weaker than the previous mapping property thanks to the uniform boundedness of s_ρ . Thus, a weaker condition is required for it to hold.

Lemma 3.4. *Suppose $s_\rho[s_\rho^{p-1}]_\rho^* \in L^\infty(\rho)$, $1 \leq p < \infty$. Then $s_\rho[\cdot]_\rho: L^p(\rho) \rightarrow L^p(\rho)$, and*

$$\|s_\rho[\cdot]_\rho\|_{L^p(\rho) \rightarrow L^p(\rho)} \leq \|s_\rho[s_\rho^{p-1}]_\rho^*\|_\infty^{1/p}.$$

Proof. Using Jensen’s inequality,

$$\begin{aligned} \|s_\rho[u]_\rho\|_{L^p(\rho)}^p &= \int_\Omega |s_\rho[u]_\rho|^p \, d\rho \leq \int_\Omega [|u|^p]_\rho s_\rho^p \, d\rho \\ &= \int_\Omega [|u|^p]_\rho s_\rho^{p-1} \, d\kappa_\rho = \int_\Omega |u|^p [s_\rho^{p-1}]_\rho^* \, d\kappa_\rho \\ &= \int_\Omega s_\rho[s_\rho^{p-1}]_\rho^* |u|^p \, d\rho \leq \|s_\rho[s_\rho^{p-1}]_\rho^*\|_\infty \|u\|_{L^p(\rho)}^p. \quad \blacksquare \end{aligned}$$

We refer to Section 3.7 for further discussion.

3.2. Reproducing kernel

For material models we often deal with the weighted averaging $s_\rho[u]_\rho$ rather than the bare averaging $[u]_\rho$. In most models the weighted average is an integral operator represented by a kernel.

Definition 3.5. A reproducing kernel of the model \mathcal{M} is a non-negative function $\phi_\rho \in L^1_+(\rho \otimes \rho)$, such that

$$\int_{\Omega} \phi_\rho(x, y) \, d\rho(y) = s_\rho(x), \quad \rho\text{-a.e.}$$

and so that for all $u \in L^\infty(\rho)$,

$$s_\rho[u]_\rho(x) = \int_{\Omega} \phi_\rho(x, y)u(y) \, d\rho(y), \quad \rho\text{-a.e.}$$

A list of examples including our core models is provided in Table 1.

Model	\mathcal{M}_{CS}	\mathcal{M}_{MT}	\mathcal{M}_β	\mathcal{M}_ϕ	\mathcal{M}_{seg}
ϕ_ρ	$\phi(x - y)$	$\frac{\phi(x - y)}{\rho_\phi(x)}$	$\frac{\phi(x - y)}{\rho_\phi^{1-\beta}(x)}$	$\int_{\Omega} \frac{\phi(x - z)\phi(y - z)}{\rho_\phi(z)} \, dz$	$\sum_{l=1}^L \frac{g_l(x)g_l(y)}{\rho(g_l)}$

Table 1. Reproducing kernels.

Generally, the kernel can be recovered from a right-stochastic reproducing kernel of the average itself,

$$\begin{aligned} \Phi_\rho \in L^1_+(\kappa_\rho \otimes \kappa_\rho), \quad \int_{\Omega} \Phi_\rho(x, y) \, d\kappa_\rho(y) = 1, \quad \kappa_\rho\text{-a.e.}, \\ [u]_\rho = \int_{\Omega} \Phi_\rho(x, y)u(y) \, d\kappa_\rho(y). \end{aligned}$$

The correspondence between the two is given by

$$\phi_\rho(x, y) = s_\rho(x)\Phi_\rho(x, y)s_\rho(y).$$

The representation of the adjoint averaging is given by

$$s_\rho(y)[v]_\rho^*(y) = \int_{\Omega} \phi_\rho(x, y)v(x) \, d\rho(x), \quad \rho\text{-a.e.} \tag{3.4}$$

Reproducing kernels are useful for many reasons. Not only do they provide more specific structure to the averaging operator, many properties of the averaging that we will

introduce later can be restated in terms of regularity of the kernel, see Section 3.7. The alignment forces that appear on all levels of description take a more conventional form:

$$s_i([v]_i - v_i) = \sum_{j=1}^N m_j \phi_{\rho^N}(x_i, x_j)(v_j - v_i), \tag{3.5a}$$

$$s_\rho([u]_\rho - v) = \int_{\Omega \times \mathbb{R}^n} \phi_\rho(x, y)(w - v)f(y, w) \, dw \, dy, \tag{3.5b}$$

$$s_\rho([u]_\rho - u) = \int_{\Omega} \phi_\rho(x, y)(u(y) - u(x)) \, d\rho(y). \tag{3.5c}$$

3.3. Conservative models and contractivity

Recall that due to (3.1) every alignment system that is based on an environmental averaging has a maximum/minimum principle and therefore tends to align. If one can quantify the rate of change of the amplitude of u based on properties of the couple $(\kappa_\rho, [\cdot]_\rho)$ one can potentially obtain an alignment $u \rightarrow \bar{u}$ to some constant velocity vector \bar{u} . However, not every model has a predetermined \bar{u} . Typically \bar{u} is uniquely defined by the initial condition if the system preserves the momentum. This property is ensured if the underlying model is conservative.

Definition 3.6. We say that the model \mathcal{M} is *conservative* if for any $\rho \in \mathcal{P}(\Omega)$, $u \in L^2(\kappa_\rho)$, we have

$$\int_{\Omega} u \, d\kappa_\rho = \int_{\Omega} [u]_\rho \, d\kappa_\rho.$$

At all levels of description (1.2), (1.8), (1.12), conservative models preserve momentum,

$$\frac{d}{dt} \bar{u} = 0, \quad \bar{u} = \int_{\Omega} \rho u \, dx.$$

Since we assume that the total mass of a flock is 1, this also predetermines the limiting average velocity from the initial condition $\bar{u} = \int_{\Omega} \rho_0 u_0 \, dx$. Non-conservative models such as \mathcal{M}_{MT} may also align, see Section 4.1 below. However, for those models the limiting velocity emerges dynamically and is not predetermined by the initial condition.

In operator terms being conservative simply means that the adjoint average $[\cdot]_\rho^*$ also preserves constants

$$[\mathbb{1}_\Omega]_\rho^* = \mathbb{1}_\Omega, \quad \kappa_\rho\text{-a.e.}, \quad \forall \rho \in \mathcal{P}(\Omega). \tag{3.6}$$

This in turn implies that the space of mean-zero fields

$$L_0^2(\kappa_\rho) = \left\{ u \in L^2(\kappa_\rho) : \int_{\Omega} u \, d\kappa_\rho = 0 \right\}$$

is invariant for both $[\cdot]_\rho$ and $[\cdot]_\rho^*$.

Together with the positivity proved in Lemma 3.1, (3.6) implies that

$$[\cdot]_\rho^*: L^\infty(\kappa_\rho) \rightarrow L^\infty(\kappa_\rho),$$

and so the adjoint model \mathcal{M}^* consisting of pairs $(\kappa_\rho, [\cdot]_\rho^*)$ fulfills all the requirements of environmental averaging.

Lemma 3.7. *If \mathcal{M} is conservative, then \mathcal{M}^* also defines a conservative model. If \mathcal{M}_1 and \mathcal{M}_2 are conservative with the identical set of strength functions, then $\mathcal{M}_2 \circ \mathcal{M}_1$ and $\frac{1}{2}(\mathcal{M}_1 + \mathcal{M}_2)$ are also conservative.*

For a material model that possesses a reproducing kernel being conservative is equivalent to Φ_ρ being doubly stochastic, or equivalently for ϕ_ρ to satisfy:

$$\int_{\Omega} \phi_\rho(x, y) \, d\rho(x) = s_\rho(y). \tag{3.7}$$

A useful reformulation of conservative property can be done in terms of contractivity.

Definition 3.8. We say that the model \mathcal{M} is *p-contractive* for $1 \leq p \leq \infty$, if for any $\rho \in \mathcal{P}(\Omega)$, $u \in L^p(\kappa_\rho)$,

$$\|[u]_\rho\|_{L^p(\kappa_\rho)} \leq \|u\|_{L^p(\kappa_\rho)}.$$

Note that straight from the definition part (ev3) all models are ∞ -contractive. It is easy to show that contractivity is equivalent to being conservative.

Lemma 3.9. *The following are equivalent:*

- (i) \mathcal{M} is conservative;
- (ii) $\|[\mathbb{1}_\Omega]_\rho^*\|_{L^\infty(\kappa_\rho)} \leq 1$ for all $\rho \in \mathcal{P}(\Omega)$;
- (iii) \mathcal{M} is *p-contractive* for all $1 \leq p \leq \infty$;
- (iv) \mathcal{M} is *1-contractive*.

Proof. (i) \Rightarrow (ii) is trivial. Conversely, assume (ii). Then we have

$$\begin{aligned} \int_{\Omega} \mathbb{1}_\Omega \, d\kappa_\rho &\geq \int_{\Omega} [\mathbb{1}_\Omega]_\rho^* \, d\kappa_\rho \\ &= \int_{\Omega} \mathbb{1}_\Omega [\mathbb{1}_\Omega]_\rho \, d\kappa_\rho = \int_{\Omega} \mathbb{1}_\Omega \, d\kappa_\rho, \end{aligned}$$

which proves that $[\mathbb{1}_\Omega]_\rho^* = \mathbb{1}_\Omega$, κ_ρ -a.e.

The implication (ii) \Rightarrow (iii) is a direct consequence of Lemma 3.3.

Since (iii) \Rightarrow (iv) is trivial, let us now assume (iv). By duality $[\cdot]_\rho^*$ is ∞ -contractive, and hence (ii) holds. ■

Contractivity also implies that the alignment force is dissipative. For example, for the pressureless Euler-alignment system (see (4.7) below), we obtain

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho |u|^2 \, dx = \int_{\Omega} [u \cdot [u]_\rho - |u|^2] \, d\kappa_\rho \leq 0. \tag{3.8}$$

3.4. Symmetric models

Most of the models on our list are in fact *symmetric*: for all $\rho \in \mathcal{P}(\Omega)$ and $u', u'' \in L^2(\kappa_\rho)$,

$$(u', [u'']_\rho)_{\kappa_\rho} = ([u']_\rho, u'')_{\kappa_\rho},$$

where we generally adopt the following notation for the inner-product relative to a measure κ :

$$(f, g)_\kappa = \int_\Omega fg \, d\kappa.$$

In other words, $[\cdot]_\rho^* = [\cdot]_\rho$. In terms of reproducing kernel, if one is available, symmetry is equivalent to Φ_ρ being symmetric. Setting $u'' = \mathbb{1}_\Omega$ we can see that every symmetric model is conservative. However, not every conservative model is automatically symmetric. Plenty of examples are provided by defining the averages with non-symmetric doubly stochastic reproducing kernels.

For symmetric models the energy law (3.8) takes a more explicit form

$$\frac{d}{dt} \int_\Omega \rho |u|^2 \, dx = - \int_\Omega \phi_\rho(x, y) |u(x) - u(y)|^2 \, d\rho(x) \, d\rho(y).$$

We can see that the dissipation burns energy for as long as communicating agents of the flock are not yet aligned. This creates a mechanism for flocking behavior to be discussed in more detail in Section 4.

If \mathcal{M} is a conservative but not symmetric model, then canonical ways to symmetrize it would be to consider the model $\frac{1}{2}(\mathcal{M} + \mathcal{M}^*)$ or $\mathcal{M}^* \circ \mathcal{M}$. According to Lemma 3.7, those define proper environmental averages.

3.5. Galilean invariance

We say that the model \mathcal{M} is Galilean invariant if for all $x \in \Omega$ and $v \in \mathbb{R}^n$,

$$\begin{aligned} \kappa_{\rho(\cdot+v)}(x) &= \kappa_\rho(x + v), \\ [u(\cdot + v)]_{\rho(\cdot+v)}(x) &= [u]_\rho(x + v). \end{aligned}$$

In terms of reproducing kernel, if one is available, the Galilean invariance is equivalent to

$$\begin{aligned} s_{\rho(\cdot+v)}(x) &= s_\rho(x + v), \\ \phi_{\rho(\cdot+v)}(x, y) &= \phi_\rho(x + v, y + v). \end{aligned}$$

For a particular differential system in which \mathcal{M} is involved, this property implies the conventional Galilean invariance with respect to the transformation

$$x \rightarrow x + tV, \quad v \rightarrow v + V, \quad u \rightarrow u + V.$$

All the models considered above except for segregation and conditional expectation ones are Galilean invariant. The segregation protocols are planted into a given geography of the map and therefore are not translation invariant.

3.6. Ball-positivity

If an operator T on a (real in our case) Hilbert space $H_{\mathbb{R}}$ is positive semi-definite, i.e.

$$(Tu, u) \geq 0, \tag{3.9}$$

geometrically this means that Tu and u lie on the same side of the hyperplane u^\perp . If Tu lies in an even more restricted location, namely, in the ball $\frac{1}{2}B_{\|u\|}(u)$, i.e.

$$\left\| Tu - \frac{1}{2}u \right\| \leq \frac{1}{2}\|u\|,$$

then we call T *ball-positive*. A more useful definition of ball-positivity can be stated equivalently as follows

$$(Tu, u) \geq \|Tu\|^2, \quad \forall u \in H_{\mathbb{R}}.$$

In other words, it is positivity (3.9) that comes with a more coercive flavor. Although, as far as we can trace, there is no standard term associated with this property in the literature, such operators appeared for instance in [63] (with $\eta = 1$) and [91].

In the context of environmental averaging models, where $H_{\mathbb{R}} = L^2(\kappa_\rho)$, and $T = [\cdot]_\rho$, the ball-positivity is stated as follows

$$(u, [u]_\rho)_{\kappa_\rho} \geq \|[u]_\rho\|_{L^2(\kappa_\rho)}^2, \quad \forall u \in L^2(\kappa_\rho). \tag{3.10}$$

This property has profound implications to flocking behavior of the system as we will see later in Section 4.4.

We identify many ball-positive models on our list with the use of a simple lemma.

Lemma 3.10. *If \mathcal{M} is symmetric, then \mathcal{M} is ball-positive if and only if it is positive semi-definite.*

Proof. The forward implication is trivial. Conversely, if \mathcal{M} is non-negative and symmetric, then $(u, v)_T = (Tu, v)$ defines a (possibly degenerate) inner product on the real Hilbert space $H_{\mathbb{R}} = L^2(\kappa_\rho)$. Hence, the Cauchy–Schwarz inequality applies

$$|(Tu, v)| \leq \sqrt{(Tu, u)}\sqrt{(Tv, v)}.$$

Taking supremum over all unit v and using the contractivity of T , we obtain the result. ■

Corollary 3.11. *If \mathcal{M} is conservative, then $\mathcal{M}^* \circ \mathcal{M}$ is ball-positive.*

Clearly, the conditional expectation model $\mathcal{M}_{\text{cond}}$ is ball-positive, because it consists of orthogonal projections. For \mathcal{M}_{seg} , we have

$$(u, [u]_\rho)_\rho = \sum_{l=1}^L \frac{\rho(ug_l)^2}{\rho(g_l)} \geq 0.$$

The classical Cucker–Smale model \mathcal{M}_{CS} is ball-positive, provided the kernel ϕ is Bochner-positive, i.e. $\phi = \psi * \psi$ for some smooth $\psi \geq 0$. We have

$$(u, [u]_\rho)_{\kappa_\rho} = \int_\Omega (u\rho) \cdot (u\rho)_\phi \, dx = \int_\Omega (u\rho)_\psi^2 \, dx \geq 0.$$

The symmetric \mathcal{M}_ϕ model is also ball-positive

$$(u, [u]_\rho)_\rho = \int_\Omega \frac{|(u\rho)_\phi|^2}{\rho_\phi} \, dx \geq 0.$$

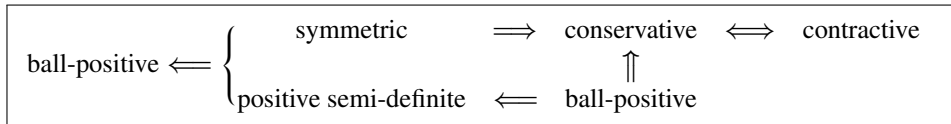
The same argument shows that all $\mathcal{M}_{\phi,p}$ -models are ball-positive.

Among symmetric but not necessarily ball-positive models are the topological models $\mathcal{M}_{CS}^{\text{topo}}$. Here, the kernel is not Bochner-positive to even imply sign definiteness of the averaging. Incidentally, ball-positivity does not imply symmetry either. This will be shown in Appendix B. So, these two properties are completely independent.

Nonetheless, ball-positivity, does imply a host of other properties including of course positivity and 2-contractivity. The 2-contractivity alone does not seem to be sufficient to imply conservation, in spite of Lemma 3.9, it is still possible to show that all ball-positive models are conservative. The proof of this result is not so straightforward. We include it in Appendix B.

Proposition 3.12. *Every ball-positive model is conservative.*

Let us summarize the list of properties, relations between them, and examples (see Table 2):



Model	conservative	symmetric	ball-positive	Galilean invariant
\mathcal{M}_I	✓	✓	✓	✓
$\mathcal{M}_{\text{glob}}$	✓	✓	✓	✓
\mathcal{M}_{CS}	✓	✓	✓ if $\phi = \psi * \psi$	✓
$\mathcal{M}_{CS}^{\text{topo}}$	✓	✓	✗	✓
\mathcal{M}_{MT}	✗	✗	✗	✓
\mathcal{M}_ϕ	✓	✓	✓	✓
\mathcal{M}_{seg}	✓	✓	✓	✗

Table 2. Functional properties of the core models.

The most important applications of ball-positivity will be seen in the context of flocking and spectral gap calculations to be discussed in Section 4.4.

3.7. Thickness, regularity, and well-posedness of microscopic systems

In order to develop a meaningful analysis of alignment models it will be necessary to make a list of continuity and regularity assumptions. We state those in terms of representing kernels and strength functions, which appears to be the most economical way.

3.7.1. Locality of communication. First and foremost we assume that representing kernels support communication at a short range, i.e.

$$\phi_\rho(x, y) \geq c_0 \mathbb{1}_{|x-y| < r_0} \quad \text{for some } r_0 > 0 \text{ and all } \rho \in \mathcal{P}(\Omega). \tag{3.11}$$

Typically, for Favre-based models, such locality follows from the corresponding locality of the defining convolution kernel ϕ :

$$\phi(r) \geq c_0 \mathbb{1}_{r < r_0}. \tag{3.12}$$

Many models in our list satisfy this condition automatically. For the classical Cucker–Smale, it simply means that $\phi > 0$ near the origin. The Motsch–Tadmor model fulfills the same via

$$\phi_\rho(x, y) = \frac{\phi(x - y)}{\rho_\phi(x)} \geq \frac{1}{\|\phi\|_\infty} \phi(x - y), \tag{3.13}$$

since $\rho_\phi(x) \leq \|\phi\|_\infty$. Similarly, for the \mathcal{M}_ϕ -model, we have

$$\phi_\rho(x, y) \geq \frac{1}{\|\phi\|_\infty} \phi * \phi(x - y) \geq c_0 \mathbb{1}_{|x-y| < r_0}.$$

The locality also holds for the segregation model \mathcal{M}_{seg} on a compact environment Ω . Indeed, since $\sum_{l=1}^L g_l(x) = 1$, for every x there exists l such that $g_l(x) \geq 1/L$. Using continuity and compactness, there exists a $r_0 > 0$ such that for any $|x - y| < r_0$, we have $g_l(y) > 1/2L$. Then, since $\rho(g_l) \leq 1$,

$$\phi_\rho(x, y) = \sum_{l=1}^L \frac{g_l(x)g_l(y)}{\rho(g_l)} \geq \frac{1}{2L^2} = c_0, \quad \forall x, y : |x - y| < r_0.$$

Thus, (3.11) is satisfied.

A better way to express (3.11) and similar conditions that follow is through the use of a smooth cut-off function. Let us fix $\chi \in C_0^\infty(B_1(0))$ such that $\chi(x) = 1$ for $x \in B_{1/2}(0)$ and $0 \leq \chi \leq 1$ throughout. We denote the rescaling of χ by $\chi_r(x) = \chi(x/r)$. Thus, (3.11) implies

$$\phi_\rho(x, y) \geq c_0 \chi_{r_0}(x - y).$$

3.7.2. Thickness. Flock with a certain weight present throughout its support or even the entire environment are called thick. One can use masses of balls, $\rho(B_r(x))$ as a measure of thickness. This concept was adopted, for example, in [68]. While useful in many situations (see Sections 4.2, 4.4) for some models, however, thickness takes more individual form

which is easier to satisfy. For example, in the \mathcal{M}_{CS} case it is more natural to measure thickness as ρ_ϕ , while for \mathcal{M}_{seg} the thickness can be measured in terms of masses of neighborhoods, $\rho(g_I)$. We adopt the following general definition.

Definition 3.13. A *thickness* of a density $\rho \in \mathcal{P}(\Omega)$ is a function $\Theta_\rho: \Omega \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (i) $\Theta(\rho, \cdot)$ is lower semi-continuous;
- (ii) $\rho(\{x : \Theta(\rho, x) = 0\}) = 0$ for all $\rho \in L^1(\Omega) \cap \mathcal{P}(\Omega)$;
- (iii) there exists $c > 0$ such that $\Theta(\rho, x) \geq c \min \rho$ for all $\rho \in \Omega$;
- (iv) continuity-in- ρ : there exists a $c > 0$ such that for all $\rho', \rho'' \in \mathcal{P}(\Omega)$,

$$\|\Theta(\rho', \cdot) - \Theta(\rho'', \cdot)\|_\infty \leq c \|\rho' - \rho''\|_{\mathcal{P}};$$

- (v) compatibility with the continuity equation: if ρ satisfies

$$\partial_t \rho + \nabla_x \cdot (u\rho) = 0,$$

then for every point $x \in \Omega$, the function $t \rightarrow \Theta(\rho(t), x)$ satisfies

$$\partial_t \Theta(\rho, x) \geq -c \|u\|_{L^2(\rho)}, \tag{3.14}$$

in a distributional sense.

Thickness of the flock over a subset $S \subset \Omega$ is defined by

$$\Theta(\rho, S) = \inf_{x \in S} \Theta(\rho, x).$$

If $S = \Omega$, we call $\Theta(\rho, \Omega)$ the *uniform thickness* of the flock.

Example 3.14. If no specific structural information is known about $\phi_\rho(x, y)$ except for locality (3.11) then a universal choice for the thickness would be the mass of a smoothed ball at a scale $0 < r < r_0$ (called *ball-thickness*):

$$\Theta(\rho, x) = \bar{\rho}_r(x) = \rho * \chi_r(x). \tag{3.15}$$

Most properties are easy to verify: for (i) we even have $\Theta \in C^\infty$, for (ii) we observe that

$$\{x : \Theta(\rho, x) = 0\} \cap \text{supp } \rho = \emptyset,$$

(iii) and (iv) are trivial, and as to (v), we have

$$\partial_t \bar{\rho}_r(x) = -\nabla_x \cdot (u\rho)_{\chi_r} = -(u\rho)_{\nabla \chi_r} \geq -c \|u\|_{L^2(\rho)}. \tag{3.16}$$

Example 3.15. Thickness associated with a local convolution-type kernel ϕ (see (3.12)) is given by

$$\Theta(\rho, x) = \rho_\phi(x).$$

Here all the properties are trivial. Note that locality (3.12) is necessary for (ii). This choice will be suitable for all Favre-based models.

Example 3.16. Another example is associated with the segregation model \mathcal{M}_{seg} :

$$\Theta(\rho, x) = \min_{l: x \in \text{supp } g_l} \rho(g_l). \tag{3.17}$$

Here we also assume for technical reasons that $|\partial(\text{supp } g_l)| = 0$.

To show the lower semi-continuity, let $x \in \Omega$ be such that $\Theta(\rho, x) > a$. Suppose l_1, \dots, l_k is the list of indexes such that $x \notin \text{supp } g_{l_i}$. Then there exists $\delta > 0$ such that $B_\delta(x) \cap \text{supp } g_{l_i} = \emptyset$. Then for all $y \in B_\delta(x)$ the list of l 's for which $y \in \text{supp } g_l$ is a subset of the list of l 's corresponding to x . So, $\Theta(\rho, y) \geq \Theta(\rho, x)$, and hence the set $\{x : \Theta(\rho, x) > a\}$ is open.

To show (ii) suppose we have $x : \Theta(\rho, x) = 0$, hence there exists l such that $x \in \text{supp } g_l$ and $\rho(g_l) = 0$. If $g_l(x) > 0$, then $\rho(B_\varepsilon(x)) = 0$ for a small ε , hence $x \notin \text{supp } \rho$. Otherwise, $x \in \partial(\text{supp } g_l)$. So,

$$\{x : \Theta(\rho, x) = 0\} \subset (\Omega \setminus \text{supp } \rho) \cup \partial(\text{supp } g_1) \cup \dots \cup \partial(\text{supp } g_L),$$

and the ρ -measure of the set on the right-hand side is 0.

(iii) and (iv) are trivial, as to (v) we have similar to (3.16)

$$\partial_t \rho(g_l) = \int_{\Omega} u \cdot \nabla g_l \, d\rho \geq -c \|u\|_{L^2(\rho)}$$

for any $l = 1, \dots, L$. So, for any fixed $x \in \Omega$ there is a finite collection of l 's such that $x \in \text{supp } g_l$. Denote it $L(x)$. Since the minimum is taken over a fixed compact set $L(x)$ at any moment of time, Rademacher's lemma applies to deduce (3.14) is distributional sense.

3.7.3. Regularity of \mathcal{M} and continuous dependence on ρ . Let us discuss now regularity and continuity-in- ρ of our models. We will encounter two types of models – ones whose regularity depends on thickness (and therefore can be violated if the density in question is not thick), and ones that are uniformly regular independently of thickness.

Before we make these definitions precise, let us make an observation – in all our models the strength is bounded from below by the native and ball-mass thicknesses: there exists an non-decreasing continuous function $s: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$s_\rho(x) \geq s(\Theta(\rho, x)), \quad s_\rho(x) \geq s(\bar{\rho}_{r_0}(x)) \quad \text{for all } x \in \Omega. \tag{3.18}$$

Let us recall the classical Kantorovich–Rubinstein distance between any two finite measures μ', μ'' over Ω :

$$W_1(\mu', \mu'') = \sup_{\text{Lip}(h) \leq 1} \left| \int_{\Omega} h(x) [d\mu'(x) - d\mu''(x)] \right|.$$

Model	\mathcal{M}_{CS}	$\mathcal{M}_{CS}^{\text{topo}}$	\mathcal{M}_β	\mathcal{M}_ϕ	\mathcal{M}_{seg}
$\Theta(\rho, x)$	$\rho_\phi(x)$	$\rho_\psi(x)$	$\rho_\phi(x)$	$\rho_\phi(x)$	$\min_{l: x \in \text{supp } g_l} \rho(g_l)$

Table 3. Associated thickness of selected models.

Definition 3.17. We say that a model \mathcal{M} is *regular* if for every $R > 0$ and $\rho, \rho', \rho'' \in \mathcal{P}(B_R)$, we have for every $k = 0, 1, \dots$,

$$\|\partial^k s_\rho\|_{L^\infty(B_R)} + \|\partial_x^k \phi_\rho\|_{L^\infty(B_R \times B_R)} + \|\partial_y^k \phi_\rho\|_{L^\infty(B_R \times B_R)} \leq C_{k,R}(\Theta(\rho, B_R)), \quad (3.19)$$

$$\|s_{\rho'} - s_{\rho''}\|_{L^\infty(B_R)} + \|\phi_{\rho'} - \phi_{\rho''}\|_{L^\infty(B_R \times B_R)} \leq C_R(\Theta(\rho', B_R), \Theta(\rho'', B_R))W_1(\rho', \rho''). \quad (3.20)$$

Definition 3.18. We say that a model \mathcal{M} is *uniformly regular* if for every $R > 0$ and $\rho, \rho', \rho'' \in \mathcal{P}(B_R)$ we have for every $k = 0, 1, \dots$,

$$\|\partial^k s_\rho\|_{L^\infty(B_R)} + \|\partial_x^k \phi_\rho\|_{L^\infty(B_R \times B_R)} + \|\partial_y^k \phi_\rho\|_{L^\infty(B_R \times B_R)} \leq C_{k,R}, \quad (3.21)$$

$$\|s_{\rho'} - s_{\rho''}\|_{L^\infty(B_R)} + \|\phi_{\rho'} - \phi_{\rho''}\|_{L^\infty(B_R \times B_R)} \leq C_R W_1(\rho', \rho''). \quad (3.22)$$

If no information is known about the thickness of one of the densities involved in inequality (3.20), some of the models still retain a level of continuity if at least the other density is thick: for every $R > 0$ and $\rho', \rho'' \in \mathcal{P}(B_R)$, one has

$$\int_\Omega |s_{\rho'}(x) - s_{\rho''}(x)|^2 d\rho''(x) \leq C_R(\Theta(\rho', B_R))W_1^2(\rho', \rho''), \quad (3.23)$$

$$\int_\Omega \int_\Omega |\phi_{\rho'}(x, y) - \phi_{\rho''}(x, y)|^2 d\rho''(x) d\rho''(y) \leq C_R(\Theta(\rho', B_R))W_1^2(\rho', \rho''). \quad (3.24)$$

This will be useful in the study of the hydrodynamic limits.

Let us go through the main examples on our list, identify their associated thicknesses and determine which level of regularity they satisfy and under which conditions. Our findings are summarized in Tables 3, 4 and 5.

Example 3.19 ($\mathcal{M}_{CS}, \mathcal{M}_{CS}^{\text{topo}}$). The Cucker–Smale model is trivially uniformly regular with $\Theta(\rho, x) = \rho_\phi$. While $\mathcal{M}_{CS}^{\text{topo}}$ is uniformly regular with $\Theta(\rho, x) = \rho_\psi$.

Example 3.20 ($\mathcal{M}_\beta, 0 \leq \beta < 1$). The model has the same associated thickness, that is, $\Theta(\rho, x) = \rho_\phi$. Under no conditions on ϕ , the model is trivially regular. If $\phi > 0$, then

$$\Theta(\rho, x) \geq \inf_{r < 2R} \phi(r) = \delta > 0 \quad (3.25)$$

on B_R for any $\rho \in \mathcal{P}(B_R)$. So, in this case the model is uniformly regular, and all the estimates are straightforward.

Model	regular	uniform	(3.23), (3.24)
\mathcal{M}_{CS}	✓	✓	✓
$\mathcal{M}_{CS}^{\text{topo}}$	✓	✓	✓
\mathcal{M}_β	✓	$\phi > 0$	$c \mathbb{1}_{ x < r_0} \leq \phi(x) \leq C \mathbb{1}_{ x < R_0}$
\mathcal{M}_ϕ	✓ on compact Ω	$\phi > 0$ on compact Ω , $\phi = \frac{1}{\langle x \rangle^{n+\gamma}}$ on \mathbb{R}^n	✓ on compact Ω
\mathcal{M}_{seg}	✓	$\text{supp } g_l = \Omega$	✓

Table 4. Regularity type of selected models.

Let us assume that ϕ is local and satisfies (2.4). We will prove that in this case the model is continuous in ρ ((3.23), (3.24)). Indeed, as to (3.23), by an elementary inequality, we have

$$|(\rho'_\phi)^\beta - (\rho''_\phi)^\beta| \leq C |\rho'_\phi|^{\beta-1} |\rho'_\phi - \rho''_\phi| \leq C(\Theta(\rho', x)) \|\nabla \phi\|_\infty W_1(\rho', \rho''),$$

and (3.23) follows. As to the kernel continuity (3.24), we have

$$\begin{aligned} & \int_\Omega \int_\Omega |\phi_{\rho'}(x, y) - \phi_{\rho''}(x, y)|^2 d\rho''(x) d\rho''(y) \\ & \leq \|\phi\|_\infty \int_\Omega \int_\Omega \left| \frac{1}{(\rho'_\phi(x))^{1-\beta}} - \frac{1}{(\rho''_\phi(x))^{1-\beta}} \right|^2 \phi(x-y) d\rho''(x) d\rho''(y) \\ & = \|\phi\|_\infty \int_\Omega \left| \frac{1}{(\rho'_\phi(x))^{1-\beta}} - \frac{1}{(\rho''_\phi(x))^{1-\beta}} \right|^2 \rho''_\phi(x) d\rho''(x) \\ & = \|\phi\|_\infty \int_\Omega \frac{|(\rho''_\phi(x))^{1-\beta} - (\rho'_\phi(x))^{1-\beta}|^2}{(\rho'_\phi(x))^{2-2\beta}} (\rho''_\phi(x))^{2\beta} \frac{d\rho''(x)}{\rho''_\phi(x)} \\ & \leq C(\Theta(\rho', B_R)) W_1^2(\rho', \rho'') \int_{B_R} \frac{d\rho''(x)}{\rho''_\phi(x)}. \end{aligned}$$

Note that

$$\begin{aligned} \int_{B_R} \frac{d\rho''(x)}{\rho''_\phi(x)} &= \frac{1}{\|\phi\|_1} \int_{B_{R+R_0}} \int_{B_R} \phi(x-y) \frac{d\rho''(x)}{\rho''_\phi(x)} dy \\ &= \frac{1}{\|\phi\|_1} \int_{B_{R+R_0}} \left(\frac{\rho''}{\rho_\phi}\right)_\phi(y) dy. \end{aligned}$$

According to (2.5), the expression inside is uniformly bounded, and hence the whole integral is bounded by a constant depending only on R, R_0 . This proves (3.24).

Finally, we note that if ϕ is local, then the ball-thickness (3.15) with $r \leq r_0$ can also be used in all the estimates. This observation will be useful in the relaxation study, see Section 8. However it should be noted that $\rho_\phi(x) \geq c\bar{\rho}_r(x)$, and so it is easier for densities to be natively thick than ball-thick.

Example 3.21 (\mathcal{M}_{seg}). The computation is quite similar for the segregation model \mathcal{M}_{seg} , where the thickness functional Θ is given by (3.17). The model is clearly uniformly thick if $\text{supp } g_l = \Omega$, $l = 1, \dots, L$, since then $\Theta(\rho, \Omega) = 1$ for any ρ . Generally, the global thickness is given by $\Theta(\rho, \Omega) = \min_l \rho(g_l)$. So, it is clear that regularity holds for this model as well. All these conclusions hold for the ball-mass thickness (3.15) where r is a small radius so that for every l there exists an $x_0 \in \Omega_l$ such that $g_l|_{B_r(x_0)} \geq c_0$ for some fixed $c_0 > 0$.

Let us establish (3.24) relative to the native thickness (the ball-thickness (3.15) will not work here)

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} |\phi_{\rho'}(x, y) - \phi_{\rho''}(x, y)|^2 d\rho''(x) d\rho''(y) \\ & \leq \int_{\Omega} \int_{\Omega} \sum_{l=1}^L g_l(x)g_l(y) \left| \frac{1}{\rho'(g_l)} - \frac{1}{\rho''(g_l)} \right|^2 \phi(x - y) d\rho''(x) d\rho''(y) \\ & \leq \sum_{l=1}^L (\rho''(g_l))^2 \left| \frac{1}{\rho'(g_l)} - \frac{1}{\rho''(g_l)} \right|^2 \\ & = \sum_{l=1}^L \left| \frac{\rho'(g_l) - \rho''(g_l)}{\rho'(g_l)} \right|^2 \\ & \leq C(\Theta(\rho', \Omega)) W_1^2(\rho', \rho''). \end{aligned}$$

Example 3.22 (\mathcal{M}_ϕ). Because of the non-local dependence on ρ_ϕ in the kernel, there does not seem to be another thickness quantity that would fulfill the local continuity and regularity assumptions. However, if we set $\Theta(\rho, x) = \rho_\phi(x)$, the model becomes regular on any compact environment Ω and for any kernel ϕ . Also on compact Ω , the model is uniformly regular when $\phi > 0$. Finally, the strong continuity-in- ρ ((3.23), (3.24)) holds as well:

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} |\phi_{\rho'}(x, y) - \phi_{\rho''}(x, y)|^2 d\rho''(x) d\rho''(y) \\ & \lesssim \int_{\Omega} \int_{\Omega} \int_{\Omega} \phi(x - z)\phi(y - z) \left| \frac{1}{\rho'_\phi(z)} - \frac{1}{\rho''_\phi(z)} \right|^2 dz d\rho''(x) d\rho''(y) \\ & = \int_{\Omega} \left| \frac{1}{\rho'_\phi(z)} - \frac{1}{\rho''_\phi(z)} \right|^2 (\rho''(z))^2 dz \\ & \leq \frac{|\Omega|}{\Theta^2(\rho', \Omega)} W_1^2(\rho', \rho''). \end{aligned}$$

On the open space, if ϕ is compactly supported then the model would fail to fulfill any regularity assumptions. However, for the integrable kernel ϕ satisfying the following conditions

$$\begin{aligned} \phi &\in W^{k,1}(\Omega), \quad \forall k \in \mathbb{N}, \\ \frac{1}{2}|y| \leq |x| \leq 2|y| &\Rightarrow \phi(x) \sim \phi(y), \end{aligned} \tag{3.26}$$

one can establish uniform regularity. The choice

$$\phi = \frac{1}{\langle x \rangle^{n+\gamma}}, \quad \gamma > 0, \quad \langle x \rangle = (1 + |x|^2)^{1/2}$$

is an example of such a kernel.

To see this, let $\rho \in \mathcal{P}(B_R)$. We have

$$\begin{aligned} \partial_x^k \phi_\rho(x, y) &= \int_\Omega \frac{\partial^k \phi(x-z)\phi(y-z)}{\rho_\phi(z)} dz \\ &= \int_{B_{2R}} \frac{\partial^k \phi(x-z)\phi(y-z)}{\rho_\phi(z)} dz + \int_{\Omega \setminus B_{2R}} \frac{\partial^k \phi(x-z)\phi(y-z)}{\rho_\phi(z)} dz. \end{aligned}$$

Inside the ball B_{2R} we have $\rho_\phi(z) \geq \delta$ by (3.25). So,

$$\int_{B_{2R}} \frac{\partial^k \phi(x-z)\phi(y-z)}{\rho_\phi(z)} dz \leq C(R) \|\phi\|_{W^{k,1}}.$$

For $z \in \Omega \setminus B_{2R}$, we have by (3.26)

$$\rho_\phi(z) = \int_{B_R} \phi(z-w) d\rho(w) \gtrsim \phi(z) \int_{B_R} d\rho(w) = \phi(z).$$

On the other hand, by the same conditions (3.26), since $y \in B_R$,

$$\phi(y-z) \lesssim \phi(z).$$

Thus,

$$\int_{\Omega \setminus B_{2R}} \frac{\partial^k \phi(x-z)\phi(y-z)}{\rho_\phi(z)} dz \lesssim \int_{\Omega \setminus B_{2R}} |\partial^k \phi(x-z)| dz \leq \|\phi\|_{W^{k,1}}.$$

Since the kernel is symmetric the same holds for $\partial_y^k \phi_\rho$. We have proved (3.21). To show that (3.22) holds, let us write

$$\begin{aligned} |\phi_{\rho'}(x, y) - \phi_{\rho''}(x, y)| &\leq \int_\Omega \phi(x-z)\phi(y-z) \frac{|\rho''_\phi(z) - \rho'_\phi(z)|}{\rho'_\phi(z)\rho''_\phi(z)} dz \\ &= \int_{B_{2R}} \phi(x-z)\phi(y-z) \frac{|\rho''_\phi(z) - \rho'_\phi(z)|}{\rho'_\phi(z)\rho''_\phi(z)} dz \\ &\quad + \int_{\Omega \setminus B_{2R}} \phi(x-z)\phi(y-z) \frac{|\rho''_\phi(z) - \rho'_\phi(z)|}{\rho'_\phi(z)\rho''_\phi(z)} dz. \end{aligned}$$

Using again that inside the ball B_{2R} , $\rho'_\phi(z), \rho''_\phi(z) \geq \delta$, we obtain

$$\lesssim W_1(\rho', \rho'') + W_1(\rho', \rho'') \int_{\Omega \setminus B_{2R}} \frac{\phi(x-z)\phi(y-z)}{\rho'_\phi(z)\rho''_\phi(z)} \sup_{w \in B_R} |\nabla_w \phi(z-w)| dz.$$

Arguing as before we conclude that $\phi(x-z)\phi(y-z)/\rho'_\phi(z)\rho''_\phi(z)$ is uniformly bounded on $\Omega \setminus B_{2R}$. At the same time, $\sup_{w \in B_R} |\nabla_w \phi(z-w)| \in L^1(dz)$. This finishes the estimate.

The native thickness in all of the above can be replaced with the ball-thickness (3.15) as well.

3.7.4. Well-posedness of agent-based systems. Let us establish basic well-posedness of the agent based system as a consequence of the uniform regularity:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = s_i([v]_i - v_i), \quad i = 1, \dots, N. \end{cases} \tag{3.27}$$

Here, Ω can be any environment. Recall that s_i and $[v]_i$ are defined in (1.6). The maximum principle implies that $\max_i |v_i| \leq \max_i |v_i(0)| := A$, and therefore, $\max |x_i| \lesssim t$ a priori. So, in order to establish global existence by the standard fix point argument it suffices to check that the right-hand side of (3.27) is locally Lipschitz on $\Omega^N \times \mathbb{R}^{nN}$.

So, let us assume that \mathcal{M} is uniformly regular. Let us fix masses m_1, \dots, m_N and two configurations

$$(x'_1, \dots, x'_N; v'_1, \dots, v'_N) \in B_R^N \times B_A^N, \quad (x''_1, \dots, x''_N; v''_1, \dots, v''_N) \in B_R^N \times B_A^N.$$

We only need to show Lipschitzness of the momentum equation. We have

$$\begin{aligned} |s'_i[v']_i - s'_i v'_i - s''_i[v'']_i + s''_i v''_i| &\leq |s'_i[v']_i - s''_i[v'']_i| + |s'_i v'_i - s''_i v''_i| \\ &= \text{I} + \text{II}. \end{aligned}$$

As for II,

$$\begin{aligned} \text{II} &\leq |s'_i - s''_i| |v'_i| + |s''_i| |v'_i - v''_i| \\ &\leq A |s'_i - s''_i| + \bar{S} |v'_i - v''_i|. \end{aligned}$$

Using (3.21) and (3.22),

$$\begin{aligned} |s'_i - s''_i| &\leq |s_{\rho'}(x'_i) - s_{\rho'}(x''_i)| + |s_{\rho'}(x''_i) - s_{\rho''}(x''_i)| \\ &\leq \bar{C}_1 |x'_i - x''_i| + \bar{C} W_1(\rho', \rho'') \\ &\leq \bar{C}_1 |x'_i - x''_i| + \bar{C} \sum_j m_j |x'_j - x''_j| \\ &\lesssim \max |x'_j - x''_j|. \end{aligned}$$

We now estimate the weighted averages term using the same regularity assumptions,

$$\begin{aligned}
 \mathbf{I} &\leq \sum_j m_j |\phi_{\rho'}(x'_i, x'_j)v'_j - \phi_{\rho''}(x''_i, x''_j)v''_j| \\
 &\leq \sum_j m_j \phi_{\rho'}(x'_i, x'_j) |v'_j - v''_j| + \sum_j m_j |\phi_{\rho'}(x'_i, x'_j) - \phi_{\rho''}(x''_i, x''_j)| |v''_j| \\
 &\leq \bar{C}_0 \max |v'_j - v''_j| + A \sum_j m_j |\phi_{\rho'}(x'_i, x'_j) - \phi_{\rho''}(x''_i, x''_j)| \\
 &\quad + A \sum_j m_j |\phi_{\rho''}(x'_i, x'_j) - \phi_{\rho''}(x''_i, x''_j)| \\
 &\leq \bar{C}_0 \max |v'_j - v''_j| + AW_1(\rho', \rho'') + 2A\bar{C}_1 \sum_j m_j |x'_j - x''_j| \\
 &\lesssim \max |v'_j - v''_j| + \max |x'_j - x''_j|.
 \end{aligned}$$

We have proved the following result.

