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Discrete Geometry

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Abstract. A number of important recent developments in various branches of discrete geometry were presented at the workshop. The presentations illustrated both the diversity of the area and its strong connections to other fields of mathematics such as convex geometry, combinatorics, or topology. Two open problem sessions highlighted the abundance of open questions and many of the results presented were obtained by young researchers, confirming the vitality of the field.

Mathematics Subject Classification (2020): 52Bxx, 52Cxx.

Introduction by the Organizers

Discrete Geometry is a classical branch of mathematics, dating back at least to etruscan polyhedra carved out of stone, and yet has remained a vital topic throughout the ages, central to mathematics as a playground for tools from other areas as well as providing those tools in return. Its main target is to study the structure and complexity of discrete objects in a geometric space ranging from finite point sets in the plane to more complex structures like arrangements of n-dimensional convex bodies. Old and well-known problems such as Kepler's conjecture, Sylvester's four-point question, and Hilbert's third problem on decomposing polyhedra, as well as classical works by mathematicians such as Minkowski, Steinitz, Hadwiger and Erdős are part of the heritage of this area.

By its nature, this area is interdisciplinary and has relations to many other vital mathematical fields, such as algebraic geometry, topology, combinatorics, computational geometry, convexity, and probability theory. At the same time it is on the cutting edge of modern applications such as geographic information systems, mathematical programming, coding theory, solid modeling, computational structural biology and crystallography.

The workshop gathered 48 participants on-site and 7 remote participants. A conscious effort was made to highlight diverse interactions of discrete geometry to other areas, in an effort to increase diversity both in science and in attendance, mixing up the people in attendance. In particular, the workshop included five distinguished survey talks that focused on such interactions and aspects of discrete geometry, explaining recent advances and open problems in geometric graph theory (speaker János Pach), convex geometry (Ramon van Handel), arithmetic geometry (Rachel Greenfeld), topological combinatorics (Andreas Holmsen), and triangulations (Gaku Liu).

A number of important recent developments in various branches of discrete geometry were also presented at the workshop. Several outstanding results were presented by junior scholars, with for instance a lecture by Rachel Greenfeld on her recent progress (with Marina Iliopoulou and Sarah Peluse) on integer distance sets, a lecture by Dmitrii Zakharov on his recent progress (with Alex Cohen and Cosmin Pohoata) on the Heilbronn distance problem and its relation to Kakeya type problems, a lecture of Edgardo Roldán-Pensado (based on works with Cuauhtémoc Gomez-Navarro and Leonardo Martínez-Sandoval) on a conjecture of Dol'nikov and a lecture by Linda Kleist on her solution (with James Davies, Chaya Keller, Shakhar Smorodinsky and Bartosz Walczak) of Ringel's circle problem. Equally, we want to highlight the innovative topological combinatorics talk by Corrine Yap (with Jason Long and Bhargav Narayanan) on topological Turán problems as well as the talk by Pavel Paták (with Martin Tancer) on NP-hardness of shellability of balls.

Other distinct highlights included the resolution of the long-standing problem on the tightness cases of the Alexandrov-Fenchel inequality (Ramon van Handel with Yair Shenfeld, Igor Pak with Swee-Hong Chan), a problem in which, prior to the presented work, not even the right conjecture was established.

Altogether, there were 24 talks presenting new connections to classical topics such as convex geometry (Henk, Oliveros), combinatorics (Zeng, Jung, Tomon), topology (Tancer) as well as new developments in classical topics such as polytope theory (Liu, Padrol), geometric graphs (Pach, Steiner), combinatorial convexity (Holmsen, McGinnis), real algebraic and tropical geometry (Brandenburg, Raz) and geometric measure theory (Avvakumov, Schnider).

Two open problem sessions took place on Tuesday and Thursday evening; the collection of open problems resulting from this session can be found in this report. The program left ample time for research and discussions.

There were several small informal sessions on specific topics of common interest. On Thursday afternoon, a substantial fraction of the participants braved the weather and joined the traditional outing to St. Roman with the black forest cherry cake, enjoying the beautiful (and humid) winter air.

Subject classification. The topics of the conference belong mainly to classes 52C and 52B in the AMS-classification scheme. They fall into category 4 (Geometry) of the International Mathematical Union (1995) classification. There is only a minor overlap with other Oberwolfach meetings like “Convex Geometry and its Applications” or “Topological and Geometric Combinatorics”.

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Workshop: Discrete Geometry

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Abstracts

Boxing inequality

Sergey Avvakumov
(joint work with Alexander Nabutovsky)

Recall the definition of *Hausdorff content*: for a subset X of a metric space, its mth Hausdorff content $\mathrm{HC}_m(X)$ is defined as the infimum of $\Sigma_i r_i^m$, where the infimum is taken over all coverings of X by a finite collection of metric balls, and r_i denote the radii of these balls.

I am going to talk about the following result [1]:

Theorem 1. *For any $m \geq 1$ there is a constant $c(m)$ such that for any Banach space B and any compact $X \subset B$, there is a homotopy $H : X \times [0,1] \to B$ with H_0 being the inclusion $X \subset B$ and $H_1(X)$ lying in an $(\lceil m \rceil - 1)$-dimensional polyhedron in B such that:*

(1) $\mathrm{HC}_m(H_1(X)) \leq c(m)\mathrm{HC}_m(X)$,
(2) $|x - H_1(x)|_B \leq c(m)(\mathrm{HC}_m(X))^{\frac{1}{m}}$ *for all* $x \in X$.

The theorem says that X can be *filled* in such a way that both HC_m of the filling and the distance from every point of the filling to X are controlled by $\mathrm{HC}_m(X)$ up to a constant independent of the ambient dimension, conditions (1) and (2) in the theorem, resp. The remarkable part here is that HC_m and not HC_{m+1} of the filling is controlled. This is, for instance, in contrast with the classical isoperimetric inequality, where for an m-submanifold $M^m \subset \mathbb{R}^n$, the $(m+1)$th-volume of the filling is controlled by the mth volume of M^m.

Theorem 1 was partially motivated by its potential applications to study of lengths of shortest periodic geodesics in essential manifolds, $\mathrm{sys}(M^m)$. In the seminal paper [3] Gromov established the inequality

$$\mathrm{sys}(M^m) \leq c(m)(\mathrm{Vol}_m(X))^{\frac{1}{m}},$$

first establishing an isoperimetric inequality in the space $B = L^\infty(M^m)$ (of continuous functions on M^m with ℓ^∞ norm) and then exploring the geometry of the filling. In the same paper he observed that $\mathrm{sys}(M^m) \leq 3 \cdot UW_{m-1}(M^m)$, where the Urysohn width $UW_{m-1}(M^m)$ measures the distance from M^m to the nearest $(m-1)$-dimensional complex.

Gromov's proof was later simplified by Wenger [7]. Its scheme was adopted [6] to prove a weaker version of Theorem 1 – that there exists a filling satisfying (2) and a weaker version of (1): $\mathrm{HC}_{m+1}(H_1(X)) \leq c(m)(\mathrm{HC}_m(X))^{\frac{m+1}{m}}$ (it is weaker because of the monotonicity $(\mathrm{HC}_{m+1}(X))^{\frac{1}{m+1}} \leq (\mathrm{HC}_m(X))^{\frac{1}{m}}$ of Hausdorff content). The result of [6], specifically part (2), proved the conjecture of Guth [5] that the Urysohn width of X can be controlled in terms of $(\mathrm{HC}_m(X))^{\frac{1}{m}}$. It also immediately implied a systolic inequality with Hausdorff content:

$$\mathrm{sys}(M^n) \leq c(m)(\mathrm{HC}_m(M^n))^{\frac{1}{m}},$$

where M^n is an essential manifold and $1 \leq m \leq n$ is an integer.

Theorem 1 improved on [6] by providing a "better" deformation to a subset of an $(\lceil m \rceil - 1)$-dimensional polyhedron.

Before all that, the special case of $B = \mathbb{R}^{m+1}$ of Theorem 1 was proved by Gustin [4] who was answering a question posed by Fleming. A simpler proof was soon found by Federer in [2], where it was used to prove a foundational result in geometric measure theory.

The main steps of the proof of Theorem 1 are

- Reducing to the case of the ambient space of finite dimension n.
- Decreasing $\mathrm{HC}_m(X)$ by a (very small) factor of $(1 - \frac{1}{3^m})$ by contracting parts of X to $(\lceil m \rceil - 1)$-dimensional polyhedrons inside several disjoint balls.
- Proving the theorem with a constant $c(m, n)$ which may depend on the ambient dimension n.

By repeating the second step a very large (but finite) number of times we make $\mathrm{HC}_m(X)$ very small, much smaller than $\frac{1}{c(m,n)}$. Then we apply the third step, which finishes the proof.

The second step is proved following the approach in [6]. To prove the third step, we first construct a suitable dyadic covering of X with so-called "good geometry", i.e., such that neighboring cubes do not differ in size by the factor of more than 2. Then we use the classical Federer–Fleming idea of inductively projecting X to lower dimensional skeleta of the covering. Unlike the classical case, however, our cubes are not of the same size which presents additional difficulties with keeping the projections continuous.

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Separating points by pieceweise linear functions: The real tropical geometry of neural networks

Marie-Charlotte Brandenburg
(joint work with Georg Loho, Guido Montúfar, and Hannah Tseran)

The combinatorics of functions which can be represented by neural networks with piecewise linear activations has received significant attention in recent years. A function represented by a *neural network* is an alternating composition of affine linear functions and a chosen activation function. Undoubtedly, one of the most successful activation functions is the *ReLU activation function* $\mathrm{ReLU}(x) = \max(0, x)$. Being a composition of affine linear functions with this activation function, a ReLU neural network always represents a piecewise linear function. It is thus no surprise that theoretical considerations of neural networks have entered the realm of discrete geometry, with a rising interest to utilize tools from combinatorial tropical geometry.

Suppose we are given a finite set $D \subset \mathbb{R}^d$ of points (the *data points*), which we seek to separate by a piecewise linear function f into two distinct sets. The function f serves as a *classifier*, separating the set D into classes $D^+(f) = \{p \in D \mid f(p) \geq 0\}$ and $D^-(f) = \{p \in D \mid f(p) \leq 0\}$. A well-studied scenario is the case of *linear classifiers*, in which f is a linear function. Parametrizing the function as $f(x) = a + \langle s, x\rangle$ yields the *parameter space* of linear functions $\Theta(d) = \{(a, s) \mid a \in \mathbb{R}, s \in \mathbb{R}^d\}$.

Given any partition of D into two sets D^+, D^-, the set of parameters of linear functions f such that $D^+(f) = D^+, D^-(f) = D^-$ is the interior of the polyhedral cone

$$C = \{(a, s) \in \Theta \mid a + \langle s, p\rangle \geq 0 \ \forall p \in D^+ \text{ and } a + \langle s, p\rangle \leq 0 \ \forall p \in D^-\},$$

Ranging over all possible partitions of D yields a complete polyhedral fan, whose cones are chambers in a hyperplane arrangement. This gives rise to a rich and well-studied combinatorial theory, which has been rediscovered multiple times by several mathematical communities (see e.g. [1]). We summarize the main important characterizations of the linear case as follows. Figure 1 illustrates an example of these characterizations.

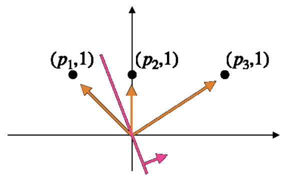
Theorem 1. *Let $D \subset \mathbb{R}^d$ be a finite data set. Then*

(i) the chambers of the hyperplane arrangement $\mathcal{H}_D = \bigcup_{p\in D}(1, p)^\perp$ subdivide the parameter space $\Theta(d)$ into regions of linear classifiers whose classification of D into D^+ and D^- agree,

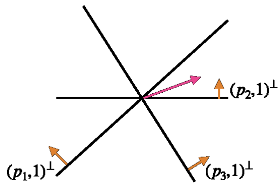
(ii) $\mathcal{H}_D$ induces the normal fan of the zonotope $P_D = \sum_{p\in D} \mathrm{conv}(0, p)$,

(iii) the possible linear classifications correspond to the covectors of a realizable oriented matroid.

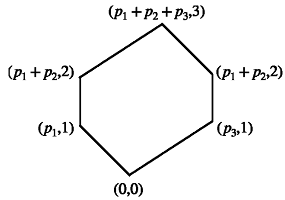
But what happens if the functions f are piecewise linear functions? Recall that any such function can be written as the difference of two convex piecewise linear functions, i.e. are of the form $f(x) = g(x) - h(x) = \max_{i\in[n]} (a_i + \langle s_i, x\rangle) - \max_{j\in[m]} (b_j + \langle x, t_j\rangle)$. A function of this shape is also called a *tropical rational*



A lifted point configuration $D \times \{1\} = \{(p_1,1),(p_2,1),(p_3,1)\}$.



The hyperplane arrangement $\mathcal{H}_D = \bigcup_{p\in D}(p,1)^\perp$.



The zonotope $P_D = \sum_{p\in D}\operatorname{conv}(0,p)$.

Figure 1. The different characterizations from Theorem 1 on an example of a 1-dimensional data set $D = \{p_1, p_2, p_3\}$.

function. Fixing the number of linear terms in such a representation, we can consider the *parameter space of tropical rational functions* as

$$\Theta(d,n,m) = \left\{ \begin{pmatrix} a_1 & \dots & a_n & b_1 & \dots & b_m \\ s_1 & \dots & b_n & t_1 & \dots & t_m \end{pmatrix} \mid a_i, b_j \in \mathbb{R}, s_i, t_j \in \mathbb{R}^d, i \in [n], j \in [m] \right\}.$$

Fixing a classification $D = D^+ \cup D^-$, the set of parameters of perfect classifiers is described by the inequalities

$$\max_{i\in[n]} a_i + \langle s_i, p\rangle \geq \max_{j\in[m]} b_j + \langle t_j, p\rangle \ \forall p \in D^+$$
$$\max_{i\in[n]} a_i + \langle s_i, p\rangle \leq \max_{j\in[m]} b_j + \langle t_j, p\rangle \ \forall p \in D^-$$

and is thus a union of polyhedral cones, which form a polyhedral fan. Interestingly, the above can be seen as a system of tropical polynomial inequalities, i.e. the set of solutions is a *tropical semialgebraic set*. Ranging over all possible partitions yields multiple polyhedral fans, and the collection of cones contained in these polyhedral fans all together form a complete fan, called the *activation fan* of D. We show that this polyhedral fan is the normal fan of the *activation polytope*, and this polytope can be written as a Minkowski sum of $(n+m-1)$-dimensional simplices. Note that the case $n = m = 1$ recovers the linear case, in which the activation polytope agrees with the zonotope from above. Also in analogy to the linear case, we may label each cone in the fan by an *activation pattern*, which is the analogue of a covector of an oriented matroid. We summarize our findings as follows:

Theorem 2. *Let $D \subset \mathbb{R}^d$ be a finite data set. Then*

(i) the cones of the activation fan subdivide the parameter space $\Theta(d,n,m)$ into regions of piecewise linear classifiers whose classification of D agree,
(ii) the activation fan is the normal fan of the activation polytope,
(iii) the possible piecewise linear classifications correspond to the possible activation patterns

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Integer distance sets

Rachel Greenfeld
(joint work with Marina Iliopoulou, Sarah Peluse)

A set $S \subset \mathbb{R}^2$ is called an *integer distance set* if the Euclidean distance $\|s' - s\|$ between any two points in $s, s' \in S$ is an integer. In 1945, Anning–Erdős [1, 5] proved that if an integer distance set S is not collinear (i.e., not contained in a single line), then it must be finite. Indeed, let P_1, P_2, P_3 be non-collinear points in S, then any point $X \in S$ must satisfy

$$|\|X - P_i\| - \|X - P_{i+1}\|| \in \{0, 1, \ldots, \|P_i - P_{i+1}\|\}, \text{ for both } i = 1, 2.$$

This means that $S \subset \mathcal{H}_1 \cap \mathcal{H}_2$ where $\mathcal{H}_i$, $i = 1, 2$, is a family of $\|P_i - P_{i+1}\| + 1$ hyperbolas with foci (P_i, P_{i+1}). As P_1, P_2, P_3 are non-collinear, the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ must be finite.

In addition, Anning–Erdős constructed two infinite families of arbitrarily large non-collinear integer distance sets: one of concyclic sets and one of sets in which all but one point are collinear. Erdős [4, page 43] then raised the question:

"Can you find n points in general position, no three on a line, no four on a circle, all distances are integers?"

All so-far-known integer distance sets have all but at most four of their points on a single line or circle. Moreover, the largest known integer distance set with no three points on a line and no four points on a circle consists of only seven points [9]. In 2003, Solymosi [10] showed that if an integer distance set $S \subset [-N, N]^2$ has no three points on a line then its size $|S|$ is at most[1] $O(N)$ (i.e., linear in N). The proof is based on further geometric analysis of the corresponding set $\mathcal{H}_1 \cap \mathcal{H}_2$ of hyperbolas intersections.

Observe that the rather simple hyperbolas-based argument relies on fixing merely three non-collinear points of S and analysing the constraints they derive on S. Clearly, any additional point of S (in general position) one fixes as a reference point, would derive further algebraic constraints on S. Therefore, to get good understanding of the structure and size of S, one would ideally aim at fixing an "optimal" number of reference points of S (in general position) and analyse the system of algebraic constraints they impose on S. This requires a new approach, which is algebraic in nature.

