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Combinatorial $*$-algebras

Organized by
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Abstract. This workshop aimed to strengthen ties and foster collaborations between different communities working on combinatorial $*$-algebras, including C^*-and pure algebraists.

Mathematics Subject Classification (2020): 16S88, 19K35, 37A55, 46L35, 46L55, 46L80.

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Introduction by the Organizers

The half-size workshop *Combinatorial $*$-algebras*, organized by Guillermo Cortiñas (Buenos Aires), Søren Eilers (Copenhagen), Elizabeth Gillaspy (Missoula) and Roozbeh Hazrat (Sydney) was well attended with 26 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds, including operator algebraists, ring theorists and semigroup theorists. The participants ranged from leading experts in the field to younger researchers and also some graduate students. Twenty-five 50min talks presented a wide range of the latest results on the theory and its applications, reflecting a good mix of nationalities and age groups.

Here is a more detailed description of the talks.

Groupoids and their full groups

Becky Armstrong considered, for an ample groupoid G with compact unit space, the representation $\rho : \mathbb{C}[F(G)] \to \mathcal{A}_{\mathbb{C}}G$ of the complex group algebra of the full group to the Steinberg algebra. She gave necessary and sufficient conditions for the injectivity/surjectivity of ρ, and discussed conditions under which the image of ρ is dense in the full and the reduced C^*-algebras of G.

Xin Li presented a construction that associates a permutative category $\mathfrak{B}G$ to any ample groupoid G with locally compact Hausdorff unit space $G^{(0)}$. His main results are that the groupoid homology of G is the (stable) homology of the K-theory spectrum of $\mathbb{K}(\mathfrak{B}K)$ and that, under additional hypothesis, the homology of the full group $F(G)$ is that of the base space of the spectrum. This result has far reaching implications including the proof of Matui's AH conjecture for a large class of groupoids.

Owen Tanner's talk was concerned with the Stein group $V(\Lambda, \ell)$ associated to a subgroup Λ of the real numbers and a positive real number ℓ. He explained how $V(\Lambda, \ell)$ can be viewed as the full group of a certain groupoid, and used this together with a result of Nekrashevych to show that the commutator subgroup $DV(\Lambda, \ell)$ is finitely generated if and only if Λ is. He then explained some particular examples, such as the Higman-Thompson groups, in more detail.

Groupoid algebras and Cartan subalgebras

Kevin Brix reported on results pertaining the description of the ideal lattice of the C^*-algebra of a Deaconu-Renault groupoid. A Deaconu-Renault system on a locally compact, second countable, Hausdorff space X is an action T of d commuting local homeomorphisms on X. Such a system has an associated groupoid G_T; a main result presented in the talk is that the lattice of ideals of $C^*(G_T)$ embeds into that of the open subsets of $X \times \mathbb{T}^d$. Another is a description of the image of the latter map under certain assumptions on T.

Anna Duwenig's talk concerned Kumjian-Renault theory for Lie groupoids. The classical theory gives a correspondence between Cartan pairs –consisting of a C^*-algebra with a Cartan subalgebra– on one hand, and groupoid twists over effective groupoids on the other. The talk introduced smooth Cartan triples and explained that these are in correspondence with Lie groupoid twists.

Kathryn McCormick talked about Steinberg's local bisection hypothesis. She showed how the latter is used to reconstruct a groupoid twist from its (twisted) Steinberg algebra. As another main result of the talk she explained that in the C^*-algebra setting, the local bisection hypothesis is equivalent to the effectivity of the groupoid.

Enrique Pardo talked about the Spielberg C^*-algebra of a left cancellative category. He explained that the latter can be regarded as the C^*-algebra of a certain tight groupoid in the sense of Exel, and how this viewpoint is useful, for example to characterize simplicity. He defined groupoid actions on left cancellative categories and their Zappa-Szép products, showed that left cancellative small categories with nice length functions can be described as such products and analyzed the structure of the tight groupoids associated to such products.

Benjamin Steinberg characterized the ample groupoids G whose Steinberg algebra $\mathcal{A}_K G$ over a field K of characteristic zero is (graded) von Neumann regular. He showed, for example, that $\mathcal{A}_K G$ is regular if and only if G is a directed union of quasi-compact open subgroupoids.

Jonathan Taylor spoke about the functoriality of groupoid C^*-algebras with respect to actors. An actor is an action of an étale groupoid G on another étale groupoid H with certain compatibility conditions, which under appropriate properness assumptions gives rise to a $*$-homomorphism from $C^*(G)$ to $C^*(H)$. A main result explained in the talk determines conditions under which the induced $*$-homomorphism arising from an actor intertwines the Cartan-like structure of groupoid C^*-algebras. Another is that the construction of the inductive limits of Xin Li generalises to inductive systems of actors, and the intermediate groupoids Li constructs from morphisms of Cartan pairs arise as transformation groupoids associated to actors.

Shift spaces and their algebras

Daniel Gonçalves' talk concerned –C^* and purely algebraic– algebras associated to a one-sided subshift not necessarily of finite type, of an arbitrary alphabet A, with focus in the case of the Ott-Tomforde-Willis subshift associated to a set F of words on A. A main result presented characterized conjugacy between Ott-Tomforde-Willis subshifts in terms of the associated algebras.

Aidan Sims considered the problem of encoding the flow space $M(\sigma)$ of a two-sided shift space (X, σ) associated to a graph E in a C^*-algebra. He associated a graph $S^\ell E$ to each positive real ℓ, and presented results on the structure of $C^*(S^\ell E)$. For $\ell = m/n$, the latter algebra is homotopy equivalent to $C^*(D_n E^m)$, the C^*-algebra of the m-power graph of the n-delayed graph of E.

Classification problems and results

Guido Arnone's talk pertained Hazrat's graded classification conjecture for Leavitt path algebras. The latter asserts that the existence of an isomorphism of pointed, ordered $\mathbb{Z}[t, t^{-1}]$-modules $K_0^{\mathrm{gr}}(L(E)) \xrightarrow{\cong} K_0^{\mathrm{gr}}(L(F))$ between the graded Grothendieck groups of two Leavitt path algebras implies that of a graded isomorphism $L(E) \xrightarrow{\cong} L(F)$. The main result presented in this talk asserts that if the graphs E and F in the conjecture are finite and primitive, then there are unital graded homomorphisms $f : L(E) \leftrightarrow L(F) : g$ such that $g \circ f$ and $f \circ g$ are graded homotopic to the respective identity maps.

Adam Dor-On reported on relations between shift equivalence of graphs (and their incidence matrices) and equivalences of their associated algebras. A variant of Hazrat's classification conjecture says that a (not necessarily pointed) ordered isomorphism between graded Grothendieck groups gives rise to a graded Morita equivalence between Leavitt path algebras. An equivalent formulation says that two square matrices A and B are shift equivalent if and only if the Leavitt path algebras are Morita equivalent. The main theorem presented in Dor-On's talk pertains the C^*-version of the latter conjecture. It states that two square matrices A and B are shift equivalent if and only if the C^*-algebras of their associated graphs are stably (gauge-) equivariantly homotopically equivalent.

Huanhuan Li talked about filtered algebraic K-theory of Leavitt path algebras and its relation to the graded Grothendieck group of the algebra. A main result of the talk was that a certain quotient of filtered K-theory, related to the filtered K-theory of the graph C^*-algebra, is encoded in the graded Grothendieck group. This was used to show that shift equivalence between matrices implies (not necessarily gauge equivariant) Morita equivalence of the C^*-algebras of their associated graphs.

Lia Vaš associated to a given graph E and a graded ideal $I \lhd L(E)$ a graph P so that $L(P)$ is graded isomorphic to the quotient algebra $L(E)/I$. Then she characterized the graphs E whose Leavitt path algebra admits a finite composition series, that is, a finite ascending filtration $L(E) = \cup_{i=0}^{n} I_n$ by graded ideals so that the succesive quotients I_{j+1}/I_j are graded simple algebras. The latter class of graphs includes all those with finitely many vertices. By the first announced result, the succesive quotients in the filtration are graded isomorphic to Leavitt path algebras, and graded simplicity implies that the underlying graphs are of four possible types. Vaš proposed to use this fact to study Hazrat's graded classification conjecture for graphs with a composition series, and carried out this program for certain families of graphs.

Efren Ruiz talked about the classification of graph C^*-algebras. These algebras carry additional structure (a diagonal, a gauge action) and the classification question can be formulated up to isomorphism or Morita equivalence, and preserving all or part of this structure. One may ask what invariants are necessary/sufficient for each of these classification questions, and whether one can connect two graphs in the same equivalence class via a finite sequence of graph moves. The talk surveyed some results and conjectures on each of these questions.