Proposition 3.23. *If \mathcal{M} is uniformly regular, then the system (3.27) is globally well-posed.*

Note that this well-posedness result is robust – the Lipschitzness is independent of the number of agents or their masses. That is why it can be extended to kinetic formulation as well, see Section 5. However, the well-posedness in a less robust form also extends to some non-regular models such as \mathcal{M}_{MT} if ϕ is finitely supported and satisfies (3.11). This is based on the fact that for any atomic ρ we have $\rho_\phi(x_i) \geq m_i \phi(0)$. So, there is a residual mass-dependent thickness of the flock left on its support. Indeed, we have in this case

$$|[v']_i - [v'']_i| \leq \frac{\|\phi\|_\infty}{\phi(0)m_i} \sum_j m_j |v'_j - v''_j| + A \sum_j m_j |\phi_{\rho'}(x'_i, x'_j) - \phi_{\rho''}(x''_i, x''_j)|,$$

and

$$\begin{aligned}
 \phi_{\rho'}(x'_i, x'_j) - \phi_{\rho''}(x''_i, x''_j) &= \frac{\phi(x'_i - x'_j)}{\sum_k m_k \phi(x'_i - x'_k)} - \frac{\phi(x''_i - x''_j)}{\sum_k m_k \phi(x''_i - x''_k)} \\
 &\lesssim \frac{1}{m_i^2} \max |x'_k - x''_k|.
 \end{aligned}$$

Similar computation works for \mathcal{M}_β . Models \mathcal{M}_{seg} and \mathcal{M}_ϕ do not seem to have good well-posedness properties when it comes to agent-based systems with purely local communication kernels.

Proposition 3.24. *The models \mathcal{M}_{MT} , \mathcal{M}_β are globally well-posed provided the defining kernel ϕ is locally supported (3.12).*

3.7.5. Uniform mapping properties. When studying well-posedness of kinetic models it will be essential to have a uniform boundedness of the weighted averages at the base

level independent of ρ . These can be readily stated in terms of Lebesgue integrability conditions on the kernel. We will isolate two such conditions.

First, the uniform boundedness on $L^2(\rho)$:

$$s_\rho[\cdot]_\rho: L^2(\rho) \rightarrow L^2(\rho) \tag{3.28}$$

can be stated using the result of Lemma 3.4. It is guaranteed to hold under a simpler condition:

$$\sup_{\rho \in \mathcal{P}} \|s_\rho[1]_\rho^*\|_\infty < \infty. \tag{3.29}$$

Recalling the action of the adjoint in terms of reproducing kernel (3.4), (3.29) can be stated as $\phi_\rho \in L_y^\infty L_x^1(\rho)$ uniformly in ρ :

$$\sup_{\rho \in \mathcal{P}(\Omega)} \sup_{y \in \Omega} \int_\Omega \phi_\rho(x, y) \, d\rho(x) < \infty. \tag{3.30}$$

This condition was first documented in the context of the \mathcal{M}_{MT} -model in [53]. It holds trivially for all conservative models, see (3.7). For \mathcal{M}_β , including the Motsch–Tadmor model \mathcal{M}_{MT} , this follows from (2.6). So, all the core models on our list satisfy (3.30).

Second, a stronger uniform boundedness

$$s_\rho[\cdot]_\rho: L^2(\rho) \rightarrow L^\infty(\rho) \tag{3.31}$$

is guaranteed by the membership $\phi_\rho \in L_x^\infty L_y^2(\rho)$ uniformly in ρ (by the Hölder inequality):

$$\sup_{\rho \in \mathcal{P}(\Omega)} \sup_{x \in \Omega} \int_\Omega |\phi_\rho(x, y)|^2 \, d\rho(y) < \infty. \tag{3.32}$$

Examples on our list include all \mathcal{M}_β and $\mathcal{M}_\beta^{\text{topo}}$ models for $\beta \geq \frac{1}{2}$, and in particular, the classical Cucker–Smale model \mathcal{M}_{CS} . Indeed, we have for \mathcal{M}_β ,

$$\int_\Omega \left| \frac{\phi(x-y)}{\rho_\phi^{1-\beta}(x)} \right|^2 \, d\rho(y) \leq \|\phi\|_\infty \frac{\rho_\phi(x)}{\rho_\phi^{2-2\beta}(x)} = \|\phi\|_\infty \rho_\phi^{2\beta-1}(x) \leq \|\phi\|_\infty^2.$$

Unfortunately, \mathcal{M}_{MT} , \mathcal{M}_ϕ , and \mathcal{M}_{seg} are not regular enough to satisfy (3.32) for arbitrary kernels. However, if $\inf \phi > 0$, that is of course the case for \mathcal{M}_ϕ and all \mathcal{M}_β , and similarly if $\text{supp } g_I = \Omega$ for \mathcal{M}_{seg} .

The results are summarized in Table 5.

Model	\mathcal{M}_{CS}	$\mathcal{M}_{\text{CS}}^{\text{topo}}$	\mathcal{M}_β	\mathcal{M}_ϕ	\mathcal{M}_{seg}
$L^2 \rightarrow L^2$	✓	✓	(2.4)	✓	✓
$L^2 \rightarrow L^\infty$	✓	✓	$\beta \geq \frac{1}{2}$ or $\inf \phi > 0$	$\inf \phi > 0$	$\text{supp } g_I = \Omega$

Table 5. Conditions under which models are uniformly bounded.

4. Flocking

4.1. The Cucker–Smale theorem

We start with an extension of the classical Cucker–Smale theorem that originally appeared in [24] for the \mathcal{M}_{CS} -model. The result declares how strong the long-range communication must be in order to ensure alignment from any initial condition. The discrete, kinetic, and hydrodynamic analogues of this result are proved in exact same way, due to essentially the same structure of the characteristic equations taking one of the forms (3.5), see [82] for a detailed account. We adhere to the context of kinetic Vlasov-alignment model

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (s_\rho(v - [u]_\rho) f), \quad \text{diam}(\text{supp } f_0) < \infty, \quad (4.1)$$

where

$$\rho(x) = \int_{\mathbb{R}^n} f(x, v) \, dv, \quad u\rho(x) = \int_{\mathbb{R}^n} v f(x, v) \, dv.$$

It incorporates the agent based dynamics as a special case of a weak solution, and does not require any particular closure assumption, for more on this see [92, 93]. The pressureless Euler-alignment system allows the same treatment if written in Lagrangian coordinates, see Theorem 4.3 below. The main idea conveyed here is that the result does not require any special properties of the model and can be extended to any general material environmental averaging that has a reproducing kernel ϕ_ρ .

We consider Ω to be an arbitrary environment, although the unbounded ones, such as \mathbb{R}^n , is where the result is most meaningful. If f is a measure-valued solution to (4.1), starting from a compactly supported initial condition f_0 , then at any point of time f is given by the push-forward of f_0 along the characteristics (see Section 5):

$$\frac{d}{dt} X(t, x, v) = V(t, x, v), \quad X(0, x, v) = x, \quad (4.2)$$

$$\frac{d}{dt} V(t, x, v) = s_\rho(X)([u]_\rho(X) - V), \quad V(0, x, v) = v. \quad (4.3)$$

We abbreviate $\omega = (x, v)$ for short. The representation formula (3.5b) gives the V -equation a more specific form (using the characteristic change of coordinates):

$$\frac{d}{dt} V(t, \omega) = \int_{\Omega \times \mathbb{R}^n} \phi_\rho(X(t, \omega), X(t, \omega')) (V(t, \omega') - V(t, \omega)) \, d f_0(\omega'),$$

from which the maximum principle for V -characteristics is evident. This fundamental principle holds even for models without a representation kernel which we prove next.

Lemma 4.1 (Maximum principle). *Suppose \mathcal{M} is a material model, and $\text{supp } f_0 \subset \Omega \times \mathbb{R}^n$ is compact. Then for any $\omega \in \text{supp } f_0$ and any $t > 0$, we have*

$$V(t, \omega) \in \text{conv sup} \left(\int_{\Omega} f_0(x, v) \, dx \right).$$

Proof. The convex hull in question can be represented as the intersection of hyperspaces:

$$\text{conv supp} \left(\int_{\Omega} f_0(x, v) \, dx \right) = \bigcap_{\ell \in F \subset \mathbb{R}^n} \{v : \ell(v) \leq c_\ell\}.$$

Let us fix an $\ell \in F$. Since the action of ℓ is just a linear combination of coordinates we have

$$\frac{d}{dt} \ell(V(t, \omega)) = s_\rho(X)([\ell(u)]_\rho(X) - \ell(V)) = s_\rho(X)[\ell(u) - \ell(V)]_\rho(X).$$

By Rademacher’s lemma we can evaluate the above at a point $w \in \text{supp } f_0$ where maximum of $\ell(V(t, \omega))$ is achieved. Looking into the field under the average we have

$$\ell(u)(y, t) - \ell(V) = \frac{\int_{\mathbb{R}^n} \ell(w - V) f(t, y, w) \, dw}{\int_{\mathbb{R}^n} f(t, y, w) \, dw}.$$

Now let ψ_δ be a standard compactly supported mollifier. We have using the transport property

$$\begin{aligned} & \int_{\mathbb{R}^n} \ell(w - V) f(t, y, w) \, dw \\ &= \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^n} \ell(w - V) \psi_\delta(y - z) f(t, z, w) \, dw \, dz \\ &= \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^n} \ell(V(t, \omega') - V(t, \omega)) \psi_\delta(y - X(\omega', t)) f_0(t, \omega') \, d\omega' \leq 0. \end{aligned}$$

Thus, $\ell(u) - \ell(V) \leq 0$ point-wise. By the order preserving property of the averages (ev3), we have

$$\frac{d}{dt} \ell(V(t, \omega)) \leq 0.$$

In other words, $\max_{\text{supp } f_0} \ell(V(t, \omega)) \leq c_\ell$ for all times. This finishes the proof. ■

As a consequence, the macroscopic velocity u of f remains bounded by the initial condition:

$$\begin{aligned} |u(x, t)| &= \lim_{\delta \rightarrow 0} \left| \frac{\int_{\Omega \times \mathbb{R}^n} V(t, z, w) \psi_\delta(x - z) f_0(z, w) \, dz \, dw}{\int_{\Omega \times \mathbb{R}^n} \psi_\delta(x - z) f_0(z, w) \, dz \, dw} \right| \\ &\leq \|V\|_{L^\infty(\text{supp } f_0)} \leq \max |\text{supp } f_0|. \end{aligned}$$

Theorem 4.2 (Kinetic Cucker–Smale). *Suppose there exists $\phi \in C^\infty$, a positive, non-increasing, radially symmetric kernel with fat tail, $\int_0^\infty \phi(r) \, dr = \infty$, such that*

$$\phi_\rho(x, y) \geq \phi(x - y), \quad \forall \rho \in \mathcal{P}. \tag{4.4}$$

Then any measure-valued solution to (4.1) starting from a compactly supported initial condition f_0 aligns and flocks exponentially fast

$$\begin{aligned} D(t) &= \max_{\omega', \omega'' \in \text{supp } f_0} |X(t, \omega') - X(t, \omega'')| < C, \quad \forall t > 0, \\ A(t) &= \max_{\omega', \omega'' \in \text{supp } f_0} |V(t, \omega') - V(t, \omega'')| \leq C e^{-\delta t}, \end{aligned} \tag{4.5}$$

where $C, \delta > 0$ depend on the initial condition and the parameters of the model. Moreover, there exists $u_\infty \in \mathbb{R}^n$ such that

$$\max_{\omega \in \text{supp } f_0} |V(t, \omega) - u_\infty| \leq C e^{-\delta t}. \tag{4.6}$$

If the model \mathcal{M} is conservative, then $u_\infty = \bar{u} = \int_{\Omega} u \rho \, dx$, the total conserved momentum.

Proof. Following characteristics let us fix at any point of time a label $\omega_{\pm} \in \text{supp } f_0$, where V^i achieves its maximum and minimum, respectively, V_{\pm}^i . So, by the Rademacher lemma, we have distributionally

$$\frac{d}{dt} V_{\pm}^i = \int_{\Omega \times \mathbb{R}^n} \phi_{\rho}(X(t, \omega_{\pm}), X(t, \omega')) (V^i(t, \omega') - V_{\pm}^i) \, d f_0(\omega').$$

In view of (4.4),

$$\begin{aligned} \frac{d}{dt} V_+^i &\leq \int_{\Omega \times \mathbb{R}^n} \phi(X(t, \omega_+) - X(t, \omega')) (V^i(t, \omega') - V_+^i) \, d f_0(\omega') \\ &\leq \phi(D) \int_{\Omega \times \mathbb{R}^n} (V^i(t, \omega') - V_+^i) \, d f_0(\omega'). \end{aligned}$$

And similarly,

$$\frac{d}{dt} V_-^i \geq \phi(D) \int_{\Omega \times \mathbb{R}^n} (V^i(t, \omega') - V_-^i) \, d f_0(\omega').$$

Subtracting the two, we obtain for the amplitude $A^i = V_+^i - V_-^i$,

$$\frac{d}{dt} A^i \leq -\phi(D) A^i.$$

Taking the Euclidean amplitude $A = \sqrt{(A^1)^2 + \dots + (A^n)^2}$, we obtain the system

$$\frac{d}{dt} D \leq A, \quad \frac{d}{dt} A \leq -\phi(D) A.$$

Following [41], we form the Lyapunov function

$$L = A + \int_0^D \phi(r) \, dr,$$

which remains bounded. Hence, in view of the fat-tail condition, D remains bounded, and going back to the A -equation, we obtain exponential decay on the amplitudes.

To conclude (4.6) let us notice that as a consequence of (4.5), we have

$$\max_{\omega \in \text{supp } f_0} |\dot{V}| \leq C e^{-\delta t}.$$

So, every characteristic $V(\omega, t)$ will converge exponentially fast to a limit $u_\infty(\omega)$. In view of (4.5), u_∞ must be a constant vector. ■

The alignment of characteristics stated in Theorem 4.2 implies corresponding behavior of the distribution f itself by transport. First, we can see that its v -marginal $f^v = \int_\Omega f(t, x, v) dx$ converges weakly to the Dirac:

$$f^v \rightarrow \delta_0(v - u_\infty).$$

Moreover, since f is a push-forward of f_0 along (4.2) and (4.3), the v -support of f will belong to an exponentially shrinking ball around u_∞ . This implies uniform convergence of the macroscopic velocity

$$u(x, t) - u_\infty = \frac{\int_{|v-u_\infty| \leq C e^{-\delta t}} (v - u_\infty) f(x, v, t) dv}{\int_{|v-u_\infty| \leq C e^{-\delta t}} f(x, v, t) dv},$$

so,

$$\sup_{x \in \text{supp } \rho} |u(x, t) - u_\infty| \leq C e^{-\delta t}.$$

And it also implies exponential alignment in the energy sense, to be discussed in greater detail in Section 4.3:

$$\begin{aligned} \bar{u} &= \int_\Omega u \rho dx, \\ \delta \mathcal{E} &:= \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v - \bar{u}|^2 f dv dx \leq C e^{-\delta t}. \end{aligned}$$

Unfortunately the result does not seem to provide much insight into behavior of the macroscopic density ρ . See, however, [85] for a convergence result to a traveling wave in the one-dimensional case.

The exact same result can be stated for the hydrodynamic alignment model without pressure, the so-called pressureless Euler-alignment system (see Section 9.1 for its derivation):

$$\rho_t + \nabla \cdot (u\rho) = 0, \quad u_t + u \cdot \nabla u = s_\rho([u]_\rho - u). \tag{4.7}$$

If passed to Lagrangian coordinates

$$\begin{aligned} \dot{x}(\alpha, t) &= v(\alpha, t) := u(x(\alpha, t), t), \quad \alpha \in \Omega, \\ \dot{v}(\alpha, t) &= \int_\Omega \phi_\rho(x(t, \alpha), x(t, \alpha'))(v(t, \alpha') - v(t, \alpha)) d\rho_0(\alpha'), \end{aligned}$$

the system is structurally similar to (4.2) and (4.3). So, the proof goes through exactly as before.

Theorem 4.3 (Hydrodynamic Cucker–Smale). *Under the assumptions of Theorem 4.2, any classical solution to the pressureless Euler-alignment system (4.7) with compactly supported initial ρ_0 aligns and flocks exponentially fast:*

$$\sup_{t \geq 0} (\text{diam}(\text{supp } \rho)) < \infty, \quad \sup_{x \in \text{supp } \rho} |u(t, x) - u_\infty| \leq C_0 e^{-\delta t}.$$

For the \mathcal{M}_{CS} -model where $\phi_\rho(x, y) = \phi(x - y)$ the statements above are classical. The kinetic and hydrodynamics versions appeared in [16] and [94], respectively.

In the Motsch–Tadmor case, we can apply the same fat-tail condition on the defining kernel ϕ due to (3.13). However, the limiting velocity u_∞ is not determined by the initial condition and emerges dynamically.

The theorem does not apply to either the over-mollified model \mathcal{M}_ϕ or the segregation model \mathcal{M}_{seg} as those are inherently local, which brings us to the next main question – what conditions guarantee emergent behavior when communication is strictly local?

4.2. Chain connectivity. The $1/t^{1/4}$ and $1/t^{1/2}$ results

It is obvious that locality (3.11) itself is insufficient for unconditional alignment of the system. In the open space \mathbb{R}^n one can simply direct two agents away from each other starting at a distance larger than communication range. On \mathbb{T}^n one can launch two agents with misaligned velocities along two parallel geodesics at a distance larger than communication range. So, it is clear that some kind of connectivity is necessary to obtain alignment. In this section we explore how to achieve this for symmetric models and for quantitatively thick flocks.

Definition 4.4. We say that the flock (u, ρ) is *chain connected at scale r* if for any two points $x', x'' \in \text{supp } \rho$, there exists a chain

$$x' = x_1, x_2, \dots, x_K = x''$$

such that $x_i \in \text{supp } \rho$ and $|x_i - x_j| < r$.

Our main result shows alignment under connectivity assumption at a sub-local scale r_0 and proper thickness rate. Here we use the term thickness to refer to the ball-thickness defined in (3.15).

Theorem 4.5. *Let $\Omega = \mathbb{R}^n$, and \mathcal{M} is a symmetric model with kernel satisfying (3.11). If the flock remains chain connected at the scale $r = r_0/8$ for all time and has thickness satisfying $\bar{\rho}_r(\text{supp } \rho) \geq c/t^{1/4}$, then the flock aligns:*

$$\sup_{\omega, \omega' \in \text{supp } f_0} |V(t, \omega) - V(t, \omega')| \lesssim \frac{1}{\sqrt{\ln t}}. \tag{4.8}$$

On the torus $\Omega = \mathbb{T}^n$, the result holds under the weaker condition $\bar{\rho}_r(\text{supp } \rho) \geq c/t^{1/2}$.

In the case of the torus we can consider a non-vacuous flock $\rho_- = \min_{\mathbb{T}^n} \rho > 0$. Such a flock remains trivially connected at any scale and is uniformly thick $\bar{\rho}_r(\text{supp } \rho) \gtrsim \rho_-$. So, one important consequence of the above theorem is a statement in terms of quantitative no-vacuum condition.

Corollary 4.6. *Let $\Omega = \mathbb{T}^n$, \mathcal{M} is symmetric and material, and the kernel satisfies (3.11). If $\rho_- \geq c/t^{1/2}$, then the flock aligns (4.8).*

Before we get to the proof, we first explore how one can reduce the number of links in a chain.

Lemma 4.7. *If the flock is chain connected at scale r , then between any pair of points there is a $3r$ -chain with the number of links limited to $K \leq 2/\bar{\rho}_r(\text{supp } \rho)$.*

If the diameter of the flock is bounded, then K can be chosen independent of thickness but dependent on the diameter, $K \leq C(\text{diam}(\text{supp } \rho))$.

Proof. Suppose we have a chain $x' = x_1, x_2, \dots, x_K = x'' \in \text{supp } \rho$ with the properties listed in the definition. We now choose a subchain in the following manner. Let $x_{i_1} = x_1$. Then let us pick $i_2 - 1$ to be the largest index $> i_1$ for which $|x_{i_1} - x_{i_2-1}| < 2r$. So, all subsequent elements will stay at a distance at least $2r$ from x_{i_1} . In particular,

$$|x_{i_1} - x_{i_2}| \geq 2r,$$

and yet since $|x_{i_2-1} - x_{i_2}| < r$, we have

$$|x_{i_1} - x_{i_2}| < 3r.$$

Pick i_3 similarly to i_2 , etc. Eventually, x_K will be selected last unconditionally.

According to the construction, we have a new chain $y_j = x_{i_j}$, $j = 1, \dots, J$, such that

$$|y_j - y_{j+1}| < 3r \quad \text{and} \quad |y_j - y_k| \geq 2r$$

for any $j \neq k < J$. Hence, the chain is connected at scale $3r$. At the same time, by disjointness

$$\bar{\rho}_r(\text{supp } \rho)(J - 1) \leq \sum_{j=1}^{J-1} \rho(B_r(y_j)) = \rho\left(\bigcup_{j=1}^{J-1} B_r(y_j)\right) \leq 1.$$

Hence, $J \leq 1 + 1/\bar{\rho}_r(\text{supp } \rho) \leq 2/\bar{\rho}_r(\text{supp } \rho)$.

Alternatively, if the flock is bounded, and the balls around y_j 's are disjoint, J is limited by volume to $c_n \text{diam}(\text{supp } \rho)^n / r^n$. This proves the lemma. ■

The primary technical use of this lemma will be in the construction of chains with thick links. Specifically, if the flock is r -connected then we find it also $3r$ -connected by chains of size $K \leq 2/\bar{\rho}_r(\text{supp } \rho)$, and since any ball $B_{4r}(x_i)$ contains the balls $B_r(x_{i-1}) \cup B_r(x_{i+1})$, then

$$\rho(B_{4r}(x_i) \cap B_{4r}(x_{i+1})) \geq \bar{\rho}_r(\text{supp } \rho). \tag{4.9}$$

Proof of Theorem 4.5. Let us assume for now that $\Omega = \mathbb{R}^n$.

By symmetry of the model we have the following energy law

$$\frac{d}{dt} \mathcal{E} = -\frac{1}{2} \int_{\Omega \times \mathbb{R}^n} \phi_\rho(x, x') |v - v'|^2 f(t, \omega') f(t, \omega) d\omega' d\omega.$$

Hence, in view of (3.11),

$$\int_0^\infty \int_{\{|x-x'| < r_0\} \times \mathbb{R}^n} |v - v'|^2 f(t, \omega') f(t, \omega) d\omega' d\omega dt < \infty.$$

Consider the averages of macroscopic momenta over balls of radius $4r$:

$$\bar{v}(x) = \frac{1}{\rho(B_{4r}(x))} \int_{\mathbb{R}^n \times B_{4r}(x)} wf(t, y, w) dw dy.$$

The quadratic deviations from the averages are all subordinated to the dissipation rate:

$$\begin{aligned} & \int_{\mathbb{R}^n \times B_{4r}(x^*)} |v - \bar{v}(x^*)|^2 f(t, x, v) dv dx \\ &= \int_{\mathbb{R}^n \times B_{4r}(x^*)} \left| \frac{1}{\rho(B_{4r}(x^*))} \int_{\mathbb{R}^n \times B_{4r}(x^*)} (v - w) f(t, y, w) dw dy \right|^2 f(t, x, v) dv dx \\ &\leq \frac{1}{\rho(B_{4r}(x^*))} \int_{\mathbb{R}^n \times B_{4r}(x^*) \times \mathbb{R}^n \times B_{4r}(x^*)} |v - w|^2 f(t, y, w) f(t, x, v) dw dy dv dx \end{aligned}$$

using that $|x - y| < 8r = r_0$,

$$\leq \frac{1}{\rho(B_{4r}(x^*))} \int_{\{|x-x'| < r_0\} \times \mathbb{R}^n} |v - v'|^2 f(t, \omega') f(t, \omega) d\omega' d\omega.$$

Thus, in view of (4.15), and the fact that $\bar{\rho}_r(\text{supp } \rho) \leq \rho(B_{4r}(x^*))$,

$$\int_0^\infty \sup_{x^* \in \Omega} \bar{\rho}_r(\text{supp } \rho) \int_{\mathbb{R}^n \times B_{4r}(x^*)} |v - \bar{v}(x^*)|^2 f(t, x, v) dv dx dt < \infty. \tag{4.10}$$

Let us now estimate the flattening near extremes. Let us fix one coordinate of $v \text{ supp } f$, say v^i and denote by $v^i_+ = V^i(t, \omega_+) = \max_{\omega \in \text{supp } f_0} V^i(t, \omega)$, and $x_+ = X(t, \omega_+)$. We drop the superindex i for shortness of notation. Then

$$\begin{aligned} \frac{d}{dt} v_+ &= \int_{\Omega \times \mathbb{R}^n} \phi_\rho(x_+, y) (w - v_+) f(t, y, w) dy dw \\ &\leq c_0 \int_{B_{4r}(x_+)} (w - v_+) f(t, y, w) dy dw \\ &= c_0 \rho(B_{4r}(x_+)) (\bar{v}(x_+) - v_+) \leq c_0 \bar{\rho}_r(\text{supp } \rho) (\bar{v}(x_+) - v_+). \end{aligned}$$

Similarly,

$$\frac{d}{dt} v_- \geq c_0 \bar{\rho}_r(\text{supp } \rho) (\bar{v}(x_-) - v_-).$$

Consequently,

$$\int_0^\infty \bar{\rho}_r(\text{supp } \rho) [(\bar{v}(x_-) - v_-) + (v_+ - \bar{v}(x_+))] dt < \infty. \tag{4.11}$$

Combining (4.10) and (4.11), and fixing an $T' > 0$ large enough, we can ensure that for any $T > 0$ there is a time $t \in [T, T + T']$ such that

$$\begin{aligned} & (\bar{v}(x_-) - v_-) + (v_+ - \bar{v}(x_+)) + \sup_{x^* \in \Omega} \int_{\mathbb{R}^n \times B_{4r}(x^*)} |v - \bar{v}(x^*)|^2 f(t, x, v) dx dv \\ & < \frac{1}{\bar{\rho}_r(\text{supp } \rho)t \ln t}. \end{aligned} \tag{4.12}$$

In particular, the extreme values are close to the averages around them. Let us now show that all the averages are close to each other, and this will finish the proof.

We have for any $x^* \in \Omega$,

$$\int_{\mathbb{R}^n \times B_{4r}(x^*)} |v - \bar{v}(x^*)|^2 f(t, x, v) dx dv \leq \frac{1}{\bar{\rho}_r(\text{supp } \rho)t \ln t}.$$

Denote $\delta = 2/\Theta_r(\rho)\sqrt{t \ln t}$. Then, by the Chebyshev inequality,

$$\begin{aligned} f(\{|v - \bar{v}(x^*)| > \delta\} \times B_{4r}(x^*)) & \leq \frac{1}{\delta^2} \int_{\mathbb{R}^n \times B_{4r}(x^*)} |v - \bar{v}(x^*)|^2 f(t, x, v) dx dv \\ & \leq \frac{1}{4} \bar{\rho}_r(\text{supp } \rho). \end{aligned} \tag{4.13}$$

Let us now consider a $3r$ -chain x_1, \dots, x_K with $K < C/\bar{\rho}_r(\text{supp } \rho)$, which connects two points x_- and x_+ . According to (4.9), $\rho(B_{4r}(x_i) \cap B_{4r}(x_{i+1})) \geq \bar{\rho}_r(\text{supp } \rho)$. Thus, $f(\mathbb{R}^n \times (B_{4r}(x_i) \cap B_{4r}(x_{i+1}))) \geq \bar{\rho}_r(\text{supp } \rho)$. Yet according to (4.13),

$$f(\left(\{|v - \bar{v}(x_i)| > \delta\} \times B_{4r}(x_i)\right) \cup \left(\{|v - \bar{v}(x_{i+1})| > \delta\} \times B_{4r}(x_{i+1})\right)) \leq \frac{1}{2} \bar{\rho}_r(\text{supp } \rho).$$

Consequently,

$$B_\delta(\bar{v}(x_i)) \times B_{4r}(x_i) \cap B_\delta(\bar{v}(x_{i+1})) \times B_{4r}(x_{i+1}) \neq \emptyset.$$

Hence,

$$|\bar{v}(x_i) - \bar{v}(x_{i+1})| \leq 2\delta.$$

Summing up over all i , we obtain

$$|\bar{v}(x_+) - \bar{v}(x_-)| \leq 2\delta K \lesssim \frac{1}{\bar{\rho}_r^2(\text{supp } \rho)\sqrt{t \ln t}} \sim \frac{1}{\sqrt{\ln t}}. \tag{4.14}$$

Combining with (4.12), we have

$$v_+ - v_- \leq \frac{1}{\sqrt{\ln t}}.$$

Since this holds at time $t < T + T'$, it must hold at time $T + T'$ by the maximum principle. But since $t > T$,

$$\frac{1}{\sqrt{\ln t}} \leq \frac{1}{\sqrt{\ln T}} \lesssim \frac{1}{\sqrt{\ln(T + T')}}.$$

Since T is arbitrary, this finishes the proof in the open space.

On the torus the diameter of the flock is uniformly bounded, and consequently, by Lemma 4.7, K remains uniformly bounded. In this case the estimate (4.14) gets improved to the following

$$|\bar{v}(x_+) - \bar{v}(x_-)| \leq 2\delta K \lesssim \frac{1}{\bar{\rho}_r(\text{supp } \rho)\sqrt{t \ln t}} \sim \frac{1}{\sqrt{\ln t}},$$

provided $\bar{\rho}_r(\text{supp } \rho) \gtrsim 1/t^{1/2}$. The rest of the proof is the same. ■

Remark 4.8. The exact same result holds for solutions of the pressureless Euler-alignment system (4.7), thanks to the fact that it has a similar form of the energy dissipation

$$\int_0^\infty \int_{\Omega \times \Omega} \phi_\rho(x, y) |u(x, t) - u(y, t)|^2 d\rho(y) d\rho(y) dt < \infty. \tag{4.15}$$

4.3. Alignment in the energy sense. Spectral gaps

The alignment of characteristics stated in Theorem 4.5 implies alignment in the energy sense. Recalling that $\bar{u} = \int_\Omega u \rho dx$, we have

$$\delta \mathcal{E} = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v - \bar{u}|^2 f dv dx \leq \int_{\Omega \times \mathbb{R}^n \times \Omega \times \mathbb{R}^n} |V - V'|^2 f_0 f'_0 d\omega d\omega' \lesssim \frac{1}{\sqrt{\ln t}}.$$

In this section we explore alignment in this weaker sense

$$\delta \mathcal{E} \rightarrow 0,$$

by appealing to the most basic energy law of the Vlasov-alignment equation (4.1).

We will not make any special assumptions on the underlying model \mathcal{M} except that \mathcal{M} is material just to make sense of the strength function in equation (4.1). In particular, the momentum \bar{u} may not be conserved.

In order to write the equation for $\delta \mathcal{E}$, let us note the identity

$$\mathcal{E} := \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 f dv dx = \delta \mathcal{E} + \frac{1}{2} |\bar{u}|^2.$$

The momentum satisfies

$$\frac{d}{dt} \frac{1}{2} |\bar{u}|^2 = (\bar{u}, [u]_\rho - u)_{\kappa_\rho},$$

and the equation for total energy is given by

$$\frac{d}{dt} \mathcal{E} = - \int_{\Omega \times \mathbb{R}^n} s_\rho |v|^2 f dv dx + (u, [u]_\rho)_{\kappa_\rho}.$$

Subtracting the two we obtain

$$\frac{d}{dt} \delta \mathcal{E} = - \int_{\Omega \times \mathbb{R}^n} s_\rho |v|^2 f \, dv \, dx + (u, [u]_\rho)_{\kappa_\rho} - (\bar{u}, [u]_\rho - u)_{\kappa_\rho}.$$

Let us further notice the identity

$$\int_{\Omega \times \mathbb{R}^n} s_\rho |v|^2 f \, dv \, dx = \int_{\Omega \times \mathbb{R}^n} s_\rho |v - \bar{u}|^2 f \, dv \, dx + 2(u, \bar{u})_{\kappa_\rho} - (\bar{u}, \bar{u})_{\kappa_\rho}.$$

Collecting the macroscopic terms together we obtain

$$\frac{d}{dt} \delta \mathcal{E} = - \int_{\Omega \times \mathbb{R}^n} s_\rho |v - \bar{u}|^2 f \, dv \, dx + (\delta u, [\delta u]_\rho)_{\kappa_\rho}, \quad \delta u = u - \bar{u}.$$

Next, let us decompose the energy on the right-hand side into the internal and macroscopic part,

$$\int_{\Omega \times \mathbb{R}^n} s_\rho |v - \bar{u}|^2 f \, dv \, dx = \int_{\Omega \times \mathbb{R}^n} s_\rho |v - u|^2 f \, dv \, dx + (\delta u, \delta u)_{\kappa_\rho}.$$

We obtain the energy law

$$\frac{d}{dt} \delta \mathcal{E} = - \int_{\Omega \times \mathbb{R}^n} s_\rho |v - u|^2 f \, dv \, dx + (\delta u, [\delta u]_\rho)_{\kappa_\rho} - (\delta u, \delta u)_{\kappa_\rho}. \tag{4.16}$$

Naturally, we will seek to relate the right-hand side back to the energy. This comes from two assumptions. First, we require that the averaging operator has a numerical range separated from 1, i.e. at any time there exists $\varepsilon = \varepsilon(t) \in (0, 1)$ such that

$$\sup \{ (u, [u]_\rho)_{\kappa_\rho} : u \in L^2(\kappa_\rho), \bar{u} = 0, \|u\|_{L^2(\kappa_\rho)} = 1 \} \leq 1 - \varepsilon. \tag{4.17}$$

This in turn implies

$$(\delta u, \delta u)_{\kappa_\rho} - (\delta u, [\delta u]_\rho)_{\kappa_\rho} \geq \varepsilon (\delta u, \delta u)_{\kappa_\rho}. \tag{4.18}$$

Second, we require the strength function to have a positive lower bound

$$\inf_{x \in \text{supp } \rho} s_\rho(x, t) = s(t).$$

Plugging these back into (4.16), we obtain

$$\begin{aligned} \frac{d}{dt} \delta \mathcal{E} &\leq - \int_{\Omega \times \mathbb{R}^n} s_\rho |v - u|^2 f \, dv \, dx - \varepsilon (\delta u, \delta u)_{\kappa_\rho} \\ &\leq -\varepsilon \left(\int_{\Omega \times \mathbb{R}^n} s_\rho |v - u|^2 f \, dv \, dx + (\delta u, \delta u)_{\kappa_\rho} \right) \\ &= -\varepsilon \int_{\Omega \times \mathbb{R}^n} s_\rho |v - \bar{u}|^2 f \, dv \, dx \leq -\varepsilon s \delta \mathcal{E}. \end{aligned}$$

This implies a general sufficient condition for alignment.

Proposition 4.9. *Let \mathcal{M} be a material model on an arbitrary environment Ω . The kinetic model (4.1) aligns in the energy sense provide the following condition holds:*

$$\int_0^\infty \varepsilon(t)s(t) dt = \infty.$$

A few remarks are in order.

Remark 4.10. Let us note that for symmetric models with $s_\rho \equiv 1$, the space of vanishing momentum $L_0^2(\rho)$ is invariant under $[\cdot]_\rho$, and the numerical range determines the range of the spectrum. So, condition (4.17) is equivalent to a spectral gap between the trivial eigenvalue 1 and the rest of the spectrum to the left

$$\text{spec}\{[\cdot]_\rho; L_0^2(\rho)\} \subset (-\infty, 1 - \varepsilon],$$

where

$$L_0^2(\rho) = \left\{ u \in L^2(\rho) : \int_\Omega u d\rho = 0 \right\}.$$

For this reason, although in general (4.17) is not a spectral property, we still refer to it as a *spectral gap*.

In general, however, conservative models leave the null-space

$$L_0^2(\kappa_\rho) = \left\{ u \in L^2(\kappa_\rho) : \int_\Omega u d\kappa_\rho = 0 \right\}$$

invariant. In this case, it is possible to relate ε to the actual spectral gap of $[\cdot]_\rho$ on $L_0^2(\kappa_\rho)$ if s_ρ is bounded from below. Details are provided in Appendix C.

Remark 4.11. Proposition 4.9 can be viewed as a generalization of Tadmor’s [92] to the non-symmetric case. The argument there is slightly different in the interpretation of the spectral gap condition (4.18). As opposed to (4.18) where all the inner products are related to the common κ_ρ -weight, one can make a more direct relation to the physical macroscopic energy, i.e. the ρ -weighted product

$$(\delta u, \delta u)_{\kappa_\rho} - (\delta u, [\delta u]_\rho)_{\kappa_\rho} \geq \lambda (\delta u, \delta u)_\rho.$$

The corresponding alignment statement in terms of λ reads

$$\int_0^\infty \min\{s(t), \lambda(t)\} dt = \infty.$$

Such λ can be expressed in variational form as the second (approximate) eigenvalue of the alignment operator

$$\mathcal{L}_\rho u = s_\rho(u - [u]_\rho).$$

We have

$$\lambda = \inf_{u \in L_0^2(\rho)} \frac{(u, \mathcal{L}_\rho u)_\rho}{(u, u)_\rho}. \tag{4.19}$$

The advantage of this approach consists in the fact that for symmetric models represented by a kernel the formulation (4.19) takes a more explicit form

$$\lambda = \inf_{u \in L^2_0(\rho), \|u\|_2=1} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 \phi_\rho(x, y) \, d\rho(y) \, d\rho(x).$$

Theorem 2 of [92] gives a kinematic estimate in terms of lower and upper bounds on the density, in the case when $\Omega = \mathbb{T}^n$. Namely,

$$\lambda \gtrsim \frac{\rho_-^2}{\rho_+}. \tag{4.20}$$

The result is proved under condition (4.21) below, however it can be recast for physically local kernels (3.11) as well. Let us reproduce the argument as it will be used later in Example 4.14.

Proof of (4.20). We obtain

$$(u, \mathcal{L}_\rho u)_\rho \geq c_0 \rho_-^2 \int_{|x-y| < r_0} |u(x) - u(y)|^2 \, dx \, dy.$$

As shown in [62, Lemma 2.1] this can be further estimated from below by

$$\geq c_0 c_1 \rho_-^2 \frac{c_0}{(2\pi)^n} \int_{\mathbb{T}^n} |u(x) - \text{Ave}(u)|^2 \, dx,$$

where $c_1 = c_1(r_0)$, and $\text{Ave}(u) = \frac{1}{(2\pi)^n} \int_\Omega u(x) \, dx$. Recalling that u has momentum zero, we finish with

$$\gtrsim \frac{\rho_-^2}{\rho_+} \int_{\mathbb{T}^n} |u(x) - \text{Ave}(u)|^2 \, d\rho(x) \geq \frac{\rho_-^2}{\rho_+} (u, u)_\rho. \quad \blacksquare$$

Estimate (4.20) shows that under global control on ρ_+ one obtains alignment under the root-assumption $\rho_- \gtrsim 1/\sqrt{t}$, the same result as proved in Corollary 4.6 under no assumption on ρ_+ . The difference between the two approaches is fundamental – dynamic vs kinematic. It appears that the dynamic approach is not sensitive to the density growth and gives a better result for symmetric models on the torus. However, as we will see later in Section 4.4 the kinematic approach, although in somewhat different form than presented here, gives estimates independent of ρ_+ as well, and in some cases can even beat the root-result, see Proposition 4.16. Any bound on the spectral gap that does not rely on ρ_+ will prove to be a crucial in the study of relaxation for kinetic Fokker–Planck models in Section 8.

Let us present two applications of Proposition 4.9 that are distinctly different from the root-result. In both cases we assume $\Omega = \mathbb{T}^n$.

Example 4.12 (\mathcal{M}_{CS} -model). Let us assume that ϕ is a mollification kernel, where $\phi \geq 0$, and

$$\int_\Omega \phi \, dx = 1,$$

local or not. Then its non-zero Fourier modes will necessarily be smaller than the unit:

$$c_0 = \sup_{k \in \mathbb{Z}^n \setminus \{0\}} |\widehat{\phi}(k)| < 1. \tag{4.21}$$

Let us compute the spectral gap as defined by (4.17). Using that $\int u \rho \, dx = 0$ by the Plancherel identity,

$$(u, [u]_\rho)_{\kappa_\rho} = \int_\Omega u \rho (u \rho)_\phi \, dx = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |\widehat{u \rho}(k)|^2 \operatorname{Re}(\widehat{\phi}(k)) \leq c_0 \int_\Omega |u \rho|^2 \, dx.$$

We now relate it back to the $L^2(\kappa_\rho)$ -norm:

$$(u, [u]_\rho)_{\kappa_\rho} \leq c_0 \int_\Omega |u|^2 \rho \rho_\phi \frac{\rho}{\rho_\phi} \, dx \leq c_0 \left\| \frac{\rho}{\rho_\phi} \right\|_\infty \|u\|_{L^2(\kappa_\rho)}^2.$$

Suppose now that $\|\rho/\rho_\phi\|_\infty < 1/c_0$. We define

$$\varepsilon = 1 - c_0 \left\| \frac{\rho}{\rho_\phi} \right\|_\infty. \tag{4.22}$$

Naturally, $\varepsilon < 1 - c_0$ since at the point of maximum of ρ we have $\rho \geq \rho_\phi$, and so the L^∞ -norm is at least 1. Also, note that if ρ is convex in a ball $B_r(x)$, where r is the range of the communication kernel, then $\rho(x) \leq \rho_\phi(x)$, and (4.22) holds if restricted to that ball. So, the spectral gap (4.22) essentially quantifies flatness of the density ρ in those regions where it is not convex.

Also, note that for $\varepsilon = 1 - c_0$, the only flock that satisfies (4.22) is the uniformly distributed one. So, the smaller the ε the more room there is for variations in distribution. However, (4.22) still ensures sufficient spread of the support across the domain (for otherwise the geodesic counterexample applies).

Now, a lower bound on s_ρ can be interpreted as a measure of thickness (see Section 3.7),

$$s(t) = \Theta(\rho, \operatorname{supp} \rho). \tag{4.23}$$

Collecting the computations above and applying Proposition 4.9, we obtain the following alignment result.

Corollary 4.13. *For the Cucker–Smale model \mathcal{M}_{CS} , a sufficient condition for alignment in the energy sense is the flatness (4.22) and thickness (4.23) to satisfy $\int_0^\infty \varepsilon(t) s(t) \, dt = \infty$.*

Example 4.14 (\mathcal{M}_{MT} -model). For the Motsch–Tadmor non-symmetric model \mathcal{M}_{MT} computation of the gap is more technical and require heavier assumptions on the density.