In a joint work with M. Iliopoulou and S. Peluse [6], we developed a new approach to study the size and structure of integer distance sets. This approach enabled us to prove a structure theorem that partially explains the above phenomenon, showing that any integer distance set in $[-N, N]^2$ has all but at most polylogarithmically many points lying on a single line or circle.

[1] For any two quantities X and Y, we write $X = O(Y)$ to mean that $|X| \leq CY$ for some absolute constant $C > 0$ and $X \asymp Y$ to mean $X \ll Y$ and $Y \ll X$.

Theorem 1 (Structure theorem)**.** *Let $S \subset [-N,N]^2$ be an integer distance set. Then there exists a line or circle $C \subset \mathbb{R}^2$ such that*
$$|S \setminus C| = O\left((\log N)^{O(1)}\right).$$

As an immediate consequence of Theorem 1, we get that integer distance sets with no three points on a line and no four points on a circle must be very sparse.

Corollary 2 (New bound on Erdős integer distance set problem)**.** *Let $S \subset [-N,N]^2$ be an integer distance set with no three points on a line and no four points on a circle. Then*
$$|S| = O\left((\log N)^{O(1)}\right).$$

Outline of the method. Let $S \subset [-N,N]^2$ be an integer distance set. Our method consists of choosing a collection of $k \asymp \log\log N$ reference points $P_1,\dots,P_k$ of S in general position, and analysing S with respect to the algebraic constraints these points impose, by using algebraic geometry and algebraic number theory tools. Below the fold we outline the main steps of our proof. We hope that this new method will open the way to further advances in additive combinatorics.

Encoding S as rational points on a surface. Using [8], we can assume, on applying translation and rotation to S, that
$$S \subset \left\{(x, y\sqrt{m}) : x, y \in \frac{1}{2M}\mathbb{Z}\right\}$$
where $m \in \mathbb{N}$ is squarefree and $M = O(N)$. Then, we fix $k \asymp \log\log N$ points of S in general position, $P_1 = (a_1, b_1\sqrt{m}), \dots, P_k = (a_k, b_k\sqrt{m})$, according to which we define the variety
$$X_k := \left\{(x, y, d_1, \dots, d_k) \in \mathbb{C}^{k+2} : Q_j(x, y, d_j) = 0,\ j = 1, \dots, k\right\},$$
where $Q_j(x,y,d_j) = (x-a_j)^2 + m(y-b_j)^2 - d_j^2$. Clearly, if $X = (x, y\sqrt{m}) \in S$ then
$$(x, y, \|X - P_1\|, \dots, \|X - P_k\|) \in X_k \cap \left(\frac{1}{2M}\mathbb{Z} \times \frac{1}{2M}\mathbb{Z} \times \mathbb{Z}^k\right).$$
Each such point corresponds to a rational point of height at most $O(N^2)$ on the projective closure $\overline{X_k} \subset \mathbb{P}^{k+2}$ of X_k. Thus, we have encoded the points of S as rational points of small height on $\overline{X_k}$. We show that $\overline{X_k}$ is an irreducible surface of degree 2^k defined over $\mathbb{Q}$.

Covering rational points of small height by few low degree curves. It is known, originally thanks to work of Heath-Brown [7], that almost all rational points of small height on an irreducible projective surface defined over $\mathbb{Q}$ lie on a small number of low degree curves. Applying a refinement of this result due to [3], we obtain that there exists a homogeneous polynomial $g \in \mathbb{Z}[x_0, \dots, x_{k+2}]$ of bounded degree (with explicit bound, depending on N, k) that vanishes at all these rational points of small height on $\overline{X_k}$. On applying appropriate projection, we show that S itself can be covered by $t = O\left((\log N)^{O(1)}\right)$ irreducible curves, $C_1, \dots, C_t$, each of which of degree at most $O\left((\log N)^{O(1)}\right)$.

Counting rational points of small height on curves. We analyse the curves $C_j, ..., C_t$ and prove, for each C_j, $1 \leq j \leq t$, that

- if C_j is not a line or a circle and $|S \cap C_j|$ is not very small, then the points of $S \cap C_j$ can be encoded as rational points of small height on a certain irreducible curve $\overline{\mathcal{C}_k} \subset \overline{X_k}$ (defined over $\mathbb{Q}$) of degree $2^k \leq d \leq 2^k \deg C_j$, with $k \asymp \log\log N$;
- if C_j is a line or a circle and $|S \setminus C_j|$ is not very small, then the points of $S \setminus C_j$ can be encoded as rational points of small height on a certain irreducible curve $\overline{\mathcal{C}_k} \subset \overline{X_k}$ (defined over $\mathbb{Q}$) of degree $2^k \leq d \leq 2^{k+1}$, with $k \asymp \log\log N$.

We then apply a refinement, due to [3, Theorem 2], of the Bombieri–Pila [2] bound on the number of points of small height on irreducible curves of certain degree, to the curve $\overline{\mathcal{C}_k} \subset \mathbb{P}^{k+2}$ in either of the above cases, and obtain that if C_j is not a line or a circle then

$$|S \cap C_j| = O\left((\log N)^{O(1)}\right),$$

and otherwise, if C_j is a line or a circle,

$$|S \setminus C_j| = O\left((\log N)^{O(1)}\right).$$

Since $t = O\left((\log N)^{O(1)}\right)$, this concludes the proof of our structure result, Theorem 1, and Corollary 2, in turn.

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Around the Alexandrov-Fenchel inequality

Ramon van Handel

The origin of the questions to be described below can be traced back to a remarkable paper of H. Minkowski [11] that celebrated its 120th birthday just prior to this workshop. This paper not only laid the foundation for the field of convex geometry, but also introduced several notions that had an major impact on other areas of mathematics—e.g., the Minkowski existence and uniqueness problems which led to much work on nonlinear PDEs in the 20th century, and one of the earliest examples of stability of geometric inequalities. At the same time, some fundamental questions that arise from this work remain open to this day.

At the heart of Minkowski's theory lies the notion of mixed volumes and the associated inequalities. To define these, we recall the basic fact that the volume of convex bodies is a homogeneous polynomial in the sense that

$$\mathrm{Vol}(\lambda_1 K_1 + \cdots \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1}\cdots\lambda_{i_n}\, \mathsf{V}(K_{i_1},\ldots,K_{i_n})$$

for all convex bodies $K_1,\ldots,K_m$ in $\mathbb{R}^n$ and $\lambda_1,\ldots,\lambda_m \geq 0$. Its coefficients $\mathsf{V}(C_1,\ldots,C_n)$, called *mixed volumes*, capture numerous geometric parameters of convex bodies as special cases. Consequently, many geometric inequalities are unified and generalized by the following fundamental result.

Theorem 1 (Alexandrov-Fenchel)**.** *For convex bodies $K, L, C_1,\ldots,C_{n-2}$ in $\mathbb{R}^n$*

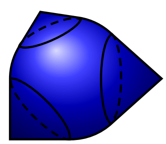
$$\mathsf{V}(K,L,C_1,\ldots,C_{n-2})^2 \geq \mathsf{V}(K,K,C_1,\ldots,C_{n-2})\,\mathsf{V}(L,L,C_1,\ldots,C_{n-2}).$$

This result was first proved by Minkowski [11] for the case $n = 3$ and by Alexandrov [1] for general n (Fenchel independently announced the result but did not publish a proof). The following question dates back to Minkowski:

Question. When does the Alexandrov-Fenchel inequality achieve equality?

Example 2. For any convex body K in $\mathbb{R}^3$

$$\underbrace{\mathsf{V}(B,K,K)}_{\text{surface area}}{}^2 \geq \underbrace{\mathsf{V}(B,B,K)}_{\text{mean width}}\,\underbrace{\mathsf{V}(K,K,K)}_{\text{volume}},$$



where B is the unit ball. Equality holds when K minimizes surface area among all bodies with a given volume and mean width. Such K are strikingly bizarre (see illustration).

This special example was considered by Minkowski [11] and Bol [2]. After that, progress stalled for several decades as the question was believed to be intractable [4, §20.5]. This would have likely remained the case if it were not for a remarkable conjectured characterization put forward by Schneider [12], which spurred new interest in the problem. Despite progress in additional special situations [13, §7.6], however, the problem has largely remained open.

The reason the problem is challenging is that proofs of the Alexandrov-Fenchel inequality rely on "nice" (e.g. smooth) bodies for which only trivial equality cases arise, and deduce the general case by approximation. Nontrivial equality cases arise in the limit due to the fact that the objects used in the proof become singular, and the heart of the matter is to understand these singularities.

A major step forward was achieved in joint work with Yair Shenfeld [16], which completely settled the problem when $C_1, \ldots, C_{n-2}$ are arbitrary convex polytopes. One central ingredient of the proof is a method to resolve the singularities.

Theorem 3 (Informal statement)**.** *When $C_1, \ldots, C_{n-2}$ are convex polytopes, the equality cases of Theorem 1 arise by a superposition of three distinct mechanisms:*

(1) translation and scaling;

(2) the relative positions of the normal cones of $\mathrm{bd}\, C_1, \ldots, \mathrm{bd}\, C_{n-2}$*;*

(3) the relative positions of the affine hulls $\mathrm{aff}\, C_1, \ldots, \mathrm{aff}\, C_{n-2}$*.*

Let us note that mechanisms (1) and (2) were conjectured by Schneider, while mechanism (3) is responsible for new equality cases. We refer to [16, §2] for a precise statement and some explicit examples.

While mixed volumes belong firmly to geometry, they lead a double life as combinatorial objects: when $K_1, \ldots, K_n$ are lattice polytopes (i.e., with vertices in $\mathbb{Z}^n$), the scaled mixed volume $n!\,\mathsf{V}(K_1, \ldots, K_n)$ is always an integer that counts, for example, the number of solutions of systems of polynomial equations [4, §27] and various other combinatorial structures [14]. The Alexandrov-Fenchel inequality therefore also gives rise to combinatorial inequalities.

One of the earliest such results is due to Stanley [14]. Let $P = \{x, y_1, \ldots, y_n\}$ be any partially ordered set, and let N_i be the number of linear extensions of P (i.e., different ways to complete the partial order to a total order) so that $x \in P$ has rank i. Stanley realized that[1] $N_i = n!\,\mathsf{V}(K[i-1], L[n+1-i])$ for suitable lattice polytopes K, L in $\mathbb{R}^n$. Thus the Alexandrov-Fenchel inequality immediately implies the following conjecture of Chung-Fishburn-Graham.

Theorem 4 (Stanley)**.** *The sequence (N_i) is log-concave, i.e., $N_i^2 \geq N_{i-1}N_{i+1}$.*

Note that the presence of an equality $N_i^2 = N_{i-1}N_{i+1}$ in a log-concave sequence corresponds precisely to a *geometric progression*. Thus Minkowski's question translates in the combinatorial setting to whether we can characterize when, where, and what kind of geometric progressions arise in log-concave combinatorial sequences, and partially ordered sets that feature such geometric progressions are combinatorial cousins of the strange bodies of Example 2.

The equality characterization of the Alexandrov-Fenchel inequality opens the door to answering such questions. In particular, the geometric progressions in Stanley's inequality were fully settled in [16, §15], and corresponding results for several related inequalities were obtained in [10, 18]. One surprising outcome of these results is that all three equality mechanisms in Theorem 3 turn out to arise naturally in combinatorial applications: these are not merely esoteric boundary cases the arise only in "weird" geometric examples!

The circle of ideas discussed above is far from complete, and continues to give rise to unexpected connections. We conclude by briefly describing three distinct themes that are motivating ongoing work on this subject.

[1] The notation $K[i-1]$ indicates that K is inserted in $i-1$ arguments of the mixed volume.

Theme 1 (general convex bodies)**.** The equality cases of the Alexandrov-Fenchel inequality remain open in full generality (beyond polytopes). The issue is that beside the *combinatorial singularities* that arise already in the polytope setting, there are *analytic singularities* that arise as the boundary of general convex bodies can be highly nonsmooth. New tools developed in [15] made it possible to surmount these issues in the setting of Minkowski's original paper [11], but a complete understanding of the analytic aspect remains challenging.

Theme 2 (beyond convexity)**.** While the Alexandrov-Fenchel inequality belongs to convex geometry, analogous inequalities turn out to appear in other areas of mathematics: in algebraic geometry [17], in complex geometry [7], and most recently in combinatorics [8, 3, 5]. In particular, there appears to be a kind of universal algebraic structure to such inequalities that is not specific to convexity. It is natural to conjecture that the structure of the equality cases is similarly universal. Initial progress on this question was recently made in [9].

Theme 3 (complexity theory)**.** While Theorem 3 provides a complete geometric characterization of the Alexandrov-Fenchel equality cases, it was shown in [6] that it may nonetheless be computationally hard to recognize when equality holds. This result and its complexity-theoretic and philosophical implications were discussed in detail in the talk of Igor Pak at this workshop.

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Polynomial bounds in Koldobsky's discrete slicing problem

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(joint work with Ansgar Freyer)

Let $\mathcal{K}^n$ be the set of convex bodies in $\mathbb{R}^n$ and the subfamily of convex bodies that are origin-symmetric is denoted by $\mathcal{K}^n_{os}$. The classical and central slicing problem in Convex Geometry due to Bourgain (see, e.g. [2]) asks for the optimal constant $b_n > 0$ such that for any $K \in \mathcal{K}^n_{os}$ there exists a central hyperplane H, i.e., a hyperplane passing through the origin, with

$$\mathrm{vol}(K) \leq b_n \mathrm{vol}_{n-1}(K \cap H)\,\mathrm{vol}(K)^{\frac{1}{n}}. \tag{1}$$

Here $\mathrm{vol}(S)$ denotes the volume, i.e., n-dimensional Lebesgue measure of $S \subset \mathbb{R}^n$, and the d-dimensional volume of a set S contained in a d-dimensional affine plane is denoted by $\mathrm{vol}_d(S)$.

It is conjectured that b_n in (1) is upper bounded by an absolute constant and the current best known bound due to a recent result of Klartag [6] is of order $O(\sqrt{\log(n)})$. This conjecture is equivalent to a multitude of other problems in Convex Geometry and Geometric Analysis such as the isotropic constant conjecture. It is considered to be one of the major open problems in Convex Geometry and for more information we refer to [3, 4].

Koldobsky considered extensions of (1) to measures others than the Lebesgue-measure (see, e.g., [7]). In particular, he also asked for the following discrete variant of the slicing problem: determine the best possible constant $d_n > 0$ such that for any $K \in \mathcal{K}^n_{os}$ with $\dim(K \cap \mathbb{Z}^n) = n$ there exists a central hyperplane $H \subset \mathbb{R}^n$ with

$$\mathrm{G}(K) \leq d_n\, \mathrm{G}(K \cap H)\mathrm{vol}(K)^{\frac{1}{n}}, \tag{2}$$

where $\mathrm{G}(K) = |K \cap \mathbb{Z}^n|$ is the lattice point enumerator. In [1] it was shown $d_n \in O(n\,2^n)$ as well as a lower bound of order $\Omega(n)$. The main reason for this exponential gap is the unfortunate circumstance that, even though K is origin-symmetric, the maximal (with respect to lattice points) hyperplane section does not need to pass through the origin (see [1]).

We prove the following theorem on central sections of arbitrary dimension of centered convex bodies, where a convex body is called centered if $\int_K x\,\mathrm{d}x = 0$.

Theorem. *Let $K \in \mathcal{K}^n$, $\dim K = n$, be centered and let $k \in \{1, \ldots, n-1\}$. There exists a k-dimensional central plane $L \subset \mathcal{R}^n$ such that*

$$\mathrm{G}(K)^{\frac{k}{n}} \leq O(\omega(n))^{n-k} O\left(\max\left\{\left(\frac{n+1}{k+1}\right)^k, \omega(k)\,k\,n\right\}\right) \mathrm{G}(K \cap L).$$

Here $\omega(n)$ is the well-known flatness constant. The best known upper bound on $\omega(n)$ is due to Reis and Rothvoss [9]:

$$\omega(n) \leq O(n \log(n)^3). \tag{3}$$

In the case of origin-symmetric convex bodies, the bound in the theorem above can be improved and together with (3), we obtain the desired polynomial upper bound for d_n in Koldobsky's discrete slicing problem (2).

Corollary. *Let $K \in \mathcal{K}^n_{os}$ with $\dim(K \cap \mathbb{Z}^n) = n$, $n \geq 2$. There exists a central hyperplane $H \subset \mathbb{R}^n$ such that*

$$\mathrm{G}(K) \leq O\left(\frac{n^2\,\omega(n)}{\log(n+1)}\right)\mathrm{G}(K \cap H)\mathrm{vol}(K)^{\frac{1}{n}}.$$

In particular,

$$\mathrm{G}(K) \leq O(n^3 \log(n)^2)\mathrm{G}(K \cap H)\mathrm{vol}(K)^{\frac{1}{n}}. \tag{4}$$

It is quite likely that the right order is linear in the dimension which would also coincide with a result of Regev [8] where by a randomized construction it is shown $d_n \in O(n)$ provided the volume of K is at most c^{n^3} for an absolute constant c.

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Topological Helly-type theorems

Andreas F. Holmsen

One of the milestones of combinatorial convexity is the (p,q)*-theorem* [2], a far-reaching generalization of Helly's theorem:

Theorem 1 (Alon–Kleitman, 1992)**.** *For all integers $p \geq q \geq d+1 > 1$ there exists a minimal integer $N = N(p,q,d)$ with the following property. For any finite family of convex sets in $\mathbb{R}^d$, where among any p of them some q intersect, there exists a set of N points that intersect every set in the family.*

Hadwiger and Debrunner, who conjectured the existence of $N(p,q,d)$, showed that for $p(d-1) < d(q-1)$, we have $N(p,q,d) = p-q+1$, while the current general bounds are $\Omega(p\log^{d-1} p) \leq N(p,d+1,d) \leq O(p^{d(d-1/2)})$.