Inverse semigroups and their actions and algebras

Pere Ara's talk concerned separated graphs and various algebraic objects associated to them. A separated graph consists of a graph E together with a family $C = \{C_v\}_{v \in E^0}$ such that C_v is a partition of the set of all edges emitted by v. In the talk an inverse semigroup $S(E, C)$ was introduced. A main result presented is a decomposition $S(E, C) = \mathcal{E} \times_\theta^r \mathbb{F}$ as a restricted semidirect product of the free group $\mathbb{F}$ on the set E^1 of edges of E, acting on the idempotent semilattice $\mathcal{E} \subset S(E, C)$. This decomposition induces analogous decompositions of the associated universal and tight groupoids $G(E, C)$ and $G_{\text{tight}}(E, C)$, and thus of both their Steinberg and C^*-algebras.

Diego Martínez spoke about Fell bundles $\{A_s\}_{s \in S}$ over a unital inverse semigroup S and the associated cross-sectional (full and reduced) C^*-algebras. He gave a sufficient condition for $C^*_{\mathrm{red}}(\{A_s\}_{s\in S})$ to be nuclear. Then he introduced an approximation property for Fell bundles. He showed that if $\{A_s\}_{s\in S}$ has the approximation property then the canonical map $C^*_{\max}(\{A_s\}_{s\in S}) \to C^*_{\mathrm{red}}(\{A_s\}_{s\in S})$ is an isomorphism and that if moreover if A_1 is nuclear, then so is $C^*_{\mathrm{red}}(\{A_s\}_{s\in S})$.

Leavitt path algebras, modules and automorphisms

Pham Ngoc Ánh's talk concerned a family of modules over the path algebra on the rose on $n \geq 2$ leaves, that is, the free algebra Λ on n noncommuting variables over a field K. He recalled that the Leavitt algebra L_n is a localization of Λ, and more generally that for any finite graph E, $L_K(E)$ is a localization of the usual path algebra KE. Then he associated a Λ-module V_λ to any $\lambda \in \Lambda$ with nonzero constant term. The main result on these modules stated in the talk are that a factorization $\lambda = \lambda_1\lambda_2$ induces a direct sum decomposition $V_\lambda = V_{\lambda_1} \oplus V_{\lambda_2}$, that V_λ is simple if and only if λ is irreducible, and that $V_\lambda \cong V_\mu$ if and only if $\Lambda/\lambda\Lambda \cong \Lambda/\mu\Lambda$.

Giang Tran Nam talked about automorphisms of $L_K(E)$, and the modules and algebras obtained from twisting by them. He introduced a new class of automorphisms of $L_K(E)$ inspired in the Anick automorphisms of the free algebra. Twisting simple Chen L_n-modules by these automorphisms he obtains new simple modules not isomorphic to any Chen module. He also considered the Zhang twist $L_K(E)^\phi$ by a graded automorphism ϕ and discussed some results, including that $L_K(E)$ is always isomorphic to a subalgebra of $L_K(E)^\phi$.

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Workshop: Combinatorial ∗-algebras

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Abstracts

Leavitt path algebras as rings of quotients and applications

Phạm Ngọc Ánh
(joint work with joint works with M. Siddoway; F. Mantese)

The definition of Leavitt path algebra $L_K(E)$ of a digraph E over a field K involves artificially the dual digraph E^* and is, in fact, asymmetric. Namely, the Leavitt path algebras of E and E^* are, in general, not isomorphic. One needs to consider the "dual" Leavitt path algebra of E^* defined by an immediate modification of Cuntz-Krieger conditions (CKI) and (CKII) for getting the required isomorphism. However, using Cohn's localization by inverting matrices, or more generally, by epimorphisms one obtains clearly the symmetry of the definition of Leavitt path algebras together with an explanation to the role of infinite emitters. For every regular vertex v in finite digraph E let A_v be the row matrix $(a_1, \cdots, a_l)$ of arrows emitting from v to not necessarily pairwise different vertices $v_1, \cdots, v_l$. Then the Leavitt path algebra $L_K(E)$ of a row-finite digraph E is precisely the localization of the quiver algebra KE inverting all matrices A_v, that is, $L_K(E)$ is generated by KE and entries a_i^* of uniquely determined column matrices $B_v =^\tau (a_1^*, \cdots, a_l^*)$ indexed by regular vertices v of E satisfying Cuntz-Krieger conditions (CKI) and (CKII). Therefore the canonical ring homomorphism $KE \to L_K(E) \leftarrow KE^*$ are clearly ring epimorphisms for a finite digraph E. Hence by Morita [4, Theorem 7.1] we have the following nice property of Leavitt path algebras.

Theorem 1. *For a finite digraph E one has the canonical isomorphisms*

$$L_K(E) \cong \mathrm{End}({}_{KE}L_K(E)) \cong \mathrm{End}(L_K(E)_{KE}) \cong$$
$$\cong \mathrm{End}({}_{KE^*}L_K(E)) \cong \mathrm{End}(L_K(E)_{KE^*}) \cong L_K(E).$$

Observing a close relation of flat epimorphisms to perfect Gabriel localizations together with Siddoway we obtain the following descriptions of Leavitt path algebras

Theorem 2. *The Leavitt path algebra $L_K(E)$ is the ring of right quotients of KE with respect to the Gabriel ideal topology defined by the ideal generated by all arrows and sinks. In particular, the generic Leavitt algebra of module type $(1, n)$ $(n \geq 2)$ is isomorphic to the ring of right quotients of the free unital algebra with respect to the Gabriel ideal topology defined by a two-sided ideal of codimension 1. This classifies the generic Leavitt algebra up to the automorphism group of the free algebra.*

Viewing Leavitt (path) algebras as localizations of corresponding quiver algebras allows to use them as a tool to a deeper study of the original quiver algebras. In particular, together with F. Mantese we use the generic Leavitt algebras to investigate polynomials in noncommuting variables. Namely, to every polynomial λ with nonzero constant term in noncommuting variables of degree ≥ 1 we assign a finite-dimensional module V_λ over a corresponding free algebra Λ consisting of its (iterated) cofactors. Hence V_λ depends only on λ, i.e., on the set of variables appearing in λ. It turns out that this finite-dimensional module V_λ encodes information on λ and its companion structures. We prove

Theorem 3. *Let $\gamma, \lambda \in \Lambda$ be polynomials with nonzero constant term of positive degree and V_γ and V_λ the finite-dimensional left Λ-modules defined by γ and λ, respectively. Then:*

(1) λ is an irreducible polynomial if and only if V_λ is a simple Λ-module.
(2) If $\lambda = \pi_1 \cdots \pi_m$ is a factorization of λ into a product of irreducible polynomials, then m is the length of V_λ and its composition factors are isomorphic to the simple modules V_{π_i}, $i = 1, \cdots m$. In particular m is an invariant of λ.
(3) $V_\lambda \cong V_\gamma$ if and only if $\Lambda/\Lambda\gamma \cong \Lambda/\Lambda\lambda$, that is, γ and λ are similar over Λ.

This result allows to find, in principle, all factorizations of λ. It extends the classical factorizzation theory of polynomials in one variable to several ones with nonzero constant terms and raises several questions connecting different areas of associative ring theory.

For further applications of this approach see [1], [3] and [2].

We note also that one can use natural Schreier bases of quiver algebras to verify well-known important properties of quiver algebras, like heredity, nonsingularity etc. and then via localization technique to study Leavitt path algebras. Moreover, for arbitrary digraphs one needs to use a weaker notion of Utumi's rings of quotient for Leavitt path algebras.

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Inverse semigroups of separated graphs and associated algebras

Pere Ara
(joint work with Alcides Buss, Ado Dalla Costa)

Inverse semigroups provide important tools in the study of several types of C^*-algebras. Close connections between inverse semigroups, étale groupoids and operator algebras were established in the work of Paterson [6], and this work was later extended by Exel [4] with the introduction of the tight spectrum and the tight C^*-algebra of an inverse semigroup.

In this talk, following [1], we will introduce the inverse semigroup $\mathcal{S}(E, C)$ of a separated graph (E, C) and we will describe the structure of the tight algebras associated with this inverse semigroup. To achieve this goal, we will examine the internal structure of the semigroup $\mathcal{S}(E,C)$, obtaining a detailed description of its semilattice of idempotents, which is a crucial component in our analysis. The inverse semigroup $\mathcal{S}(E,C)$ generalizes the graph inverse semigroup $\mathcal{S}(E)$ first introduced by Ash and Hall in [3], which has been widely analized by several authors.