Let us assume that the defining kernel ϕ is local (see (3.11)) and $\int \phi \, dx = 1$. We have

$$(u, u)_\rho - (u, [u]_\rho)_\rho = \int_{\Omega \times \Omega} u(x) \cdot (u(x) - u(y)) \rho(x) \rho(y) \frac{\phi(x - y)}{\rho_\phi(x)} \, dy \, dx,$$

symmetrizing in x and y ,

$$\begin{aligned}
 &= \frac{1}{2} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 \rho(x) \rho(y) \frac{\phi(x - y)}{\rho_\phi(x)} \, dy \, dx \\
 &\quad + \frac{1}{2} \int_{\Omega \times \Omega} u(y) \cdot (u(x) - u(y)) \rho(x) \rho(y) \left(\frac{1}{\rho_\phi(x)} - \frac{1}{\rho_\phi(y)} \right) \phi(x - y) \, dy \, dx. \tag{4.24}
 \end{aligned}$$

Now, using that $\rho_\phi(x) \leq \|\phi\|_\infty$ we bound the first term from below by a multiple of $(u, \mathcal{L}_\rho u)_\rho$, which by (4.20) is bounded from below by $c(\rho_-^2/\rho_+) \|u\|_2^2$. As to the second term, note that the component with the dot-product $u(y) \cdot u(x)$ vanishes by symmetry, and hence we are left with

$$\begin{aligned}
 &-\frac{1}{2} \int_{\Omega \times \Omega} |u(y)|^2 \rho(y) \rho(x) \left(\frac{1}{\rho_\phi(x)} - \frac{1}{\rho_\phi(y)} \right) \phi(x - y) \, dy \, dx \\
 &= -\frac{1}{2} \int_{\Omega \times \Omega} |u(y)|^2 \rho(y) \left(\frac{\rho}{\rho_\phi} \right)_\phi(y) \, dy + \frac{1}{2} \|u\|_2^2 \\
 &= \frac{1}{2} \int_{\Omega \times \Omega} |u(y)|^2 \rho(y) \left(1 - \frac{\rho}{\rho_\phi} \right)_\phi(y) \, dy.
 \end{aligned}$$

We now impose the following condition on the smallness of variation:

$$\rho_+ - \rho_- \leq c \frac{\rho_-^3}{\rho_+}. \tag{4.25}$$

Then

$$\left(1 - \frac{\rho}{\rho_\phi} \right)_\phi(y) \leq \frac{\rho_+ - \rho_-}{\rho_-} \leq c \frac{\rho_-^2}{\rho_+}.$$

Consequently, this term becomes less than half of the main dissipation term (4.24),

$$(u, u)_\rho - (u, [u]_\rho)_\rho \geq \frac{c}{2} \frac{\rho_-^2}{\rho_+} (u, u)_\rho.$$

So, similar to the symmetric case under the flatness assumption (4.25), the size of the spectral gap is still estimated at $\lambda = \varepsilon \gtrsim \rho_-^2/\rho_+$.

Corollary 4.15. *There exists a $c > 0$ which depends only on the parameters of the model such that any solution to the kinetic equation (4.1) on \mathbb{T}^n governed by the Motsch–Tadmor averaging aligns in the energy sense, provided*

$$\rho_+ - \rho_- \leq c \frac{\rho_-^3}{\rho_+}, \quad \int_0^\infty \frac{\rho_-^2}{\rho_+} \, ds = \infty.$$

4.4. Spectral gap of a ball-positive model. Low energy method

As we have seen the spectral gap condition (4.17) plays a central role in alignment dynamics and will be important in the study of relaxation, see Section 8. It will be essential to

find bounds on ε that are independent of ρ_+ , since the growth of the density cannot be controlled away from equilibrium. In this section we present the so-called *low energy method* which allows one to obtain such bounds for ball-positive models on \mathbb{T}^n .

To describe the method let us first discuss energetics of ball-positive models. Since $\|[\cdot]_\rho\|_{L^2(\kappa_\rho)} \leq 1$, we obtain a streak of three inequalities,

$$(u, u)_{\kappa_\rho} \geq (u, [u]_\rho)_{\kappa_\rho} \geq ([u]_\rho, [u]_\rho)_{\kappa_\rho}.$$

This defines the hierarchy of three κ -energies (not to be confused with the physical ρ -energies)

$$\mathcal{E}_0 = (u, u)_{\kappa_\rho}, \quad \mathcal{E}_1 = (u, [u]_\rho)_{\kappa_\rho}, \quad \mathcal{E}_2 = ([u]_\rho, [u]_\rho)_{\kappa_\rho}.$$

As seen from (3.8) the difference between the first two energies $\mathcal{A}_0 = \mathcal{E}_0 - \mathcal{E}_1$ controls the rate of alignment in collective systems. The next difference $\mathcal{A}_1 = \mathcal{E}_1 - \mathcal{E}_2$ is also non-negative by the very definition of ball-positivity, and in fact by the Cauchy–Schwarz inequality one has the relation

$$\mathcal{A}_0 \geq \mathcal{A}_1.$$

So, it is clear that the strength of ball-positivity measured by \mathcal{A}_1 bears direct relevance to alignment.

To adopt it for spectral gap calculations, we note that the spectral gap condition (4.17) can be expressed directly in terms of top tier energies

$$\mathcal{A}_0 \geq \varepsilon \mathcal{E}_0, \quad \forall u \in L^2(\kappa_\rho), \quad \bar{u} = \int_\Omega u \rho \, dx = 0. \tag{4.26}$$

The lower energy method seeks to achieve (4.26) through comparison between the two terms down in the hierarchy (low energies)

$$\mathcal{A}_1 \geq \varepsilon \mathcal{E}_1, \quad \forall u \in L^2(\kappa_\rho), \quad \bar{u} = 0. \tag{4.27}$$

Indeed, let us observe that (4.27) is equivalent to

$$(1 - \varepsilon)(u, [u]_\rho)_{\kappa_\rho} \geq ([u]_\rho, [u]_\rho)_{\kappa_\rho}, \quad \forall u \in L^2(\kappa_\rho), \quad \bar{u} = 0, \tag{4.28}$$

and hence

$$\|[u]_\rho\|_{L^2(\kappa_\rho)} \leq (1 - \varepsilon)\|u\|_{L^2(\kappa_\rho)}, \quad \forall u \in L^2(\kappa_\rho), \quad \bar{u} = 0,$$

which implies that (4.17) \sim (4.26).

One can see from (4.28) that the method is necessarily restricted to the class of ball-positive models. It turns out that estimating the low energy gap (4.27) sometimes gives substantial improvements over the direct approach (4.26) in the sense of giving a bound independent of ρ_+ . Let us present several examples from our list.

Throughout we assume that the kernel in question is local ((3.11), (3.12)), and the environment is periodic $\Omega = \mathbb{T}^n$. The summary of estimates to be obtained below is given in the following proposition.

Proposition 4.16. *For each of the ball-positive models \mathcal{M}_{CS} , \mathcal{M}_ϕ , \mathcal{M}_{seg} , Table 6 shows the bounds from below on the spectral gap up to a constant multiple. In particular, if the kernel is all-to-all, $\inf \phi_\rho > 0$, then the spectral gap is automatically uniform.*

Model	\mathcal{M}_{CS}	\mathcal{M}_ϕ	\mathcal{M}_{seg}
spectral gap	$\bar{\rho}_{r_0/2}^3(\Omega)$	$\bar{\rho}_{r_0/2}(\Omega)$	$\bar{\rho}_{r_{seg}}^{2L}(\Omega)$, see (4.37)

Table 6. Lower bounds on spectral gaps.

Proof of Proposition 4.16 for the \mathcal{M}_ϕ -model. For the \mathcal{M}_ϕ -model the following formula was proved in [83]:

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{2} \int_{\Omega \times \Omega} \rho_{\phi\phi}(x, y) |u_F(x) - u_F(y)|^2 dx dy, \\ \rho_{\phi\phi}(x, y) &= \int_{\Omega} \phi(x - \xi)\phi(y - \xi)\rho(\xi) d\xi, \end{aligned} \tag{4.29}$$

where u_F is the Favre-filtration given by \mathcal{M}_{MT} . The proof goes as follows:

$$\begin{aligned} \mathcal{A}_1 &= \int_{\Omega} (\rho_\phi |u_F|^2 - \rho |(u_F)_\phi|^2) dx \\ &= \int_{\Omega} (\rho_\phi u_F \cdot u_F - \rho (u_F)_\phi \cdot (u_F)_\phi) dx \\ &= \int_{\Omega} (\rho_\phi u_F - \rho (u_F)_\phi) \cdot u_F dx \\ &= \int_{\Omega \times \Omega} \phi(x - \xi)\rho(\xi)(u_F(x) - (u_F)_\phi(\xi)) \cdot u_F(x) d\xi dx \\ &= \int_{\Omega \times \Omega \times \Omega} \phi(x - \xi)\phi(y - \xi)\rho(\xi)(u_F(x) - u_F(y)) \cdot u_F(x) d\xi dx dy \\ &= \int_{\Omega \times \Omega} \rho_{\phi\phi}(x, y)(u_F(x) - u_F(y)) \cdot u_F(x) dx dy \\ &= \frac{1}{2} \int_{\Omega \times \Omega} \rho_{\phi\phi}(x, y) |u_F(x) - u_F(y)|^2 dx dy, \end{aligned}$$

where in the last step we performed symmetrization in x, y .

We now estimate $\rho_{\phi\phi}$ from below: let $|x - y| < r_0/2$, then

$$\begin{aligned} \rho_{\phi\phi}(x, y) &= \int_{\Omega} \phi(x - y + \xi)\phi(\xi)\rho(y - \xi) d\xi \\ &\geq \int_{|\xi| < r_0/2} \phi(x - y + \xi)\phi(\xi)\rho(y - \xi) d\xi \\ &\geq c_0^2 \int_{|\xi| < r_0/2} \rho(y - \xi) d\xi \geq c_0^2 \bar{\rho}_{r_0/2}(\Omega). \end{aligned}$$

Thus,

$$\rho_\phi(x, y) \gtrsim \bar{\rho}_{r_0/2}(\Omega) \mathbb{1}_{|x-y| < r_0/2}. \tag{4.30}$$

With this at hand, we have

$$\mathcal{A}_1 \gtrsim \bar{\rho}_{r_0/2}(\Omega) \int_{|x-y| < r_0/2} |u_F(x) - u_F(y)|^2 dx dy,$$

and by [62, Lemma 2.1],

$$\gtrsim \bar{\rho}_{r_0/2}(\Omega) \int_{\Omega} |u_F - \text{Ave}(u_F)|^2 dx \gtrsim \bar{\rho}_{r_0/2}(\Omega) \int_{\Omega} \rho_\phi |u_F - \text{Ave}(u_F)|^2 dx.$$

Using the vanishing momentum, $\text{Ave}((u\rho)_\phi) = 0$, we continue:

$$= \bar{\rho}_{r_0/2}(\Omega) \left(\int_{\Omega} \rho_\phi |u_F|^2 dx - \underbrace{2 \text{Ave}(u_F) \cdot \text{Ave}((u\rho)_\phi)}_{=0} + (2\pi)^n |\text{Ave}(u_F)|^2 \right).$$

Noting that $\int_{\Omega} \rho_\phi |u_F|^2 dx = \mathcal{E}_1$, we conclude:

$$\geq \bar{\rho}_{r_0/2}(\Omega) \mathcal{E}_1.$$

So, we have a bound

$$\varepsilon \geq c \bar{\rho}_{r_0/2}(\Omega),$$

where $c > 0$ depends only on the parameters of the model. ■

We obtain the following improvement over the general root-result of Corollary 4.6.

Corollary 4.17. *Under the \mathcal{M}_ϕ -averaging protocol a solution to the Vlasov-alignment equation (4.1) aligns if $\rho_- \gtrsim 1/t$.*

Let us note that under this weak assumption on the density the only known alignment result was established in [87] for singular topological models. And in one dimension it was proved to hold automatically for any non-vacuous solutions to the Euler-alignment system (9.2) based on the metric or topological Cucker–Smale averaging protocol. For the system based on the \mathcal{M}_ϕ -model such a bound is unknown a priori.

Proof of Proposition 4.16 for the \mathcal{M}_{CS} -model. By the assumptions of ball-positivity and locality, $\phi = \psi * \psi$, where ψ is a non-negative smooth kernel satisfying

$$\psi(x) \geq c_0 \mathbb{1}_{|x| \leq r_0}. \tag{4.31}$$

Let us apply the low energy method. We aim to prove the following bound:

$$\varepsilon \gtrsim \bar{\rho}_{r_0/2}^3(\Omega). \tag{4.32}$$

To prove (4.32) we will quantify the alignment term \mathcal{A}_1 in a way similar to the previous example. To achieve this we notice that for the Bochner-positive ϕ the \mathcal{M}_{CS} -averaging is nothing but a nested application of two distinct Favre filtrations. Indeed, let us denote

$$v = \frac{(u\rho)_\psi}{\rho_\psi}, \quad \varrho = \rho_\psi.$$

Then denoting $v_F = (v\varrho)_\psi / \varrho_\psi$, we obtain

$$[u]_\rho = \frac{(u\rho)_\phi}{\rho_\phi} = \frac{((u\rho)_\psi)_\psi}{\varrho_\psi} = \frac{(((u\rho)_\psi / \rho_\psi)\rho_\psi)_\psi}{\varrho_\psi} = v_F. \tag{4.33}$$

Observe that

$$\mathcal{A}_1 = \int_\Omega (u\rho)_\psi^2 dx - \int_\Omega |[u]_\rho|^2 \rho \rho_\phi dx = \int_\Omega |v|^2 \varrho \rho_\psi dx - \int_\Omega |v_F|^2 \rho \varrho_\psi dx.$$

Let us examine the second term now: $|v_F|^2 \rho \varrho_\psi$. We use the fact that the Favre-filtration with respect to ψ, ϱ is a symmetric operation relative to the measure $\varrho \varrho_\psi$. So, we can write

$$\begin{aligned} \int_\Omega |v_F|^2 \rho \varrho_\psi dx &= \int_\Omega v_F \cdot \left(v_F \frac{\rho}{\varrho}\right) \varrho \varrho_\psi dx \\ &= \int_\Omega v \cdot \left(v_F \frac{\rho}{\varrho}\right)_F \varrho \varrho_\psi dx = \int_\Omega v \cdot (v_F \rho)_\psi \varrho dx. \end{aligned}$$

Now let us factor out the common $v\varrho$ term:

$$\begin{aligned} \mathcal{A}_1 &= \int_\Omega \varrho v \cdot (\rho_\psi v - (v_F \rho)_\psi) dx \\ &= \int_{\Omega^2} \varrho(x) \rho(y) v(x) \cdot (v(x) - v_F(y)) \psi(x - y) dy dx \end{aligned}$$

expanding further in $v_F(y)$, we obtain

$$\begin{aligned} &= \int_{\Omega^2} \frac{\varrho(x) \rho(y)}{\varrho_\psi(y)} v(x) \cdot (v(x) \varrho_\psi(y) - (v\varrho)_\psi(y)) \psi(x - y) dy dx \\ &= \int_{\Omega^3} \frac{\varrho(x) \rho(y) \varrho(z)}{\varrho_\psi(y)} v(x) \cdot (v(x) - v(z)) \psi(z - y) \psi(x - y) dz dy dx, \end{aligned}$$

symmetrizing in x, z ,

$$= \frac{1}{2} \int_{\Omega^3} \frac{\varrho(x) \rho(y) \varrho(z)}{\varrho_\psi(y)} |v(x) - v(z)|^2 \psi(z - y) \psi(x - y) dz dy dx.$$

Notice that the integral in y represents the application of the variable doubling convolution to ρ/ρ_ϕ as in (4.29) using kernel ψ . So we obtain the following exact formula for \mathcal{A}_1 :

$$\mathcal{A}_1 = \frac{1}{2} \int_{\Omega^2} \varrho(x) \varrho(z) \left(\frac{\rho}{\rho_\phi}\right)_{\psi\psi}(x, z) |v(x) - v(z)|^2 dz dx.$$

Since $\rho_\phi \leq c_1$ pointwise, we have, using (4.30),

$$\left(\frac{\rho}{\rho_\phi}\right)_{\psi\psi} \geq c_1 \rho_{\psi\psi} \gtrsim \bar{\rho}_{r_0/2}(\Omega) \mathbb{1}_{|x-z|<r_0/2}.$$

So,

$$\begin{aligned} \mathcal{A}_1 &\gtrsim \bar{\rho}_{r_0/2}(\Omega) \int_{|x-z|<r_0/2} \varrho(x)\varrho(z)|v(x) - v(z)|^2 \, dz \, dx \\ &\gtrsim \bar{\rho}_{r_0/2}^3(\Omega) \int_{|x-z|<r_0/2} |v(x) - v(z)|^2 \, dz \, dx, \end{aligned}$$

proceeding as for the \mathcal{M}_ϕ -model,

$$\begin{aligned} &\geq \bar{\rho}_{r_0/2}^3(\Omega) \int_{\Omega} |v(x) - \text{Ave}(v)|^2 \, dx \gtrsim \bar{\rho}_{r_0/2}^3(\Omega) \int_{\Omega} \varrho|v(x) - \text{Ave}(v)|^2 \, dx \\ &\geq \bar{\rho}_{r_0/2}^3(\Omega) \int_{\Omega} \varrho|v(x)|^2 \, dx = \bar{\rho}_{r_0/2}^3(\Omega) \int_{\Omega} \frac{(u\rho)_{\psi}^2}{\rho_{\psi}} \, dx \\ &\gtrsim \bar{\rho}_{r_0/2}^3(\Omega) \int_{\Omega} (u\rho)_{\psi}^2 \, dx = \bar{\rho}_{r_0/2}^3(\Omega) \mathcal{E}_1. \end{aligned}$$

We arrive at (4.32). ■

Proof of Proposition 4.16 for the \mathcal{M}_{seg} -model. Since this model is symmetric and non-negative definite it is automatically ball-positive by Lemma 3.10. So, it is natural to apply the low-energy approach. We start with the analogue (4.29) which in this case reads

$$\mathcal{A}_1 = \frac{1}{2} \sum_{l,l'} \rho(g_l g_{l'}) \left| \frac{\rho(ug_l)}{\rho(g_l)} - \frac{\rho(ug_{l'})}{\rho(g_{l'})} \right|^2.$$

Indeed,

$$\begin{aligned} \mathcal{A}_1 &= \sum_l \frac{(\rho(ug_l))^2}{\rho(g_l)} - \int_{\Omega} \left(\sum_l g_l \frac{\rho(ug_l)}{\rho(g_l)} \right)^2 \rho \, dx \\ &= \sum_l \frac{(\rho(ug_l))^2}{\rho(g_l)} - \sum_{l,l'} \rho(g_l g_{l'}) \frac{\rho(ug_l)}{\rho(g_l)} \frac{\rho(ug_{l'})}{\rho(g_{l'})} \\ &= \sum_l \rho(ug_l) \left(\frac{\rho(ug_l)}{\rho(g_l)} - \sum_{l'} \frac{\rho(g_l g_{l'})}{\rho(g_l)} \frac{\rho(ug_{l'})}{\rho(g_{l'})} \right), \end{aligned}$$

noting that the coefficients $\rho(g_l g_{l'})/\rho(g_l)$ add up to 1 over l' ,

$$\begin{aligned} &= \sum_l \rho(ug_l) \sum_{l'} \frac{\rho(g_l g_{l'})}{\rho(g_l)} \left(\frac{\rho(ug_l)}{\rho(g_l)} - \frac{\rho(ug_{l'})}{\rho(g_{l'})} \right) \\ &= \sum_{l,l'} \rho(g_l g_{l'}) \frac{\rho(ug_l)}{\rho(g_l)} \left(\frac{\rho(ug_l)}{\rho(g_l)} - \frac{\rho(ug_{l'})}{\rho(g_{l'})} \right), \end{aligned}$$

symmetrizing over l, l' ,

$$= \frac{1}{2} \sum_{l,l'} \rho(g_l g_{l'}) \left| \frac{\rho(ug_l)}{\rho(g_l)} - \frac{\rho(ug_{l'})}{\rho(g_{l'})} \right|^2.$$

The formula indicates that the energy keeps dissipating as long as discrepancies remain between local averages in adjacent and connected neighborhoods, $\rho(g_l g_{l'}) > 0$. To extract a working criterion out of it, we rewrite \mathcal{A}_1 in a different way:

$$\mathcal{A}_1 = \sum_l \frac{(\rho(ug_l))^2}{\rho(g_l)} - \sum_{l,l'} G_{ll'} \frac{\rho(ug_l)}{\sqrt{\rho(g_l)}} \frac{\rho(ug_{l'})}{\sqrt{\rho(g_{l'})}},$$

where

$$G_{ll'} = \frac{\rho(g_l g_{l'})}{\sqrt{\rho(g_l)\rho(g_{l'})}}.$$

Considering those as entries of the symmetric matrix $G = \{G_{ll'}\}_{l,l'=1}^L$ and denoting the vector

$$X = \left(\frac{\rho(ug_1)}{\sqrt{\rho(g_1)}}, \dots, \frac{\rho(ug_L)}{\sqrt{\rho(g_L)}} \right),$$

the above expression can be written as

$$\mathcal{A}_1 = |X|^2 - \langle GX, X \rangle.$$

The vanishing momentum condition means that the vector X belongs to the hyperplane orthogonal to the vector of roots $Y = (\sqrt{\rho(g_1)}, \dots, \sqrt{\rho(g_L)})$, denoted Y^\perp . Such plane remains invariant under the action of G , while $GY = Y$. So, the low-energy bound (4.27) becomes equivalent to the spectral gap condition on G :

$$\text{spec}\{G; Y^\perp\} \leq 1 - \varepsilon. \tag{4.34}$$

It is not easy, however, to compute the spectrum of G exactly. A more practical approach to (4.34) would be to find a condition on the entries of G that implies a bound like (4.34). To this end, let us assume that non-zero entries are uniformly bounded from below, i.e. the neighborhoods have ‘‘populated intersections’’:

$$\rho(g_l g_{l'}) \geq \delta \sqrt{\rho(g_l)\rho(g_{l'})}, \quad \forall l, l' : \text{supp } g_l \cap \text{supp } g_{l'} \neq \emptyset, \tag{4.35}$$

for some $\delta > 0$.

Under this condition let us consider the eigenvalue problem

$$(1 - \varepsilon)X = GX, \quad X \cdot Y = 0.$$

Renormalizing $X = (X_1, \dots, X_L)$ via $x_l = X_l / \sqrt{\rho(g_l)}$, we obtain the system

$$(1 - \varepsilon)x_l = \sum_{l': \text{supp } g_{l'} \cap \text{supp } g_l \neq \emptyset} \frac{\rho(g_{l'} g_l)}{\rho(g_l)} x_{l'}. \tag{4.36}$$

Note that the sum on the right represents a convex combination of coordinates.

Denote $x^+ = x_{l^+}$ the positive maximal and $x^- = x_{l^-}$ the negative minimal values. Since $X \in Y^\perp$, those must be strictly signed. Since g 's form a partition of unity, there is a sequence of indexes $l^+ = l_0, l_1, \dots, l_p = l^-$ with $p \leq L$ such that

$$\text{supp } g_{l_i} \cap \text{supp } g_{l_{i+1}} \neq \emptyset.$$

Let us start with (4.36) at $l = l_0$. Then l_1 is one of the neighbors. We can assume without loss of generality that $x_{l_1} < x^+$ for otherwise, we relabel and start with the first index l_1 having this property.

We leave the l_1 -term unchanged, and estimate the rest of the x 's by x^+ to obtain

$$(1 - \varepsilon)x^+ \leq \left(1 - \frac{\rho(g_{l_0}g_{l_1})}{\rho(g_{l_0})}\right)x^+ + \frac{\rho(g_{l_0}g_{l_1})}{\rho(g_{l_0})}x_{l_1}.$$

Solving for x_{l_1} , we obtain

$$x_{l_1} \geq \left(1 - \frac{\varepsilon}{\rho(g_{l_0}g_{l_1})/\rho(g_{l_0})}\right)x^+.$$

Since $x_{l_1} < x^+$ it implies in particular that $\varepsilon > 0$. It also follows from (4.35) that

$$\frac{\rho(g_{l_0}g_{l_1})}{\rho(g_{l_0})} \geq \delta^2,$$

and hence

$$x_{l_1} \geq \left(1 - \frac{\varepsilon}{\delta^2}\right)x^+.$$

By the same computation centered this time at x_{l_1} and with ε reset to ε/δ^2 , we obtain

$$x_{l_2} \geq \left(1 - \frac{\varepsilon}{\delta^4}\right)x^+.$$

Continuing the process to the last term we obtain

$$x^- \geq \left(1 - \frac{\varepsilon}{\delta^{2p}}\right)x^+ \geq \left(1 - \frac{\varepsilon}{\delta^{2L}}\right)x^+.$$

Recalling that $x^- < 0$, it implies $\varepsilon \geq \delta^{2L}$. Thus, the spectral gap is estimated to be at least

$$\varepsilon = \delta^{2L}.$$

To estimate δ in terms of thickness, let us observe that by continuity in any overlapping neighborhoods there exists a ball of fixed radius $r_{\text{seg}} > 0$ such that

$$B_{r_{\text{seg}}}(x) \subset \text{supp}(g_l) \cap \text{supp}(g_{l'}) \tag{4.37}$$

and such that $g_l, g_{l'} \geq c_1$ on $B_{r_{\text{seg}}}(x)$ for some fixed $c_1 > 0$. Thus, we have using that $\rho(g_l) \leq 1$,

$$\frac{\rho(g_l g_{l'})}{\sqrt{\rho(g_l)\rho(g_{l'})}} \geq c_1^2 \bar{\rho}_{r_{\text{seg}}}(\Omega).$$

So, $\delta \gtrsim \bar{\rho}_{r_{\text{seg}}}(\Omega)$. ■

4.4.1. Application of the low energy method to non-ball-positive models. For non-ball-positive models such as Motsch–Tadmor, or more generally, for \mathcal{M}_β the low energy method can still produce estimates on spectral gap for almost uniformly distributed densities,

$$\left\| \rho - \frac{1}{|\Omega|} \right\|_1 \leq \delta. \tag{4.38}$$

Here, we make the same Bochner positivity assumption on the defining kernel $\phi = \psi * \psi$ and the locality (4.31).

Let us start as in Example 4.14 by symmetrizing and using cancellation:

$$\begin{aligned} & (u, u)_{\kappa_\rho} - (u, [u]_\rho)_{\kappa_\rho} \\ &= \int_{\Omega \times \Omega} u(x) \cdot (u(x) - u(y)) \rho(x) \rho(y) \frac{\phi(x - y)}{\rho_\phi^{1-\beta}(x)} \, dy \, dx \\ &= \frac{1}{2} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 \rho(x) \rho(y) \frac{\phi(x - y)}{\rho_\phi^{1-\beta}(x)} \, dy \, dx \\ &\quad - \frac{1}{2} \int_{\Omega \times \Omega} |u(y)|^2 \rho(x) \rho(y) \left(\frac{1}{\rho_\phi^{1-\beta}(x)} - \frac{1}{\rho_\phi^{1-\beta}(y)} \right) \phi(x - y) \, dy \, dx \\ &= \text{I} + \text{II}. \end{aligned}$$

First, note that

$$\begin{aligned} \text{I} &\geq c_1 \int_{\Omega \times \Omega} u(x) \cdot (u(x) - u(y)) \rho(x) \rho(y) \phi(x - y) \, dy \, dx \\ &= c_1 [(u, u)_{\rho_\phi \rho} - (u, [u]_\rho)_{\rho_\phi \rho}], \end{aligned}$$

which is exactly the spectral gap form that appears for the \mathcal{M}_{CS} model. So, using Proposition 4.16 and (4.38), we obtain

$$\text{I} \geq c_2 \bar{\rho}_{r_0/2}^3(\Omega) (u, u)_{\rho_\phi \rho} \geq c_2 (c_3 - \delta)^3 (c_4 - \delta)^{1-\beta} (u, u)_{\rho_\phi \rho} \geq c_5 (u, u)_{\kappa_\rho},$$

provided $\delta < \frac{1}{2} \min\{c_3, c_4\}$. Next,

$$\text{II} = \frac{1}{2} \int_{\Omega \times \Omega} |u(y)|^2 \rho(y) \rho_\phi^\beta(y) \left[1 - \frac{1}{\rho_\phi^\beta} \left(\frac{\rho}{\rho_\phi^{1-\beta}} \right)_\phi \right] \, dy.$$

Using again (4.38),

$$\begin{aligned} \frac{1}{\rho_\phi^\beta} \left(\frac{\rho}{\rho_\phi^{1-\beta}} \right)_\phi &\leq \frac{\rho_\phi}{c_4 - \delta} \leq \frac{c_4 + \delta}{c_4 - \delta} \leq 1 + \frac{2\delta}{c_4 - \delta}, \\ \frac{1}{\rho_\phi^\beta} \left(\frac{\rho}{\rho_\phi^{1-\beta}} \right)_\phi &\geq \frac{\rho_\phi}{c_4 + \delta} \geq \frac{c_4 - \delta}{c_4 + \delta} \geq 1 - \frac{2\delta}{c_4 + \delta}. \end{aligned}$$

So, if δ is small enough we have

$$\left| 1 - \frac{1}{\rho_\phi^\beta} \left(\frac{\rho}{\rho_\phi^{1-\beta}} \right)_\phi \right| \leq c_6 \delta.$$

We arrive at

$$|\text{II}| \leq c_6 \delta (u, u)_{\kappa_\rho}.$$

Combining the two together, we obtain

$$(u, u)_{\kappa_\rho} - (u, [u]_\rho)_{\kappa_\rho} \geq (c_5 - c_6 \delta)(u, u)_{\kappa_\rho} \geq c_7 (u, u)_{\kappa_\rho},$$

provided $\delta < c_8$, where c_8 is an absolute constant depending only on the parameters of the model. We have thus proved a version of Proposition 4.16 for \mathcal{M}_β models.

Proposition 4.18. *There exist constants $\delta, c_0 > 0$ depending only on the parameters of the model \mathcal{M}_β , $0 \leq \beta \leq 1$, such that for any density satisfying (4.38) the size of the spectral gap is estimated as $\varepsilon_0 > c_0$.*

5. Deterministic mean-field limit

In this section we consider either the periodic or open environments $\Omega = \mathbb{T}^n, \mathbb{R}^n$.

The goal of this section will be to derive the Vlasov-alignment equation (4.1), as the weak limit of empirical measures

$$\mu_t^N = \sum_{i=1}^N m_i \delta_{x_i(t)} \otimes \delta_{v_i(t)}, \tag{5.1}$$

where $(x_i, v_i)_{i=1}^N$ solve the agent based system (3.27). We will focus on the measure-valued solutions with bounded support. Although this is not a necessary assumption, it simplifies some of the technical issues considerably.

Definition 5.1. We say that $\{\mu_t\}_{0 \leq t < T} \in C_{w*}([0, T]; \mathcal{P}(B_R \times B_R))$ is a measure-valued solution to (4.1) with initial condition μ_0 if for any test-function $g \in C^\infty([0, T] \times \Omega \times \mathbb{R}^n)$ one has, for all $0 < t < T$,

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^n} g(t, x, v) d\mu_t(x, v) &= \int_{\Omega \times \mathbb{R}^n} g(0, x, v) d\mu_0(x, v) \\ &+ \int_0^t \int_{\Omega \times \mathbb{R}^n} (\partial_s g + v \cdot \nabla_x g + s_{\rho_s}([u_s]_{\rho_s} - v) \cdot \nabla_v g) d\mu_s(x, v) ds. \end{aligned} \tag{5.2}$$

The definition makes sense provided s_{ρ_s} and $s_{\rho_s}[u_s]_{\rho_s}$ are bounded and continuous functions in (s, x) . This typically can be derived from the regularity of the model as stipulated in Section 3.7. But since we cannot rely on any a priori thickness of solutions we must assume that the model \mathcal{M} is uniformly regular.

With this assumption, the continuity of $s_{\rho_s}(x)$ in x follows from (3.21). Continuity in s follows from (3.22):

$$\|s_{\rho_{s'}} - s_{\rho_{s''}}\|_{\infty} \lesssim W_1(\rho_{s'}, \rho_{s''}) \leq W_1(\mu_{s'}, \mu_{s''}).$$

Since for compactly supported measures W_1 determines the weak*-convergence, the claim follows. As to the weighted averages, we have

$$s_{\rho_s}[u_s]_{\rho_s} = \int_{B_R \times B_R} \phi_{\rho_s}(x, y)v \, d\mu_s(y, v).$$

So, again the continuity in x follows from (3.21). In terms of time, we use (3.22) to obtain

$$\begin{aligned} & \int_{B_R \times B_R} \phi_{\rho_{s'}}(x, y)v \, d\mu_{s'}(y, v) - \int_{B_R \times B_R} \phi_{\rho_{s''}}(x, y)v \, d\mu_{s''}(y, v) \\ &= \int_{B_R \times B_R} \phi_{\rho_{s'}}(x, y)v [d\mu_{s'}(y, v) - d\mu_{s''}(y, v)] \\ & \quad + \int_{B_R \times B_R} [\phi_{\rho_{s'}}(x, y) - \phi_{\rho_{s''}}(x, y)]v \, d\mu_{s''}(y, v) \\ & \leq \bar{C}_1 W_1(\mu_{s'}, \mu_{s''}) + \bar{C} R W_1(\rho_{s'}, \rho_{s''}) \lesssim W_1(\mu_{s'}, \mu_{s''}). \end{aligned}$$

The crucial and elementary observation is that the empirical measure given in (5.1) satisfies (5.2) if and only if $\{(x_i, v_i)\}_i$ solve the agent-based system (3.27). As a consequence, solutions to (3.27) fall naturally into the framework of the Vlasov-alignment equation. Our goal will be to prove the following theorem by showing contractivity of the map $\mu_0 \rightarrow \mu_t$ on any finite time interval.

Theorem 5.2. *Suppose \mathcal{M} is uniformly regular. Let $\mu_0 \in \mathcal{P}(\Omega \times \mathbb{R}^n)$ be any measure with compact support. Then for any $T > 0$ there exists a unique measure-valued solution $\{\mu_t\}_{0 \leq t < T} \in C_w^*([0, T]; \mathcal{P}(B_R(T)))$ to (4.1), which can be reconstructed from solutions to (3.27) as follows. Let all $(x_i^0, v_i^0) \in \mathcal{O}$, where \mathcal{O} is some fixed neighborhood of $\text{supp } \mu_0$ and such that $\mu_0^N \rightarrow \mu_0$ weakly. Then $\mu_t^N \rightarrow \mu_t$ weakly uniformly on $[0, T]$.*

As a corollary, we obtain validity of the mean-field limit in all the cases listed in the last row of Table 4.

The theorem will be proved via a Lagrangian approach using the transport structure of (4.1). To this end, we introduce the characteristic flow

$$\begin{aligned} \frac{d}{dt} X(t, s, x, v) &= V(t, s, x, v), & X(s, s, x, v) &= x, \\ \frac{d}{dt} V(t, s, x, v) &= s_{\rho}(X)([u]_{\rho}(X) - V), & V(s, s, x, v) &= v. \end{aligned} \tag{5.3}$$

We also denote $X(t, 0, x, v) = X(t, x, v)$, $V(t, 0, x, v) = V(t, x, v)$, and $(x, v) = \omega$. Note that the right-hand side of (5.3) is Lipschitz in (X, V) , so the flow is well defined on $[0, T]$.

Define the test-function $g(s, \omega) = h(X(t, s, \omega), V(t, s, \omega))$ for some $h \in C_0^\infty(\mathbb{R}^{2n})$, for which we have

$$\partial_s g + v \cdot \nabla_x g + s_\rho([u]_\rho - v) \cdot \nabla_v g = 0.$$

So, plugging it into (5.2) we obtain

$$\int_{\Omega \times \mathbb{R}^n} h(\omega) \, d\mu_t(\omega) = \int_{\Omega \times \mathbb{R}^n} h(X(t, \omega), V(t, \omega)) \, d\mu_0(\omega). \tag{5.4}$$

This means that μ_t is a *push-forward* of the initial measure μ_0 along the flow-map (X, V) , $\mu_t = (X, V)\#\mu_0$.

The proof of the mean-field limit consists of two steps: establishing control over the deformation $(\nabla X, \nabla V)$ on a given time interval, and proving Lipschitzness of the push-forward map in the W_1 -metric.

So, let us assume that on a time interval $[0, T]$ we have a solution $\mu_t \in \mathcal{P}(B_R)$. By the maximum principle of Lemma 4.1,

$$\|V(t)\|_{L^\infty(\mathcal{O})} \leq \max_{(x,v) \in \mathcal{O}} |v| \leq \text{diam } \mathcal{O}. \tag{5.5}$$

Let us fix a compact domain \mathcal{O} with $\text{supp } \mu_0 \subset \mathcal{O}$. Then

$$\frac{d}{dt} \|\nabla X\|_{L^\infty(\mathcal{O})} \leq \|\nabla V\|_{L^\infty(\mathcal{O})}.$$

Next,

$$\frac{d}{dt} \nabla V \leq \nabla X^\top \nabla(s_\rho[u]_\rho)(X) + \nabla X^\top \nabla s_\rho(X)V + s_\rho(X)\nabla V,$$

so, in view of (ev4), (5.5), and (3.21), we obtain the inequality up to a constant depending only on $R, m, \mathcal{O}, \bar{S}$,

$$\frac{d}{dt} \|\nabla V\|_{L^\infty(\mathcal{O})} \leq \|\nabla X\|_{L^\infty(\mathcal{O})} + \|\nabla V\|_{L^\infty(\mathcal{O})}.$$

We thus conclude that

$$\sup_{[0,T]} \|\nabla X\|_{L^\infty(\mathcal{O})} + \|\nabla V\|_{L^\infty(\mathcal{O})} \leq C(R, m, \mathcal{O}, T). \tag{5.6}$$

Let us now proceed to continuity estimates. Let us fix two measures $\mu'_t, \mu''_t \in \mathcal{P}(B_R)$ for all $t \in [0, T]$. We also fix a common initial domain \mathcal{O} , $\text{supp } \mu'_0 \cup \text{supp } \mu''_0 \subset \mathcal{O}$. Clearly,

$$\frac{d}{dt} \|X' - X''\|_{L^\infty(\mathcal{O})} \leq \|V' - V''\|_{L^\infty(\mathcal{O})}. \tag{5.7}$$

For velocities, we have

$$\frac{d}{dt} (V' - V'') = s_{\rho'}(X')[u']_{\rho'}(X') - s_{\rho''}(X'')[u'']_{\rho''}(X'') + s_{\rho''}(X'')V'' - s_{\rho'}(X')V'.$$

So, from (3.21) and (3.22), we have

$$\begin{aligned} & |s_{\rho'}(X')[u']_{\rho'}(X') - s_{\rho''}(X'')[u'']_{\rho''}(X'')| \\ & \lesssim W_1(\rho', \rho'') + W_1(u'\rho', u''\rho'') + \|X' - X''\|_{L^\infty(\Omega)}, \end{aligned}$$

and

$$|s_{\rho'}(X') - s_{\rho''}(X'')| \lesssim W_1(\rho', \rho'') + \|X' - X''\|_{L^\infty(\Omega)}.$$

Thus,

$$\begin{aligned} \frac{d}{dt} \|V' - V''\|_{L^\infty(\mathcal{O})} & \lesssim W_1(\rho', \rho'') + W_1(u'\rho', u''\rho'') \\ & \quad + \|V' - V''\|_{L^\infty(\mathcal{O})} + \|X' - X''\|_{L^\infty(\Omega)}. \end{aligned}$$

But for any $\|g\|_{\text{Lip}} \leq 1$, we have

$$\begin{aligned} \int_{\Omega} g(x)(d\rho'_t - d\rho''_t) & = \int_{\Omega \times \mathbb{R}^n} g(x)(d\mu'_t - d\mu''_t) \\ & = \int_{\Omega \times \mathbb{R}^n} g(X') d\mu'_0 - \int_{\Omega \times \mathbb{R}^n} g(X'') d\mu''_0 \\ & = \int_{\Omega \times \mathbb{R}^n} g(X')(d\mu'_0 - d\mu''_0) + \int_{\Omega \times \mathbb{R}^n} (g(X') - g(X'')) d\mu'_0 \\ & \leq \|\nabla X'\|_{L^\infty(\Omega)} W_1(\mu'_0, \mu''_0) + \|X' - X''\|_{L^\infty(\Omega)}. \end{aligned}$$

In view of (5.6) we conclude that

$$W_1(\rho'_t, \rho''_t) \lesssim W_1(\mu'_0, \mu''_0) + \|X' - X''\|_{L^\infty(\Omega)}.$$

Similarly, for any $\|g\|_{\text{Lip}} \leq 1$, we have

$$\begin{aligned} & \int_{\Omega} g(x)(d(u'\rho'_t) - d(u''\rho''_t)) \\ & = \int_{\Omega \times \mathbb{R}^n} g(x)v(d\mu'_t - d\mu''_t) \\ & = \int_{\Omega \times \mathbb{R}^n} g(X')V' d\mu'_0 - \int_{\Omega \times \mathbb{R}^n} g(X'')V'' d\mu''_0 \\ & = \int_{\Omega \times \mathbb{R}^n} g(X')V'(d\mu'_0 - d\mu''_0) + \int_{\Omega \times \mathbb{R}^n} (g(X')V' - g(X'')V'') d\mu'_0 \\ & \leq (\text{diam } \mathcal{O} \|\nabla X'\|_{L^\infty(\mathcal{O})} + \|g\|_{L^\infty(B_R)} \|\nabla V'\|_{L^\infty(\mathcal{O})}) W_1(\mu'_0, \mu''_0) \\ & \quad + m \text{diam } \mathcal{O} \|X' - X''\|_{L^\infty(\mathcal{O})} + \|g\|_{L^\infty(B_R)} \|V' - V''\|_{L^\infty(\mathcal{O})}. \end{aligned}$$

In view of (5.6), we conclude that

$$W_1(u'\rho', u''\rho'') \lesssim W_1(\mu'_0, \mu''_0) + \|X' - X''\|_{L^\infty(\mathcal{O})} + \|V' - V''\|_{L^\infty(\mathcal{O})}.$$

Thus, we obtain

$$\frac{d}{dt} \|V' - V''\|_{L^\infty(\mathcal{O})} \lesssim W_1(\mu'_0, \mu''_0) + \|X' - X''\|_{L^\infty(\mathcal{O})} + \|V' - V''\|_{L^\infty(\mathcal{O})}.$$

Combining with (5.7) we conclude that

$$\|X' - X''\|_{L^\infty(\mathcal{O})} + \|V' - V''\|_{L^\infty(\mathcal{O})} \leq C(R, T)W_1(\mu'_0, \mu''_0). \tag{5.8}$$

Let us now fix a function h with $\text{Lip}(h) \leq 1$, and use the transport identity (5.4):

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^n} h(\omega) d\mu'_t - \int_{\Omega \times \mathbb{R}^n} h(\omega) d\mu''_t \\ &= \int_{\Omega \times \mathbb{R}^n} h(X', V') d\mu'_0 - \int_{\Omega \times \mathbb{R}^n} h(X'', V'') d\mu''_0 \\ &= \int_{\Omega \times \mathbb{R}^n} h(X', V')(d\mu'_0 - d\mu''_0) + \int_{\Omega \times \mathbb{R}^n} [h(X', V') - h(X'', V'')] d\mu''_0 \\ &\leq \text{Lip}_\mathcal{O}(h(X', V'))W_1(\mu'_0, \mu''_0) + \|X_\mu - X_\nu\|_{L^\infty(\mathcal{O})} + \|V_\mu - V_\nu\|_{L^\infty(\mathcal{O})}. \end{aligned}$$

Using that

$$\text{Lip}_\mathcal{O}(h(X', V')) \leq \|\nabla V'\|_{L^\infty(\mathcal{O})} + \|\nabla X'\|_{L^\infty(\mathcal{O})},$$

and applying (5.6), (5.8), we conclude the following bounds

$$W_1(\mu'_t, \mu''_t) \leq C(R, \mathcal{O}, T)W_1(\mu'_0, \mu''_0). \tag{5.9}$$

This immediately implies uniqueness and stability of measure-valued solutions.

So, we start now with an arbitrary measure μ_0 , and approximate it weakly with a sequence of empirical measures

$$\mu_0^N = \sum_{i=1}^N m_i \delta_{x_i} \otimes \delta_{v_i},$$

with all $(x_i, v_i) \in \mathcal{O}$, where \mathcal{O} is some fixed neighborhood of $\text{supp } \mu_0$. Then let us run the agent-based alignment model alignment (3.27). For any time T , we have

$$\text{supp } \mu_t^N \subset B_{|\mathcal{O}|+TA_0} \times B_{A_0}, \quad t < T.$$

Thus, according to (5.9), μ_t^N is weakly Cauchy, and hence $\mu_t^N \rightarrow \mu_t$ for some μ_t . To finish the proof Theorem 5.2 we now prove a short lemma showing that the limit solves the Vlasov-alignment equation weakly.