An active research direction, initiated by Alon et al. [1], is to identify classes of set systems for which a (p,q)-theorem is valid. (This is closely related to the notion of χ-boundedness in graph theory.) A first step in this direction was their generalization of the (p,q)-theorem to *good covers* in $\mathbb{R}^d$, i.e. families of subsets of $\mathbb{R}^d$ where the intersection of any subfamily is empty or contractible:

Theorem 2 (Alon–Kalai–Matoušek–Meshulam, 2002)**.** *The assertion of the (p,q)-theorem remains valid for all finite good covers in $\mathbb{R}^d$.*

The following generalization of a good cover in $\mathbb{R}^d$ was introduced by Goaoc et al. [3]. For a family F of sets in $\mathbb{R}^d$, define the *k-level homological complexity* of F as

$$\mathrm{HC}_k(F) = \sup\big\{\widetilde{\beta}_i\big(\bigcap_{S\in G} S\big) : G \subsetneq F, 0 \leq i < k\big\}.$$

Here $\widetilde{\beta}_i$ denotes the ith reduced Betti-number with $\mathbb{Z}_2$-coefficients. (For a good cover F in $\mathbb{R}^d$ we have $\mathrm{HC}_d(F) = 0$.) Goaoc et al. [3] showed that bounding the following Helly-type theorem:

Theorem 3 (Goaoc–Paták–Patáková–Tancer–Wagner, 2017)**.** *Let F be a finite family of sets in $\mathbb{R}^d$ with $HC_{\lfloor d/2\rfloor}(F) \leq C$. If any m or fewer members of F have a point in common, then there is a point in common to all members of F, where m is a constant depending only on C and d.*

A natural question (which we answer below) is whether a (p,q)-theorem is valid for set systems in $\mathbb{R}^d$ with bounded homological complexity.

Combinatorial aspects. The Alon–Kleitman proof of the (p,q)-theorem has been scrutinized over the years, revealing a number of crucial combinatorial properties at work. Consider a set system $\mathcal{C}$ on a ground set X. For a subset $Y \subset X$ let $c(Y) = \bigcap_{S\in\mathcal{C}} S$, and let $c(Y) = X$ if no set in $\mathcal{C}$ contains Y. For a finite family $F \subset \mathcal{C}$ let $\tau(F)$ denote its *transversal number*, and let $\tau^*(F)$ denote its *fractional transversal number*. If $\tau(F) = 1$, then F is called an *intersecting* family. We define the following properties of $\mathcal{C}$:

(1) $\mathcal{C}$ has the *Radon property* if there exists an integer r such that any set $Y \subset X$, with $|Y| = r$, has a partition $Y = A \cup B$ where $c(A) \cap c(B) \neq \emptyset$. (The minimal such r is called the *Radon number* of $\mathcal{C}$.)

(2) $\mathcal{C}$ has the *colorful Helly property* if there exists an integer m such that in any collection of finite families $F_1, \ldots, F_m \subset \mathcal{C}$, where $\bigcap_{i=1}^m S_i \neq \emptyset$ for all choices $S_1 \in F_1, \ldots, S_m \in F_m$, one of the F_i is intersecting. (The minimal such m is called the *colorful Helly number* of $\mathcal{C}$.)

(3) $\mathcal{C}$ has the *fractional Helly-property for k-tuples* if there exists a function $f : (0,1) \to (0,1)$ such that in any finite family $F \subset \mathcal{C}$, where at least $\alpha\binom{|F|}{k}$ of the subfamilies of size k are intersecting, there is an intersecting subfamily of size at least $f(\alpha)|F|$. (The minimal k for which $\mathcal{C}$ has the fractional Helly property is called the *fractional Helly number* of $\mathcal{C}$.)

(4) $\mathcal{C}$ has the *weak ε-net property* if there exists a function $g : \mathbb{Q} \to \mathbb{N}$ such that for any finite family $F \subset \mathcal{C}$ we have $\tau(F) \leq g\big(\tau^*(F)\big)$.

It is known that the system of all convex sets in $\mathbb{R}^d$ satisfies each of these properties. The proof method Alon and Kleitman reveals the following:

Theorem 4. *If a set system $\mathcal{C}$ satisfies properties* (3) *and* (4)*, then assertion of the (p, q)-theorem is valid for all finite $F \subset \mathcal{C}$ whenever $p \geq q \geq k$, where k is the fractional Helly number.*

A set system $\mathcal{C}$ is *intersection-closed* if the intersection of any sets in $\mathcal{C}$ is contained in $\mathcal{C}$. Examples include e.g. convex sets in $\mathbb{R}^d$, convex lattice sets in $\mathbb{R}^d$, and abstract convexity spaces. Recent developments have culminated in the following:

Theorem 5. *If $\mathcal{C}$ is an intersection-closed set system, then properties* (1), (2), (3)*, and* (4) *are equivalent.*

The implication (3) ⇒ (4) was shown in [1], (1) ⇒ (2) in [6], (2) ⇒ (3) in [5], and (4) ⇒ (1) in [7].

Topological set systems. The key step in extending Theorem 3 to a (p, q)-theorem is the following result due to Patákova [8]:

Theorem 6 (Patákova, 2020)**.** *Let F be a family of subsets of $\mathbb{R}^d$. If $\mathrm{HC}_{\lfloor d/2 \rfloor}(F) \leq C$, then F has the Radon property, where the Radon number depends only on C and d.*

For a family F of subsets in $\mathbb{R}^d$, let $F^\cap$ denote its *intersection-closure*, i.e. $F^\cap = \{\bigcap_{S \in G} S : G \subset F\}$. Note that F and $F^\cap$ have the same Radon number and that $\mathrm{HC}_k(F) = \mathrm{HC}_k(F^\cap)$. Therefore, Theorems 4, 5, and 6 imply that a (p, q)-theorem is valid for any family F of sets in $\mathbb{R}^d$ where $\mathrm{HC}_{\lfloor d/2 \rfloor}(F) \leq C$. The important caveat here is that we require $p \geq q \geq k$, for some rapidly increasing function $k = k(C, d)$.

The proof of Theorem 6 uses homological non-embeddability (homological minors) and the technique of "constrained chain maps" originally developed in the proof of Theorem 3. Recently, Goaoc et al. [4] extended these techniques further to improve the bound on the fractional Helly number of topological set systems in $\mathbb{R}^d$:

Theorem 7 (Goaoc–Holmsen–Patáková, 2021). *Let F be a family of sets in $\mathbb{R}^d$. If $\mathrm{HC}_{\lfloor d/2 \rfloor}(F) \leq C$, then F has fractional Helly number at most $d+1$.*

Consequently, the assertion of the (p, q)-theorem is valid for such families whenever $p \geq q \geq d+1$.

There are interesting examples from geometric transversal theory that indicate that a constant bound on the homological complexity is not a necessary condition for a (p, q)-theorem. Motivated by earlier conjectures of Kalai and Meshulam we propose the following: Fix constants c and t, and consider families F of sets in $\mathbb{R}^d$ which satisfy

$$\sum_{i=1}^{\lfloor d/2 \rfloor} \widetilde{\beta}_i\Big(\bigcap_{S \in G} S\Big) \leq c \cdot |G|^t \ , \quad \text{for all finite } G \subset F.$$

We conjecture that such families satisfy a (p, q)-theorem for all $p \geq q \geq d+1$.

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A purely combinatorial proof of $(\aleph_0, q)$-theorems

Attila Jung

(joint work with Dömötör Pálvölgyi)

Let $\mathcal{K}_d$ be the hypergraph whose vertices are the compact convex sets in $\mathbb{R}^d$ and edges represent intersecting families of convex sets. Many results of combinatorial convexity can be stated as properties of this hypergraph.

For a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ be its vertex set and for an $S \subset V(\mathcal{H})$, let $\mathcal{H}[S]$ be the subhypergraph spanned by S. We will denote the number of edges by $e(\mathcal{H})$, and the q-uniform part of the hypergraph by $\mathcal{H}^{(q)}$.

The celebrated Alon-Kleitman theorem can be stated as follows.

Theorem 1 (Alon and Kleitman [1]). *For every finite $p \geq d+1$ there exists a $D < \infty$ with the property that if $S \subset V(\mathcal{K}_d)$ is such that $\mathcal{K}_d^{(d+1)}[S]$ does not contain independent sets of size p, then S can be covered with D edges of $\mathcal{K}_d$.*

One of the main ingredients in its proof, the fractional Helly theorem of Katchalski and Liu can be phrased as follows.

Theorem 2 (Katchalski and Liu [2]). *If $S \subset V(\mathcal{K}_d)$ is a finite subset and $e(\mathcal{K}_d^{(d+1)}[S]) \geq a\binom{|S|}{d+1}$ with some $a > 0$, then there exists a clique of $\mathcal{K}_d^{(d+1)}[S]$ of size $b|S|$ with some $b(a,d) > 0$.*

For a q-uniform hypergraph $\mathcal{H}$, we say that $\mathcal{H}$ satisfies the fractional Helly property, if for all $a > 0$ there exists a $b > 0$ such that if $e(\mathcal{H}[S]) \geq a\binom{|S|}{q}$ for some finite $S \subset V(\mathcal{H})$, then there exists a clique of $\mathcal{H}[S]$ of size $b|S|$.

For $0 \leq k < d$, let $\mathcal{B}_{d,k}$ be the hypergraph whose vertices are balls from $\mathbb{R}^d$ and edges represent families of balls which can be intersected with a single k-dimensional affine subspace. Keller and Perles proved the following infinite variant of the Alon-Kleitman theorem for k-flats intersecting Euclidean balls.

Theorem 3 (Keller and Perles [3, 4]). *If $S \subset V(\mathcal{B}_{d,k})$ is such that $\mathcal{B}_{d,k}^{(k+2)}[S]$ has no infinitely large independent set, then S can be covered with a finite number of edges of $\mathcal{B}_{d,k}$.*

We prove that such an infinite variant of the Alon-Kleitman theorem always follows from the corresponding finite version and a fractional Helly theorem. Thus we obtain the following analog of the result of Keller and Perles as one of the corollaries.

Corollary 4 (Jung and Pálvölgyi). *If $S \subset V(\mathcal{K}_d)$ is such that $\mathcal{K}_d^{(d+1)}[S]$ has no infinitely large independent set, then S can be covered with a finite number of edges of $\mathcal{K}_d$.*

We actually prove that if our hypergraph satisfies the fractional Helly property, then the condition of the infinite version of the Alon-Kleitman theorem implies the condition of the finite version. We state that in the contrapositive form as follows.

Theorem 5 (Jung and Pálvölgyi). *If a q-uniform hypergraph satisfies the fractional Helly property and has arbitrarily large finite independent sets, then it has an infinitely large independent set.*

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A solution to Ringel's circle problem

Linda Kleist
(joint work with James Davies, Chaya Keller, Shakhar Smorodinsky, Bartosz Walczak)

A *constellation* is a finite collection of circles in the plane in which no three circles are tangent at the same point. The *tangency graph* $G(\mathcal{C})$ of a constellation $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and edges comprising the pairs of tangent circles in $\mathcal{C}$.

Jackson and Ringel [5] discussed four problems regarding the chromatic number of constellations. The problems are illustrated in Figure 1.

(a) *The penny problem.* What is the maximum chromatic number of a constellation of non-overlapping unit circles?
(b) *The coin problem.* What is the maximum chromatic number of a constellation of non-overlapping circles (of arbitrary radii)?

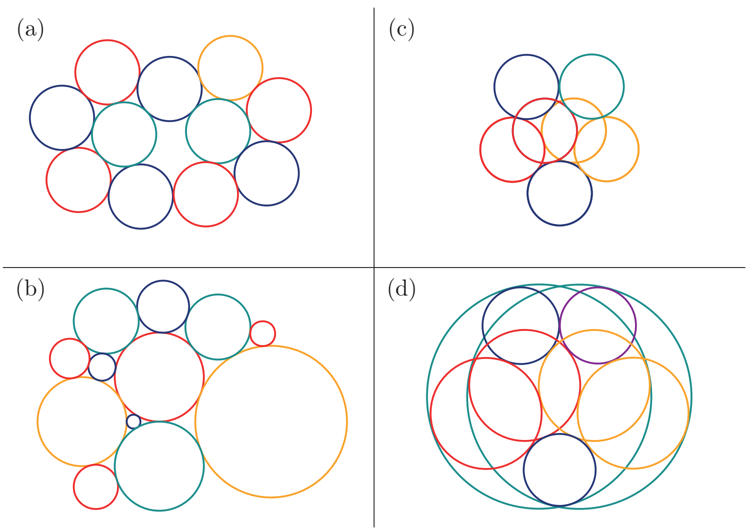

Figure 1. An illustration of the four coloring problems of tangency graphs of constellations: (a) a penny graph, (b) a coin graph, (c) an overlapping penny graph, and (d) a general constellation as in the circle problem.

(c) *The overlapping penny problem.* What is the maximum chromatic number of a (possibly overlapping) constellation of unit circles?
(d) *The circle problem.* What is the maximum chromatic number of a general constellation of circles?

Jackson and Ringel provided a simple proof that the answer to the *penny problem* is 4. The claim that the answer for the *coin problem* is also 4 is equivalent to the *four color theorem* [1, 2]. Indeed, on the one hand, if the circles are non-overlapping, then $G(\mathcal{C})$ is planar and thus 4-colorable by the four-color theorem. On the other hand, by the Koebe-Andreev-Thurston *circle packing theorem* [8], every planar graph can be realized as $G(\mathcal{C})$ for some constellation $\mathcal{C}$ of non-overlapping circles, and hence, the assertion that every such constellation $\mathcal{C}$ is 4-colorable implies the four color theorem.

The *overlapping penny problem* is equivalent to the celebrated Hadwiger-Nelson problem, which asks what is the minimum number of colors needed for a coloring of the plane such that no two points at distance 1 get the same color. Indeed, if all circles in $\mathcal{C}$ have a radius of $1/2$, then two circles are tangent if and only if the distance between their centers is 1. For this setting, Isbell observed about 60 years ago that 7 colors suffice [11], and much more recently de Grey [4] showed that 4 colors are not always sufficient, and hence, the chromatic number of the plane lies between 5 and 7.

Unlike for the first three problems, in which a finite upper bound was known already when they were stated, for the *circle problem* no finite upper bound was known. This open problem was introduced for the first time by Ringel [10] in 1959 and appeared in several places as either a question (e.g., [5, 6, 9]) or a conjecture that there is a finite upper bound (e.g., [7]). For lower bounds, Jackson and Ringel [5] presented an example that requires 5 colors; see Figure 1(d). Another such example follows from de Grey's 5-chromatic unit distance graph. No construction requiring more than 5 colors has been known so far.

We solve Ringel's circle problem in a strong sense by showing that the chromatic number is unbounded, even if we require high girth.

Theorem 1. *There exist constellations of circles in the plane with arbitrarily large girth and chromatic number.*

The constellation condition (that no three circles are tangent at a point) is crucial for Ringel's circle problem to be interesting—otherwise one could drive the chromatic number arbitrarily high by taking a set of circles all tangent at one point. In Theorem 1, however, the condition is redundant because it follows from the stronger condition that the girth of the tangency graph is greater than 3. Actually, we prove an even stronger statement in which we additionally forbid pairs of internally tangent circles.

To prove Theorem 1, we use a "sparse" version of Gallai's theorem. In order to guarantee that there are no "unwanted" tangencies in the resulting collection of circles, we develop a refined "sparse" version of Gallai's theorem with additional (polynomial) constraints. We believe that this version may be applicable to obtaining lower bound constructions for other geometric coloring problems, in which some specific form of algebraic independence is requested.

Tangent circles can be thought of as circles intersecting at zero angle. We extend Theorem 1 to graphs defined by pairs of circles intersecting at an arbitrary fixed angle. Specifically, we say that two intersecting circles C_1 and C_2 intersect at angle θ if at any intersection point of C_1 and C_2, the (smaller) angle between the tangent line to C_1 and the tangent line to C_2 equals θ. For any $\theta \in [0, \pi/2]$, the *θ-graph* $G_\theta(\mathcal{C})$ of a collection of circles $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ and edges comprising the pairs of circles in $\mathcal{C}$ that intersect at angle θ. In particular, the 0-graph is the tangency graph. We extend Theorem 1 as follows.

Theorem 2. *For every $\theta \in [0, \pi/2]$, there exist θ-graphs of circles in the plane with arbitrarily large girth and chromatic number.*

The proof of Theorem 2 for $\theta > 0$ is significantly simpler than the proof for $\theta = 0$ corresponding to Theorem 1. For full details, we refer to our paper on arxiv or the SoCG proceedings [3].

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Triangulations

Gaku Liu

We survey work on geometric triangulations from the past 30 years. We cover the following topics:

(1) Geometric bistellar flips: Results on connectivity and non-connectivity of spaces of triangulations through flips.
(2) Asymptotic enumeration of the number of triangulations with a given number of vertices.
(3) Lattice triangulations and unimodular triangulations: Dilation theorem of Kempf–Knudsen–Mumford–Waterman and related results.

We give several open problems, including the following:

(1) Oda's factorization conjecture: Do any two triangulations with the same support have a common iterated stellar subdivision?
(2) Are any two point sets in $\mathbb{R}^3$ and $\mathbb{R}^4$ connected by bistellar flips?
(3) Is there a constant c_d such that for any lattice polytope P in $\mathbb{R}^d$, $c_d P$ has a unimodular triangulation?