Recall that a separated graph is a pair (E,C) consisting of a (directed) graph $E=(s,r\colon E^1\to E^0)$ and a separation C on E, meaning a partition $C=\bigsqcup_{v\in E^0} C_v$ of its edges such that each C_v is a partition of $s^{-1}(v)$ for all $v\in E^0$. The inverse semigroup $\mathcal{S}(E,C)$ is defined as follows. Let (E,C) be a separated graph and let $\hat{E}$ be the extended (or double) graph of E. The *inverse semigroup* of (E,C) is the universal inverse semigroup $\mathcal{S}(E,C)$ generated by $E^0\cup\hat{E}^1=E^0\cup E^1\cup E^{-1}$ subject to the following relations:

(1) $vw\equiv\delta_{v,w}v$ for all $v,w\in E^0$;
(2) $s(x)x\equiv x$ for all $x\in\hat{E}^1$;
(3) $xr(x)\equiv x$ for all $x\in\hat{E}^1$;
(4) $e^{-1}f\equiv\delta_{e,f}r(e)$ for all $e,f\in X$ with $X\in C$.

Our main goal is to study the internal structure of the inverse semigroup $\mathcal{S}(E,C)$ and its associated ∗-algebras, as well as C^*-algebras. As should be expected, the ∗-algebra $K[\mathcal{S}(E,C)]$ is a "tame" quotient of the Cohn algebra $\mathcal{C}_K(E,C)$ introduced in [2], namely the quotient by the ideal generated by the commutators $[e(x),e(y)]=e(x)e(y)-e(y)e(x)$ for x,y in the ∗-subsemigroup of $\mathcal{C}_K(E,C)$ generated by E^1. We denote this quotient ∗-algebra by $\mathcal{C}_K^{\mathrm{ab}}(E,C)$. Similarly, we can define a tame quotient C^*-algebra $\mathcal{T}(E,C)$ of the C^*-envelope of $\mathcal{C}_{\mathbb{C}}(E,C)$. Using these notations, a standard argument using their universal defining relations shows that we have canonical isomorphisms :

$$K[\mathcal{S}(E,C)]\cong\mathcal{C}_K^{\mathrm{ab}}(E,C),\qquad C^*(\mathcal{S}(E,C))\cong\mathcal{T}(E,C).\tag{1}$$

In a similar way, we can define tame versions of the Leavitt path ∗-algebra $\mathcal{L}_K(E,C)$ introduced in [2] and its enveloping C^*-algebra $C^*(E,C)$, and we denote these tame quotients by $\mathcal{L}_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$, respectively. We show that these algebras can be described in terms of the tight algebras of the inverse semigroup $\mathcal{S}(E,C)$, more precisely we have canonical isomorphisms factoring (1):

$$K_{\mathrm{tight}}[\mathcal{S}(E,C)]\cong\mathcal{L}_K^{\mathrm{ab}}(E,C),\qquad C^*_{\mathrm{tight}}(\mathcal{S}(E,C))\cong\mathcal{O}(E,C).\tag{2}$$

The tight algebra of an inverse semigroup S can be understood once we have a good knowledge of its canonical action on the spectrum $\hat{\mathcal{E}}$ (i.e. the space of filters) of the semilattice of idempotents $\mathcal{E}=\mathcal{E}(S)$, and on the tight spectrum $\hat{\mathcal{E}}_{\mathrm{tight}}$, which is the closure in $\hat{\mathcal{E}}$ of the space $\hat{\mathcal{E}}_\infty$ of maximal filters. Taking germs of these actions of S on $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}_{\mathrm{tight}}$, one gets the universal groupoid $\mathcal{G}(S)$ of S, and its tight quotient $\mathcal{G}_{\mathrm{tight}}(S)$, with associated C^*-algebras $C^*(\mathcal{G}(S))\cong C^*(S)$ and $C^*(\mathcal{G}_{\mathrm{tight}}(S))\cong C^*_{\mathrm{tight}}(S)$, respectively, and similarly for their reduced C^*-algebras, or their abstract ∗-algebras over the ring K.

We show that elements of $\mathcal{S}(E,C)$ have a standard form, which we call the *Scheiblich normal form*, because it is similar to the well-known Scheiblich normal form for the elements of the free inverse semigroup [5, Chapter 6]. More precisely, we show that every non-trivial element of $\mathcal{S}(E,C)$ can be represented as an expression of the form

$$(\gamma_1\gamma_1^{-1})\cdots(\gamma_n\gamma_n^{-1})\lambda \tag{3}$$

for certain words γ_i, λ in the free group $\mathbb{F} = \mathbb{F}(E^1)$ generated by the edge set E^1. Moreover, we show that this representation is unique if we add some natural conditions on the words appearing in the above representation. The Scheiblich normal form is essentially equivalent to a representation of the inverse semigroup $\mathcal{S}(E,C)$ as a (restricted) semidirect product of the form

$$\mathcal{S}(E,C) \cong \mathcal{Y} \times_\theta^r \mathbb{F}, \tag{4}$$

where $\mathcal{Y}$ is an isomorphic copy of the semilattice $\mathcal{E} = \mathcal{E}(\mathcal{S}(E,C))$, which can be described in different forms in terms of certain special words in $\mathbb{F}$ (and the vertices) that take into account the structure of the separated graph (E,C). Here θ is a certain special partial action of $\mathbb{F}$ on $\mathcal{Y}$ by partial semilattice isomorphisms between ideals of $\mathcal{Y}$. We also show that this partial action can be described internally inside $\mathcal{S}(E,C)$ via the canonical partial representation $\mathbb{F} \to \mathcal{S}(E,C)$ that extends the inclusion map of E^1 into $\mathcal{S}(E,C)$. In particular this shows that $\mathcal{S}(E,C)$ is a well-behaved inverse semigroup: it is strongly E^*-unitary, which means that it admits an idempotent pure partial homomorphism $f\colon \mathcal{S}(E,C)^\times \to \mathbb{F}$, which we view as a 'grading'. This map reads off the element $\lambda \in \mathbb{F}$ in the Scheiblich normal form (3).

The isomorphism (4) then leads immediately to a similar representation of the groupoid $\mathcal{G}(E,S) := \mathcal{G}(\mathcal{S}(E,S))$ and its tight quotient as semidirect products of the form

$$\mathcal{G}(E,S) \cong \hat{\mathcal{Y}} \times \mathbb{F}, \qquad \mathcal{G}_{\text{tight}}(E,S) \cong \hat{\mathcal{Y}}_{\text{tight}} \times \mathbb{F}$$

with respect to the (dual) partial action of $\mathbb{F}$ on $\hat{\mathcal{Y}}$ induced from θ. We also describe both $\hat{\mathcal{Y}}$ and $\hat{\mathcal{Y}}_{\text{tight}}$ in terms of certain subsets of words in $\mathbb{F}$ and the vertices. All this implies analogous decompositions of the several tame algebras associated with (E,C) as partial crossed products by partial actions of $\mathbb{F}$:

$$\mathcal{C}_K^{\text{ab}}(E,C) \cong C_K(\hat{\mathcal{Y}}) \times \mathbb{F}, \qquad \mathcal{T}(E,C) \cong C_0(\hat{\mathcal{Y}}) \times \mathbb{F}$$

and

$$\mathcal{L}_K^{\text{ab}}(E,C) \cong C_K(\hat{\mathcal{Y}}_{\text{tight}}) \times \mathbb{F}, \qquad \mathcal{O}(E,C) \cong C_0(\hat{\mathcal{Y}}_{\text{tight}}) \times \mathbb{F}.$$

Here $C_K(X)$ denotes the commutative K-algebra of compactly supported locally constant functions $X \to K$ from a totally disconnected locally compact Hausdorff space X, and $C_0(X)$ the commutative C^*-algebra of continuous functions $X \to \mathbb{C}$ vanishing at infinity.

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Representing topological full groups in Steinberg algebras and C*-algebras

Becky Armstrong
(joint work with Lisa Orloff Clark, Mahya Ghandehari, Eun Ji Kang, and Dilian Yang)

Topological full groups of ample Hausdorff groupoids were introduced by Matui [11] as a generalisation of the topological full groups associated to Cantor minimal systems studied by Giordano, Putnam, and Skau [5]. In addition to providing a useful groupoid invariant (see [12, Theorem 3.10]), topological full groups give presentations of Thompson's groups [10, 12, 13] and have been used to solve several important open problems in group theory [8, 9, 14].

Steinberg algebras are a purely algebraic analogue of groupoid C*-algebras that generalise both Leavitt path algebras and Kumjian–Pask algebras. Introduced by Steinberg [15] and also independently by Clark, Farthing, Sims, and Tomforde [4], Steinberg algebras often given valuable insight into problems involving groupoid C*-algebras (for instance, they were used in [3] to characterise simplicity of C*-algebras of Hausdorff étale groupoids).

