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On the convergence of critical points of the Ambrosio–Tortorelli functional

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Abstract. This work is devoted to studying the asymptotic behavior of critical points $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ of the Ambrosio–Tortorelli functional. Under a uniform energy bound assumption, the usual Γ-convergence theory ensures that $(u_\varepsilon, v_\varepsilon)$ converges in the L^2-sense to some $(u_*, 1)$ as $\varepsilon \to 0$, where u_* is a special function of bounded variation. Assuming further that the Ambrosio–Tortorelli energy of $(u_\varepsilon, v_\varepsilon)$ converges to the Mumford–Shah energy of u_*, the latter is shown to be a critical point with respect to inner variations of the Mumford–Shah functional. As a by-product, the second inner variation is also shown to pass to the limit. To establish these convergence results, interior ($\mathcal{C}^\infty$) regularity and boundary regularity for Dirichlet boundary conditions are first obtained for a fixed parameter $\varepsilon > 0$. The asymptotic analysis is then performed by means of varifold theory in the spirit of scalar phase transition problems.

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary ($N \geq 1$) and $g \in H^{\frac{1}{2}}(\partial\Omega)$ be prescribed Dirichlet boundary data on $\partial\Omega$. For infinitesimal parameters $\varepsilon \to 0$ and $\eta_\varepsilon \to 0$ with $0 < \eta_\varepsilon \ll \varepsilon$, we consider the *Ambrosio–Tortorelli functional* defined by

$$\mathrm{AT}_\varepsilon(u, v) := \int_\Omega (\eta_\varepsilon + v^2)|\nabla u|^2 \,\mathrm{d}x + \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\,\mathrm{d}x,$$

for all pairs $(u, v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ satisfying $(u, v) = (g, 1)$ on $\partial\Omega$. This functional, originally introduced in [3], can be interpreted as a *phase-field regularization* of the Mumford–Shah functional which sends (u, v) to

$$\begin{cases} \mathrm{MS}(u) := \displaystyle\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{N-1}(J_u) + \mathcal{H}^{N-1}(\partial\Omega \cap \{u \neq g\}) & \text{if } \begin{cases} u \in SBV^2(\Omega), \\ v = 1 \text{ in } \Omega, \end{cases} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1}$$

The Mumford–Shah functional is well known as a theoretical tool to approach image segmentation [35, 37, 38]. It is also at the heart of the Francfort–Marigo model in fracture

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mechanics [15], and the numerical implementation of this model heavily relies on Ambrosio–Tortorelli-type functionals [8]. The use of such a phase-field approximation in numerics is usually justified through Γ-convergence theory. In terms of the functionals defined above, it states that AT_ε Γ-converges in the $[L^2(\Omega)]^2$-topology as $\varepsilon \to 0$ towards the Mumford–Shah functional (see e.g. the seminal paper [4]).

As a consequence, the fundamental theorem of Γ-convergence ensures the convergence of global minimizers $(u_\varepsilon, v_\varepsilon)$ of AT_ε to $(u, 1)$ as $\varepsilon \to 0$, where $u \in \mathrm{SBV}^2(\Omega)$ is a global minimizer of MS. This result is of course of importance, but it is somehow not fully satisfactory. Beyond the fact that the use of global minimizers in the models mentioned above remains under debate, this convergence result does not really provide a rigorous justification of the numerical simulations based on the Ambrosio–Tortorelli functional. One particular feature of AT_ε is its lack of convexity due to the nonconvex coupling term $v^2|\nabla u|^2$ with respect to the pair (u, v). This is a high obstacle to reach global minimizers through a numerical method. An idea employed in the context of image segmentation or fracture mechanics consists in performing an alternate minimization algorithm; see [9]. Each iteration of the scheme is well posed since AT_ε is continuous, coercive, and separately strictly convex. Letting the number of steps go to infinity, the sequence of iterates turns out to converge to a critical point of the energy AT_ε (see [13] and also [8, Theorem 1]), but this critical point might fail to be a global minimizer. Consequently, the original target of numerically approximating global minimizers of the Mumford–Shah functional might be lost. These issues motivate the question of convergence as $\varepsilon \to 0$ of critical points of the Ambrosio–Tortorelli functional and it constitutes the main goal of this article, continuing a task initiated in [14, 24] in dimension $N = 1$. In higher dimensions, a fundamental issue in such an analysis is the regularity of critical points of AT_ε. It is also of importance for numerics as the efficiency of the numerical methods crucially rests on it. Here we fully resolve this last question, showing smoothness of arbitrary critical points according to the smoothness of $\partial\Omega$ and the Dirichlet boundary data.

The reason why here we consider Dirichlet boundary conditions and not Neumann ones (with a so-called fidelity term as in the standard Mumford–Shah functional [38]) is that we have in mind possible applications to fracture mechanics. We nevertheless confess that the results of the present paper do not directly apply to fracture since an irreversibility constraint on the crack set (or on the phase field variable v at the regularized level), see [9, 15], does not allow any competitors to be taken as we do. However, our results can be seen as a first step in that direction.

A critical point $(u_\varepsilon, v_\varepsilon)$ of the Ambrosio–Tortorelli functional is a weak (distributional) solution of the nonlinear elliptic system

$$\begin{cases} -\operatorname{div}((\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon) = 0 & \text{in } \Omega, \\ -\varepsilon\Delta v_\varepsilon + \dfrac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon|\nabla u_\varepsilon|^2 = 0 & \text{in } \Omega, \\ (u_\varepsilon, v_\varepsilon) = (g, 1) & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

To be more precise, critical points of AT_ε are defined as follows.

Definition 1.1. Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $g \in H^{\frac{1}{2}}(\partial\Omega)$. A pair

$$(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega) := \big\{(u,v) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)] : (u,v) = (g,1) \text{ on } \partial\Omega\big\}$$

is a *critical point of the Ambrosio–Tortorelli functional* if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{AT}_\varepsilon(u + t\phi, v + t\psi) = 0 \quad \text{for all } (\phi,\psi) \in H^1_0(\Omega) \times [H^1_0(\Omega) \cap L^\infty(\Omega)],$$

that is,

$$\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \cdot \nabla\phi \,\mathrm{d}x = 0 \quad \text{for all } \phi \in H^1_0(\Omega) \tag{1.3}$$

and

$$\varepsilon \int_\Omega \nabla v_\varepsilon \cdot \nabla\psi \,\mathrm{d}x + \int_\Omega \Big(\frac{v_\varepsilon - 1}{4\varepsilon} + v_\varepsilon|\nabla u_\varepsilon|^2\Big)\psi \,\mathrm{d}x = 0 \tag{1.4}$$

for all $\psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$. By density, test functions (ϕ,ψ) in (1.3)–(1.4) can equivalently be chosen in $[\mathcal{C}^\infty_c(\Omega)]^2$, and (1.2) holds in the sense of distributions in Ω.

One may expect that critical points of AT_ε with uniformly bounded energy converge along some subsequence $\varepsilon \to 0$ to a limit satisfying some first-order criticality conditions for MS. Unfortunately, the theory of Γ-convergence does not provide convergence of critical points towards critical points of the limiting functional. Even for local minimizers such a result usually fails. We refer to [23, Remark 4.5] and [10, Example 3.5.1] for counterexamples. However, it has been proved in some specific examples that critical points do converge to critical points, possibly under the assumption of convergence of critical values. This is the case for the Allen–Cahn (or Modica–Mortola) functional from phase transitions approximating the $(N-1)$-dimensional area functional [17, 20, 22, 39, 45, 46], the Ginzburg–Landau functional approximating the $(N-2)$-dimensional area functional [2, 6, 32, 40, 44], and the Dirichlet energy of manifold-valued stationary harmonic maps [28–31, 33]. These functionals share many features with AT_ε, and we shall take advantage of the existing theory to develop our asymptotic analysis of critical points of AT_ε. In particular, we shall make essential use of both outer and inner variations of the energy, a common approach in all these studies.

1.1. Outer and inner variations

Definition 1.1 is simply saying that the first outer variation of AT_ε vanishes at $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ in any direction (ϕ,ψ). In the case of a smooth functional like AT_ε, outer variations coincide with Gâteaux differentials. For $(u,v) \in \mathcal{A}_g(\Omega)$ and $(\phi,\psi) \in H^1_0(\Omega) \times [H^1_0(\Omega) \cap L^\infty(\Omega)]$ as before, we introduce the following notation for the first and second outer variations of AT_ε (see Lemma A.1 for explicit formulas):

$$\mathrm{dAT}_\varepsilon(u,v)[\phi,\psi] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{AT}_\varepsilon(u + t\phi, v + t\psi), \tag{1.5}$$

$$\mathrm{d}^2\mathrm{AT}_\varepsilon(u,v)[\phi,\psi] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{AT}_\varepsilon(u + t\phi, v + t\psi). \tag{1.6}$$

Concerning the Mumford–Shah functional, the notion of critical points requires some definition and notation. Before doing so, let us first comment on the functional MS in (1.1) we are considering. Contrary to AT_ε, the admissible u's for MS are not required to agree with g on $\partial\Omega$ in the sense of traces. In turn, the additional term $\mathcal{H}^{N-1}(\partial\Omega \cap \{u \neq g\})$ in the expression of $\mathrm{MS}(u)$ penalizes "boundary jumps" where the inner trace of u (still denoted by u) differs from g. The expression $u \neq g$ on $\partial\Omega$ is also intended in the sense of traces. In the sequel, we shall often use the following compact notation:

$$\mathrm{MS}(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \mathcal{H}^{N-1}(\hat{J}_u), \quad u \in \mathrm{SBV}^2(\Omega),$$

where $\hat{J}_u = J_u \cup (\partial\Omega \cap \{u \neq g\})$, so that

$$\hat{J}_u = J_{\hat{u}} \text{ with } \hat{u} := u\mathbf{1}_\Omega + G\mathbf{1}_{\mathbf{R}^N \setminus \Omega} \in \mathrm{SBV}^2(\mathbf{R}^N),$$

and $G \in H^1(\mathbf{R}^N)$ is an arbitrary extension of g.

Unlike AT_ε, the Mumford–Shah functional is not smooth, and outer variations must be accordingly defined (see e.g. [1, Section 7.4]). Given $u, \phi \in \mathrm{SBV}^2(\Omega)$ such that $\hat{J}_\phi \subset \hat{J}_u$, the first and second outer variations of MS at u in the direction ϕ are respectively defined and given by

$$\mathrm{dMS}(u)[\phi] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{MS}(u + t\phi) = 2\int_\Omega \nabla u \cdot \nabla\phi \, \mathrm{d}x,$$
$$\mathrm{d}^2\mathrm{MS}(u)[\phi] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{MS}(u + t\phi) = 2\int_\Omega |\nabla\phi|^2 \, \mathrm{d}x.$$

In this definition, the requirement $\hat{J}_\phi \subset \hat{J}_u$ ensures the differentiability at $t = 0$ of the function $t \mapsto \mathrm{MS}(u + t\phi)$, since $\mathcal{H}^{N-1}(\hat{J}_{u+t\phi})$ remains constantly equal to $\mathcal{H}^{N-1}(\hat{J}_u)$. As a consequence, these differentials provide only information on the "regular part" of the function u, and not on the jump set $\hat{J}_u$. Note also that the second-order condition $\mathrm{d}^2\mathrm{MS}(u)[\phi] \geq 0$ is obviously satisfied at any u, ϕ as above. On the other hand, the condition $\hat{J}_\phi \subset \hat{J}_u$ also implies that the direction ϕ must agree with g on $\partial\Omega \cap \{u = g\}$, in agreement with the notion of a Dirichlet boundary condition.

It is clear that outer variations are not sufficient to define the notion of a critical point for MS since admissible perturbations leave the "singular part" $\mathcal{H}^{N-1}(\hat{J}_u)$ unchanged. The way to complement outer variations is to consider *inner variations*, i.e. variations under domain deformations. In doing so (up to the boundary), we shall assume that $\partial\Omega$ is at least of class $\mathcal{C}^2$.

Given a vector field $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$ satisfying $X \cdot \nu_\Omega = 0$ on $\partial\Omega$ (here ν_Ω denotes the outward unit normal field on $\partial\Omega$), we consider its flow map $\Phi \colon \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$, i.e. for every $x \in \mathbf{R}^N$, $t \mapsto \Phi(t, x)$ is defined as the unique solution of the system of ODEs

$$\begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}t}(t, x) = X(\Phi(t, x)), \\ \Phi(0, x) = x. \end{cases} \tag{1.7}$$

According to standard Cauchy–Lipschitz theory, $\Phi \in \mathcal{C}^1(\mathbf{R} \times \mathbf{R}^N; \mathbf{R}^N)$ is well defined, and $\{\Phi_t\}_{t\in\mathbf{R}}$ with $\Phi_t := \Phi(t,\cdot)$ is a one-parameter group of $\mathcal{C}^1$-diffeomorphisms of $\mathbf{R}^N$ into itself satisfying $\Phi_0 = \mathrm{Id}$. Then the requirement $X \cdot \nu_\Omega = 0$ on $\partial\Omega$ implies that $\Phi_t(\partial\Omega) = \partial\Omega$ for every $t \in \mathbf{R}$. Hence (the restriction of) Φ_t is a $\mathcal{C}^1$-diffeomorphism of $\partial\Omega$ into itself, and a $\mathcal{C}^1$-diffeomorphism of Ω into itself.

Definition 1.2. Let $u \in \mathrm{SBV}^2(\Omega)$, $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$, and $G \in H^1(\mathbf{R}^N)$ satisfy $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$. Setting $\{\Phi_t\}_{t\in\mathbf{R}}$ to be the integral flow of X and

$$u_t := u \circ \Phi_t^{-1} - G \circ \Phi_t^{-1} + G \in \mathrm{SBV}^2(\Omega), \tag{1.8}$$

the first and second inner variations of MS at u are defined by

$$\delta\mathrm{MS}(u)[X,G] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{MS}(u_t), \quad \delta^2\mathrm{MS}(u)[X,G] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{MS}(u_t).$$

It can be checked that, provided $\partial\Omega$, g, and G are smooth enough, the above derivatives exist and they can be explicitly computed (see Lemma A.4). Analogously, we define inner variations of the Ambrosio–Tortorelli functional.

Definition 1.3. Let $(u,v) \in \mathcal{A}_g(\Omega)$, $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$, and $G \in H^1(\mathbf{R}^N)$ satisfy $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$. We set

$$(u_t, v_t) := (u \circ \Phi_t^{-1} - G \circ \Phi_t^{-1} + G, v \circ \Phi_t^{-1}) \in \mathcal{A}_g(\Omega).$$

We define the first and second inner variations of AT_ε at (u,v) by

$$\begin{aligned} \delta\mathrm{AT}_\varepsilon(u,v)[X,G] &:= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{AT}_\varepsilon(u_t,v_t), \\ \delta^2\mathrm{AT}_\varepsilon(u,v)[X,G] &:= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{AT}_\varepsilon(u_t,v_t). \end{aligned} \tag{1.9}$$

Once again, the limits in (1.9) exist whenever $\partial\Omega$, g, and G are sufficiently smooth, and one can compute them explicitly (see Lemmas A.2 and A.3).

Remark 1.1. In Definition 1.2, we could similarly consider a competitor of the form $u_t = u \circ \Phi_t^{-1}$ instead of (1.8) since, in the case of the Mumford–Shah functional, the Dirichlet boundary condition may fail to be satisfied at the expense of paying a boundary energy penalization. If $\partial\Omega$, g, and G are smooth enough, the expressions of the first variation $\delta\mathrm{MS}$ and of the second variation $\delta^2\mathrm{MS}$ obtained in Lemma A.4 remain unchanged. This is in contrast with the inner variations of the Ambrosio–Tortorelli functional in Definition 1.3, for which it is necessary for the competitors (u_t, v_t) to match the exact boundary condition $(g, 1)$.

We emphasize that in Definitions 1.2 and 1.3 we are considering deformations *up to the boundary*. Compared to the usual deformations involving compactly supported perturbations in Ω of the original maps, it requires the additional test function G. This is of fundamental importance for the MS functional to recover information at the boundary, since the Dirichlet boundary condition is implemented in the functional as a penalization. Of course, the type of deformations we are using includes as a particular case the usual ones defined only through a vector field X compactly supported in Ω; see Remark A.1.

1.2. First-order criticality conditions for MS

In view of the discussion above, the nonsmooth character of MS forces the appropriate notion of a critical point to involve both outer and inner variations. In other words, a critical point of the Mumford–Shah functional is a critical point with respect to both outer and inner variations, a property obviously satisfied by global (and even local) minimizers.

Definition 1.4. Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with boundary of class at least $\mathcal{C}^2$ and $g \in \mathcal{C}^2(\partial\Omega)$. A function $u_* \in \mathrm{SBV}^2(\Omega)$ is a critical point of the Mumford–Shah functional if

$$\mathrm{dMS}(u_*)[\phi] = 0 \quad \text{for all } \phi \in \mathrm{SBV}^2(\Omega) \text{ with } \hat{J}_\phi \subset \hat{J}_{u_*}, \tag{1.10}$$

and

$$\delta\mathrm{MS}(u_*)[X, G] = 0 \tag{1.11}$$

for all $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$ and $G \in \mathcal{C}^2(\mathbf{R}^N)$ satisfying $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$.

From these criticality conditions, one can derive a set of Euler–Lagrange equations, which can be written in a strong form if the smoothness of u_* and $\hat{J}_{u_*}$ allow it. First specializing condition (1.10) to $\phi \in \mathcal{C}^\infty_c(\Omega)$ yields

$$\mathrm{div}(\nabla u_*) = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.12}$$

Then, if $\hat{J}_{u_*}$ is regular enough, one can choose test functions ϕ in (1.10) with a nontrivial jump set but smooth up to $\hat{J}_{u_*}$ from both sides. It leads to the homogeneous Neumann condition

$$\partial_\nu u_* = 0 \quad \text{on } \hat{J}_{u_*}; \tag{1.13}$$

see [1, formula (7.42)]. In other words, allowing test functions ϕ in (1.10) with $\hat{J}_\phi \subset \hat{J}_{u_*}$ (and not only in $\phi \in \mathcal{C}^\infty_c(\Omega)$) provides the weak formulation of (1.13) which complements (1.12).

Computing $\delta\mathrm{MS}(u_*)[X, G]$ (see formula (A.16)) and using equation (1.12), the stationarity condition (1.11) appears to be independent of the test function G and it reduces to

$$\int_\Omega (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, \mathrm{d}x + \int_{\hat{J}_{u_*}} \mathrm{div}^{\hat{J}_{u_*}} X \, \mathrm{d}\mathcal{H}^{N-1}$$
$$= -2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X \cdot \nabla_\tau g) \, \mathrm{d}\mathcal{H}^{N-1} \quad \text{for all } X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N) \text{ with } X \cdot \nu_\Omega = 0 \text{ on } \partial\Omega. \tag{1.14}$$

Here $\operatorname{div}^{\hat{J}_{u_*}} X = \operatorname{tr}((\mathrm{Id} - \nu_{u_*} \otimes \nu_{u_*})DX)$ is the tangential divergence of X on the countably $\mathcal{H}^{N-1}$-rectifiable set $\hat{J}_{u_*}$ with ν_{u_*} the approximate unit normal to that set. The boundary term on the right-hand side of (1.14) is interpreted in the sense of duality by (1.12), and $\nabla_\tau g$ denotes the tangential derivative of g. If J_{u_*} and u_* are regular enough, then (1.14) provides the coupling equation

$$H_{u_*} + [|\nabla u_*|^2]^{\pm} = 0 \quad \text{on } J_{u_*},$$

where H_{u_*} denotes the scalar mean curvature of J_{u_*} with respect to the normal ν_{u_*} and $[|\nabla u_*|^2]^{\pm}$ the (accordingly oriented) jump of $|\nabla u_*|^2$ across J_{u_*} (see [1, Chapter 7, Section 7.4]).

Remark 1.2 (One-dimensional case). In the one-dimensional case $N = 1$, if $\Omega = (0, L)$ for some $L > 0$, we can see that if $u \in \mathrm{SBV}^2(0, L)$ satisfies conditions (1.12)–(1.13), then u is either piecewise constant with a finite number of jumps or u is a globally affine function (with no jump). Indeed, the very definition of $\mathrm{SBV}^2(0, L)$ shows that u has a finite number of jumps. Then condition (1.12) implies that u is affine in between two consecutive jump points, and (1.13) implies that the slope of all affine functions must be zero. However, condition (1.14) does not play any role because it only implies that $|u'|$ is constant in $(0, L)$, where u' is the approximate derivative of u. From this, we just deduce that u is a piecewise affine function with equal slopes in absolute value, and it is not sufficient by itself to prove that u is piecewise constant. It indicates that the use of SBV^2-test functions in (1.10) cannot be relaxed to a class of smooth functions (in any dimension).

1.3. Main results

As already mentioned, the main purpose of this article is to investigate the asymptotic behavior of critical points of the Ambrosio–Tortorelli functional as $\varepsilon \to 0$. In view of the Γ-convergence result, one may expect that critical points converge to critical points, possibly under the assumption of convergence of energies. Without fully resolving this question, our analysis provides the first answer in this direction in arbitrary dimensions showing that a limit of critical points of AT_ε must at least be a critical point of MS with respect to inner variations, i.e. a stationary point of MS. If a critical point $(u_\varepsilon, v_\varepsilon)$ of AT_ε is smooth enough, then it is easy to see that it is also stationary, i.e. $\delta\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ (see Lemma A.2). Hence, if regularity of critical points AT_ε holds, proving the convergence of the first inner variations implies the announced stationarity of the limit. This is the path we have followed, and the regularity issue is the object of our first main theorem.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $g \in H^{\frac{1}{2}}(\partial\Omega)$. If $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ is a critical point of AT_ε, then $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$ and the following regularity up to the boundary holds:*

(i) *If $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$, then $u_\varepsilon \in L^\infty(\Omega)$.*

(ii) *If $\partial\Omega$ is of class $\mathcal{C}^{k\vee 2,1}$ and $g \in \mathcal{C}^{k,\alpha}(\partial\Omega)$ with $k \geq 1$ and $\alpha \in (0,1)$, then* $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{k,\alpha}(\overline{\Omega}) \times \mathcal{C}^{k\vee 2,\alpha}(\overline{\Omega})$.