A Complex Analogue of the Goodman-Pollack-Wenger Theorem

Daniel McGinnis

The well-known Helly's theorem states that if a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ has the property that any choice of $d+1$ or less sets in $\mathcal{F}$ have a non-empty intersection, then there is a point in common to all the sets in $\mathcal{F}$ (see [1] for surveys on Helly's theorem and related results). A k-*transversal* is a k-dimensional affine space that intersects each set of $\mathcal{F}$, so Helly's theorem provides a necessary and sufficient condition for $\mathcal{F}$ to have a 0-transversal. In 1935, Vincensini was interested in the natural extension of Helly's theorem of finding necessary and sufficient conditions for a finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ to have a k-transversal for $k > 0$. In particular, Vincensini asked if there exists some constant $r = r(k, d)$ such that if every choice of r or fewer sets in $\mathcal{F}$ has a k-transversal, then $\mathcal{F}$ has a k-transversal. However, Santaló provided examples showing that such a constant r does not exist for any $k > 0$ [7].

In 1957, Hadwiger made the first positive progress toward this extenson of Helly's theorem considered by Vincensini by proving the following theorem.

Theorem 1 (Hadwiger [4])**.** *A finite family of pairwise disjoint convex sets in* $\mathbb{R}^2$ *has a* 1*-transversal if and only if the sets in the family can be linearly ordered such that any three sets have a* 1*-transversal consistent with the ordering.*

Hadwiger's theorem has been generalized in different ways, eventually resulting in an encompassing result for $(d-1)$-transversals in $\mathbb{R}^d$ [6]. Despite the previous work on the existence of $(d-1)$-transversals, no necessary and sufficient conditions for the existence of k-transversals in $\mathbb{R}^d$ for $0 < k < d-1$ have been proven or conjectured. Our attempts to find such a condition for $(d-2)$-transversals in

$\mathbb{R}^d$ eventually led us to instead a necessary and sufficient condition for complex $(d-1)$-transversals in $\mathbb{C}^d$. The statement of this result appears in Section 2.

1. Hyperplane transversals revisited

Here we will describe the result of Pollack and Wenger on $(d-1)$-transversals in $\mathbb{R}^d$ as presented in [1], then we will discuss an equivalent rephrasing of this theorem to put our main result in Section 2 into context.

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ and let P be a subset of points in $\mathbb{R}^k$ for some k. We say that $\mathcal{F}$ *separates consistently* with P if there exists a map $\phi : \mathcal{F} \to P$ such that for any two subfamilies $\mathcal{F}_1,\ \mathcal{F}_2 \subset \mathcal{F}$, we have that

$$\operatorname{conv}(\mathcal{F}_1) \cap \operatorname{conv}(\mathcal{F}_2) = \emptyset \implies \operatorname{conv}(\phi(\mathcal{F}_1)) \cap \operatorname{conv}(\phi(\mathcal{F}_2)) = \emptyset.$$

Here we mean $\operatorname{conv}(\mathcal{F}_i)$ to be $\operatorname{conv}(\cup_{F\in\mathcal{F}_i} F)$. Another way to think about this condition is that if the sets of $\mathcal{F}_1$ can be separated from the sets of $\mathcal{F}_2$ by a hyperplane in $\mathbb{R}^d$, then the sets of points $\phi(\mathcal{F}_1)$ and $\phi(\mathcal{F}_2)$ can be separated by a hyperplane in $\mathbb{R}^k$. We also note that $\mathcal{F}$ separates consistently with P if and only if

$$\operatorname{conv}(\mathcal{F}_1) \cap \operatorname{conv}(\mathcal{F}_2) = \emptyset \implies \operatorname{conv}(\phi(\mathcal{F}_1)) \cap \operatorname{conv}(\phi(\mathcal{F}_2)) = \emptyset.$$

whenever $|\mathcal{F}_1|+|\mathcal{F}_2| \le k+2$. This is a consequence of the well-known Kirchberger's theorem [5], which states that if U and V are finite point sets in $\mathbb{R}^k$ such that for every set of $k+2$ points $S \subset U \cup V$, we have that $\operatorname{conv}(S\cap U)\cap\operatorname{conv}(S\cap V)=\emptyset$, then $\operatorname{conv}(U)\cap\operatorname{conv}(V)=\emptyset$.

We now have the terminology to state the Goodman-Pollack-Wenger theorem.

Theorem 2 (Goodman-Pollack-Wenger theorem [6]). *A finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ has a $(d-1)$-transversal if and only if $\mathcal{F}$ separates consistently with a set $P \subset \mathbb{R}^{d-1}$.*

The condition in our main result of Section 2 is quite similar to the condition in Theorem 2, and we will first provide a slight rephrasing of the definition for $\mathcal{F}$ to separate consistently with P in order to make this similarity more apparent. By taking the contrapositive of the implication in the definition of separating consistently, we may equivalently say that $\mathcal{F}$ separates consistently with P if there exists a map $\phi : \mathcal{F} \to P \subset \mathbb{R}^k$ such that

$$\operatorname{conv}(\phi(\mathcal{F}_1)) \cap \operatorname{conv}(\phi(\mathcal{F}_2)) \neq \emptyset \implies \operatorname{conv}(\mathcal{F}_1) \cap \operatorname{conv}(\mathcal{F}_2) \neq \emptyset.$$

In other words, the existence of an affine dependence

$$\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} a_F = 0, \quad \sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} a_F\phi(F) = 0$$

where $a_F \ge 0$ for all $F \in \mathcal{F}_1$ (not all 0) and $a_F \le 0$ for all $F \in \mathcal{F}_2$ implies the existence of points $p_F \in F$ and real numbers $r_F \ge 0$ such that

$$\sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} r_F a_F = 0, \quad \sum_{F\in\mathcal{F}_1\cup\mathcal{F}_2} (r_F a_F) p_F = 0$$

is an affine dependence of the points p_F and the numbers $r_F a_F$ are not all 0.

2. Main result

In this section, we provide a necessary and sufficient condition for a finite family $\mathcal{F}$ of convex sets in $\mathbb{C}^d$ to have a *complex* $(d-1)$-transversal, where a complex $(d-1)$-transversal here is a complex $(d-1)$-dimensional affine subspace of $\mathbb{C}^d$ that intersects each set in $\mathcal{F}$.

First, following our discussion from Section 1, we make the following definition in order to articulate our main theorem, Theorem 4.

Definition 1. *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{C}^d$, and let $P \subset \mathbb{C}^k$. We say that $\mathcal{F}$ is dependency-consistent with P if there exists a map $\phi : \mathcal{F} \to P$ such that for every subfamily $\mathcal{F}' \subset \mathcal{F}$ and every affine dependence*

$$\sum_{F \in \mathcal{F}'} a_F = 0, \sum_{F \in \mathcal{F}'} a_F \phi(F) = 0$$

for complex numbers a_F, there exist real numbers $r_F \geq 0$ and points $p_F \in F$ for $F \in \mathcal{F}'$ such that

$$\sum_{F \in \mathcal{F}'} r_F a_F = 0, \sum_{F \in \mathcal{F}'} (r_F a_F) p_F = 0$$

where not all of the values $r_F a_F$ are 0.

Remark 3. For the purpose of Theorem 4, we could add the additional restriction that $|\mathcal{F}'| \leq 2k+3$ in Definition 1, and the statement of Theorem 4 still holds. This is due to the following reasoning. By associating the points $(a_F \phi(F), a_F)$ with points in $\mathbb{R}^{2k+2}$, we have that the set of points $\{(a_F \phi(F), a_F)\}_{F \in \mathcal{F}'}$ contains $0 \in \mathbb{R}^{2k+2}$ in its convex hull. Therefore, by Carathéodory's Theorem, there exist $m \leq 2k+3$ sets $F_1, \ldots, F_m \in \mathcal{F}'$ and real numbers $s_i > 0$ such that $\sum_{i=1}^{m} s_i (a_{F_i} \phi(F_i), a_{F_i}) = 0$. In other words, there is the complex affine dependence

$$\sum_{i=1}^{m} s_i a_{F_i} = 0, \sum_{i=1}^{m} (s_i a_{F_i}) \phi(F_i) = 0$$

among the points $\phi(F_1), \ldots, \phi(F_m)$.

Theorem 4 (Main theorem)**.** *A finite family of convex sets $\mathcal{F}$ in $\mathbb{C}^d$ has a complex $(d-1)$-transversal if and only if $\mathcal{F}$ is dependency-consistent with a set $P \subset \mathbb{C}^{d-1}$.*

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Borsuk and Vázsonyi problems through Reuleaux polyhedra

Déborah Oliveros
(joint work with Gyivan López-Campos, Jorge Ramírez Alfonsín)

The Borsuk partition problem and *the frequent large distance problem* are two well-known problems in discrete and combinatorial geometry, both based on the notion of *diameter* in bounded sets. The *diameter* of a bounded set $S \subset \mathbb{R}^d$ is defined as $\text{diam}(S) := \sup_{x,y \in S} ||x - y||$. If S is a finite set of points, the diameter would be the maximum euclidean distance between any two points of S.

In 1933, Borsuk [1] proposed the following question (sometime known as the *Borsuk conjecture*):

> *does every set $S \subset \mathbb{R}^d$ with finite diameter diam(S) is the union of at most $d+1$ sets of diameter less than diam(S)?*

It is known to be true for $d = 2$ (see [1]) and for $d = 3$ (see [14], [4] and [6] for a simpler proof). Proved to be false for $d > 63$ see [19] for a survey on the Borsuk conjecture, and the problem still open for $4 \leq d \leq 63$.

Given a set $S \subset \mathbb{R}^d$, the *Borsuk number*, denoted by $a(S)$, is the smallest number of subsets that S can be partitioned, in such way that each subset has a smaller diameter than S. The *diameter graph* Diam_V of finite $V \subset \mathbb{R}^3$ is the graph with a set of vertices V and two vertices are joined by an edge if their distance is a diameter. It is not difficult to observe that $\chi(\text{Diam}_V) = a(V)$, where $\chi(G)$ denotes the chromatic number of the graph G.

In the early 1990's Boltyanski characterized all the sets in $\mathbb{R}^2$ having Borsuk number 3 as those ones that have a unique *completion* to a body of constant width ([3] for the original proof in Russian or [2, pp-245] for English). In the case of $\mathbb{R}^3$, the same argument does not work, it is enough to observe, that four points in tetrahedral position has Borsuk number 4 but its completition to a body of constant width is not unique (see for example [15], [16]).

Although some attempts to find the characterization of sets with Borsuk number 4 were made for sets up to 7 points, (see [9]) no real progress was found until now. In this talk, we gave a full characterization of all finite sets in $\mathbb{R}^3$ with Borsuk number 4 via the frequent large distance problem (Main Theorem).

The *frequent large distance problem*, one of the oldest problems in discrete and combinatorial geometry. Was first proposed in 1934 in the plane by Hopf and Pannwitz [8] and later generalized to all dimensions:

Given $0 < d < n$, what is the maximum number of diameters over all the sets of n points in $\mathbb{R}^d$?

It is well known that in the plane the maximum is n and how all the extremal configurations (set of points that achieve such maximum) look like (see [13, pp 213-214], [10]). For $d = 3$, the problem is better known as *the Vázsonyi problem* in honor to Vázsonyi, who conjecture that the maximum must be $2n - 2$. Proved to be true by Grünbaum [5], Heppes [7] and Straszewicz [20]. See also a totally diferent proof by K. Swanepoel in [21].

In 2010, Kupitz, Martini and Perles characterize all the extremal configurations, pointing out several important facts about such characterization. (For details see [11]) Among this facts, we recall that a set of points V in $\mathbb{R}^3$ is an extremal configuration if the 1-skeleton of the *ball set* denoted by $\mathcal{SF}(\mathcal{B}(V))$ were $\mathcal{B}(V)$ is defined as $\mathcal{B}(V) = \{y \in \mathbb{R}^3 : \forall x \in V, \|x - y\| \leq 1\}$, and behaves as follows:

a) The set is tight, (every vertex is essential, or its removal changes $\mathcal{SF}(\mathcal{B}(V))$)
b) Is planar and 2-connected (but not necessarily the 1-skeleton of a 3-polytope)
c) posseses an *involutory self-duality* that is, an order reversing map, $\varphi : \mathcal{SF}(\mathcal{B}(V)) \to \mathcal{SF}(\mathcal{B}(V))$ of order two ($\varphi^2 = Id$), that sends every vertex $v \in \mathcal{SF}(\mathcal{B}(V))$ to its corresponding *dual face* $F_v \in \mathcal{SF}(\mathcal{B}(V))$

A self-dual polyhedron G admitting an involution is called an *involutive* polyhedron (see [18]). Note that $\tau(v)$ can be thought as a face of G (called *dual* face of v). Let $G = (V, E)$ be an involutive polyhedron and let $a, x \in V$. We say that $[a, x]$ is a *diagonal* of G if $x \in \tau(a)$. Next, Given an involutive polyhedron $G = (V, E)$, we define the *diagonal* graph Diag_G arising from G, as the graph where the set of vertices is V and set of edges consisting on the set of all the diagonals of G. During this talk we outline the proof of the following Lemma using interesting embeddings of self dual graphs presented in [17].

Key Lemma: [G. López-Campos., D.O. J. Ramírez Alfonsín (2023) [12]]
If G is an involutive polyhedron. Then, Diam_G is 4-critical.
($\chi(\text{Diam}_G) = 4$ and $\chi(\text{Diam}_G \setminus v) = 3$ for every vertex $v \in V$).

We say that V is *strongly critical*, if V is an extremal configuration for the Vázsonyi problem, any point of V is adjacent to at least 3 diameters and V does not have an extremal configuration subset.

The Key Lemma allows us to prove the following conjecture posted in [11].

Conjecture: An extremal set $V \subset \mathbb{R}^3$ is a Reuleaux polyhedron $\mathcal{B}(V)$
(i.e. the 1-skeleton of $\mathcal{B}(V)$ is the 1-skeleton of a 3-polytope)
if and only if V is strongly critical.

Moreover the Key lemma together with all the previews work presented in [11] allows us to present a full characterization of all finite sets in $\mathbb{R}^3$ with Borsuk number 4 as follows:

Main Theorem: [G. López-Campos., D.O. J. Ramírez Alfonsín (2023) [12]]
Let $V \subset \mathbb{R}^3$ be a finite set of points with $\mid V \mid= n \geq 4$. The following three statements are equivalent:

(1) V is strongly critical for the Vázsonyi problem.
(2) diam_V is 4-critical.
(3) $\mathcal{B}(V)$ is a Reuleaux polyhedron.

For further details of the results shown in this extended abstract see [12]

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Enumeration of intersection graphs

János Pach
(joint work with Jacob Fox, Andrew Suk)

Given a collection C of n geometric objects, define their intersection graph $G(C)$ as follows. Let the vertex set of $G(C)$ be C, and connect two elements of C by an edge if and only if they have a point in common. The total number of graphs on n labeled vertices is $2^{\binom{n}{2}}$. How many of them are intersection graphs of connected arcs ("strings") in the plane? Pach and Tóth proved that the answer is $2^{(3/4+o(1))\binom{n}{2}}$. If we restrict our attention to intersection graphs of strings, any pair of which intersect at most k times, for a fixed k, then the number becomes $2^{o(n^2)}$. On the other hand, it was shown by Pach and Solymosi that the number of segment intersection graphs on n vertices is $2^{(4+o(1))n \log n}$. After giving a whirlwind tour of enumeration results and methods of this kind, we present some recent results by Jacob Fox, Andrew Suk, and the speaker.

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Moser's shadow problem in high dimensions

Arnau Padrol

In a famous list of open problems in combinatorial geometry from 1966 (see [2] and [5]), Moser asked for the largest $\mathfrak{sh}(n)$ such that every 3-polytope with n vertices has a 2-dimensional projection with at least $\mathfrak{sh}(n)$ vertices. The solution to this problem, popularly known as *Moser's shadow problem*, was implicit in the work of Chazelle, Edelsbrunner and Guibas in 1989 [1] but went unnoticed until recently [4].

The results of [1] concerned *silhouettes* from arbitrary light sources, but they can be easily adapted to show that $\mathfrak{sh}(n) = \theta\left(\log n / \log \log n\right)$. The lower bound was derived from a related result concerning stabbing numbers of convex subdivisions of the plane by lines, that Tóth generalized to stabbing numbers of convex subdivisions of $\mathbb{R}^d$ by lines [6]. He showed that for each convex subdivision of $\mathbb{R}^d$ into n regions there is a line stabbing $\Omega\left((\log n / \log\log n)^{1/(d-1)}\right)$ cells. In the polytopal set-up, this implies that every d-polytope with n-vertices has a 2-dimensional shadow with at least $\Omega\left((\log n / \log\log n)^{1/(d-2)}\right)$ vertices.

We consider the same questions with lines and 2-dimensional shadows replaced by k-flats and k-shadows (open problems formulated in [6] and [4], respectively).