Lemma 5.3. *Suppose a sequence of solutions $\mu^N \in C_{w^*}([0, T]; \mathcal{P}(B_R))$ converges weakly pointwise, i.e. $\mu_t^N \rightarrow \mu_t$ for all $0 \leq t < T$. Then $\mu \in C_{w^*}([0, T]; \mathcal{P}(B_R))$ is a weak solution to (4.1).*

Proof. The weak*-continuity of the limit will follow immediately from (5.2) once it is established. Clearly, all the linear terms in (5.2) converge to their natural limits. As to the force let us note for any $s < T$, we have (by the computations done above)

$$W_1(\rho_s^N, \rho_s^M) + W_1(u_s^N \rho_s^N, u_s^M \rho_s^M) \leq C W_1(\mu_s^N, \mu_s^M) \leq C W_1(\mu_0^N, \mu_0^M),$$

since both are solutions to the Vlasov-alignment equation. Sending $M \rightarrow \infty$, we obtain

$$W_1(\rho_s^N, \rho_s) + W_1(u_s^N \rho_s^N, u_s \rho_s) \leq C W_1(\mu_0^N, \mu_0),$$

which by continuity (3.22) implies that

$$\|s_{\rho_s^N}([u_s^N]_{\rho_s^N} - v) - s_{\rho_s}([u_s]_{\rho_s} - v)\|_{L^\infty(B_R)} \rightarrow 0$$

uniformly in s . Together with the weak convergence assumed for μ_s^N , we obtain

$$\int_0^t \int_{\Omega \times \mathbb{R}^n} (s_{\rho_s^N}([u_s^N]_{\rho_s^N} - v)) \, d\mu_s^N(x, v) \, ds \rightarrow \int_0^t \int_{\Omega \times \mathbb{R}^n} (s_{\rho_s}([u_s]_{\rho_s} - v)) \, d\mu_s(x, v) \, ds.$$

This finishes the proof. ■

Finally, let us discuss the implementation of Theorem 5.2 to global well-posedness of smooth solutions. Since all solutions are transported according to (5.4) regularity of a solution will depend on the regularity of initial data and the parameters of the model. First, let us notice that the Jacobian of the characteristic map, by the Liouville formula, is given by

$$\det \nabla_\omega(X, V)(t, \omega) = \exp \left\{ -n \int_0^t s_\rho(X(s, \omega)) \, ds \right\}.$$

Then if $\mu_0 = f_0 \, d\omega$, with $f_0 \in C^k$, $k \in \mathbb{N}$ and compactly supported, then for any $t > 0$,

$$f(t, X(t, \omega), V(t, \omega)) = f_0(\omega) \exp \left\{ n \int_0^t s_\rho(X(s, \omega)) \, ds \right\}.$$

Inverting the flow and noting that (X, V) and s_ρ are C^k implies $f \in C^k$ at all times with support in v being confined to its original bounds and support in x growing at most linearly.

Theorem 5.4. *Suppose the model \mathcal{M} is uniformly regular. Let $f_0 \in C_0^k(\Omega \times \mathbb{R}^n)$ be any compactly supported distribution. Then for any $T > 0$, there exists a unique solution $f \in L^\infty([0, T]; C_0^k)$ to (4.1), which is supported on $\text{supp } f_0 + B_{tA_0} \times \{0\}$, where A_0 is the maximal initial velocity.*

6. Stochastic mean-field limit

As discussed in Section 4.1 one of the main obstacles for alignment on the torus \mathbb{T}^n is existence of so-called locked states: solutions with agents locked on periodic orbits that

stay at a positive distance greater than the communication length scale r_0 . A natural way to avoid such unstable states is to introduce stochastic noise:

$$\begin{aligned} dx_i &= v_i dt, \\ dv_i &= s_i([v]_i - v_i) dt + \sqrt{2\sigma s_i} dB_i, \end{aligned} \tag{6.1}$$

where the B_i 's are independent Brownian motions in \mathbb{R}^n . Note that the noise here is assumed to be “material”, i.e. it places stochasticity only within the influence of the flock. As $N \rightarrow \infty$ and assuming that the agents are indistinguishable, i.e. $m_1 = \dots = m_N = 1/N$, the system comes in natural correspondence with what we call the *Fokker–Planck-alignment equation*

$$\partial_t f + v \cdot \nabla_x f = \sigma s_\rho \Delta_v f + \nabla_v (s_\rho (v - [u]_\rho) f). \tag{6.2}$$

A major advantage of using material noise is that the kinetic model (6.2) possesses a family of thermodynamic equilibria

$$\mu_{\sigma, \bar{u}} = \frac{1}{|\Omega|(2\pi\sigma)^{n/2}} e^{-|v-\bar{u}|^2/2\sigma}.$$

If the underlying model \mathcal{M} is conservative every solution is centered around the constant averaged momentum \bar{u} , which predetermines the corresponding equilibrium and opens a possibility for potential relaxation towards that distribution. The collective behavior interpretation of this result would say that, as expected, the noise disrupts the locked states and redistributes initial velocities symmetrically around the mean value \bar{u} . Alignment is then restored in the sense of the vanishing noise limit:

$$\lim_{\sigma \rightarrow 0} \lim_{t \rightarrow \infty} f^\sigma(t) = \frac{1}{|\Omega|} \delta_{v=\bar{u}} \otimes dx.$$

The problem of relaxation and hypocoercivity will be discussed in Section 8. In this section we provide a rigorous derivation of equation (6.2) as a mean-field limit of solutions to the stochastic system (6.1). To make this statement precise, let us consider f a solution to (6.2) on a time interval $[0, T]$ with initial distribution f_0 . Consider now N independent identically distributed random variables (x_i^0, v_i^0) , $i \leq N$, with $f_0 = \text{law}(x_i^0, v_i^0)$, and let (x_i, v_i) solve (6.1). Form the empirical measure-valued random variables

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)} \otimes \delta_{v_i(t)}.$$

The mean-field limit consist of showing that for all $t \leq T$, we have $\mu_t^N \rightarrow f_t$ in law, i.e. for any Lipschitz function h on $\Omega \times \mathbb{R}^n$,

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N h(x_i(t), v_i(t)) - \int_{\Omega \times \mathbb{R}^n} h(x, v) f(t, x, v) dx dv \right|^2 \rightarrow 0. \tag{6.3}$$

Note that $f dx dv$ in this context is considered as a constant random measure.

In general, the convergence (6.3) is equivalent to propagation of chaos, see [90]: if f^N denotes the joint probability distribution of the process $(x_1, v_1, \dots, x_N, v_N)$ solving (6.1), then for any $k \geq 1$, the k -th marginal $f^{(k)}$ converges weakly to the product of k copies of f , $f^{\otimes k}$, as $N \rightarrow \infty$:

$$\begin{aligned} &\langle f^{(k)}, \varphi_1 \otimes \dots \otimes \varphi_k \rangle \\ &= \langle f^N, \varphi_1 \otimes \dots \otimes \varphi_k \otimes 1 \otimes \dots \otimes 1 \rangle \rightarrow \prod_{j=1}^k \langle f, \varphi_j \rangle, \quad \varphi_j \in C_b(\mathbb{R}^{2n}). \end{aligned}$$

The strategy of proving (6.3) is based on the classical coupling method. Note that if the (x_i, v_i) 's were independent and identically distributed by f , then (6.3) would have been nothing but the law of large numbers. So, to achieve the limit we couple (6.1) with another system of separate N copies of the characteristic processes for (6.2):

$$\begin{aligned} d\bar{x}_i &= \bar{v}_i dt, \\ d\bar{v}_i &= s_\rho(\bar{x}_i)([u]_\rho(\bar{x}_i) - \bar{v}_i) dt + \sqrt{2\sigma s_\rho(\bar{x}_i)} dB_i, \end{aligned} \tag{6.4}$$

with initial condition (x_i^0, v_i^0) . Here, ρ and u are the macroscopic values of f . Note that because the equations are decoupled, the pairs (\bar{x}_i, \bar{v}_i) remain independent and identically distributed. By the Itô formula, f is their common law.

To establish (6.3), one can add and subtract the intermediate average of h with $\bar{x}_i(t), \bar{v}_i(t)$ pairs:

$$\begin{aligned} &\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N h(x_i(t), v_i(t)) - \int_{\mathbb{R}^{2n}} h(x, v) f(t, x, v) dx dv \right|^2 \\ &\leq \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N h(x_i(t), v_i(t)) - \frac{1}{N} \sum_{i=1}^N h(\bar{x}_i(t), \bar{v}_i(t)) \right|^2 \\ &\quad + \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N h(\bar{x}_i(t), \bar{v}_i(t)) - \int_{\mathbb{R}^{2n}} h(x, v) f(t, x, v) dx dv \right|^2. \end{aligned}$$

The second term goes to zero by the law of large numbers, while the first can be estimated using symmetry by

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N h(x_i(t), v_i(t)) - \frac{1}{N} \sum_{i=1}^N h(\bar{x}_i(t), \bar{v}_i(t)) \right|^2 \leq \|\nabla h\|_\infty \mathbb{E}[|x_1 - \bar{x}_1|^2 + |v_1 - \bar{v}_1|^2].$$

So the proof of (6.3) reduces to obtaining control over separation of characteristics:

$$E(t) = \mathbb{E}[|x_i - \bar{x}_i|^2 + |v_i - \bar{v}_i|^2] \rightarrow 0 \quad \text{as } N \rightarrow \infty. \tag{6.5}$$

This approach was carried out by Bolley et al. [9] in the case of convolution-type alignment systems and with additive noise (no strength s_ρ thermalization). We now provide a

proper extension that includes general environmental averaging models and material noise as stated.

Let us also note, following [9], that a bound on (6.5) entails a bound on the rate of decorrelation $f^{(k)} \rightarrow f^{\otimes k}$. Indeed,

$$W_2^2(f^{(k)}, f^{\otimes k}) \leq \mathbb{E} \left[\sum_{i=1}^k |x_i - \bar{x}_i|^2 + |v_i - \bar{v}_i|^2 \right] = kE(t) \rightarrow 0,$$

where W_2 is the Wasserstein-2 distance.

6.1. Law of large numbers

We will work on the torus $\Omega = \mathbb{T}^n$ and assume that \mathcal{M} is uniformly regular in the sense of Definition 3.18 with a minor modification. For all our averaging models the W_1 -metric used to define continuity in ρ can in fact be replaced with a weaker W_1 semi-metric determined by finitely many fixed Lipschitz functions: for $h_1, \dots, h_K \in \text{Lip}(\Omega)$ with $\|h_k\|_{\text{Lip}} \leq 1$,

$$W_1^{h_1, \dots, h_K}(\rho', \rho'') = \max_{k=1, \dots, K} \left| \int_{\Omega} h_k(x) [d\rho'(x) - d\rho''(x)] \right|. \tag{6.6}$$

Such is the case for all Favre-based models where $h = \phi$, or for \mathcal{M}_{seg} where $h_l = g_l$. Thus, the uniform continuity can be understood as follows:

$$\|s_{\rho'} - s_{\rho''}\|_{\infty} + \|\phi_{\rho'} - \phi_{\rho''}\|_{\infty} \leq C W_1^{h_1, \dots, h_K}(\rho', \rho''). \tag{6.7}$$

Let us now discuss consequences of the assumed regularity of the model on the law of large numbers. The basic idea is that the model is compatible with the law of large numbers in the averaged sense. Let us recall the classical law first, see [90]: for a sequence of i.i.d. random variables $X_j: \Sigma \rightarrow \mathbb{R}^d$ with bounded second momentum $\mathbb{E}|X_j|^2 \leq E_0$ and mean $\mathbb{E}X_j = m$, we have

$$\mathbb{E} \left| \frac{1}{N} \sum_{j=1}^N X_j - m \right|^2 \leq \frac{E_0}{N}.$$

Consequently, if $h \in C_b(\mathbb{R}^d)$ and μ is the law of X_j 's, then in terms of μ the above reads

$$\int_{\mathbb{R}^{Nd}} \left| \frac{1}{N} \sum_{j=1}^N h(\omega_j) - \int_{\mathbb{R}^d} h(\omega) \mu(\omega) \right|^2 d\mu(\omega_1) \cdots d\mu(\omega_N) \leq \frac{\|h\|_{\infty}^2}{N}. \tag{6.8}$$

We will encounter the inequality (6.8) in two interpretations. Namely, for any $h \in C_b(\Omega)$ and $f \in \mathcal{P}(\Omega \times \mathbb{R}^n)$, we have

$$\int_{\Omega^N} \left| \frac{1}{N} \sum_{j=1}^N h(y_j) - \int_{\Omega} h(z) d\rho(z) \right|^2 d\rho(y_1) \cdots d\rho(y_N) \leq \frac{C \|h\|_{\infty}^2}{N}, \tag{6.9}$$

$$\begin{aligned} & \int_{\Omega^N \times \mathbb{R}^{nN}} \left| \frac{1}{N} \sum_{j=1}^N v_j h(y_j) - \int_{\Omega} h(z) u(z) \, d\rho(z) \right|^2 \, df(y_1, v_1) \cdots \, df(y_N, v_N) \\ & \leq \frac{C \mathcal{E}(f) \|h\|_{\infty}^2}{N}, \end{aligned} \tag{6.10}$$

where $\mathcal{E}(f) = \int_{\Omega \times \mathbb{R}^n} |v|^2 \, df$.

The next two lemmas show that the analogue of these two laws of large numbers also holds with respect to the components of the model \mathcal{M} .

Lemma 6.1. *We have*

$$\alpha_N = \sup_{\rho \in \mathcal{P}(\Omega)} \int_{\Omega^N} |s_{\rho^N}(y_i) - s_{\rho}(y_i)|^2 \, d\rho(y_1) \cdots \, d\rho(y_N) \lesssim \frac{1}{N}, \tag{6.11}$$

where $\rho^N = \frac{1}{N} \sum_{j=1}^N \delta_{y_j}$. Note that α_N is independent of i by symmetry.

Proof. To see this, we have by (6.7) and (6.9) that

$$\begin{aligned} & \int_{\Omega^N} |s_{\rho^N}(y_i) - s_{\rho}(y_i)|^2 \, d\rho(y_1) \cdots \, d\rho(y_N) \\ & \lesssim \sum_{k=1}^K \int_{\Omega^N} \left| \int_{\Omega} h_k(z) [d\rho^N - d\rho] \right|^2 \, d\rho(y_1) \cdots \, d\rho(y_N) \\ & = \sum_{k=1}^K \int_{\Omega^N} \left| \frac{1}{N} \sum_{j=1}^N h_k(y_j) - \int_{\Omega} h_k(z) \, d\rho(z) \right|^2 \, d\rho(y_1) \cdots \, d\rho(y_N) \\ & \leq \frac{C}{N} (\|h_1\|_{\infty} + \cdots + \|h_K\|_{\infty}). \quad \blacksquare \end{aligned}$$

Lemma 6.2. *We have*

$$\begin{aligned} \beta_N = \sup_{f: \mathcal{E}(f) \leq \varepsilon_0} \int_{\Omega^N \times \mathbb{R}^{nN}} & |s_{\rho^N}(y_i) [u^N]_{\rho^N}(y_i) - s_{\rho}(y_i) [u]_{\rho}(y_i)|^2 \\ & \, df(y_1, v_1) \cdots \, df(y_N, v_N) \lesssim \frac{1}{N}, \end{aligned} \tag{6.12}$$

where ρ^N is as before, $u^N = \sum_{j=1}^N v_j \mathbb{1}_{\{y_j\}}$, and ρ, u are the macroscopic density and velocity of f .

Proof. Let us assume $i = 1$ for definiteness. We have

$$\begin{aligned} & s_{\rho^N}(y_1) [u^N]_{\rho^N}(y_1) - s_{\rho}(y_1) [u]_{\rho}(y_1) \\ & = \int_{\Omega} (\phi_{\rho^N}(y_1, z) - \phi_{\rho}(y_1, z)) u^N(z) \, d\rho^N(z) \\ & \quad + \int_{\Omega} \phi_{\rho}(y_1, z) [u^N(z) \, d\rho^N(z) - u(z) \, d\rho(z)] = \text{I} + \text{II}. \end{aligned} \tag{6.13}$$

Let us examine I first. We have by (6.6),

$$|I| \leq W_1^{h_1, \dots, h_K}(\rho^N, \rho) \frac{1}{N} \sum_{j=1}^N |v_j|.$$

Thus,

$$\begin{aligned} & \int_{\Omega^N \times \mathbb{R}^{nN}} |I|^2 \, df(y_1, v_1) \cdots df(y_N, v_N) \\ & \leq \frac{1}{N} \sum_{j=1}^N \int_{\Omega^N \times \mathbb{R}^{nN}} |v_j|^2 (W_1^{h_1, \dots, h_K}(\rho^N, \rho))^2 \, df(y_1, v_1) \cdots df(y_N, v_N), \end{aligned}$$

and by symmetry in j ,

$$\begin{aligned} & = \int_{\Omega^N \times \mathbb{R}^{nN}} |v_1|^2 (W_1^{h_1, \dots, h_K}(\rho^N, \rho))^2 \, df(y_1, v_1) \cdots df(y_N, v_N) \\ & \lesssim \sum_{k=1}^K \int_{\Omega^N \times \mathbb{R}^{nN}} |v_1|^2 \left| \frac{1}{N} \sum_{j=1}^N h_k(y_j) - \int_{\Omega} h_k(z) \, d\rho(z) \right|^2 \, df(y_1, v_1) \cdots df(y_N, v_N). \end{aligned}$$

Let us focus on one k th term. We single out the $j = 1$ term from the rest:

$$\begin{aligned} & \frac{1}{N^2} \int_{\Omega^N \times \mathbb{R}^{nN}} |v_1|^2 |h_k(y_1)|^2 \, df(y_1, v_1) \cdots df(y_N, v_N) \\ & \quad + \int_{\Omega^N \times \mathbb{R}^{nN}} |v_1|^2 \left| \frac{1}{N} \sum_{j=2}^N h_k(y_j) - \int_{\Omega} h_k(z) \, d\rho(z) \right|^2 \, df(y_1, v_1) \cdots df(y_N, v_N) \\ & \leq \frac{1}{N^2} \|h_k\|_{\infty}^2 \mathcal{E}_0 \\ & \quad + \mathcal{E}_0 \int_{\Omega^{N-1} \times \mathbb{R}^{n(N-1)}} \left| \frac{1}{N} \sum_{j=2}^N h_k(y_j) - \int_{\Omega} h_k(z) \, d\rho(z) \right|^2 \, d\rho(y_2) \cdots d\rho(y_N). \end{aligned}$$

The latter integral is $\lesssim 1/N$ with a minor adjustment to $N \rightarrow N - 1$ in the average. Thus, by (6.9),

$$\int_{\Omega^N \times \mathbb{R}^{nN}} |I|^2 \, df(y_1, v_1) \cdots df(y_N, v_N) \lesssim \frac{1}{N}.$$

It remains to analyze II. We will treat y_1 as a parameter, and let us denote $h_{y_1}(z) = \phi_{\rho}(y_1, z)$. By the regularity assumption, $h_{y_1} \in C_b(\Omega)$ and is even Lipschitz. We have

$$II = \frac{1}{N} \sum_{j=1}^N v_j h_{y_1}(y_j) - \int_{\Omega} h_{y_1}(z) u(z) \, d\rho(z).$$

Again, we single out the $j = 1$ term, and by (6.10),

$$\begin{aligned} & \int_{\Omega^N \times \mathbb{R}^{nN}} |\Pi|^2 \, df(y_1, v_1) \cdots \, df(y_N, v_N) \\ & \leq \frac{1}{N^2} \int_{\Omega^N \times \mathbb{R}^{nN}} |v_1|^2 |h_{y_1}(y_1)|^2 \, df(y_1, v_1) \cdots \, df(y_N, v_N) \\ & + \int_{\Omega^N \times \mathbb{R}^{nN}} \left| \int_{\Omega} \frac{1}{N} \sum_{j=2}^N v_j h_{y_1}(y_j) - \int_{\Omega} h_{y_1}(z) u(z) \, d\rho(z) \right|^2 \, df(y_1, v_1) \cdots \, df(y_N, v_N) \\ & \leq \frac{1}{N^2} \|\phi_{\rho}\|_{\infty}^2 \mathcal{E}_0 + \frac{1}{N} \|\phi_{\rho}\|_{\infty}^2 \mathcal{E}_0 \end{aligned}$$

with a minor adjustment to the index $N \rightarrow N - 1$ in the latter. ■

6.2. Main result

As discussed earlier we now focus on obtaining an estimate on separations of characteristics to achieve (6.5). The result holds on a finite time interval $[0, T]$ where f is a smooth solution to (6.2) by which we mean existence of sufficiently many derivatives in weighted Sobolev spaces to sufficient to understand (6.2) classically, see Section 7.

Theorem 6.3. *Suppose \mathcal{M} is uniformly regular satisfying (6.7). Let f be a classical solution to the Fokker–Planck-alignment equation (6.2) on a time interval $[0, T]$ satisfying*

$$\Theta(\rho, \Omega) \geq \delta, \quad \forall 0 \leq t \leq T, \tag{6.14}$$

and

$$\int_{\Omega \times \mathbb{R}^n} e^{a|v|^2} f(x, v, t) \, dx \, dv \leq c_8, \quad \forall 0 \leq t \leq T. \tag{6.15}$$

Then for any solution to the particle system (6.1) and (6.4) on the time interval $[0, T]$ with i.i.d. initial datum (x_i^0, v_i^0) distributed according to the law f_0 one has the following estimate

$$\mathbb{E}[|x_i - \bar{x}_i|^2 + |v_i - \bar{v}_i|^2] \leq C_1 \frac{1}{N e^{-c_2 t}},$$

for some $C_1, C_2 > 0$ depending on T and all the constants involved in the assumptions above. Consequently, the mean-field limit (6.3) holds.

Proof. We set $\sigma = 1$ for simplicity. First, we notice that the solution has a uniformly bounded energy on $[0, T]$ and thus (6.12) applies uniformly on $[0, T]$.

Let us denote

$$\begin{aligned} \mathbb{E} &= \mathbb{E}_x + \mathbb{E}_v, \\ \mathbb{E}_x &= \mathbb{E}[|x_i - \bar{x}_i|^2], \quad \mathbb{E}_v = \mathbb{E}[|v_i - \bar{v}_i|^2]. \end{aligned}$$

Taking the derivative of the x -component, we obviously obtain

$$\frac{d}{dt} \mathbb{E}_x = 2\mathbb{E}[(x_i - \bar{x}_i) \cdot (v_i - \bar{v}_i)] \leq \mathbb{E}.$$

For the velocity component we use the Itô formula,

$$\frac{d}{dt}\mathbb{E}_v = \mathbb{E}\left[(v_i - \bar{v}_i) \cdot (s_i([v]_i - v_i) - s_\rho(\bar{x}_i)([u]_\rho(\bar{x}_i) - \bar{v}_i))\right] + \mathbb{E}\left|\sqrt{2s_i} - \sqrt{2s_\rho(\bar{x}_i)}\right|^2.$$

Let us start with the noise term using (6.14) and (3.18),

$$\mathbb{E}\left|\sqrt{2s_i} - \sqrt{2s_\rho(\bar{x}_i)}\right|^2 = 2\mathbb{E}\left|\frac{s_i - s_\rho(\bar{x}_i)}{\sqrt{s_i} + \sqrt{s_\rho(\bar{x}_i)}}\right|^2 \leq C \mathbb{E}|s_i - s_\rho(\bar{x}_i)|^2.$$

Recalling that $s_i = s_{\rho^N}(x_i)$, where $\rho^N = \frac{1}{N} \sum_{j=1}^N \delta_{x_j}$, and denoting $\bar{\rho}^N = \frac{1}{N} \sum_{j=1}^N \delta_{\bar{x}_j}$, we add and subtract intermediate terms

$$\mathbb{E}|s_i - s_\rho(\bar{x}_i)|^2 \lesssim \mathbb{E}|s_{\rho^N}(x_i) - s_{\bar{\rho}^N}(\bar{x}_i)|^2 + \mathbb{E}|s_{\bar{\rho}^N}(\bar{x}_i) - s_\rho(\bar{x}_i)|^2. \quad (6.16)$$

By regularity of the strength function (3.21) and symmetry,

$$\mathbb{E}|s_{\rho^N}(x_i) - s_{\bar{\rho}^N}(\bar{x}_i)|^2 \lesssim \mathbb{E}_x + \mathbb{E}\left[\frac{1}{N} \sum_{j=1}^N |x_j - \bar{x}_j|^2\right] = C \mathbb{E}_x.$$

The second term is bounded by α_N as defined in (6.11), since ρ is the law of \bar{x}_i and the latter are independent,

$$\mathbb{E}|s_{\bar{\rho}^N}(\bar{x}_i) - s_\rho(\bar{x}_i)|^2 = \int_{\Omega^N} \left|s_{\frac{1}{N} \sum_{j=1}^N \delta_{y_j}}(y_i) - s_\rho(y_i)\right|^2 d\rho(y_1) \cdots d\rho(y_N) \leq \alpha_N.$$

In conclusion, we obtain

$$\mathbb{E}\left|\sqrt{2s_i} - \sqrt{2s_\rho(\bar{x}_i)}\right|^2 \leq C \mathbb{E} + \alpha_N.$$

Let us now turn to the alignment term. By adding and subtracting several intermediate terms we expand it as follows:

$$\begin{aligned} & \mathbb{E}\left[(v_i - \bar{v}_i) \cdot (s_i([v]_i - v_i) - s_\rho(\bar{x}_i)([u]_\rho(\bar{x}_i) - \bar{v}_i))\right] \\ &= -\mathbb{E}[s_i |v_i - \bar{v}_i|^2] + \mathbb{E}\left[(v_i - \bar{v}_i) \cdot (s_i[v]_i - s_\rho(\bar{x}_i)[u]_\rho(\bar{x}_i))\right] \\ & \quad + \mathbb{E}\left[(v_i - \bar{v}_i) \cdot \bar{v}_i (s_\rho(\bar{x}_i) - s_i)\right]. \end{aligned}$$

The first term is non-positive, so we simply drop it. Let us estimate the last term. We fix an $R > 0$ to be determined later and split the integrand as follows:

$$\begin{aligned} \mathbb{E}\left[(v_i - \bar{v}_i) \cdot \bar{v}_i (s_\rho(\bar{x}_i) - s_i)\right] &= \mathbb{E}\left[(v_i - \bar{v}_i) \cdot \bar{v}_i \mathbb{1}_{|\bar{v}_i| < R} (s_\rho(\bar{x}_i) - s_i)\right] \\ & \quad + \mathbb{E}\left[(v_i - \bar{v}_i) \cdot \bar{v}_i \mathbb{1}_{|\bar{v}_i| \geq R} (s_\rho(\bar{x}_i) - s_i)\right] \\ &\leq R^2 \mathbb{E}|v_i - \bar{v}_i|^2 + \mathbb{E}|s_i - s_\rho(\bar{x}_i)|^2 + C \mathbb{E}|v_i - \bar{v}_i|^2 \\ & \quad + C \mathbb{E}\left[|\bar{v}_i|^2 \mathbb{1}_{|\bar{v}_i| \geq R}\right], \end{aligned} \quad (6.17)$$

where in the last term we simply used the global boundedness of the strength functions.

For the second term we use the same estimate as before (6.16), hence continuing

$$\leq (R^2 + C)\mathbb{E} + \alpha_N + C\mathbb{E}[|\bar{v}_i|^2 \mathbb{1}_{|\bar{v}_i| \geq R}].$$

Now,

$$\mathbb{E}[|\bar{v}_i|^2 \mathbb{1}_{|\bar{v}_i| \geq R}] \leq \mathbb{E}^{1/2}[|\bar{v}_i|^4] \mathbb{E}^{1/2}[\mathbb{1}_{|\bar{v}_i| \geq R}].$$

Here the first term is bounded by the fourth moment of f , which is clearly bounded from the assumption (6.15). And using again (6.15), we estimate the last term by

$$\mathbb{E}[\mathbb{1}_{|\bar{v}_i| \geq R}] = \int_{\Omega} \int_{|v| \geq R} f_t(x, v) dv dx \leq \frac{c_4}{e^{aR^2}}.$$

The latter remains bounded on the interval $[0, T]$ by a constant by assumption. We thus obtained

$$\mathbb{E}[(v_i - \bar{v}_i) \cdot \bar{v}_i (s_{\rho}(\bar{x}_i) - s_i)] \leq (R^2 + C)\mathbb{E}(t) + \alpha_N + c_4 e^{-aR^2/2}.$$

Lastly, let us estimate the second term on the right-hand side of (6.17). We have, denoting $u^N = \sum_{j=1}^N v_j \mathbb{1}_{\{x_j\}}$ and $\bar{u}^N = \sum_{j=1}^N \bar{v}_j \mathbb{1}_{\{\bar{x}_j\}}$,

$$\begin{aligned} & \mathbb{E}[(v_i - \bar{v}_i) \cdot (s_i[v]_i - s_{\rho}(\bar{x}_i)[u]_{\rho}(\bar{x}_i))] \\ & \leq \mathbb{E}_v + \mathbb{E}|s_{\rho^N}(x_i)[u^N]_{\rho^N}(x_i) - s_{\bar{\rho}^N}(x_i)[\bar{u}^N]_{\bar{\rho}^N}(x_i)|^2 \\ & \quad + \mathbb{E}|s_{\bar{\rho}^N}(x_i)[\bar{u}^N]_{\bar{\rho}^N}(x_i) - s_{\bar{\rho}^N}(\bar{x}_i)[\bar{u}^N]_{\bar{\rho}^N}(\bar{x}_i)|^2 \\ & \quad + \mathbb{E}|s_{\bar{\rho}^N}(\bar{x}_i)[\bar{u}^N]_{\bar{\rho}^N}(\bar{x}_i) - s_{\rho}(\bar{x}_i)[u]_{\rho}(\bar{x}_i)|^2. \end{aligned} \tag{6.18}$$

The last term here is bounded by β_N , see (6.12). The elements in the first term are evaluated at the same point x_i . So, by a similar computation as in (6.13), we have

$$\begin{aligned} & \mathbb{E}|s_{\rho^N}(x_i)[u^N]_{\rho^N}(x_i) - s_{\bar{\rho}^N}(x_i)[\bar{u}^N]_{\bar{\rho}^N}(x_i)|^2 \\ & \lesssim \mathbb{E} \left[\left(\frac{1}{N} \sum_{j=1}^N |\bar{v}_j|^2 \right) W_1^2(\rho^N, \bar{\rho}^N) \right] + \mathbb{E}[W_1^2(u^N \rho^N, \bar{u}^N \bar{\rho}^N)] \\ & \leq \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}[|\bar{v}_j|^2 |x_i - \bar{x}_i|^2] + \frac{1}{N} \sum_{i=1}^N \mathbb{E}[|\bar{v}_i|^2 |x_i - \bar{x}_i|^2] + \mathbb{E}_v. \end{aligned} \tag{6.19}$$

Each term here will be estimated by the same splitting method as before:

$$\begin{aligned} \mathbb{E}[|\bar{v}_j|^2 |x_i - \bar{x}_i|^2] &= \mathbb{E}[|\bar{v}_j|^2 \mathbb{1}_{|\bar{v}_j| < R} |x_i - \bar{x}_i|^2] + \mathbb{E}[|\bar{v}_j|^2 \mathbb{1}_{|\bar{v}_j| \geq R} |x_i - \bar{x}_i|^2] \\ &\leq R^2 \mathbb{E}_x + \mathbb{E}^{1/2}[|\bar{v}_j|^4] \mathbb{E}^{1/2}[\mathbb{1}_{|\bar{v}_j| \geq R}] \leq R^2 \mathbb{E} + C e^{-aR^2/2}, \end{aligned}$$

and similarly for the middle term. Thus,

$$(6.19) \leq R^2 \mathbb{E} + C e^{-aR^2/2}.$$

It remains to estimate the middle term in (6.18). Here we use the regularity of the kernel (3.21) and obtain

$$\mathbb{E} |s_{\bar{\rho}^N}(x_i)[\bar{u}^N]_{\bar{\rho}^N}(x_i) - s_{\bar{\rho}^N}(\bar{x}_i)[\bar{u}^N]_{\bar{\rho}^N}(\bar{x}_i)|^2 \lesssim \frac{1}{N} \sum_{j=1}^N \mathbb{E}[|x_i - \bar{x}_i|^2 |\bar{v}_j|^2].$$

This term becomes exactly as the previous one. So, the same estimate applies.

Putting the above estimates together and denoting $r = aR^2/2$ and $\gamma_N = \alpha_N + \beta_N$, we arrive at

$$\frac{d}{dt} \mathbb{E} \leq C_1(r + 1)\mathbb{E} + C_2\gamma_N + C_3e^{-r}. \tag{6.20}$$

Inequality (6.20) is exactly the one that appeared in [9]. Let us recap the conclusion for completeness. First, by choosing $r = 1$ we see that \mathcal{E} remains uniformly bounded on $[0, T]$, $\mathbb{E} \leq \mathbb{E}_0$. Thus, $-\ln(\mathbb{E}/e\mathbb{E}_0) \geq 1$. Denoting $v = \mathbb{E}/e\mathbb{E}_0$ and picking $r = -\ln v$, we obtain

$$v' \leq -c_1 v \ln v + c_2 \gamma_N \leq -cv \ln v + c\gamma_N,$$

where $c = \max\{c_1, c_1\}$. Rescaling time $u(t) = v(t/c)$, we further obtain

$$u' \leq -u \ln u + \gamma_N.$$

Letting $w = u\gamma_N^{-1}e^{-t}$, we conclude

$$w' \leq -w \ln w + 1 \leq e^{-1} + 1.$$

Thus, $w \leq T(e^{-1} + 1) = C_T$, and hence unwrapping the notation, $\mathbb{E} \leq C_1\gamma_N^{-C_2t}$, as claimed. ■

7. Fokker–Planck–alignment equation

In this section we develop a well-posedness theory of classical solutions to Fokker–Planck–alignment (FPA) equations (6.2) that is suitable for applications to flocking. This means that in addition to the standard regularity questions we will pay close attention to thickness as related to the spectral gap computations discussed in Section 4. We will restrict ourselves to the periodic domain $\Omega = \mathbb{T}^n$ as that is the setting where most of our results will be used in the sequel. We also set $\sigma = 1$ as it plays no role in the analysis. So, we consider the FPA equation

$$\partial_t f + v \cdot \nabla_x f = s_\rho \Delta_v f + \nabla_v \cdot (s_\rho(v - [u]_\rho) f). \tag{7.1}$$

Classical solutions to (7.1) are defined to be solutions that belong to a high regularity weighted Sobolev class. For reasons that will be clarified later it is essential to distribute velocity weights in the manner defined as follows:

$$H_t^k(\Omega \times \mathbb{R}^n) = \left\{ f : \sum_{k' \leq k} \sum_{|k'|=k'} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l+2(k-k')} |\partial_{x,v}^{k'} f|^2 dv dx < \infty \right\}, \tag{7.2}$$

where $\langle v \rangle = (1 + |v|^2)^{1/2}$. Some remarks are in order to elaborate on this choice. First, we note that the alignment term in (6.2) prevents the persistence of a sub-Gaussian bound $f \leq C\mu$ if it holds initially. So, setting the problem traditionally in sub-Gaussian Hölder classes (cf. [2, 48]), is not natural for the FPA equations. One exception is the class of perturbative solutions developed for particular models in [18, 32]. Inclusion of the weights in (7.2) is necessary to achieve uniqueness primarily due to, again, the presence of alignment components; see, however, [97] for the classical, much weaker result. The use of progressively increasing weights for lower order terms is required to control terms coming from the inhomogeneity in front of the Fokker–Planck operator, which prevents closing a priori estimates for any single-weight choice. Single weight spaces, however, would have been sufficient for models with $s_\rho = 1$.

7.1. Local well-posedness

Let us first discuss local well-posedness for thick data on compact domain.

Theorem 7.1. *Suppose that the model \mathcal{M} is regular in the sense of Definition 3.17, and $\Omega = \mathbb{T}^n$. Let $f_0 \in H_1^k(\Omega \times \mathbb{R}^n)$, $k, l \geq n + 3$, be an initial condition such that*

$$\Theta(\rho_0, \Omega) > 0.$$

Then there exists a unique local solution to (7.1) on a time interval $[0, T)$, where $T > 0$ depends only on the initial energy \mathcal{E}_0 and thickness $\Theta(\rho_0, \Omega)$, in the regularity class

$$f \in C_w([0, T); H_1^k), \quad \nabla_v f \in L^2([0, T]; H_1^k). \tag{7.3}$$

Moreover, if $f \in L_{loc}^\infty([0, T); H_1^k)$ is a given solution such that

$$\inf_{[0, T)} \Theta(\rho, \Omega) > 0, \tag{7.4}$$

then f can be extended to an interval $[0, T + \varepsilon)$ in the same class.

We can view the right-hand side of (7.1) as a sum of a weighted Fokker–Planck operator and a smooth drift

$$\partial_t f + v \cdot \nabla_x f = s_\rho \mathcal{L}_{FP} f + w_\rho \cdot \nabla_v f, \tag{7.5}$$

where

$$\mathcal{L}_{FP} f = \nabla_v \cdot (\nabla_v f + v f), \quad w_\rho = -s_\rho [u]_\rho.$$

Let us first disassociate the weights s_ρ and w_ρ from the solution and consider the linear problem

$$\partial_t f + v \cdot \nabla_x f = s(x, t) \mathcal{L}_{FP} f + w(x, t) \cdot \nabla_v f, \tag{7.6}$$

where s, w is a given smooth set of data on $\Omega \times [0, T]$ with uniform bounds

$$s \geq c_0 > 0, \quad \|s\|_{C^k} + \|w\|_{C^k} < C_0 \quad \text{on } \Omega \times [0, T]. \tag{7.7}$$

Lemma 7.2. *Under the assumptions (7.7) for any initial condition $f_0 \in H_l^k$, there exists a unique solution to (7.6) on $[0, T]$ with $f \in C_w([0, T]; H_l^k)$, $\nabla_v f \in L_{loc}^2([0, T]; H_l^k)$, and moreover*

$$\|f\|_{H_l^k} \leq \|f_0\|_{H_l^k} e^{Ct},$$

where C depends only on c_0 and C_0 .

Proof. To construct a solution to (7.6) from initial data $f_0 \in H_l^k$ one first considers a fully viscous regularization

$$\partial_t f + v \cdot \nabla_x f = s\mathcal{L}_{FP} f + w \cdot \nabla_v f + \varepsilon \Delta_{x,v} f. \tag{7.8}$$

A local solution to (7.8) on a time interval $[0, T_\varepsilon]$ is obtained via the standard fixed point argument, see [57]. In order to extend it to all of $[0, \infty)$ we provide a priori estimates for (7.6) which automatically apply to (7.8) independently of ε . As a result we obtain a bound on $\|f\|_{H_l^k}$, which depends only on its initial value, on c_0 and C^k -norms of s and w . So, we have a family of solutions f^ε uniformly in $C([0, T]; H_l^k)$, and clearly also in $f_t^\varepsilon \in L^\infty([0, T]; L^2)$. By the Aubin–Lions compactness lemma we can pass to the limit $\varepsilon \rightarrow 0$ in any $H_{l'}^{k'}$ for $k' < k$, $l' < l$ and weakly in H_l^k extracting a subsequence converging to a solution to (7.6). Weak continuity in H_l^k also follows classically.

Thus, the problem reduced to obtaining proper a priori bounds for solutions to (7.6).

Let us estimate the top v -derivative $\partial_v^k f$ first (here and further on we use a less formal notation for the partials, only keeping track of the order):

$$\partial_t \partial_v^k f + v \cdot \nabla_x \partial_v^k f + \partial_v^{k-1} \partial_x f = s\mathcal{L}_{FP} \partial_v^k f + s\partial_v^k f + w \cdot \nabla_v \partial_v^k f.$$

Testing with $\langle v \rangle^l \partial_v^k f$, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^k f|^2 \, dv \, dx + \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l v \cdot \nabla_x |\partial_v^k f|^2 \, dv \, dx \\ & + \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_v^{k-1} \partial_x f \partial_v^k f \, dv \, dx \\ & = - \int_{\Omega \times \mathbb{R}^n} s \langle v \rangle^l |\partial_v^{k+1} f|^2 \, dv \, dx + \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} s \Delta_v (\langle v \rangle^l) |\partial_v^k f|^2 \, dv \, dx \\ & + \int_{\Omega \times \mathbb{R}^n} s \partial_v (\langle v \rangle^l v) |\partial_v^k f|^2 \, dv \, dx + \int_{\Omega \times \mathbb{R}^n} s \langle v \rangle^l |\partial_v^k f|^2 \, dv \, dx \\ & - \int_{\Omega \times \mathbb{R}^n} w \cdot \nabla_v (\langle v \rangle^l) |\partial_v^k f|^2 \, dv \, dx \end{aligned}$$

While the first integral on the left-hand side vanishes, the second is bounded by $\|f\|_{H_l^k}^2$. All the terms on the right-hand side, using that $\partial_v \langle v \rangle^p \lesssim \langle v \rangle^{p-1}$, are also bounded by $\|f\|_{H_l^k}^2$ except for the dissipation which has a uniform bound from below by (7.7). Thus,

$$\frac{d}{dt} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^k f|^2 \, dv \, dx \leq C \|f\|_{H_l^k}^2 - c_0 \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k+1} f|^2 \, dv \, dx,$$

where C is a constant depending on all the L^∞ -norms of s, w .

Let us now estimate the rest of the other top derivatives $\partial_v^{k-k'} \partial_x^{k'} f$, $k' > 0$. Sparing the tedious details, most of terms are all bounded by $C \|f\|_{H_t^k}^2$, where C is an upper bound for the C^k -norms of s, w . The rest is given by

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'} \partial_x^{k'} f|^2 dv dx \\ & \lesssim C \|f\|_{H_t^k}^2 - c_0 \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'+1} \partial_x^{k'} f|^2 dv dx \\ & \quad + \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_x s \Delta_v \partial_v^{k-k'} \partial_x^{k'-1} f \partial_v^{k-k'} \partial_x^{k'} f dv dx \\ & \quad + \int_{\Omega \times \mathbb{R}^n} \partial_x s \langle v \rangle^l v \cdot \nabla_v \partial_v^{k-k'} \partial_x^{k'-1} f \partial_v^{k-k'} \partial_x^{k'} f dv dx. \end{aligned}$$

For the penultimate term, we have

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_x s \Delta_v \partial_v^{k-k'} \partial_x^{k'-1} f \partial_v^{k-k'} \partial_x^{k'} f dv dx \\ & \leq - \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_x s \nabla_v \partial_v^{k-k'} \partial_x^{k'-1} f \cdot \nabla_v \partial_v^{k-k'} \partial_x^{k'} f dv dx + C \|f\|_{H_t^k}^2 \\ & = -\frac{1}{2} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_x s \partial_x |\nabla_v \partial_v^{k-k'} \partial_x^{k'-1} f|^2 dv dx + C \|f\|_{H_t^k}^2 \\ & = \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l \partial_x^2 s |\partial_v^{k-k'+1} \partial_x^{k'-1} f|^2 dv dx + C \|f\|_{H_t^k}^2 \leq C \|f\|_{H_t^k}^2. \end{aligned}$$

In the remaining term we take advantage of the dissipation and the higher weight assigned to the lower order derivatives. Integrating by parts in v , we have

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}^n} \partial_x s \langle v \rangle^l v \cdot \nabla_v \partial_v^{k-k'} \partial_x^{k'-1} f \partial_v^{k-k'} \partial_x^{k'} f dv dx \\ & = - \int_{\Omega \times \mathbb{R}^n} \partial_x s \nabla_v \cdot (\langle v \rangle^l v) \partial_v^{k-k'} \partial_x^{k'-1} f \partial_v^{k-k'} \partial_x^{k'} f dv dx \\ & \quad - \int_{\Omega \times \mathbb{R}^n} \partial_x s \langle v \rangle^l \partial_v^{k-k'} \partial_x^{k'-1} f v \cdot \nabla_v \partial_v^{k-k'} \partial_x^{k'} f dv dx \\ & \leq C \|f\|_{H_t^k}^2 + \frac{c_0}{2} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'+1} \partial_x^{k'} f|^2 dv dx \\ & \quad + C \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l+2} |\partial_v^{k-k'} \partial_x^{k'-1} f|^2 dv dx \\ & \leq C \|f\|_{H_t^k}^2 + \frac{c_0}{2} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'+1} \partial_x^{k'} f|^2 dv dx. \end{aligned}$$

As a result, we obtain

$$\frac{d}{dt} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'} \partial_x^{k'} f|^2 dv dx \leq C \|f\|_{H_t^k}^2 - \frac{c_0}{2} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^l |\partial_v^{k-k'+1} \partial_x^{k'} f|^2 dv dx.$$

The same argument works to estimate any positive order derivatives, each time taking advantage of the higher weight put on one order below. It remains to estimate the 0-th order term

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l+2k} |f|^2 \, dv \, dx \\ &= \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} s \Delta_v \langle v \rangle^{l+2k} f^2 \, dv \, dx - \int_{\Omega \times \mathbb{R}^n} s \langle v \rangle^{l+2k} |\nabla_v f|^2 \, dv \, dx \\ &\quad - \int_{\Omega \times \mathbb{R}^n} s \nabla_v \langle v \rangle^{l+2k} \cdot v f^2 \, dv \, dx - \int_{\Omega \times \mathbb{R}^n} s \nabla_v \cdot (\langle v \rangle^{l+2k} v) f^2 \, dv \, dx \\ &\quad - \int_{\Omega \times \mathbb{R}^n} (w \cdot \nabla_v \langle v \rangle^{l+2k}) f^2 \, dv \, dx \\ &\leq C \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l+2k} |f|^2 \, dv \, dx - c_0 \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l+2k} |\nabla_v f|^2 \, dv \, dx. \end{aligned}$$

So, the estimate on the 0-th order term closes on itself.