We emphasize that the regularity in Theorem 1.1 is nontrivial since the second equation in (1.2) is of the form $\Delta v = f$ with $f \in L^1$ and standard linear elliptic theory does not directly apply (see however Remark 3.1). Instead, we shall rely on arguments borrowed from the regularity theory for harmonic maps into a manifold, or more generally for variational nonlinear elliptic systems; see e.g. [18]. The key issue is to prove Hölder continuity of v_ε, which we achieve by proving that it belongs to a suitable Morrey–Campanato space. We treat interior regularity and boundary regularity in a similar way through a reflection argument of independent interest originally devised in [42].

In our second main theorem, we show that, under the assumption of convergence of energies, limits (up to a subsequence) of critical points of AT_ε are critical points of MS for the inner variations.

Theorem 1.2. *Assume that $\Omega \subset \mathbf{R}^N$ is a bounded open set of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g(\Omega)$ be a family of critical points of the Ambrosio–Tortorelli functional. Then the following properties hold:*

(i) *If the energy bound*

$$\sup_{\varepsilon>0} \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty \tag{1.15}$$

is satisfied, up to a subsequence, $u_\varepsilon \to u_*$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$ for some $u_* \in \mathrm{SBV}^2(\Omega) \cap L^\infty(\Omega)$ satisfying $\nabla u_* \cdot \nu_\Omega \in L^2(\partial\Omega)$, and $\mathrm{dMS}(u_*)[\phi] = 0$ for all $\phi \in \mathcal{C}_c^\infty(\Omega)$, i.e.

$$\operatorname{div}(\nabla u_*) = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

(ii) *If, further, the energy convergence*

$$\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathrm{MS}(u_*) \tag{1.16}$$

is satisfied, then $\delta\mathrm{MS}(u_*) = 0$, i.e.

$$\int_\Omega (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \, \mathrm{d}x + \int_{\hat{J}_{u_*}} \operatorname{div}^{\hat{J}_{u_*}} X \, \mathrm{d}\mathcal{H}^{N-1}$$
$$= -2\int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X \cdot \nabla_\tau g) \, \mathrm{d}\mathcal{H}^{N-1} \tag{1.17}$$

for all vector fields $X \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$.

Remark 1.3. At this stage, it is still open whether or not u_* is a critical point of MS as we do not know whether the outer variation $\mathrm{dMS}(u_*)$ also vanishes on arbitrary functions $\phi \in \mathrm{SBV}^2(\Omega)$ satisfying $\hat{J}_\phi \subset \hat{J}_{u_*}$ (and not only on $\mathcal{C}_c^\infty(\Omega)$). In other words, the weak form of the homogeneous Neumann condition (1.13) on $\hat{J}_{u_*}$ remains to be established. This is the only missing ingredient to obtain that u_* is a critical point of MS.

An assumption of convergence of energies similar to (1.16) has been used in [25–27, 34] to prove that critical points of the Allen–Cahn functional (from phase transitions) converge towards critical points of the perimeter functional, hence to minimal surfaces. The analysis without this assumption was first carried out in [22], and it shows that critical points converge (in the sense of inner variations) towards integer multiplicity stationary varifolds, a measure-theoretic generalization of minimal surfaces allowing for multiplicities. Interfaces with multiplicities do appear as limits of critical points of the Allen–Cahn energy and cannot be excluded; see e.g. [22, Section 6.3]. In our context, a similar phenomenon may appear, so that assumption (1.16) is probably necessary.

In [25–27], convergence of energies is also used to pass to the limit in the second inner variation. Following the same path, (1.16) allows us to pass to the limit in the second inner variation of AT_ε. It shows that the second inner variations of AT_ε *do not* converge to the second inner variation of MS, but to the second inner variation plus a residual additional term. As a by-product, it follows that limits of stable critical points of AT_ε satisfy an "augmented" second-order minimality condition. Second-order minimality criteria for MS have been addressed in [7, 11]. We also note that the convergence of the second inner variation for the Allen–Cahn functional without the assumption of convergence of energies has been studied in [16]; see also [21]. Convergence of second inner variations is our third and last main result.

Theorem 1.3. *Assume that $\Omega \subset \mathbf{R}^N$ is a bounded open set of class $\mathcal{C}^{3,1}$ and $g \in \mathcal{C}^{3,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g(\Omega)$ be a family of critical points of the Ambrosio–Tortorelli functional and $u_* \in \mathrm{SBV}^2(\Omega) \cap L^\infty(\Omega)$ be as in Theorem 1.2, satisfying the convergence of energy (1.16). Then the following properties hold:*

(i) *For all $X \in \mathcal{C}^2_c(\mathbf{R}^N; \mathbf{R}^N)$ and all $G \in \mathcal{C}^3(\mathbf{R}^N)$ with $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$,*

$$\lim_{\varepsilon\to 0} \delta^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \delta^2 \mathrm{MS}(u_*)[X, G] + \int_{\hat{J}_{u*}} |DX : (\nu_{u_*} \otimes \nu_{u_*})|^2 \, \mathrm{d}\mathcal{H}^{N-1}.$$

(ii) *If $(u_\varepsilon, v_\varepsilon)$ is a stable critical point of AT_ε, i.e.*

$$\mathrm{d}^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[\phi, \psi] \geq 0 \quad \textit{for all } (\phi, \psi) \in [\mathcal{C}^\infty_c(\Omega)]^2,$$

then u_* satisfies the second-order inequality

$$\delta^2 \mathrm{MS}(u_*)[X, G] + \int_{\hat{J}_{u*}} |DX : (\nu_{u_*} \otimes \nu_{u_*})|^2 \, \mathrm{d}\mathcal{H}^{N-1} \geq 0 \tag{1.18}$$

for all $X \in \mathcal{C}^2_c(\mathbf{R}^N; \mathbf{R}^N)$ and all $G \in \mathcal{C}^3(\mathbf{R}^N)$ with $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$.

In the one-dimensional case, the asymptotic analysis as $\varepsilon \to 0$ of critical points of the Ambrosio–Tortorelli functional has already been carried out in [14, 24] for different sets of boundary conditions. In [14], a homogeneous Neumann boundary condition is assumed for the phase field variable v. The authors proved that if $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ is a family of critical points of the Ambrosio–Tortorelli functional satisfying (1.15), then, up to a subsequence, $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $[L^2(\Omega)]^2$ with $u \in \mathrm{SBV}^2(\Omega)$ that is either globally affine or piecewise constant with a finite number of jumps; see Remark 1.2. This result is extended in [24] to the Ambrosio–Tortorelli functional with a fidelity term. Note that our present analysis also applies in the presence of a fidelity term, but we do not consider this case here in order not to add useless difficulties. In a short note [5], we have also carried out the one-dimensional analysis in our setting, i.e. with the Dirichlet boundary condition on the v variable. In this case, we have established a convergence result for critical points without assuming the convergence of the energy (1.16), but proving (1.16) as a consequence of the energy bound (1.15). It allows us to exhibit nonminimizing critical points of AT_ε satisfying our energy convergence assumption (1.16) (see [5, Remark 1.2]).

1.4. Ideas of the convergence proof

The proof of Theorem 1.2 relies on the classical compactness argument and the lower bound inequality for the Ambrosio–Tortorelli functional. Indeed, the energy bound for a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset \mathcal{A}_g(\Omega)$ of critical points for AT_ε implies the $L^2(\Omega)$-convergence (up to a subsequence) of u_ε to a limit $u_* \in \mathrm{SBV}^2(\Omega)$, together with a Γ-liminf inequality $\mathrm{MS}(u_*) \leq \liminf_\varepsilon \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)$. Our energy convergence assumption (1.16) leads to the *equipartition* of phase field energy, as well as the convergence of the bulk energy. Then, as in [22], we associate an $(N-1)$-varifold V_ε to the phase field variable v_ε, which converges (again up to a subsequence) to a limiting varifold V_*. The energy convergence (1.16) allows us to identify the mass of V_*, that is, $\|V_*\| = \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$. Next we use the equations satisfied by $(u_\varepsilon, v_\varepsilon)$ in their conservative form to pass to the limit, and find an equation satisfied by u_* and V_*. The idea is then to employ a blow-up argument similar to [2] to identify (the first moment of) V_*, and show that it is the rectifiable varifold associated to $\hat{J}_{u_*}$ with multiplicity one.

To prove Theorem 1.3, we argue as in [25–27]. We observe that the convergence $V_\varepsilon \rightharpoonup V_*$ in the sense of varifolds and the identification of V_* implies the convergence of quadratic terms $\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \rightharpoonup \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ in the sense of measures. This information is precisely what is needed to pass to the limit in the second inner variation of AT_ε, and we infer from a stability condition on $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ a stability condition on the limit $u_* \in \mathrm{SBV}^2(\Omega)$.

The paper is organized as follows. Section 2 collects notation that will be used throughout the paper. In Section 3 we study the regularity theory for critical points of the Ambrosio–Tortorelli functional, proving first smoothness in the interior of the domain, and then smoothness at the boundary. In Section 4 we prove compactness of a family $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ satisfying a uniform energy bound $\sup_\varepsilon \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) < \infty$. The regularity result allows one to derive the conservative form of the equations satisfied by these critical points which itself provides bounds on the normal traces of u_ε and v_ε on $\partial\Omega$. Then, in Section 5, we improve the previous results by assuming the energy convergence $\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathrm{MS}(u_*)$. From this assumption we obtain equipartition of the phase field part of the energy. Then we employ a reformulation in terms of varifolds to pass to the limit in the inner variational equations satisfied by critical points of AT_ε to prove that the weak limit u_* of u_ε is a stationary point of the Mumford–Shah energy. The asymptotic behavior of the second inner variations is investigated in Section 6.

2. Notation and preliminaries

2.1. Measures

The Lebesgue measure in $\mathbf{R}^N$ is denoted by $\mathcal{L}^N$, and the k-dimensional Hausdorff measure by $\mathcal{H}^k$. We shall sometime write ω_k for the $\mathcal{L}^k$-measure of the k-dimensional unit ball in $\mathbf{R}^k$.

If $X \subset \mathbf{R}^N$ is a locally compact set and Y a Euclidean space, we denote by $\mathcal{M}(X;Y)$ the space of Y-valued bounded Radon measures in X endowed with the norm $\|\mu\| = |\mu|(X)$, where $|\mu|$ is the variation of the measure μ. If $Y = \mathbf{R}$, we simply write $\mathcal{M}(X)$ instead of $\mathcal{M}(X;\mathbf{R})$. By the Riesz representation theorem, $\mathcal{M}(X;Y)$ can be identified with the topological dual of $\mathcal{C}_0(X;Y)$, the space of continuous functions $f: X \to Y$ such that $\{|f| \geq \varepsilon\}$ is compact for all $\varepsilon > 0$. The weak* topology of $\mathcal{M}(X;Y)$ is defined using this duality.

2.2. Functional spaces

We use standard notation for Lebesgue, Sobolev, and Hölder spaces. Given a bounded open set $\Omega \subset \mathbf{R}^N$, the space of functions of bounded variation is defined by

$$\mathrm{BV}(\Omega) = \big\{u \in L^1(\Omega) : Du \in \mathcal{M}(\Omega;\mathbf{R}^N)\big\}.$$

We shall also consider the subspace $\mathrm{SBV}(\Omega)$ of special functions of bounded variation made of functions $u \in \mathrm{BV}(\Omega)$ whose distributional derivative can be decomposed as $Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$. In the previous expression, ∇u is the Radon–Nikodým derivative of Du with respect to $\mathcal{L}^N$, and it is called the approximate gradient of u. The Borel set J_u is the (approximate) jump set of u. It is a countably $\mathcal{H}^{N-1}$-rectifiable subset of Ω oriented by the (approximate) normal direction of jump $\nu_u : J_u \to \mathbf{S}^{N-1}$, and $u^\pm$ are the one-sided approximate limits of u on J_u according to ν_u. Finally we define

$$\mathrm{SBV}^2(\Omega) = \big\{u \in \mathrm{SBV}(\Omega) : \nabla u \in L^2(\Omega;\mathbf{R}^N) \text{ and } \mathcal{H}^{N-1}(J_u) < \infty\big\}.$$

2.3. Varifolds

Let us recall several basic ingredients of the theory of varifolds (see [43] for a detailed description). We denote by $\mathbf{G}_{N-1}$ the Grassmannian manifold of all $(N-1)$-dimensional linear subspaces of $\mathbf{R}^N$. The set $\mathbf{G}_{N-1}$ is as usual identified with the set of all orthogonal projection matrices onto $(N-1)$-dimensional linear subspaces of $\mathbf{R}^N$, i.e. $N \times N$ symmetric matrices A such that $A^2 = A$ and $\operatorname{tr}(A) = N-1$, in other words, matrices of the form $A = \mathrm{Id} - e \otimes e$ for some $e \in \mathbf{S}^{N-1}$.

An $(N-1)$-varifold in X (a locally compact subset of $\mathbf{R}^N$) is a bounded Radon measure on $X \times \mathbf{G}_{N-1}$. The class of $(N-1)$-varifolds in X is denoted by $\mathbf{V}_{N-1}(X)$. The mass of $V \in \mathbf{V}_{N-1}(X)$ is simply the measure $\|V\| \in \mathcal{M}(X)$ defined by $\|V\|(B) = V(B \times \mathbf{G}_{N-1})$ for all Borel sets $B \subset X$. We define the first variation of an $(N-1)$-varifold in V in an open set $U \subset \mathbf{R}^N$ by

$$\delta V(\varphi) = \int_{U \times \mathbf{G}_{N-1}} D\varphi(x) : A \, \mathrm{d}V(x, A) \quad \text{for all } \varphi \in \mathcal{C}^1_c(U; \mathbf{R}^N).$$

We say that an $(N-1)$-varifold is stationary in U if $\delta V(\varphi) = 0$ for all φ in $\mathcal{C}^1_c(U; \mathbf{R}^N)$. We recall that such a varifold satisfies the monotonicity formula

$$\frac{\|V\|(B_\varrho(x_0))}{\varrho^{N-1}} = \frac{\|V\|(B_r(x_0))}{r^{N-1}} + \int_{(B_\varrho(x_0) \setminus B_r(x_0)) \times \mathbf{G}_{N-1}} \frac{|P_{A^\perp}(x - x_0)|^2}{|x - x_0|^{N+1}} \, \mathrm{d}V(x, A)$$

for all $x_0 \in U$ and $0 < r < \varrho$ with $B_\varrho(x_0) \subset U$, where $P_{A^\perp}$ is the orthogonal projection onto the one-dimensional space $A^\perp$ (see [43, paragraph 40]).

2.4. Tangential divergence

Let Γ be a countably $\mathcal{H}^{N-1}$-rectifiable set and let $T_x\Gamma$ be its approximate tangent space defined for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$. We consider an orthonormal basis $\{\tau_1(x), \dots, \tau_{N-1}(x)\}$ of $T_x\Gamma$ and denote by $\nu(x)$ a normal vector to $T_x\Gamma$. If $\zeta \colon \mathbf{R}^N \to \mathbf{R}^N$ is a smooth vector field, we denote by

$$\operatorname{div}^\Gamma \zeta := \sum_{i=1}^{N-1} \tau_i \cdot \partial_{\tau_i} \zeta = (\mathrm{Id} - \nu \otimes \nu) : D\zeta$$

the tangential divergence, and $(\partial_{\tau_i} \zeta)^\perp = ((\partial_{\tau_i} \zeta) \cdot \nu)\nu = \partial_{\tau_i} \zeta - \sum_{j=1}^{N-1} (\tau_j \cdot \partial_{\tau_i} \zeta)\tau_j$.

3. Regularity theory for critical points of the Ambrosio–Tortorelli energy

In this section we investigate interior and boundary regularity properties of critical points of the Ambrosio–Tortorelli functional AT_ε for a parameter $\varepsilon > 0$ which is kept fixed.

3.1. Interior regularity

We first establish interior regularity following ideas used by Rivière [41] to prove the regularity of harmonic maps with values into a revolution torus.

Theorem 3.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set. If $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ satisfies (1.3)–(1.4), then $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^\infty(\Omega)]^2$.*

Proof. For simplicity, we drop the subscript ε in $(u_\varepsilon, v_\varepsilon)$ and instead write (u, v). We also assume $N \geq 2$ since, in the case $N = 1$, the regularity of (u, v) a solution of (1.3)–(1.4) is elementary.

By (1.3), u weakly solves

$$-\operatorname{div}((\eta_\varepsilon + v^2)\nabla u) = 0 \quad \text{in } \Omega. \tag{3.1}$$

Setting $M := \|v\|_{L^\infty(\Omega)}$, the matrix field $(\eta_\varepsilon + v^2)\mathrm{Id}$ has bounded measurable coefficients and it satisfies $\eta_\varepsilon \mathrm{Id} \leq (\eta_\varepsilon + v^2)\mathrm{Id} \leq (\eta_\varepsilon + M^2)\mathrm{Id}$ a.e. in Ω in the sense of quadratic forms. It is therefore uniformly elliptic and the De Giorgi–Nash–Moser regularity theorem applies to equation (3.1). It provides the existence of $\alpha \in (0, 1)$ such that $u \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$, together with the estimate

$$K(\omega) := \sup_{\substack{x_0 \in \omega,\, \varrho > 0 \\ B_\varrho(x_0) \subset \omega}} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} |\nabla u|^2 \, \mathrm{d}x < \infty \tag{3.2}$$

for every open subset ω such that $\bar{\omega} \subset \Omega$ (see e.g. [18, Theorem 8.13 and Eq. (8.18)]).

Now we claim that the function v belongs to $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$. Before proving this claim, we complete the proof of the theorem. Assuming the claim to be true, we can use the Schauder estimates (see e.g. [18, Theorem 5.19]) to derive from equation (3.1) that $u \in \mathcal{C}^{1,\alpha}_{\mathrm{loc}}(\Omega)$. On the other hand, by (1.4), v weakly solves

$$-\varepsilon \Delta v = \frac{1 - v}{4\varepsilon} - |\nabla u|^2 v \quad \text{in } \Omega. \tag{3.3}$$

Since the right-hand-side of (3.3) belongs to $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$, it follows from standard Schauder estimates that $v \in \mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\Omega)$. By a classical bootstrap, it now follows from equations (3.1) and (3.3) that both u and v are of class $\mathcal{C}^\infty$ in Ω.

Hence, it only remains to show the claim $v \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$. To this purpose, we fix an arbitrary ball $\bar{B}_{2R}(x_0) \subset \Omega$, and we aim to prove that $v \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$. Consider $v_1 \in H^1(B_{2R}(x_0))$ to be the unique weak solution of

$$\begin{cases} -\Delta v_1 = \dfrac{1 - v}{4\varepsilon^2} & \text{in } B_{2R}(x_0), \\ v_1 = v & \text{on } \partial B_{2R}(x_0). \end{cases} \tag{3.4}$$

Since $\Delta v_1 \in L^\infty(B_{2R}(x_0))$, the Calderón–Zygmund estimates yield $v_1 \in W^{2,p}_{\mathrm{loc}}(B_{2R}(x_0))$ for every $p < \infty$. By Sobolev embedding, it follows that $v_1 \in C^{1,\beta}_{\mathrm{loc}}(B_{2R}(x_0))$ for every $\beta \in (0, 1)$. In particular, we have $v_1 \in L^\infty(B_R(x_0))$.

Set $v_2 := v - v_1 \in H^1_0(B_{2R}(x_0))$. By (3.3) and (3.4), v_2 is a weak solution of

$$-\Delta v_2 = -\frac{1}{\varepsilon} |\nabla u|^2 v \quad \text{in } B_{2R}(x_0). \tag{3.5}$$

To show that $v_2 \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$, the Morrey–Campanato theorem (see e.g. [18, Theorem 5.7]) ensures that it suffices to prove the following Morrey-type estimate:

$$\sup_{\substack{y\in B_R(x_0)\\ \varrho\in(0,R)}} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(y)} |\nabla v_2|^2 \,\mathrm{d}x < \infty. \tag{3.6}$$

Let $y \in B_R(x_0)$ and $r \in (0, R)$ be arbitrary. We denote by $w \in v_2 + H^1_0(B_r(y))$ the harmonic extension of v_2 in the ball $B_r(y)$, i.e. the unique (weak) solution of

$$\begin{cases} -\Delta w = 0 & \text{in } B_r(y),\\ w = v_2 & \text{on } \partial B_r(y). \end{cases} \tag{3.7}$$

Since $v_2 = v - v_1 \in L^\infty(B_R(x_0))$, we have $|w| \le \|v_2\|_{L^\infty(B_R(x_0))}$ on $\partial B_r(y)$, and the weak maximum principle implies $w \in L^\infty(B_r(y))$ with $\|w\|_{L^\infty(B_r(y))} \le \|v_2\|_{L^\infty(B_R(x_0))}$. Moreover, $|\nabla w|^2$ being subharmonic in $B_r(y)$, we get that for every $\varrho < r$,

$$\int_{B_\varrho(y)} |\nabla w|^2 \,\mathrm{d}x \le \left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla w|^2 \,\mathrm{d}x.$$

Recalling that w also minimizes the Dirichlet integral among all functions agreeing with v_2 on $\partial B_r(y)$, we infer that

$$\begin{aligned}
\int_{B_\varrho(y)} |\nabla v_2|^2 \,\mathrm{d}x &\le 2\int_{B_\varrho(y)} |\nabla w|^2 \,\mathrm{d}x + 2\int_{B_\varrho(y)} |\nabla (w - v_2)|^2 \,\mathrm{d}x\\
&\le 2\left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla w|^2 \,\mathrm{d}x + 2\int_{B_r(y)} |\nabla (w - v_2)|^2 \,\mathrm{d}x\\
&\le 2\left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla v_2|^2 \,\mathrm{d}x + 2\int_{B_r(y)} |\nabla (w - v_2)|^2 \,\mathrm{d}x
\end{aligned}$$

for every $\varrho < r$. Since $w - v_2 = 0$ on $\partial B_r(y)$, (3.5) and (3.7) lead to

$$\begin{aligned}
\int_{B_r(y)} |\nabla (w - v_2)|^2 \,\mathrm{d}x &= \frac{1}{\varepsilon}\int_{B_r(y)} |\nabla u|^2 v (w - v_2) \,\mathrm{d}x\\
&\le \frac{2}{\varepsilon}\|v\|_{L^\infty(\Omega)} \|v_2\|_{L^\infty(B_R(x_0))} \int_{B_r(y)} |\nabla u|^2 \,\mathrm{d}x.
\end{aligned}$$

In view of (3.2), we have thus proved that for every $y \in B_R(x_0)$ and $0 < \varrho \le r < R$,

$$\int_{B_\varrho(y)} |\nabla v_2|^2 \,\mathrm{d}x \le 2\left(\frac{\varrho}{r}\right)^N \int_{B_r(y)} |\nabla v_2|^2 \,\mathrm{d}x + C_1 r^{N-2+2\alpha},$$

with

$$C_1 := \frac{4}{\varepsilon}\|v\|_{L^\infty(\Omega)} \|v_2\|_{L^\infty(B_R(x_0))} K(B_{2R}(x_0)).$$

By using a classical iteration lemma (see e.g. [18, Lemma 5.13]), we infer that for every $y \in B_R(x_0)$ and $0 < \varrho < R$,

$$\int_{B_\varrho(y)} |\nabla v_2|^2 \, dx \le C_\alpha \varrho^{N-2+2\alpha}\left(\frac{1}{R^{N-2+2\alpha}} \int_{B_{2R}(x_0)} |\nabla v_2|^2 \, dx + C_1\right),$$

for a constant C_α depending only on α and N. Hence v_2 satisfies the Morrey estimate (3.6), and thus $v_2 \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$. In turn, $v = v_1 + v_2 \in \mathcal{C}^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$ and the proof of the claim is complete. ■

Remark 3.1. Note that, in dimension $N = 2$, an alternative and simpler proof can be obtained using Meyers's regularity result [36]. Indeed, the first equation of (1.2) for u_ε is a standard elliptic equation in divergence form for u_ε with measurable coefficients. Thus Meyers's regularity result applies, and ensures the existence of $p > 1$ (possibly depending on ε) such that $\nabla u_\varepsilon \in L^{2p}_{\mathrm{loc}}(\Omega)$. Inserting this information into the second equation of (1.2) for v_ε and appealing to the Calderon–Zygmund estimates yields $v_\varepsilon \in W^{2,p}_{\mathrm{loc}}(\Omega)$. Then the Sobolev embedding implies that ∇v_ε is in $L^q_{\mathrm{loc}}(\Omega)$ for every $1 \le q \le Np/(N-p)$. One can check that when $N = 2$, if $p > 1$ we have automatically that $2p/(2-p) > 2$. Hence, using Sobolev–Morrey embedding, we find that v_ε is actually Hölder continuous. A bootstrap argument based on the Schauder estimates yields the interior regularity result stated in Theorem 3.1. Unfortunately, this argument fails in dimensions higher than 2 since in this case it is not true that $Np/(N-p)$ is strictly larger than N for $p > 1$. This is in contrast with our alternative proof above which is independent of the dimension.