We define a *k-shadow* as an affine projection onto a k-dimensional subspace, and the *shadow number* $\mathfrak{sh}(n, d, k)$ to be the largest number such that every d-polytope with n vertices has a k-shadow with at least $\mathfrak{sh}(n, d, k)$ vertices. By polarity, this is equivalent to the largest number such that for every d-polytope P with n facets containing the origin in its interior there is a k-dimensional subspace F through the origin so that the intersection $P \cap F$ has at least $\mathfrak{sh}(n,d,k)$ facets. Similarly, we define the *silhouette number* $\mathfrak{si}(n,d,k)$ to be the largest number such that every d-polytope with n vertices admits a projective transformation with a k-shadow with at least $\mathfrak{si}(n,d,k)$ vertices. By polarity, $\mathfrak{si}(n,d,k)$ is the largest number such that for every d-polytope with n facets there is an affine k-dimensional subspace that induces a section with at least $\mathfrak{si}(n,d,k)$ facets. Finally, we define the *stabbing number* $\mathfrak{st}(n,d,k)$ as the largest number such that for every subdivision of $\mathbb{R}^d$ into n convex cells, there is a k-dimensional affine subspace that intersects at least $\mathfrak{st}(n,d,k)$ cells.

These functions are related by the following inequalities

$$\mathfrak{st}(\lceil n/2 \rceil, d-1, k-1) \leq \mathfrak{sh}(n,d,k) \leq \mathfrak{si}(n,d,k).$$

The second inequality is straightforward from the definition, as having more freedom (with the choice of a projective transformation or, in the polar, passing from linear to affine flats), can only increase this minimax parameter. For the first inequality, fix a polytope P and assume that the fixed point is the origin. After a perturbation and a reflection if needed, we can assume that at least half of the facets are lower (their normal vector has negative last coordinate). A central projection from the origin onto the $x_d = -1$ hyperplane will send lower facets to convex cells, and linear k-subspaces intersecting these facets to affine $(k-1)$-flats stabbing the convex cells. A detailed proof can be found in [1, Lemma 5.1]. It is formulated for the $d = 3$ and $k = 2$ case, but the generalization is straightforward.

In [3], in joint work with Alfredo Hubard, we proved that for every convex subdivision of $\mathbb{R}^d$ into n cells there exists a k-flat stabbing $\Omega\big((\log n/\log\log n)^{1/(d-k)}\big)$ of its cells, showing that :

Theorem 1. *Let $k \leq d-1$ be fixed, then*

$$\Omega\Big((\log n/\log\log n)^{1/(d-k)}\Big) \leq \mathfrak{st}(\lceil n/2 \rceil, d-1, k-1) \leq \mathfrak{sh}(n,d,k) \leq \mathfrak{si}(n,d,k)$$

as $n \to \infty$.

I presented a new construction a family of convex polytopes in $\mathbb{R}_d$ with at least $\exp(\alpha_d n^{\lceil \frac{d-k}{k-1} \rceil} \log n)$ facets such that no affine k-flat can simultaneously intersect more than $\beta_d n$ of them, for some constants α_d, β_d depending on the dimension. This provides a new upper bound for the shadow, silhouette and stabbing numbers that simultaneously generalizes the particular cases settled in [1] and [6]. It is asymptotically tight for 2- and $(d-1)$-shadows and silhouettes, and for stabbing numbers for lines and hyperplanes.

Theorem 2. *Let $k \leq d-1$ be fixed, then*

$$\mathfrak{st}(\lceil n/2 \rceil, d-1, k-1) \leq \mathfrak{sh}(n,d,k) \leq \mathfrak{si}(n,d,k) \leq O\left((\log n/\log\log n)^{\frac{1}{\lceil \frac{d-k}{k-1} \rceil}}\right)$$

as $n \to \infty$.

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Equality cases of the Alexandrov–Fenchel inequality

Igor Pak

Describing the equality conditions of the *Alexandrov–Fenchel inequality* has been a major open problem for decades. Recently, Shenfeld and van Handel gave a geometric characterization of the equality conditions for the case of convex polytopes. From the computational point of view, this characterization is not explicit and can be hard to efficiently verify. We show that this difficulty is inherent to the problem.

Formally, we prove that in the case of totally uniform convex polytopes, the description of equality cases is not in the polynomial hierarchy unless the polynomial hierarchy collapses to a finite level. This is the first hardness result for the problem, and is a complexity counterpart of the above mentioned result by Shenfeld and van Handel.

In the talk, we give an introduction to the problem, give a quick overview of the earlier work, and describe our result in the context. We also emphasize the applications of our work to stability of geometric inequality, and the problem of finding combinatorial interpretations. The proof involves Stanley's *order polytopes* and employs poset theoretic technology, but we will limit ourselves to a brief sketch.

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Shellability of balls in 3D

Pavel Paták
(joint work with Martin Tancer)

A simplicial complex K is called *purely k-dimensional*, if all its facets have dimension k.

A purely k-dimensional simplicial complex is called *shellable*, if there exists an ordering $F_1, \dots, F_k$ of its facets such that for all $k > 1$, $F_k \cap \bigcup_{i<k} F_i$ is a non-empty union of $(k-1)$ simplices. Shellability plays an important role in theory of convex polyhedra, however it also find applications in algebra or poset theory.

If K is a simplicial complex, and σ is a face that is contained in a uninque facets $\tau \supseteq \sigma$ different from σ, the *elementary collapse* of K along σ consist in deleting all simplices $\rho \supseteq \sigma$ from K. A complex is called *collapsible*, if it can be transformed into a single point by a series of elementary collapses. Collapsibility is a combinatorial analogue of of contractibility.

We use the reduction from planar rectilinear monotone 3-SAT, to show that the following problems are NP-complete:

(1) Deciding whether a 3D simplicial ball is shellable.
(2) Deciding whether a purely 2D simplicial complex in $\mathbb{R}^3$ is shellable.
(3) Deciding whether a 3D simplicial complex in $\mathbb{R}^3$ is collapsible.

For shellability of purely 2D simplicial complexes in $\mathbb{R}^3$, the construction refines the previous construction from [1]: We replace the original clause gadgets, which did not fit into $\mathbb{R}^3$ with a modified version of the 3-turbine from [2], which can be embedded into $\mathbb{R}^3$. Moreover, instead of using a reduction from a general 3-SAT formula, we use the reduction from planar rectilinear mononote 3-SAT, which allows us to keep the whole construction in $\mathbb{R}^3$. In other words, we have a polynomial time procedure that turns each instance φ of planar rectilinear monotone 3-SAT into a purely 2-dimensional complex K_φ such that φ is satisfiable if and only if K_φ is shellable.

For the result about collapsibility, we replace the variable gadget in the construction for shellability with a 3-dimensional piece: the join of a segment with a boundary of a triangle. Otherwise the construction is the same.

For the main result about balls, we thicken the construction and replace 2D pieces with their thickened 3D versions. However, this thickening has to be done carefully and is description is thus rather technical. The main problem is to maintain the shellability if the instance φ is satisfiable and to not introduce any unwanted shellings if the instance φ is not. For more details, see [3].

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Discretized Elekes-Ronyai theorems and projection theory

Orit Raz
(joint work with Josh Zahl)

In the talk I talked about a recent result with Josh Zahl where we show how to use ideas from discrete geometry in order to prove results in projection theory. Specifically, we prove the following theorem:

Theorem 1. *Let $E \subseteq \mathbb{R}^2$ be a Borel set of Hausdorff dimension α. Let $\gamma \subseteq \mathbb{R}^2$ be a smooth curve with non-vanishing curvature. Then*

$$\dim_H \{q \in \mathbb{R}^2 \mid \dim \Delta_q(E) \leq \frac{\alpha}{2} + c\} = 0,$$

where $c = c(\alpha) > 0$ and $\Delta_q(E)$ is the set of distances spanned between the point q and the set E.

To prove this we show a discretized analogue of a question from discrete geometry about distances of a set from 3 fixed points.

On Dol'nikov's Conjecture.

Edgardo Roldán-Pensado
(joint work with Cuauhtémoc Gomez-Navarro, Leonardo Martínez-Sandoval)

This work is focused on a conjecture by Dol'nikov and its relation to the colorful Helly theorem. The colorful Helly theorem states the following.

Theorem 1 (Colorful Helly theorem)**.** *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be families of convex bodies in $\mathbb{R}^d$, such that for any choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty. Then, there exists an index i for which $\bigcap \mathcal{F}_i \neq \emptyset$.*

Dol'nikov's conjecture is similar to the colorful Helly theorem but there are less intersecting sets overall. To compensate, the sets are required to be translates of each other.

Conjecture 1 (Dol'nikov's Conjecture)**.** *Let K be a compact convex set in $\mathbb{R}^2$. Consider $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ as finite families of translates of K. If for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$, the intersection $A \cap B$ is non-empty, then there exists an $\mathcal{F}_j$ that can be pierced by 3 points.*

In 2015 Jerónimo-Castro, Magazinov and Soberón [2] validated this conjecture when K is centrally symmetric or a triangle. Moreover, they established a stronger result when K is a circle.

Further progress was made in 2023, through a collaboration with Gomez-Navarro [1]. We proved Dol'nikov's conjecture to encompass additional convex sets in a more general setting.

Theorem 2. *Let K be a convex body in $\mathbb{R}^2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be finite families of translates of K such that $A \cap B$ is non-empty for every $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. If K is of constant width or has a Banach-Mazur distance of at most* 1.1178 *to a disk, then either $\mathcal{F}_1$ or $\mathcal{F}_2$ can be pierced by* 3 *points.*

The proof of this theorem is related to a strenghening of the Colorful Helly Theorem. This result was proved previously in in collaboration with Martínez-Sandoval and Rubin [3].

Theorem 3. *Let $\mathcal{F}_1, \dots, \mathcal{F}_d$ be finite families of convex sets in $\mathbb{R}^d$, such that for every choice of sets $C_1 \in \mathcal{F}_1, \dots, C_d \in \mathcal{F}_d$, their intersection $\bigcap_{i=1}^d C_i$ is non-empty. Then one of the following statements holds:*

(1) there is a family $\mathcal{F}_j$ that can be pierced by $f(d)$ points, or
(2) the combined family $\bigcup_i \mathcal{F}_i$ can be intersected by $g(d)$ lines.

Describing the pairs $(f(d), g(d))$ for which this theorem holds is an open problem. In particular, we do not know if the theorem is applicable for $f(d) = 1$ with a sufficiently large $g(d)$. In collaboration with Gómez-Navarro [1], we improved the values for $f(2)$ and $g(2)$.

Theorem 4. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of convex sets in $\mathbb{R}^2$, where $n \geq 2$. Assume that for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$, the intersection $A \cap B$ is non-empty. Then one of the following statements holds:*

(1) there exists an index j such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 1 *point, or*
(2) the family $\bigcup_i \mathcal{F}_i$ can be crossed by 2 *lines.*

The numbers in this theorem are best possible. A simple proof of this result uses the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem and is adapted from a result by McGinnis and Zerbib [5]. Additionally, an alternative proof exists that works in higher dimensions, but uses hyperplanes in place of lines.

For translates we managed to improve the numbers in an unexpected way.

Theorem 5. *Let K be a convex body in $\mathbb{R}^2$. Consider $\mathcal{F}_1, \dots, \mathcal{F}_n$ as finite families of translates of K such that $A \cap B$ is non-empty for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

(1) there exists an index j such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 *point, or*
(2) the family $\bigcup_i \mathcal{F}_i$ can be crossed by 1 *line.*

This theorem plays an important role in our proof of Theorem 2, especially in establishing the existence of a line transversal to $F_1 \cup F_2$. Using Theorem 5 and building on the same foundational ideas, we can demonstrate Dol'nikov's conjecture with 8 points instead of 3. More recently, through collaboration with Leonardo Martínez-Sandoval [4], we managed to further reduce this number to 4 points.

Theorem 6. *Let K be a compact convex set in $\mathbb{R}^2$, and let $F_1, \dots, F_n$ be finite families of translates of K. Suppose that $A \cap B$ is non-empty for every $A \in F_i$ and $BinF_j$ with $i \neq j$, then there exists an index j such that $\bigcup_{i \neq j} F_i$ can be pierced by* 4 *points.*

The novel idea used to prove this theorem is a new result related to the Banach-Mazur distance between a convex body K and the square.

Lemma 7. *Let K be a convex body in $\mathbb{R}^2$ and let u be a given direction. Then there exists a parallelogram $P \subset K$ with one of its sides aligned with u, and a translated copy $2P$ that fully contains K.*

There are many open problems described in [1] and [3] which remain unsolved. We hope that these will spark further research and discussions in the study of discrete geometry and its applications.

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Depth Measures for Hyperplane Arrangements

Patrick Schnider

(joint work with Pablo Soberón)

A central topic in combinatorial geometry and computational geometry is the study of structural properties of finite families of points in Euclidean spaces. Studying which sets can be separated from others by hyperplanes is a natural question, which leads us to study combinatorial properties of convex sets. Classic results, such as Tverberg's theorem [11] and Rado's centerpoint theorem [6] follow from this line of thought.

In some cases, instead of being provided our data as a finite set of points in $\mathbb{R}^d$, we might receive it as a set of hyperplanes. Understanding which results for families of points transfer to families of hyperplanes is a natural question.

Given a hyperplane arrangement A in $\mathbb{R}^d$ and a point q, we first consider the depth of q with respect to A as follows.

Definition 1. *The regression depth of a query point q with respect to hyperplane arrangement A, denoted by RD(A,q), is the minimum number of hyperplanes in A intersected by or parallel to any ray emanating from q.*

Given an arrangement A of n hyperplanes, the existence of points with regression depth at least $n/(d+1)$ has been established by Amenta, Bern, Eppstein, and Teng [1], and later by Mizera [5] as well as Karasev [2]. This can be considered a hyperplane version of Rado's centerpoint theorem [6]. In the full version [9] we prove that the centerpoint theorem for regression depth is the consequence of a Tverberg-type theorem, confirming a conjecture of Rousseeuw and Hubert [7].

Theorem 1. *Let r, d be positive integers and A be an arrangement of at least $(r-1)(d+1)+1$ hyperplanes in $\mathbb{R}^d$. Then, there exists a point q in $\mathbb{R}^d$ and a partition of A into r parts such that q has positive regression depth with respect to each of the r parts.*

This was previously known when $d = 2$ [7] or when r is a prime power [3, 4]. The version for prime powers by Karasev holds with a slightly more restrictive version of regression depth. Based on this result, we define the *hyperplane Tverberg depth* of a point.

Definition 2. *The hyperplane Tverberg depth of a query point q with respect to hyperplane arrangement A, denoted by HTvD(A,q), is the maximum r such that there is a partition of A into r parts such that q has positive regression depth with respect to each part.*

1. Correspondence to depth measures for point sets

For an arrangement A and a query point q, we define the *dual of A at q*, denoted by A_q^*, as follows. For each hyperplane $h \in A$, let $p(h)$ be the unique point on h that is closest to q. We define A_q^* as the set formed by all these points, that is, $A_q^* := \{p(h) \mid h \in A\}$. Note that if q lies on k hyperplanes, then those k dual points coincide with q in A_q^*.

Using this duality, for every depth measure ρ on point sets we can define a corresponding depth measure ρ^* on hyperplane arrangements and vice versa, by setting $\rho^*(A, q) = \rho(A_q^*, q)$. We have the following observation.

Observation 1.

(1) a ray r emanating from q intersects a hyperplane h if and only if the half-space $r^\perp$ defined by the hyperplane thorugh q orthogonal to r, oriented such that it contains r, contains $p(h)$;

(2) the point q has positive regression depth with respect to $h_1, \ldots, h_n$ if and only if it is in the convex hull of $p(h_1), \ldots, p(h_n)$.

(3) the point q lies in the simplex defined by $h_1 \ldots, h_{d+1}$ if and only if it is in the interior of the convex hull of $p(h_1), \ldots, p(h_{d+1})$.

The depth measures for hyperplane arrangements defined above all have natural corresponding depth measures for point sets that follow immediately from Observation 1. For regression depth, the corresponding depth measure is *Tukey depth* (TD), which is defined as the minimum number of data points contained in any closed half-space containing the query point q [10]. For hyperplane Tverberg depth we get Tverberg depth (TvD), which is defined as the maximum r for which there exists an r-partition of the data points containing the query point q in their intersection.

2. Axioms for hyperplane depth

Let $A^{\mathbb{R}^d}$ denote the family of all finite arrangements of hyperplanes in $\mathbb{R}^d$. A depth measure for hyperplanes is a function $\rho : (A^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ which assigns to each pair (A, q) consisting of a hyperplane arrangement A and a query point q a value, which describes how deep the query point q lies within the arrangement A. A depth measure is called *combinatorial* if it is the same for all points in a face of A. Similar to [8], we introduce some axioms, that reasonable depth measures for hyperplane arrangements should satisfy.

We say that a combinatorial depth measure for hyperplanes is *super-additive* if it satisfies the following four conditions.

(i) for all $A \in A^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ and any hyperplane h we have $|\rho(A, q) - \rho(A \cup \{h\}, q)| \leq 1$,
(ii) for all $A \in A^{\mathbb{R}^d}$ we have $\rho(A, q) = 0$ if q is in an unbounded cell of A,
(iii) for all $A \in A^{\mathbb{R}^d}$ we have $\rho(A, q) \geq 1$ if q is in a bounded cell or if q lies on a hyperplane of A,
(iv) for any disjoint subsets $A_1, A_2 \subseteq A$ and $q \in \mathbb{R}^d$ we have $\rho(A, q) \geq \rho(A_1, q) + \rho(A_2, q)$.