We obtained

$$\frac{d}{dt} \|f\|_{H_t^k}^2 \leq C \|f\|_{H_t^k}^2 - \frac{c_0}{2} \|\nabla_v f\|_{H_t^k}^2,$$

and the estimate stated in the lemma follows. It also proves uniqueness since the equation is linear. ■

Proof of Theorem 7.1. To construct solutions to the fully non-linear problem (7.5) we use iteration scheme based on solving a sequence of linear problems

$$\begin{aligned} \partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} &= s_{\rho^m} \mathcal{L}_{FP} f^{m+1} + w_{\rho^m} \cdot \nabla_v f^{m+1}, \\ f^{m+1}(0) &= f_0 \end{aligned} \tag{7.9}$$

for $m = 0, 1, \dots$. In order to pass to the limit as $m \rightarrow \infty$ we need to ensure that f^{m+1} remain uniformly bounded in H_t^k on a fixed time interval $[0, T]$. According to Lemma 7.2 a bound on f^{m+1} depends on smoothness of s_{ρ^m} and w_{ρ^m} and a lower bound on s_{ρ^m} . Thanks to the regularity of \mathcal{M} and (3.18), these conditions can be controlled by the thickness $\Theta(\rho^m, \Omega)$ and the energies

$$\mathcal{E}^m = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 f^m \, dv \, dx.$$

Let us show that there exists a common time interval $[0, T]$ on which all the energies are uniformly bounded and the all the densities ρ^m are uniformly thick.

Starting with the energy, testing (7.9) with $\frac{1}{2}|v|^2$ we can see that the Fokker–Planck component yields a bound $\|s_{\rho^m}\|_{\infty} \mathcal{E}^{m+1} \leq \bar{S} \mathcal{E}^{m+1}$. Let us denote $2\delta = \Theta(\rho_0, \Omega)$. Assuming for a moment that the m -th flock remains thick $\Theta(\rho^m, \Omega) \geq \delta$ and then using the bound

$$\|w_{\rho^m}\|_{\infty} \leq C_0(\delta) \sqrt{\mathcal{E}^m},$$

we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}^{m+1} &\leq C - \int_{\Omega \times \mathbb{R}^n} w^n \cdot v f^{m+1} \, dv \, dx \\ &= C + \int_{\Omega} s^m [u^m]_{\rho^m} \cdot u^{m+1} \rho^{m+1} \, dx \\ &\leq C + C_0(\delta) \sqrt{\mathcal{E}^m} \int_{\Omega} |u^{m+1}| \rho^{m+1} \, dx \leq C + C_0(\delta) \sqrt{\mathcal{E}^m \mathcal{E}^{m+1}} \\ &\leq C + C_0(\delta) \max\{\mathcal{E}^m, \mathcal{E}^{m+1}\}. \end{aligned}$$

Hence, denoting $\bar{\mathcal{E}}^m = \max\{\mathcal{E}^0, \mathcal{E}^1, \dots, \mathcal{E}^m\}$, we obtain from the above

$$\frac{d}{dt} \bar{\mathcal{E}}^{m+1} \leq C + C_0(\delta) \bar{\mathcal{E}}^{m+1}. \tag{7.10}$$

At the same time, by (3.14),

$$\partial_t \Theta(\rho^m, \Omega) \geq -c \|u^m\|_{L^2(\rho^m)} \geq -c \sqrt{\bar{\mathcal{E}}^{m+1}}. \tag{7.11}$$

Let us argue by induction. The initial interval of existence for $m = 0$ is $T_0 = \infty$. On this interval, $\Theta(\rho_0, \Omega) = 2\delta > \delta$. Then from (7.10), we have

$$\bar{\mathcal{E}}^1 \leq \mathcal{E}_0 e^{C_0(\delta)t} + C_1(\delta) e^{C_0(\delta)t}.$$

So, for $t \leq (\ln 2)/C_0(\delta)$, we have

$$\bar{\mathcal{E}}^1(t) \leq 2\mathcal{E}_0 + 2C_1(\delta).$$

Using (7.11) we conclude on the same time interval (recall that $\rho^m(0) = \rho^0$ initially)

$$\begin{aligned} \Theta(\rho^1, \Omega) &\geq \Theta(\rho_0, \Omega) - t C_1(\delta) \sqrt{2\mathcal{E}_0 + 2C_1(\delta)} \\ &\geq 2\delta - t C_1(\delta) \sqrt{2\mathcal{E}_0 + 2C_1(\delta)}. \end{aligned}$$

Consequently, for $t < \delta/C_1(\delta) \sqrt{2\mathcal{E}_0 + 2C_1(\delta)}$, we have $\Theta(\rho^1, \Omega) \geq \delta$.

Setting

$$T = \min \left\{ \frac{\ln 2}{C_0(\delta)}, \frac{\delta}{C_1(\delta) \sqrt{2\mathcal{E}_0 + 2C_1(\delta)}} \right\},$$

we obtain exact same estimates for the next elements in the sequence:

$$\bar{\mathcal{E}}^2(t) \leq 2\mathcal{E}_0 + 2C_1(\delta), \quad t < T$$

and

$$\Theta(\rho^2, \Omega) \geq 2\delta - t C_1(\delta) \sqrt{2\mathcal{E}_0 + 2C_1(\delta)} \geq \delta, \quad t < T.$$

Continuing in the same manner it follows that $\mathcal{E}^m \leq 2\mathcal{E}_0 + 2C_1(\delta)$ and $\Theta(\rho^m, \Omega) \geq \delta$ on the same time interval $[0, T]$ for all $m \in \mathbb{N}$.

Lemma 7.2 implies that each solution in the sequence f^m will exist and be uniformly bounded in class $C_w([0, T]; H_l^k)$. By compactness we conclude that there exists a converging subsequence in any lower regularity class, and that the limit solves the equation (7.5) classically by continuity properties of the model (3.20).

From the above we see that the local time of existence T depends only on the initial energy \mathcal{E}_0 and thickness δ . With this observation let us assume that we are given a solution on an interval $[0, T')$ in the Sobolev class H_l^k and such that (7.4) holds for all $t < T'$. Then the estimate analogous to (7.10) shows that the energy $\mathcal{E}(t)$ remains bounded on $[0, T')$ by a constant depending only on δ and \mathcal{E}_0 . Starting from $T' - \varepsilon$ where $\varepsilon > 0$ is small we construct a solution on a time interval $[T' - \varepsilon, T' - \varepsilon + T)$ where T depends only on δ and E_0 and not on ε . This extends the solution beyond T' by uniqueness, which we address next.

Let us have two thick solutions f and \tilde{f} in class (7.3) starting from the same initial condition f_0 . Denote $g = f - \tilde{f}$. We will estimate evolution of this difference in the weighted class $L_l^2 = H_l^0$ and show that estimates close if l is large enough. Note that according to definition of H_l^k for k large as assumed, we have $\nabla_v f, \nabla_v \tilde{f}, \mathcal{L}_{FP} f, \mathcal{L}_{FP} \tilde{f} \in L_l^2$, uniformly.

Let us take the difference

$$\partial_t g + v \cdot \nabla_x g = s_\rho \mathcal{L}_{FP} g + (s_\rho - s_{\tilde{\rho}}) \mathcal{L}_{FP} \tilde{f} + w_\rho \cdot \nabla_v g + (w_\rho - w_{\tilde{\rho}}) \cdot \nabla_v \tilde{f}.$$

Testing with $\langle v \rangle^l g$ and integrating the x -transport term drops out. The rest of the terms are estimated using continuity assumption (3.20) and the usual energy estimates

$$\begin{aligned} \frac{d}{dt} \|g\|_{L_l^2}^2 &\lesssim \|g\|_{L_l^2}^2 + \|\rho - \tilde{\rho}\|_1 \|\mathcal{L}_{FP} \tilde{f}\|_{L_l^2} \|g\|_{L_l^2} \\ &\quad + (\|\rho u - \tilde{\rho} \tilde{u}\|_1 + \|\rho - \tilde{\rho}\|_1 \mathcal{E}) \|\nabla_v \tilde{f}\|_{L_l^2} \|g\|_{L_l^2}. \end{aligned}$$

Here, we replaced the W_1 -metrics with L^1 since this is not essential. The \tilde{f} components and the energy are uniformly bounded as noted above. So, we have

$$\frac{d}{dt} \|g\|_{L_l^2}^2 \lesssim \|g\|_{L_l^2}^2 + \|\rho - \tilde{\rho}\|_1 \|g\|_{L_l^2} + \|\rho u - \tilde{\rho} \tilde{u}\|_1 \|g\|_{L_l^2}$$

Now,

$$\|\rho - \tilde{\rho}\|_1 \leq \int_{\Omega \times \mathbb{R}^n} |g| \, dv \, dx = \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l/2} |g| \langle v \rangle^{-l/2} \, dv \, dx \lesssim \|g\|_{L_l^2},$$

provided $l > n$. Similarly,

$$\|\rho u - \tilde{\rho} \tilde{u}\|_1 \leq \int_{\Omega \times \mathbb{R}^n} |v| |g| \, dv \, dx = \int_{\Omega \times \mathbb{R}^n} \langle v \rangle^{l/2} |g| \langle v \rangle^{-l/2+1} \, dv \, dx \lesssim \|g\|_{L_l^2},$$

provided $l > n + 2$. So, we arrive at

$$\frac{d}{dt} \|g\|_{L_l^2}^2 \lesssim \|g\|_{L_l^2}^2,$$

and uniqueness follows. ■

7.2. Spread of positivity

In order to extend local solutions globally it is clear that we have to generate a lower bound on the macroscopic density. Since regularity of the local solution can deteriorate propagation of the thickness is impossible to prove with the local existence estimates. Instead we resort to what is called spread of positivity.

Solutions to many kinetic equations tend to develop instantaneous spread of support across the domain, in the sense of gaining a Gaussian bound

$$f(t, x, v) \geq be^{-a|v|^2}, \tag{7.12}$$

see [2, 2, 28, 30, 36, 46, 47, 49, 56, 71]. The constants a, b , however, depend on either the regularity of the solution on a given time interval or bounds on macroscopic quantities such as the mass-density, energy-density and entropy-density. Such bounds may deteriorate in time which puts constants a, b in dependence on time as well. With a view towards flocking and regularity the primary purpose of a bound like (7.12) would be to translate into a global lower bound on the density $\rho \geq \rho_-$ dependent only on the basic quantities such as drift and entropy. At this point we are essentially using the advantage of a compact environment. As a consequence, for those models where the drift and entropy can be controlled in time we can develop global existence and relaxation results.

So, our primary goal in this section will be to establish the Gaussian bound (7.12) with parameters that depend only on the entropy/energy and the drift.

Proposition 7.3. *For a given classical solution $f \in C_w([0, T]; H_1^k(\mathbb{T}^n))$ of (7.1) on a time interval $[0, T)$ there exists $a, b > 0$ which depend only on the parameters of the model \mathcal{M} , time T , and*

$$\begin{aligned} \bar{W} &= \sup_{t \in [0, T)} \|s_\rho[u]_\rho\|_\infty, \\ \bar{H} &= \sup_{t \in [0, T)} \int_{\mathbb{T}^n \times \mathbb{R}^n} |v|^2 f \, dv \, dx + \int_{\mathbb{T}^n \times \mathbb{R}^n} f |\log f| \, dv \, dx, \end{aligned}$$

such that

$$f(t, x, v) \geq be^{-a|v|^2}, \quad \forall (t, x, v) \in \mathbb{T}^n \times \mathbb{R}^n \times [T/2, T).$$

Central to our proof will be the weak Harnack inequality proved in [38]. To state it we need to introduce some notation.

We will be looking at solutions on kinetic cylinders defined by, for $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$,

$$Q_r(z_0) = \{z : -r^2 < t - t_0 \leq 0, |x - x_0 - (t - t_0)v_0| < r^3, |v - v_0| < r\}.$$

One can define the Lie-group action on triplets z by

$$z_0 \circ z = (t_0 + t, x_0 + x + tv_0, v_0 + v).$$

Then

$$z^{-1} = (-t, -x + tv, -v).$$

And we define the kinetic multiplication by a scalar as

$$rz = (r^2t, r^3x, rv).$$

The cylinders $Q_r(z_0)$ can then be considered as the shift and rescaling of the 0-centered cylinder $Q_r = Q_r(0)$

$$Q_r(z_0) = z_0 \circ Q_r.$$

And by scaling, $rQ_1 = Q_r$.

We consider super-solutions to the following general Fokker–Planck equation

$$\partial_t f + v \cdot \nabla_x f \geq \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f. \tag{7.13}$$

The equation has natural scaling invariance. If

$$f_{r,z_0}(z) = f(z_0 \circ rz),$$

then f_{r,z_0} satisfies

$$\partial_t f_{r,z_0} + v \cdot \nabla_x f_{r,z_0} \geq \nabla_v \cdot (A_{r,z_0} \nabla_v f_{r,z_0}) + B_{r,z_0} \cdot \nabla_v f_{r,z_0}, \tag{7.14}$$

where

$$A_{r,z_0}(z) = A(z_0 \circ rz), \quad B_{r,z_0}(z) = rB(z_0 \circ rz).$$

Thus, the following rule applies.

Claim 7.4. If f solves (7.13) on $Q_{r'}(z')$, then f_{r,z_0} solves the rescaled equation (7.14) on $Q_{r'/r}((z_0^{-1} \circ z')/r)$. Furthermore, if A, B satisfy

$$\lambda \mathbb{I} \leq A \leq \Lambda \mathbb{I}, \quad |B| \leq \Lambda, \quad z \in Q_{r'}(z'), \tag{7.15}$$

for some $\lambda, \Lambda > 0$, then the new coefficients A_{r,z_0}, B_{r,z_0} satisfy the same bounds on

$$Q_{r'/r}((z_0^{-1} \circ z')/r),$$

provided $r < 1$.

For $\omega > 0$ small let us introduce the following two non-overlapping cylinders

$$Q_\omega^+ = Q_\omega(1, 0, 0), \quad Q_\omega^- = Q_\omega(\omega^2, 0, 0).$$

That is, Q_ω^+ is attached to the top of the basic cylinder $Q_1(1, 0, 0)$, and Q_ω^- is lying on the bottom.

Theorem 7.5 (Weak Harnack inequality, [38]). *There are constants $R_0, \omega_0, p, C_0 > 0$, which depend only on λ, L, n satisfying the following property. If f is a super-solution to (7.13), in the cylinder $Q^{R_0} = [0, 1] \times \{|x| < R_0\} \times \{|v| < R_0\}$ with A, B satisfying (7.15), then whenever*

$$\left(\int_{Q_{\omega_0}^-} f^p \, dz \right)^{1/p} \leq C_0 \inf_{Q_{\omega_0}^+} f.$$

We now turn to proving Proposition 7.3. The proof goes in several steps. First, we rewrite the FPA (7.5) as follows:

$$\partial_t f + v \cdot \nabla_x f = s_\rho \Delta_v f + (s_\rho v + w_\rho) \cdot \nabla_v f + ns_\rho f.$$

Since the last term is non-negative, f is a super-solution to the truncated equation

$$\partial_t f + v \cdot \nabla_x f \geq s_\rho \Delta_v f + (s_\rho v + w_\rho) \cdot \nabla_v f,$$

which has the structure of (7.13). We will be mindful of the fact, however, that $B = s_\rho v + w_\rho$ is unbounded in v , and this will be taken into account in due course.

In the subsequent course of the proof, the various constants denoted

$$c_0, c_1, \dots, \quad \omega_0, \omega_1, \dots, \quad T_0, T_1, \dots, \quad r_0, r_1, \dots, \quad R_0, R_1, \dots$$

depend only on the parameters of the model, T , and \bar{W}, \bar{H} . We call such constants *admissible*.

Step 1: Choosing domain of ellipticity. Let us recall from (3.18) that the strength function is supported from below by a measure of ball-thickness at scale r_0 across the domain $\Omega = \mathbb{T}^n$. Since \mathbb{T}^n has finite volume by a covering argument, there exists a constant c_1 depending on n , and there exists $x' \in \mathbb{T}^n$ such that

$$\bar{\rho}_{r_0/4}(0, x') \geq c_1, \tag{7.16}$$

Consequently,

$$\bar{\rho}_{r_0}(0, x) \geq c_2, \quad \forall x \in B_{r_0/2}(x').$$

Next, notice that $\bar{\rho}_{r_1}$ satisfies the following equation:

$$\partial_t \bar{\rho}_{r_0} = -\nabla_x \cdot (u\rho)_{x_{r_0}} = -(u\rho)_{\nabla x_{r_0}} \geq -c_3 \|u\|_{L^2(\rho)} \geq -c_3 \bar{H}.$$

So, for any $t > 0$, and any $x \in B_{r_0/2}(x')$, we have

$$\bar{\rho}_{r_0}(t, x) \geq c_2 - tc_3 \bar{H}.$$

This implies that on the time interval $t \in [0, T_1]$, where $T_1 = (c_2/2c_3 \bar{H}) \wedge T$, we have

$$\bar{\rho}_{r_0}(t, x) \geq c_2/2, \quad \forall x \in B_{r_0/2}(x'),$$

and in view of (3.18),

$$s_\rho(t, x) \geq s(c_2/2) = \lambda, \quad \forall (t, x) \in [0, T_1] \times B_{r_0/2}(x').$$

Let us come back to (7.16) and extract a thick subdomain for f not too far in v -direction. We have

$$\begin{aligned} c_1 &\leq \int_{B_{r_0/4}(x')} \int_{\mathbb{R}^n} f(0, x, v) \, dv \, dx \\ &= \int_{B_{r_0/4}(x')} \int_{|v| < R} f(0, x, v) \, dv \, dx + \int_{B_{r_0/4}(x')} \int_{|v| \geq R} f(0, x, v) \, dv \, dx \\ &\leq \int_{B_{r_0/4}(x')} \int_{|v| < R} f(0, x, v) \, dv \, dx + \frac{\bar{H}}{R^2}, \end{aligned}$$

for $R = R_1 = 1 \vee \sqrt{2\bar{H}/c_1}$, we have

$$\int_{B_{r_0/4}(x')} \int_{|v| < R_1} f(0, x, v) \, dv \, dx \geq \frac{c_1}{2} = c_3. \tag{7.17}$$

Let us define our domain of ellipticity $\bar{\Omega} = [0, T_1] \times B_{r_0/2}(x') \times B_{2R_1}(0)$, where we have

$$\lambda \leq s_\rho \leq \Lambda, \quad |s_\rho v| + |w_\rho| \leq \Lambda, \tag{7.18}$$

where $\Lambda = \max\{\bar{S}, 2\bar{S}R_1 + \bar{W}\}$, and \bar{S} is the common bound on the strength function by (ev4). The constants λ, Λ determine $R_0, \omega_0, p, C_0 > 0$ from Theorem 7.5, which depend only on λ, Λ, n , so they are admissible.

Step 2: Finding the initial plateau. We want to find a center of inflation $(0, x_0, v_0)$ in such a way that the point (x_0, v_0) lies within the interior subdomain $B_{r_0/4}(x') \times B_{R_1}(0)$ and a small ω -cylinder around it has a substantial presence of f . That cylinder will be blown into $Q_{\omega_0}^-$ resulting in f having a substantial L^p -mass in it. At the same time the domain of ellipticity $\bar{\Omega}$ will be blown to engulf the needed wide cylinder Q^{R_0} to fulfill the assumptions of Theorem 7.5. The theorem then applies to obtain an admissible lower bound on f at a later time.

Thanks to (7.17) by the standard covering argument, for any small ω , one can find a point $(x_0, v_0) \in B_{r_0/4}(x') \times B_{R_1}(0)$ such that

$$\int_{B_{\omega^3}(x_0) \times B_\omega(v_0)} f(0, x, v) \, dv \, dx \geq c_3 c_4 |B_{\omega^3}(x_0) \times B_\omega(v_0)| = c_5 \omega^{4n}, \tag{7.19}$$

We will choose ω later. Let us prove now that the initial weight of f in a cylinder as in (7.19) stretches in time on the natural scale ω^2 .

Lemma 7.6. *Suppose initially*

$$\int_{B_{\omega^3}(x_0) \times B_\omega(v_0)} f(0, x, v) \, dv \, dx \geq c_5 \omega^{4n}.$$

Then

$$\int_{Q_{2\omega}(4\omega^2, x_0, v_0)} f(z) dz \geq c_6 \omega^{8n+2}. \tag{7.20}$$

Proof. Let us fix a smooth cut-off function $h(r) = \mathbb{1}_{r < 1}$ and $h(r) = 0$ for $r \geq 2$, bounded by 1. Let

$$h_\omega(x, v) = h\left(\frac{x}{\omega^3}\right)h\left(\frac{v}{\omega}\right).$$

Define the kinetic convolution

$$g(t) = \int_{\Omega \times \mathbb{R}^n} f(t, x_0 + x + tv_0, v + v_0) h_\omega(x, v) dv dx$$

Then initially, $g(0) \geq c_5 \omega^{4n}$. Let us compute the derivative

$$\begin{aligned} \frac{d}{dt} g &= \int_{\Omega \times \mathbb{R}^n} (\partial_t f + v_0 \cdot \nabla_x f) h_\omega dv dx \\ &= \int_{\Omega \times \mathbb{R}^n} (\partial_t f + (v_0 + v) \cdot \nabla_x f) h_\omega dv dx - \int_{\Omega \times \mathbb{R}^n} v \cdot \nabla_x f h_\omega dv dx. \end{aligned}$$

Note that

$$\left| \int_{\Omega \times \mathbb{R}^n} v \cdot \nabla_x f h_\omega dv dx \right| = \left| \int_{\Omega \times \mathbb{R}^n} f v \cdot \nabla_x h_\omega dv dx \right| \leq c \|\nabla h\|_\infty \omega^{-2}.$$

So,

$$\begin{aligned} \frac{d}{dt} g &\geq \int_{\Omega \times \mathbb{R}^n} [s_\rho \Delta_v f + (s_\rho(v + v_0) + w_\rho) \cdot \nabla_v f] h_\omega dv dx - c \omega^{-2} \\ &= \int_{\Omega \times \mathbb{R}^n} f [s_\rho \Delta_v h_\omega - n s_\rho h_\omega - (s_\rho(v + v_0) + w_\rho) \cdot \nabla_v h_\omega] dv dx - c \omega^{-2} \\ &\geq -\Lambda \omega^{-2} - n \Lambda - (\Lambda 2R_1 + \bar{W}) \omega^{-1} - c \omega^{-2} \geq -c_7 \omega^{-2}. \end{aligned}$$

Hence,

$$g(t) \geq c_5 \omega^{4n} - c_7 \omega^{-2} t.$$

Integrating again, we obtain

$$\int_0^t g(s) ds \geq c_5 \omega^{4n} t - c_7 \omega^{-2} \frac{t^2}{2}$$

Setting $t = c_5 \omega^{4n+2} c_7 \ll \omega^2$, we obtain

$$\int_0^{\omega^2} g(s) ds \geq c_6 \omega^{8n+2}.$$

Noting that h_ω is supported on $B_{8\omega^3} \times B_{2\omega}$ and bounded by 1, we obtain the desired result. ■

We now make a transformation

$$z \rightarrow z_0 \circ rz, \quad z_0 = (0, x_0, v_0), \quad r = \frac{\omega_1}{\omega_0}, \quad \omega_1 = 2\omega. \tag{7.21}$$

This ensures that, whatever ω is, the box $Q_{\omega_0}^-$ gets transformed into our $Q_{2\omega}(4\omega^2, x_0, v_0)$. We now choose ω such that the ambient domain Q^{R_0} transforms inside our the lower half of the domain of ellipticity. Given that x_0 is within $B_{r_0/4}(x')$ and $v_0 \in B_{R_1}(0)$ it suffices to choose

$$\omega_1 = \omega_0 \min \left\{ \sqrt{\frac{T_1}{4}}, \sqrt{\frac{r_0}{16R_1}}, \frac{R_1}{2R_0} \right\}.$$

Under so defined rescaling the we have

$$Q_{\omega_0}^- \rightarrow Q_{\omega_1}(\omega_1^2, x_0, v_0), \quad Q_{\omega_0}^+ \rightarrow Q_{\omega_1}((\omega_1/\omega_0)^2, x_0 + v_0(\omega_1/\omega_0)^2, v_0),$$

and moreover $2Q^{R_0} = [0, 2] \times B_{2R_0} \times B_{2R_0}$ gets transformed inside the domain of ellipticity

$$2Q^{R_0} \hookrightarrow [0, T/2] \times B_{r_1/2}(x') \times B_{2R_1}(0) \subset \bar{\Omega}.$$

At the same time, the ellipticity bounds (7.18) remain the same (and in fact improve on the drift). Observe also that all the parameters involved so far are admissible.

In order to apply the weak Harnack inequality, we need to essentially interpolate the L^1 -information on f expressed by (7.20) between L^p and $L \log L$ in order to extract information on the L^p level.

Since $\omega_1 = 2\omega$ has been picked already and it is dependent only on the parameters of the model, and T, \bar{W}, \bar{H} , let us write (7.20) as follows

$$\int_{Q_{\omega_1}(\omega_1^2, x_0, v_0)} f(z) \, dz \geq c_8.$$

We have

$$\int_{Q_{\omega_1}(\omega_1^2, x_0, v_0)} f |\log f| \, dz \leq \omega_1^2 \bar{H}.$$

Thus,

$$\int_0^\infty |\{f \geq \alpha\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| (|\log \alpha| + \text{sgn}(\alpha - 1)) \, d\alpha \leq \omega_1^2 \bar{H}.$$

Consequently, for $\alpha_0 > 1$,

$$\int_{\alpha_0}^\infty |\{f \geq \alpha\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| \, d\alpha \leq \frac{1}{\log \alpha_0} \omega_1^2 \bar{H}.$$

Choosing $\alpha_0 = \exp\{4\omega_1^2 \bar{H}/c_8\}$, we have

$$\int_{\alpha_0}^\infty |\{f \geq \alpha\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| \, d\alpha \leq \frac{c_8}{4}.$$

At the same time, for $\alpha_1 = c_8/4\omega_1^{2+4n}$, we have

$$\int_0^{\alpha_1} |\{f \geq \alpha\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| \, d\alpha \leq \alpha_1 |Q_{\omega_1}(\omega_1^2, x_0, v_0)| = \alpha_1 \omega_1^{2+4n} = \frac{c_8}{4}.$$

Consequently,

$$\int_{\alpha_1}^{\alpha_0} |\{f \geq \alpha\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| \, d\alpha \geq \frac{c_8}{4}.$$

This implies that

$$|\{f \geq \alpha_1\} \cap Q_{\omega_1}(\omega_1^2, x_0, v_0)| \geq \frac{c_8}{4(\alpha_0 - \alpha_1)} := c_9.$$

Note again that all the constants depend only on the parameters of the model, and T, \bar{W}, \bar{H} .

Using transformation (7.21), which has Jacobian $(\omega_1/\omega_0)^{4n+2}$, we obtain

$$|\{f_{r,z_0} \geq \alpha_1\} \cap Q_{\omega_0}^-| \geq (\omega_1/\omega_0)^{4n+2} c_9 := c_{10}.$$

Hence, by the Chebychev inequality,

$$\left(\int_{Q_{\omega_0}^-} f_{r,z_0}^p \, dz \right)^{1/p} \geq (\alpha_1^p |\{f_{r,z_0} \geq \alpha_1\} \cap Q_{\omega_0}^-|)^{1/p} \geq \alpha_1 c_{10}^{1/p} := c_{11}.$$

Theorem 7.5 applies to show that

$$\inf_{Q_{\omega_0}^+} f_{r,z_0} \geq c_{12},$$

or in terms of the original function f ,

$$\inf_{Q_{\omega_1}((\omega_1/\omega_0)^2, x_0+v_0(\omega_1/\omega_0)^2, v_0)} f \geq c_{12}.$$

Step 3: Harnack chains. It will be more efficient, in terms of notation, to remain in the new system of coordinates defined by (7.21). Since the transformation involves only admissible parameters, any bound on f obtained in the new system will translate into an admissible bound in the old system.

So, in the new coordinates, f satisfies

$$\partial_t f + v \cdot \nabla_x f \geq s(t, x) \Delta_v f + (b(t, x)v + w(t, x)) \cdot \nabla_v f.$$

We make another time-shift to make notation even simpler $z \rightarrow (1, 0, 0) \circ z$. Thus, we have

$$\lambda \leq s \leq \Lambda, \quad |bv| + |w| \leq \Lambda, \tag{7.22}$$

on the new wide domain of ellipticity

$$\bar{\Omega} = [-1, 1] \times B_{2R_0} \times B_{2R_0}.$$

Notice that the new quantities \bar{W}, \bar{H} turn into another pair of admissible constants.

On the previous step we established a bound, which in the new coordinate frame reads

$$\inf_{Q_{\omega_0}} f \geq c_0, \tag{7.23}$$

where c_0 is admissible. The goal now is to show that by the time $t = 1$ the solution spreads across the entire torus Ω .

It will also be more accommodating to use Theorem 7.5, where $Q_{\omega_0}^-$ is replaced by Q_{ω_0} . This is clearly achievable by a slight rescaling and a shift which is allowed by our enlarged ellipticity domain $\bar{\Omega}$. Also, notice that by rescaling the theorem also applies to the cylinders $Q_{\omega_0/2}^\pm$ with C_0 being replaced with an absolute multiple of C_0 , also admissible.

Now let us proceed with the construction of Harnack chains. The original idea goes back to [4,5] and has seen more recent adaptations for Fokker–Planck equation in [2]. Our construction will be similar in spirit to the latter, although quite different in two technical aspects. First, we produce a chain that reaches the targeted velocity field in fewer steps, thus achieving the exact Gaussian tail on the first run. And second, the estimates along the chain will take into account the loss of information that comes with the use of a weaker version of the Harnack inequality.

Lemma 7.7. *Let (7.23) hold. There exist admissible constants $a, b > 0$ such that*

$$f(t, x, v) \geq be^{-a|v|^2}$$

for all $(\omega_0/4)^2 \leq t \leq (\omega_0/2)^2$, $|x| \leq (\omega_0/2)^3$ and all $v \in \mathbb{R}^n$.

Proof. Let us fix an $N \in \mathbb{N}$ to be determined later, and let $r = 2|v|/\omega_0 N$. Denote $\hat{v} = v/|v|$. Let us define the sequence of points

$$z_0 = 0, \quad z_{l+1} = z_l \circ r \left(1, 0, \frac{\omega_0}{2} \hat{v} \right), \quad l = 0, \dots, N - 1.$$

In other words,

$$z_l = \left(lr^2, lr^3 \frac{\omega_0}{2} \hat{v}, lr \frac{\omega_0}{2} \hat{v} \right) := (t_l, x_l, v_l).$$

Notice that the end-point

$$z_N = \left(\frac{4|v|^2}{N\omega_0^2}, \frac{4|v|^3}{N^2\omega_0^2} \hat{v}, v \right)$$

reaches the target velocity vector v by cost of a small shift in time-space variable.

Also notice the following embeddings of cylinders

$$z_1 \circ rQ_{\omega_0/2} \subset rQ_{\omega_0}(1, 0, 0), \tag{7.24}$$

which follows by direct verification. Applying $z_l \circ$ from the left, we obtain

$$z_{l+1} \circ rQ_{\omega_0/2} \subset z_l \circ rQ_{\omega_0}(1, 0, 0). \tag{7.25}$$

We will be looking at the rescalings

$$f_l(z) = f(z_l \circ rz).$$

All these functions can be thought of as defined on the same domain $\bar{\Omega}$ with the same ellipticity constants. Indeed, if $z = (s, y, w) \in \bar{\Omega}$, then

$$z_l \circ rz = (t_l + r^2s, x_l + r^3y + r^2sv_l, rw + v_l).$$

We have

$$|t_l + r^2s| \leq \frac{4|v|^2}{\omega_0^2 N} + \frac{4|v|^2}{\omega_0^2 N^2} \leq \frac{8|v|^2}{\omega_0^2 N} < 1$$

provided $N \geq 8|v|^2/\omega_0^2$. Next,

$$|x_l + r^3y + r^2sv_l| \leq \frac{4|v|^3}{\omega_0^2 N^2} + 2R_0 \frac{8|v|^3}{\omega_0^3 N^3} + \frac{4|v|^3}{\omega_0^2 N^2} \leq (2R_0 + 1) \frac{16|v|^3}{\omega_0^3 N^2} \leq 2R_0,$$

provided $N^2 \geq (2R_0 + 1/2R_0) \cdot (16|v|^3/\omega_0^3)$. This puts the (t, x) pair into the box $[-1, 1] \times B_{2R_0}$, and so, the ellipticity for $s(z_l \circ rz)$ enjoys the same bounds (7.22). As to the drift term, which gets rescaled to

$$B_l = b(t_l + r^2s, x_l + r^3y + r^2sv_l)r(rw + v_l) + rw(t_l + r^2s, x_l + r^3y + r^2sv_l),$$

notice that

$$|rw + v_l| \leq 2R_0r + |v|$$

so,

$$|B_l| \leq \Lambda r(2R_0r + |v|) + r\Lambda < \Lambda,$$

provided $N \geq c_1|v|^2$, where c_1 is admissible.

The conclusion is that all functions f_l if considered defined on $\bar{\Omega}$ satisfy the equation with the same ellipticity constants provided

$$N \geq c_2\langle v \rangle^2,$$

where c_2 is admissible.

Let us now start iteration of the weak Harnack inequality. We have for $f_0(z) = f(rz)$ from the assumption (7.23), and since $rQ_{\omega_0} \subset Q_{\omega_0}$,

$$\left(\int_{Q_{\omega_0}} f_0^p(z) dz \right)^{1/p} \geq c_0 \omega_0^{(4n+2)/p}.$$

According to Theorem 7.5,

$$\inf_{Q_{\omega_0}(1,0,0)} f_0 \geq C_0^{-1} c_0 (\omega_0/2)^{(4n+2)/p},$$

(we artificially divided ω_0 by 2 in order to fit with the general pattern later). According to (7.24), we have in particular

$$\inf_{Q_{\omega_0/2}} f_1 \geq C_0^{-1} c_0 \left(\frac{\omega_0}{2}\right)^{(4n+2)/p}.$$

Then by restricting to the cylinder $Q_{\omega_0/2}$,

$$\left(\int_{Q_{\omega_0}} f_1^p(z) dz\right)^{1/p} \geq C_0^{-1} c_0 \left(\frac{\omega_0}{2}\right)^{2 \cdot (4n+2)/p}.$$

According to Theorem 7.5,

$$\inf_{Q_{\omega_0}(1,0,0)} f_1 \geq C_0^{-2} c_0 \left(\frac{\omega_0}{2}\right)^{2 \cdot (4n+2)/p}.$$

We proceed in the same manner using (7.25) and applying repeatedly Theorem 7.5.

On the last step we achieve the following bound

$$\inf_{Q_{\omega_0/2}} f_N \geq c_0 \left[\left(\frac{\omega_0}{2}\right)^{(4n+2)/p} C_0^{-1}\right]^N = c_0 c_3^N.$$

In particular, at the origin we obtain

$$f_N(0, 0, 0) = f(z_N) = f(t_N, x_N, v) \geq c_0 c_3^N.$$

Let us now fix a pair (t, x) such that $(\omega_0/4)^2 \leq t \leq (\omega_0/2)^2$, $|x| \leq (\omega_0/2)^3$ and consider the function

$$g(z) = f((t - t_N, x - x_N, 0) \circ z).$$

This function satisfies the equation on the slightly shrunken domain of ellipticity

$$[-0.9, 0.9] \times B_{1.9R_0} \times B_{1.9R_0}.$$

At the same time

$$\inf_{(t_N-t, x_N-x, 0) \circ Q_{\omega_0}} g \geq c_0.$$

The same holds on the subcylinder $Q_{\omega_0/2} \subset (t_N - t, x_N - x, 0) \circ Q_{\omega_0}$ (the inclusion follows from the assumptions on (t, x)). Applying the above proof to the new function g , we obtain

$$g(t_N, x_N, v) = f(t, x, v) \geq c_0 c_4^N.$$

Picking the minimal N under which the above holds we find $N = c_5 \langle v \rangle^2$. Hence,

$$f(t, x, v) \geq c_0 e^{N \ln c_4} = c_0 e^{-c_5 |\ln c_4| \langle v \rangle^2},$$

and the proof is complete. ■

Step 4: Spread of positivity in x. Let us fix any point of time $(\omega_0/4)^2 \leq t \leq (\omega_0/2)^2$ and reset it to 0. So, at the moment, we have

$$f(0, x, v) \geq be^{-a|v|^2}, \tag{7.26}$$

for all $|x| \leq (\omega_0/2)^3 := r_3$ and all $v \in \mathbb{R}^n$. Note that r_3 is admissible.

The next goal is to establish spread of positivity across the entire periodic domain. Recall that after the rescaling (7.21) our distribution f is defined on $L_0\mathbb{T}^n \times \mathbb{R}^n$, where L_0 is an admissible new period. Also, recall that since the scaling parameter $r < 1$, we still have global bounds on the coefficients

$$|s| \leq \bar{S}, \quad |b| \leq \bar{S}, \quad |w| \leq \bar{W}.$$

First, let us adopt a barrier construction from [2] to our situation.

Lemma 7.8. *Suppose*

$$f(0, x, v) \geq \delta \mathbb{1}_{\{|x| < r, |v| < R\}}.$$

Then, for any $\tau > 0$, we have

$$f(t, x, v) \geq \frac{\delta}{4} \mathbb{1}_{\{|x-tv| < r/2, |v| < R/2\}}.$$

for

$$t \leq t_1 := \min \left\{ 1, \tau, \frac{1}{8} \cdot \frac{1}{n\bar{S}(1/r^2 + 1/R^2) + (\bar{S}R + \bar{W})(\tau/r + 1/R)} \right\}. \tag{7.27}$$

Proof. Let us fix $A > 0$ to be determined later and consider the barrier function

$$\chi = -At + \delta \left(1 - \frac{|x - tv|^2}{r^2} - \frac{|v|^2}{R^2} \right).$$

Note that $f(0, x, v) \geq \chi(0, x, v)$, and also for all $t > 0$, $f(t, x, v) \geq \chi(t, x, v) = 0$, on the boundary

$$1 = \frac{|x - tv|^2}{r^2} + \frac{|v|^2}{R^2}.$$

So, we have $f \geq \chi$ on the parabolic boundary in question. We now need to show that χ is a sub-solution inside the ellipsoid

$$1 \geq \frac{|x - tv|^2}{r^2} + \frac{|v|^2}{R^2}.$$

By the classical comparison principle it implies $f \geq \chi$ on the same region.

So, differentiating we obtain

$$\begin{aligned} \chi_t + v \cdot \nabla_x \chi &= -A, \\ |s\Delta_v \chi| &= s\delta \left| \frac{2t^2n}{r^2} + \frac{2n}{R^2} \right| \leq 2n\delta\bar{S} \left(\frac{1}{r^2} + \frac{1}{R^2} \right), \end{aligned}$$

$$\begin{aligned} |(bv + w) \cdot \nabla_v \chi| &\leq \delta(\bar{S}R + \bar{W}) \left(2t \frac{|x - tv|}{r^2} + \frac{2|v|}{R^2} \right) \\ &\leq \delta(\bar{S}R + \bar{W}) \left(\frac{2\tau}{r} + \frac{2}{R} \right). \end{aligned}$$

Let

$$A = 2n\delta\bar{S} \left(\frac{1}{r^2} + \frac{1}{R^2} \right) + \delta(\bar{S}R + \bar{W}) \left(\frac{2\tau}{r} + \frac{2}{R} \right).$$

In view of the bounds above this implies that χ is a sub-solution.

It remains to observe that as long as $t \leq \delta/4A$ and $|x - tv| < r/2$, $|v| < R/2$, we have $\chi \geq \delta/4$. ■

We will be applying Lemma 7.8 for $r = r_3$. Let us pick τ and R now. Our aim is to make sure that the time limitation giving by the bound (7.27) is long enough that every corner of the torus $L_0\mathbb{T}^n$ is reachable in that time with velocities from the ball $|v| \leq R/4$. In other words, we ask for $t_1 R \geq 4L_0$, or

$$\tau R \geq 4L_0, \quad R \geq 4L_0 \tag{7.28}$$

$$R \geq 32L_0 \left[n\bar{S} \left(\frac{1}{r_3^2} + \frac{1}{R^2} \right) + (\bar{S}R + \bar{W}) \left(\frac{\tau}{r_3} + \frac{1}{R} \right) \right]. \tag{7.29}$$

So, first we fix $\tau = r_3/2\bar{S}$. This ensures that the leading order term in (7.29) has coefficient 1/2. Next, we fix the minimal $R = R_1$ satisfying both (7.28) and (7.29). Note that R_1 is admissible.

Setting $\delta = be^{-aR_1^2}$, which is also admissible, in view of (7.26) we have

$$f(0, x, v) \geq \delta \mathbb{1}_{\{|x| < r_1, |v| < R_1\}}.$$

Then

$$f(t, x, v) \geq \frac{\delta}{4} \mathbb{1}_{\{|x - tv| < r_3/2, |v| < R_1/2\}}, \quad t \leq t_1.$$

Fix any $x_0 \in L_0\mathbb{T}^n$. Then at time t_1 there exists $|v_0| < R_1/4$ such that $t_1 v_0 = x_0$. Notice that if

$$|x - x_0| < r_3/4, \quad |v - v_0| < r_3/4,$$

then $|x - t_1 v| = |x - x_0 + t_1(v_0 - v)| < r_3/2$, and certainly, $|v| < R_1/2$. So,

$$f(t_1, x, v) \geq \frac{\delta}{4} \mathbb{1}_{\{|x - x_0| < r_3/4, |v - v_0| < r_3/4\}}.$$

Let us recall that we have started from any point of time $(\omega_0/4)^2 \leq t \leq (\omega_0/2)^2$, and obtained a time t_1 independent of t . So, we found that for any $x_0 \in L_0\mathbb{T}^n$ there exists a v_0 , $|v_0| < R_1/4$, which depends only on x_0 such that

$$f(t, x, v) \geq \frac{\delta}{4} \mathbb{1}_{\{(\omega_0/4)^2 < t - t_1 < (\omega_0/2)^2, |x - x_0| < r_3/4, |v - v_0| < r_3/4\}}. \tag{7.30}$$

In particular,

$$\rho(t, x_0) = \int_{\mathbb{R}^n} f(t, x_0, v) \, dv \geq \int_{|v-v_0| < r_3/4} f(t, x_0, v) \, dv \geq \lambda_1,$$

where λ_1 is admissible, and $(\omega_0/4)^2 \leq t - t_1 \leq (\omega_0/2)^2$. So, for all such times, the density has a uniform lower bound λ_1 . At the same time there exists an admissible Λ_1 such that

$$s(t, x) + |b(t, x)v + w(t, x)| \leq \Lambda_1$$

for all $(t, x, v) \in [(\omega_0/4)^2 + t_1, (\omega_0/2)^2 + t_1] \times L_0\mathbb{T}^n \times B_{4R_1} = \bar{\Omega}_1$.