3.2. Maximum principle and boundary regularity

We first show a (standard) maximum principle which says that v_ε takes values between 0 and 1, and that u_ε is bounded whenever the boundary condition g is.

Lemma 3.1 (Maximum principle). *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ satisfy (1.3)–(1.4). If $v_\varepsilon = 1$ on $\partial\Omega$, then $0 \le v_\varepsilon \le 1$ a.e. in Ω. In addition, if $u_\varepsilon = g$ on $\partial\Omega$ for a function $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$, then $u_\varepsilon \in L^\infty(\Omega)$ and $\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$.*

Proof. For a generic function $f \in L^1(\Omega)$, we set $f^+ := (f + |f|)/2$ and $f^- := (|f| - f)/2$. For simplicity, we drop the subscript ε in $(u_\varepsilon, v_\varepsilon)$ and instead write (u, v).

Since $v - 1 \in H^1_0(\Omega) \cap L^\infty(\Omega)$, it follows that $(v-1)^+ \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $\nabla(v-1)^+ = \nabla v \mathbf{1}_{\{v \ge 1\}}$. A classical argument using $(v-1)^+$ as a test function in (1.4) leads to $v \le 1$ a.e. in Ω. Next, since $v = 1$ on $\partial\Omega$, we have $-v^- \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and the same argument with $-v^-$ as a test function in (1.4) shows that $-v = 0$ a.e. in Ω, that is, $v \ge 0$ a.e. in Ω.

Now we assume that $u = g$ on $\partial\Omega$ with $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$ and we set $M := \|g\|_{L^\infty(\partial\Omega)}$. Since $|g| \le M$ on $\partial\Omega$, we have $(u - M)^+ \in H^1_0(\Omega)$ and $\nabla(u - M)^+ = \nabla u \mathbf{1}_{\{u \ge M\}}$. Using $(u - M)^+$ as a test function in (1.3) yields $\nabla(u - M)^+ = 0$ a.e. in Ω

which implies that $(u-M)^+$ is constant. Since $(u-M)^+ \in H^1_0(\Omega)$, it follows that $(u-M)^+ = 0$ a.e. in Ω, that is, $u \le M$ a.e. in Ω. The same argument applied to $(u+M)^- \in H^1_0(\Omega)$ shows that $u \ge -M$ a.e. in Ω, and thus $\|u\|_{L^\infty(\Omega)} \le M$. ∎

Next we study the boundary regularity of a critical point $(u_\varepsilon, v_\varepsilon)$ of the Ambrosio–Tortorelli energy. Our strategy is to use a global reflexion argument to extend $(u_\varepsilon, v_\varepsilon)$ across the boundary. The extension will then satisfy a modified system of PDEs for which we can apply an interior regularity result (similar to that of Theorem 3.1). The reflexion argument originates in [42] and we follow the arguments in [12]. In contrast with the classical method consisting in locally flattening the boundary, this method has the advantage of providing a global extension through the construction of a diffeomorphism mapping a small internal neighborhood of $\partial\Omega$ onto an external neighborhood of the boundary.

Theorem 3.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set whose boundary is of class $\mathcal{C}^{k\vee 2,1}$ and $g \in \mathcal{C}^{k,\alpha}(\partial\Omega)$ for some $k \ge 1$ and $\alpha \in (0,1)$. If $(u_\varepsilon, v_\varepsilon) \in (g,1) + H^1_0(\Omega) \times [H^1_0(\Omega) \cap L^\infty(\Omega)]$ satisfies (1.3)–(1.4), then $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{k,\alpha}(\overline{\Omega}) \times \mathcal{C}^{k\vee 2,\alpha}(\overline{\Omega})$.*

Proof. We start by describing the reflexion method that we use to extend functions across $\partial\Omega$. Since $\partial\Omega$ is (at least) of class $\mathcal{C}^{2,1}$, we can find a small $\delta_0 > 0$ such that the nearest point projection on $\partial\Omega$, denoted by $\boldsymbol{\pi}_\Omega$, is well defined and (at least) of class $\mathcal{C}^{1,1}$ in a tubular neighborhood of size $2\delta_0$ of $\partial\Omega$. For $\delta \in (0, 2\delta_0]$, we set

$$\begin{cases} U_\delta := \{x \in \mathbf{R}^N : \operatorname{dist}(x, \partial\Omega) < \delta\}, \\ U^{\text{in}}_\delta := \Omega \cap U_\delta, \\ U^{\text{ex}}_\delta := U_\delta \setminus \overline{\Omega}. \end{cases}$$

The *geodesic reflexion* across $\partial\Omega \cap U_{2\delta_0}$ is denoted by $\boldsymbol{\sigma}_\Omega : U_{2\delta_0} \to \mathbb{R}^N$ and it is defined by

$$\boldsymbol{\sigma}_\Omega(x) := 2\boldsymbol{\pi}_\Omega(x) - x \quad \text{for all } x \in U_{2\delta_0}.$$

The mapping $\boldsymbol{\sigma}_\Omega$ is an involutive $\mathcal{C}^{1,1}$-diffeomorphism (onto its image), which satisfies $\boldsymbol{\sigma}_\Omega(x) = x$ for all $x \in \partial\Omega \cap U_{2\delta_0}$. Reducing the value of δ_0 if necessary, we have

$$\boldsymbol{\sigma}_\Omega(U^{\text{ex}}_\delta) = U^{\text{in}}_\delta \text{ and } \boldsymbol{\sigma}_\Omega(U^{\text{in}}_\delta) = U^{\text{ex}}_\delta \quad \text{for } \delta \in (0, 2\delta_0).$$

Next we consider the bounded open set

$$\widetilde{\Omega} := \Omega \cup U_{\delta_0} = \overline{\Omega} \cup U^{\text{ex}}_{\delta_0}. \tag{3.8}$$

Differentiating the relation $\boldsymbol{\sigma}_\Omega(\boldsymbol{\sigma}_\Omega(x)) = x$, for $x \in U_{\delta_0}$, yields

$$D\boldsymbol{\sigma}_\Omega(x) D\boldsymbol{\sigma}_\Omega(\boldsymbol{\sigma}_\Omega(x)) = \mathrm{Id},$$

and thus

$$D\boldsymbol{\sigma}_\Omega(\boldsymbol{\sigma}_\Omega(x)) = (D\boldsymbol{\sigma}_\Omega(x))^{-1} \quad \text{for every } x \in U_{\delta_0}. \tag{3.9}$$

For $x \in \partial\Omega \cap \widetilde{\Omega}$, one has $(D\boldsymbol{\sigma}_\Omega(x))^\mathsf{T} v = 2\mathbf{p}_x(v) - v$ for all $v \in \mathbf{R}^N$, where $\mathbf{p}_x$ is the orthogonal projection from $\mathbf{R}^N$ onto the tangent space $T_x(\partial\Omega)$ to $\partial\Omega$ at x, i.e. $(D\boldsymbol{\sigma}_\Omega(x))^\mathsf{T}$ is the reflexion matrix across the hyperplane $T_x(\partial\Omega)$. In particular,

$$(D\boldsymbol{\sigma}_\Omega(x))^\mathsf{T} D\boldsymbol{\sigma}_\Omega(x) = (D\boldsymbol{\sigma}_\Omega(x))^\mathsf{T}(D\boldsymbol{\sigma}_\Omega(x))^\mathsf{T} = \mathrm{Id} \quad \text{for every } x \in \partial\Omega \cap \widetilde{\Omega}. \tag{3.10}$$

Now, for $x \in \widetilde{\Omega}$, we define

$$j(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ |\det D\boldsymbol{\sigma}_\Omega(x)| & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases}$$

and

$$A(x) := \begin{cases} \mathrm{Id} & \text{if } x \in \Omega, \\ j(x)[D\boldsymbol{\sigma}_\Omega(\boldsymbol{\sigma}_\Omega(x))]^\mathsf{T} D\boldsymbol{\sigma}_\Omega(\boldsymbol{\sigma}_\Omega(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega. \end{cases}$$

In view of (3.10), j and A are Lipschitz continuous in $\widetilde{\Omega}$ and A is uniformly elliptic, i.e. there exist two constants $0 < \lambda_\Omega \le \Lambda_\Omega$ such that

$$\lambda_\Omega |\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda_\Omega |\xi|^2 \quad \text{for every } (x, \xi) \in \widetilde{\Omega} \times \mathbf{R}^N.$$

With these geometrical preliminaries, we are now ready to provide the extension of $(u_\varepsilon, v_\varepsilon)$ to $\widetilde{\Omega}$. We define, for $x \in \widetilde{\Omega}$,

$$\begin{aligned} \hat{u}_\varepsilon(x) &:= \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega, \\ u_\varepsilon(\boldsymbol{\sigma}_\Omega(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \\ \hat{v}_\varepsilon(x) &:= \begin{cases} v_\varepsilon(x) & \text{if } x \in \Omega, \\ v_\varepsilon(\boldsymbol{\sigma}_\Omega(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \tilde{u}_\varepsilon(x) &:= \begin{cases} u_\varepsilon(x) & \text{if } x \in \Omega, \\ 2g(\boldsymbol{\pi}_\Omega(x)) - u_\varepsilon(\boldsymbol{\sigma}_\Omega(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \\ \tilde{v}_\varepsilon(x) &:= \begin{cases} v_\varepsilon(x) & \text{if } x \in \Omega, \\ 2 - v_\varepsilon(\boldsymbol{\sigma}_\Omega(x)) & \text{if } x \in \widetilde{\Omega} \setminus \Omega, \end{cases} \end{aligned} \tag{3.12}$$

By the chain rule in Sobolev spaces and the fact that the traces of these functions coincide on both sides of $\partial\Omega \cap \widetilde{\Omega}$, each one of them belongs to $H^1(\widetilde{\Omega})$. In addition, $\hat{v}_\varepsilon$ and $\tilde{v}_\varepsilon$ also belong to $L^\infty(\widetilde{\Omega})$ since $v_\varepsilon \in L^\infty(\Omega)$. We finally set

$$\tilde{g} := g \circ \boldsymbol{\pi}_\Omega \in \mathcal{C}^{1,\alpha}(\widetilde{\Omega}).$$

Now we show that these extensions satisfy suitable equations in the domain $\widetilde{\Omega}$.

Lemma 3.2. *We have*

$$-\operatorname{div}((\eta_\varepsilon + \hat{v}_\varepsilon^2)A\nabla\tilde{u}_\varepsilon) = -2\operatorname{div}(\mathbf{1}_{\widetilde{\Omega}\setminus\overline{\Omega}}(\eta_\varepsilon + \hat{v}_\varepsilon^2)A\nabla\tilde{g}) \quad \text{in } \mathcal{D}'(\widetilde{\Omega}) \tag{3.13}$$

and

$$-\varepsilon\operatorname{div}(A\nabla\tilde{v}_\varepsilon) = (\mathbf{1}_{\widetilde{\Omega}\cap\Omega} - \mathbf{1}_{\widetilde{\Omega}\setminus\overline{\Omega}})\Big(\frac{j}{4\varepsilon}(1-\hat{v}_\varepsilon) - (A\nabla\hat{u}_\varepsilon\cdot\nabla\hat{u}_\varepsilon)\hat{v}_\varepsilon\Big) \quad \text{in } \mathcal{D}'(\widetilde{\Omega}). \tag{3.14}$$

Proof. Again, for simplicity, we drop the subscript ε. We fix an arbitrary test function $\varphi \in \mathcal{D}(\widetilde{\Omega})$. Since equations (3.13) and (3.14) are clearly satisfied in Ω, there is no loss of generality in supposing that $\operatorname{supp}(\varphi) \subset U_{\delta_0}$. We define for $x \in U_{\delta_0}$ the *symmetric* and *anti-symmetric* parts of φ,

$$\varphi^s(x) := \frac{1}{2}(\varphi(x) + \varphi\circ\boldsymbol{\sigma}_\Omega(x)), \quad \varphi^a(x) := \frac{1}{2}(\varphi(x) - \varphi\circ\boldsymbol{\sigma}_\Omega(x)).$$

The functions φ^s and φ^a belong to $\mathcal{C}^{1,1}(U_{\delta_0})$ and, by construction, $\varphi^s\circ\boldsymbol{\sigma}_\Omega = \varphi^s$ and $\varphi^a\circ\boldsymbol{\sigma}_\Omega = -\varphi^a$.

Step 1: Proof of (3.13). We start with the identity

$$\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{u})\cdot\nabla\varphi^s\,\mathrm{d}x = -\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla(u\circ\boldsymbol{\sigma}_\Omega))\cdot\nabla(\varphi^s\circ\boldsymbol{\sigma}_\Omega)\,\mathrm{d}x + 2\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{g})\cdot\nabla\varphi^s\,\mathrm{d}x. \tag{3.15}$$

Using relation (3.9) and changing variables yields

$$\begin{aligned}\int_{\widetilde{\Omega}\setminus\overline{\Omega}} &(\eta_\varepsilon + \hat{v}^2)(A\nabla(u\circ\boldsymbol{\sigma}_\Omega))\cdot\nabla(\varphi^s\circ\boldsymbol{\sigma}_\Omega)\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)\nabla u(\boldsymbol{\sigma}_\Omega)\cdot\nabla\varphi^s(\boldsymbol{\sigma}_\Omega)j\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}\cap\Omega} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{u})\cdot\nabla\varphi^s\,\mathrm{d}x.\end{aligned} \tag{3.16}$$

Therefore, combining (3.15) and (3.16) yields

$$\int_{\widetilde{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{u})\cdot\nabla\varphi^s\,\mathrm{d}x = 2\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{g})\cdot\nabla\varphi^s\,\mathrm{d}x.$$

In the same way, we have

$$\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{u})\cdot\nabla\varphi^a\,\mathrm{d}x = \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla(u\circ\boldsymbol{\sigma}_\Omega))\cdot\nabla(\varphi^a\circ\boldsymbol{\sigma}_\Omega)\,\mathrm{d}x + 2\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{g})\cdot\nabla\varphi^a\,\mathrm{d}x.$$

Since $\varphi^a = 0$ on $\widetilde{\Omega} \cap \partial\Omega$, we have $\varphi^a \in H^1_0(\Omega)$. Hence, we can use the first equation in (1.2) to infer that

$$\begin{aligned}\int_{\widetilde{\Omega}\setminus\overline{\Omega}} &(\eta_\varepsilon + \hat{v}^2)(A\nabla(u \circ \boldsymbol{\sigma}_\Omega)) \cdot \nabla(\varphi^a \circ \boldsymbol{\sigma}_\Omega)\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)\nabla u(\boldsymbol{\sigma}_\Omega) \cdot \nabla\varphi^a(\boldsymbol{\sigma}_\Omega)\, j\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}\cap\Omega} (\eta_\varepsilon + v^2)\nabla u \cdot \nabla\varphi^a\,\mathrm{d}x = 0.\end{aligned}$$

Consequently,

$$\int_{\widetilde{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{u}) \cdot \nabla\varphi\,\mathrm{d}x = 2\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (\eta_\varepsilon + \hat{v}^2)(A\nabla\tilde{g}) \cdot \nabla\varphi\,\mathrm{d}x,$$

and (3.13) follows.

Step 2: Proof of (3.14). We proceed as above, starting with

$$\begin{aligned}\varepsilon\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (A\nabla\tilde{v}) \cdot \nabla\varphi^s\,\mathrm{d}x &= -\varepsilon\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (A\nabla(v \circ \boldsymbol{\sigma}_\Omega)) \cdot \nabla(\varphi^s \circ \boldsymbol{\sigma}_\Omega)\,\mathrm{d}x \\ &= -\varepsilon\int_{\widetilde{\Omega}\cap\Omega} (A\nabla\tilde{v}) \cdot \nabla\varphi^s\,\mathrm{d}x,\end{aligned}$$

which yields

$$\varepsilon\int_{\widetilde{\Omega}} (A\nabla\tilde{v}) \cdot \nabla\varphi^s\,\mathrm{d}x = 0. \tag{3.17}$$

On the other hand,

$$\begin{aligned}\varepsilon\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (A\nabla\tilde{v}) \cdot \nabla\varphi^a\,\mathrm{d}x &= \varepsilon\int_{\widetilde{\Omega}\setminus\overline{\Omega}} (A\nabla(v \circ \boldsymbol{\sigma}_\Omega)) \cdot \nabla(\varphi^a \circ \boldsymbol{\sigma}_\Omega)\,\mathrm{d}x \\ &= \varepsilon\int_{\widetilde{\Omega}\cap\Omega} \nabla v \cdot \nabla\varphi^a\,\mathrm{d}x.\end{aligned} \tag{3.18}$$

Since $\varphi^a \in H^1_0(\Omega)$, we can apply the second equation in (1.2) to deduce that

$$\varepsilon\int_{\widetilde{\Omega}\cap\Omega} \nabla v \cdot \nabla\varphi^a\,\mathrm{d}x = -\int_{\widetilde{\Omega}\cap\Omega} |\nabla u|^2 v\varphi^a\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_{\widetilde{\Omega}\cap\Omega} (1-v)\varphi^a\,\mathrm{d}x. \tag{3.19}$$

Summing (3.17), (3.18), and (3.19), and using that $\boldsymbol{\sigma}_\Omega$ is an involution leads to

$$\begin{aligned}\varepsilon\int_{\widetilde{\Omega}} (A\nabla\tilde{v}) \cdot \nabla\varphi\,\mathrm{d}x = &-\int_{\widetilde{\Omega}\cap\Omega} |\nabla\hat{u}|^2\hat{v}\varphi\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_{\widetilde{\Omega}\cap\Omega} (1-\hat{v})\varphi\,\mathrm{d}x \\ &+ \int_{\widetilde{\Omega}\cap\Omega} |\nabla\hat{u}|^2\hat{v}\varphi \circ \boldsymbol{\sigma}_\Omega\,\mathrm{d}x - \frac{1}{4\varepsilon}\int_{\widetilde{\Omega}\cap\Omega} (1-\hat{v})\varphi \circ \boldsymbol{\sigma}_\Omega\,\mathrm{d}x.\end{aligned}$$

Changing variables in the two last integrals, we obtain

$$\varepsilon \int_{\widetilde{\Omega}} (A\nabla \tilde{v}) \cdot \nabla \varphi \, dx = -\int_{\widetilde{\Omega}} (\mathbf{1}_{\widetilde{\Omega}\cap\Omega} - \mathbf{1}_{\widetilde{\Omega}\setminus\overline{\Omega}})((A\nabla \hat{u}) \cdot \nabla \hat{u})\hat{v}\varphi \, dx + \frac{1}{4\varepsilon} \int_{\widetilde{\Omega}} (\mathbf{1}_{\widetilde{\Omega}\cap\Omega} - \mathbf{1}_{\widetilde{\Omega}\setminus\overline{\Omega}})(1-\hat{v})\varphi j \, dx,$$

and (3.14) follows. ■

We now provide a general regularity result generalizing the argument used in the proof of the interior regularity.

Lemma 3.3. *Let $A\colon \widetilde{\Omega} \to \mathbf{M}^{N\times N}_{\mathrm{sym}}$ be a Lipschitz field of symmetric $N \times N$ matrices which is uniformly elliptic (i.e. there exist $0 < \lambda < \Lambda$ such that $\lambda|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2$ for all $(x,\xi) \in \widetilde{\Omega} \times \mathbf{R}^N$) and $f \in L^1(\widetilde{\Omega})$ satisfy*

$$\sup_{B_\varrho(x_0)\subset\widetilde{\Omega}} \frac{1}{\varrho^{N-2+\gamma}} \int_{B_\varrho(x_0)} |f| \, dx < \infty, \tag{3.20}$$

for some $\gamma \in (0,2]$. If $z \in H^1(\widetilde{\Omega}) \cap L^\infty(\widetilde{\Omega})$ solves

$$-\operatorname{div}(A\nabla z) = f \quad \text{in } \mathcal{D}'(\widetilde{\Omega}), \tag{3.21}$$

then for every open set $\omega \subset\subset \widetilde{\Omega}$,

$$\sup_{\substack{x_0\in\overline{\omega}\\ B_\varrho(x_0)\subset\subset\widetilde{\Omega}}} \frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} |\nabla z|^2 \, dx < \infty, \tag{3.22}$$

and $z \in \mathcal{C}^{0,\alpha}(\overline{\omega})$ for every $\alpha \in (0,\gamma/2)$.