Following [8] we get

Theorem 2. *Let ρ be a super-additive depth measure for hyperplanes. Then for all $A \in A^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $RD(A, q) \geq \rho(A, q) \geq HTvD(A, q) \geq \frac{1}{d} RD(A, q)$.*

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A logarithmic bound for simultaneous embeddings of planar graphs

Raphael Steiner

Given a planar graph G, a *straight-line embedding* of G in the plane is an injective mapping π from the vertex-set V of G to $\mathbb{R}^2$ such that adding in all the straight-line segments with endpoints $\pi(u)$ and $\pi(v)$ for every edge uv in G yields a crossing-free drawing of G in the plane. Given a planar graph G and a point set $P \subseteq \mathbb{R}^2$ in the plane, we say that G *has a straight-line embedding on* P or equivalently that it *embeds straight-line on* P if there exists a straight-line embedding of G in the plane in which every vertex of G is mapped to a distinct point of P. If G is a *labelled planar graph* with vertices numbered as $\{v_1, \ldots, v_n\}$, and if P is a *labelled point set*, that is, its elements are numbered as $P = \{p_1, \ldots, p_n\}$, then we say that G has a *label-preserving straight-line embedding on* P if the bijection $\pi : V(G) \to P$, $\pi(v_i) := p_i$, forms a straight-line embedding of G.

The study of straight-line embeddings of planar graphs is a classical area in graph drawing. For instance, one of the most fundamental results on this topic, the Fáry-Wagner-Theorem [7], states that every planar graph G admits a straight-line embedding in the plane on *some* point set. However, it is not true that a planar graph on n vertices can be embedded on any given point set of size n. In fact, for a fixed planar graph G on n vertices, only a small fraction of all potential n-point sets may allow a straight-line embedding of G. Thus, many interesting questions in graph drawing arise from considering the embeddability of (restricted classes of) planar graphs on (restricted types of) point sets. One of the biggest branches of research in this direction concerns *simultaneous embeddings* of sets of planar graphs, we refer to [4] for a survey on this topic. Given a set $\mathcal{G}$ of planar graphs and a point set P in the plane, we say that $\mathcal{G}$ is *simultaneously embeddable on* P if every member $G \in \mathcal{G}$ admits a straight-line embedding on P. If $n \in \mathbb{N}$ and $\mathcal{G}$ is a set of planar graphs, each on n vertices, we say that $\mathcal{G}$ is *simultaneously embeddable* (without mapping) if there exists a point set $P \subseteq \mathbb{R}^2$ of size n such that $\mathcal{G}$ is simultaneously embeddable on P.

There are two major open problems in geometric graph theory related to the notions introduced above. First, there is the so-called the *universal set problem*, which asks to find the asymptotics of the function $f(n)$, defined as the smallest size of a point set P in the plane such that the set of all n-vertex planar graphs is simultaneously embeddable on P (such a set is called *n-universal*). Currently there is still a large gap in our understanding of this problem, with the best asymptotic estimates being $f(n) \leq \frac{1}{4}n^2 + O(n)$ by Bannister et al. [3] and $f(n) \geq (1.293 - o(1))n$ by Scheucher et al. [10].

Second, there is a major open problem concering the simultaneous embeddability of small collections of planar graphs, a systematic study of which was initiated by Brass et al. [5]. In particular, they raised the following intriguing open problem, which remains unsolved.

Problem 1 (cf. [5])**.** Is there a set $\mathcal{G} = \{G_1, G_2\}$ consisting of two planar graphs of the same order such that $\mathcal{G}$ is not simultaneously embeddable?

Following the terminology of [6, 8, 10], let us call a set $\mathcal{G}$ of planar graphs, all of the same order $n \in \mathbb{N}$, a *conflict collection* if $\mathcal{G}$ is not simultaneously embeddable. Addressing small values of n, Cardinal et al. [6] proved that for $n \leq 10$, there exists no conflict collection consisting of n-vertex planar graphs. In contrast, they showed that for every $n \geq 15$ a conflict collection *does* exist. Motivated by Problem 1, it is natural to study the value $\sigma(n)$, defined as the smallest size of a conflict collection of n-vertex planar graphs (if such a collection exists). By the Fáry-Wagner-Theorem, we have $\sigma(n) \geq 2$ for every n, and Problem 1 is equivalent to the question whether there exists some n such that $\sigma(n) = 2$. Approaching this question, Cardinal et al. [6] constructed a relatively small conflict collection on 35-vertex graphs, proving that $\sigma(35) \leq 7393$. A significantly smaller conflict collection consisting of 11-vertex graphs was found by Scheucher et al. [10], showing that $\sigma(11) \leq 49$. Regarding the general asymptotic bounds on the function $\sigma(n)$, it was recently proved by Goenka et al. [8] that $\sigma(n) \leq O(n \cdot 4^{n/11}) < O(1.135^n)$, by an explicit general construction of a conflict collection on n-vertex planar graphs. While this bound is exponential in n, we give a short probabilistic proof that $\sigma(n) = O(\log n)$, and thus for large enough n much smaller conflict collections of n-vertex graphs of only logarithmic size in n exist.

Theorem 1. *It holds that* $\sigma(n) \leq (3 + o(1)) \log_2(n)$.

Using the same technique, but with a more careful analysis, we obtain the following upper bounds, which improve upon the benchmark of 49 for the size of the previously smallest known conflict collection of planar graphs [10]. In contrast to the heavily computer-assisted proof of the bound 49 in [10], our proof of this new bound is computer-free, elementary and self-contained.

Theorem 2. *For every* $n \in \{107, 108, \dots, 193\}$, *we have* $\sigma(n) \leq 30$. *In particular, there exists a conflict collection consisting of* 30 *planar graphs.*

The proofs of Theorem 1 and 2 rely on the probabilistic method and thus unfortunately do not provide explicit constructions of the asserted conflict collections. Motivated by this, we also present a different, fully explicit construction of a conflict collection of less than n^6 planar n-vertex graphs for every large enough n. This still improves significantly over the explicit construction of size $(21n+552)4^{(n+37)/11}$ given by Goenka et al. [8], reducing the size of the constructed conflict collection from exponential to polynomial.

Theorem 3. *For every* $n \geq 7! = 5040$, *there exists an explicit construction of a conflict collection consisting of* $n^{\underline{6}} + 1 = n(n-1)\cdots(n-5) + 1$ *planar* n*-vertex graphs.*

Very brief proof overview. The main new idea to prove Theorems 1 and 2 is to construct a probability distribution on the set of n-vertex labelled planar graphs, more specifically on a certain class of stacked triangulations, such that a random planar graph $\mathbf{G}$ drawn at random from this distribution satisfies the following properties:

- For every fixed point set $P \subseteq \mathbb{R}^2$ of size n, the probability that $\mathbf{G}$ embeds on P is at most $2^{-(1-o(1))n}$, and
- For every fixed labelled point set $P \subseteq \mathbb{R}^2$ of size n, the probability that $\mathbf{G}$ admits a label-preserving straight-line embedding on P is at most $\frac{1}{n!2^{(1-o(1))n}}$.

The idea is then to construct a conflict collection of planar graphs by independently sampling several planar graphs from this distribution. For any fixed point set P, the probability that all the sampled planar graphs embed on P will be very very small, and thus this can be weighed in a union bound against the total number of distinct order types of n points in the plane in general position. Denoting by $t_s(n,2)$ the number of labelled order types of n-point sets in general position and working out the details of this argument, one obtains that

$$\sigma(n) \leq \left\lfloor \frac{\log_2(t_s(n,2)) - (n-4) - \log_2((n-3)!)}{n - \log_2(16n(n-1)(n-2))} \right\rfloor + 2$$

for every $n \geq 16$. Plugging in the asymptotically tight estimate $t_s(n,2) = n^{(4+o(1))n}$ due to Alon [2], one then obtains the upper bound in Theorem 1. To get the bounds in Theorem 2, one has to work a little harder to obtain preciser estimates of $t_s(n,2)$ for moderately small values of n. This can be achieved by using Warren's inequality [11] which gives an upper bound on the number of sign-patterns defined by a collection of multivariate polynomials. This yields

$$t_s(n) \leq 2 \cdot 16^n \cdot \sum_{k=0}^{2n} 2^k \binom{\binom{n}{3}}{k}$$

for every $n \geq 3$, which then can be plugged into the above upper bound on $\sigma(n)$, yielding Theorem 2.

Finally, to prove Theorem 3, a totally different combinatorial construction is used. Let $n \geq 5040$ be given. Let us start by considering the set

$$\mathcal{S}_n := \left\{ (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) \in \mathbb{N} \cup \{0\} \,\middle|\, \sum_{i=1}^{8} n_i = n - 6 \right\}$$

of ordered partitions of the number $n-6$ into 8 non-negative integers. We then have

$$|\mathcal{S}_n| = \binom{(n-6)+8-1}{8-1} = \binom{n+1}{7} = \frac{(n+1)^{\underline{7}}}{7!} = \frac{(n+1)n^{\underline{6}}}{7!} > n^{\underline{6}},$$

where we used that $n \geq 5040 = 7!$ in the last line. It is therefore possible to select (explicitly) a subset $\mathcal{S}'_n \subset \mathcal{S}_n$ such that $|\mathcal{S}'_n| = n^{\underline{6}} + 1$. We now define a collection $\mathcal{G}_n$ of planar triangulations on n vertices as follows:

Let H denote the octahedron graph, which is a triangulation with 6 vertices a, b, c, d, e, f and 8 faces $f_1, \ldots, f_8$.

Now, for every element $(n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) \in \mathcal{S}'_n$, construct a planar triangulation by (1) adding for every one of the 8 faces f_i of H a set of exactly n_i

additional vertices that are placed into the corresponding face, and (2) triangulating each of the faces f_i together with the n_i new vertices placed in it.

Clearly, there may be many different ways to triangulate, but for the sake of this construction, we just choose and fix for every given $\mathbf{x} \in \mathcal{S}'_n$ one possible way to do this, and call the resulting triangulation $T_n(\mathbf{x})$.

Finally, we define $\mathcal{G}_n = \{T_n(\mathbf{x}) | (\mathbf{x}) \in \mathcal{S}'_n\}$.

The claim is now that the family $\mathcal{G}_n$ of planar n-vertex graphs is not simultaneously embeddable, which then yields Theorem 3. The idea of the proof is simple: For any fixed point set P of n points, there are at most $n^{\underline{6}}$ ways of choosing the location of the 6 vertices of the octahedron in a straight-line embedding. Since $|\mathcal{S}'_n| > n^{\underline{6}}$, the pigeon-hole principle implies that in any simultaneous embedding of $\mathcal{G}_n$, there would need to be two distinct triangulation $T_n(\mathbf{x}), T_n(\mathbf{x}')$ with $\mathbf{x} \neq \mathbf{x}'$ whose straight-line embeddings on P embed the octahedron H contained in them in the same way. However, counting points that are in the faces of the octahedron now is easily seen to imply that $\mathbf{x} = \mathbf{x}'$, a contradiction.

Open problems. One open problem that would be very interesting to settle (in particular with regards to Problem 1) is the following: For large n, what is the probability that two independently sampled random planar graphs $\mathbf{G}_1, \mathbf{G}_2$ from the probability distribution used in the proofs of Theorem 1 and 2 admit a simultaneous embedding?

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Pach's animal problem within the bounding box

Martin Tancer

A collection of unit cubes with integer coordinates in $\mathbb{R}^3$ is an *animal* if its union is homeomorphic to the 3-ball. Pach's animal problem asks whether any animal can be transformed to a single cube by adding or removing cubes one by one in such a way that any intermediate step is an animal as well. In the talk we have provided an example of an animal that cannot be transformed to a single cube this way within its bounding box.

In more detail, a *grid cube* is a subset of $\mathbb{R}^3$ that can be written as $[a, a+1] \times [b, b+1] \times [c, c+1]$ where a, b and c are integers. A *grid complex* is a 3-dimensional polytopal complex formed by a finite collection of grid cubes and the faces of the cubes in this collection. A grid complex is an animal if the union of cubes in the complex is homeomorphic to a 3-ball. In 1988 Pach asked whether any animal can be transformed to a single cube by adding or removing cubes one by one in such a way that any intermediate step is an animal as well [6]. This question is known as *Pach's animal problem* and has been reproduced in several other venues; see, e.g., notes in Chapter 8 of [10] or [3, 5]. In the following text, when we consider cube removals or additions, we always mean that each intermediate step is an animal.

Surprisingly, this innocent-looking question is actually very complex and resistant. On the one hand, there are examples of animals that cannot be transformed to a single cube by removals only: The first one (the author is aware of) is Furch's "knotted hole ball" from 1924 [4] (see also [9]). Another one from 1964 is a 3-dimensional variant of famous Bing's house with two rooms [1].[1] After Pach asked about the animal problem, Shermer obtained a particularly small such animals independently of the earlier results [7]. (See also [5].) On the other hand, allowing also cube additions adds much more flexibility how to transform animals. If we replace "cube removals" and "cube additions" with closely related "collapses" and "anticollapses", it follows from a classical result of Whitehead [8] that any animal can be reduced to a point (or a cube) by collapses and anticollapses. But the integer grid does not seem to be flexible enough to emulate all possible anticollapses. Altogether, adding geometric restrictions coming from the integer grid to classical setting in topology makes the question interesting.

Pach's animal problem within the bounding box. For all the aforementioned examples, even if they cannot be transformed to a single cube by removals, it is extremely easy to transform them to a single cube if we also allow additions of cubes. All the aforementioned examples can be built by gradual removals of cubes from the *bounding box* (i. e., the smallest grid-aligned box containing the animal) while each intermediate step is an animal. If we revert this process, each of the aforementioned examples can be transformed to the bounding box (by cube additions) and then to a single cube (by cube removals).

[1] The aim of the constructions of Furch and Bing is to obtain so called non-shellable balls. But for an animal non-shellable exactly means that the animal cannot be transformed to a single cube by removals only.

Dumitrescu and Hilscher [2] provided an example of an animal which cannot be transformed to the bounding box by cube additions but this example is essentially the complement of Shermer's construction. Thus the cost is that this animal can be easily transformed to a single cube by removals.

In principle it should be possible to combine (and possibly iterate) two types of the aforementioned constructions which would require alternating cube additions and cube removals in order to transform the animal into a single cube within the bounding box. But this would still leave the hope that there is an algorithm for Pach's animal problem which gradually simplifies the "innermost" part of the animal (or its complement) eventually reaching a single cube.

We provide a new significantly stronger construction showing that this hope is vain.

Theorem 1. *There is an animal A such that it cannot be transformed to a single cube by additions or removals of cubes which are inside the bounding box of A. In fact, if we remove a cube from A or add a cube to A contained inside the bounding box, we never obtain an animal.*

Part of our motivation for proving Theorem 1 is also that we find it realistic that this construction (or the ideas beyond it) could be a part of a construction of a counterexample to original Pach's animal problem (without any restriction coming from the bounding box), of course, only if such a counterexample exists.

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Exponential Erdős-Szekeres theorem for matrices

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(joint work with Recep Altar Çiçeksiz, Zhihan Jin, Eero Räty)

The Erdős-Szekeres theorem [5] from 1935 is one of the cornerstone results of Ramsey theory, with countless applications in analysis, combinatorics, geometry and logic. It states that any sequence of $(n-1)^2+1$ real numbers contains a monotone increasing or decreasing subsequence of length n, and this bound is the best possible. See Steele [14] for several different proofs and applications.

Since then, many generalizations and extensions of the Erdős-Szekeres theorem are proposed [3, 4, 7, 9, 10, 12, 13, 15], among which one of the most natural is due to Fishburn and Graham [7]. Say that a matrix is *row-monotone* if every row is monotone increasing or every row is monotone decreasing, and define *column-monotone* analogously. A matrix is *monotone* if it is both row- and column-monotone. With this notation in our hand, the result of Fishburn and Graham states that for every n there exists a smallest number $N = M_2(n)$ such that every $N \times N$ real matrix contains an $n \times n$ monotone submatrix. In particular, they proved that $M_2(n) \leq \text{tw}_5(O(n))$, where the *tower function* $tw_k(x)$ is defined recursively as $\text{tw}_1(x) := x$ and $\text{tw}_k(x) := 2^{\text{tw}_{k-1}(x)}$. On the other hand, the best known lower bound, due to a simple probabilistic argument, gives $M_2(n) \geq n^{n/2}$. Recently, the upper bound was greatly improved by Bucić, Sudakov, and Tran [2] to $M_2(n) < 2^{2^{O(n)}}$. Lichev [11] slightly improved their upper bound, but the improved bound is still of the order $2^{2^{O(n)}}$. Therefore, it remained a puzzling open problem whether $M_2(n)$ grows exponentially or double-exponentially. We prove the following bound, which answers this question:

$$M_2(n) \leq 2^{O(n^4(\log n)^2)}.$$

In [2], in order to prove a double-exponential upper bound on $M_2(n)$, a key idea is to show that any $2n \times N$ matrix contains an $n \times n$ row-monotone submatrix if $N > (n-1)^{2^{2n}}$. A natural idea would be to show that this bound on N can be significantly improved. Unfortunately, this is not possible due to a construction of Burkill and Mirsky [3], see also Lichev [11]. Our key contribution is showing that if

$$N \geq 2^{cn^4(\log_2 n)^2},$$

then every $8n^2 \times N$ matrix contains a row-monotone $n \times n$ matrix. This tells us that roughly n^2 rows are enough to guarantee an $n \times n$ row-monotone submatrix if the number of columns is exponential. It comes as a surprise that there is a very sharp transition for this phenomenon around n^2. We also prove there exists an $\lfloor n^2/6 \rfloor \times 2^{2^{\lfloor n/2 \log_2 n \rfloor}}$ matrix with no $n \times n$ row-monotone submatrix.