This implies that we have another initial plateau (7.30), but now around an arbitrary point $x_0 \in L_0\mathbb{T}^n$, and inside a large domain of ellipticity $\bar{\Omega}_1$. Applying Lemma 7.7 to shifted and if necessary rescaled solution f , we find a time $t_2 < (\omega_0/2)^2 + t_1$ and admissible $\omega_1, a_1, b_1 > 0$ such that

$$f(t_2, x, v) \geq b_1 e^{-a_1|v|^2} \mathbb{1}_{|x-x_0| < \omega_1}.$$

The obtained admissible constants are independent of x_0 by virtue of the argument on Step 3. Thus,

$$f(t_2, x, v) \geq b_1 e^{-a_1|v|^2}. \tag{7.31}$$

Now, let us go back to Step 1 and recall that we started with time 0 and found an admissible time $0 < t_2 < 1/2$ such that (7.31) holds. Starting at any other initial time $1 - t_2 > t > 0$, we find that (7.31) holds at $t + t_2$. This finishes the proof. ■

7.3. Entropy and global well-posedness

The main implication of Proposition 7.3 can be expressed in terms of lower bound on the density.

Corollary 7.9. *For a given classical solution $f \in C_w([0, T]; H_1^k(\mathbb{T}^n))$ of (7.1) on a time interval $[0, T)$ there exists ρ_- which depends only on the parameters of the model \mathcal{M} , time T , and \bar{W}, \bar{H} such that*

$$\rho(t, x) \geq \rho_-, \quad \forall (t, x) \in [T/2, T) \times \mathbb{T}^n.$$

So, controlling \bar{W} and \bar{H} over any finite time interval prevents formation of vacuum, which by Theorem 7.1 implies global extension. For a special class of our models control over \bar{W} and \bar{H} can indeed be given a priori in terms of energy. We start with \bar{H} .

First we recall that \bar{H} is controlled by the true entropy

$$\mathcal{H} = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 f \, dv \, dx + \int_{\Omega \times \mathbb{R}^n} f \log f \, dv \, dx.$$

Indeed, by the classical inequality, [37, 64], there is an absolute constant $C > 0$ such that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^n} |f \log f| \, dv \, dx &\leq \int_{\Omega \times \mathbb{R}^n} f \log f \, dv \, dx + \frac{1}{4} \int_{\Omega \times \mathbb{R}^n} |v|^2 f \, dv \, dx + C \\ &\leq \mathcal{H} + C. \end{aligned}$$

So,

$$\bar{H} \leq 2\mathcal{H} + C.$$

We also have control over the energy

$$\mathcal{E} \leq 2\mathcal{H} + C. \tag{7.32}$$

Lemma 7.10. *Suppose \mathcal{M} satisfies (3.28). Then \bar{H} is finite on any finite time interval. Moreover, if the model \mathcal{M} is conservative, then \bar{H} is globally bounded by initial data*

$$\bar{H} \leq 2\mathcal{H}_0 + C. \tag{7.33}$$

Proof. We have directly from the equation

$$\frac{d}{dt} \mathcal{H} = - \int_{\Omega \times \mathbb{R}^n} s_\rho \left[\frac{|\nabla_v f|^2}{f} + 2(v - [u]_\rho) \cdot \nabla_v f + v \cdot (v - [u]_\rho) f \right] dv dx. \tag{7.34}$$

Using the identities

$$\int_{\Omega \times \mathbb{R}^n} s_\rho [u]_\rho \cdot \nabla_v f \, dx \, dv = 0, \quad \int_{\Omega \times \mathbb{R}^n} s_\rho v \cdot [u]_\rho f \, dx \, dv = (u, [u]_\rho)_{\kappa_\rho},$$

and replace $[u]_\rho$ with u in the second term and compute the third as follows:

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^n} s_\rho v \cdot (v - [u]_\rho) f \, dv \, dx &= \int_{\Omega \times \mathbb{R}^n} s_\rho v \cdot (v - u) f \, dv \, dx \\ &\quad + \int_{\Omega \times \mathbb{R}^n} s_\rho v \cdot (u - [u]_\rho) f \, dv \, dx \\ &= \int_{\Omega \times \mathbb{R}^n} s_\rho |v - u|^2 f \, dv \, dx + \|u\|_{L^2(\kappa_\rho)}^2 - (u, [u]_\rho)_{\kappa_\rho}. \end{aligned}$$

We obtain

$$\frac{d}{dt} \mathcal{H} = - \int_{\Omega \times \mathbb{R}^n} s_\rho \frac{|\nabla_v f + (v - u) f|^2}{f} \, dv \, dx - \|u\|_{L^2(\kappa_\rho)}^2 + (u, [u]_\rho)_{\kappa_\rho}. \tag{7.35}$$

We can see that $\mathcal{H} \leq \mathcal{H}_0$ for conservative models and (7.33) follows.

Under (3.28), we use (7.32) to conclude that $\dot{\mathcal{H}} \leq C_1 \mathcal{H} + C_2$. The conclusion follows. ■

Immediately from Lemma 7.10, we obtain control over \bar{W} as well under $L^2 \rightarrow L^\infty$ -boundedness on the averages.

Lemma 7.11. *Suppose \mathcal{M} satisfies (3.31). Then \bar{H}, \bar{W} are finite on any finite time interval. Moreover, if the model \mathcal{M} is conservative, then \bar{H}, \bar{W} are uniformly bounded by a constant depending only on the initial condition.*

Theorem 7.12. *Suppose the model \mathcal{M} is regular and satisfies (3.31). Then any local solution f to the Fokker–Planck-alignment equation (7.1) in class $H_l^k(\Omega \times \mathbb{R}^n)$ extends globally in time. Consequently, (7.1) is globally well-posed for thick data*

$$f_0 \in H_l^k(\Omega \times \mathbb{R}^n), \quad k, l \geq n + 3, \quad \Theta(\rho_0, \Omega) > 0.$$

If in addition \mathcal{M} is conservative, then there exists a $\rho_- > 0$ depending only on the initial entropy \mathcal{H}_0 and the parameters of the model, such that

$$\rho(t, x) \geq \rho_-, \quad \forall (t, x) \in [1, \infty) \times \mathbb{T}^n. \tag{7.36}$$

A simple rescaling argument shows that in fact for any time $t_0 > 0$, there exists $\rho_- > 0$ depending on the initial entropy \mathcal{H}_0 , t_0 , and the parameters of the model such that

$$\rho(t, x) \geq \rho_-, \quad \forall (t, x) \in [t_0, \infty) \times \mathbb{T}^n.$$

So, the vacuum disappears instantaneously.

As shown on the third row of Table 4 all our models are regular on compact environment, and hence the corresponding FPA are globally well-posed for thick data. In addition, the models \mathcal{M}_{CS} , \mathcal{M}_ϕ , and \mathcal{M}_{seg} , due to being conservative, also enjoy the uniform bound from below on the density (7.36).

8. Global relaxation and hypocoercivity

The discussion in this section will be taking place on the compact domain $\Omega = \mathbb{T}^n$. The Fokker–Planck-alignment equation

$$\partial_t f + v \cdot \nabla_x f = \sigma s_\rho \Delta_v f + \nabla_v \cdot (s_\rho (v - [u]_\rho) f), \tag{8.1}$$

has an obvious equilibrium

$$\mu_{\sigma, \bar{u}} = \frac{1}{|\Omega| (2\pi\sigma)^{n/2}} e^{-|v - \bar{u}|^2 / 2\sigma},$$

for any constant vector \bar{u} . In this section we demonstrate relaxation towards such equilibrium for large data.

There are several issues that arise when comparing this result to the classical linear Fokker–Planck relaxation, see [97]. First, the non-linear alignment force pumps energy into the system as will be seen from (8.3), which prevents direct sliding of the solution towards global Maxwellian. Second, the degeneracy of thermalization σs_ρ needs to be avoided in order to retain uniform hypoellipticity of the equation. And third, since we are not assuming that \mathcal{M} is conservative, it is not immediately clear that the time dependent momentum \bar{u} settles to a limiting vector u_∞ .

We settle these issues in steps. Our first general result lists all the necessary functional requirements on the solution to ensure relaxation towards a moving Maxwellian. We then

examine how these requirements are met in the context of regularity properties stated in Section 3.7 for specific classes of models and how the stabilization of momentum can be deduced.

Proposition 8.1. *Suppose \mathcal{M} is a material model. Let $f \in H_1^k(\Omega \times \mathbb{R}^n)$ be a classical solution to (6.2) with density ρ satisfying the following conditions uniformly in time*

- (i) *there exist constants $c_0, c_1, c_2 > 0$ such that $c_0 \leq s_\rho \leq c_1$ and $\|\nabla s_\rho\|_\infty \leq c_2$ for all $\rho \in \mathcal{D}$;*
- (ii) *there exists a constant $\varepsilon_0 > 0$ such that*

$$\sup\{(u, [u]_\rho)_{\kappa_\rho} : u \in L^2(\kappa_\rho), \bar{u} = 0, \|u\|_{L^2(\kappa_\rho)} = 1\} \leq 1 - \varepsilon_0.$$

- (iii) $\|s_\rho[\cdot]_\rho\|_{L^2(\rho) \rightarrow L^2(\rho)} + \|\nabla_x(s_\rho[\cdot]_\rho)\|_{L^2(\rho) \rightarrow L^2(\rho)} \leq c_3.$

Then f relaxes to the corresponding Maxwellian exponentially fast,

$$\|f(t) - \mu_{\sigma, \bar{u}(t)}\|_{L^1(\Omega \times \mathbb{R}^n)} \leq c_4 \sqrt{\sigma^{-1} \mathcal{I}(f_0)} e^{-c_5 \sigma t},$$

where $c_4, c_5 > 0$ depend only on the parameters of the model \mathcal{M} and $c_0, c_1, c_2, c_3, \varepsilon_0$. Here, $\mathcal{I}(f_0)$ is the Fisher information defined in (8.7), and

$$\bar{u} = \int_{\Omega \times \mathbb{R}^n} v f(t, x, v) \, dv \, dx.$$

Proof of Proposition 8.1. We seek to estimate the relative entropy defined by

$$\mathcal{H} = \sigma \int_{\Omega \times \mathbb{R}^n} f \log \frac{f}{\mu_{\sigma, \bar{u}}} \, dv \, dx.$$

By the Csiszár–Kullback inequality, we have

$$c\sigma \|f - \mu_{\sigma, \bar{u}}\|_1^2 \leq \mathcal{H}$$

for some absolute c . So, an exponential decay of the entropy would imply the desired result. Let us also recall that Sobolev densities $f \in H_1^k(\Omega \times \mathbb{R}^n)$ have finite Fisher information (see below) which in turns control \mathcal{H} , see [95, Lemma 1]. We can therefore analyze \mathcal{H} classically.

Since the model at hand is not assumed to be Galilean invariant or conservative the mean velocity \bar{u} is time dependent and generally may not be assumed 0 without changing the equation. It will, however, be beneficial to pass to the reference frame centered at \bar{u} . So, we consider the change of variables

$$\tilde{f}(x, v, t) = f(x, v + \bar{u}, t), \quad \tilde{u} = u - \bar{u}, \quad \tilde{\rho} = \rho.$$

In the new variables the equation becomes a system (dropping tildas)

$$\begin{aligned} \partial_t f + (v + \bar{u}) \cdot \nabla_x f &= \bar{u}_t \cdot \nabla_v f + \sigma s_\rho \Delta_v f + \nabla_v \cdot (s_\rho(v - [u]_\rho) f), \\ \bar{u}_t &= \int_\Omega ([u]_\rho - u) \, d\kappa_\rho, \quad \int_\Omega u \rho \, dx = 0. \end{aligned} \tag{8.2}$$

We also denote $\mu_\sigma = \mu_{\sigma,0}$. Again, let us note that the extra transport term $\bar{u}_t \cdot \nabla_v f$ appears because we do not assume that our model is conservative. We keep in mind that \bar{u}_t is a constant vector at any point of time.

The starting point in the proof is two forms of the entropy law. One is (7.35), which after σ -rescaling reads

$$\frac{d}{dt} \mathcal{H} = - \int_{\Omega \times \mathbb{R}^n} s_\rho \frac{|\sigma \nabla_v f + (v - u) f|^2}{f} dv dx - \|u\|_{L^2(\kappa_\rho)}^2 + (u, [u]_\rho)_{\kappa_\rho}.$$

Using the spectral gap assumption (ii), we conclude

$$\frac{d}{dt} \mathcal{H} \leq -\varepsilon_0 \|u\|_{L^2(\kappa_\rho)}^2.$$

And another form of entropy law follows directly from (7.34) (note that the extra transport term $\bar{u}_t \cdot \nabla_v f$ does not effect either of them):

$$\frac{d}{dt} \mathcal{H} = - \int_{\Omega \times \mathbb{R}^n} s_\rho \frac{|\sigma \nabla_v f + v f|^2}{f} dv dx + (u, [u]_\rho)_{\kappa_\rho}. \tag{8.3}$$

Although this form is not dissipative, it gives access to the partial Fisher information

$$\mathcal{I}_{vv} = \int_{\Omega \times \mathbb{R}^n} \frac{|\sigma \nabla_v f + v f|^2}{f} dv dx.$$

In view of (i) and (iii), we have

$$\frac{d}{dt} \mathcal{H} \leq -c_0 \mathcal{I}_{vv} + (u, [u]_\rho)_{\kappa_\rho} \leq -c_0 \mathcal{I}_{vv} + c \|u\|_{L^2(\kappa_\rho)}^2. \tag{8.4}$$

Combining with the previous form (8.4), we obtain

$$\frac{d}{dt} \mathcal{H} \lesssim -\mathcal{I}_{vv} - \|u\|_{L^2(\kappa_\rho)}^2. \tag{8.5}$$

The next stage of the proof consists of showing that the classical hypocoercivity of the linear Fokker–Planck equation extends to the fully non-linear alignment model. In contrast to the \mathcal{M}_ϕ -model analyzed in [83] the general system (8.2) requires special attention due to presence of several additional ingredients such as inhomogeneity of diffusion and \bar{u} -shift in the transport term. These result in the slower exponential rate σ , as opposed to $\sigma^{1/2}$ for the \mathcal{M}_ϕ -model.

Let us write the equation for the new distribution

$$\begin{aligned} h &= \frac{f}{\mu_\sigma}, \\ h_t + (v + \bar{u}) \cdot \nabla_x h &= \bar{u}_t \cdot \nabla_v h - \frac{v}{\sigma} \cdot \bar{u}_t h + s_\rho (\sigma \Delta_v h - v \cdot \nabla_v h) \\ &\quad + s_\rho (\sigma^{-1} ([u]_\rho \cdot v) h - [u]_\rho \cdot \nabla_v h). \end{aligned} \tag{8.6}$$

The Fokker–Planck part of the equation (8.6) has the traditional structure of an evolution semi-group. Denoting

$$B = (v + \bar{u}) \cdot \nabla_x, \quad A = \nabla_v, \quad A^* = \left(\frac{v}{\sigma} - \nabla_v \right),$$

where A^* is understood relative to the inner product of the weighted space $L^2(\mu_\sigma)$, we can write

$$h_t = -\sigma s_\rho A^* A h - B h + s_\rho A^* ([u]_\rho h) - A^*(h \bar{u}_t).$$

We consider Fisher information functionals

$$\begin{aligned} \mathcal{I}_{vv}(h) &= \sigma^2 \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla_v h|^2}{h} \, d\mu_\sigma, \\ \mathcal{I}_{xv}(h) &= \sigma^{3/2} \int_{\Omega \times \mathbb{R}^n} \frac{\nabla_x h \cdot \nabla_v h}{h} \, d\mu_\sigma, \\ \mathcal{I}_{xx}(h) &= \sigma \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla_x h|^2}{h} \, d\mu_\sigma, \end{aligned}$$

where $d\mu_\sigma = \mu_\sigma \, dv \, dx$. The full Fisher information defined by

$$\mathcal{I} = \mathcal{I}_{vv} + \mathcal{I}_{xx} \tag{8.7}$$

dominates the relative entropy by the classical log-Sobolev inequality

$$\mathcal{I}_{vv} + \mathcal{I}_{xx} \geq \lambda \mathcal{H}.$$

We now differentiate each of these functionals and obtain estimates on the obtained equations. The coercivity will be restored by putting them together in a proper linear combination along with the entropy law (8.5).

We will use the following notation: $(g)_\mu = \int_{\Omega \times \mathbb{R}^n} g \, d\mu_\sigma$.

Lemma 8.2. *We have*

$$\frac{d}{dt} \mathcal{I}_{vv}(h) \leq -2\sigma^3 \mathcal{D}_{vv} - 2c_0 \mathcal{I}_{vv} - 2\sigma^{1/2} \mathcal{I}_{xv} + 2\|u\|_{L^2(\kappa_\rho)}^2,$$

where

$$\mathcal{D}_{vv} = (s_\rho h |\nabla_v^2 \bar{h}|^2)_\mu, \quad \bar{h} = \log h.$$

Proof. Let us write $\mathcal{I}_{vv} = (\nabla_v h \cdot \nabla_v \bar{h})_\mu$. Computing the derivative, we obtain

$$\frac{1}{\sigma^2} \frac{d}{dt} \mathcal{I}_{vv} = 2(\nabla_v h_t \cdot \nabla_v \bar{h})_\mu - (|\nabla_v \bar{h}|^2 h_t)_\mu = J_A + J_B + J_u + J_{\bar{u}},$$

where

$$\begin{aligned} J_A &= -2\sigma (s_\rho \nabla_v A^* A h \cdot \nabla_v \bar{h})_\mu + \sigma (s_\rho |\nabla_v \bar{h}|^2 A^* A h)_\mu, \\ J_B &= -2(\nabla_v B h \cdot \nabla_v \bar{h})_\mu + (|\nabla_v \bar{h}|^2 B h)_\mu, \end{aligned}$$

$$\begin{aligned} J_u &= 2(s_\rho \nabla_v A^*([u]_\rho h) \cdot \nabla_v \bar{h})_\mu - (s_\rho |\nabla_v \bar{h}|^2 A^*([u]_\rho h))_\mu, \\ J_{\bar{u}} &= -2(\nabla_v A^*(h \bar{u}_t) \cdot \nabla_v \bar{h})_\mu + (\nabla_v |\nabla_v \bar{h}|^2 \cdot \bar{u}_t h)_\mu. \end{aligned}$$

Let us start with the most straightforward transport term B . We have

$$\begin{aligned} J_B &= -2(\nabla_x h \cdot \nabla_v \bar{h})_\mu - 2((v + \bar{u}) \cdot \nabla_x h_{v_i}) \bar{h}_{v_i} \mu + (|\nabla_v \bar{h}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu \\ &=: J_B^1 + J_B^2 + J_B^3. \end{aligned}$$

Observe that

$$J_B^1 = -2\sigma^{-3/2} \mathcal{I}_{xv}.$$

Next, as to the second term:

$$\begin{aligned} J_B^2 &= -2((v + \bar{u}) \cdot \nabla_x h_{v_i} h_{v_i} h^{-1})_\mu = -((v + \bar{u}) \cdot \nabla_x |h_{v_i}|^2 h^{-1})_\mu \\ &= -(|h_{v_i}|^2 (v + \bar{u}) \cdot \nabla_x h h^{-2})_\mu = -(|\bar{h}_{v_i}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu = -J_B^3, \end{aligned}$$

and so the two cancel. We obtain

$$J_B = -2\sigma^{-3/2} \mathcal{I}_{xv}.$$

Let us turn to the dissipation term J_A . Using the identity

$$\partial_{v_i} (A^* A h) = A^* A h_{v_i} + \sigma^{-1} h_{v_i},$$

we have

$$J_A = -2\sigma (s_\rho A^* A h_{v_i} \bar{h}_{v_i})_\mu - 2(s_\rho \nabla_v h \cdot \nabla_v \bar{h})_\mu + \sigma (s_\rho |\nabla_v \bar{h}|^2 A^* A h)_\mu.$$

Note that the term in the middle is bounded above by $-2c_0 \sigma^{-2} \mathcal{I}_{vv}$ in view of (i). In the other two we switch A^* to the opposite side,

$$\begin{aligned} J_A &\leq -2\sigma (s_\rho A h_{v_i} \cdot A \bar{h}_{v_i})_\mu - 2c_0 \sigma^{-2} \mathcal{I}_{vv} + \sigma (s_\rho A |\nabla_v \bar{h}|^2 \cdot A h)_\mu \\ &= -2\sigma (s_\rho h A \bar{h}_{v_i} \cdot A \bar{h}_{v_i})_\mu - 2\sigma (s_\rho \bar{h}_{v_i} A h \cdot A \bar{h}_{v_i})_\mu \\ &\quad - 2c_0 \sigma^{-2} \mathcal{I}_{vv} + 2\sigma (s_\rho \bar{h}_{v_i} A \bar{h}_{v_i} \cdot A h)_\mu. \end{aligned}$$

The second and last terms cancel, while the first is exactly $-2\sigma \mathcal{D}_{vv}$:

$$J_A \leq -2\sigma \mathcal{D}_{vv} - 2c_0 \sigma^{-2} \mathcal{I}_{vv}.$$

For the alignment term, we obtain the following the exact identity:

$$J_u = 2\sigma^{-2} ([u]_\rho \cdot u)_{\kappa_\rho}. \tag{8.8}$$

We note, however, that there is no advantage of keeping the low energy here as the full energy will emerge later in the proof. So, we replace it with the full energy

$$J_u \leq 2(1 - \varepsilon_0) \sigma^{-2} \|u\|_{L^2(\kappa_\rho)}^2 \leq 2\sigma^{-2} \|u\|_{L^2(\kappa_\rho)}^2.$$

To prove (8.8), we manipulate with the formula for J_u as follows:

$$\begin{aligned}
 J_u &= 2(s_\rho \nabla_v A^*([u]_\rho h) \cdot \nabla_v \bar{h})_\mu - (s_\rho |\nabla_v \bar{h}|^2 A^*([u]_\rho h))_\mu \\
 &= 2(s_\rho \nabla_v (\sigma^{-1} v \cdot [u]_\rho h - [u]_\rho \cdot \nabla_v h) \cdot \nabla_v \bar{h})_\mu - (s_\rho \nabla_v |\nabla_v \bar{h}|^2 \cdot [u]_\rho h)_\mu \\
 &= 2\sigma^{-1} (s_\rho [u]_\rho h \cdot \nabla_v \bar{h})_\mu + 2\sigma^{-1} (s_\rho (v \cdot [u]_\rho) \nabla_v h \cdot \nabla_v \bar{h})_\mu \\
 &\quad - 2(s_\rho \nabla_v^2 h [u]_\rho \cdot \nabla_v \bar{h})_\mu - 2(s_\rho \nabla_v^2 \bar{h} (\nabla_v \bar{h}) \cdot [u]_\rho h)_\mu \\
 &=: J_u^1 + J_u^2 + J_u^3 + J_u^4,
 \end{aligned}$$

where $\nabla_v^2 h$ is the Hessian matrix of h .

Observe that the first term is exactly the lower energy

$$J_u^1 = 2\sigma^{-1} (s_\rho [u]_\rho \cdot \nabla_v h)_\mu = 2\sigma^{-2} (s_\rho [u]_\rho \cdot v h)_\mu = 2\sigma^{-2} ([u]_\rho \cdot u)_{\kappa_\rho}.$$

Now comes the crucial observation that the remaining terms that cannot be controlled cancel altogether

$$J_u^2 + J_u^3 + J_u^4 = 0.$$

Indeed, using

$$h_{v_i v_j} = h \bar{h}_{v_i v_j} + \frac{1}{h} h_{v_i} h_{v_j}$$

let us compute J_u^3 ,

$$\begin{aligned}
 J_u^3 &= -2(s_\rho h_{v_i v_j} [u_j]_\rho \bar{h}_{v_i})_\mu = -2(s_\rho h \bar{h}_{v_i v_j} [u_j]_\rho \bar{h}_{v_i})_\mu - 2\left(s_\rho \frac{1}{h} h_{v_i} h_{v_j} [u_j]_\rho \bar{h}_{v_i}\right)_\mu \\
 &= J_u^4 - 2(s_\rho \nabla_v h \cdot [u]_\rho |\nabla_v \bar{h}|^2)_\mu.
 \end{aligned}$$

Also note that

$$J_u^2 = 2\sigma^{-1} (s_\rho (v \cdot [u]_\rho h) |\nabla_v \bar{h}|^2)_\mu.$$

Then we have

$$\begin{aligned}
 J_u^2 + J_u^3 &= J_u^4 + 2\sigma^{-1} (s_\rho (v \cdot [u]_\rho h) |\nabla_v \bar{h}|^2)_\mu - 2(s_\rho \nabla_v h \cdot [u]_\rho |\nabla_v \bar{h}|^2)_\mu \\
 &= J_u^4 + 2(s_\rho A^*([u]_\rho h) |\nabla_v \bar{h}|^2)_\mu.
 \end{aligned}$$

Switching A^* in the last term we obtain

$$2(s_\rho A^*([u]_\rho h) |\nabla_v \bar{h}|^2)_\mu = 2(s_\rho h [u]_\rho \cdot \nabla_v |\nabla_v \bar{h}|^2)_\mu = -2J_u^4.$$

The obtained terms sum up to zero.

Finally, we show that the momentum term vanishes $J_{\bar{u}} = 0$. Let us expand

$$\begin{aligned}
 J_{\bar{u}} &= 2(\nabla_v^2 h \nabla_v \bar{h} \cdot \bar{u}_t)_\mu - 2\left(\frac{v}{\sigma} \cdot \bar{u}_t \frac{|\nabla_v h|^2}{h}\right)_\mu + (\nabla_v |\nabla_v \bar{h}|^2 \cdot \bar{u}_t h)_\mu \\
 &=: J_{\bar{u}}^1 + J_{\bar{u}}^2 + J_{\bar{u}}^3.
 \end{aligned}$$

Let us look into the first term,

$$\begin{aligned} J_{\bar{u}}^1 &= \left(\frac{\nabla_v |\nabla_v h|^2}{h} \cdot \bar{u}_t \right)_\mu = \left(A \left(\frac{|\nabla_v h|^2}{h} \right) \cdot \bar{u}_t \right)_\mu + \left(\frac{|\nabla_v h|^2}{h^2} \nabla_v h \cdot \bar{u}_t \right)_\mu \\ &= \left(\frac{|\nabla_v h|^2}{h} \frac{v \cdot \bar{u}_t}{\sigma} \right)_\mu + (|\nabla_v \bar{h}|^2 \nabla_v h \cdot \bar{u}_t)_\mu. \end{aligned}$$

So,

$$\begin{aligned} J_{\bar{u}}^1 + J_{\bar{u}}^2 &= - \left(\frac{|\nabla_v h|^2}{h} \frac{v \cdot \bar{u}_t}{\sigma} \right)_\mu + (|\nabla_v \bar{h}|^2 \nabla_v h \cdot \bar{u}_t)_\mu \\ &= - (|\nabla_v \bar{h}|^2 A^*(h \bar{u}_t))_\mu = - (\nabla_v |\nabla_v \bar{h}|^2 h \bar{u}_t)_\mu = - J_{\bar{u}}^3. \end{aligned}$$

Thus, $J_{\bar{u}}^1 + J_{\bar{u}}^2 + J_{\bar{u}}^3 = 0$. ■

Lemma 8.3. *We have*

$$\begin{aligned} \frac{d}{dt} \mathcal{I}_{xv}(h) &\leq -\frac{1}{4} \sigma^{1/2} \mathcal{I}_{xx} + c(\sigma^{-1/2} + 1) \mathcal{I}_{vv} \\ &\quad + \frac{1}{2} \sigma^3 \mathcal{D}_{vv} + \frac{1}{2} \sigma^2 \mathcal{D}_{xv} + c(\sigma^{-1/2} + 1) \|u\|_2^2, \end{aligned}$$

where

$$\mathcal{D}_{xv} = (s_\rho h |\nabla_v \nabla_x \bar{h}|^2)_\mu.$$

Proof. Let us write

$$\begin{aligned} \frac{1}{\sigma^{3/2}} \frac{d}{dt} \mathcal{I}_{xv}(h) &= (\nabla_x h_t \cdot \nabla_v \bar{h})_\mu + (\nabla_x \bar{h} \cdot \nabla_v h_t)_\mu - (h_t \nabla_v \bar{h} \cdot \nabla_x \bar{h})_\mu \\ &:= J_A + J_B + J_u + J_{\bar{u}}, \end{aligned}$$

where, as before, $J_A, J_B, J_u, J_{\bar{u}}$ collect contributions from A, B , and alignment components, respectively.

For the B -term, we have

$$\begin{aligned} J_B &= -(\nabla_x((v + \bar{u}) \cdot \nabla_x h) \cdot \nabla_v \bar{h})_\mu - (\nabla_x \bar{h} \cdot \nabla_v((v + \bar{u}) \cdot \nabla_x h))_\mu \\ &\quad + (((v + \bar{u}) \cdot \nabla_x h) \nabla_v \bar{h} \cdot \nabla_x \bar{h})_\mu \\ &:= J_B^1 + J_B^2 + J_B^3. \end{aligned}$$

For the middle term, we expand

$$J_B^2 = -(\nabla_x \bar{h} \cdot \nabla_x h)_\mu - (\bar{h}_{x_i} (v_j + \bar{u}_j) h_{x_j v_i})_\mu.$$

The first term is exactly $-\sigma^{-1} \mathcal{I}_{xx}$ and in the second integrating by parts in x_j , we obtain

$$= -\sigma^{-1} \mathcal{I}_{xx} + (\bar{h}_{x_i x_j} (v_j + \bar{u}_j) h_{v_i})_\mu,$$

and using that $\bar{h}_{x_i x_j} = h^{-1} h_{x_i x_j} - \bar{h}_{x_i} \bar{h}_{x_j}$,

$$\begin{aligned} &= -\sigma^{-1} \mathcal{I}_{xx} + (h_{x_i x_j} (v_j + \bar{u}_j) \bar{h}_{v_i})_\mu - (\bar{h}_{x_i} \bar{h}_{x_j} (v_j + \bar{u}_j) h_{v_i})_\mu \\ &= -\sigma^{-1} \mathcal{I}_{xx} - J_B^1 - J_B^3. \end{aligned}$$

Hence,

$$J_B = -\sigma^{-1} \mathcal{I}_{xx}. \quad (8.9)$$

Let us look into the J_A -term:

$$\begin{aligned} \frac{1}{\sigma} J_A &= -(\nabla_x (s_\rho A^* A h) \cdot \nabla_v \bar{h})_\mu - (s_\rho \nabla_x \bar{h} \cdot \nabla_v A^* A h)_\mu + (s_\rho A^* A h \nabla_v \bar{h} \cdot \nabla_x \bar{h})_\mu \\ &= J_A^1 + J_A^2 + J_A^3. \end{aligned}$$

For J_A^1 , we obtain

$$\begin{aligned} J_A^1 &= -(s_\rho A^* A h_{x_i} \bar{h}_{v_i})_\mu - ((s_\rho)_{x_i} A^* A h \bar{h}_{v_i})_\mu \\ &= -(s_\rho \nabla_v h_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - ((s_\rho)_{x_i} \nabla_v h \nabla_v \bar{h}_{v_i})_\mu \\ &= -(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - (s_\rho \bar{h}_{x_i} \nabla_v h \cdot \nabla_v \bar{h}_{v_i})_\mu - \left(\frac{(s_\rho)_{x_i}}{s_\rho^{1/2}} \frac{\nabla_v h}{h^{1/2}} \cdot s_\rho^{1/2} h^{1/2} \nabla_v \bar{h}_{v_i} \right)_\mu. \end{aligned}$$

In view of assumption (i),

$$J_A^1 \leq -(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - (s_\rho \bar{h}_{x_i} \nabla_v h \cdot \nabla_v \bar{h}_{v_i})_\mu + c \sigma^{-1} \sqrt{\mathcal{I}_{vv} \mathcal{D}_{vv}}.$$

For J_A^2 , we obtain

$$\begin{aligned} J_A^2 &= -\sigma^{-1} (s_\rho \nabla_x \bar{h} \cdot \nabla_v h)_\mu - (s_\rho \bar{h}_{x_i} A^* A h_{v_i})_\mu \\ &= -\sigma^{-1} (s_\rho \nabla_x \bar{h} \cdot \nabla_v h)_\mu - (s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu - (s_\rho \bar{h}_{v_i} \nabla_v \bar{h}_{x_i} \cdot \nabla_v h)_\mu. \end{aligned}$$

The two add up to

$$\begin{aligned} J_A^1 + J_A^2 &= -\sigma^{-1} \left(s_\rho \frac{\nabla_x h}{h^{1/2}} \cdot \frac{\nabla_v h}{h^{1/2}} \right)_\mu - 2(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu \\ &\quad - (s_\rho A h \cdot A (\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu + c \sigma^{-1} \sqrt{\mathcal{I}_{vv} \mathcal{D}_{vv}} \\ &\leq -c_0 \sigma^{-5/2} \sqrt{\mathcal{I}_{xx} \mathcal{I}_{vv}} + \sqrt{\mathcal{D}_{xv} \mathcal{D}_{vv}} + c \sigma^{-1} \sqrt{\mathcal{I}_{vv} \mathcal{D}_{vv}} - J_A^3. \end{aligned}$$

Thus,

$$J_A \leq c_0 \sigma^{-3/2} \sqrt{\mathcal{I}_{xx} \mathcal{I}_{vv}} + \sigma \sqrt{\mathcal{D}_{xv} \mathcal{D}_{vv}} + c \sqrt{\mathcal{I}_{vv} \mathcal{D}_{vv}}.$$

Let us examine the alignment term now,

$$\begin{aligned} J_u &= (\nabla_x (s_\rho A^* ([u]_\rho h)) \cdot \nabla_v \bar{h})_\mu + (\nabla_x \bar{h} \cdot \nabla_v (s_\rho A^* ([u] h)))_\mu \\ &\quad - (s_\rho A^* ([u]_\rho h) \nabla_v \bar{h} \cdot \nabla_x \bar{h})_\mu \end{aligned}$$

$$\begin{aligned}
 &= ((s_\rho)_{x_i} A^*([u]_\rho h) \bar{h}_{v_i})_\mu + (s A^*([u]_\rho)_{x_i} h \bar{h}_{v_i})_\mu + (s_\rho A^*([u]_\rho h_{x_i}) \bar{h}_{v_i})_\mu \\
 &\quad + (s_\rho \bar{h}_{x_i} A^*([u]_\rho h_{v_i}))_\mu + \sigma^{-1} (s_\rho h \nabla_x \bar{h} \cdot [u]_\rho)_\mu - (s_\rho h [u]_\rho \cdot \nabla_v (\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu \\
 &= (h (s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu + (s_\rho h [u]_\rho \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu + (s_\rho h \nabla_v \bar{h}_{x_i} \cdot [u]_\rho \bar{h}_{v_i})_\mu \\
 &\quad + \sigma^{-1} (s_\rho h \nabla_x \bar{h} \cdot [u]_\rho)_\mu - (s_\rho h [u]_\rho \cdot \nabla_v (\nabla_v \bar{h} \cdot \nabla_x \bar{h}))_\mu
 \end{aligned}$$

We can see that the second, third, and fifth terms cancel, and we arrive at

$$J_u = (h (s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{v_i})_\mu + \sigma^{-1} (s_\rho h \nabla_x \bar{h} \cdot [u]_\rho)_\mu = J_u^1 + J_u^2.$$

We estimate J_u^1 using the assumption (iii), and the fact that $L^2(\kappa_\rho)$ - and $L^2(\rho)$ -norms are equivalent under (i),

$$J_u^1 \leq c \|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{vv}}$$

And again, by (iii),

$$J_u^2 \leq \sigma^{-3/2} \|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{I}_{xx}} \leq \frac{1}{2} \sigma^{-1} \mathcal{I}_{xx} + \sigma^{-2} \|u\|_{L^2(\kappa_\rho)}^2.$$

Noticing that $\frac{1}{2} \sigma^{-1} \mathcal{I}_{xx}$ is absorbed into (8.9) and summing up all the terms we arrive at

$$\begin{aligned}
 \frac{d}{dt} \mathcal{I}_{xv}(h) &\leq -\frac{1}{2} \sigma^{1/2} \mathcal{I}_{xx} + c_0 \sqrt{\mathcal{I}_{xx} \mathcal{I}_{vv}} + c \sigma^{3/2} \sqrt{\mathcal{I}_{vv} \mathcal{D}_{vv}} + \sigma^{5/2} \sqrt{\mathcal{D}_{xv} \mathcal{D}_{vv}} \\
 &\quad + c \sigma^{3/2} \|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{vv}} + \sigma^{-1/2} \|u\|_{L^2(\kappa_\rho)}^2 \\
 &\leq -\frac{1}{4} \sigma^{1/2} \mathcal{I}_{xx} + c(\sigma^{-1/2} + 1) \mathcal{I}_{vv} \\
 &\quad + \frac{1}{2} \sigma^3 \mathcal{D}_{vv} + \frac{1}{2} \sigma^2 \mathcal{D}_{xv} + c(\sigma^{-1/2} + 1) \|u\|_{L^2(\kappa_\rho)}^2.
 \end{aligned}$$

Finally, let us look at the momentum term

$$\begin{aligned}
 J_{\bar{u}} &= -(A^*(\bar{u}_t \partial_{x_i} h) \partial_{v_i} \bar{h})_\mu - (\partial_{x_i} \bar{h} \partial_{v_i} A^*(\bar{u}_t h))_\mu + (A^*(\bar{u}_t h) \partial_{v_i} \bar{h} \partial_{x_i} \bar{h})_\mu \\
 &:= J_{\bar{u}}^1 + J_{\bar{u}}^2 + J_{\bar{u}}^3.
 \end{aligned}$$

Let us note the identity

$$\partial_{v_i} A^* = A^* \partial_{v_i} + \frac{1}{\sigma} \text{Id},$$

and expand on $J_{\bar{u}}^2$:

$$J_{\bar{u}}^2 = -(\partial_{x_i} \bar{h} A^*(\bar{u}_t \partial_{v_i} h))_\mu - \sigma^{-1} (\partial_{x_i} h \bar{u}_t)_\mu.$$

The last term vanishes since \bar{u}_t is a constant vector. Thus,

$$J_{\bar{u}}^2 = -(\nabla_v \partial_{x_i} \bar{h} \cdot \bar{u}_t \partial_{v_i} h)_\mu.$$

In the other two terms we switch A^* as well

$$J_{\bar{u}}^1 = -(\partial_{x_i} h \bar{u}_t \cdot \nabla_v \partial_{v_i} \bar{h})_\mu, \quad J_{\bar{u}}^3 = (h \bar{u}_t \cdot \nabla_v (\partial_{v_i} \bar{h} \partial_{x_i} \bar{h}))_\mu.$$

The sum of the three is clearly zero by the product rule. So,

$$J_{\bar{u}} = 0. \quad \blacksquare$$

Lemma 8.4. *We have*

$$\frac{d}{dt} \mathcal{I}_{xx}(h) \leq c \mathcal{I}_{vv} - \sigma^2 \mathcal{D}_{xv} + c \|u\|_{L^2(\kappa_\rho)}^2.$$

Proof. We have

$$\frac{1}{\sigma} \frac{d}{dt} \mathcal{I}_{xx}(h) = 2(\nabla_x h_t \cdot \nabla_x \bar{h})_\mu - (|\nabla_x \bar{h}|^2 h_t)_\mu := J_A + J_B + J_u + J_{\bar{u}}.$$

The B -term cancels entirely,

$$\begin{aligned} J_B &= -2(\nabla_x((v + \bar{u}) \cdot \nabla_x h) \cdot \nabla_x \bar{h})_\mu + (|\nabla_x \bar{h}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu \\ &= -2(((v + \bar{u}) \cdot \nabla_x h_{x_i}) h_{x_i} h^{-1})_\mu + (|\nabla_x \bar{h}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu \\ &= -((v + \bar{u}) \cdot \nabla_x |\nabla_x h|^2 h^{-1})_\mu + (|\nabla_x \bar{h}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu \\ &= -((v + \bar{u}) \cdot \nabla_x h |\nabla_x h|^2 h^{-2})_\mu + (|\nabla_x \bar{h}|^2 (v + \bar{u}) \cdot \nabla_x h)_\mu = 0. \end{aligned}$$

So $J_{\bar{u}}$ is

$$\begin{aligned} J_{\bar{u}} &= -2(\nabla_x A^*(\bar{u}_t h) \cdot \nabla_x \bar{h})_\mu + (|\nabla_x \bar{h}|^2 A^*(\bar{u}_t h))_\mu \\ &= -2(\partial_{x_i} h \bar{u}_t \cdot \nabla_v \partial_{x_i} \bar{h})_\mu + (\nabla_v |\nabla_x \bar{h}|^2 \cdot \bar{u}_t h)_\mu = 0. \end{aligned}$$

The A -term is given by

$$\begin{aligned} J_A &= -2(\nabla_x (s_\rho A^* Ah) \cdot \nabla_x \bar{h})_\mu + (s_\rho |\nabla_x \bar{h}|^2 A^* Ah)_\mu \\ &= -2((s_\rho)_{x_i} Ah \cdot A \bar{h}_{x_i})_\mu - 2(s_\rho Ah_{x_i} \cdot A \bar{h}_{x_i})_\mu + (s_\rho A |\nabla_x \bar{h}|^2 \cdot Ah)_\mu \\ &\leq c \sigma^{-1} \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2(s_\rho \nabla_v (h \bar{h}_{x_i}) \cdot \nabla_v \bar{h}_{x_i})_\mu + (s_\rho \nabla_v |\nabla_x \bar{h}|^2 \cdot \nabla_v h)_\mu \\ &= c \sigma^{-1} \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2(s_\rho h \nabla_v \bar{h}_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu - 2(s_\rho \bar{h}_{x_i} \nabla_v h \cdot \nabla_v \bar{h}_{x_i})_\mu \\ &\quad + (s_\rho \nabla_v |\nabla_x \bar{h}|^2 \cdot \nabla_v h)_\mu \\ &= c \sigma^{-1} \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2\mathcal{D}_{xv} - (s_\rho \nabla_v |\nabla_x \bar{h}|^2 \cdot \nabla_v h)_\mu + (s_\rho \nabla_v |\nabla_x \bar{h}|^2 \cdot \nabla_v h)_\mu \\ &= c \sigma^{-1} \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2\mathcal{D}_{xv}. \end{aligned}$$

Thus,

$$J_A \leq c \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2\sigma \mathcal{D}_{xv}.$$

Finally, the alignment term is given by

$$J_u = 2(\nabla_x (s_\rho A^*([u]_\rho h)) \cdot \nabla_x \bar{h})_\mu - (s_\rho |\nabla_x \bar{h}|^2 A^*([u]_\rho h))_\mu.$$

In the second term, we switch the operator A^* :

$$-(s_\rho |\nabla_x \bar{h}|^2 A^*([u]_\rho h))_\mu = -(s_\rho \nabla_v |\nabla_x \bar{h}|^2 [u]_\rho h)_\mu. \quad (8.10)$$

For the first term, we obtain

$$2(\nabla_x(s_\rho A^*([u]_\rho h)) \cdot \nabla_x \bar{h})_\mu = 2(h(s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu + 2(s_\rho h \bar{h}_{x_i} [u]_\rho \cdot \nabla_v \bar{h}_{x_i})_\mu.$$

We can see that the last term cancels with (8.10). We thus obtain, using assumption (iii),

$$J_u = 2(h(s_\rho [u]_\rho)_{x_i} \cdot \nabla_v \bar{h}_{x_i})_\mu \leq c \|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{xv}}.$$

Putting together the obtained bounds we have

$$\begin{aligned} \frac{d}{dt} \mathcal{I}_{xx}(h) &\leq c\sigma \sqrt{\mathcal{D}_{xv} \mathcal{I}_{vv}} - 2\sigma^2 \mathcal{D}_{xv} + c\sigma \|u\|_{L^2(\kappa_\rho)} \sqrt{\mathcal{D}_{xv}} \\ &\leq c \mathcal{I}_{vv} - \sigma^2 \mathcal{D}_{xv} + c \|u\|_{L^2(\kappa_\rho)}^2. \end{aligned}$$

To conclude the proof, let us combine together all the Fisher functionals in the following format

$$\tilde{\mathcal{I}} = \mathcal{I}_{vv} + \delta\sigma^{1/2} \mathcal{I}_{xv} + \frac{c_0}{c} \mathcal{I}_{xx},$$

where $\delta > 0$ is small, but dependent only on the parameters of the assumptions (i), (ii), and (iii). Since $\sigma < 1$, for δ small enough we have $\tilde{\mathcal{I}} \sim \mathcal{I}$. Then,

$$\frac{d}{dt} \tilde{\mathcal{I}} \leq -c_0 \mathcal{I}_{vv} - c_1 \sigma \mathcal{I}_{xx} + C \|u\|_2^2.$$

Invoking the entropy inequality (8.5), we obtain with a properly chosen constant $C > 0$,

$$\frac{d}{dt} (\tilde{\mathcal{I}} + C \mathcal{H}) \leq -c_0 \mathcal{I}_{vv} - c_1 \sigma \mathcal{I}_{xx} \leq -c_2 \sigma (\mathcal{I}_{vv} + \mathcal{I}_{xx}) \sim -\sigma (\tilde{\mathcal{I}} + C \mathcal{H}).$$

Hence,

$$\tilde{\mathcal{I}} + C \mathcal{H} \leq c_3 \mathcal{I}(f_0) e^{-c_2 \sigma t},$$

and the result follows. ■

8.1. Applications

Let us now explore how Proposition 8.1 implies relaxation for various situations.