Proof. Throughout the proof, we fix an exponent $\alpha \in (0,\gamma/2)$ and we set $\beta := \gamma - 2\alpha > 0$. We also denote by K an upper bound for $\|z\|_{L^\infty(\widetilde{\Omega})}$, and by M an upper bound for (3.20). Then $C > 0$ will stand for a constant (which may vary from line to line) depending only on N, α, γ, λ, Λ, K, M, and the Lipschitz constant of A.

Let $\omega \subset\subset \widetilde{\Omega}$. Fix $x_0 \in \overline{\omega}$ and $\varrho > 0$ such that $B_\varrho(x_0) \subset\subset \widetilde{\Omega}$. Consider $w \in H^1(B_\varrho(x_0))$ the unique (weak) solution of

$$\begin{cases} -\operatorname{div}(A(x_0)\nabla w) = 0 & \text{in } B_\varrho(x_0), \\ w = z & \text{on } \partial B_\varrho(x_0). \end{cases} \tag{3.23}$$

Recalling that

$$\int_{B_\varrho(x_0)} A(x_0)\nabla w \cdot \nabla w \, dx \le \int_{B_\varrho(x_0)} A(x_0)\nabla \overline{w} \cdot \nabla \overline{w} \, dx \quad \text{for every } \overline{w} \in w + H^1_0(B_\varrho(x_0)),$$

we have

$$\lambda \int_{B_\varrho(x_0)} |\nabla w|^2 \, \mathrm{d}x \le \int_{B_\varrho(x_0)} A(x_0)\nabla w \cdot \nabla w \, \mathrm{d}x \le \int_{B_\varrho(x_0)} A(x_0)\nabla z \cdot \nabla z \, \mathrm{d}x \le \Lambda \int_{B_\varrho(x_0)} |\nabla z|^2 \, \mathrm{d}x. \tag{3.24}$$

Moreover, according to the maximum principle, $\|w\|_{L^\infty(B_\varrho(x_0))} \le \|z\|_{L^\infty(B_\varrho(x_0))} \le K$.

First, we infer from the triangle inequality,

$$\begin{aligned} &\left(\frac{1}{(\varrho/2)^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A\nabla z \cdot \nabla z \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad \le \left(\frac{2^{N-2+2\alpha}}{\varrho^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A\nabla w \cdot \nabla w \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\qquad + \left(\frac{2^{N-2+2\alpha}}{\varrho^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A(\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad \le \left(\frac{2^{N-2+2\alpha}}{\varrho^{N-2+2\alpha}} \int_{B_{\frac{\varrho}{2}}(x_0)} A\nabla w \cdot \nabla w \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\qquad + C\left(\frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} A(\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, \mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{3.25}$$

We start by estimating the first term in the right-hand side of (3.25) using (3.24), and the fact that A is Lipschitz continuous and uniformly elliptic. It yields

$$\begin{aligned} \int_{B_{\frac{\varrho}{2}}(x_0)} A\nabla w \cdot \nabla w \, \mathrm{d}x &\le (1 + C\varrho) \int_{B_\varrho(x_0)} A(x_0)\nabla w \cdot \nabla w \, \mathrm{d}x \\ &\le (1 + C\varrho) \int_{B_\varrho(x_0)} A(x_0)\nabla z \cdot \nabla z \, \mathrm{d}x \\ &\le (1 + C\varrho) \int_{B_\varrho(x_0)} A\nabla z \cdot \nabla z \, \mathrm{d}x. \end{aligned} \tag{3.26}$$

To estimate the second term in the right-hand side of (3.25), we make use of equation (3.21) to write

$$\begin{aligned} &\int_{B_\varrho(x_0)} A(\nabla z - \nabla w) \cdot (\nabla z - \nabla w) \, \mathrm{d}x \\ &\quad = \int_{B_\varrho(x_0)} A\nabla z \cdot \nabla(z - w) \, \mathrm{d}x - \int_{B_\varrho(x_0)} A\nabla w \cdot \nabla(z - w) \, \mathrm{d}x \\ &\quad = \int_{B_\varrho(x_0)} f(z - w) \, \mathrm{d}x - \int_{B_\varrho(x_0)} A\nabla w \cdot \nabla(z - w) \, \mathrm{d}x. \end{aligned}$$

Using assumption (3.20) on f, we infer that

$$\left|\frac{1}{\varrho^{N-2+2\alpha}}\int_{B_\varrho(x_0)} f(z-w)\,\mathrm{d}x\right| \le 2KM\varrho^\beta. \tag{3.27}$$

On the other hand, equation (3.23) satisfied by w implies that

$$\begin{aligned}\int_{B_\varrho(x_0)} A\nabla w\cdot\nabla(z-w)\,\mathrm{d}x &= \int_{B_\varrho(x_0)} (A-A(x_0))\nabla w\cdot\nabla(z-w)\,\mathrm{d}x\\ &\le C\varrho\int_{B_\varrho(x_0)}[|\nabla w|^2+|\nabla z|^2]\,\mathrm{d}x\\ &\le C\varrho\int_{B_\varrho(x_0)}|\nabla z|^2\,\mathrm{d}x\\ &\le C\varrho\int_{B_\varrho(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x,\end{aligned} \tag{3.28}$$

where we used (3.24) again, together with the ellipticity of A.

Gathering (3.25), (3.26), (3.27), and (3.28), we get

$$\begin{aligned}&\left(\frac{1}{(\varrho/2)^{N-2+2\alpha}}\int_{B_{\frac{\varrho}{2}}(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x\right)^{1/2}\\ &\qquad\le (1+C\sqrt{\varrho})\left(\frac{1}{\varrho^{N-2+2\alpha}}\int_{B_\varrho(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x\right)^{1/2} C\varrho^{\beta/2}.\end{aligned}$$

We now choose $\varrho=\varrho_k=2^{-(k+1)}$ for $k\in\mathbf{N}$ large enough, and we obtain

$$\begin{aligned}&\left(\frac{1}{\varrho_{k+1}^{N-2+2\alpha}}\int_{B_{\varrho_{k+1}}(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x\right)^{1/2}\\ &\qquad\le (1+C\,2^{-(k+1)/2})\left(\frac{1}{\varrho_k^{N-2+2\alpha}}\int_{B_{\varrho_k}(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x\right)^{1/2} + C2^{-\beta(k+1)/2}.\end{aligned}$$

Next we observe that if $(\theta_k)_{k\in\mathbf{N}}$, $(\sigma_k)_{k\in\mathbf{N}}$, and $(y_k)_{k\in\mathbf{N}}$ are real sequences such that $\theta_k\in(1,\infty)$, $\theta:=\prod_{k=0}^\infty\theta_k<\infty$, $\sigma_k\in(0,\infty)$, $\sigma:=\sum_{k=0}^\infty\sigma_k<\infty$, and satisfying $y_{k+1}\le\theta_k y_k+\sigma_k$ for all $k\in\mathbf{N}$, then $y_k\le\theta(y_0+\sigma)$. Applying this principle with

$$\begin{aligned} y_k &= \left(\frac{1}{\varrho_k^{N-2+2\alpha}}\int_{B_{\varrho_k}(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x\right)^{1/2},\\ \theta_k &= 1+C\,2^{-(k+1)/2},\\ \sigma_k &= C2^{-\beta(k+1)/2},\end{aligned}$$

yields

$$\frac{1}{\varrho_k^{N-2+2\alpha}}\int_{B_{\varrho_k}(x_0)} A\nabla z\cdot\nabla z\,\mathrm{d}x \le Ce^C\left(\int_{\widetilde{\Omega}} A\nabla z\cdot\nabla z\,\mathrm{d}x+1\right)\quad \text{for all } k\in\mathbf{N}$$

(we have also used the elementary estimates $\theta \le e^C$ and $\sigma \le C$). Since for all $\varrho > 0$ small, there exists a unique $k \in \mathbf{N}$ such that $\varrho_{k+1} < \varrho \le \varrho_k$ and $\frac{1}{\varrho} < \frac{1}{\varrho_{k+1}} \le \frac{2}{\varrho_k}$, we conclude that

$$\frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} A\nabla z \cdot \nabla z \, dx \le Ce^C \left(\int_{\tilde{\Omega}} A\nabla z \cdot \nabla z \, dx + 1 \right) \quad \text{for all } \varrho > 0 \text{ small.}$$

Finally, by ellipticity of A and the arbitrariness of x_0, we conclude that (3.22) holds with

$$\frac{1}{\varrho^{N-2+2\alpha}} \int_{B_\varrho(x_0)} |\nabla z|^2 \, dx \le Ce^C \left(\int_{\tilde{\Omega}} A\nabla z \cdot \nabla z \, dx + 1 \right)$$

for all $x_0 \in \bar{\omega}$ and $\varrho > 0$ small with $B_\varrho(x_0) \subset\subset \tilde{\Omega}$. By Morrey's theorem (see e.g. [18, Theorem 5.7]), it then follows that $v \in \mathcal{C}^{0,\alpha}(\bar{\omega})$. ∎

We are now ready to prove the boundary regularity result in Theorem 3.2.

Proof of Theorem 3.2 completed. We consider $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ satisfying (1.3)–(1.4), and we consider the extensions $\hat{u}_\varepsilon$, $\tilde{u}_\varepsilon$, $\hat{v}_\varepsilon$, and $\tilde{v}_\varepsilon$ to the domain $\tilde{\Omega}$ provided by (3.11)–(3.12) and (3.8). Again, for simplicity, we drop the subscript ε.

We first improve the regularity of $\tilde{u}$ which satisfies (3.13). We aim to apply the De Giorgi–Nash–Moser theorem to infer that $\tilde{u}$ is locally Hölder continuous in $\tilde{\Omega}$ and that a suitable Morrey estimate holds for $\nabla\hat{u}$. Since equation (3.13) is linear with respect to $\tilde{u}$, we first observe that

$$\int_{\tilde{\Omega}} f(x, \nabla\tilde{u}) \, dx \le \int_{\tilde{\Omega}} f(x, \nabla w) \, dx \quad \text{for all } w \in H^1(\Omega) \text{ such that } \operatorname{supp}(w - \tilde{u}) \subset \tilde{\Omega},$$

with

$$f(x,\xi) := \frac{1}{2}(\eta_\varepsilon + \hat{v}_\varepsilon^2(x))A(x)\xi \cdot \xi - h(x) \cdot \xi \quad \text{for a.e. } x \in \tilde{\Omega} \text{ and all } \xi \in \mathbf{R}^N,$$

and

$$h := 2\mathbf{1}_{\tilde{\Omega}\setminus\overline{\Omega}}(\eta_\varepsilon + \hat{v}^2)A\nabla\tilde{g} \in L^\infty(\tilde{\Omega}; \mathbf{R}^N).$$

The function f is a Carathéodory function, and since A is uniformly elliptic and the functions $\hat{v}$ and h are essentially bounded, we can find positive constants c_1, c_2, and c_3 such that

$$c_1|\xi|^2 - c_3 \le f(x,\xi) \le c_2|\xi|^2 + c_3 \quad \text{for a.e. } x \in \tilde{\Omega} \text{ and all } \xi \in \mathbf{R}^N.$$

Hence we can apply the De Giorgi–Nash–Moser theorem (see [19, Theorems 7.5 and 7.6]) to deduce the existence of some $\beta \in (0,1)$ such that $\tilde{u} \in \mathcal{C}^{0,\beta}_{\mathrm{loc}}(\tilde{\Omega})$. From [19, Theorem 7.7] and [18, Lemma 5.13], we also obtain the Morrey estimate

$$\sup_{B_\varrho(x_0) \subset \tilde{\Omega}} \frac{1}{\varrho^{N-2+2\beta}} \int_{B_\varrho(x_0)} |\nabla\tilde{u}|^2 \, dx < \infty. \tag{3.29}$$

Next we consider equation (3.14) satisfied by $\tilde{v}$ restricted to $B_{\delta_0/2}(x_0)$, which we write

$$-\operatorname{div}(A\nabla\tilde{v}) = H \quad \text{in } \mathcal{D}'(\widetilde{\Omega}),$$

with

$$H := \frac{1}{\varepsilon}(\mathbf{1}_{\widetilde{\Omega}\cap\Omega} - \mathbf{1}_{\widetilde{\Omega}\setminus\overline{\Omega}})\Big(\frac{j}{4\varepsilon}(1-\hat{v}) - (A\nabla\hat{u}\cdot\nabla\hat{u})\hat{v}\Big).$$

Since $\hat{u} = 2\tilde{g} - \tilde{u}$ and $\nabla\tilde{g}$, A, j, and $\hat{v}$ are essentially bounded, we infer from (3.29) that

$$\sup_{B_\varrho(x_0)\subset\widetilde{\Omega}} \frac{1}{\varrho^{N-2+2\beta}} \int_{B_\varrho(x_0)} |H|\,\mathrm{d}x < \infty.$$

Applying Lemma 3.3, we deduce that $\tilde{v} \in \mathcal{C}^{0,\gamma}_{\mathrm{loc}}(\widetilde{\Omega})$ for every $\gamma \in (0,\beta)$. In particular, we have $v \in \mathcal{C}^{0,\gamma}(\overline{\Omega})$ for every $\gamma \in (0,\beta)$. Using equation (1.2) satisfied by u together with the (up to the boundary) Schauder estimate (see [18, Theorem 5.21]), we obtain that $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$ for every $\gamma \in (0,\beta)$. Then, in view of equation (1.2) satisfied by v, and owing to classical elliptic regularity at the boundary, we obtain $v \in \mathcal{C}^{2,\gamma}(\overline{\Omega})$ for every $\gamma \in (0,\beta)$. Going back to equation (1.2) satisfied by u, elliptic regularity at the boundary now tells us that $u \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ in the case $g \in \mathcal{C}^{1,\alpha}(\partial\Omega)$, and in turn $v \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ still by (1.2). If $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ then $v \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and, once again, elliptic boundary regularity implies that $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$.

If $\partial\Omega$ is of class $\mathcal{C}^{k,\alpha}$ and $g \in \mathcal{C}^{k,\alpha}(\partial\Omega)$ with $k \geq 3$, one can iterate the preceding argument using elliptic boundary regularity to conclude that u and v belong to $\mathcal{C}^{k,\alpha}(\overline{\Omega})$. ■

A similar argument shows the validity of a localized version of Theorem 3.2.

Theorem 3.3. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set and $(u_\varepsilon, v_\varepsilon) \in H^1(\Omega) \times [H^1(\Omega) \cap L^\infty(\Omega)]$ satisfy (1.3)–(1.4). Assume that in some ball $B_{4R}(x_0)$ with $x_0 \in \partial\Omega$, the boundary portion $\partial\Omega \cap B_{4R}(x_0)$ is of class $\mathcal{C}^{k\vee 2,1}$ and $(u_\varepsilon, v_\varepsilon) = (g, 1)$ on $\partial\Omega \cap B_{4R}(x_0)$ for a function $g \in \mathcal{C}^{k,\alpha}(\partial\Omega \cap B_{4R}(x_0))$ with $k \geq 1$ and $\alpha \in (0,1)$. Then $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{k,\alpha}(\overline{\Omega} \cap B_{\theta R}(x_0)) \times \mathcal{C}^{k\vee 2,\alpha}(\overline{\Omega} \cap B_{\theta R}(x_0))$ for some constant $\theta \in (0,1)$.*

4. Compactness results

We start by recalling a weak compactness result, in the spirit of the compactness argument in the Γ-convergence analysis, only under the assumption of a uniform energy bound (1.15). The result is a direct application of the standard lower bound inequality considering the extension of a pair $(u,v) \in \mathcal{A}_g(\Omega)$ to a larger bounded open set $\Omega' \supset \overline{\Omega}$ of the form $(u,v) = (g',1)$ in $\Omega' \setminus \Omega$ for some arbitrary extension $g' \in H^1(\Omega') \cap L^\infty(\Omega')$ of g.

Proposition 4.1 (Weak compactness). *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary, $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$, and $\varepsilon_n \to 0^+$ be an arbitrary sequence. Assume that*

$(u_n, v_n) := (u_{\varepsilon_n}, v_{\varepsilon_n}) \in \mathcal{A}_g(\Omega)$ satisfies $0 \le v_n \le 1$ a.e. in Ω, $\|u_n\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\partial\Omega)}$, and the uniform energy bound $\sup_n \mathrm{AT}_{\varepsilon_n}(u_n, v_n) < \infty$. There exist a (not relabeled) subsequence and $u_* \in \mathrm{SBV}^2(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} (u_n, v_n) \to (u_*, 1) & \text{strongly in } [L^2(\Omega)]^2, \\ v_n \nabla u_n \rightharpoonup \nabla u_* & \text{weakly in } L^2(\Omega; \mathbf{R}^N). \end{cases}$$

Moreover,

$$\begin{aligned} \int_\Omega |\nabla u_*|^2 \,\mathrm{d}x &\le \liminf_{n\to\infty} \int_\Omega v_n^2 |\nabla u_n|^2 \,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_\Omega (\eta_{\varepsilon_n} + v_n^2)|\nabla u_n|^2 \,\mathrm{d}x \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathcal{H}^{N-1}(J_{u_*} \cup (\partial\Omega \cap \{u_* \ne g\})) &\le \liminf_{n\to\infty} \int_\Omega (1 - v_n)|\nabla v_n| \,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_\Omega \Big(\varepsilon_n |\nabla v_n|^2 + \frac{(v_n - 1)^2}{4\varepsilon_n}\Big) \,\mathrm{d}x. \end{aligned} \tag{4.2}$$

The regularity of solutions established in Theorem 1.1 allows us to prove that critical points of the Ambrosio–Tortorelli functional satisfy a Noether-type conservation law, which is the starting point of our asymptotic analysis.

Proposition 4.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with boundary of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. If $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ is a critical point of AT_ε, then for all $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$,*

$$\begin{aligned} &\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)[2\nabla u_\varepsilon \otimes \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \mathrm{Id}] : DX \,\mathrm{d}x \\ &\quad + \int_\Omega \Big[2\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon - \Big(\varepsilon |\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon}\Big)\mathrm{Id}\Big] : DX \,\mathrm{d}x \\ &\quad = \int_{\partial\Omega} \big[(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2 - (\eta_\varepsilon + 1)|\nabla_\tau g|^2\big](X \cdot \nu_\Omega) \,\mathrm{d}\mathcal{H}^{N-1} \\ &\quad + 2(\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g) \,\mathrm{d}\mathcal{H}^{N-1}, \end{aligned} \tag{4.3}$$

where $X_\tau := X - (X \cdot \nu_\Omega)\nu_\Omega$ is the tangential part of X, and $\nabla_\tau g$ is the tangential gradient of g on $\partial\Omega$.

Proof. Let us fix an arbitrary $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$. By Theorem 1.1, $(u_\varepsilon, v_\varepsilon) \in [\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$ and (1.2) is satisfied in the classical sense. Multiplying the first equation of (1.2) by

$X \cdot \nabla u_\varepsilon$ (which belongs to $\mathcal{C}^1(\overline{\Omega})$) and by integration by parts, a standard computation yields

$$
\begin{aligned}
0 = &\int_\Omega (\eta_\varepsilon + v_\varepsilon^2)\Big[(\nabla u_\varepsilon \otimes \nabla u_\varepsilon) - \frac{1}{2}|\nabla u_\varepsilon|^2 \mathrm{Id}\Big] : DX \,\mathrm{d}x - \int_\Omega v_\varepsilon (X \cdot \nabla v_\varepsilon)|\nabla u_\varepsilon|^2 \,\mathrm{d}x \\
&- \frac{\eta_\varepsilon + 1}{2}\int_{\partial\Omega} (\partial_\nu u_\varepsilon)^2 (X \cdot \nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} + \frac{\eta_\varepsilon + 1}{2}\int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} \\
&- (\eta_\varepsilon + 1)\int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g)\,\mathrm{d}\mathcal{H}^{N-1}. \qquad (4.4)
\end{aligned}
$$

Similarly, multiplying the second equation in (1.2) by $X \cdot \nabla v_\varepsilon$ (which belongs to $\mathcal{C}^1(\overline{\Omega})$) and performing a similar integration by parts leads to

$$
\begin{aligned}
0 = \varepsilon &\int_\Omega \Big[(\nabla v_\varepsilon \otimes \nabla v_\varepsilon) - \frac{1}{2}\Big(|\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon}(v_\varepsilon - 1)^2\Big)\mathrm{Id}\Big] : DX \,\mathrm{d}x \\
&+ \int_\Omega v_\varepsilon (X \cdot \nabla v_\varepsilon)|\nabla u_\varepsilon|^2 \,\mathrm{d}x - \frac{\varepsilon}{2}\int_{\partial\Omega} (\partial_\nu v_\varepsilon)^2 (X \cdot \nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1}, \qquad (4.5)
\end{aligned}
$$

since $v_\varepsilon = 1$ on $\partial\Omega$. Then the conclusion follows by summing (4.4) and (4.5). ■

Remark 4.1. The fact that critical points $(u_\varepsilon, v_\varepsilon)$ enjoy the higher regularity $[\mathcal{C}^{2,\alpha}(\overline{\Omega})]^2$ allows one to obtain a strong form of the conservative equations for $(u_\varepsilon, v_\varepsilon)$. In particular, some information on the boundary is recovered since the vector field X does not need to be tangential on $\partial\Omega$. This additional information will be instrumental in Section 5 to characterize the boundary term occurring in the first inner variation of the Mumford–Shah functional.

Owing to the previous results, we get the following property for the weak limit u_* as $\varepsilon \to 0$ of a converging sequence of critical points of the Ambrosio–Tortorelli functional.