In my talk I presented necessary and sufficient conditions under which the natural representation of the topological full group of an ample Hausdorff groupoid with compact unit space is injective and/or surjective in the groupoid's complex Steinberg algebra. I also discussed conditions under which the image of this representation is dense in the full and reduced groupoid C*-algebras. This work complements related work by Brix and Scarparo [2] studying the extension of this representation to the full C*-algebra of the topological full group as well as connections to C*-simplicity of certain topological full groups.

Let $\mathcal{G}$ be a Hausdorff étale groupoid with compact totally disconnected unit space $\mathcal{G}^{(0)}$. Then $\mathcal{G}$ is *ample*; that is, its topology has a basis of compact open bisections. These compact open bisections form an inverse semigroup $B^{\mathrm{co}}(\mathcal{G})$, with multiplication given by

$$(B_1, B_2) \mapsto B_1B_2 \coloneqq \{\gamma_1\gamma_2 : \gamma_1 \in B_1,\, \gamma_2 \in B_2,\, s(\gamma_1) = r(\gamma_2)\},$$

and inversion given by $B \mapsto B^* \coloneqq \{\gamma^{-1} : \gamma \in B\}$. We say that a compact open bisection $B \in B^{\mathrm{co}}(\mathcal{G})$ is *full* if $r(B) = s(B) = \mathcal{G}^{(0)}$. The *topological full group* of $\mathcal{G}$ is the subgroup

$$F(\mathcal{G}) \coloneqq \{B \in B^{\mathrm{co}}(\mathcal{G}) : B \text{ is full}\}$$

of $B^{\mathrm{co}}(\mathcal{G})$. By extending the multiplication and inversion operations of $B^{\mathrm{co}}(\mathcal{G})$ to the free vector space $\mathbb{C}B^{\mathrm{co}}(\mathcal{G}) = \operatorname{span}\{\delta_B : B \in B^{\mathrm{co}}(\mathcal{G})\}$, we obtain a $*$-algebra with $*$-subalgebra $\mathbb{C}F(\mathcal{G})$.

The *(complex) Steinberg algebra* of $\mathcal{G}$ is the $*$-algebra obtained by equipping the vector space

$$A(\mathcal{G}) \coloneqq \operatorname{span}\{1_B : B \in B^{\mathrm{co}}(\mathcal{G})\}$$

of characteristic functions on compact open bisections of $\mathcal{G}$ with a multiplication given by $1_{B_1}1_{B_2} = 1_{B_1B_2}$ and an involution given by $1_B^* = 1_{B^{-1}}$. It follows that

$$A(\mathcal{G}) = \{f \in C_c(\mathcal{G}) : f \text{ is locally constant}\},$$

and that $A(\mathcal{G})$ is dense in both the full and reduced groupoid C*-algebras, $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$, respectively (see [4, Proposition 4.2]). The Steinberg algebra $A(\mathcal{G})$ is a quotient of the $*$-algebra $\mathbb{C}B^{\mathrm{co}}(\mathcal{G})$ (see [15, Page 699]). Since $F(\mathcal{G}) \subseteq B^{\mathrm{co}}(\mathcal{G})$, $\mathbb{C}F(\mathcal{G})$ embeds in $\mathbb{C}B^{\mathrm{co}}(\mathcal{G})$, and so it is natural to ask what the relationship is between $\mathbb{C}F(\mathcal{G})$ and $A(\mathcal{G})$. This is the question addressed in [1] and in my talk.

There is a $*$-homomorphism $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ satisfying $\pi(\delta_B) = 1_B$ for all $B \in F(\mathcal{G})$, which we call the *representation* of $F(\mathcal{G})$ in $A(\mathcal{G})$. In my talk, I presented the following theorem relating to injectivity and surjectivity of this representation, which appears as two separate results in our paper [1].

Theorem ([1, Theorem 3.2 and Corollary 4.4]). *Let $\mathcal{G}$ be an ample Hausdorff groupoid with compact unit space $\mathcal{G}^{(0)}$. Denote the isotropy of $\mathcal{G}$ by $\operatorname{Iso}(\mathcal{G})$. The representation $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ is surjective if and only if $\mathcal{G}$ is a group, and it is injective if and only if*

(1) $\mathcal{G} = \operatorname{Iso}(\mathcal{G})$ and $\mathcal{G}$ has at most one nontrivial isotropy group; or
(2) $\mathcal{G} \neq \operatorname{Iso}(\mathcal{G})$ and $\left|\mathcal{G} \setminus \mathcal{G}^{(0)}\right| < 3$.

Even when π is not surjective, it is possible that the image of π is dense in the full or reduced C*-algebras of $\mathcal{G}$. For example, if $\mathcal{G}$ is the Cuntz groupoid (that is, the boundary-path groupoid of the directed graph with a single vertex and two loops), then $F(\mathcal{G})$ is Thompson's group V_2, and $\pi\colon \mathbb{C}F(\mathcal{G}) \to A(\mathcal{G})$ extends to a surjective representation of $F(\mathcal{G})$ in the Cuntz algebra $\mathcal{O}_2 = C^*(\mathcal{G})$; see [2, Remark 4.7] and [7, Proposition 5.3].

In [1, Section 5] we study the image of the representation π in the setting where the groupoid $\mathcal{G}$ is discrete and has finite unit space $\mathcal{G}^{(0)} = \{a_1, \dots, a_n\}$. In this setting, there is a $*$-representation $T\colon A(\mathcal{G}) \to M_n(\mathbb{C})$ given by

$$T(f)_{ij} \coloneqq \sum_{\gamma \in \mathcal{G}_{a_j}^{a_i}} f(\gamma), \qquad \text{for each } i, j \in \{1, \dots, n\},$$

where $\mathcal{G}_{a_j}^{a_i} \coloneqq \{\gamma \in \mathcal{G} : r(\gamma) = a_i,\, s(\gamma) = a_j\}$. We then have that for each $f \in \pi(\mathbb{C}F(\mathcal{G}))$, there exists $c_f \in \mathbb{C}$ such that all row and column sums of the matrix $T(f)$ are c_f (see [1, Corollary 5.2]). We use this fact to prove the following theorem, which I presented in my talk.

Theorem ([1, Theorem 5.3]). *Let $\mathcal{G}$ be a discrete groupoid with finite unit space. Then $\pi(\mathbb{C}F(\mathcal{G}))$ is dense in $C^*(\mathcal{G})$ if and only if $\mathcal{G}$ is a group. Thus the extension*

$\overline{\pi}_{\max}\colon C^*(F(\mathcal{G})) \to C^*(\mathcal{G})$ *of* π *is an isomorphism if and only if* $\mathcal{G}$ *is a group. In particular, if* $\mathcal{G}$ *is not a group, then* $1_\gamma \notin \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\max}}$ *for any* $\gamma \in \mathcal{G}$*.*

Interestingly, the above theorem does not hold in the reduced setting, as the following example shows.

Example ([1, Example 5.5])**.** Let $\mathcal{G} = \mathbb{F}_2 \sqcup \mathbb{F}_2 = \{(t,i) : t \in \mathbb{F}_2, i \in \{1,2\}\}$, where $\mathbb{F}_2 = \{g_1, g_2, g_3, \dots\}$ is the free group with two generators, with elements listed in increasing order of their lengths (so $|g_i| \le |g_{i+1}|$ for all $i \ge 1$). Then $F(\mathcal{G}) \cong \mathbb{F}_2 \times \mathbb{F}_2$, and $\overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\mathrm{r}}} = C_r^*(\mathcal{G})$. To see this, it suffices by symmetry to show that for each $t \in \mathbb{F}_2$, we have $1_{\{(t,1)\}} \in \overline{\pi(\mathbb{C}F(\mathcal{G}))}^{\|\cdot\|_{\mathrm{r}}}$. For this, we fix $t \in \mathbb{F}_2$ and define a sequence of functions $(\phi_n)_{n=1}^\infty \subseteq \pi(\mathbb{C}F(\mathcal{G})) \subseteq A(\mathcal{G})$ by

$$\phi_n := \pi\Big(\sum_{i=1}^n \delta_{\{(t,1),(g_i,2)\}}\Big) = 1_{\{(t,1)\}} + \frac{1}{n}\sum_{i=1}^n 1_{\{(g_i,2)\}}.$$

We then prove that $\phi_n \to 1_{\{(t,1)\}}$ in $C_r^*(\mathcal{G})$ by showing that $\frac{1}{n}\sum_{i=1}^n \delta_{g_i} \to 0$ in $C_r^*(\mathbb{F}_2)$. For this, we make use of Haagerup's inequality [6, Lemma 1.5], which says that for each $f \in C_c(\mathbb{F}_2)$, we have $\|f\|_r^2 \le 4\sum_{s\in\mathbb{F}_2} |f(s)|^2 \left(1 + |s|^4\right)$.