Our main results have further implications about *lexicographic matrices* as well. Say that a matrix M is *lex-increasing* if the following is satisfied: $M(a,b) \leq M(a',b')$ holds whenever $a < a'$, or $a = a'$ and $b < b'$. Say that a matrix is *lex-monotone* if one can rotate and/or mirror it to get a lex-increasing matrix. Fishburn and Graham [7] proved that there exists a smallest $N = L_2(n)$ such that every $N \times N$ matrix contains an $n \times n$ lex-monotone submatrix. As every lex-monotone matrix is also monotone, we trivially have $M_2(n) \leq L_2(n)$. On the other hand, Fishburn and Graham showed that

$$L_2(n) \leq M_2(2n^2 - 5n + 4),$$

which combined with the result of [2] gave the previously best known upper bound $L_2(n) \leq 2^{2^{O(n^2)}}$. However, our main result immediately implies the following improvement, which also has the right order on an exponential scale:

$$L_2(n) \leq 2^{O(n^8(\log n)^2)}.$$

Fishburn and Graham [7] also established higher dimensional analogues of the Erdős-Szekeres theorem. A *d-dimensional array* is a function $A : S_1 \times \cdots \times S_d \to \mathbb{R}$, where $S_1, \ldots, S_d$ are finite subsets of integers. Say that A is *monotone* if $f(x) = A(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_d)$ is a monotone function for every fixed $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d$, and whether it is increasing or decreasing only depends on i. Fishburn and Graham proved that for every d and n, there exists a smallest number $N = M_d(n)$ such that every d-dimensional array of size $N \times \cdots \times N$ contains an $n \times \cdots \times n$ sized d-dimensional monotone subarray. Observe that the Erdős-Szekeres theorem is equivalent to the statement that $M_1(n) = (n-1)^2 + 1$.

The first proofs of Fishburn and Graham [7] of the existence of $M_d(n)$ gave Ackermann-type upper bounds of order d for $d \geq 4$. On the other hand, the best known lower bound for every $d \geq 2$ is

$$M_d(n) \geq n^{(1-1/d)n^{d-1}},$$

due to a simple probabilistic argument. Bucić, Sudakov, and Tran [2] greatly improved the upper bounds in every dimension by showing that

$$M_3(n) \leq 2^{2^{O(n^2)}}, \text{ and } M_d(n) \leq \mathrm{tw}_4(O_d(n^{d-1})) \text{ for } d \geq 4.$$

Girão, Kronenberg, and Scott [8] removed one exponential for $d \geq 4$, and established the inequality $M_d(n) \leq 2^{2^{O_d(n^{d-1})}}$, which is then the best known upper bound for every $d \geq 3$. They proved this bound by considering a more general problem about the Ramsey properties of the Cartesian products of graphs. Despite the recent progress, there is still an exponential gap between the best known lower and upper bound for every $d \geq 3$. Unfortunately, there seem to be multiple obstacles if one tries to adapt our argument already for $d = 3$.

One can also consider a d-dimensional generalization of lexicographically ordered matrices. A d-dimensional array A is *lex-monotone* if there exists a permutation $\sigma \in S_d$ and a sign vector $s \in \{-1, 1\}^d$ such that

$$A(a_1, \ldots, a_d) < A(b_1, \ldots, b_d) \Leftrightarrow$$
$$(s(\sigma(1)) \cdot a_{\sigma(1)}, \ldots, s(\sigma(d)) \cdot a_{\sigma(d)}) <_{LEX} (s(\sigma(1)) \cdot b_{\sigma(1)}, \ldots, s(\sigma(d)) \cdot b_{\sigma(d)}).$$

Here $<_{LEX}$ denotes the lexicographic ordering, that is, $(x_1, ..., x_d) <_{LEX} (y_1, ..., y_d)$ if $x_b < y_b$, where b is the smallest index such that $x_b \neq y_b$. Fishburn and Graham [7] proved that there exists a smallest $N = L_d(n)$ such that every d-dimensional $N \times \cdots \times N$ array contains a d-dimensional $n \times \cdots \times n$ lex-monotone subarray.

This result has found applications in poset dimension theory [6] and computational complexity theory [1].

Bucić, Sudakov, and Tran [2] proved the following relationship between the functions M_d and L_d for $d \geq 3$:

$$L_d(n) \leq M_d(2^{O_d(n^{d-2})}).$$

Combining this with the result of Girão, Kronenberg, and Scott [8], we get the best known upper bound $L_d(n) \leq \mathrm{tw}_4(O_d(n^{d-2}))$ for $d \geq 3$.

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A Topological Turán Theorem

Corrine Yap

(joint work with Jason Long, Bhargav Narayanan)

Extremal combinatorics revolves around finding the discrete structures that maximize or minimize a certain quantity. For example, what is the maximum number of edges that an n-vertex graph with no triangles can have? By Mantel's Theorem [4], the answer is $\frac{n^2}{4}$. We can replace "triangle" with a more complicated graph H, and this is often called the *Turán problem.* The answer is well-understood in all cases except when H is a bipartite graph. We may generalize this problem to hypergraphs, where a *hypergraph* $\mathcal{H}$ on vertex set V is a collection of subsets of V, called (hyper)edges. We say $\mathcal{H}$ is *k-uniform* if every edge has size k. Observe that a 2-uniform hypergraph is simply a graph.

By taking the closure of a hypergraph under subset inclusion, one can obtain an abstract simplicial complex and thus introduce a topological version of Turán's problem. Nati Linial [1, 2] first posed the following question in 2008 regarding facets (maximal simplices) in a simplicial complex rather than edges in a graph: given a k-dimensional simplicial complex ("k-complex") $\mathcal{S}$, what is the maximum number of facets in an n-vertex k-complex that contains no homeomorphic copy of $\mathcal{S}$?

The maximum number of facets in a k-dimensional simplicial complex is $\binom{n}{k+1}$, which is asymptotically on the order of n^{k+1}, so we are interested in finding some $\lambda(\mathcal{S})$ such that $n^{k+1-\lambda(\mathcal{S})}$ facets guarantees a homeomorph of $\mathcal{S}$. It is essentially folklore that for every $\mathcal{S}$, there does exist some such $\lambda(\mathcal{S}) > 0$. One could instead ask for a *universal exponent* λ_k that depends only on the dimension k and not the choice of $\mathcal{S}$. In particular, does there exist $\lambda_k > 0$ such that for any k-complex $\mathcal{S}$, every n-vertex k-complex with $n^{k+1-\lambda_k}$ facets contains a homeomorphic copy of $\mathcal{S}$?

Previous work had been done for $k = 1, 2$. Indeed, when $k = 1$, our k-complexes are simply graphs and a theorem of Mader [5] shows that linearly many edges is enough to guarantee the existence of a subdivision of every graph, thus implying that $\lambda_1 = 1$. For $k = 2$, Brown, Erdős, and Sós proved that $\lambda(S^2) = \frac{1}{2}$ where S^2 is the 2-sphere. Much more recently, Kupavskii, Polyanskii, Tomon, and Zakharov [8] proved that $\frac{1}{2}$ is the exponent not only for the sphere but also for all orientable closed surfaces, and Sankar [9] extended these results to all non-orientable surfaces. Regarding the universal exponent, however, a complete answer is yet unknown. Keevash, Long, Narayanan, and Scott [6] provided the first known bounds of $\lambda_2 \geq \frac{1}{5}$.

We show for the first time the existence of the universal exponent.

Theorem (Long–Narayanan–Y. [3]). *For all $k \in \mathbb{N}$, there exists $\lambda_k \geq k^{-2k^2}$ such that for any k-dimensional simplicial complex $\mathcal{S}$, every n-vertex k-dimensional simplicial complex with at least $n^{k+1-\lambda_k}$ facets must contain a homeomorphic copy of $\mathcal{S}$.*

We prove this by first reducing from a topological statement to a strictly combinatorial statement by associating the desired k-complex $\mathcal{S}$ with a specific homeomorph called the *barycentric subdivision* of $\mathcal{S}$. The $(k+1)$-uniform hypergraph $\mathcal{H}$ corresponding to this homeomorph is not only $(k+1)$-partite but possesses a property we define as *d-trace-boundedness*, which roughly refers to the degrees of the vertices being bounded within certain traces of $\mathcal{H}$. This reduces our problem from a topological one to a strictly combinatorial one: our goal becomes to bound the so-called *Turán exponent* for this class of hypergraphs. To do this, we utilize a nonstandard version of a probababilistic technique called *dependent random choice*. Given a dense $(k+1)$-unifornm hypergraph $\mathcal{G}$, we prove the existence of a large subset of vertices whose small subsets have a large common neighborhood. This property allows us to embed a copy of $\mathcal{H}$ into $\mathcal{G}$ greedily.

Several open questions in this area remain, primarily: what is the exact value of λ_2 and can we obtain better bounds on λ_k? A possible approach to the latter would be considering the sphere S^k in higher dimensions. The techniques of Brown-Erdős-Sós can be generalized to provide bounds but not exact values, and for example, $\lambda(S^3)$ remains unknown.

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Variation of no-three-in-line problem

Ji Zeng

(joint work with Andrew Suk, Yaobin Chen, Xizhi Liu, Jiaxi Nie)

The famous *no-three-in-line problem* raised by Dudeney [6] in 1917: Is it true that one can select $2n$ points in $[n]^2$ such that no three are collinear? Clearly, $2n$ is an upper bound as any vertical line must contain at most 2 points. For small values of n, many authors have published solutions to this problem obtaining the bound of $2n$ (e.g. see [7]), but for large n, the best known general construction is due to Hall et al. [8] with slightly fewer than $3n/2$ points.

As a generalization of this problem, we consider the quantity $\alpha_{d,k}(n)$, defined as the maximum size of a subset $S \subset [n]^d$ such that no $k+2$ points of S is contained in a k-dimensional affine subspace. The original no-three-in-line problem is equivalent to asking whether $\alpha_{2,1}(n) = 2n$. Since $[n]^d$ can be covered by n^{d-k} many copies of $[n]^k$, we have the trivial upper bound $\alpha_{d,k}(n) \leq (k+1)n^{d-k}$. For certain fixed values of d and k, and n tends to infinity, this bound is known to be asymptotically tight: Many authors [13, 3, 10] noticed that $\alpha_{d,d-1}(n) = \Theta(n)$ by looking at the modular moment curve over a finite field $\mathbb{F}_p$; In [11], Pór and Wood proved that $\alpha_{3,1}(n) = \Theta(n^2)$. On the other hand, Lefmann [10] (see also [9]) showed that $\alpha_{d,k}(n) \le O\left(n^{\frac{d}{\lfloor (k+2)/2 \rfloor}}\right)$, which behaves differently from $\Theta(n^{d-k})$ for infinitely many values of d and k. The first result of this talk is an improvement of Lefmann's upper bound.

Theorem 1 (Suk–Zeng [15]). *For fixed d and k, as $n \to \infty$, we have*

$$\alpha_{d,k}(n) \le O\left(n^{\frac{d}{2\lfloor (k+2)/4 \rfloor}\left(1-\frac{1}{2\lfloor (k+2)/4 \rfloor d+1}\right)}\right).$$

In particular, this theorem tells us that, when 4 divides $k+2$, $\alpha_{d,k}(n)$ only behaves like $\Theta(n^{d-k})$ if $k = d-1$. For example, we have $\alpha_{4,2}(n) \le O(n^{\frac{16}{9}})$, and this is quite interesting compared to the result on $\alpha_{3,1}(n)$ by Pór and Wood [11]. The proof of this theorem is a modification of the arguments of Cilleruelo and Timmons [5] regarding "k-fold Sidon sets" in additive combinatorics. Essentially, we observed that any $S \subset [n]^d$ without the hypothesized affine dependency must be a higher dimensional analog of "k-fold Sidon sets".

Let us note that Lefmann [10] showed that $\alpha_{d,k}(n) \ge \Omega\left(n^{\frac{d}{k+1}-k-\frac{k}{k+1}}\right)$, which is the current best lower bound for this quantity for general values of d and k.

Problem 1. Determine the asymptotic magnitude of $\alpha_{4,2}(n)$, and more generally, of $\alpha_{d,k}(n)$ for all fixed d and k.

As another variation of the no-three-in-line problem, we consider the quantity $\alpha(\mathbb{F}_q^2, p)$, defined as the maximum size of a collinear-triple-free subset S in a p-random set of $\mathbb{F}_q^2$, that is, the plane over the finite field of order q. Here, a p-random set is a random subset where each point is sampled uniformly independently with probability p. The second result of this talk characterizes $\alpha(\mathbb{F}_q^2, p)$ up to polylogarithmic factors for all possible values of p.

Theorem 2 (Chen–Liu–Nie–Zeng [4]). *As the prime power $q \to \infty$, asymptotically almost surely, we have*

$$\alpha(\mathbb{F}_q^2, p) = \begin{cases} \Theta(pq^2), & q^{-2+o(1)} \le p \le q^{-3/2-o(1)}, \\ q^{1/2+o(1)}, & q^{-3/2-o(1)} \le p \le q^{-1/2+o(1)}, \\ \Theta(pq), & q^{-1/2+o(1)} \le p \le 1. \end{cases}$$

Moreover, all $q^{o(1)}$ factors here are polylogarithmic.

This theorem is partially established by Roche-Newton–Warren [12] and Bhowmick–Roche-Newton [2] previously for p within some smaller ranges. Following their approach, our proof is based on the container method. Specifically, we proved that there always exists a collection $\mathcal{C}$ of subsets (called containers) of $\mathbb{F}_q^2$ such that: (i) every collinear-triple-free subset of $\mathbb{F}_q^2$ is contained in some $C \in \mathcal{C}$; (ii) $|C| \le 9q$ for all $C \in \mathcal{C}$; and (iii) $|\mathcal{C}| \le \exp\left(O(q^{1/2}(\log q)^2)\right)$. We remark that container theorems of this type can also be applied to counting problems of collinear-triple-free subsets and their variants, see [12, 2, 4] for details.

One of the key tools in our proof is the well-known hypergraph container lemma, proved independently by Balogh–Morris–Samotij [1] and Saxton–Thomason [14].

To obtain the desired quantitative bounds of our containers, we apply the hypergraph container lemma together with a "balanced supersaturation" result of collinear triples in $\mathbb{F}_q^2$. And we leverage the pseudorandomness of the point-line incidence bipartite graph of $\mathbb{F}_q^2$ to achieve such a "balanced supersaturation". This is the technical reason that we considered this problem over finite fields, hence a future direction is to study the same problem for the n-by-n grid.

Problem 2. Characterize the quantity $\alpha([n]^2, p)$, defined as the maximum size of a collinear-triple-free subset S in a p-random set of the grid $[n]^2$.

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Lower bounds for incidences

Dmitrii Zakharov
(joint work with Alex Cohen, Cosmin Pohoata)

Consider the following problem: let $p_1, \dots, p_n \in [0,1]^2$ be a set of points and for every $i = 1, \dots, n$, let ℓ_i be a line passing through p_i. What is the smallest possible distance $\delta = d(p_i, \ell_j)$ between a point p_i and a line ℓ_j for $i \neq j$? Our 'goal' is to make δ as large as possible. Here are some examples:

- Let $p_1, \ldots, p_n$ be equally spaced points on the segment $[0,1] \times \{0\}$ and $\ell_i = p_i + \langle e_2 \rangle$ be the vertical line through p_i. Then $\delta \sim n^{-1}$, achieved on adjacent indices $j = i+1$.
- Let $p_1, \ldots, p_n$ be equally spaced on the unit circle and ℓ_i is the tangent line to p_i. Then $\delta \sim n^{-2}$.
- Let $p_1, \ldots, p_n$ form a $\sqrt{n} \times \sqrt{n}$ square grid. Then it is possible to choose lines ℓ_i so that $\delta \sim n^{-1}$ and Minkowski's theorem implies that this is best possible.
- Let $p_1, \ldots, p_n$ be a random set of points in $[0,1]^2$. Then with high probability we can choose ℓ_i so that $\delta \sim \frac{\log n}{n}$.

These examples suggest that perhaps $\delta \leq n^{\varepsilon - 1}$ for any configuration of points and lines p_i, ℓ_i. If true, this would be best possible and would have some nice consequences.

The Heilbronn's triangle problem asks to find the smallest area triangle among an arbitrary set of n points in the unit square $[0,1]^2$. Using a pigeonhole argument, it is easy to find triangles of area $O(n^{-1})$ but beating this bound turned out to be quite challenging. We can find a smaller triangle using a bound on δ as follows. First, any set of n points contains $m \sim n$ pairwise disjoint pairs p_i, q_i, $i = 1, \ldots, m$ such that $d(p_i, q_i) \leq Cn^{-1/2}$. Second, let ℓ_i be the line passing through the points p_i, q_i. So we get a collection of $\sim n$ points p_i and each point has a line through it. Then we can find $i \neq j$ such that $d(p_i, \ell_j) \leq \delta$. But then we have

$$Area(p_i, p_j, q_j) = \frac{1}{2} d(p_j, q_j) d(q_i, \ell_j) \leq Cn^{-1/2}\delta.$$

This gives a better bound than n^{-1} as long as $\delta \ll n^{-1/2}$ and if we assume the best possible bound $\delta \leq n^{\varepsilon-1}$, then this gives triangles of area $n^{\varepsilon - 3/2}$. The best current upper bound on the Heilbronn's triangle problem is $Area(x,y,z) \leq n^{-\frac{8}{7} - \frac{1}{2000}}$ due to Cohen, Pohoata and the author [1], very far away from this optimistic bound. On the other hand, there are point sets with all triangles having area at least $n^{-2+o(1)}$.