First, we have convergence near equilibrium for all models whose spectral gap is under control for densities near uniform.

Proposition 8.5. *Suppose that \mathcal{M} is regular and has a uniformly positive spectral gap for any densities close to uniform*

$$\inf\{\varepsilon_0(\rho) : \|\rho - 1/|\Omega|\|_1 \leq \delta_0\} > 0.$$

Then there exists a constant $c > 0$ depending only on the parameters of the model such that for any initial condition $f_0 \in H_1^k(\Omega)$ satisfying

$$\mathcal{I}(f_0) \leq c\sigma\delta_0,$$

there exists a global classical solution converging to the Maxwellian exponentially fast.

Proof. By Definition 3.13 (iii), (iv) we can further reduce the size of δ_0 if necessary to have not only the uniform spectral gap condition but also the uniform thickness condition satisfied, $\Theta(\rho, \Omega) > c_4$. By the regularity assumption (3.19), the assumption on the spectral gap, and (3.18), such densities fulfill all the conditions (i), (ii), (iii) of Proposition 8.1, with constants $c_0, c_1, c_2, c_3, \varepsilon_0$ depending only on δ_0 . And according to Theorem 7.1 such data give rise to local classical solutions $f \in H_I^k$.

If (8.11) holds, then by the Csiszár–Kullback inequality

$$\left\| \rho_0 - \frac{1}{|\Omega|} \right\|_{L^1} \leq c_5 c \delta_0$$

for some absolute $c_5 > 0$. If $c < 1/2c_5$, then by continuity we have

$$\left\| \rho(t) - \frac{1}{|\Omega|} \right\|_{L^1} < \delta_0$$

at least on some short time interval $[0, T)$. Let T be the maximal time of existence of the local solution which satisfies the above. Note that the solution cannot blowup before it reaches the equality $\|\rho(t) - 1/|\Omega|\|_{L^1} = \delta_0$, due to the continuation criterion (7.4). Hence, if T is finite it is only because we have $\|\rho(T) - 1/|\Omega|\|_{L^1} = \delta_0$ for the first time.

The Proposition 8.1 then applies on $[0, T]$. As a consequence,

$$\left\| \rho(T) - \frac{1}{|\Omega|} \right\|_{L^1} \leq c_6 c \delta_0$$

for all $t \leq T$ and some c_6 depending on the parameters of the model only. Assuming further that $c < 1/2c_6$ we conclude that T cannot be finite. Thus, the solution exists globally and satisfies $\|\rho(t) - 1/|\Omega|\|_{L^1} < \delta_0$ for all time. Proposition 8.1 applies again to conclude the result. ■

Before we state application of this result to particular models, let us address the issue of the time-dependent momentum for non-conservative case. It turns out that for all our core non-conservative models \mathcal{M}_β , including the Motsch–Tadmor model, the average momentum stabilizes.

Lemma 8.6. *Suppose \mathcal{M} satisfies the assumptions of Proposition 8.5 and*

$$[1]_{1/|\Omega|}^* = 1.$$

For any solution f that relaxes at an exponential rate in the sense of relative entropy there exists $u_\infty \in \mathbb{R}^n$ such that $\bar{u}(t) \rightarrow u_\infty$ exponentially fast, and consequently,

$$\|f(t) - \mu_{\sigma, u_\infty}\|_{L^1(\Omega \times \mathbb{R}^n)} \leq c_9 e^{-c_{10} t}.$$

In particular, the conclusion applies to all conservative and all \mathcal{M}_β models.

Proof. By the Csiszár–Kullback inequality, we have

$$\left\| \rho - \frac{1}{|\Omega|} \right\|_1 \lesssim e^{-ct}.$$

Thus, as we argued in the proof of Proposition 8.5, the density eventually enters into a class satisfying all the functional requirements of Proposition 8.1 uniformly in time.

We have

$$\begin{aligned} \bar{u}_t &= \int_{\Omega} \int_{\Omega} \phi_{\rho}(x, y) u(y) \rho(y) \, dy \rho(x) \, dx - \int_{\Omega} s_{\rho}(y) u(y) \rho(y) \, dy \\ &= \int_{\Omega} \int_{\Omega} (\phi_{\rho}(x, y) - \phi_{1/|\Omega|}(x, y)) u(y) \rho(y) \, dy \rho(x) \, dx \\ &\quad + \int_{\Omega} \int_{\Omega} \phi_{1/|\Omega|}(x, y) u(y) \rho(y) \, dy \left[\rho(x) - \frac{1}{|\Omega|} \right] dx \\ &\quad + \int_{\Omega} \left[\frac{1}{|\Omega|} \int_{\Omega} \phi_{1/|\Omega|}(x, y) \, dx - s_{1/|\Omega|}(y) \right] u(y) \rho(y) \, dy \\ &\quad + \int_{\Omega} [s_{1/|\Omega|}(y) - s_{\rho}(y)] u(y) \rho(y) \, dy. \end{aligned}$$

By continuity assumptions all the terms on the right-hand side are bounded by a constant multiple of $\|\rho - 1/|\Omega|\|_1 \sqrt{\mathcal{E}}$. Since the energy remains uniformly bounded all these terms are exponentially decaying. This proves the exponential convergence $\bar{u}(t) \rightarrow u_{\infty}$ for some $u_{\infty} \in \mathbb{R}^n$.

Next, we have

$$\int_{\Omega \times \mathbb{R}^n} f \log \frac{f}{\mu_{\sigma, u_{\infty}}} \, dv \, dx = \int_{\Omega \times \mathbb{R}^n} f \log \frac{f}{\mu_{\sigma, \bar{u}}} \, dv \, dx + \int_{\Omega \times \mathbb{R}^n} f \log \frac{\mu_{\sigma, \bar{u}}}{\mu_{\sigma, u_{\infty}}} \, dv \, dx.$$

The last term is a constant multiple of

$$\int_{\Omega \times \mathbb{R}^n} f (|u_{\infty} - v|^2 - |\bar{u} - v|^2) \, dv \, dx = |u_{\infty}|^2 - |\bar{u}|^2 + 2 \int_{\Omega} (\bar{u} - u_{\infty}) \cdot u \rho \, dx \lesssim e^{-ct}.$$

This finishes the proof. ■

According to our computations of spectral gaps stated in Proposition 4.16 and Proposition 4.18, we can apply the above results to conclude local relaxation for all core models. Let us gather all this in one statement.

Theorem 8.7 (Relaxation near equilibrium). *Suppose that \mathcal{M} is a regular model satisfying*

$$\inf \left\{ \varepsilon_0(\rho) : \left\| \rho - \frac{1}{|\Omega|} \right\|_1 \leq \delta_0 \right\} > 0,$$

and

$$[1]_{1/|\Omega|}^* = 1.$$

Then there exists a constant $c > 0$ depending only on the parameters of the model such that for any initial condition $f_0 \in H_1^k(\Omega)$ satisfying

$$\mathcal{I}(f_0) \leq c\sigma\delta_0, \tag{8.11}$$

there exists a global classical solution f and there exists $u_\infty \in \mathbb{R}^n$ such that

$$\|f(t) - \mu_{\sigma, u_\infty}\|_{L^1(\Omega \times \mathbb{R}^n)} \leq c_9 e^{-c_{10}t}.$$

In particular, the conclusion applies to all core models \mathcal{M}_{CS} , \mathcal{M}_{MT} , \mathcal{M}_β , \mathcal{M}_ϕ , \mathcal{M}_{seg} .

We now gather a set of conditions which guarantees global relaxation.

Theorem 8.8 (Global relaxation). *Suppose \mathcal{M} is a regular conservative model satisfying (3.31) and such that*

$$\inf\{\varepsilon_0(\rho) : \bar{\rho}_r(\Omega) > \delta_0\} > 0$$

for some $r, \delta_0 > 0$. Then any classical global solution to the Fokker–Planck-alignment equation (8.1) relaxes to equilibrium as stated in Proposition 8.1 with $\bar{u} = \bar{u}_0$.

In particular, global relaxation holds for the following models:

- the Cucker–Smale model \mathcal{M}_{CS} with a Bochner-positive kernel $\phi = \psi * \psi$;
- the \mathcal{M}_ϕ -model with $\inf \phi > 0$;
- the segregation model \mathcal{M}_{seg} with $\text{supp } g_l = \Omega$ for all $l = 1, \dots, L$.

Proof. According to Theorem 7.12 under the given conditions on \mathcal{M} any classical solution gains a uniform bound on the density from below $\rho(x, t) \geq \rho_-$, for $(x, t) \in \Omega \times [1, \infty)$. This automatically puts the solution into a class satisfying the assumptions of Proposition 8.1 uniformly. ■

Let us remark that the only requirement that prevents global relaxation for the \mathcal{M}_ϕ and \mathcal{M}_{seg} models with general local kernels is the uniform $L^2 \rightarrow L^\infty$ -boundedness (3.31), which is needed to control \bar{W} . However, this control can be regained if the solution is known to have uniformly bounded macroscopic velocities

$$\sup_{t \geq 0} \|u(t)\|_\infty < \infty.$$

This is precisely the result obtained for \mathcal{M}_ϕ in [83].

9. Hydrodynamic limits

Supplementing the Vlasov equation (4.1) with a strong penalization force

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (s_\rho(v - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon) + \frac{1}{\varepsilon} F(f^\varepsilon), \tag{9.1}$$

one can achieve regimes in which the distribution f asymptotically takes a special form explicitly expressible in terms of the macroscopic quantities u, ρ . The limiting system satisfied by u, ρ is called the Euler-alignment system (EAS) (1.12), in which the pressure law depends on the particular force F used in the limit. In this study we will cover two types of limits – monokinetic and Maxwellian.

The monokinetic limit is achieved by enforcing strong local alignment

$$F = \nabla_v \cdot [(v - u^\varepsilon) f^\varepsilon].$$

The force penalizes deviation from the Dirac concentrated on u^ε , which drives the solution towards monokinetic distribution $f = \rho(x, t) \delta_{u(x,t)}(v)$, where ρ, u solve the pressureless EAS

$$\partial_t \rho + \nabla \cdot (u \rho) = 0, \quad \partial_t u + u \cdot \nabla u = s_\rho([u]_\rho - u). \tag{9.2}$$

Solutions to (9.2) will always be understood in smooth regularity classes such as

$$(u, \rho) \in C_w([0, T]; H^m \times (H^k \cap L^1_+)) \cap \text{Lip}([0, T]; H^{m-1} \times (H^{k-1} \cap L^1_+))$$

for $m \geq k + 1 > n/2 + 2$. Local and global well-posedness theory for such solutions can be established for a variety of models and data, see [82] for a detailed analysis. Because of the maximum principle on u which applies to solutions of (9.2) any initially compact flock $\text{supp } \rho_0 \subset B_{R_0}$ will remain compactly supported on any finite time interval

$$\text{supp } \rho(t) \subset B_{R(t)}, \quad R(t) \leq R_0 + A_0 t. \tag{9.3}$$

The history of this limit goes back to [52, 66] where the alignment term in (4.1) is considered centered around zero velocity. In the settings of the classical Cucker–Smale model the hydrodynamic limit was studied in [35]. In both studies the force $F = \nabla_v \cdot [(v - u^\varepsilon) f^\varepsilon]$ includes the rough macroscopic field u^ε causing issues with uniqueness of characteristics of (9.1) and subsequently the transport of f^ε . These issues have been dealt with in [35] by imposing no vacuum condition $\rho > 0$ and restricting analysis to the periodic domain. A more recent remake of Figalli-Kang’s argument done in [82] avoids all these issues by replacing u^ε with a mollified version if it, u^ε_δ , based on the \mathcal{M}_ϕ -protocol. Such change allows to extend the limit to vacuum and compactly supported flocks on either \mathbb{T}^n or \mathbb{R}^n .

In the context of the general environmental averaging models this result can be broadly extended to include all uniformly regular models. Moreover, in contrast to the previous studies the convergence $f^\varepsilon \rightarrow f$ can be upgrade quantitatively to Wasserstein-2 metric, see Theorem 9.2.

In the Maxwellian regime the force F is given by the Fokker–Planck-alignment operator

$$F = \Delta_v f^\varepsilon + \nabla_v \cdot [(v - u^\varepsilon) f^\varepsilon].$$

The local thermodynamic equilibrium becomes the Maxwellian

$$f = \frac{\rho(x, t)}{(2\pi)^{n/2}} e^{-|v-u(x,t)|^2/2},$$

and so the corresponding macroscopic model is given by the EAS with isothermal pressure

$$\begin{aligned} \partial_t \rho + \nabla \cdot (u\rho) &= 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho &= \rho s_\rho([u]_\rho - u). \end{aligned}$$

In the Cucker–Smale settings, this limit was justified in [55] via the relative entropy method. Again, because of the roughness of u^ε the result had to be cast in the settings of a special class of weak solutions established in [53], see also [54] for the justification of a local alignment limit. The work [83] implemented similar method to prove hydrodynamic limit in the context of the \mathcal{M}_ϕ -model.

Now, we can cast the Maxwellian limit in the framework of general environmental averaging models with the additional implementation of the mollified local alignment field u_δ^ε – the same methodology we will be using in the monokinetic case. This allows to work in the class of classical solutions as stated in Theorem 7.1 and Theorem 7.12. The limiting solution must be non-vacuous and the domain is restricted to the torus $\Omega = \mathbb{T}^n$. Theorem 9.6 shows convergence $f^\varepsilon \rightarrow f$ in the relative entropy sense, which implies stronger convergence in L^1 by the Csiszár–Kullback inequality.

9.1. Monokinetic limit

In this section we discuss the monokinetic limit. The analysis will be carried out on any environment Ω , compact or not under the assumption of uniform regularity of \mathcal{M} .

Let us consider solutions to the following Vlasov model with forced local alignment

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \nabla_v \cdot (s_{\rho^\varepsilon}(v - [u^\varepsilon]_{\rho^\varepsilon})f^\varepsilon) + \frac{1}{\varepsilon} \nabla_v \cdot ((v - u_\delta^\varepsilon)f^\varepsilon), \tag{9.4}$$

where subscript δ designates a special mollification. To define it, let us fix a smooth mollifier $\psi_\delta(x) = \frac{1}{\delta^n} \psi(x/\delta)$, where $\psi > 0$ on Ω and in the case of $\Omega = \mathbb{R}^n$ we assume that ψ satisfies the algebraic decay condition (3.26). Then let u_δ be the average of u based on the $\mathcal{M}_{\psi_\delta}$ -protocol,

$$u_\delta = \left(\frac{(u\rho)\psi_\delta}{\rho\psi_\delta} \right)_{\psi_\delta}.$$

Formally, (9.4) corresponds to the Vlasov equation (4.1) based on the model given by

$$s_\rho^\varepsilon = s_\rho + \frac{1}{\varepsilon}, \quad [u]_{\rho^\varepsilon}^{\varepsilon,\delta} = \frac{\varepsilon s_\rho}{\varepsilon s_\rho + 1} [u]_\rho + \frac{1}{\varepsilon s_\rho + 1} u_\delta.$$

Clearly, since $\mathcal{M}_{\psi_\delta}$ and \mathcal{M} are uniformly regular, then so is the model above. Consequently, the global existence of classical compactly supported solutions to (9.4) is warranted in this case by Theorem 5.4. Moreover, characteristics of (9.4) satisfy the usual maximum principle for velocities Lemma 4.1. Hence, $|X^\varepsilon(t)| \leq R_0 + tA_0$, where R_0 is the initial radius of the support in x and A_0 is the maximal initial velocity. Thus, on any time interval $[0, T]$, the family f^ε will be supported on a bounded region uniformly in ε if initial f_0 is compactly supported.

Before we focus on the main convergence result let us go back to the defined mollification u_δ and note another remarkable approximation property – if u is a smooth field, then u_δ approximates u with a quantitative bound *independent* of any regularity of ρ . This allows to implement it to situations where the only information known on ρ is its mass. The following is a generalization of such approximation property presented in [82, Lemma 5.1].

Lemma 9.1. *For any $u \in \text{Lip}$ and for any $1 \leq p < \infty$, one has*

$$\|u_\delta - u\|_{L^p(\rho)} \leq C\delta \|u\|_{\text{Lip}}, \tag{9.5}$$

where $C > 0$ is a constant depending only on the kernel ψ and p . The estimate also holds for all $1 \leq p \leq \infty$ with C independent of p if ψ is compactly supported.

Proof. Let us fix a test-function $f \in L^q(\rho)$, where $q^{-1} + p^{-1} = 1$. Then, let us split

$$\int_{\Omega} f(u_\delta - u)\rho \, dx = \int_{\Omega} f(u_\delta - u_{\psi_\delta})\rho \, dx + \int_{\Omega} f(u_{\psi_\delta} - u)\rho \, dx := I_1 + I_2.$$

For I_2 , we simply use the standard approximation property of mollification

$$I_2 \leq \delta \|u\|_{\text{Lip}} \|f\|_{L^1(\rho)} \leq \delta \|u\|_{\text{Lip}} \|f\|_{L^q(\rho)}.$$

For I_1 , using the Minkowskii and Hölder inequality, we have

$$\begin{aligned} I_1 &= \int_{\Omega} (f\rho)_{\psi_\delta} \frac{(u\rho)_{\psi_\delta}}{\rho_{\psi_\delta}} \, dx + \int_{\Omega} (f\rho)_{\psi_\delta} \frac{u\rho_{\psi_\delta}}{\rho_{\psi_\delta}} \, dx \\ &= \int_{\Omega} \frac{(f\rho)_{\psi_\delta} ((u\rho)_{\psi_\delta} - u\rho_{\psi_\delta})}{\rho_{\psi_\delta}} \, dx = \int_{\Omega} \frac{(f\rho)_{\psi_\delta}}{\rho_{\psi_\delta}^{1/p}} \frac{(u\rho)_{\psi_\delta} - u\rho_{\psi_\delta}}{\rho_{\psi_\delta}^{1/q}} \, dx \\ &\leq \left(\int_{\Omega} \frac{|(f\rho)_{\psi_\delta}|^q}{\rho_{\psi_\delta}^{q/p}} \, dx \right)^{1/q} \left(\int_{\Omega} \frac{|(u\rho)_{\psi_\delta} - u\rho_{\psi_\delta}|^p}{\rho_{\psi_\delta}^{p/q}} \, dx \right)^{1/p} \\ &\leq \left(\int_{\Omega} (|f|^q \rho)_{\psi_\delta} \, dx \right)^{1/q} \left(\int_{\Omega \times \Omega} |u(y) - u(x)|^p \rho(y) \psi_\delta(x - y) \, dy \, dx \right)^{1/p} \\ &\leq \|f\|_{L^q(\rho)} \|u\|_{\text{Lip}} \left(\int_{\Omega \times \Omega} |x - y|^p \rho(y) \psi_\delta(x - y) \, dy \, dx \right)^{1/p} \\ &= \delta \|f\|_{L^q(\rho)} \|u\|_{\text{Lip}} C_{p,\psi}^{1/p}, \end{aligned}$$

where $C_{p,\psi} = \int_{\Omega} |x|^p \psi(x) \, dx$. This implies (9.5) for all $p < \infty$. If however ψ is compactly supported, then $C_{p,\psi} \leq (\text{diam supp } \psi)^p$, and so the estimate holds also in the limit as $p \rightarrow \infty$. ■

The main convergence result of this section will be quantified in terms of W_2 -metric:

$$W_2^2(f_1, f_2) = \inf_{\gamma \in \Pi(f_1, f_2)} \int_{\Omega^2 \times \mathbb{R}^{2n}} |\omega_1 - \omega_2|^2 \, d\gamma(\omega_1, \omega_2),$$

where $\Pi(f_1, f_2)$ is the set of probability measures with marginals f_1 and f_2 , respectively.

Theorem 9.2. Suppose \mathcal{M} is a uniformly regular model. Let (ρ, u) be a classical solution to (9.2) on the time interval $[0, T)$ with compact support (9.3), and let $f = \rho(x, t)\delta_{u(x, t)}(v)$. Suppose $f_0^\varepsilon \in C_0^k(\Omega \times \mathbb{R}^n)$ is a family of initial conditions satisfying

- (i) $\text{supp } f_0^\varepsilon \subset \{|w| < R_0\}$;
- (ii) $W_2(f_0^\varepsilon, f_0) \leq \varepsilon$.

Then there exists a constant C such that for all $t < T$, one has

$$W_2(f_t^\varepsilon, f_t) \leq C \sqrt{\varepsilon + \frac{\delta}{\varepsilon}}. \tag{9.6}$$

Remark 9.3. Let us note that the scaling regime $\delta = \varepsilon^2$ appears to be the most optimal: if $\delta \ll \varepsilon^2$, the model becomes over-resolved without improvement on convergence rate of solutions, if $\delta \gg \varepsilon^2$, the model is under-resolved and the convergence rate slows down. We obtain in this case the optimal rate of $\sqrt{\varepsilon}$:

$$W_2(f^\varepsilon, f) \leq C \sqrt{\varepsilon}.$$

Remark 9.4. Note that $W_2(f^\varepsilon, f) \rightarrow 0$ also implies convergence of densities, simply because ρ 's are marginals of f 's: $W_2(\rho^\varepsilon, \rho) \leq W_2(f^\varepsilon, f)$. Similarly, since all distributions are confined to a bounded set, we also have $W_1(u^\varepsilon \rho^\varepsilon, u\rho) \leq C W_1(f^\varepsilon, f) \leq C W_2(f^\varepsilon, f)$. So, this also implies the convergence of momenta.

Remark 9.5. The theorem applies to a range of core models listed in Table 4. However, we also note that the uniform regularity is only needed to facilitate global existence of solutions. The actual assumptions that are needed to run the argument for a given family of solutions are (3.28), (3.23) and (3.24), where $\rho' = \rho$ is the limiting density, and $\|\partial_y \phi_\rho\|_\infty < C$. Thus, if the limiting density is known to be thick and the model is simply regular and satisfies (3.28), (3.23) and (3.24), then the theorem applies just as well to putative solutions and extends to a much wider class of models listed in the last row of Table 4.

Proof. Let us first note that since all densities are compactly supported the model satisfies all the estimates of (3.21) and (3.22) for ρ and ρ^ε uniformly on $[0, T]$.

Denoting

$$\mathcal{E}_\varepsilon = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 f^\varepsilon(x, v) \, dx \, dv,$$

we have the following energy balance relation for solutions of (9.4):

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\varepsilon &= - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} |v|^2 f^\varepsilon \, dv \, dx + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa_{\rho^\varepsilon}} \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega} \frac{|(\rho^\varepsilon u^\varepsilon)_{\psi_\delta}|^2}{\rho_{\psi_\delta}^\varepsilon} \, dx - \frac{2}{\varepsilon} \mathcal{E}_\varepsilon. \end{aligned}$$

Noting that

$$\int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} |v|^2 f^\varepsilon \, dv \, dx \geq (u^\varepsilon, u^\varepsilon)_{\kappa_{\rho^\varepsilon}},$$

we obtain

$$\frac{d}{dt} \mathcal{E}_\varepsilon \leq (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon)_{\kappa_{\rho^\varepsilon}} + \frac{1}{\varepsilon} \int_{\Omega} \frac{|(\rho^\varepsilon u^\varepsilon)_{\psi_\delta}|^2}{\rho_\psi^\varepsilon} \, dx - \frac{2}{\varepsilon} \mathcal{E}_\varepsilon. \tag{9.7}$$

Obviously the last two terms store a lot of dissipative information. The crucial observation is that they control internal energies of f^ε both the native one relative to the local field u^ε and relative to the filtered field u_δ^ε . To see this, let us note the following two identities:

$$\begin{aligned} e(f^\varepsilon | u^\varepsilon) &:= \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v - u^\varepsilon|^2 f^\varepsilon \, dv \, dx = \mathcal{E}_\varepsilon - \frac{1}{2} \int_{\mathbb{R}^n} \rho^\varepsilon |u^\varepsilon|^2 \, dx, \\ e(f^\varepsilon | u_\delta^\varepsilon) &:= \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v - u_\delta^\varepsilon|^2 f^\varepsilon \, dv \, dx = \mathcal{E}_\varepsilon - \int_{\Omega} \frac{|(\rho^\varepsilon u^\varepsilon)_{\psi_\delta}|^2}{\rho_\psi^\varepsilon} \, dx + \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u_\delta^\varepsilon|^2 \, dx. \end{aligned}$$

Summing up, we obtain

$$e(f^\varepsilon | u^\varepsilon) + e(f^\varepsilon | u_\delta^\varepsilon) = 2\mathcal{E}_\varepsilon - \int_{\Omega} \frac{|(\rho^\varepsilon u^\varepsilon)_{\psi_\delta}|^2}{\rho_\psi^\varepsilon} \, dx + \frac{1}{2} \int_{\mathbb{R}^n} \rho^\varepsilon |u_\delta^\varepsilon|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} \rho^\varepsilon |u^\varepsilon|^2 \, dx,$$

and since the \mathcal{M}_ϕ -model is contractive, the last two terms add up to a non-positive value. Thus,

$$2\mathcal{E}_\varepsilon - \int_{\Omega} \frac{|(\rho^\varepsilon u^\varepsilon)_{\psi_\delta}|^2}{\rho_\psi^\varepsilon} \, dx \geq e(f^\varepsilon | u^\varepsilon) + e(f^\varepsilon | u_\delta^\varepsilon).$$

Consequently, plugging this back into (9.7) we obtain

$$\frac{d}{dt} \mathcal{E}_\varepsilon \leq (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon)_{\kappa_{\rho^\varepsilon}} - \frac{1}{\varepsilon} [e(f^\varepsilon | u^\varepsilon) + e(f^\varepsilon | u_\delta^\varepsilon)]. \tag{9.8}$$

The energy inequality (9.8) already shows that the solution concentrates to a mono-kinetic form near its own macroscopic field. However, the quantity that controls how far that concentration is from u , is the modulated kinetic energy:

$$e(f^\varepsilon | u) = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v - u|^2 f^\varepsilon \, dv \, dx.$$

This quantity plays a key role in the argument. It should be noted that it controls the corresponding macroscopic relative entropy

$$\begin{aligned} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx &= \int_{\Omega} (|u^\varepsilon|^2 \rho^\varepsilon - 2u^\varepsilon \cdot u \rho^\varepsilon + |u|^2 \rho^\varepsilon) \, dx \\ &\leq \int_{\Omega \times \mathbb{R}^n} (|v|^2 f^\varepsilon - 2u \cdot v f^\varepsilon + |u|^2 f^\varepsilon) \, dx \, dv = e(f^\varepsilon | u). \end{aligned} \tag{9.9}$$

According to (ii) we can fix an initial $\gamma_0 \in \Pi(f_0^\varepsilon, f_0)$ such that

$$\int_{\Omega^2 \times \mathbb{R}^{2n}} |\omega_1 - \omega_2|^2 d\gamma_0(\omega_1, \omega_2) \leq 2\varepsilon^2.$$

Let us now propagate γ_0 along the direct product of characteristic maps of (9.4) and (4.1), i.e. let γ_t be the measure-valued solution to the transport equation

$$\begin{aligned} & \partial_t \gamma + v_1 \cdot \nabla_{x_1} \gamma + v_2 \cdot \nabla_{x_2} \gamma \\ & + \nabla_{v_1} \left[\gamma \left(s_{\rho^\varepsilon}(v_1 - [u^\varepsilon]_{\rho^\varepsilon}) + \frac{1}{\varepsilon}(v_1 - u_\delta^\varepsilon) \right) \right] + \nabla_{v_2} [\gamma s_\rho(v_2 - [u]_\rho)] = 0. \end{aligned}$$

Integrating upon pairs (x_1, v_1) and (x_2, v_2) we can see that the marginals of γ satisfy the same transport equations as f and f^ε , respectively. Consequently, by uniqueness, $\gamma_t \in \Pi(f_t^\varepsilon, f_t)$ for all time. This means that the cost of γ_t dominates the W_2 -distance at any time,

$$W := \int_{\Omega^2 \times \mathbb{R}^{2n}} |\omega_1 - \omega_2|^2 d\gamma_t(\omega_1, \omega_2) \geq W_2^2(f^\varepsilon, f).$$

Let us split W into potential and kinetic components

$$W = \int_{\Omega^2 \times \mathbb{R}^{2n}} |v_1 - v_2|^2 d\gamma + \int_{\Omega^2 \times \mathbb{R}^{2n}} |x_1 - x_2|^2 d\gamma := W_v + W_x.$$

Evolution of the potential component is easily estimated using the transport of γ

$$\begin{aligned} \frac{d}{dt} W_x &= \frac{d}{dt} \int_{\Omega^2 \times \mathbb{R}^{2n}} |X^\varepsilon(\omega_1, t) - X(\omega_2, t)|^2 d\gamma_0 \\ &= 2 \int_{\Omega^2 \times \mathbb{R}^{2n}} (X^\varepsilon(\omega_1, t) - X(\omega_2, t)) \cdot (V^\varepsilon(\omega_1, t) - V(\omega_2, t)) d\gamma_0 \leq W_x + W_v. \end{aligned}$$

Instead of writing the evolution equation for W_v we subordinate it to the internal energy, and trace its evolution. Let us make the following estimate:

$$\begin{aligned} W_v &\leq \int_{\Omega^2 \times \mathbb{R}^{2n}} |v_1 - u(x_1)|^2 d\gamma + \int_{\Omega^2 \times \mathbb{R}^{2n}} |u(x_1) - u(x_2)|^2 d\gamma \\ &\quad + \int_{\Omega^2 \times \mathbb{R}^{2n}} |u(x_2) - v_2|^2 d\gamma \\ &\leq \int_{\Omega \times \mathbb{R}^n} |v - u(x)|^2 f^\varepsilon(x, v) dv dx + C \int_{\Omega^2 \times \mathbb{R}^{2n}} |x_1 - x_2|^2 d\gamma + 0, \end{aligned}$$

where the last term is canceled thanks to the monokinetic nature of f ,

$$= e(f^\varepsilon | u) + C W_x.$$

So far, we have obtained

$$\frac{d}{dt} W_x \leq e(f^\varepsilon | u) + c_1 W_x, \quad W_v \leq e(f^\varepsilon | u) + c_2 W_x. \tag{9.10}$$

To complete this system we now investigate evolution of the internal energy itself.

Before we write the equation for the modulated energy $e(f^\varepsilon|u)$, let us recall that we are dealing with smooth solutions to both so all the computations are legitimate. From (9.4) we can read off the macroscopic system for the ε -density and momentum

$$\begin{cases} \rho_t^\varepsilon + \nabla \cdot (\rho^\varepsilon u^\varepsilon) = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + \nabla_x \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \mathcal{R}_\varepsilon) = \kappa_{\rho^\varepsilon}([u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon) + \frac{1}{\varepsilon} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon), \end{cases}$$

where the Reynolds stress is given by

$$\mathcal{R}_\varepsilon = \int_{\mathbb{R}^n} (v - u^\varepsilon) \otimes (v - u^\varepsilon) f^\varepsilon(x, v, t) dv.$$

Let us expand $e(f^\varepsilon|u)$ into three parts:

$$e(f^\varepsilon|u) = \mathcal{E}_\varepsilon - \int_{\mathbb{R}^n} \rho^\varepsilon u^\varepsilon \cdot u \, dx + \frac{1}{2} \int_{\mathbb{R}^n} \rho^\varepsilon |u|^2 \, dx.$$

From the energy inequality (9.8) we will only retain the alignment component (to be used later) and the native internal energy

$$\frac{d}{dt} \mathcal{E}_\varepsilon \leq (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon)_{\kappa_{\rho^\varepsilon}} - \frac{1}{\varepsilon} e(f^\varepsilon|u^\varepsilon). \tag{9.11}$$

Let us work out the equation for the macroscopic part:

$$\frac{d}{dt} \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot u \, dx = \int_{\Omega} \partial_t (\rho^\varepsilon u^\varepsilon) \cdot u \, dx + \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot \partial_t u \, dx \tag{9.12}$$

$$\begin{aligned} &= \int_{\mathbb{R}^n} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon + \mathcal{R}_\varepsilon) : \nabla u \, dx - \int_{\Omega} \rho^\varepsilon u^\varepsilon \otimes u : \nabla u \, dx \\ &\quad + ([u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon, u)_{\kappa_{\rho^\varepsilon}} + \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot ([u]_\rho - u) s_\rho \, dx \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon) \cdot u \, dx \end{aligned}$$

$$\frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u|^2 \, dx = \int_{\Omega} \rho^\varepsilon u \cdot \partial_t u \, dx + \frac{1}{2} \int_{\Omega} \partial_t \rho^\varepsilon |u|^2 \, dx \tag{9.13}$$

$$\begin{aligned} &= - \int_{\Omega} \rho^\varepsilon u \otimes u : \nabla u \, dx + \int_{\Omega} \rho^\varepsilon u \otimes u^\varepsilon : \nabla u \, dx \\ &\quad + \int_{\Omega} \rho^\varepsilon u \cdot ([u]_\rho - u) s_\rho \, dx. \end{aligned}$$

Putting the two equations together and collecting all the inertia terms and using (9.9), we obtain

$$- \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \otimes (u^\varepsilon - u) : \nabla u \, dx \leq \|\nabla u\|_\infty \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx \lesssim e(f^\varepsilon|u).$$

The Reynolds stress is estimated similarly

$$\int_{\Omega} \mathcal{R}_\varepsilon : \nabla u \, dx \leq \|\nabla u\|_\infty \int_{\Omega \times \mathbb{R}^n} |v - u^\varepsilon(x, t)|^2 f^\varepsilon(x, v, t) \, dx \, dv \lesssim e(f^\varepsilon|u^\varepsilon).$$

As to the local alignment term, we use the symmetry and approximation property of the \mathcal{M}_ϕ -averaging used to define u_δ^ε , which is crucially independent of regularity of ρ^ε . Namely, by Lemma 9.1 with $p = 2$, we have

$$\begin{aligned} \int_{\Omega} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon) \cdot u \, dx &= \int_{\Omega} \rho^\varepsilon u_\delta^\varepsilon \cdot u \, dx - \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot u \, dx \\ &= \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot u_\delta \, dx - \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot u \, dx \\ &= \int_{\Omega} \rho^\varepsilon u^\varepsilon \cdot (u_\delta - u) \, dx \leq C \|u^\varepsilon\|_{L^2(\rho^\varepsilon)} \delta \|\nabla u\|_\infty. \end{aligned}$$

Thus, the local alignment term can be estimated by

$$A_{\text{loc}} = \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon) \cdot u \, dx \lesssim \|u^\varepsilon\|_{L^2(\rho^\varepsilon)} \frac{\delta}{\varepsilon}.$$

Note that the energy $\|u^\varepsilon\|_{L^2(\rho^\varepsilon)}$ remains uniformly bounded in ε , so

$$A_{\text{loc}} \lesssim \frac{\delta}{\varepsilon}. \tag{9.14}$$

Let us collect the obtained estimates (9.11), (9.12), (9.13), and simplify the native alignment components

$$\begin{aligned} \frac{d}{dt} e(f^\varepsilon|u) &\lesssim e(f^\varepsilon|u) + \frac{\delta}{\varepsilon} + (u^\varepsilon - u, [u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon)_{\kappa_{\rho^\varepsilon}} \\ &\quad + \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot ([u]_\rho - u) s_\rho \, dx. \end{aligned}$$

It remains to estimate the alignment terms. Let us rearrange them as follows:

$$\begin{aligned} \delta A &= (u^\varepsilon - u, [u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon)_{\kappa_{\rho^\varepsilon}} + \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot ([u]_\rho - u) s_\rho \, dx \\ &= \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot (s_{\rho^\varepsilon} [u^\varepsilon]_{\rho^\varepsilon} - s_\rho [u]_\rho + s_\rho u - s_{\rho^\varepsilon} u^\varepsilon) \, dx \\ &= (u^\varepsilon - u, [u^\varepsilon - u]_{\rho^\varepsilon})_{\kappa_{\rho^\varepsilon}} + \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot (s_{\rho^\varepsilon} [u]_{\rho^\varepsilon} - s_\rho [u]_\rho) \, dx \\ &\quad - \int_{\Omega} \rho^\varepsilon |u - u^\varepsilon|^2 s_{\rho^\varepsilon} \, dx + \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot u (s_{\rho^\varepsilon} - s_\rho) \, dx := \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Note that I and III would add up to a non-positive constant had we assumed that our model was contractive. Instead, we simply drop III and use the uniform boundedness (3.28), which is implied by (3.21) to estimate I by the macroscopic relative entropy

$$\text{I} \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx.$$

Since $u \in W^{1,\infty}$, using (3.22) the second term is bounded by

$$\begin{aligned} & \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot (s_{\rho^\varepsilon}[u]_{\rho^\varepsilon} - s_\rho[u]_\rho) \, dx \\ & \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx \\ & \quad + \int_{\Omega} \left| \int_{\Omega} (\phi_{\rho^\varepsilon}(x, y) \rho^\varepsilon(y) - \phi_\rho(x, y) \rho(y)) u(y) \, dy \right|^2 \rho^\varepsilon(x) \, dx \\ & \leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + \int_{\Omega} \left| \int_{\Omega} (\phi_{\rho^\varepsilon}(x, y) - \phi_\rho(x, y)) \rho^\varepsilon(y) u(y) \, dy \right|^2 \rho^\varepsilon(x) \, dx \\ & \quad + \int_{\Omega} \left| \int_{\Omega} \phi_\rho(x, y) (\rho^\varepsilon(y) - \rho(y)) u(y) \, dy \right|^2 \rho^\varepsilon(x) \, dx \\ & \leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + \|u\|_\infty^2 \int_{\Omega} \int_{\Omega} |\phi_{\rho^\varepsilon}(x, y) - \phi_\rho(x, y)|^2 \rho^\varepsilon(y) \rho^\varepsilon(x) \, dy \, dx \\ & \quad + \|\partial_y(\phi_\rho u)\|_\infty^2 W_1^2(\rho^\varepsilon, \rho), \end{aligned}$$

which by (3.21) and (3.22) (or in fact by a weaker assumption (3.23) and (3.24)) is bounded further by

$$\lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + W_1^2(\rho^\varepsilon, \rho).$$

Finally, by (3.22) the last term is bounded by the same quantity

$$\begin{aligned} \int_{\Omega} \rho^\varepsilon (u - u^\varepsilon) \cdot u (s_{\rho^\varepsilon} - s_\rho) \, dx & \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + \int_{\Omega} \rho^\varepsilon |s_{\rho^\varepsilon} - s_\rho|^2 \, dx \\ & \leq \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + W_1^2(\rho^\varepsilon, \rho). \end{aligned}$$

In summary, the alignment term is bounded by

$$\delta A \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + W_1^2(\rho^\varepsilon, \rho) \leq e(f^\varepsilon|u) + W_2^2(\rho^\varepsilon, \rho). \tag{9.15}$$

Collecting all the estimates together, we obtain

$$\frac{d}{dt} e(f^\varepsilon|u) \lesssim e(f^\varepsilon|u) + \frac{\delta}{\varepsilon} + W_2^2(\rho^\varepsilon, \rho).$$

Note that since the (x_1, x_2) -marginal of γ belongs to $\Pi(\rho^\varepsilon, \rho)$, we further find that

$$W_2^2(\rho^\varepsilon, \rho) \leq W_x.$$

So, we have obtained the system

$$\frac{d}{dt} W_x \leq e(f^\varepsilon|u) + c_1 W_x, \quad \frac{d}{dt} e(f^\varepsilon|u) \leq c_2 \left(e(f^\varepsilon|u) + W_x + \frac{\delta}{\varepsilon} \right).$$

Note that the initial value of $e(f^\varepsilon|u) + W_x$ is bounded by a constant multiple of ε in view of the choice of γ_0 for W_x (even ε^2 in this case), and

$$\begin{aligned} e(f_0^\varepsilon|u_0) &= \int_{\Omega \times \mathbb{R}^n} |v - u_0|^2 \, d f_0^\varepsilon \\ &= \int_{\Omega \times \mathbb{R}^n} |v - u_0|^2 [d f_0^\varepsilon - d f_0] \\ &\leq C W_1(f_0^\varepsilon, f_0) \leq C W_2(f_0^\varepsilon, f_0) \leq \varepsilon. \end{aligned}$$

Grönwall’s lemma implies $e(f^\varepsilon|u) + W_x \lesssim \varepsilon + \delta/\varepsilon$, and thanks to (9.10),

$$W_v \leq \varepsilon + \frac{\delta}{\varepsilon}.$$

We have established (9.6). ■

9.2. Maxwellian limit

In this section we provide a derivation of the Euler-alignment system with isothermal pressure for material models on the torus $\Omega = \mathbb{T}^n$,

$$\begin{aligned} \rho_t + \nabla \cdot (u\rho) &= 0 \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \rho &= \rho s_\rho([u]_\rho - u). \end{aligned} \tag{9.16}$$

Well-posedness of this system has been established for non-vacuous solutions for various models, see [19, 23].

As outlined in the beginning of this section our strategy will be to consider the equation with strong Fokker–Planck penalization force

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} [\Delta_v f^\varepsilon + \nabla_v \cdot ((v - u_\delta^\varepsilon) f^\varepsilon)] + \nabla_v \cdot (s_{\rho^\varepsilon}(v - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon), \tag{9.17}$$

where u^ε is the macroscopic velocity field associated with f^ε , and u_δ^ε is the same mollification as defined in the previous monokinetic study.