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Graded bivariant K-theory and the classification of Leavitt path algebras

Guido Arnone

We consider a variant of Hazrat's graded classification conjecture for Leavitt path algebras [6, Conjecture 1], replacing graded algebra isomorphisms by graded homotopy equivalences. The latter are algebra homomorphisms which become isomorphisms modulo the graded homotopy relation we shall presently define. An *elementary homotopy* between two $\mathbb{Z}$-graded algebra homomorphisms $f_0, f_1 \colon A \to B$ is an algebra homomorphism $h \colon A \to B[t]$ such that $ev_0 \circ h = f_0$ and $ev_1 \circ h = f_1$. Here $B[t]$ is the ring of polynomials with coefficients in B, where t is homoengeous of degree zero, and ev_0, ev_1 are the evaluation maps at $t = 0$ and $t = 1$ respectively. Elementary homotopies define a reflexive, symmetric relation on graded algebra homomorphisms $A \to B$. We define *polynomial graded homotopy* as the transitive closure of this relation and denote it by $\sim$.

Recall that a graph is primitive if there exists $N \geq 1$ such that every pair of vertices can be connected by a path of length N. Our main classification result ([2, Theorem 8.1]) reads as follows:

Theorem 1. *Let ℓ be a field and E, F two finite primitive graphs. The following two conditions are equivalent:*

(i) *There exists a $\mathbb{Z}$-equivariant isomorphism of ordered groups*

$$K_0^{gr}(L_\ell(E)) \xrightarrow{\sim} K_0^{gr}(L_\ell(F))$$

mapping $[L_\ell(E)]$ to $[L_\ell(F)]$.

(ii) *There exist unital graded algebra homomorphisms $f \colon L_\ell(E) \longleftrightarrow L_\ell(F) \colon g$ such that $g \circ f \sim \mathrm{id}_{L_\ell(E)}$ and $f \circ g \sim \mathrm{id}_{L_\ell(F)}$.*

The primitivity hypothesis is in fact a simplicity hypothesis: namely, we use that for each edge e of a primitive graph E, the idempotent ee^* of $L_\ell(E)$ is full as an element of the subalgebra $L_\ell(E)_0$ of homogeneous elements of degree zero.

Our main tool to prove the aforementioned classification result is graded bivariant algebra K-theory, introduced by Ellis in [5]. It consists of a triangulated category kk^{gr} equipped with a functor $j \colon \mathrm{Alg}_\ell^{gr} \to kk^{gr}$ from $\mathbb{Z}$-graded ℓ-algebras, which is universal with respect to excision, homotopy invariance, and (graded) matricial stability. We point out that the objects of kk^{gr} are all $\mathbb{Z}$-graded algebras and j is the identity on objects.

In previous work joint with Guillermo Cortiñas [4], we study Leavitt path algebras as objects of kk^{gr}. In particular, we show that $kk^{gr}(\ell, L_\ell(E)) \cong K_0^{gr}(L_\ell(E))$

for all graphs E, and that there is a "universal coefficient theorem" which states that there is a short exact sequence of the following form:
(1)
$$K_0^{gr}(L_\ell(E_t))\otimes_{\mathbb{L}} K_1^{gr}(L_\ell(F))\hookrightarrow kk^{gr}(L_\ell(E),L_\ell(F))\twoheadrightarrow \hom_{\mathbb{L}}(K_0^{gr}(L_\ell(E)),K_0^{gr}(L_\ell(F)))$$

Here $\mathbb{L}=\mathbb{Z}[t,t^{-1}]$ and E_t is the dual graph of E. Using these results, we studied the comparison map

$$\bar{j}\colon \hom_{\mathrm{Alg}_\ell^{gr}}(L_\ell(E),L_\ell(F))\to kk^{gr}(L_\ell(E),L_\ell(F))$$

induced by j, and the extent to which it is surjective or injective. Namely, we showed that $\bar{j}$ satisfies the following two properties.

(a) If $\xi\in kk^{gr}(L_\ell(E),L_\ell(F))$ is such that the induced map

$$\xi_*:=kk^{gr}(\ell,\xi)\colon K_0^{gr}(L_\ell(E))\to K_0^{gr}(L_\ell(F))$$

is a $\mathbb{Z}$-equivariant isomorphism of ordered groups mapping $[L_\ell(E)]$ to $[L_\ell(F)]$, then $\xi=\bar{j}(f)$ for some unital graded algebra homomorphism $f\colon L_\ell(E)\to L_\ell(F)$.

(b) If $\bar{j}(f)=\bar{j}(g)$, then there exists a unit $u\in L_\ell(F)$ which is homogeneous of degree zero and such that $f\sim ugu^{-1}$.

We use (1) to reduce our problem to one involving K_0^{gr} and K_1^{gr}. Condition (a) was then proved using a previous lifting result, which says that K_0^{gr} is a full functor when restricted to Leavitt path algebras ([3, 7]). To obtain condition (b), the main step was computing K_1^{gr} of Leavitt path algebras as a $\mathbb{Z}[t,t^{-1}]$-module, and in particular understaind its $\mathbb{Z}$-action. We do this by proving an analogue of a result of Ara and Pardo for K_0^{gr} ([1, Lemma 3.6]), which characterizes the $\mathbb{Z}$-action on $K_0^{gr}(L_\ell(E))$ in terms of a corner skew Laurent polynomial ring structure of $L_\ell(E)$.

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Ideal structure of Deaconu–Renault groupoid C*-algebras

Kevin Aguyar Brix
(joint work with Toke Meier Carlsen and Aidan Sims)

An ideal in a C*-algebra is always assumed to be closed and two-sided here. The collection of all such ideals in a C*-algebra form a lattice, and it is a fundamental (albeit very difficult) problem in C*-algebra theory to understand and describe the ideal structure completely. An ideal is primitive if it is the kernel of an irreducible *-representation, and the collection of primitive ideals may be equipped with the hull-kernel topology. It transpires that the ideal lattice of a separable C*-algebra is isomorphic to the lattice of open subsets of the primitive ideal space, so the general problem may be reinterpreted as understanding the topology on the primitive ideals.

I this talk, I discussed recent work of Carlsen, Sims, and myself [1], in which we aim to describe the ideal structure of C*-algebras constructed from dynamical systems. A Deaconu–Renault system is an action T of d commuting local homeomorphisms on a (second-countable) locally compact Hausdorff space X. The Deaconu–Renault groupoid of T is then a (second-countable) locally compact Hausdorff and étale groupoid G_T given as

$$G_T = \bigcup_{k,l\in\mathbb{N}^d} \{(x, k-l, y) \in X \times \mathbb{Z}^d \times X : T^k x = T^l y\},$$

and the groupoid C*-algebra $C^*(G_T)$ is our object of interest. (Since G_T is amenable we do not distinguish between reduced or full C*-algebras).

This class of C*-algebras is wide and includes those coming from directed graphs [3][2], Katsura's topological graphs [4], one-dimensional subshifts, crossed products by $\mathbb{Z}^d$ [6], higher-rank graphs, and some multidimensional symbolic systems. Unfortunately, the available results for ideal structure that cover directed graphs and topological graphs do not readily extend to the multidimensional case, e.g. higher-rank graphs.

Inspired by a cornerstone result for crossed products by abelian groups (see [6]), Sims and Williams [5] describe a surjective function

$$\pi\colon X \times \mathbb{T}^d \to \mathrm{Prim}(C^*(G_T))$$

and an equivalence relation $\sim$ on $X \times \mathbb{T}^d$ such that $\pi(x, z) = \pi(x', z')$ precisely when $(x, z) \sim (x', z')$. As opposed to the case of actions by abelian groups, π is not open in general and does not descend to a homeomorphism; we merely obtain a bijection $\tilde{\pi}\colon (X \times \mathbb{T})^d/\sim \to \mathrm{Prim}(C^*(G_T))$. Our first main result is that the map π is continuous. We prove this using families of (homogeneous) open bisections indexed over the isotropy of every unit $x \in X$. Such bisections always exist because G_T is étale.

The fact that π is continuous provides an injective lattice homomorphism θ from the ideal lattice in $C^*(G_T)$ into the open subsets of $X \times \mathbb{T}^d$. A description of the range of this map thus constitutes a complete description of the ideal lattice. In general, the image of θ is not well understood.