Lemma 4.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with boundary of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. Along a sequence $\varepsilon \to 0^+$, let $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ be a critical point of AT_ε satisfying the uniform energy bound (1.15) and the conclusion of Proposition 4.1. If u_* denotes the weak limit of u_ε as $\varepsilon \to 0$, then $\nabla u_* \in L^2(\Omega; \mathbf{R}^N)$ satisfies $\mathrm{div}(\nabla u_*) = 0$ in $\mathcal{D}'(\Omega)$, its normal trace $\nabla u_* \cdot \nu_\Omega$ belongs to $L^2(\partial\Omega)$, and $\partial_\nu u_\varepsilon \rightharpoonup \nabla u_* \cdot \nu_\Omega$ weakly in $L^2(\partial\Omega)$ as $\varepsilon \to 0$. Moreover, up to a subsequence, there exists a nonnegative Radon measure $\lambda_* \in \mathcal{M}^+(\partial\Omega)$ such that*

$$[(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2]\mathcal{H}^{N-1} \llcorner \partial\Omega \overset{*}{\rightharpoonup} \lambda_* \quad \textit{weakly* in } \mathcal{M}(\partial\Omega).$$

Proof. We first claim that $(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \rightharpoonup \nabla u_*$ weakly in $L^2(\Omega; \mathbf{R}^N)$. Indeed, on the one hand we have

$$\|\eta_\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega;\mathbf{R}^N)} \le \sqrt{\eta_\varepsilon}\Big\|\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon\Big\|_{L^2(\Omega;\mathbf{R}^N)} \le C\sqrt{\eta_\varepsilon} \to 0,$$

and on the other hand, for all $\varphi \in \mathcal{C}_c^\infty(\Omega;\mathbf{R}^N)$,

$$\left|\int_\Omega v_\varepsilon \nabla u_\varepsilon \cdot \varphi \, \mathrm{d}x - \int_\Omega v_\varepsilon^2 \nabla u_\varepsilon \cdot \varphi \, \mathrm{d}x\right| \leq \|\varphi\|_{L^\infty(\Omega;\mathbf{R}^N)} \|v_\varepsilon \nabla u_\varepsilon\|_{L^2(\Omega;\mathbf{R}^N)} \|v_\varepsilon - 1\|_{L^2(\Omega)} \to 0.$$

Using the previous two points and Proposition 4.1 leads to $(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \rightharpoonup \nabla u_*$ weakly* in $\mathcal{D}'(\Omega;\mathbf{R}^N)$. Since the sequence $\{(\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon\}$ is bounded in $L^2(\Omega;\mathbf{R}^N)$, its weak L^2-convergence follows. We can thus pass to the limit in (1.3) in the sense of distributions and conclude that $\operatorname{div}(\nabla u_*) = 0$ in $\mathcal{D}'(\Omega)$.

Since ∇u_* belongs to $L^2(\Omega;\mathbf{R}^N)$, and $\operatorname{div}(\nabla u_*) = 0$, the normal trace $\nabla u_* \cdot \nu_\Omega$ is well defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$. Recalling that $v_\varepsilon = 1$ on $\partial\Omega$, we get

$$(\eta_\varepsilon + 1)\partial_\nu u_\varepsilon = (\eta_\varepsilon + v_\varepsilon^2)\nabla u_\varepsilon \cdot \nu \rightharpoonup \nabla u_* \cdot \nu_\Omega \quad \text{weakly in } H^{-\frac{1}{2}}(\partial\Omega).$$

We now improve this convergence into a weak convergence in $L^2(\partial\Omega)$. For that, let us consider a test function $X \in \mathcal{C}_c^1(\mathbf{R}^N;\mathbf{R}^n)$ such that $X = \nu$ on $\partial\Omega$ in relation (4.3). Using that the left-hand side of (4.3) is clearly controlled by the Ambrosio–Tortorelli energy (see (1.15)), we infer that

$$\sup_{\varepsilon>0} \int_{\partial\Omega} [(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] \, \mathrm{d}\mathcal{H}^{N-1} < \infty.$$

On the one hand, we obtain that $\{\partial_\nu u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\partial\Omega)$, hence $\partial_\nu u_\varepsilon \rightharpoonup \nabla u_* \cdot \nu_\Omega$ weakly in $L^2(\partial\Omega)$. On the other hand, there exists a nonnegative Radon measure $\lambda_* \in \mathcal{M}^+(\partial\Omega)$ such that $[(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2]\mathcal{H}^{N-1} \llcorner \partial\Omega \overset{*}{\rightharpoonup} \lambda_*$ weakly* in $\mathcal{M}(\partial\Omega)$. ■

Remark 4.2. Our choice of Dirichlet boundary conditions for both u and v in (1.2) allows one to obtain an ε-dependent boundary term which is nonnegative in the boundary integral involving $X \cdot \nu_\Omega$ in (4.3). This sign information is essential to get a limit boundary term which is a measure λ_* concentrated on $\partial\Omega$. If we had chosen a Neumann condition for v and a Dirichlet condition for u as in [14], one would have obtained a more involved boundary term which would lead to a first-order distribution on $\partial\Omega$ in the $\varepsilon \to 0$ limit. It is not clear in this case how to perform the analysis in Section 5 (in particular Lemma 5.3).

5. Convergence of critical points

Our objective is to show that u_* is a critical point of the Mumford–Shah functional. We now improve the convergence results established in the previous section by additionally assuming convergence of the energy (1.16), i.e. $\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathrm{MS}(u_*)$. Under this stronger assumption, we can improve the above-established convergences and in particular obtain the equipartition of the phase-field energy.

Proposition 5.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $g \in H^{\frac{1}{2}}(\partial\Omega) \cap L^\infty(\partial\Omega)$. Let us consider a critical point $(u_\varepsilon, v_\varepsilon)$ of the Ambrosio–Tortorelli functional satisfying the energy convergence (1.16) and let $u_* \in \mathrm{SBV}^2(\Omega)$ be given by Proposition 4.1. Then, up to a further subsequence,*

$$\sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \to \nabla u_*, \quad v_\varepsilon \nabla u_\varepsilon \to \nabla u_* \quad \textit{strongly in } L^2(\Omega; \mathbf{R}^N). \tag{5.1}$$

Moreover, setting $\Phi(t) := t - t^2/2$, and

$$w_\varepsilon := \Phi(v_\varepsilon), \tag{5.2}$$

then

$$\begin{cases} \nabla w_\varepsilon \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} 0 & \textit{weakly* in } \mathcal{M}(\Omega; \mathbf{R}^N), \\ |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \textit{weakly* in } \mathcal{M}(\overline{\Omega}), \end{cases} \tag{5.3}$$

where we recall that $\hat{J}_{u_*} = J_{u_*} \cup (\partial\Omega \cap \{u_* \neq g\})$. Finally, there is equipartition of the phase-field energy, i.e.

$$\lim_{\varepsilon \to 0} \int_\Omega \left| \varepsilon |\nabla v_\varepsilon|^2 - \frac{1}{4\varepsilon}(1 - v_\varepsilon)^2 \right| \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega \big| 2\varepsilon |\nabla v_\varepsilon|^2 - |\nabla w_\varepsilon| \big| \, \mathrm{d}x = 0. \tag{5.4}$$

Proof. According to the convergence of energy assumption (1.16) and the lower semicontinuity properties (4.1)–(4.2) established in Proposition 4.1 (which applies by Lemma 3.1), we deduce that

$$\int_\Omega |\nabla u_*|^2 \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon^2 |\nabla u_\varepsilon|^2 \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega (\eta_\varepsilon + v_\varepsilon^2) |\nabla u_\varepsilon|^2 \, \mathrm{d}x \tag{5.5}$$

and

$$\mathcal{H}^{N-1}(\hat{J}_{u_*}) = \lim_{\varepsilon \to 0} \int_\Omega (1 - v_\varepsilon) |\nabla v_\varepsilon| \, \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega \left(\varepsilon |\nabla v_\varepsilon|^2 + \frac{(v_\varepsilon - 1)^2}{4\varepsilon} \right) \mathrm{d}x. \tag{5.6}$$

Convergence (5.5) combined with the weak L^2-convergence of $v_\varepsilon \nabla u_\varepsilon$ to ∇u_* implies that $v_\varepsilon \nabla u_\varepsilon \to \nabla u_*$ strongly in $L^2(\Omega; \mathbf{R}^N)$. Moreover, it follows from (5.5) that $\sqrt{\eta_\varepsilon} \nabla u_\varepsilon \to 0$ strongly in $L^2(\Omega; \mathbf{R}^N)$. Hence $\sqrt{\eta_\varepsilon + v_\varepsilon^2} \nabla u_\varepsilon \to \nabla u_*$ strongly in $L^2(\Omega; \mathbf{R}^N)$ as well.

Next, setting $w_\varepsilon = \Phi(v_\varepsilon)$, with $\Phi(t) = t - t^2/2$, and using that $v_\varepsilon \to 1$ strongly in $L^2(\Omega)$ yields $w_\varepsilon \to 1/2$ strongly in $L^1(\Omega)$. Furthermore, owing to the chain rule in Sobolev spaces, we have $\nabla w_\varepsilon = \Phi'(v_\varepsilon) \nabla v_\varepsilon = (1 - v_\varepsilon) \nabla v_\varepsilon$. In view of (5.6), we deduce that $\{\nabla w_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^1(\Omega; \mathbf{R}^N)$, hence $\nabla w_\varepsilon \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} 0$ weakly* in $\mathcal{M}(\Omega; \mathbf{R}^N)$. Moreover, localizing the conclusion of Proposition 4.1, we get that for every open set $U \subset \Omega' \subset \mathbf{R}^N$, we have $\mathcal{H}^{N-1}(\hat{J}_{u_*} \cap U) \leq \liminf_\varepsilon \int_U |\nabla w_\varepsilon| \, \mathrm{d}x$, and, using (5.6) it shows that

$$\int_\Omega |\nabla w_\varepsilon| \, \mathrm{d}x \to \mathcal{H}^{N-1}(\hat{J}_{u_*}). \tag{5.7}$$

Thus [1, Proposition 1.80] implies that $|\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ weakly* in $\mathcal{M}(\overline{\Omega})$.

The equipartition of energy is obtained by observing that

$$\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_\Omega (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2\,\mathrm{d}x + \int_\Omega |\nabla w_\varepsilon|\,\mathrm{d}x + \int_\Omega \Big|\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big|^2\,\mathrm{d}x,$$

and by using (1.16), (5.5), and (5.7). Indeed, we get

$$\int_\Omega \Big|\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big|^2\,\mathrm{d}x \to 0,$$

and then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_\Omega \Big|\varepsilon|\nabla v_\varepsilon|^2 - \frac{1}{4\varepsilon}(1-v_\varepsilon)^2\Big|\,\mathrm{d}x &\le \Big\|\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big\|_{L^2(\Omega)} \\ &\quad\times \Big\|\sqrt{\varepsilon}|\nabla v_\varepsilon| + \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big\|_{L^2(\Omega)} \\ &\le C\Big\|\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big\|_{L^2(\Omega)} \to 0.\end{aligned}$$

Finally, using again that $|\nabla w_\varepsilon| = (1-v_\varepsilon)|\nabla v_\varepsilon|$, we observe that

$$\begin{aligned}\int_\Omega \big|2\varepsilon|\nabla v_\varepsilon|^2 - |\nabla w_\varepsilon|\big|\,\mathrm{d}x &= \int_\Omega \Big|\varepsilon|\nabla v_\varepsilon|^2 - \frac{(1-v_\varepsilon)^2}{4\varepsilon} \\ &\qquad + \Big(\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big)^2\Big|\,\mathrm{d}x \\ &\le \int_\Omega \Big|\varepsilon|\nabla v_\varepsilon|^2 - \frac{(1-v_\varepsilon)^2}{4\varepsilon}\Big|\,\mathrm{d}x \\ &\quad + \int_\Omega \Big|\sqrt{\varepsilon}|\nabla v_\varepsilon| - \frac{1}{2\sqrt{\varepsilon}}(1-v_\varepsilon)\Big|^2\,\mathrm{d}x \\ &\to 0.\end{aligned}$$

This implies (5.4). ∎

The proof of Theorem 1.2 is based on (geometric) measure-theoretic arguments. Let us define the $(N-1)$-varifold $V_\varepsilon \in \mathbf{V}_{N-1}(\overline{\Omega})$ associated to the phase-field variable $v_\varepsilon \in H^1(\Omega)$ by

$$\langle V_\varepsilon, \varphi\rangle := \int_{\Omega\cap\{\nabla w_\varepsilon \neq 0\}} \varphi\Big(x, \mathrm{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\Big)|\nabla w_\varepsilon|\,\mathrm{d}x \quad \text{for all } \varphi \in \mathcal{C}(\overline{\Omega}\times \mathbf{G}_{N-1}),$$

where $w_\varepsilon := \Phi(v_\varepsilon)$, and $\Phi(t) = t - t^2/2$ for $t \in [0,1]$. By the coarea formula, this definition is equivalent to the definition of a varifold associated to a function in [22]. By standard compactness of bounded Radon measures, at the expense of extracting a further subsequence, there exists a varifold $V_* \in \mathbf{V}_{N-1}(\overline{\Omega})$ such that $V_\varepsilon \overset{*}{\rightharpoonup} V_*$ weakly* in $\mathcal{M}(\overline{\Omega}\times \mathbf{G}_{N-1})$.

Note that $\|V_\varepsilon\| \overset{*}{\rightharpoonup} \|V_*\|$ weakly* in $\mathcal{M}(\overline{\Omega})$ by definition of the mass of a varifold and thanks to the compactness of $\mathbf{G}_{N-1}$. Recalling the definition of w_ε in (5.2), we observe that $\|V_\varepsilon\| = |\nabla w_\varepsilon|\mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$ weakly* in $\mathcal{M}(\overline{\Omega})$ according to (5.3), and it follows that $\|V_*\| = \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$. According to the disintegration theorem ([1, Theorem 2.28]), there exists a weak* $\mathcal{H}^{N-1}$-measurable mapping $x \in \overline{\Omega} \mapsto V_x \in \mathcal{M}(\mathbf{G}_{N-1})$ of probability measures such that $V_* = (\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}) \otimes V_x$, i.e. for all $\varphi \in \mathcal{C}(\overline{\Omega} \times \mathbf{G}_{N-1})$,

$$\int_{\overline{\Omega} \times \mathbf{G}_{N-1}} \varphi(x, A)\, \mathrm{d}V_*(x, A) = \int_{\hat{J}_{u_*}} \left(\int_{\mathbf{G}_{N-1}} \varphi(x, A)\, \mathrm{d}V_x(A) \right) \mathrm{d}\mathcal{H}^{N-1}(x). \tag{5.8}$$

For $\mathcal{H}^{N-1}$ almost every $x \in \hat{J}_{u_*}$, we set

$$\bar{A}(x) := \int_{\mathbf{G}_{N-1}} A\, \mathrm{d}V_x(A). \tag{5.9}$$

Owing to our various convergence results, we are now in a position to pass to the limit in the inner variation equation (4.3). For now, the limit expression depends on the abstract limit varifold V_* through its first moment $\bar{A}$ of V_x, and the abstract boundary measure λ_* introduced in Lemma 4.1.

Lemma 5.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set of class $\mathcal{C}^{2,1}$ and $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$. Let $u_* \in \mathrm{SBV}^2(\Omega)$ be a limit of critical points of the Ambrosio–Tortorelli functional as in Proposition 5.1. For all $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$, we have*

$$\begin{aligned} &\int_\Omega \left(|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_* \right) : DX\, \mathrm{d}x + \int_{\hat{J}_{u_*}} \bar{A} : DX\, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad = - \int_{\partial\Omega} (X \cdot \nu_\Omega)\, \mathrm{d}\lambda_* + \int_{\partial\Omega} |\nabla_\tau g|^2 (X \cdot \nu_\Omega)\, \mathrm{d}\mathcal{H}^{N-1} \\ &\qquad - 2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X_\tau \cdot \nabla_\tau g)\, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{5.10}$$

Proof. Using the strong convergence (5.1) established above, it is easy to pass to the limit in the first integral and in the left-hand side of (4.3). We get, for all $X \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$,

$$\begin{aligned} &\int_\Omega \left(2\left(\sqrt{\eta_\varepsilon + v_\varepsilon^2}\, \nabla u_\varepsilon \right) \otimes \left(\sqrt{\eta_\varepsilon + v_\varepsilon^2}\, \nabla u_\varepsilon \right) - (\eta_\varepsilon + v_\varepsilon^2)|\nabla u_\varepsilon|^2 \mathrm{Id} \right) : DX\, \mathrm{d}x \\ &\quad \underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega (2\nabla u_* \otimes \nabla u_* - |\nabla u_*|^2 \mathrm{Id}) : DX\, \mathrm{d}x. \end{aligned} \tag{5.11}$$

According to Lemma 4.1 we can also pass to the limit in the boundary integrals in the right-hand side of (4.3); we get

$$\begin{aligned} &\int_{\partial\Omega} [(\eta_\varepsilon + 1)(\partial_\nu u_\varepsilon)^2 + \varepsilon(\partial_\nu v_\varepsilon)^2] X \cdot \nu\, \mathrm{d}\mathcal{H}^{N-1} \\ &\qquad - (1 + \eta_\varepsilon) \int_{\partial\Omega} X \cdot \nu |\nabla_\tau g|^2\, \mathrm{d}\mathcal{H}^{N-1} + 2(\eta_\varepsilon + 1) \int_{\partial\Omega} (\partial_\nu u_\varepsilon)(X_\tau \cdot \nabla_\tau g)\, \mathrm{d}\mathcal{H}^{N-1} \end{aligned}$$

$$\underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\Omega} (X\cdot\nu_\Omega)\,\mathrm{d}\lambda_* - \int_{\partial\Omega} |\nabla_\tau g|^2 (X\cdot\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} + 2\int_{\partial\Omega} (\nabla u_*\cdot\nu_\Omega)(X_\tau\cdot\nabla_\tau g)\,\mathrm{d}\mathcal{H}^{N-1}. \tag{5.12}$$

It remains to pass to the limit in the second integral in the left-hand side of (4.3). Using the chain rule, we have $\nabla w_\varepsilon = \Phi'(v_\varepsilon)\nabla v_\varepsilon$. The equipartition of energy (5.4) thus implies that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \Big(2\varepsilon\nabla v_\varepsilon\otimes\nabla v_\varepsilon - \varepsilon|\nabla v_\varepsilon|^2\mathrm{Id} - \frac{(v_\varepsilon-1)^2}{4\varepsilon}\mathrm{Id}\Big) : DX\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{\nabla v_\varepsilon\neq 0\}} 2\varepsilon|\nabla v_\varepsilon|^2\Big(\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|}\otimes\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} - \mathrm{Id}\Big) : DX\,\mathrm{d}x\\
&\quad= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{\nabla w_\varepsilon\neq 0\}} |\nabla w_\varepsilon|\Big(\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\otimes\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} - \mathrm{Id}\Big) : DX\,\mathrm{d}x\\
&\quad= -\int_{\overline{\Omega}\times\mathbf{G}_{N-1}} A : DX(x)\,\mathrm{d}V_*(x,A) = -\int_{\hat{J}_{u_*}} \bar{A} : DX\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned} \tag{5.13}$$

Gathering (5.11), (5.12), and (5.13), we infer that (5.10) holds. ∎

Let us now identify the first moment $\bar{A}$ of the measure V_x. We first establish some algebraic properties of this matrix.

Lemma 5.2. *For $\mathcal{H}^{N-1}$-almost every $x\in\hat{J}_{u_*}$, the matrix $\bar{A}(x)$ satisfies*

$$\bar{A}(x)\geq 0,\quad \mathrm{tr}(\bar{A}(x)) = N-1,\quad \rho(\bar{A}(x)) = 1,$$

where ρ denotes the spectral radius.

Proof. To simplify notation, we set

$$A_\varepsilon := \Big(\mathrm{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\otimes\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\Big).$$

The matrix A_ε is well defined on the set $\Omega\cap\{\nabla w_\varepsilon\neq 0\}$; it is a symmetric matrix corresponding to the orthogonal projection on $\{\nabla w_\varepsilon\}^\perp$. It satisfies $A_\varepsilon\geq 0$, $\mathrm{tr}(A_\varepsilon) = N-1$, and $\rho(A_\varepsilon) = 1$ in $\Omega\cap\{\nabla w_\varepsilon\neq 0\}$. For all $\varphi\in\mathcal{C}^0(\overline{\Omega})$, we have

$$\begin{aligned}
\int_{\hat{J}_{u_*}} \mathrm{tr}(\bar{A})\varphi\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\overline{\Omega}\times\mathbf{G}_{N-1}} \mathrm{tr}(A)\varphi(x)\,\mathrm{d}V_*(x,A)\\
&= \lim_{\varepsilon\to 0}\int_{\Omega\times\mathbf{G}_{N-1}} \mathrm{tr}(A)\varphi(x)\,\mathrm{d}V_\varepsilon(x,A)\\
&= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{\nabla w_\varepsilon\neq 0\}} \mathrm{tr}(A_\varepsilon)\varphi|\nabla w_\varepsilon|\,\mathrm{d}x\\
&= (N-1)\lim_{\varepsilon\to 0}\int_\Omega \varphi|\nabla w_\varepsilon|\,\mathrm{d}x = (N-1)\int_{\hat{J}_{u_*}} \varphi\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

which shows that $\operatorname{tr}(\bar{A}) = (N-1)$, $\mathcal{H}^{N-1}$-a.e. on $\hat{J}_{u_*}$. If, further, $\varphi \geq 0$ and $z \in \mathbf{R}^N$ then

$$\begin{aligned}\int_{\hat{J}_{u_*}} (\bar{A}z \cdot z)\varphi \, \mathrm{d}\mathcal{H}^{N-1} &= \int_{\overline{\Omega}\times\mathbf{G}_{N-1}} (Az \cdot z)\varphi(x) \, \mathrm{d}V_*(x, A) \\ &= \lim_{\varepsilon\to 0} \int_{\overline{\Omega}\times\mathbf{G}_{N-1}} (Az \cdot z)\varphi(x) \, \mathrm{d}V_\varepsilon(x, A) \\ &= \lim_{\varepsilon\to 0} \int_{\Omega\cap\{\nabla w_\varepsilon \neq 0\}} (A_\varepsilon z \cdot z)\varphi|\nabla w_\varepsilon| \, \mathrm{d}x \geq 0.\end{aligned}$$

As a consequence, for all $z \in \mathbf{R}^N$, we have $\bar{A}z \cdot z \geq 0$, $\mathcal{H}^{N-1}$-a.e. on $\hat{J}_{u_*}$, from which we deduce that $\bar{A}$ is a nonnegative matrix $\mathcal{H}^{N-1}$-a.e. on $\hat{J}_{u_*}$.