Let us briefly discuss regularity of (9.17). In what follows we will study solutions of (9.17) that exist on a common time interval $[0, T]$ independent of ε . Unfortunately the local existence result alone stated in Theorem 7.1 will not provide such solutions, because the energy bounds (or entropy for that matter) will deteriorate with ε . So, the only way to ensure common existence is to guarantee global well-posedness of (9.17). According to Theorem 7.12 the equation is globally well-posed for thick data if both models – the native \mathcal{M} and the mollification u_δ based on \mathcal{M}_ψ – are regular and satisfy (3.31). Assuming that $\text{supp } \psi = \Omega$ the model \mathcal{M}_ψ will fulfill these conditions, and as to the defining model \mathcal{M} , we will make it as an assumption. The focus will now be turned to establishing convergence of the hydrodynamic limit for a given family of solutions.

Let us write out the corresponding macroscopic system

$$\begin{aligned} \rho_t^\varepsilon + \nabla \cdot (u^\varepsilon \rho^\varepsilon) &= 0, \\ (\rho^\varepsilon u^\varepsilon)_t + \nabla \cdot (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla \rho^\varepsilon + \nabla_x \cdot \mathcal{R}_\varepsilon &= \rho^\varepsilon s_{\rho^\varepsilon}([u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon) + \frac{1}{\varepsilon} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon), \\ \mathcal{R}_\varepsilon &= \int_{\mathbb{R}^n} ((v - u^\varepsilon) \otimes (v - u^\varepsilon) - \mathbb{I}) f^\varepsilon \, dv. \end{aligned}$$

Here, \mathbb{I} is the identity matrix.

We measure the distance between pairs $(u^\varepsilon, \rho^\varepsilon)$ and (u, ρ) by using the relative entropy between the corresponding local Maxwellians:

$$\mu = \frac{\rho(x, t)}{(2\pi)^{n/2}} e^{-|v-u(x,t)|^2/2}, \quad \mu^\varepsilon = \frac{\rho^\varepsilon(x, t)}{(2\pi)^{n/2}} e^{-|v-u^\varepsilon(x,t)|^2/2}.$$

In fact, such entropy is encoded into the total relative entropy between f^ε and μ :

$$\mathcal{H}(f^\varepsilon | \mu) = \int_{\Omega \times \mathbb{R}^n} f^\varepsilon \log \frac{f^\varepsilon}{\mu} \, dv \, dx.$$

Indeed, the following identity holds,

$$\mathcal{H}(f^\varepsilon | \mu) = \mathcal{H}(f^\varepsilon | \mu^\varepsilon) + \mathcal{H}(\mu^\varepsilon | \mu), \tag{9.18}$$

$$\mathcal{H}(\mu^\varepsilon | \mu) = \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + \int_{\Omega} \rho^\varepsilon \log \left(\frac{\rho^\varepsilon}{\rho} \right) \, dx. \tag{9.19}$$

So, if $\mathcal{H}(f^\varepsilon | \mu) \rightarrow 0$, then also $\mathcal{H}(\mu^\varepsilon | \mu) \rightarrow 0$. Recall that by the classical Csiszár–Kullback inequality (see, for example, [82]), the relative entropy controls L^1 -distance between the probability densities,

$$\mathcal{H}(f | g) \geq c \|f - g\|_{L^1}^2.$$

So, vanishing of the relative entropy $\mathcal{H}(\mu^\varepsilon | \mu) \rightarrow 0$ implies strong limits

$$\rho^\varepsilon \rightarrow \rho, \quad \rho^\varepsilon u^\varepsilon \rightarrow \rho u, \quad \rho^\varepsilon |u^\varepsilon|^2 \rightarrow \rho |u|^2$$

in $L^1(\Omega)$.

Theorem 9.6. *Suppose \mathcal{M} is a regular model on \mathbb{T}^n satisfying (3.23), (3.24) and (3.31). Let (u, ρ) be a given smooth non-vacuous solution to (9.16) on a time interval $[0, T]$. Suppose that initial distributions $f_0^\varepsilon \in H_1^k$ converge to μ_0 in the sense of entropies as $\varepsilon \rightarrow 0$:*

$$\mathcal{H}(f_0^\varepsilon | \mu_0) \rightarrow 0.$$

Then for all ε small enough there exists a unique global solution $f^\varepsilon \in H_1^k$, and as long as $\delta = o(\varepsilon)$, we have

$$\sup_{t \in [0, T]} \mathcal{H}(f^\varepsilon | \mu) \rightarrow 0.$$

Remark 9.7. Going back to the discussion of Section 3.7 we can see that the theorem applies to many core models on our list. Specifically, we have it for \mathcal{M}_{CS} and \mathcal{M}_{CS}^{topo} unconditionally, for \mathcal{M}_β we have it for all local kernels if $\beta \geq 1/2$ and for all-to-all kernels $\phi > 0$ for any β , the \mathcal{M}_ϕ requires $\phi > 0$ as well, and \mathcal{M}_{seg} requires $\text{supp } g_I = \Omega$.

Remark 9.8. We also note that in the course of the proof, just like in the monokinetic case, the regularity and (3.31) conditions are only needed to facilitate global existence of solutions, while the bounds (3.28), (3.23) and (3.24) are used in the actual estimates. So, for putative classical solutions to (9.17) the result extends to a wider range of models listed on the last row of Table 4.

Proof. First let us notice that by the Csiszár–Kullback inequality,

$$\mathcal{H}(f_0^\varepsilon | \mu_0) \geq \|\rho_0^\varepsilon - \rho_0\|_1^2.$$

Since, $\rho_0 > 0$ on Ω , it implies that $\Theta(\rho_0, \Omega) > 0$ by Definition 3.13 (iii), and by (iv) we have

$$|\Theta(\rho_0^\varepsilon, \Omega) - \Theta(\rho_0, \Omega)| \leq c \|\rho_0^\varepsilon - \rho_0\|_1 \rightarrow 0,$$

so starting from some ε_0 we have $\Theta(\rho_0^\varepsilon, \Omega) > \delta > 0$, for $\varepsilon < \varepsilon_0$. Such initial conditions give rise to global solutions by Theorem 7.12.

Let us break down the relative entropy into kinetic and macroscopic parts:

$$\begin{aligned} \mathcal{H}(f^\varepsilon | \mu) &= \mathcal{H}_\varepsilon + \mathcal{G}_\varepsilon, \\ \mathcal{H}_\varepsilon &= \int_{\Omega \times \mathbb{R}^n} \left(f^\varepsilon \log f^\varepsilon + \frac{1}{2} |v|^2 f^\varepsilon \right) dv dx + \frac{n}{2} \log(2\pi), \\ \mathcal{G}_\varepsilon &= \int_{\Omega} \left(\frac{1}{2} \rho^\varepsilon |u|^2 - \rho^\varepsilon u^\varepsilon \cdot u - \rho^\varepsilon \log \rho \right) dx. \end{aligned}$$

Let us state the energy bounds for each component. In the sequel we denote $\kappa^\varepsilon = \kappa_{\rho^\varepsilon}$ for short.

Lemma 9.9. *There are constants c_1, c_2, c_3 that depend only on the model such that we have the following entropy law:*

$$\mathcal{H}_\varepsilon, \mathcal{G}_\varepsilon \in L^\infty([0, T]) \text{ uniformly in } \varepsilon, \tag{9.20}$$

$$\frac{d}{dt} \mathcal{H}_\varepsilon \leq -\frac{1}{\varepsilon} \mathcal{I}_\varepsilon + c_2 \varepsilon e(f^\varepsilon | u^\varepsilon) - \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon}, \tag{9.21}$$

where

$$\mathcal{I}_\varepsilon = \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla_v f^\varepsilon + (1 + \varepsilon S_{\rho^\varepsilon}/2)(v - u^\varepsilon) f^\varepsilon|^2}{f^\varepsilon} dv dx.$$

Proof. Differentiating,

$$\begin{aligned} \frac{d}{dt} \mathcal{H}_\varepsilon &= -\frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \left[\frac{|\nabla_v f^\varepsilon|^2}{f^\varepsilon} + 2\nabla_v f^\varepsilon \cdot (v - u_\delta^\varepsilon) + |v - u_\delta^\varepsilon|^2 f^\varepsilon \right] dv dx \\ &\quad - \frac{1}{\varepsilon} [(u_\delta^\varepsilon, u^\varepsilon)_{\rho^\varepsilon} - (u_\delta^\varepsilon, u_\delta^\varepsilon)_{\rho^\varepsilon}] \\ &\quad - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} [\nabla_v f^\varepsilon \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) + v \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon] dv dx. \end{aligned} \tag{9.22}$$

To prove (9.20), we simply dismiss the first information term, and recall that the δ -mollification constitutes the \mathcal{M}_ϕ -averaging which is ball-positive. So, the second term, according to (3.10) is also non-negative and we dismiss it too. We estimate the third term as follows:

$$\begin{aligned} & - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} [\nabla_v f^\varepsilon \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) + v \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon] dv dx \\ &= n \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} f^\varepsilon dv dx - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} |v|^2 f^\varepsilon dv dx + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon} \\ &\leq C - \int_{\Omega} s_{\rho^\varepsilon} \rho^\varepsilon |u^\varepsilon|^2 dx + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon} \\ &= c_1 - \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon}. \end{aligned}$$

Now, according to (3.28) the averaging operators are uniformly bounded on $L^2(\kappa_{\rho^\varepsilon})$. So, we obtain

$$\frac{d}{dt} \mathcal{H}_\varepsilon \leq c_1 + c_2 \mathcal{E}_\varepsilon, \quad \mathcal{E}_\varepsilon = \frac{1}{2} \int_{\Omega \times \mathbb{R}^n} |v|^2 f^\varepsilon dv dx,$$

and according to (7.32),

$$\frac{d}{dt} \mathcal{H}_\varepsilon \leq c_3 + c_4 \mathcal{H}_\varepsilon.$$

This proves (9.20).

To show (9.21), we replace all the macroscopic velocities in (9.22) with the native one u^ε . Indeed, in the information term we have

$$\begin{aligned} & -\frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \left[\frac{|\nabla_v f^\varepsilon|^2}{f^\varepsilon} + 2\nabla_v f^\varepsilon \cdot (v - u_\delta^\varepsilon) + |v - u_\delta^\varepsilon|^2 f^\varepsilon \right] dv dx \\ &= -\frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \left[\frac{|\nabla_v f^\varepsilon|^2}{f^\varepsilon} + 2\nabla_v f^\varepsilon \cdot (v - u^\varepsilon) + |v - u^\varepsilon|^2 f^\varepsilon + |u^\varepsilon - u_\delta^\varepsilon|^2 f^\varepsilon \right] dv dx \\ &\leq -\frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \left[\frac{|\nabla_v f^\varepsilon|^2}{f^\varepsilon} + 2\nabla_v f^\varepsilon \cdot (v - u^\varepsilon) + |v - u^\varepsilon|^2 f^\varepsilon \right] dv dx \\ &= -\frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla_v f^\varepsilon + (v - u^\varepsilon) f^\varepsilon|^2}{f^\varepsilon} dv dx. \end{aligned}$$

For the alignment term, we obtain similarly

$$\begin{aligned} & - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} [\nabla_v f^\varepsilon \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) + v \cdot (v - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon] \, dv \, dx \\ &= - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} \nabla_v f^\varepsilon \cdot (v - u^\varepsilon) \, dv \, dx \\ & \quad - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} [v \cdot (v - u^\varepsilon) f^\varepsilon + v \cdot (u^\varepsilon - [u^\varepsilon]_{\rho^\varepsilon}) f^\varepsilon] \, dv \, dx \\ &= - \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} (\nabla_v f^\varepsilon \cdot (v - u^\varepsilon) + |v - u^\varepsilon|^2 f^\varepsilon) \, dv \, dx - \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon}. \end{aligned}$$

Combing the two expressions and completing the squares:

$$\begin{aligned} \frac{d}{dt} \mathcal{H}_\varepsilon &\leq - \frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^n} \frac{|\nabla_v f^\varepsilon + (1 + \varepsilon s_{\rho^\varepsilon}/2)(v - u^\varepsilon) f^\varepsilon|^2}{f^\varepsilon} \, dv \, dx \\ & \quad + \frac{\varepsilon}{4} \int_{\Omega \times \mathbb{R}^n} s_{\rho^\varepsilon} |v - u^\varepsilon|^2 f^\varepsilon \, dv \, dx - \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon} \\ &\leq - \frac{1}{\varepsilon} \mathcal{I}_\varepsilon + c\varepsilon e(f^\varepsilon |u^\varepsilon) - \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 + (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon} \end{aligned}$$

We have obtained (9.21). ■

Lemma 9.10. *We have the following inequality:*

$$\frac{d}{dt} \mathcal{G}_\varepsilon \leq C \mathcal{H}(f^\varepsilon | \mu) + C \sqrt{\mathcal{I}_\varepsilon} + C\varepsilon + \frac{\delta}{\varepsilon} + \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 - (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon}, \quad (9.23)$$

where C is independent of ε .

Proof. Let us compute the derivative of each component of \mathcal{G}_ε

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\Omega} \rho^\varepsilon |u|^2 \, dx &= \int_{\Omega} [\rho^\varepsilon (u^\varepsilon - u) \cdot \nabla u \cdot u - \rho^\varepsilon u \cdot \nabla \log \rho + \rho^\varepsilon s_\rho ([u]_\rho - u) \cdot u] \, dx \\ \frac{d}{dt} \int_{\Omega^n} \rho^\varepsilon u^\varepsilon \cdot u \, dx &= \int_{\Omega} \left[\rho^\varepsilon (u^\varepsilon - u) \cdot \nabla u \cdot u^\varepsilon + \rho^\varepsilon \nabla \cdot u - \rho^\varepsilon u^\varepsilon \cdot \nabla \log \rho - \nabla u : \mathcal{R}_\varepsilon \right. \\ & \quad \left. + \rho^\varepsilon s_{\rho^\varepsilon} ([u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon) \cdot u \right. \\ & \quad \left. + \rho^\varepsilon s_\rho ([u]_\rho - u) \cdot u^\varepsilon + \frac{1}{\varepsilon} \rho^\varepsilon (u_\delta^\varepsilon - u^\varepsilon) \cdot u \right] \, dx \\ \frac{d}{dt} \int_{\Omega^n} \rho^\varepsilon \log \rho \, dx &= \int_{\Omega} [\rho^\varepsilon u^\varepsilon \cdot \nabla \log \rho - \rho^\varepsilon u \cdot \nabla \log \rho - \rho^\varepsilon \nabla \cdot u] \, dx. \end{aligned}$$

Thus,

$$\frac{d}{dt} \mathcal{G}_\varepsilon = \int_{\Omega} [\nabla u : \mathcal{R}_\varepsilon - \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla u \cdot (u^\varepsilon - u)] \, dx + A + A_{\text{loc}},$$

where A_{loc} is the same local alignment terms as appeared in the previous section, and

$$\begin{aligned} A &= \int_{\Omega} [\rho^\varepsilon s_\rho([u]_\rho - u) \cdot u - \rho^\varepsilon s_{\rho^\varepsilon}([u^\varepsilon]_{\rho^\varepsilon} - u^\varepsilon) \cdot u - \rho^\varepsilon s_\rho([u]_\rho - u) \cdot u^\varepsilon] \, dx \\ &= \delta A + \|u^\varepsilon\|_{L^2(\kappa^\varepsilon)}^2 - (u^\varepsilon, [u^\varepsilon]_{\rho^\varepsilon})_{\kappa^\varepsilon}, \end{aligned}$$

where δA is again the same alignment term that appeared in the previous section. We estimate A_{loc} and δA as before using (9.14) and the intermediate estimate in (9.15). We recall that only (3.23) and (3.24) and regularity of the kernel ϕ_ρ are necessary to prove (9.15). Since $\rho \geq \rho_- > 0$ we have both by the assumptions. Thus,

$$\delta A + A_{\text{loc}} \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + W_1^2(\rho^\varepsilon, \rho) + \frac{\delta}{\varepsilon}.$$

Keeping in mind that both the macroscopic relative entropy $\int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx$ and

$$W_1^2(\rho^\varepsilon, \rho) \leq \|\rho^\varepsilon - \rho\|_1^2$$

are controlled by $\mathcal{H}(f^\varepsilon|\mu)$ (see (9.18), (9.19)), we obtain

$$\delta A + A_{\text{loc}} \lesssim \mathcal{H}(f^\varepsilon|\mu) + \frac{\delta}{\varepsilon}.$$

Next, given that u is smooth we have

$$\left| \int_{\Omega} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla u \cdot (u^\varepsilon - u) \, dx \right| \lesssim \int_{\Omega} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx \leq \mathcal{H}(f^\varepsilon|\mu).$$

As to the Reynolds stress, we will use a well-known estimate from [66] that establishes a bound in terms of information and energy. Let us rerun this argument to account for the ε -correction. We simply note that

$$\mathcal{R}_\varepsilon = \int_{\mathbb{R}^n} [2\nabla_v \sqrt{f^\varepsilon} + (v - u^\varepsilon) \sqrt{f^\varepsilon}] \otimes [(v - u^\varepsilon) \sqrt{f^\varepsilon}] \, dv,$$

then we reinsert the ε -correction to obtain

$$\begin{aligned} \mathcal{R}_\varepsilon &= \int_{\mathbb{R}^n} \left[2\nabla_v \sqrt{f^\varepsilon} + \left(1 + \frac{\varepsilon S \rho^\varepsilon}{2}\right) (v - u^\varepsilon) \sqrt{f^\varepsilon} \right] \otimes [(v - u^\varepsilon) \sqrt{f^\varepsilon}] \, dv \\ &\quad - \varepsilon S \rho^\varepsilon / 2 \int_{\mathbb{R}^n} (v - u^\varepsilon) \otimes (v - u^\varepsilon) f^\varepsilon \, dv. \end{aligned}$$

So,

$$\int_{\Omega} |\mathcal{R}_\varepsilon| \, dx \lesssim \sqrt{e(f^\varepsilon|u^\varepsilon) \mathcal{I}_\varepsilon} + \varepsilon e(f^\varepsilon|u^\varepsilon) \lesssim \sqrt{\mathcal{I}_\varepsilon} + \varepsilon.$$

Collecting the obtained estimates together we obtain (9.23). ■

Combining the equations on \mathcal{H}_ε and \mathcal{G}_ε , (9.21), (9.23), we see that the residual alignment-energy terms cancel, and we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{H}(f^\varepsilon|\mu) &\lesssim \mathcal{H}(f^\varepsilon|\mu) - \frac{1}{\varepsilon} \mathcal{I}_\varepsilon + \varepsilon + \frac{\delta}{\varepsilon} + \sqrt{\mathcal{I}_\varepsilon} \\ &\leq \mathcal{H}(f^\varepsilon|\mu) - \frac{1}{2\varepsilon} \mathcal{I}_\varepsilon + 2\varepsilon + \frac{\delta}{\varepsilon} \lesssim \mathcal{H}(f^\varepsilon|\mu) + \varepsilon + \frac{\delta}{\varepsilon}. \end{aligned}$$

By Grönwall’s lemma, we obtain

$$\mathcal{H}(f^\varepsilon|\mu) \leq \mathcal{H}(f_0^\varepsilon|\mu_0) e^{CT} + C \left(\varepsilon + \frac{\delta}{\varepsilon} \right) e^{CT}, \quad \forall t \leq T,$$

where C depends only on the parameters of the model and the regularity of (u, ρ) . This finishes the proof. ■

Remark 9.11. The same observation can be made here as in the monokinetic case. If we quantify the initial entropy

$$\mathcal{H}(f_0^\varepsilon|\mu_0) \leq \varepsilon,$$

then the proof produces the bound

$$\mathcal{H}(f^\varepsilon|\mu) \leq \varepsilon + \frac{\delta}{\varepsilon}.$$

So, again, the optimal convergence is achieved when $\delta \sim \varepsilon^2$. However, unlike in the monokinetic case, here we do not loose on the magnitude of the entropy at positive times.

9.3. Remarks on the pressureless Euler-alignment system

We will leave discussion of the well-posedness of macroscopic systems that arise from general models \mathcal{M} to future research; see [15, 44, 60, 70, 82, 94] for the literature on this problem, specifically for smooth communication models. The most clear-cut result obtained in [15] pertains to the regularity of the one-dimensional pressureless EAS based on the Cucker–Smale protocol

$$\partial_t u + uu_x = \rho_\phi u - (u\rho)_\phi.$$

Here, one finds an additional conserved quantity

$$e = u_x + \rho_\phi,$$

which controls u_x , and hence regularity of the system. In fact, e satisfies

$$\partial_t e + \partial_x(ue) = 0,$$

or in Lagrangian coordinates associated with u ,

$$\frac{d}{dt} e = e\rho_\phi - e^2 = e(\rho_\phi - e),$$

which is a non-homogeneous logistic ordinary differential equation. The critical threshold for regularity becomes $e_0 \geq 0$.

In multiple dimensions, the law of e is given by

$$e = \operatorname{div} u + \rho_\phi, \quad \partial_t e + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \operatorname{Tr}[(\nabla u)^2]. \tag{9.24}$$

Although the right-hand side in this case involves ∇u , in some cases this still allows one to obtain partial regularity results in multiple dimensions, for example, for small data or for unidirectional flocks; see [44, 60, 94]. The latter are solutions of the form

$$u = (u(x_1, \dots, x_n), 0, \dots, 0).$$

For these the right-hand side of (9.24) vanishes.

While the existence of e is attributed to the particular commutator structure of the alignment forcing of the Cucker–Smale model, in general, it can be seen as a consequence of another property of the model, namely, transport of the specific strength function s_ρ itself. Indeed, let us notice that in the \mathcal{M}_{CS} -case we have

$$\partial_t s_\rho + \nabla \cdot (s_\rho [u]_\rho) = 0, \tag{9.25}$$

simply because ρ_ϕ is transported by the Favre-filtration $u_F = (u\rho)_\phi / \rho_\phi$. This turns out to be the general reason for the conservation of e .

Lemma 9.12. *If for any solution of the pressureless EAS (9.2) the strength function satisfies (9.25), then $e = \operatorname{div} u + s_\rho$ satisfies (9.24). In particular, e is conserved for all solutions in one dimension and unidirectional solutions in multiple dimensions.*

Proof. By direct verification. ■

The above observation motivates to consider a system where the strength is not fixed, but rather evolves according to the “natural law” (9.25), whereby the strength itself becomes another unknown. This leads to the following system:

$$\partial_t \rho + \nabla \cdot (u\rho) = 0, \quad \partial_t s + \nabla \cdot (s[u]_\rho) = 0, \quad \partial_t u + u \cdot \nabla u = s([u]_\rho - u).$$

All such systems will satisfy the e -law by design, where $e = \operatorname{div} u + s$.

For example, if we start from the initial Favre-based model, $[u]_\rho = u_F$ and set $s_0 = 1$, like for instance in the \mathcal{M}_{MT} -model, the future value of strength will be determined by the transport along the averaged velocity $[u]_\rho$, rather than being forcefully set at $s = 1$ for all times. Given that both s and ρ_ϕ solve the same continuity equation in this case, we also have transport of the ratio

$$\partial_t \frac{s}{\rho_\phi} + u_F \cdot \nabla_x \frac{s}{\rho_\phi} = 0.$$

This implies that

$$c_1 \rho_\phi \leq s \leq c_2 \rho_\phi, \quad \forall (t, x) \in [0, \infty) \times \Omega,$$

if initially so. In particular, s remains uniformly bounded regardless of the regularity of u_F (!).

A thorough study of this model has been recently completed for Favre-based modes in [89] during the review of this present work.

A. Proof of (2.6)

We start as in [53, Lemma 5.2]. Let us fix $x \in \Omega$ and consider a cover of the ball $B_R(x)$ by balls of radius $r/2$:

$$B_R(x) \subset \bigcup_{i=1}^I B_{r/2}(x_i),$$

where N depends only on n and R/r . Then assuming (2.4), we obtain

$$\begin{aligned} \left(\frac{\rho}{\rho_\phi^{1-\beta}}\right)_\phi(x) &= \int_{B_R(x)} \frac{\rho(y)\phi(x-y)}{\left(\int_\Omega \rho(z)\phi(y-z) dz\right)^{1-\beta}} dy \\ &\leq \sum_{i=1}^I \int_{B_{r/2}(x_i)} \frac{\rho(y)\phi(x-y)}{\left(\int_{B_{r/2}(x_i)} \rho(z)\phi(y-z) dz\right)^{1-\beta}} dy. \end{aligned}$$

Since $y, z \in B_{r/2}(x_i)$, we have $|y-z| \leq r$, and by the lower bound on the kernel we obtain

$$\left(\frac{\rho}{\rho_\phi^{1-\beta}}\right)_\phi(x) \lesssim \sum_{i=1}^I \int_{B_{r/2}(x_i)} \frac{\rho(y)\phi(x-y)}{\left(\int_{B_{r/2}(x_i)} \rho(z) dz\right)^{1-\beta}} dy.$$

If $\beta = 0$, we remove the kernel by $\|\phi\|_\infty$, and the rest adds up to I .

In the case where $\beta > 0$, we have

$$\left(\frac{\rho}{\rho_\phi^{1-\beta}}\right)_\phi(x) \lesssim \sum_{i=1}^I \int_{B_{r/2}(x_i)} \frac{\rho(y)\phi(x-y)}{\int_{B_{r/2}(x_i)} \rho(z) dz} \left(\int_{B_{r/2}(x_i)} \rho(z) dz\right)^\beta dy.$$

Treating

$$\mathbb{1}_{B_{r/2}(x_i)}(y) \frac{\rho(y)}{\int_{B_{r/2}(x_i)} \rho(z) dz} dy$$

as a probability measure for each integral, by the Hölder inequality, we obtain

$$\begin{aligned} \left(\frac{\rho}{\rho_\phi^{1-\beta}}\right)_\phi(x) &\lesssim \sum_{i=1}^I \left(\int_{B_{r/2}(x_i)} \frac{\rho(y)\phi^{1/\beta}(x-y)}{\int_{B_{r/2}(x_i)} \rho(z) dz} \int_{B_{r/2}(x_i)} \rho(z) dz dy\right)^\beta \\ &= \sum_{i=1}^I \left(\int_{B_{r/2}(x_i)} \rho(y)\phi^{1/\beta}(x-y) dy\right)^\beta \\ &\leq \|\phi\|_\infty^{1-\beta} \sum_{i=1}^I \left(\int_{B_{r/2}(x_i)} \rho(y)\phi(x-y) dy\right)^\beta \leq I \|\phi\|_\infty^{1-\beta} \rho_\phi^\beta(x). \end{aligned}$$

B. Averagings on finite sets and proof of Proposition 3.12

In this section we will prove Proposition 3.12.

To achieve this, we first study properties of models on finite sets – to which, as we will see, the result is reduced. Below, we will use the notation of Example 2.11.

For models on finite sets, being conservative is equivalent to

$$A^\top \kappa = \kappa, \quad \kappa = (\kappa_1, \dots, \kappa_N).$$

Denoting $K = \text{diag}\{\kappa_1, \dots, \kappa_N\}$, one can see that being symmetric is equivalent to the matrix KA being symmetric,

$$(KA)^\top = KA.$$

Similarly, the ball-positivity is equivalent to the matrix A being ball-coercive relative to the inner product $(\cdot, \cdot)_K = (K\cdot, \cdot)$:

$$(Au, u)_K \geq (Au, Au)_K.$$

Lemma B.1. *If \mathcal{M} is ball-positive on a 2-point set, then \mathcal{M} is symmetric.*

Proof. The result reduces to showing that $\kappa_1 a_{12} = \kappa_2 a_{21}$ for any ball-coercive model. Coercivity is equivalent to

$$\begin{aligned} & \kappa_1 a_{11} u_1^2 + (\kappa_1 a_{12} + \kappa_2 a_{21}) u_1 u_2 + \kappa_2 a_{22} u_2^2 \\ & \geq \kappa_1 (a_{11} u_1 + a_{12} u_2)^2 + \kappa_2 (a_{21} u_1 + a_{22} u_2)^2. \end{aligned}$$

Collecting coefficients in front of each monomial we obtain

$$\alpha u_1^2 + \beta u_1 u_2 + \gamma u_2^2 \geq 0,$$

where

$$\begin{aligned} \alpha &= \kappa_1 a_{11} - \kappa_1 a_{11}^2 - \kappa_2 a_{21}^2, \\ \beta &= \kappa_1 a_{12} + \kappa_2 a_{21} - 2\kappa_1 a_{11} a_{12} - 2\kappa_2 a_{21} a_{22}, \\ \gamma &= \kappa_2 a_{22} - \kappa_1 a_{12}^2 - \kappa_2 a_{22}^2. \end{aligned}$$

This means that the determinant of the quadratic form is non-negative

$$4\alpha\gamma \geq \beta^2.$$

Using stochasticity of A and after a long but elementary computation, the above condition reduces to

$$(\kappa_1 a_{12} - \kappa_2 a_{21})^2 \leq 0,$$

which proves the result. ■

Proof of Proposition 3.12. Since the averages act coordinatewise it is sufficient to prove the result for scalar fields u .

Let us fix ρ . Let us pick any partitioning of Ω into two sets A, B and assume that $v(A), v(B) > 0$. Let us denote

$$\begin{aligned} a_{11} &= \frac{1}{\kappa_\rho(A)} \int_A [\mathbb{1}_A]_\rho \, d\kappa_\rho, & a_{12} &= \frac{1}{\kappa_\rho(A)} \int_A [\mathbb{1}_B]_\rho \, d\kappa_\rho; \\ a_{21} &= \frac{1}{\kappa_\rho(B)} \int_B [\mathbb{1}_A]_\rho \, d\kappa_\rho, & a_{22} &= \frac{1}{\kappa_\rho(B)} \int_B [\mathbb{1}_B]_\rho \, d\kappa_\rho. \end{aligned}$$

Note that the matrix $A = (a_{ij})_{i,j=1}^2$ is right stochastic. Denoting $\kappa_1 = \kappa_\rho(A), \kappa_2 = \kappa_\rho(B)$ and verifying coercivity on functions of the form $u = u_1 \mathbb{1}_A + u_2 \mathbb{1}_B$, we obtain

$$\kappa_1 a_{11} u_1^2 + (\kappa_1 a_{12} + \kappa_2 a_{21}) u_1 u_2 + \kappa_2 a_{22} u_2^2 \geq \int_\Omega |u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho|^2 \, d\kappa_\rho.$$

Breaking down the integral and using the Hölder inequality, we obtain

$$\begin{aligned} &\int_\Omega |u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho|^2 \, d\kappa_\rho \\ &= \int_A |u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho|^2 \, d\kappa_\rho + \int_B |u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho|^2 \, d\kappa_\rho \\ &\geq \frac{1}{\kappa_\rho(A)} \left| \int_A (u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho) \, d\kappa_\rho \right|^2 + \frac{1}{\kappa_\rho(B)} \left| \int_B (u_1 [\mathbb{1}_A]_\rho + u_2 [\mathbb{1}_B]_\rho) \, d\kappa_\rho \right|^2 \\ &= \kappa_1 (a_{11} u_1 + a_{12} u_2)^2 + \kappa_2 (a_{21} u_1 + a_{22} u_2)^2, \end{aligned}$$

which implies that the 2-point model with A and κ defined above is ball-positive. The previous lemma implies that

$$\int_A [\mathbb{1}_B]_\rho \, d\kappa_\rho = \int_B [\mathbb{1}_A]_\rho \, d\kappa_\rho. \tag{B.1}$$

We further conclude

$$\begin{aligned} \int_\Omega [\mathbb{1}_A]_\rho \, d\kappa_\rho &= \int_A [\mathbb{1}_A]_\rho \, d\kappa_\rho + \int_B [\mathbb{1}_A]_\rho \, d\kappa_\rho \\ &= \int_A [\mathbb{1}_A]_\rho \, d\kappa_\rho + \int_A [\mathbb{1}_B]_\rho \, d\kappa_\rho = \int_A [\mathbb{1}_\Omega]_\rho \, d\kappa_\rho = \kappa_\rho(A). \end{aligned}$$

In other words, the conservative property holds for all characteristic functions. Since it is also linear and the average, by our assumption, is a bounded operator on $L^2(\kappa_\rho)$, we obtain the result by the standard approximation. ■

It would seem like (B.1) is suggestive of symmetry as it holds for any pair of partitioning sets. However, to prove general symmetry one would have to make the same conclusion for any pair of disjoint sets not necessarily partitioning Ω , or for any triple

of partitioning sets. The above argument fails to do it, and in fact the implication “ball-positive \Rightarrow symmetric” is generally not true. A finite-dimensional example can be found via a 3-point construction.

Example B.2. Let us assume for simplicity that $\kappa = \mathbb{1} = (1, 1, 1)$. Then we are looking for a matrix that is non-symmetric yet doubly stochastic, $A\mathbb{1} = A^\top\mathbb{1} = \mathbb{1}$, and ball-positive.

Thanks to stochasticity, A leaves the space $X = \mathbb{1}^\perp$ invariant, and so it is enough to properly define A on the two-dimensional space X only. Let us fix a non-orthogonal basis in X : $e_1 = (1, -1, 0)$, $e_2 = (1, 0, -1)$, and complement it to $e_3 = \mathbb{1}$. We define

$$Ae_1 = \lambda_1 e_1, \quad Ae_2 = \lambda_2 e_2,$$

where $1 > \lambda_i > 0$ and $\lambda_1 \neq \lambda_2$. This choice guarantees that the matrix A is not symmetric. Now, we need to make sure that A is ball-positive. Again, by stochasticity, ball-positivity reduces to that of the restriction $A|_X$. The latter is equivalent to the condition

$$(e_1 + te_2) \cdot (\lambda_1 e_1 + t\lambda_2 e_2) \geq |\lambda_1 e_1 + t\lambda_2 e_2|^2$$

for all $t \in \mathbb{R}$. Expanding, we obtain

$$(\lambda_2 - \lambda_1^2)t^2 + \left[\frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_1\lambda_2 \right]t + \lambda_1 - \lambda_1^2 \geq 0.$$

This is equivalent to

$$(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)^2 \leq 16(\lambda_2 - \lambda_1^2)(\lambda_1 - \lambda_1^2). \tag{B.2}$$

In addition, we need to ensure that all the entries of the matrix A in the original system of coordinates are non-negative. We can write down these entries explicitly:

$$A = \frac{1}{3} \begin{pmatrix} 1 + \lambda_1 + \lambda_2 & 1 + \lambda_2 - 2\lambda_1 & 1 + \lambda_1 - 2\lambda_2 \\ 1 - \lambda_1 & 1 + 2\lambda_1 & 1 - \lambda_1 \\ 1 - \lambda_2 & 1 - \lambda_2 & 1 + 2\lambda_2 \end{pmatrix}.$$

So the only conditions to guarantee are

$$1 + \lambda_2 - 2\lambda_1 \geq 0, \quad 1 + \lambda_1 - 2\lambda_2 \geq 0. \tag{B.3}$$

There are plenty of choices to fulfill both (B.2) and (B.3). For example, $\lambda_1 = 1/2, \lambda_2 = 1/3$. This concludes the construction.

C. On spectral gaps

With regard to the discussion of Remark 4.10, we prove a lemma that establishes equivalence of numerical ranges on the space of zero-momenta and the mean-zero functions.

Lemma C.1. *Suppose \mathcal{M} is conservative and satisfies the following*

$$c_0 \leq s_\rho(x) \leq c_1, \quad \forall x \in \text{supp } \rho, \tag{C.1}$$

$$\sup\{(u, [u]_\rho)_{\kappa_\rho} : u \in L^2_0(\kappa_\rho), \|u\|_2 = 1\} \leq 1 - \varepsilon.$$

Then

$$\sup\{(u, [u]_\rho)_{\kappa_\rho} : u \in L^2(\kappa_\rho), \bar{u} = 0, \|u\|_2 = 1\} \leq 1 - \varepsilon \frac{c_0}{c_0 + c_1}.$$

Conversely, if

$$\sup\{(u, [u]_\rho)_{\kappa_\rho} : u \in L^2(\kappa_\rho), \bar{u} = 0, \|u\|_2 = 1\} \leq 1 - \delta,$$

then

$$\sup\{(u, [u]_\rho)_{\kappa_\rho} : u \in L^2_0(\kappa_\rho), \|u\|_2 = 1\} \leq 1 - \delta \frac{c_0}{c_0 + c_1}.$$

Proof. First let us observe that the bounds on s_ρ in (C.1), imply bounds on κ_ρ -masses

$$c_0 \leq \kappa_\rho(\Omega) \leq c_1. \tag{C.2}$$

Let us denote $\mathbb{P}: L^2(\kappa_\rho) \rightarrow \mathbb{R}^n$ the orthogonal projection onto the space of constant fields. For all u with $\bar{u} = 0$, we have

$$\left| \int_\Omega (u - \mathbb{P}u) \rho \, dx \right| = |\mathbb{P}u| = \frac{1}{\sqrt{\kappa_\rho(\Omega)}} \|\mathbb{P}u\|_2.$$

On the other hand, by (i),

$$\left| \int_\Omega (u - \mathbb{P}u) \rho \, dx \right| = \left| \int_\Omega (u - \mathbb{P}u) \frac{1}{s_\rho} \, d\kappa_\rho \right| \leq \frac{\sqrt{\kappa_\rho(\Omega)}}{c_0} \|u - \mathbb{P}u\|_2.$$

Using compatibility of masses (C.2),

$$\|u - \mathbb{P}u\|_2 \geq \frac{c_0}{c_1} \|\mathbb{P}u\|_2.$$

Hence,

$$\|u\|_2^2 = \|u - \mathbb{P}u\|_2^2 + \|\mathbb{P}u\|_2^2 \geq \left(1 + \frac{c_0}{c_1}\right) \|\mathbb{P}u\|_2^2,$$

or

$$\|\mathbb{P}u\|_2^2 \leq \frac{c_1}{c_0 + c_1} \|u\|_2^2. \tag{C.3}$$

Now, let us compute the numerical range, noting that $[u]_\rho = \mathbb{P}u$,

$$(u, [u]_\rho)_{\kappa_\rho} = (u - \mathbb{P}u, [u - \mathbb{P}u]_\rho)_{\kappa_\rho} + (u - \mathbb{P}u, \mathbb{P}u)_{\kappa_\rho} \\ + (\mathbb{P}u, [u - \mathbb{P}u]_\rho)_{\kappa_\rho} + \|\mathbb{P}u\|_2^2.$$

The second term vanishes due to orthogonality. For the third term we observe that due to the conservative property of the average integrating against a constant field produces the same result as integrating without the average. So,

$$(\mathbb{P}u, [u - \mathbb{P}u]_\rho)_{\kappa_\rho} = (\mathbb{P}u, u - \mathbb{P}u)_{\kappa_\rho} = 0.$$

Using the spectral gap condition for the first term and (C.3) for the last one, we obtain

$$\begin{aligned} (u, [u])_{\kappa_\rho} &\leq (1 - \varepsilon_0) \|u - \mathbb{P}u\|_2^2 + \|\mathbb{P}u\|_2^2 = (1 - \varepsilon_0) \|u\|_2^2 + \varepsilon_0 \|\mathbb{P}u\|_2^2 \\ &\leq \left(1 - \varepsilon_0 + \varepsilon_0 \frac{c_1}{c_0 + c_1}\right) \|u\|_2^2 = \left(1 - \varepsilon_0 \frac{c_0}{c_0 + c_1}\right) \|u\|_2^2. \end{aligned}$$

To obtain the converse statement, apply the same argument replacing the roles of ρ and κ_ρ , and note that $1/c_1 \leq 1/s_\rho \leq 1/c_0$. ■

D. Categorical considerations

Environmental averagings form an “ecosystem” of models. On a more formal level they can be thought of as a category of objects and we can discuss relationships between them.

For a couple of models \mathcal{M}' , \mathcal{M}'' defined over Ω' and Ω'' , respectively, a morphism $\mathcal{M}' \rightarrow \mathcal{M}''$ is defined by a volume preserving homeomorphism $\tau: \Omega' \rightarrow \Omega''$ such that if $\rho'' \circ \tau = \rho'$ and $u'' \circ \tau = u'$, then

$$[u'']'_{\rho''} \circ \tau = [u']'_{\rho'},$$

and there exist two constants $c, C > 0$ such that

$$c \, d\kappa'_{\rho'} \leq d\kappa''_{\rho''} \circ \tau \leq C \, d\kappa'_{\rho'}.$$

For material models, the latter can be restated in terms of specific strengths

$$c s'_{\rho'} \leq s''_{\rho''} \circ \tau \leq C s'_{\rho'}.$$

We have tacitly employed this concept in Appendix B when discussing models on finite sets.

On a given environment Ω , all models can be partially ordered in several ways. The most straightforward definition of $\mathcal{M}' \preceq \mathcal{M}''$ is

$$[[u]']'_\rho = [[u]']''_\rho = [u]'_\rho, \quad \forall u \in L^\infty(\Omega).$$

For example, among rough segregation models we have $\mathcal{M}_{\mathcal{F}'} \preceq \mathcal{M}_{\mathcal{F}''}$ provided $\mathcal{F}' \subset \mathcal{F}''$. The identity model $\mathcal{M}_{\mathbb{I}}$ is the finest of all material ones (although if we defined it to be $[u] = u$ irrelevant of the supp ρ , then it would have become the finest of all). At the same time $\mathcal{M}_{\text{glob}}$ is the coarsest among all conservative ones with $s_\rho = 1$.

A more refined definition of order can be given on classes of equivalence where we say $\mathcal{M}' \sim \mathcal{M}''$ if there exist intermediate averagings $\mathcal{M}_1, \dots, \mathcal{M}_n$ such that for any $\rho \in \mathcal{P}$, there exist $\rho_1, \dots, \rho_n \in \mathcal{P}$ such that

$$[\dots [[u]''_{\rho_1}]^1 \dots]_{\rho_n}^n = [u]'_{\rho}, \quad \forall u \in L^\infty(\Omega),$$

and there exist intermediate averagings $\mathcal{M}_{n+1}, \dots, \mathcal{M}_{n+m}$ such that for any $\rho \in \mathcal{P}$, there exist $\rho_{n+1}, \dots, \rho_{n+m} \in \mathcal{P}$ such that

$$[\dots [[u]'_{\rho_{n+1}}]^{n+1} \dots]_{\rho_{n+m}}^{n+m} = [u]''_{\rho}, \quad \forall u \in L^\infty(\Omega).$$

Then for a pair of models representing their equivalence classes we say $\mathcal{M}' \preceq \mathcal{M}''$ if only one half of the definition above holds, namely, there exist intermediate averagings $\mathcal{M}_1, \dots, \mathcal{M}_n$ such that for any $\rho \in \mathcal{P}$, there exist $\rho_1, \dots, \rho_n \in \mathcal{P}$ such that

$$[\dots [[u]''_{\rho_1}]^1 \dots]_{\rho_n}^n = [u]'_{\rho}, \quad \forall u \in L^\infty(\Omega).$$

Under this partial ordering, more subtle examples emerge. For instance, for Cucker–Smale models with Bochner-positive kernels, it can be seen from the identity (4.33) that if $\phi = \psi * \psi$, and assuming that $\int \psi = 1$, then the \mathcal{M}_{CS} -model based on ψ is finer than that based on ϕ , $\mathcal{M}_{CS}^\phi \preceq \mathcal{M}_{CS}^\psi$. The same applies for \mathcal{M}_{MT} -models as those are based on the same averaging.

One can build new averaging models from old ones by superimposing averages as long as they are defined over the same strength measures. So, if

$$\mathcal{M}_i = \{(\kappa_\rho, [\cdot]_\rho^i) : \rho \in \mathcal{P}(\Omega)\}, \quad i = 1, 2$$

are two averaging models, then

$$\mathcal{M}_2 \circ \mathcal{M}_1 = \{(\kappa_\rho, [[\cdot]_\rho^1]_\rho^2) : \rho \in \mathcal{P}(\Omega)\}$$

defines another averaging model.

Certain compositions preserve special properties. For example, if \mathcal{M}_i are ball-positive and symmetric the conjugation $(\kappa_\rho, [[[\cdot]_\rho^1]_\rho^2]_\rho^1)$ is also ball-positive and symmetric.

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