In this project, we made progress by assuming the existence of harmonious bisection families in G_T. A bisection family $\mathcal{B}$ at a unit x (again indexed over the isotropy of x as above) is harmonious if it satisfies extra compatibility conditions. These conditions allow us to make sense of $\mathcal{B}$-saturation for subsets of $X \times \mathbb{T}^d$, and we prove that the image of θ consists of those open subsets that are $\mathcal{B}$-saturated in our sense. We formulate this result in various ways: as a description of the image of θ, as a neighbourhood basis for each primitive ideal, and in terms of convergence in the primitive ideal space.

We verify that examples classes such as directed graphs, actions by $\mathbb{Z}^d$, local homeomorphisms, and — importantly — all row-finite 2-graphs without sources all do admit harmonious bisection families. We emphasise that our results for the ideal lattice provide the first description that covers all row-finite 2-graphs without sources. At the moment, we do not know if all Deaconu–Renault groupoids admit harmonious bisection families.

Therefore, it remains to determine precisely what the image of θ is for general actions of commuting local homeomorphisms and whether harmonious bisection families always exist.

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Shift equivalences through the lens of C*-bimodules

Adam Dor-On

(joint work with Boris Bilich, Efren Ruiz)

In a foundational 1973 paper [21], Williams recast conjugacy and eventual conjugacy problems for SFTs purely in terms of equivalence relations between adjacency matrices of directed graphs. These are called Strong Shift Equivalence (SSE) and Shift Equivalence (SE), respectively. It was shown by Kim and Roush [15, 16] that shift equivalence is decidable, but the problem of decidability of SSE remains a fundamental open problem in symbolic dynamics. Williams expected SSE and SE to coincide, but after around 25 years the last hope for a positive answer was extinguished by Kim and Roush [17, 18].

Definition 1 (Williams)**.** Let A and B be square matrices with entries in $\mathbb{N}$. We say A and B are

(1) *shift equivalent* with lag $m \in \mathbb{N} \setminus \{0\}$ if there are rectangular matrices R and S with entries in $\mathbb{N}$ such that

$$A^m = RS, \quad B^m = SR,$$
$$SB = AS, \quad AR = RB.$$

(2) *elementary shift related* if they are shift equivalent with lag 1.
(3) *strong shift equivalent* if they are equivalent in the transitive closure of elementary shift relation.

In tandem with early attacks on Williams' problem, Cuntz and Krieger [12, 11] created a link between symbolic dynamics and operator algebras, where several equivalence relations between SFTs are expressed through associated C*-algebras. Cuntz and Krieger showed that SSE implies that the Cuntz-Krieger C*-algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic in a way preserving their gauge actions γ^A and γ^B together with their diagonal subalgebras $\mathcal{D}_A$ and $\mathcal{D}_B$. On the other hand, by a theorem of Krieger [19], we know that the dimension group triples of SFTs are isomorphic if and only if the associated matrices are SE. Thus, as a corollary to Krieger's theorem we get that if $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic in a way preserving their gauge actions γ^A and γ^B, then A and B are SE, this is known as Krieger's corollary. For the purpose of distinguishing SFTs up to SSE, it is important to determine whether the converse of Krieger's corollary holds.

Decades later, the works on Cuntz-Krieger C*-algebras inspired a systematic study of their purely algebraic counterparts [4, 1, 3], called Leavitt path algebras, creating new interactions between the areas of pure algebra and analysis. In his work Hazrat [14] proved an algebraic analog to Krieger's corollary, and it is similarly important to determine if the algebraic analog of the converse of Krieger's corollary is true. In the purely algebraic setting, this is now known as Hazrat's Conjecture.

Conjecture 2 (Hazrat)**.** *Let A and B be essential square matrices with entries in $\mathbb{N}$. Then the following are equivalent:*

(1) The matrices A and B are shift equivalent.
(2) The Leavitt path algebras L_A and L_B are graded Morita equivalent.

Inspired by previous work of the author with Carlsen and Eilers on shift equivalences involving relations between identifications of matrix multiplication [8], we prove that a strengthening of SE, called Aligned Shift Equivalence (ASE), coincides with SSE. Hence, the decidability question for SSE becomes equivalent to the one for ASE. This also led us to consider a "quantized" module version of ASE. Each rectangular $V \times W$ matrix C with entries in $\mathbb{N} \setminus \{0\}$ can be associated with a C*-bimodule $X(C)$ in such a way that fibered products correspond to C*-bimodule balanced tensor products.

Definition 3. Let A and B be square matrices with entries in $\mathbb{N}$ indexed by V and W respectively. Suppose $m \in \mathbb{N} \setminus \{0\}$, and let R and S be matrices over

$V \times W$ and $W \times V$ (respectively) with entries in $\mathbb{N}$. Suppose further that there are isometric bimodule isomorphisms (with balanced tensor product)

$$\Phi_R : X(A) \otimes_{c_0(V)} X(R) \to X(R) \otimes_{c_0(W)} X(B),$$
$$\Phi_S : X(B) \otimes_{c_0(W)} X(S) \to X(S) \otimes_{c_0(V)} X(A),$$
$$\Psi_A : X(R) \otimes_{c_0(W)} X(S) \to X(A^m), \quad \Psi_B : X(S) \otimes_{c_0(V)} X(R) \to X(B^m).$$

We say that A and B are *module aligned shift equivalent (MASE)* of lag m via R and S together with $\Phi_R, \Phi_S, \Psi_A, \Psi_B$ if

$$(\Psi_A \otimes \mathrm{id}_{X(A)})(\mathrm{id}_{X(R)} \otimes \Phi_S)(\Phi_R \otimes \mathrm{id}_{X(S)}) = (\mathrm{id}_{X(A)} \otimes \Psi_A),$$
$$(\Psi_B \otimes \mathrm{id}_{X(B)})(\mathrm{id}_{X(S)} \otimes \Phi_R)(\Phi_S \otimes \mathrm{id}_{X(R)}) = (\mathrm{id}_{X(B)} \otimes \Psi_B).$$

We use C*-bimodule theory to detect more refined information from MASE. More precisely, using the theory of bicategories of C*-correspondences developed by Meyer and his students [2], [20], we are able to show that MASE implies that $\mathcal{O}(G_A)$ and $\mathcal{O}(G_B)$ are stably isomorphic in a way preserving their gauge actions. Through a weakening of MASE, the strength of considering bimodule isomorphisms as opposed to mere matrix identifications is revealed. Indeed, this allows us to view SE in an equivalent way as a *homotopic* version of MASE (with equality in Definition 3 replaced by homotopy), which then provides a characterization of SE in terms of *homotopy equivalence* of graph C*-algebras.

Theorem 4 (Bilich, Dor-On & Ruiz)**.** *Let A and B be square matrices with entries in $\mathbb{N}$. Then A and B are SE if and only if the graph C*-algebras $\mathcal{O}(G_A)$ and $\mathcal{O}(G_B)$ are stably equivariantly homotopically equivalent.*

A variation of homotopy has been considered in the context of Leavitt path algebras in the ungraded situation [9, 10] and fairly recently also in the graded situation when the matrices A and B are assumed to be primitive [6]. However, due to the lack of an underlying topology for the algebras, the notion of homotopy is quite different in the purely algebraic setting. Our approach via homotopy equivalence uses the underlying topology of the C*-algebras, and completely avoids K-theoretic classification techniques used in [9, 10] and [7, 6].

In a recent work of Abrams, Ruiz, and Tomforde [5], the authors proved that an algebraic version of MASE implies that the Leavitt path algebras are graded Morita equivalent. Our techniques are likely to lead to an algebraic analog of Theorem 4 which will directly recover and strengthen known results on graded homotopy classification of Leavitt path algebras.

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Smooth Cartan triples and Lie groupoid twists

Anna Duwenig
(joint work with Aidan Sims)

The duality theorem of Gel'fand and Naimark provides a two-way street between the realm of locally compact Hausdorff (LCH) spaces and that of commutative C*-algebras. One of the most studied expansion of this duality is *Kumjian–Renault theory*: it describes the way back from C*-algebras that are not themselves commutative but at least contain certain MASAs, to groupoids. In [2], we analyzed whether this reconstruction of groupoids can be refined to take geometry into account, and this extended abstract will serve as a brief overview of our results.