Since, for all $\varphi \in \mathcal{C}^0(\overline{\Omega})$, we have

$$\begin{aligned}\int_{\Omega\cap\{\nabla w_\varepsilon\neq 0\}} |\nabla w_\varepsilon|\Big(\mathrm{Id} - \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}\Big)\varphi \, \mathrm{d}x \to \int_{\overline{\Omega}\times\mathbf{G}_{N-1}} A\varphi(x) \, \mathrm{d}V_*(x, A) \\ = \int_{\hat{J}_{u_*}} \bar{A}\varphi \, \mathrm{d}\mathcal{H}^{N-1},\end{aligned}$$

we deduce that

$$A_\varepsilon|\nabla w_\varepsilon|\mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \bar{A}\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \text{weakly* in } \mathcal{M}(\overline{\Omega}; \mathbf{M}^{N\times N}). \tag{5.14}$$

Using that the spectral radius ρ is a convex, continuous, and positively 1-homogeneous function on the set of symmetric matrices, it follows from the Reshetnyak continuity theorem (see [1, Theorem 2.39]) that, for all $\varphi \in \mathcal{C}^0(\overline{\Omega})$,

$$\int_{\hat{J}_{u_*}} \varphi \, \mathrm{d}\mathcal{H}^{N-1} = \lim_{\varepsilon\to 0} \int_\Omega \varphi|\nabla w_\varepsilon| \, \mathrm{d}x = \lim_{\varepsilon\to 0} \int_\Omega \varphi\rho(A_\varepsilon)|\nabla w_\varepsilon| \, \mathrm{d}x = \int_{\hat{J}_{u_*}} \varphi\rho(\bar{A}) \, \mathrm{d}\mathcal{H}^{N-1},$$

hence $\rho(\bar{A}) = 1$, $\mathcal{H}^{N-1}$-a.e. on $\hat{J}_{u_*}$. ∎

We now focus on the interior structure of the varifold V_*.

Lemma 5.3. *For $\mathcal{H}^{N-1}$-a.e. x in $J_{u_*} = \hat{J}_{u_*} \cap \Omega$, we have $\bar{A}(x) = \mathrm{Id} - \nu_{u_*}(x) \otimes \nu_{u_*}(x)$, where ν_{u_*} is the approximate normal to the countably $\mathcal{H}^{N-1}$-rectifiable set J_{u_*}.*

Proof. Step 1: Let us show that for $\mathcal{H}^{N-1}$-a.e. x in J_{u_*}, $\bar{A}(x)$ is a projection matrix onto a $(N-1)$-dimensional hyperplane. To this aim, we perform a blow-up argument on the first variation equation (5.10). Let $x_0 \in J_{u_*}$ be such that

(1) x_0 is a Lebesgue point of $\bar{A}$ with respect to $\mathcal{H}^{N-1} \llcorner J_{u_*}$;

(2) J_{u_*} admits an approximate tangent space at x_0 given by $T_{x_0} = \{\nu_{u_*}(x_0)\}^\perp$;

(3) $\displaystyle\lim_{\varrho\to 0} \frac{\mathcal{H}^{N-1}(J_{u_*} \cap B_\varrho(x_0))}{\omega_{N-1}\varrho^{N-1}} = 1$;

(4) $\displaystyle\lim_{\varrho\to 0} \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} |\nabla u_*|^2 \, \mathrm{d}x = 0$.

It turns out that $\mathcal{H}^{N-1}$-almost every point $x_0 \in J_{u_*}$ satisfies these properties. Indeed, (1) is a consequence of the Besicovitch differentiation theorem, (2) and (3) are consequences of the rectifiability of J_{u_*} (see [1, Theorems 2.63 and 2.83]), while condition (4) is a consequence of (3) together with the fact that the measure $|\nabla u_*|^2\mathcal{L}^N \llcorner \Omega$ is singular with respect to $\mathcal{H}^{N-1} \llcorner J_{u_*}$.

Let $x_0 \in J_{u_*}$ be such a point and let $\varrho > 0$ be such that $\overline{B_\varrho(x_0)} \subset \Omega$. For $\phi \in \mathcal{C}_c^\infty(\mathbf{R}^N;\mathbf{R}^N)$ with $\mathrm{supp}(\phi) \subset B_1$, we set $\phi_{x_0,\varrho}(x) := \phi(\frac{x-x_0}{\varrho})$ for $x \in \mathbf{R}^N$, so that $\phi_{x_0,\varrho} \in \mathcal{C}_c^\infty(\mathbf{R}^N;\mathbf{R}^N)$ and $\mathrm{supp}(\phi_{x_0,\varrho}) \subset B_\varrho(x_0)$. Taking $\phi_{x_0,\varrho}$ as a test vector field in (5.10) (note that $\phi_{x_0,\varrho} = 0$ in a neighborhood of $\partial\Omega$) yields

$$\int_{J_{u_*}\cap B_\varrho(x_0)} \bar{A} : D\phi_{x_0,\varrho}\, \mathrm{d}\mathcal{H}^{N-1} = -\int_{B_\varrho(x_0)} (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi_{x_0,\varrho}\, \mathrm{d}x.$$

Dividing this identity by ϱ^{N-2} yields

$$\begin{aligned}\frac{1}{\varrho^{N-1}} &\int_{J_{u_*}\cap B_\varrho(x_0)} \bar{A} : D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}\mathcal{H}^{N-1} \\ &= \frac{-1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}x.\end{aligned}$$

We first show that the right-hand side of the previous equality tends to zero as $\varrho \to 0$. Indeed, thanks to our choice of x_0, we have

$$\begin{aligned}&\left|\frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0)} (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : D\phi\Big(\frac{x - x_0}{\varrho}\Big)\, \mathrm{d}x\right| \\ &\qquad \le C\, \frac{\|D\phi\|_{L^\infty(B_1;\mathbf{M}^{N\times N})}}{\varrho^{N-1}} \int_{B_\varrho(x_0)} |\nabla u_*|^2\, \mathrm{d}x \to 0,\end{aligned}$$

for some constant $C > 0$. Concerning the left-hand side, using first that x_0 is a Lebesgue point of $\bar{A}$ and then item (3), we get

$$\begin{aligned}&\left|\frac{1}{\varrho^{N-1}} \int_{J_{u_*}\cap B_\varrho(x_0)} (\bar{A} - \bar{A}(x_0)) : D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}\mathcal{H}^{N-1}\right| \\ &\qquad \le \frac{\|D\phi\|_{L^\infty(B_1;\mathbf{M}^{N\times N})}}{\varrho^{N-1}} \int_{J_{u_*}\cap B_\varrho(x_0)} |\bar{A} - \bar{A}(x_0)|\, \mathrm{d}\mathcal{H}^{N-1} \to 0,\end{aligned}$$

so that

$$\begin{aligned}\lim_{\varrho\to 0} &\frac{1}{\varrho^{N-1}} \int_{J_{u_*}\cap B_\varrho(x_0)} \bar{A} : D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}\mathcal{H}^{N-1} \\ &= \bar{A}(x_0) : \lim_{\varrho\to 0} \frac{1}{\varrho^{N-1}} \int_{J_{u_*}\cap B_\varrho(x_0)} D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

Using next that J_{u_*} admits an approximate tangent space that we denote by T_{x_0} at x_0, we obtain

$$\lim_{\varrho\to 0} \frac{1}{\varrho^{N-1}} \int_{J_{u_*}\cap B_\varrho(x_0)} D\phi\Big(\frac{\cdot - x_0}{\varrho}\Big)\, \mathrm{d}\mathcal{H}^{N-1} = \int_{T_{x_0}\cap B_1} D\phi\, \mathrm{d}\mathcal{H}^{N-1}.$$

Hence,

$$\int_{T_{x_0}\cap B_1} \bar{A}(x_0) : D\phi \, \mathrm{d}\mathcal{H}^{N-1} = 0 \quad \text{for all } \phi \in \mathcal{C}_c^\infty(B_1; \mathbf{R}^N). \tag{5.15}$$

Let $t \in (0,1)$ be such that $t < \sqrt{N}^{-N}$; the measure $\nu := \frac{1}{\omega_{N-1}} \mathcal{H}^{N-1} \llcorner T_{x_0}$ satisfies

$$t^{N-1} \le \nu(\bar{B}_t) \le \nu(B_1) \le 1.$$

According to [2, Lemma 3.9] with $\beta = s = N-1$, we get that the matrix $\bar{A}(x_0)$ has at most $N-1$ nonzero eigenvalues. Recalling that $\mathrm{tr}(\bar{A}(x_0)) = N-1$ and that all eigenvalues of $\bar{A}(x_0)$ belong to $[0,1]$, this implies that $\bar{A}(x_0)$ has exactly $N-1$ eigenvalues which are equal to 1, and one eigenvalue which is zero. Hence, there exists $e \in \mathbf{S}^{N-1}$ such that $A = \mathrm{Id} - e \otimes e$.

Step 2: Let us show that $e = \pm\nu_{u_*}(x_0)$. Let us consider the varifold

$$\widetilde{V} := \mathcal{H}^{N-1} \llcorner T_{x_0} \otimes \delta_{\bar{A}(x_0)} \in \mathcal{M}(B_1 \times \mathbf{G}_{N-1}),$$

whose action is given by

$$\int_{B_1\times\mathbf{G}_{N-1}} \varphi(x,A) \, \mathrm{d}\widetilde{V}(x,A) = \int_{B_1\cap T_{x_0}} \varphi(x, \bar{A}(x_0)) \, \mathrm{d}\mathcal{H}^{N-1}(x)$$

for all $\varphi \in \mathcal{C}_c^0(B_1 \times \mathbf{G}_{N-1})$. Since $\bar{A}(x_0)$ is a projection matrix onto the hyperplane $e^\perp$, it follows that $\widetilde{V} \in \mathbf{V}_{N-1}(B_1)$ is an $(N-1)$-varifold in B_1 with $\|\widetilde{V}\| = \mathcal{H}^{N-1} \llcorner T_{x_0}$. Moreover, condition (5.15) shows that $\widetilde{V}$ is a stationary varifold; cf. Section 2.3. It follows from the monotonicity formula (see e.g. [43, formula (40.3), p. 236]) that, for all $x \in T_{x_0} \cap B_1$ and all $\varrho > 0$ such that $B_\varrho(x) \subset B_1$,

$$\frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_\varrho(x))}{\varrho^{N-1}} = \frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_r(x))}{r^{N-1}} + \int_{T_{x_0}\cap B_\varrho(x)\setminus B_r(x)} \frac{|e\cdot(y-x)|^2}{|y-x|^{N+1}} \, \mathrm{d}\mathcal{H}^{N-1}(y)$$

for all $0 < r < \varrho$. Since

$$\frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_r(x))}{r^{N-1}} = \frac{\mathcal{H}^{N-1}(T_{x_0} \cap B_\varrho(x))}{\varrho^{N-1}} = \omega_{N-1},$$

we deduce that

$$\int_{T_{x_0}\cap B_\varrho(x)\setminus B_r(x)} \frac{|e\cdot(y-x)|^2}{|y-x|^{N+1}} \, \mathrm{d}\mathcal{H}^{N-1}(y) = 0.$$

Choosing $x = 0$, $\varrho = 1$, and letting $r \to 0^+$, we infer that $y \cdot e = 0$ for $\mathcal{H}^{N-1}$-a.e. $y \in T_{x_0} \cap B_1$, which implies that $T_{x_0} = e^\perp$, hence $e = \pm\nu_{u_*}(x_0)$. ■

Next we focus on boundary points.

Lemma 5.4. *For $\mathcal{H}^{N-1}$-a.e. $x \in \hat{J}_{u_*} \cap \partial\Omega$, we have $\bar{A}(x) = \mathrm{Id} - \nu_\Omega(x) \otimes \nu_\Omega(x)$, where ν_Ω is the outward unit normal to $\partial\Omega$.*

Proof. We perform again a blow-up argument, this time at boundary points. Let $x_0 \in \hat{J}_{u_*} \cap \partial\Omega$ be such that

(1) x_0 is a Lebesgue point of $\bar{A}$ with respect to $\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$;

(2) $\hat{J}_{u_*}$ admits an approximate tangent space at x_0 which coincides with the (usual) tangent space of $\partial\Omega$ at x_0 (this in particular implies that $\nu_{u_*}(x_0) = \pm\nu_\Omega(x_0)$);

(3) $\displaystyle\lim_{\varrho\to 0} \frac{\mathcal{H}^{N-1}(\hat{J}_{u_*} \cap B_\varrho(x_0))}{\omega_{N-1}\varrho^{N-1}} = 1$;

(4) $\displaystyle\lim_{\varrho\to 0} \frac{1}{\varrho^{N-1}} \int_{B_\varrho(x_0)\cap\Omega} |\nabla u_*|^2 \,\mathrm{d}x = 0$;

(5) $\displaystyle\lim_{\varrho\to 0} \frac{\lambda_*(B_\varrho(x_0))}{\varrho^{N-2}} = 0, \quad \lim_{\varrho\to 0} \frac{1}{\varrho^{N-2}} \int_{\partial\Omega\cap B_\varrho(x_0)} |\nabla u_* \cdot \nu| \,\mathrm{d}\mathcal{H}^{N-1} = 0.$

It turns out that $\mathcal{H}^{N-1}$ almost every point $x_0 \in \hat{J}_{u_*} \cap \partial\Omega$ satisfies these properties. Indeed, (1) is a consequence of the Besicovitch differentiation theorem while (2) comes from the rectifiability of $\hat{J}_{u_*}$ (see [1, Theorem 2.83]), together with the locality of approximate tangent spaces (see [1, Proposition 2.85]). Condition (3) is again a consequence of the rectifiability of $\hat{J}_{u_*}$ and the Besicovitch–Marstrand–Mattila theorem (see [1, Proposition 2.63]). Condition (4) is a consequence of (3), together with the fact that the measure $|\nabla u_*|^2 \mathcal{L}^N \llcorner \Omega$ is singular with respect to $\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$. To justify (5) we define, for $x \in \partial\Omega$,

$$\Theta(x) := \limsup_{\varrho\to 0} \frac{\lambda_*(B_\varrho(x))}{\varrho^{N-2}}.$$

According to [1, Theorem 2.56], we have $t\,\mathcal{H}^{N-2}(\{\Theta \geq t\}) \leq \lambda_*(\{\Theta \geq t\}) < \infty$ for all $t > 0$. Hence $\mathcal{H}^{N-1}(\{\Theta \geq t\}) = 0$ for all $t > 0$. As a consequence, $\mathcal{H}^{N-1}(\{\Theta > 0\}) = 0$. The second property of (5) can be obtained similarly, by replacing λ_* by $|\nabla u_* \cdot \nu| \mathcal{H}^{N-1} \llcorner \partial\Omega$.

We choose such a point $x_0 \in \partial\Omega \cap \hat{J}_{u_*}$ and we take $\varrho > 0$.

Step 1: We first prove that $\nu_\Omega(x_0)$ is an eigenvector of $\bar{A}(x_0)$. Consider first a test vector field ϕ of the form $\phi(x) := \varphi(\frac{x-x_0}{\varrho})\tilde{\tau}(x)$ for $x \in B_\varrho(x_0)$, where $\varphi \in \mathcal{C}_c^\infty(B_1)$ and $\tilde{\tau} \in \mathcal{C}_c^1(\mathbf{R}^N; \mathbf{R}^N)$ is such that $\tilde{\tau} \cdot \nu_\Omega = 0$ on $\partial\Omega$. Plugging ϕ into (5.10) and using estimates similar to the proof of Lemma 5.3, we obtain

$$\int_{T_{x_0}\cap B_1} \bar{A}(x_0) : (\tilde{\tau}(x_0) \otimes \nabla\varphi) \,\mathrm{d}\mathcal{H}^{N-1} = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(B_1). \tag{5.16}$$

Note that to get (5.16), the boundary term is cancelled thanks to the second property of (5). Let $\{\tau_1, \ldots, \tau_{N-1}\}$ be an orthonormal basis of T_{x_0}, and $\nu := \nu_\Omega(x_0)$ be the outward unit normal to Ω at x_0 (i.e., ν is a normal vector to T_{x_0}). We choose the vector field $\tilde{\tau}$ in such a way that $\tilde{\tau}(x_0) = \tau_i$, and we decompose $\nabla\varphi$ along the orthonormal basis $\{\tau_1, \ldots, \tau_{N-1}, \nu\}$

of $\mathbf{R}^N$ as $\nabla\varphi = \sum_{j=1}^{N-1}(\partial_j\varphi)\tau_j + (\partial_\nu\varphi)\nu$. Since $\int_{T_{x_0}\cap B_1}\partial_j\varphi\,\mathrm{d}\mathcal{H}^{N-1} = 0$ for all $1 \le j \le N-1$, we infer from (5.16) that

$$((\bar{A}(x_0)\tau_i)\cdot\nu)\int_{T_{x_0}\cap B_1}\partial_\nu\varphi\,\mathrm{d}\mathcal{H}^{N-1} = 0.$$

From the arbitrariness of φ, it follows that $(\bar{A}(x_0)\tau_i)\cdot\nu = 0$ for all $1 \le i \le N-1$. Since $\bar{A}(x_0)$ is symmetric, we deduce that $\bar{A}(x_0)\nu \in T_{x_0}^{\perp}$, that is, $\bar{A}(x_0)\nu = c\nu$ for some $c \in [0,1]$ (recall that all eigenvalues of $\bar{A}(x_0)$ belong to $[0,1]$ by Lemma 5.2). Thus ν is an eigenvector of $\bar{A}(x_0)$, and by the spectral theorem, we can also assume without loss of generality that $\tau_1,\dots,\tau_{N-1}$ are also eigenvectors of $\bar{A}(x_0)$.

Step 2: We next show that $\bar{A}(x_0)$ is the projection matrix onto the tangent space to $\partial\Omega$ at x_0. We now consider a test vector field ϕ of the form $\phi(x) := \tilde{\nu}(x)\varphi(\frac{x-x_0}{\varrho})$ for $x \in B_\varrho(x_0)$, where $\varphi \in \mathcal{C}_c^\infty(B_1)$ and $\tilde{\nu} \in \mathcal{C}_c^1(\mathbf{R}^N;\mathbf{R}^N)$ is such that $\tilde{\nu}(x)$ is a normal vector to $\partial\Omega$ at $x \in \partial\Omega$ satisfying $\tilde{\nu}(x_0) = \nu_\Omega(x_0)$. Using estimates in a similar way to the proof of Lemma 5.3 (this time, the boundary term is cancelled thanks to the first property of (5)), we obtain

$$\int_{T_{x_0}\cap B_1}\bar{A}(x_0) : (\nu\otimes\nabla\varphi)\,\mathrm{d}\mathcal{H}^{N-1} = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(B_1),$$

and thus, by Step 1, $c\int_{T_{x_0}\cap B_1}\partial_\nu\varphi\,\mathrm{d}\mathcal{H}^{N-1} = 0$. By the arbitrariness of φ, this last equality shows that $c = 0$. As a consequence, there exist real numbers $c_1,\dots,c_{N-1} \in [0,1]$ (the eigenvalues of $\bar{A}(x_0)$ associated to the eigenvectors $\tau_1,\dots,\tau_{N-1}$) such that $\bar{A}(x_0) = \sum_{i=1}^{N-1}c_i\tau_i\otimes\tau_i$. According to Lemma 5.2, $\mathrm{tr}(\bar{A}(x_0)) = c_1+\dots+c_{N-1} = N-1$, and we deduce that $c_1 = \dots = c_{N-1} = 1$. Hence, $\bar{A}(x_0) = \sum_{i=1}^{N-1}\tau_i\otimes\tau_i = \mathrm{Id} - \nu\otimes\nu$ as announced. ■

We can now complete the proof of our second main theorem.

Proof of Theorem 1.2. (i) This point is a consequence of Lemma 4.1.

(ii) Using that $\nu_{u_*} = \pm\nu_\Omega$, $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega\cap\hat{J}_{u_*}$, and gathering Lemmas 5.3 and 5.4, yields $\bar{A} = \mathrm{Id} - \nu_{u_*}\otimes\nu_{u_*}$, $\mathcal{H}^{N-1}$-a.e. in $\hat{J}_{u_*}$. Thus, according to (5.8) and (5.9), we get

$$\int_{\hat{J}_{u_*}}\bar{A} : DX\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\hat{J}_{u_*}}(\mathrm{Id} - \nu_{u_*}\otimes\nu_{u_*}) : DX\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then Lemma 5.1 implies that

$$\begin{aligned}\int_\Omega(|\nabla u_*|^2\mathrm{Id} - 2\nabla u_*\otimes\nabla u_*) : DX\,\mathrm{d}x + \int_{\hat{J}_{u_*}}(\mathrm{Id} - \nu_{u_*}\otimes\nu_{u_*}) : DX\,\mathrm{d}\mathcal{H}^{N-1}\\ = -\int_{\partial\Omega}(X\cdot\nu_\Omega)\,\mathrm{d}\lambda_* + \int_{\partial\Omega}|\nabla_\tau g|^2(X\cdot\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1}\\ - 2\int_{\partial\Omega}(\nabla u_*\cdot\nu_\Omega)(X_\tau\cdot\nabla_\tau g)\,\mathrm{d}\mathcal{H}^{N-1}\end{aligned}$$

for all $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$. Specifying this identity to vector fields $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$ satisfying $X \cdot \nu_\Omega = 0$ on $\partial\Omega$ leads to

$$\int_\Omega (|\nabla u_*|^2 \mathrm{Id} - 2\nabla u_* \otimes \nabla u_*) : DX \,\mathrm{d}x + \int_{\hat{J}_{u_*}} (\mathrm{Id} - \nu_{u_*} \otimes \nu_{u_*}) : DX \,\mathrm{d}\mathcal{H}^{N-1}$$
$$= -2 \int_{\partial\Omega} (\nabla u_* \cdot \nu_\Omega)(X \cdot \nabla_\tau g) \,\mathrm{d}\mathcal{H}^{N-1},$$

and (1.17) follows from the definition of the tangential divergence of X on the countably $\mathcal{H}^{N-1}$-rectifiable set $\hat{J}_{u_*}$. ■

The results of this section also give the following convergences that will be used in Section 6.

Corollary 5.1. *Let $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$, w_ε given by (5.2), and $u_* \in \mathrm{SBV}^2(\Omega) \cap L^\infty(\Omega)$ be as in Theorem 1.2. Then*

$$\frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} \otimes \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|} |\nabla w_\varepsilon| \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*}$$
$$\textit{weakly* in } \mathcal{M}(\overline{\Omega}; \mathbf{M}^{N\times N}), \quad (5.17)$$

$$\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \textit{weakly* in } \mathcal{M}(\overline{\Omega}; \mathbf{M}^{N\times N}). \quad (5.18)$$

Proof. The first point (5.17) follows from (5.14) together with Lemmas 5.3 and 5.4 by observing that $\nu_{u_*} = \pm\nu_\Omega$, $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega \cap \hat{J}_{u_*}$. The second point (5.18) follows from the first one, the equipartition of energy (5.4), and the fact that $\frac{\nabla v_\varepsilon}{|\nabla v_\varepsilon|} = \frac{\nabla w_\varepsilon}{|\nabla w_\varepsilon|}$. ■

6. Passing to the limit in the second inner variation

The aim of this section is to complement Theorem 1.2, also proving the convergence of the second inner variation of AT_ε. As a consequence, we shall deduce that if the limit u_* comes from stable critical points of AT_ε, then u_* satisfies a certain stability condition for MS. Our analysis and result parallel completely those in [25–27] for the Allen–Cahn-type energies arising in phase transitions problems.