Consider a variant of the problem: given n points in $[0,1]^2$, what is the smallest area of a 4-gon determined by these points? Here again a simple pigeonhole argument gives a 4-gon of area Cn^{-1} but the best lower bound construction only avoids 4-gons of area $n^{-3/2+o(1)}$. Presently we cannot even show that there always exists a 4-gon of area $o(n^{-1})$. However, if we assume that $\delta \leq n^{\varepsilon-1}$ holds in the point-line problem, then we can find 4-gons of area at most $n^{-9/8+C\varepsilon}$, a polynomial improvement provided that ε is small enough. The argument is similar in spirit to the $n^{-3/2+\varepsilon}$ bound in case of triangles. One can go further and show an upper bound $n^{-1-c_k+C_k\varepsilon}$ for the area of the smallest k-gon determined by n points.

Another neat application of an upper bound on δ is the following 'point-line incidence dichotomy'. For a set of points $P \subset [0,1]^2$ and a set of lines L let us define $I(\delta; P, L)$ to be the number of pairs $p \in P$, $\ell \in L$ such that $d(p, \ell) \leq \delta$. This is a 'blurred' version of the number of incidences between points and lines in the plane and understanding this quantity well has many applications to additive combinatorics, discrete geometry and analysis. The most studied questions are different 'blurred' variants of Szemeredi–Trotter theorem: given some distributional constraints on P and L, how large can the number of incidences $I(\delta, P, L)$ be? A remarkable example of this type of question is the Furstenberg set problem very recently resolved by Ren and Wang [2] completing a long line of research.

We ask the opposite question: how *small* can the number of incidences $I(\delta, P, L)$ be? Of course, it can be 0, just draw all lines far away from points in P. On the other hand, if P and L were 'random', then we would have $I(\delta, P, L) \approx \delta|P||L|$. If we know the bound $\delta \leq n^{\varepsilon-1}$ for our point-line problem then, in some sense, these are the only two possibilities.

More precisely, suppose that $\delta \leq n^{\varepsilon-1}$ holds for the point-line problem. Let $\delta > 0$ and P, L be arbitrary sets of points and lines in $[0,1]^2$. Then one of the two possibilities holds:

- We have $I(\delta; P, L) \geq \delta^{1+\varepsilon}|P||L|$,
- There exist subsets $P' \subset P$ and $L' \subset L$ such that $|P'| \geq (1-\delta^{\varepsilon})|P|$ and $|L'| \geq (1-\delta^{\varepsilon})|L|$ and $I(\delta, P', L') = 0$.

But what can we actually prove? A trivial upper bound is $\delta \leq Cn^{-1/2}$ follows from a packing argument. We can improve this by a polynomial factor to something like $\delta \leq n^{-4/7+\varepsilon}$ (and perhaps even better) using harmonic analysis and multi-scale arguments. Our method has a hard barrier at the exponent $\delta = n^{-2/3}$ though. The reason is the following 'Szemeredi–Trotter' example

$$P = \sqrt{n} \times \sqrt{n} \text{ grid inside, } [0,1]^2, \ L = \text{ the set of } n^{1/3}\text{-rich lines for } P,$$

it then turns out that: $|P|, |L| \sim n$ and

$$I(\delta; P, L) \sim \begin{cases} \delta|P||L|, \ \delta \geq n^{-1/3}, \\ n^{4/3}, \ \delta \leq n^{-1/3}, \end{cases}$$

i.e. there is a change of behavior of the incidence count below the scale $n^{-2/3}$. Because of this, our method cannot control incidences below this scale. In some sense, proving the bound $\delta \leq n^{\varepsilon-1}$ (or even anything better than $n^{-2/3}$) would have to rule out 'Szemeredi–Trotter examples' where the number of incidences is much smaller than expected.

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Open Problems in Discrete Geometry

Collected by Nikita Gladkov

Problem 1 (Sergey Avvakumov)**.** A *grid graph* is a graph drawn in the plane such that each of its vertices has integer coordinates and each of its edges has unit length. A drawing of a graph is *1–thick* if the distance between x and y is at least 1 whenever x and y are disjoint vertices, disjoint edges, or disjoint vertex and edge.

Is there a constant C such that for any n any grid graph with n vertices admits a 1–thick drawing in a square of side length $C\sqrt{n}$?

This question is based on the "Sponge problem", see [1].

[1] Larry Guth, The Width-Volume Inequality *Geometric and Functional Analysis* **17** (2007), pp. 1139–1179.

Problem 2 (Luis Montejano)**.** For what topological groups G, does the collection of cosets of closed subgroups satisfy a Helly theorem?

It is known that the answer is yes for S^1 and R^1.

The answer is probably yes for S^3 and $SO(3)$.

Problem 3 (Günter Rote)**.** What is the probability p_3 that a random face in a random simple pseudoline arrangement is a triangle, in the limit as the number n of pseudolines goes to infinity?

Here, a *random pseudoline arrangement* is a uniform choice among the combinatorial types of simple pseudoline arrangements, and a *random face* is a uniform choice among the $\binom{n-1}{2}$ bounded faces of the arrangement. (The answer would be unchanged if we including the $2n$ unbounded faces.)

The same question can be asked for the probability p_4 of a quadrilateral, etc. Computational experiments suggest the values $p_3 \approx 0.3$, $p_4 \approx 0.45$, $p_5 \approx 0.2$, $p_6 \approx 0.04$. It is not even proved that these probabilities converge as $n \to \infty$. However, it is known that the *average* face size converges to 4 [1]. The answer is the same if we ask for the size of the convex hull in a random *abstract order type.*

The question can also be asked for *line* arrangements with a random combinatorial type. It is not clear that the answer is the same. One may also generate n random lines in some way and consider the resulting line arrangement.

[1] Xavier Goaoc and Emo Welzl, Convex hulls of random order types. *Journal of the ACM* **70** (2023), Article No. 8: 47 pp.

Problem 4 (Karim Adiprasito)**.** A Moore graph is the graph which attains the smallest girth with the given diameter.

$$\text{girth} = 2\,\text{diam} - 1$$

Except for K_n and C_{2n+1} there are just sporadic regular Moore graphs. What about regular Moore graphs satisfying

$$\text{girth} = (2 - c)\,\text{diam}?$$

Is their number also finite?

What if I have a Riemanian manifold with

$$\frac{\text{systole}}{\text{diam}} \leq 2?$$

Gromov showed that this bound is attained only at $\mathbb{RP}^n$. Is the number of topologies for $2-\varepsilon$ finite? As shown by Buser Somak, this ratio is infinitely often greater than some $c > 0$.

Problem 5 (Patrick Schnider)**.** Let P be a finite set of n points in $\mathbb{R}^d$ (maybe in general position or even in strongly general position). Let $q \in \mathbb{R}^d$ be another point. We define the *Tverberg depth* of q with respect to P, denoted by $\mathrm{TvD}(P, q)$, as the maximum k such that there is a partition of P into k pairwise disjoint subsets $P_1, \ldots, P_k \subset P$ with $\bigcap_{i=1}^k \mathrm{conv}(P_i) \neq \emptyset$. In this language, Tverberg's theorem says that there is a point q with $\mathrm{TvD}(P, q) \geq \frac{n}{d+1}$. Consider the region D_k of points in $\mathbb{R}^d$ that have Tverberg depth at least k, that is, $D_k = \{q \in \mathbb{R}^d \mid \mathrm{TvD}(P, q) \geq k\}$. The question is the following:

Question 1. *For what values of k is D_k contractible for every point set P of size n?*

Note that D_1 is just the convex hull of P, which is of course contractible. On the other hand, the set of Tverberg points is in general not even connected, so D_k is generally not connected for $k \geq \frac{n}{d+1}$ (and is of course empty for large k). In $\mathbb{R}^2$ it is not hard to show that D_k is convex and thus contractible for $k \leq \lfloor \frac{n}{3} \rfloor$.

Problem 6 (Raphael Steiner)**.** This problem was asked by Pavel Valtr at another workshop, and I (Raphael Steiner) am only restating it.

Does there exist an absolute constant $c \in \mathbb{N}$ such that for every $d \in \mathbb{N}$ the hypercube graph Q_d admits a topological drawing in the plane such that there exists no collection of c pairwise crossing edges?

This relates to another open question, namely whether the maximum number of edges in an n-vertex k-quasi-planar graph is bounded linearly in n.

Problem 7 (Dima Zakharov)**.** Let $n \geq k \geq 2$ and $A \subset S^n$ be a measurable subset which does not contain k pairwise orthogonal vectors. Show that $\mu(A) \lesssim e^{-cn/k}$ for some $c > 0$. A double cap construction shows that this would be best possible. We know this only in the special cases $k = 2$ or $k \gtrsim n$, and in general $\mu(A) \lesssim e^{-\sqrt{n/k}}$. See [1] for details and further references.

[1] Dmitrii Zakharov, Spherical sets avoiding orthogonal bases. arXiv:2310.06821, 2023, 7 pp.

Problem 8 (Xavier Goaoc)**.** Let S be an arbitrary set of n great circles in $\mathbb{S}^2$ such that no three have a point in common. Let c be a 2-dimensional cell chosen equiprobably from the arrangement of S. Let $S' \subset S$ be the subset of great circles *not* touching c, and let c' be the 2-dimensional cell of the arrangement of S' that contains c. Let x denote the number of edges of c'. Note that x is a random variable whose distribution depends on the arrangement of S.

How large can the expectation of x be?

Motivation. The maximum (over the choice of S) expectation of x bounds from above the average number of points on the second onion layer of a realizable order type chosen uniformly at random.

Problem 9 (Ji Zeng)**.** Let B_t be the bipartite graph with bipartition $L \sqcup R$ such that $L = \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_t\} \cup \{z_1, \ldots, z_t\}$ and $R = \{w_{ijk}; i, j, k \in \{1, \ldots, t\}\}$, and the edges of B_t are $x_i w_{ijk}$, $y_j w_{ijk}$, and $z_k w_{ijk}$ for all indices i, j, k.

Are there constants $\sigma < 3$ and positive C such that any large enough bipartite graph $G = (A \sqcup B, E(G))$ with $|A|^\sigma < |B|$ and $|E(G)| > C|B|$ must contain a copy of B_t?

Problem 10 (Igor Pak)**.** Let X be a finite set of points in the plane in general position. Denote by $c(X)$ the number of triangulations of the convex hull of X with vertices in X. Prove that for every sufficiently large k there exists a set X with $c(X) = k$.

Problem 11 (Imre Bárány)**.** Let X be a set of n points in the plane in general position. Three points $a, b, c \in X$ determine an *empty triangle* if $\text{conv}\,\{a, b, c\} \cap V = \{a, b, c\}$. Let $f(X)$ denote the number of empty triangles in X.

The first question is about $f(n)$, the minimum of $f(X)$ taken over all n-element sets X in general position in the plane.

Question 2. *Show that* $f(n) > 1.001n^2$.

It is known that

$$n^2 + \Omega(\log n)^{2/3}) < f(n) < 1.62n^2.$$

The lower bound comes from [1], the upper one from [3]

For the second question define the *degree*, $\deg(a, b)$, of $a, b \in X$ as the number of $c \in X$ such that a, b, c form an empty triangle in X and let $\deg X$ be the maximum of $\deg(a, b)$, when $a, b \in X$.

Question 3. *Show that* $\deg X$ *tends to infinity when* $|X| = n \to \infty$.

This question appeared first in [4] and later in [2]. A construction with $\deg X = O(\sqrt{n})$ is given in [3].

[1] O. Aichholzer, M. Balko, T. Hackl, J. Kynčl, I. Parada, M. Scheucher, P. Valtr, B. Vogtenhuber, A superlinear lower bound on the number of 5-holes. J. Combin. Theory Ser. A 173 (2020), 105236, 31 pp.

[2] I. Bárány and Gy. Károlyi, Problems and results around the Erdős-Szekeres convex polygon theorem. Discrete and computational geometry (Tokyo, 2000), 91–105, Lecture Notes in Comput. Sci., 2098, Springer, Berlin, 2001.

[3] I. Bárány and P. Valtr, Planar point sets with a small number of empty convex polygons. Studia Sci. Math. Hungar. 41 (2004), no. 2, 243–266.

[4] P. Erdős, On some unsolved problems in elementary geometry (in Hungarian), Mat. Lapok 2 (1992), 1–10.

Problem 12 (Boris Bukh). A *k-hole* in a set $X \subset \mathbb{R}^2$ is a set of vertices of an empty convex k-gon. Erdős asked if, for every fixed k, every sufficiently large X contains a k-hole. Horton constructed a counterexample, a family of arbitrarily large sets not containing a 7-hole.

Horton's construction is inductive: Let H_0 be a one-point set. Given H_i, the set H_{i+1} is obtained by taking the union of H_i and $H_i + p$ where the point p has very large y-coordinate and small non-zero x-coordinate.

We would like to know if Horton's construction is essentially unique. Specifically, must every sufficiently large 100-hole-free set in general position contain a copy of H_3, i.e., 8 points whose order type is the same as that of H_3?

Problem 13 (Alexander Barvinok). What is the maximum number of edges that a centrally symmetric 4-dimensional polytope with n vertices can have?

Problem 14 (Eran Nevo). Let $c(n)$ be the maximal number of faces that a cubical 4-dimensional polytope on n vertices can have. Is $c(n) = o(n^2)$?

Remarks:

1. That $c(n) = O(n^2)$ is easy (also for higher dimensional cubical polytopes.
2. Joswig-Ziegler [1] showed that $c(n) = \Omega(n \lg(n))$.
3. It is not hard to see that any upper bound on the following quantity $t(n)$ is also valid, up to a multiplicative constant, for $c(n)$ above.

[1] M. Joswig and G. M. Ziegler. Neighborly Cubical Polytopes. *Discrete & Computational Geometry* **24** (2000), pp. 325–344.

Problem 15 (Eran Nevo). Let $t(n)$ be the maximal number of *geometric* triangles in $\mathbb{R}^3$ whose vertices form a subset of size n, such that for every two of the triangles their intersection is either empty or consists of a single point which is a vertex in each. Is $t(n) = o(n^2)$?

Remarks:

1. For combinatorial / piecewise-linear triangles, $\binom{n}{2}$ triangles are possible, as demonstrated by Steiner triple systems (PL-embedded in $\mathbb{R}^3$).
2. Károlyi-Solymosi [1] showed that $t(n) = \Omega(n^{3/2})$.

[1] Károlyi and Solymosi. Almost Disjoint Triangles in 3-Space. *Discrete & Computational Geometry* **28** (2002), pp. 577–583.

Problem 16 (Dima Zakharov). Let $X \subset \mathbb{R}^n$ be a finite set such that no 3 points of X determine an angle more than $\frac{\pi}{2}(1 - 10^{-10000})$. Is it true that $|X| \leq 1.99^n$? If you forbid angles less than $\pi/2$ then it is known that $|X| \leq 2^n$ and this is essentially sharp. But it could be the case that decreasing the angles a little bit already forces a significant drop in size.

We know that such X has to have size at most $(2 - 10^{-20000})^n$ and there is a construction of size $(\sqrt{2} - 0.01)^n$. See [1] for precise formulation and more details.

[1] Andrey Kupavskii and Dmitriy Zakharov. The right acute angles problem? *European Journal of Combinatorics* **89** (2020), 7 pp.

Problem 17 (János Pach). Let $f(n)$ be the largest number such that no matter how we choose n red and n blue points in the plane, in general position, we can always find at least $m(n)$ monochromatic empty triangles. (A triangle determined by three of our points is *empty* if there is no other point in its interior.) Is it true that $f(n) = o(n^2)$?

In a joint paper with Géza Tóth, I showed a long time ago that $f(n) \geq cn^{4/3}$ for a positive constant c. See Discret. Appl. Math. 161(9): 1259-1261 (2013). On the other hand, Bárány and Valtr constructed several (uncolored) n-element point sets with only $O(n^2)$ empty triangles.

[1] Pach, J., & Tóth, G. (2013). Monochromatic empty triangles in two-colored point sets. Discrete Applied Mathematics, 161(9), 1259-1261.

Problem 18 (Karim Adiprasito)**.** For what maximal ε_d is there an infinite number of d-polytopes with all dihedral angles $\leq \pi - \varepsilon_d$?

Problem 19 (Bartosz Walczak)**.** Does there exist a constant $c > 0$ such that every intersection graph of n axis-parallel boxes in $\mathbb{R}^3$ with clique number 2 (i.e., no three meeting at a common point) has an independent set of size at least cn?

For rectangles in $\mathbb{R}^2$, a stronger statement holds—every triangle-free intersection graph of axis-parallel rectangles has chromatic number at most 6 [1]. Such a stronger statement fails in $\mathbb{R}^3$—Burling constructed triangle-free intersection graphs of axis-parallel boxes with arbitrarily large chromatic number [2]. Furthermore, the answer to the weighted analogue of the problem above is negative, because the graphs constructed by Burling are known to have arbitrarily large fractional chromatic number [3].

[1] E. Asplund, B. Grünbaum. On a coloring problem, *Math. Scand.* **8**, 181–188, 1960
[2] J. P. Burling. On coloring problems of families of polytopes. PhD thesis, University of Colorado, Boulder, 1965
[3] B. Walczak. Triangle-free geometric intersection graphs with no large independent sets, *Discrete Comput. Geom.* **53**, 221–225, 2015

Problem 20 (Arnau Padrol)**.** What is the maximum number of vertices that the convex hull of a subset of k-barycenters of a set of n points in $\mathbb{R}^d$ can have?

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