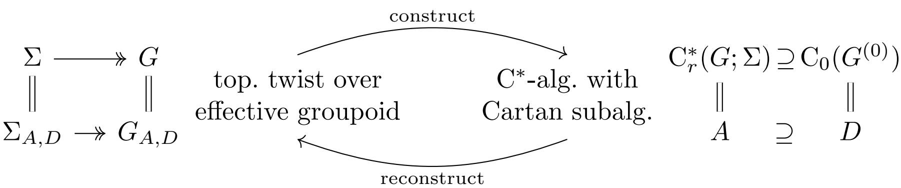
Kumjian–Renault theory. Given a LCH étale groupoid G, we can construct the reduced groupoid C*-algebra $A = \mathrm{C}^*_r(G)$ as completion of the space $\mathrm{C}_c(G)$ with the usual convolution and involution formulas, in the reduced norm built from the regular representations associated to point masses. If G is *effective*, meaning that the interior of its isotropy is reduced to being just the unit space $G^{(0)}$, then the commutative subalgebra $D = \mathrm{C}_0(G^{(0)})$ of A is *Cartan*: it is maximal abelian, its normalizers span A, and it admits a faithful conditional expectation $P\colon A \to D$.

The remarkable insight of Kumjian [3] and Renault [4] is that the functional analytic datum of a Cartan pair $D \subseteq A$ is sufficient to *reconstruct* the groupoid input—with a 'twisted caveat': there is a central groupoid extension

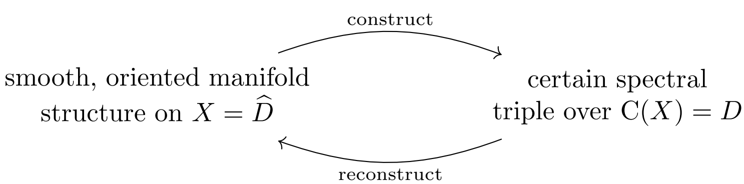
$$\mathbb{T} \times G^{(0)} \overset{\iota}{\hookrightarrow} \Sigma_{A,D} \overset{q}{\twoheadrightarrow} G_{A,D} \tag{1}$$

consisting of LCH groupoids $\Sigma := \Sigma_{A,D}$ and $G := G_{A,D}$, with G effective and étale, such that there is an isomorphism from A to the twisted groupoid C*-algebra $\mathrm{C}^*_r(G;\Sigma)$ which carries D to the canonical Cartan $\mathrm{C}_0(G^{(0)})$. The main ingredient here is the collection of *normalizers* of D in A: elements $n \in A$ which satisfy $n^*Dn, nDn^* \subseteq D$. The open G-support U_n of any such n is a bisection of G, and one can reconstruct not only the partial homeomorphism $r \circ s|_{U_n}^{-1}\colon s(U_n) \to r(U_n)$ from n but also a local trivialization $\psi_n\colon \mathbb{T} \times U_n \approx q^{-1}(U_n) \subseteq \Sigma$ of the twist. The upshot is that, from $G^{(0)}$, one is able to build G and Σ, both their algebraic and their topological structure.

In very hand-wavy terms, the reconstruction theorems of Kumjian and Renault 'add more algebra' to the Gel'fand–Naimark duality between LCH spaces and commutative C*-algebras, and can be diagrammatically summed up as follows:



Adding geometry. In [1], Connes expanded on Gel'fand–Naimark duality in a different direction: he identified properties of a commutative C*-algebra that ensure that its Gel'fand spectrum is, in fact, a smooth manifold, and he then reconstructed the C^∞-structure from these properties. Roughly speaking:



The natural question that arises is: Can such a reconstruction also be done for C^∞-structures on *groupoids* as in the set-up of Kumjian–Renault? Conversely, suppose we are given an étale *Lie* groupoid, i.e., the topological groupoid G has the structure of a smooth manifold with respect to which $G^{(0)}$ is a submanifold; the maps $r, s\colon G \to G^{(0)}$ are not just local homeo- but even diffeomorphisms; and all structure maps are smooth. What additional information about the inclusion of $\mathrm{C}_0(G^{(0)})$ in $\mathrm{C}^*_r(G)$ is required to capture this C^∞-structure of G? Courtesy of Connes' theorem, we may take the manifold structure on $G^{(0)}$–and with it the subalgebra D^∞ of smooth elements in $D = \mathrm{C}_0(G^{(0)})$–for granted.

Reconstructing Lie groupoids. Each of the local homeomorphisms r and s allows us to equip G with a manifold structure by deeming either r or s a local *diffeomorphism.* These two structures are compatible exactly if the local homeomorphisms $s(U) \approx r(U)$ of the manifold $G^{(0)}$ are diffeomorphisms, in which case the structure maps of G are easily seen to be smooth. The bottom line:

Proposition 1. *If G is an étale groupoid with $G^{(0)}$ a manifold, then G admits a smooth structure with respect to which it is an étale Lie groupoid if and only if*

(N^∞) there exists a topology base consisting of bisections U for which $r \circ s|_U^{-1}$ is a diffeomorphism onto its image.

Translating from twisted groupoids to Cartan pairs of C*-algebras, this condition becomes:

(N*) There exists a collection $\mathcal{N}$ of normalizers of D in A that densely spans A, normalizes the smooth subalgebra D^∞, and such that $n^*n, nn^* \in D^\infty$.

But what about the twist? A priori, the right definition of a 'Lie twist' should be a topological twist in which both groupoids are Lie and "all maps in sight are smooth". The most concise definition turns out to be [2, Definition 4.2.]:

Definition 2. A (topological) twist as in (1) is *Lie* if G and Σ are Lie groupoids, $q\colon \Sigma \to G$ is a submersion, and $\iota\colon \mathbb{T} \times G^{(0)} \to \Sigma$ is smooth.

In this scenario, Σ turns out to be a smooth principal $\mathbb{T}$-bundle, meaning that there exist *smooth* trivializations $\psi\colon \mathbb{T} \times U \approx q^{-1}(U)$, and ι is a diffeomorphism onto its image. The forward implication of the following theorem is thus trivial; one of our main observations in [2, Theorem 4.21] is the backwards implication.

Theorem 3. *Let Σ be a topological twist over a Lie groupoid G. Then Σ is a* smooth *principal $\mathbb{T}$-bundle iff there exists a base $\{U_n\}_{n\in\mathcal{N}}$ of open bisections of G and continuous trivializations $\psi_n\colon \mathbb{T}\times U_n \approx q^{-1}(U_n)$ of Σ such that for all n, m, k:*

(U^∞) *ψ_n is the identity on $\mathbb{T} \times G^{(0)}$; and*

(S^∞) *the partial function $G \to \mathbb{T} \times G^{(0)}$ which maps g to $\iota^{-1}\big(\psi_m(g)\psi_k(g)^{-1}\big)$, is smooth.*

The C^∞*-structure on Σ is unique such that the given family of trivializations is smooth. Moreover, Σ is a* Lie *twist iff one can choose this family $\{\psi_n\}_n$ such that*

(M^∞) *the partial function $G^{(2)} \to \mathbb{T} \times G^{(0)}$ which maps a composable pair (g, h) to $\iota^{-1}\big(\psi_n(g)\psi_m(h)\psi_k(gh)^{-1}\big)$, is smooth.*

Translating these into the realm of Cartan pairs, we arrive at the next definition, where P denotes the conditional expectation and Ph the phase of a function:

Definition 4 ([2, Definition 5.9])**.** A *smooth Cartan triple* $(A, D, \mathcal{N})$ consists of a Cartan pair $D \subseteq A$ with $\widehat{D}$ a manifold, and a family $\mathcal{N}$ of normalizers of D which satisfies $D^\infty_c \subseteq \mathcal{N} \cap D \subseteq D^\infty$, Condition (N*), and for all $n, m, k \in \mathcal{N}$:

(U*) $\mathrm{Ph}(P(n))$ multiplies $D_c^\infty(n) := D^\infty \cap C_c(\mathrm{supp}^\circ(n^*n))$;
(S*) $\mathrm{Ph}(P(mk^*))$ multiplies $D_c^\infty(mk^*)$;
(M*) $\mathrm{Ph}(P(nmk^*))$ multiplies $D_c^\infty(nmk^*)$.

Our main theorem is then summed up by this diagram, where X is a smooth manifold:

Theorem 5 ([2, Theorem 5.17.]).

Lie twist $\Sigma \to G$ *with unit space* $G^{(0)} = X$ $\xrightarrow{\text{construct}}$ *Smooth Cartan triple* $(A, D, \mathcal{N})$ *with spectrum* $\widehat{D} = X$

Smooth Cartan triple $(A, D, \mathcal{N})$ *with spectrum* $\widehat{D} = X$ $\xrightarrow{\text{reconstruct}}$ *Lie twist* $\Sigma \to G$ *with unit space* $G^{(0)} = X$

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Subshift algebras

Daniel Gonçalves
(joint work with Giuliano Boava, Gilles Gonçalves de Castro, Danilo Royer, Daniel W. van Wyk)

Let A be a non-empty set, called an *alphabet*. The *shift map* on $A^{\mathbb{N}}$ is the map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ given by $\sigma(x) = (y_n)$, where $x = (x_n)$ and $y_n = x_{n+1}$. Elements of $A^* := \bigcup_{k=0}^{\infty} A^k$ are called *blocks* or *words*, and ω stands for the empty word.