Proof of Theorem 1.3. Assume that $\partial\Omega$ is of class $\mathcal{C}^{3,1}$ and $g \in \mathcal{C}^{3,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$. By Theorem 3.2, if $(u_\varepsilon, v_\varepsilon)$ is a critical point of the Ambrosio–Tortorelli functional then it belongs to $[\mathcal{C}^{3,\alpha}(\overline{\Omega})]^2$. To prove the convergence of the second inner variation, we use Lemma A.3 and formula (A.6). From Proposition 5.1, we know that

$$\begin{cases} \sqrt{\eta_\varepsilon + v_\varepsilon^2}\nabla u_\varepsilon \to \nabla u_* & \text{strongly in } L^2(\Omega;\mathbf{R}^N), \\ \varepsilon|\nabla v_\varepsilon|^2 \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \frac{1}{2}\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \text{weakly* in } \mathcal{M}(\overline{\Omega}), \\ \dfrac{(v_\varepsilon - 1)^2}{4\varepsilon} \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \frac{1}{2}\mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} & \text{weakly* in } \mathcal{M}(\overline{\Omega}). \end{cases}$$

On the other hand, Corollary 5.1 ensures that

$$\varepsilon \nabla v_\varepsilon \otimes \nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega \overset{*}{\rightharpoonup} \frac{1}{2} \nu_{u_*} \otimes \nu_{u_*} \mathcal{H}^{N-1} \llcorner \hat{J}_{u_*} \quad \text{weakly* in } \mathcal{M}(\overline{\Omega}; \mathbf{M}^{N\times N}).$$

Let $X \in \mathcal{C}^2_c(\mathbf{R}^N; \mathbf{R}^N)$ and $G \in \mathcal{C}^2(\mathbf{R}^N)$ be such that $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$, and set $Y := (DX)X$. Observing that $|DX^\mathsf{T} \nabla v_\varepsilon|^2 = (DX(DX)^\mathsf{T}) : (\nabla v_\varepsilon \otimes \nabla v_\varepsilon)$, we can pass to the limit in all the terms of $\delta^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G]$ in (A.6) to find that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \delta^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G]
&= \int_\Omega (|\nabla u_*|^2 \mathrm{Id} - 2(\nabla u_* \otimes \nabla u_*)) : DY \,\mathrm{d}x \\
&\quad + \int_{\hat{J}_{u_*}} \mathrm{div}^{\hat{J}_{u_*}} Y \,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + \int_\Omega |\nabla u_*|^2((\mathrm{div}\, X)^2 - \mathrm{tr}(DX)^2) - 4((\nabla u_* \otimes \nabla u_*) : DX)\,\mathrm{div}\, X \,\mathrm{d}x \\
&\quad + \int_\Omega [4(\nabla u_* \otimes \nabla u_*) : (DX)^2 + 2|DX^\mathsf{T} \nabla u_*|^2] \,\mathrm{d}x \\
&\quad + 4\int_\Omega [\nabla u_* \cdot \nabla(X \cdot \nabla G)\,\mathrm{div}\, X - (\nabla u_* \otimes \nabla(X \cdot \nabla G)) : (DX + (DX)^\mathsf{T})] \,\mathrm{d}x \\
&\quad + 2\int_\Omega \nabla u_* \cdot \nabla(X \cdot \nabla(X \cdot G)) \,\mathrm{d}x + 2\int_\Omega |\nabla(X \cdot \nabla G)|^2 \,\mathrm{d}x \\
&\quad + \int_{\hat{J}_{u_*}} [(\mathrm{div}\, X)^2 - \mathrm{tr}(DX)^2 - 2((\nu_{u_*} \otimes \nu_{u_*}) : DX)\,\mathrm{div}\, X] \,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad + 2\int_{\hat{J}_{u_*}} [(\nu_{u_*} \otimes \nu_{u_*}) : (DX)^2 + |DX^\mathsf{T} \nu_{u_*}|^2] \,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned} \tag{6.1}$$

Using the geometric formulas stated in [25, proof of Theorem 1.1, pp. 1851–1852], we infer that

$$\begin{aligned}
&(\mathrm{div}\, X)^2 - \mathrm{tr}[(DX)^2] - 2((\nu_{u_*} \otimes \nu_{u_*}) : DX)\,\mathrm{div}\, X + 2(\nu_{u_*} \otimes \nu_{u_*}) : (DX)^2 \\
&\qquad + |DX^\mathsf{T} \nu_{u_*}|^2 \\
&\quad = (\mathrm{div}^{J_{u_*}} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X)(\tau_j \cdot \partial_{\tau_i} X) \\
&\qquad + ((\nu_{u_*} \otimes \nu_{u_*}) : DX)^2.
\end{aligned} \tag{6.2}$$

According to the expression of the inner second variation of the Mumford–Shah energy stated in Lemma A.4, (6.1), and (6.2), we infer that

$$\lim_{\varepsilon\to 0} \delta^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \delta^2 \mathrm{MS}(u_*)[X, G] + \int_{\hat{J}_{u_*}} ((\nu_{u_*} \otimes \nu_{u_*}) : DX)^2 \,\mathrm{d}\mathcal{H}^{N-1}.$$

Now assume that $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ is a stable critical point of AT_ε, i.e.,

$$\mathrm{d}^2 \mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[\phi, \psi] \geq 0 \quad \text{for all } (\phi, \psi) \in H^1_0(\Omega) \times [H^1_0(\Omega) \cap L^\infty(\Omega)], \tag{6.3}$$

where $\mathrm{d}^2\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)$ is the second outer variation of AT_ε at $(u_\varepsilon, v_\varepsilon)$ given by formula (A.1).

Let us fix an arbitrary vector field $X \in \mathcal{C}^2_c(\mathbf{R}^N; \mathbf{R}^N)$ and an arbitrary function $G \in \mathcal{C}^3(\mathbf{R}^N)$ satisfying $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$. According to Lemma A.2, we have

$$\delta^2\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \mathrm{d}^2\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(u_\varepsilon - G), X \cdot \nabla v_\varepsilon] \\ + \mathrm{dAT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)), X \cdot \nabla(X \cdot \nabla v_\varepsilon)].$$

Since $(u_\varepsilon, v_\varepsilon) = (g, 1)$ and $X \cdot \nu_\Omega = 0$ on $\partial\Omega$, we have $X \cdot \nabla(u_\varepsilon - G) = X \cdot \nabla v_\varepsilon = 0$ on $\partial\Omega$. As a consequence, the functions $X \cdot \nabla(u_\varepsilon - G)$ and $X \cdot \nabla v_\varepsilon$ belong to $\mathcal{C}^2(\overline{\Omega})$ and also vanish on $\partial\Omega$. Therefore, $X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)) = X \cdot \nabla(X \cdot \nabla v_\varepsilon) = 0$ on $\partial\Omega$. Next, $(u_\varepsilon, v_\varepsilon)$ being a critical point of AT_ε, we have

$$\mathrm{dAT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(X \cdot \nabla(u_\varepsilon - G)), X \cdot \nabla(X \cdot \nabla v_\varepsilon)] = 0.$$

Back to (6.3), it follows that

$$\delta^2\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X, G] = \mathrm{d}^2\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon)[X \cdot \nabla(u_\varepsilon - G), X \cdot \nabla v_\varepsilon] \geq 0.$$

Now passing to the limit in the second inner variation yields (1.18), and the proof of Theorem 1.3 is now complete. ∎

Remark 6.1. In [7, 11], the authors explore second-order minimality conditions for the Mumford–Shah functional in the case where the jump set is regular enough. Such conditions could be derived in our context, taking care of the Dirichlet boundary data and thus of the fact that the jump set can charge the boundary. We do not develop this point here and refer to [11, Theorem 3.6], where the authors provide another expression for $\delta^2\mathrm{MS}(u)$ defined for smooth vector fields X compactly supported in Ω (see Remark A.1). But we indicate that, as a consequence of Theorem 1.3, it can be seen that, if $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ is a stable critical point of AT_ε such that, up to a subsequence, $u_\varepsilon \to u_*$ in $L^2(\Omega)$ and (1.16) hold, then u_* satisfies the second-order minimality condition for the Mumford–Shah functional derived in [7, 11], provided $\hat{J}_{u_*}$ is sufficiently smooth. This follows by choosing $X \in \mathcal{C}^\infty_c(\mathbf{R}^N; \mathbf{R}^N)$ of the form $X = \varphi\nu_{u_*} \circ \Pi$ in a neighborhood of $\hat{J}_{u_*}$ and satisfying $\nu_{u_*} \cdot (DX\nu_{u_*}) = 0$ on $\hat{J}_{u_*}$, where Π denotes the nearest point projection onto $\hat{J}_{u_*}$ and φ is an arbitrary smooth scalar function.

7. Conclusion

We have proved that critical points of the Ambrosio–Tortorelli functional converge to critical points of the Mumford–Shah functional if the convergence of energies $\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathrm{MS}(u_*)$ holds true.

The same assumption has been used before in [25–27, 34] to prove that critical points of the Allen–Cahn energy converge to minimal surfaces. However, the results of these papers are different and more general. Indeed, in those references the authors prove the following stronger result: if $z_\varepsilon \in H^1(\Omega)$ is such that $z_\varepsilon \to z$ in $L^1(\Omega)$ with $z \in \mathrm{BV}(\Omega;\{1,-1\})$ and

$$\mathrm{AC}_\varepsilon(z_\varepsilon) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla z_\varepsilon|^2 + \frac{(1-z_\varepsilon^2)^2}{2\varepsilon}\Big)\,\mathrm{d}x \to P(z) := \frac{4}{3}|Dz|(\Omega),$$

then, provided the interface $\Gamma := \partial\{z=1\}\cap\Omega$ is such that $\bar{\Gamma}$ is of class $\mathcal{C}^2$,

$$\begin{aligned}&\delta\mathrm{AC}_\varepsilon(z_\varepsilon)[X] \to \delta P(z)[X],\\ &\delta^2\mathrm{AC}_\varepsilon(z_\varepsilon)[X] \to \delta^2 P(z)[X] + \frac{4}{3}\int_\Gamma |DX : (\nu\otimes\nu)|^2\,\mathrm{d}\mathcal{H}^{N-1}\end{aligned}$$

for all smooth vector fields $X : \overline{\Omega} \to \mathbf{R}^N$ that are tangent on the boundary. In particular, this result holds true even if z_ε is not a critical point of the Allen–Cahn energy. The argument rests on the observation that the first and second inner variations of AC_ε can be written as functionals of the measure $|1-z_\varepsilon^2|\nabla z_\varepsilon \mathcal{L}^N \llcorner \Omega$ which, thanks to the convergence of energy assumption, converges in a strict sense to $\frac{4}{3}\nu_\Gamma \mathcal{H}^{N-1}\llcorner\Gamma$. The passage to the limit in $\delta\mathrm{AC}_\varepsilon(z_\varepsilon)[X]$ and $\delta^2\mathrm{AC}_\varepsilon(z_\varepsilon)[X]$ is then an immediate consequence of the Reshetnyak continuity theorem (see [1, Theorem 2.39]).

In our case, if $(u_\varepsilon, v_\varepsilon) \in \mathcal{A}_g(\Omega)$ and $u_* \in \mathrm{SBV}^2(\Omega)$ are such that $\mathrm{AT}_\varepsilon(u_\varepsilon, v_\varepsilon) \to \mathrm{MS}(u_*)$, the analogous object to consider is the measure $(1-v_\varepsilon)\nabla v_\varepsilon \mathcal{L}^N \llcorner \Omega$, which only converges weakly* in the sense of measures to 0, while its variation $(1-v_\varepsilon)|\nabla v_\varepsilon|\mathcal{L}^N\llcorner\Omega$ converges to $\mathcal{H}^{N-1}\llcorner \hat{J}_{u_*}$ (see Proposition 5.1). In particular, no similar strict convergence result holds. This is related to a cancellation property: in the Allen–Cahn case, jumps correspond to a single (positive or negative) transition of the phase field variable z_ε between the wells -1 and 1, while in the Ambrosio–Tortorelli case, jumps are the result of a first (negative) transition of the phase field variable v_ε from 1 to 0, and a second (positive) one from 0 to 1. It turns out that both transitions cancel out due to the opposite signs. However, taking the variation, the transitions sum in absolute values and the jump set appears as the limit of the mass. It formally explains why our result is really about critical points, and why we had to use additional ingredients based on PDE and varifold approaches.

A. First and second variations

In this appendix we give explicit expressions for outer and inner variations of the Ambrosio–Tortorelli and Mumford–Shah functionals. First, we recall the expression of the first and second outer variations of AT_ε defined by (1.5) and (1.6).

Lemma A.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with Lipschitz boundary and $g \in H^{\frac{1}{2}}(\partial\Omega)$. For all $(u,v) \in \mathcal{A}_g(\Omega)$ and all $(\phi,\psi) \in H^1_0(\Omega)\times[H^1_0(\Omega)\cap L^\infty(\Omega)]$,*

$$\mathrm{dAT}_\varepsilon(u,v)[\phi,\psi] = 2\int_\Omega \Big[(\eta_\varepsilon + v^2)\nabla u\cdot\nabla\phi + \varepsilon\nabla v\cdot\nabla\psi + |\nabla u|^2 v\psi + \frac{(v-1)\psi}{4\varepsilon}\Big]\,\mathrm{d}x,$$

$$\mathrm{d}^2\mathrm{AT}_\varepsilon(u,v)[\phi,\psi] = \int_\Omega \Big[(\eta_\varepsilon + v^2)|\nabla\phi|^2 + 4v\psi\nabla u\cdot\nabla\phi + \varepsilon|\nabla\psi|^2 + |\nabla u|^2\psi^2 + \frac{\psi^2}{4\varepsilon}\Big]\,\mathrm{d}x. \tag{A.1}$$

The computations of inner variations rely on one-parameter groups of diffeomorphisms over $\overline{\Omega}$, or equivalently on their infinitesimal generators. More precisely, assuming that $\partial\Omega$ is of class $\mathcal{C}^{k+1}$ with $k \geq 1$, and given a vector field $X \in \mathcal{C}^k_c(\mathbb{R}^N;\mathbb{R}^N)$ satisfying $X\cdot\nu_\Omega = 0$ on $\partial\Omega$, we consider the integral flow $\{\Phi_t\}_{t\in\mathbf{R}}$ of X defined through the resolution of ODE (1.7) for every $x \in \mathbf{R}^N$. Then $\Phi_0 = \mathrm{Id}$ and the flow rule asserts that $\Phi_{t+s} = \Phi_t \circ \Phi_s$. Since $X\cdot\nu_\Omega = 0$ on $\partial\Omega$, $\{\Phi_t\}_{t\in\mathbf{R}}$ is a one-parameter group of $\mathcal{C}^k$-diffeomorphisms from Ω into itself, and from $\partial\Omega$ into itself.

Given (sufficiently smooth) boundary data g, we consider an arbitrary (smooth) extension G of g to $\overline{\Omega}$ to define a one-parameter family of deformations $\{(u_t,v_t)\}_{t\in\mathbf{R}} \subset \mathcal{A}_g(\Omega)$ satisfying $(u_0,v_0) = (u,v)$ for a given pair $(u,v) \in \mathcal{A}_g(\Omega)$ by setting $u_t := u\circ\Phi_{-t} - G\circ\Phi_{-t} + G$ and $v_t := v\circ\Phi_{-t}$. The first and second inner variations $\delta\mathrm{AT}_\varepsilon$ and $\delta^2\mathrm{AT}_\varepsilon$ of AT_ε at (u,v) are then defined by (1.9).

Remark A.1. We emphasize that $\delta\mathrm{AT}_\varepsilon(u,v)[X,G]$ and $\delta^2\mathrm{AT}_\varepsilon(u,v)[X,G]$ depend on both the vector field X and the extension G of the boundary data g, because the family of deformations $\{(u_t,v_t)\}_{t\in\mathbf{R}}$ depends on X and G. It allows one to perform inner variations of the energy *up to the boundary*. This type of deformation includes the more usual variation $\{(u\circ\Phi_{-t}, v\circ\Phi_{-t})\}_{t\in\mathbf{R}}$ with X compactly supported in Ω. Indeed, in this case we may choose an extension G supported in a small neighborhood of $\partial\Omega$ in such a way that $\operatorname{supp} G \cap \operatorname{supp} X = \emptyset$. Then $G\circ\Phi_{-t} = G$, and thus $u_t = u\circ\Phi_{-t}$.

If the pair (u,v) and $\partial\Omega$ are smooth enough, one can compute the first and second inner variations of AT_ε at (u,v) using the Taylor expansion of (u_t,v_t) with respect to the parameter t. One may for instance follow the general setting of [27, Lemma 2.2 and Corollary 2.3].

Lemma A.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with boundary of class $\mathcal{C}^2$, $g \in \mathcal{C}^2(\partial\Omega)$, and $(u,v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^2(\overline{\Omega})]^2$.*

(i) *Then for every vector field $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$ and every extension $G \in \mathcal{C}^2(\mathbf{R}^N)$ satisfying $X\cdot\nu_\Omega = 0$ and $G = g$ on $\partial\Omega$,*

$$\delta\mathrm{AT}_\varepsilon(u,v)[X,G] = -\,\mathrm{d}\mathrm{AT}_\varepsilon(u,v)[X\cdot\nabla(u-G), X\cdot\nabla v].$$

(ii) *If further $\partial\Omega$ is of class $\mathcal{C}^3$, $g \in \mathcal{C}^3(\partial\Omega)$, and $(u,v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^3(\overline{\Omega})]^2$, then for every vector field $X \in \mathcal{C}^2_c(\mathbf{R}^N;\mathbf{R}^N)$ and every extension $G \in \mathcal{C}^3(\mathbf{R}^N)$ satisfying $X\cdot\nu_\Omega = 0$ and $G = g$ on $\partial\Omega$,*

$$\delta^2\mathrm{AT}_\varepsilon(u,v)[X,G] = \mathrm{d}^2\mathrm{AT}_\varepsilon(u,v)[X\cdot\nabla(u-G), X\cdot\nabla v] + \mathrm{d}\mathrm{AT}_\varepsilon(u,v)[X\cdot\nabla(X\cdot\nabla(u-G)), X\cdot\nabla(X\cdot\nabla v)].$$

Proof. Define $\bar{u}_t := u \circ \Phi_{-t} - G \circ \Phi_{-t}$. Since (u, v) and G belong to $\mathcal{C}^2(\overline{\Omega})$, we can differentiate (u_t, v_t) with respect to t and use (1.7) with the flow rule $\Phi_{t+s} = \Phi_t \circ \Phi_s$ to find

$$(\dot{u}_t, \dot{v}_t) := \frac{\mathrm{d}}{\mathrm{d}t}(u_t, v_t) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\bar{u}_{t+s}, v_{t+s}) = (-X \cdot \nabla \bar{u}_t, -X \cdot \nabla v_t) \in [\mathcal{C}^1(\overline{\Omega})]^2.$$

In particular, we have

$$(\dot{u}_0, \dot{v}_0) = (-X \cdot \nabla(u - G), -X \cdot \nabla v). \tag{A.2}$$

If $\partial\Omega$ is of class $\mathcal{C}^3$, $g \in \mathcal{C}^3(\partial\Omega)$, and $(u, v) \in \mathcal{A}_g(\Omega) \cap [\mathcal{C}^3(\overline{\Omega})]^2$, then we can differentiate $(\dot{u}_t, \dot{v}_t)$ with respect to t to obtain

$$\ddot{u}_t := \frac{\mathrm{d}}{\mathrm{d}t}\dot{u}_t = -X \cdot \nabla \dot{u}_t = X \cdot \nabla(X \cdot \nabla \bar{u}_t),$$
$$\ddot{v}_t := \frac{\mathrm{d}}{\mathrm{d}t}\dot{v}_t = -X \cdot \nabla \dot{v}_t = X \cdot \nabla(X \cdot \nabla v_t).$$

Hence,

$$\ddot{u}_0 = X \cdot \nabla(X \cdot \nabla(u - G)) \quad \text{and} \quad \ddot{v}_0 = X \cdot \nabla(X \cdot \nabla v). \tag{A.3}$$

Next, elementary computations yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{AT}_\varepsilon(u_t, v_t) = \mathrm{dAT}_\varepsilon(u_t, v_t)[\dot{u}_t, \dot{v}_t]$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{AT}_\varepsilon(u_t, v_t) = \mathrm{d}^2\mathrm{AT}_\varepsilon(u_t, v_t)[\dot{u}_t, \dot{v}_t] + \mathrm{dAT}_\varepsilon(u_t, v_t)[\ddot{u}_t, \ddot{v}_t],$$

so that the conclusion follows from (A.2)–(A.3) evaluating those derivatives at $t = 0$. ■

In the case that the pair (u, v) only belongs to the energy space $\mathcal{A}_g(\Omega)$, we can compute the first and second inner variations of AT_ε by making the change of variables $y = \Phi_t(x)$ in the integrals defining $\mathrm{AT}_\varepsilon(u_t, v_t)$. Then one expands the result with respect to t using a Taylor expansion of Φ_t. If $X \in \mathcal{C}^2_c(\mathbf{R}^N; \mathbf{R}^N)$, the second-order Taylor expansion near $t = 0$ of the flow map Φ_t induced by X is given by

$$\Phi_t = \mathrm{Id} + tX + \frac{t^2}{2}Y + o(t^2), \tag{A.4}$$

where $Y \in \mathcal{C}^1_c(\mathbf{R}^N; \mathbf{R}^N)$ denotes the vector field $Y := (DX)X$, DX being the Jacobian matrix of X (i.e. $(DX)_{ij} = \partial_j X_i$, with i the row index and j the column index), and $o(s)$ denotes a quantity satisfying $o(s)/s \to 0$ as $s \to 0$ uniformly with respect to $x \in \mathbf{R}^N$.

Lemma A.3. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set with boundary of class $\mathcal{C}^2$, $g \in \mathcal{C}^2(\partial\Omega)$ and $(u, v) \in \mathcal{A}_g(\Omega)$.*

(i) *Then for every vector field* $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$ *and every extension* $G \in \mathcal{C}^2(\mathbf{R}^N)$ *satisfying* $X \cdot \nu_\Omega = 0$ *and* $G = g$ *on* $\partial\Omega$,

$$\begin{aligned}\delta\mathrm{AT}_\varepsilon(u,v)[X,G] &= \int_\Omega (\eta_\varepsilon + v^2)[|\nabla u|^2\mathrm{Id} - 2\nabla u \otimes \nabla u] : DX\,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\Big(\varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\mathrm{Id} - 2\varepsilon\nabla v \otimes \nabla v\Big] : DX\,\mathrm{d}x \\ &\quad + 2\int_\Omega (\eta_\varepsilon + v^2)\nabla u \cdot \nabla(X \cdot \nabla G)\,\mathrm{d}x. \qquad (A.5)\end{aligned}$$

(ii) *If, further,* $\partial\Omega$ *is of class* $\mathcal{C}^3$, $g \in \mathcal{C}^3(\partial\Omega)$, *then for every vector field* $X \in \mathcal{C}^2_c(\mathbf{R}^N;\mathbf{R}^N)$ *and every extension* $G \in \mathcal{C}^3(\mathbf{R}^N)$ *satisfying* $X \cdot \nu_\Omega = 0$ *and* $G = g$ *on* $\partial\Omega$,

$$\begin{aligned}&\delta^2\mathrm{AT}_\varepsilon(u,v)[X,G] \\ &= \int_\Omega (\eta_\varepsilon + v^2)(|\nabla u|^2\mathrm{Id} - 2(\nabla u \otimes \nabla u)) : DY\,\mathrm{d}x \\ &\quad + \int_\Omega (\eta_\varepsilon + v^2)\big\{|\nabla u|^2\big((\mathrm{div}\,X)^2 - \mathrm{tr}((DX)^2)\big) \\ &\qquad\qquad - 4\big((\nabla u \otimes \nabla u) : DX\big)\,\mathrm{div}\,X + 4(\nabla u \otimes \nabla u) : (DX)^2 \\ &\qquad\qquad + 2|DX^{\mathsf{T}}\nabla u|^2\big\}\,\mathrm{d}x \\ &\quad + 4\int_\Omega (\eta_\varepsilon + v^2)\big[(\nabla u \cdot \nabla(X \cdot \nabla G))\,\mathrm{div}\,X \\ &\qquad\qquad - (\nabla u \otimes \nabla(X \cdot \nabla G)) : (DX + (DX)^{\mathsf{T}})\big]\,\mathrm{d}x \\ &\quad + 2\int_\Omega (\eta_\varepsilon + v^2)\nabla u \cdot \nabla(X \cdot \nabla(X \cdot \nabla G))\,\mathrm{d}x \\ &\quad + 2\int_\Omega (\eta_\varepsilon + v^2)|\nabla(X \cdot \nabla G)|^2\,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\Big(\varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\mathrm{Id} - 2\varepsilon\nabla v \otimes \nabla v\Big] : DY\,\mathrm{d}x \\ &\quad + \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\big((\mathrm{div}\,X)^2 - \mathrm{tr}((DX)^2)\big)\,\mathrm{d}x \\ &\quad - 4\int_\Omega \varepsilon((\nabla v \otimes \nabla v) : DX)\,\mathrm{div}\,X\,\mathrm{d}x \\ &\quad + 4\int_\Omega \varepsilon(\nabla v \otimes \nabla v) : (DX)^2\,\mathrm{d}x + 2\int_\Omega \varepsilon|DX^{\mathsf{T}}\nabla v|^2\,\mathrm{d}x, \qquad (A.6)\end{aligned}$$

with $Y := (DX)X$.

Remark A.2. From (A.5) we see that if $(u,v) \in \mathcal{A}_g(\Omega)$ is a critical point of AT_ε in the sense that $\mathrm{dAT}_\varepsilon(u,v)[\phi,\psi] = 0$ for all admissible (ϕ,ψ), then $\delta\mathrm{AT}_\varepsilon(u,v)[X,G]$ is independent of the extension G. Indeed, an integration by parts and the first equation in

(1.2) show that, in this case,

$$\int_\Omega (\eta_\varepsilon + v^2)\nabla u \cdot \nabla (X \cdot \nabla G)\,\mathrm{d}x = (\eta_\varepsilon + 1)\int_{\partial\Omega} \partial_\nu u (X \cdot \nabla_\tau g)\,\mathrm{d}\mathcal{H}^{N-1},$$

since $X \cdot \nu_\Omega = 0$ on $\partial\Omega$ (∇_τ is the tangential gradient on $\partial\Omega$). For the second inner variation (A.6), even if $(u, v) \in \mathcal{A}_g(\Omega)$ satisfies $\mathrm{dAT}_\varepsilon(u, v)[\phi, \psi] = 0$ for $(\phi, \psi) \in [\mathcal{C}_c^\infty(\Omega)]^2$, the expression $\delta^2\mathrm{AT}_\varepsilon(u, v)[X, G]$ does depend on the extension G, and not only on the boundary data g, because of the terms

$$\int_\Omega (\eta_\varepsilon + v^2)[(\nabla u \cdot \nabla(X \cdot \nabla G))\mathrm{Id} - 2\nabla u \otimes \nabla(X \cdot \nabla G)] : DX\,\mathrm{d}x \\ + \int_\Omega (\eta_\varepsilon + v^2)|\nabla(X \cdot \nabla G)|^2\,\mathrm{d}x.$$

If we take $X \in \mathcal{C}_c^\infty(\Omega; \mathbf{R}^N)$ and $G \in \mathcal{C}^2(\mathbf{R}^N)$ an extension of g such that $\operatorname{supp} G \cap \operatorname{supp} X = \emptyset$, and if we assume $(u, v) \in \mathcal{A}_g(\Omega)$ to be a critical of AT_ε, then the expression of the second inner variation (A.6) simplifies. Indeed, the terms that contain $Y = (DX)X$ disappear, since by regularity we have $\delta\mathrm{AT}_\varepsilon(u, v)[Y, G] = 0$, and all terms containing G disappear. In this case, we end up with

$$\begin{aligned}
&\delta^2\mathrm{AT}_\varepsilon(u, v)[X, G] \\
&\quad = \int_\Omega (\eta_\varepsilon + v^2)\big\{\big((\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)\big) - 4((\nabla u \otimes \nabla u) : DX)\operatorname{div} X \\
&\qquad\qquad + 4(\nabla u \otimes \nabla u) : (DX)^2 + 2|DX^\mathsf{T}\nabla u|^2\big\}\,\mathrm{d}x \\
&\qquad + \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\big((\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)\big)\,\mathrm{d}x \\
&\qquad - 4\int_\Omega \varepsilon((\nabla v \otimes \nabla v) : DX)\operatorname{div} X\,\mathrm{d}x + 4\int_\Omega \varepsilon(\nabla v \otimes \nabla v) : (DX)^2\,\mathrm{d}x \\
&\qquad + 2\int_\Omega \varepsilon|DX^\mathsf{T}\nabla v|^2\,\mathrm{d}x.
\end{aligned}$$

Proof of Lemma A.3. For simplicity, we assume that $\partial\Omega$ is of class $\mathcal{C}^3$, $g \in \mathcal{C}^3(\partial\Omega)$, and we observe that the computation of $\delta\mathrm{AT}_\varepsilon$ below only requires $\mathcal{C}^2$ regularity. We fix $X \in \mathcal{C}_c^2(\mathbf{R}^N; \mathbf{R}^N)$ and $G \in \mathcal{C}^3(\overline{\Omega})$ satisfying $X \cdot \nu = 0$ and $G = g$ on $\partial\Omega$. We set $\hat{u}_t := u \circ \Phi_{-t}$ and $G_t := G \circ \Phi_{-t}$. Since $u_t = \hat{u}_t - (G_t - G)$, we have

$$\begin{aligned}
\mathrm{AT}_\varepsilon(u_t, v_t) = {} & \mathrm{AT}_\varepsilon(\hat{u}_t, v_t) - 2\int_\Omega (\eta_\varepsilon + v_t^2)\nabla\hat{u}_t \cdot \nabla(G_t - G)\,\mathrm{d}x \\
& + \int_\Omega (\eta_\varepsilon + v_t^2)|\nabla(G_t - G)|^2\,\mathrm{d}x \\
=: {} & \mathcal{A}(t) + \mathcal{B}(t) + \mathcal{C}(t).
\end{aligned} \tag{A.7}$$

We aim to compute the first and second derivatives at $t = 0$ of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, starting with $\mathcal{A}$.

By the chain rule in Sobolev spaces, we have

$$\nabla \hat{u}_t = [D\Phi_{-t}]^{\mathsf{T}} \nabla u(\Phi_{-t}) = [D\Phi_t(\Phi_{-t})]^{-\mathsf{T}} \nabla u(\Phi_{-t}). \tag{A.8}$$

On the other hand, in view of (A.4), we have

$$[D\Phi_t]^{-1} = I - tDX - \frac{t^2}{2} Y + t^2 (DX)^2 + o(t^2) \tag{A.9}$$

and

$$\det(D\Phi_t) = 1 + t \operatorname{div} X + \frac{t^2}{2}[\operatorname{div} Y + (\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)] + o(t^2). \tag{A.10}$$

Using the change of variables $x = \Phi_t(y)$, classical computations (see e.g. [27]) yield

$$\begin{aligned}\mathcal{A}'(0) = &\int_\Omega (\eta_\varepsilon + v^2)[|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u] : DX \,\mathrm{d}x \\ &+ \int_\Omega \Big[\Big(\varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\mathrm{Id} - 2\varepsilon \nabla v \otimes \nabla v\Big] : DX \,\mathrm{d}x\end{aligned} \tag{A.11}$$

and

$$\begin{aligned}\mathcal{A}''(0) = &\int_\Omega (\eta_\varepsilon + v^2)\big\{|\nabla u|^2 \big(\operatorname{div} Y + (\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)\big) \\ &\qquad - 4((\nabla u \otimes \nabla u) : DX) \operatorname{div} X \\ &\qquad - 2(\nabla u \otimes \nabla u) : (DY - 2(DX)^2) + 2|DX^{\mathsf{T}} \nabla u|^2\big\} \,\mathrm{d}x \\ &+ \int_\Omega \Big(\varepsilon |\nabla v|^2 + \frac{(v-1)^2}{4\varepsilon}\Big)\big(\operatorname{div} Y + (\operatorname{div} X)^2 - \operatorname{tr}((DX)^2)\big) \\ &- 4\varepsilon((\nabla v \otimes \nabla v) : DX) \operatorname{div} X \,\mathrm{d}x \\ &- 2\int_\Omega \varepsilon (\nabla v \otimes \nabla v) : (DY - 2(DX)^2) \,\mathrm{d}x + 2\int_\Omega \varepsilon |DX^{\mathsf{T}} \nabla v|^2 \,\mathrm{d}x.\end{aligned} \tag{A.12}$$

Next we compute the derivatives of $\mathcal{B}$. To this purpose, we observe that

$$\nabla \hat{u}_{t+s} = [D\Phi_s(\Phi_{-s})]^{-\mathsf{T}} \nabla \hat{u}_t(\Phi_{-s}), \qquad \nabla G_{t+s} = [D\Phi_s(\Phi_{-s})]^{-\mathsf{T}} \nabla G_t(\Phi_{-s}),$$

and by (A.9),

$$[D\Phi_s]^{-\mathsf{T}} \nabla G_t - \nabla G(\Phi_s) = \nabla(G_t - G) - s(DX)^{\mathsf{T}} \nabla(G_t - G) - s\nabla(X \cdot \nabla G) + o(s).$$

Using the change of variables $x = \Phi_s(y)$ and (A.9)–(A.10) again, we obtain

$$\begin{aligned}&\int_\Omega (\eta_\varepsilon + v_{t+s}^2) \nabla \hat{u}_{t+s} \cdot \nabla(G_{t+s} - G) \,\mathrm{d}x \\ &\quad = \int_\Omega (\eta_\varepsilon + v_t^2)([D\Phi_s]^{-\mathsf{T}} \nabla \hat{u}_t) \cdot ([D\Phi_s]^{-\mathsf{T}} \nabla G_t - \nabla G(\Phi_s)) \det(D\Phi_s) \,\mathrm{d}y \\ &\quad = \int_\Omega (\eta_\varepsilon + v_t^2) \nabla \hat{u}_t \cdot \nabla(G_t - G) \,\mathrm{d}x - s\int_\Omega (\eta_\varepsilon + v^2) \nabla \hat{u}_t \cdot \nabla(X \cdot \nabla G) \,\mathrm{d}x \\ &\qquad + s\int_\Omega (\eta_\varepsilon + v_t^2)[(\nabla \hat{u}_t \cdot \nabla(G_t - G)) \operatorname{div} X \\ &\qquad\qquad - (\nabla \hat{u}_t \otimes \nabla(G_t - G)) : (DX + (DX)^{\mathsf{T}})] \,\mathrm{d}x + o(s).\end{aligned}$$

Consequently,

$$\mathcal{B}'(t) = 2\int_\Omega (\eta_\varepsilon + v_t^2)\nabla\hat{u}_t \cdot \nabla(X\cdot\nabla G)\,\mathrm{d}x$$
$$- 2\int_\Omega (\eta_\varepsilon + v_t^2)[(\nabla\hat{u}_t \cdot \nabla(G_t - G))\,\mathrm{div}\, X$$
$$- (\nabla\hat{u}_t \otimes \nabla(G_t - G)) : (DX + (DX)^{\mathsf{T}})]\,\mathrm{d}x$$

and, in particular,

$$\mathcal{B}'(0) = 2\int_\Omega (\eta_\varepsilon + v^2)\nabla u \cdot \nabla(X\cdot\nabla G)\,\mathrm{d}x. \tag{A.13}$$

To compute $\mathcal{B}''(0)$, we write $\mathcal{B}'(t) =: \mathrm{I}(t) - \mathrm{II}(t)$, and we set, for simplicity, $H := X\cdot\nabla G$. Since

$$\nabla H(\Phi_t) = \nabla H + t\nabla(X\cdot\nabla H) - t(DX)^{\mathsf{T}}\nabla H + o(t),$$

we can change variables $y = \Phi_t(x)$ and use (A.8)–(A.9)–(A.10) again to find

$$\mathrm{I}(t) = 2\int_\Omega (\eta_\varepsilon + v^2)\nabla u\cdot\nabla H\,\mathrm{d}x + 2t\int_\Omega (\eta_\varepsilon + v^2)\nabla u\cdot\nabla(X\cdot\nabla H)\,\mathrm{d}x$$
$$+ 2t\int_\Omega (\eta_\varepsilon + v^2)[(\nabla u\cdot\nabla H)\,\mathrm{div}\,X - (\nabla u\otimes\nabla H) : (DX + (DX)^{\mathsf{T}})]\,\mathrm{d}x + o(t).$$

Since $G_t - G = -tH + o(t)$, we easily infer that

$$\mathrm{II}(t) = -2t\int_\Omega (\eta_\varepsilon + v^2)[(\nabla u\cdot\nabla H)\,\mathrm{div}\,X - (\nabla u\otimes\nabla H) : (DX + (DX)^{\mathsf{T}})]\,\mathrm{d}x + o(t),$$

and consequently,

$$\mathcal{B}''(0) = 2\int_\Omega (\eta_\varepsilon + v^2)\nabla u\cdot\nabla(X\cdot\nabla H)\,\mathrm{d}x$$
$$+ 4\int_\Omega (\eta_\varepsilon + v^2)[(\nabla u\cdot\nabla H)\,\mathrm{div}\,X$$
$$- (\nabla u\otimes\nabla H) : (DX + (DX)^{\mathsf{T}})]\,\mathrm{d}x. \tag{A.14}$$

Similarly,

$$\mathcal{C}(t) = t^2\int_\Omega (\eta_\varepsilon + v^2)|\nabla H|^2\,\mathrm{d}x + o(t^2),$$

so that

$$\mathcal{C}'(0) = 0 \quad \text{and} \quad \mathcal{C}''(0) = 2\int_\Omega (\eta_\varepsilon + v^2)|\nabla H|^2\,\mathrm{d}x. \tag{A.15}$$

In view of (A.7), gathering (A.11)–(A.13)–(A.15) or (A.12)–(A.14)–(A.15) leads to the announced formula for $\delta\mathrm{AT}_\varepsilon(u,v)[X,G]$ and $\delta^2\mathrm{AT}_\varepsilon(u,v)[X,G]$ respectively. ■

Similar computations, together with the well-known first and second inner variations of a countably $\mathcal{H}^{N-1}$-rectifiable set (see e.g. [43, Chapter 2, Section 9]), lead to explicit expressions for the first and second inner variations of the Mumford–Shah functional.

Lemma A.4. *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set of class $\mathcal{C}^2$, $g \in \mathcal{C}^2(\partial\Omega)$, and $u \in$* $\mathrm{SBV}^2(\Omega)$.

(i) *Then for all vector fields $X \in \mathcal{C}^1_c(\mathbf{R}^N;\mathbf{R}^N)$ and every extension $G \in \mathcal{C}^2(\mathbf{R}^N)$ with $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$, we have*

$$\delta \mathrm{MS}(u)[X,G] = \int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DX \, \mathrm{d}x \\ + \int_{\hat{J}_u} \mathrm{div}^{\hat{J}_u} X \, \mathrm{d}\mathcal{H}^{N-1} + 2\int_\Omega \nabla u \cdot \nabla(X \cdot \nabla G) \, \mathrm{d}x. \quad \text{(A.16)}$$

(ii) *If, further, Ω is of class $\mathcal{C}^3$ and $g \in \mathcal{C}^3(\partial\Omega)$, then for all vector fields $X \in \mathcal{C}^2_c(\mathbf{R}^N;\mathbf{R}^N)$ and every extension $G \in \mathcal{C}^3(\mathbf{R}^N)$ with $X \cdot \nu_\Omega = 0$ and $G = g$ on $\partial\Omega$, we have*

$$\begin{aligned} &\delta^2 \mathrm{MS}(u)[X,G] \\ &= \int_\Omega (|\nabla u|^2 \mathrm{Id} - 2\nabla u \otimes \nabla u) : DY \, \mathrm{d}x + \int_{\hat{J}_u} \mathrm{div}^{\hat{J}_u} Y \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_\Omega \big[|\nabla u|^2 \big((\mathrm{div}\, X)^2 - \mathrm{tr}((DX)^2) \big) - 4((\nabla u \otimes \nabla u) : DX) \, \mathrm{div}\, X \big] \, \mathrm{d}x \\ &\quad + \int_\Omega [4(\nabla u \otimes \nabla u) : (DX)^2 + 2|DX^\top \nabla u|^2] \, \mathrm{d}x \\ &\quad + \int_{\hat{J}_u} \Big[(\mathrm{div}^{\hat{J}_u} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X)(\tau_j \cdot \partial_{\tau_i} X) \Big] \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + 2\int_\Omega \nabla u \cdot \nabla(X \cdot \nabla(X \cdot G)) \, \mathrm{d}x + 2\int_\Omega |\nabla(X \cdot \nabla G)|^2 \, \mathrm{d}x \\ &\quad + 4\int_\Omega [(\nabla u \cdot \nabla(X \cdot \nabla G)) \, \mathrm{div}\, X \\ &\qquad - (\nabla u \otimes \nabla(X \cdot \nabla G)) : (DX + (DX)^\top)] \, \mathrm{d}x, \end{aligned}$$

where $(\tau_1, \ldots, \tau_{N-1})$ is a basis of the tangent plane to $\hat{J}_u$ at a given point $x \in \hat{J}_u$.

Proof. The second inner variation of the part $\int_\Omega |\nabla u|^2 \, \mathrm{d}x$ is computed exactly as in the proof of Lemma A.3, recalling that the chain rule still holds for the approximate gradient $\nabla(u \circ \Phi_{-t}) = [D\Phi_{-t}]^\top \nabla u(\Phi_{-t})$. For the singular part of the energy, we use that $\mathcal{H}^{N-1}(\hat{J}_{u_t}) = \mathcal{H}^{N-1}(J_{\hat{u}_t})$, where $\hat{u}_t = \hat{u} \circ \Phi_{-t}$ and $\hat{u} = u\mathbf{1}_\Omega + G\mathbf{1}_{\mathbf{R}^N \setminus \Omega}$. The second variation of such a functional is computed with the area formula as in [43, Chapter 2], together with the geometric formula

$$\begin{aligned} &(\mathrm{div}\, X)^2 - \mathrm{tr}[(\nabla X)^2] - 2((\nu_u \otimes \nu_u) : DX) \, \mathrm{div}\, X + 2(\nu_u \otimes \nu_u) : (DX)^2 + |DX^\top \nu_u|^2 \\ &= (\mathrm{div}^{J_u} X)^2 + \sum_{i=1}^{N-1} |(\partial_{\tau_i} X)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot \partial_{\tau_j} X)(\tau_j \cdot \partial_{\tau_i} X) + ((\nu_u \otimes \nu_u) : \nabla X)^2, \end{aligned}$$

stated in [25, proof of Theorem 1.1, pp. 1851–1852]. ∎

Remark A.3. As in Remark A.2, $\delta\mathrm{MS}(u)[X,G]$ is independent of the extension G when u is a critical point for the outer variations of MS, while $\delta^2\mathrm{MS}(u)[X,G]$ does depend on the extension G in general. If u satisfies $\delta\mathrm{MS}(u)[X,G]=0$, then for all $X\in\mathcal{C}_c^\infty(\Omega,\mathbf{R}^N)$ with $\operatorname{supp} G\cap\operatorname{supp} X=\emptyset$, the formula of the second inner variation reduces to

$$\begin{aligned}\delta^2\mathrm{MS}(u)[X,G] &= \int_\Omega |\nabla u|^2\big((\operatorname{div} X)^2-\operatorname{tr}((DX)^2)\big)-4((\nabla u\otimes\nabla u):DX)\operatorname{div} X\,\mathrm{d}x\\ &\quad+\int_\Omega 4\nabla u\otimes\nabla u:(DX)^2+2|DX^{\mathsf T}\nabla u|^2\,\mathrm{d}x\\ &\quad+\int_{\hat J_u}\Big[(\operatorname{div}^{\hat J_u} X)^2+\sum_{i=1}^{N-1}|(\partial_{\tau_i}X)^\perp|^2\\ &\qquad-\sum_{i,j=1}^{N-1}(\tau_i\cdot\partial_{\tau_j}X)(\tau_j\cdot\partial_{\tau_i}X)\Big]\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

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