Given $F \subseteq A^*$, we define the *subshift* $X_F \subseteq A^{\mathbb{N}}$ as the set of all sequences x in $A^{\mathbb{N}}$ such that no word of x belongs to F. The language of X_F, denoted $\mathcal{L}$, consists of all blocks that appear in some element of X_F. Given a subshift X over an alphabet A and $\alpha, \beta \in \mathcal{L}$, we define

$$C(\alpha, \beta) := \{\beta x \in X : \alpha x \in X\}.$$

In particular, $Z_\beta := C(\omega, \beta)$ is called the *cylinder set* of β, and $F_\alpha = C(\alpha, \omega)$ the *follower set* of α.

Let $\mathcal{U}$ be the Boolean algebra of subsets of X generated by all $C(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{L}$, and R a commutative unital ring. The *unital subshift algebra* $\widetilde{A}_X$ is the universal unital R-algebra with generators $\{p_A : A \in \mathcal{U}\}$ and $\{s_a, s_a^* : a \in A\}$, subject to the relations:

(i) $p_X = 1$, $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ and $p_\emptyset = 0$, for every $A, B \in \mathcal{U}$;
(ii) $s_a s_a^* s_a = s_a$ and $s_a^* s_a s_a^* = s_a^*$ for all $a \in A$;
(iii) $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all $\alpha, \beta \in \mathcal{L}$, where $s_\omega := 1$ and, for $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{L}$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$ and $s_\alpha^* := s_{\alpha_n}^* \cdots s_{\alpha_1}^*$.

In the analytical setting, the definition above, replacing the ring R with the complex numbers $\mathbb{C}$ and taking the universal C*-algebra instead of the universal R-algebra, yields a C*-algebra which, in the finite alphabet setting, coincides with the one defined by Carlsen in [3].

Ott-Tomforde-Willis subshifts are introduced in [4]. Given a subshift X_F and an indecomposable ring R, we have proved, in the algebraic setting, that $\text{span}_R\{s_\alpha s_\alpha^* : \alpha \in \mathcal{L}\}$ is isomorphic to the algebra of R-valued, locally constant functions on the Ott-Tomforde-Willis subshift associated with X_F. In the analytical setting, $\overline{\text{span}}_{\mathbb{C}}\{s_\alpha s_\alpha^* : \alpha \in \mathcal{L}\}$ is isomorphic to the continuous functions on the Ott-Tomforde-Willis subshift associated with X_F.

In our main result, we describe conjugacy between two Ott-Tomforde-Willis subshifts (and isometric conjugacy of subshifts with the usual metric that induces the product topology) via certain isomorphisms of the associated algebras, in terms of conjugacy of the cover spaces (which are the Stone duals of the corresponding boolean algebras), and in terms of isomorphism of the associated topological groupoids, see [1, Theorem 7.6] and [2, Theorem 6.11].

We propose a non-unital version of a subshift algebra, for which it is necessary to consider the Boolean algebra $\mathcal{B}$ of subsets of X generated by all $C(\alpha, \beta)$, for $\alpha, \beta \in \mathcal{L}$ not both simultaneously equal to ω. In this case, the *non-necessarily unital subshift C*-algebra* is defined as the universal C*-algebra generated by projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in A\}$ subject to the relations:

(i) $p_{A\cap B} = p_A p_B$, $p_{A\cup B} = p_A + p_B - p_{A\cap B}$, and $p_\emptyset = 0$, for every $A, B \in \mathcal{B}$;
(ii) $s_\beta s_\alpha^* s_\alpha s_\beta^* = p_{C(\alpha,\beta)}$ for all $\alpha, \beta \in \mathcal{L} \setminus \{\omega\}$, where for $\alpha = \alpha_1 \dots \alpha_n \in \mathcal{L} \setminus \{\omega\}$, $s_\alpha := s_{\alpha_1} \cdots s_{\alpha_n}$.
(iii) $s_\alpha^* s_\alpha = p_{C(\alpha,\omega)}$ for all $\alpha \in \mathcal{L} \setminus \{\omega\}$;
(iv) $s_\beta s_\beta^* = p_{C(\omega,\beta)}$ for all $\beta \in \mathcal{L} \setminus \{\omega\}$.

The algebraic version of the above algebra is defined analogously. Moreover, when the above algebra is unital it coincides with the subshift algebra we first defined. If it is not unital then its unitization coincides with the algebra we first defined.

Finally, at the end of the presentation, we provide a brief description of ongoing joint work with Danilo Royer about the socle of a subshift algebra. Recalling that the socle consists of the sum of all left minimal ideals of the algebra, we show that the minimal left ideals are associated with irrational paths and describe a graph associated with a subshift in a manner that the socle of the subshift algebra is graded isomorphic to the Leavitt path algebra of the graph.

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Graded and filtered K-theories for Leavitt path algebras

Huanhuan Li

(joint work with Pere Ara, Roozbeh Hazrat)

The theory of Leavitt path algebras is intrinsically related, via graphs, to the theory of symbolic dynamics and C^*-algebras where the major classification programs have been a domain of intense research in the last 50 years. However, it is not yet clear what is the right invariant for the classification of Leavitt path algebras, and for that matter, graph C^*-algebras [13]. In the case of simple graph C^*-algebras (i.e., algebras with no nontrivial ideals), it is now established that K-theory functors K_0 and K_1 can classify these algebras completely [11, 12]. Following the early work of Rørdam [10] and Restorff [9], it became clear that one way to preserve enough information in the presence of ideals in a C^*-algebra, is to further consider the K-groups of the ideals, their subquotients and how they are related to each other via the six-term sequence. Over the next ten years since [10, 9] this approach, which is now called *filtered K-theory*, was subsequently investigated and further developed by Eilers, Restorff, Ruiz and Sørensen [2, 3], where it was shown that the sublattice of gauge invariant prime ideals and their subquotient K-groups can be used as an invariant. In a major work [4] it was shown that filtered K-theory is a complete invariant for unital graph C^*-algebras. In [5] the four authors introduced the filtered K-theory in the purely algebraic setting and showed that if two Leavitt path algebras with coefficients in complex numbers $\mathbb{C}$ have isomorphic filtered algebraic K-theory then the associated graph C^*-algebras have isomorphic filtered K-theory.

Graded K-theory was initiated as a capable invariant for the classification of graph algebras in [6] and further studied in [1, 7, 8]. We show that in the setting of graph algebras, graded K-theory determines a large portion of filtered K-theory. To be precise, we show that for two Leavitt path algebras over a field, if their graded Grothendieck groups K_0^{gr} are isomorphic, then certain precisely defined quotients of their filtered K-theories are also isomorphic. This shows the richness of a graded Grothendieck group as an invariant. Namely, the single group $K_0^{\mathrm{gr}}(L_k(E))$ of the Leavitt path algebra $L_k(E)$ associated to a graph E, with coefficients in a field k, contains all the information about the K_0 groups and the quotient reduced groups $\overline{K}_1$ of K_1 of the subquotients of graded ideals of $L_k(E)$, and how they are related via the long exact sequence of K-theory. For the sake of precision, let us point out that the graded Grothendieck group K_0^{gr} is a *graded invariant* of Leavitt path algebras, but not an algebra invariant: isomorphic Leavitt path algebras may have non-isomorphic graded Grothendieck groups.

During the talk, the speaker first recalled the definition of algebraic filtered K-theory and pointed out the motivation of the talk, to find filtered K-theory as an invariant up to the isomorphism of graded Grothendieck groups for Leavitt path algebras. Then the speaker introduced the definition of reduced filtered K-theory for Leavitt path algebras. Finally the speaker stated the main result and gave the sketch of the proof for it.

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Ample groupoids, topological full groups, algebraic K-theory spectra and infinite loop spaces

Xin Li

Topological full groups have recently attracted attention because they led to solutions of several outstanding open problems in group theory (see [7], [8] and [16]).

Topological groupoids and their topological full groups arise in a variety of settings, for instance from Cantor minimal systems, from shifts of finite type, or more generally, from graphs (see for instance [12]), from self-similar groups or actions and from higher rank graphs (see for instance [13, 15]). In this context, there is an interesting connection to C*-algebra theory because topological groupoids serve as models for C*-algebras (see [18]) such as Cuntz algebras, Cuntz-Krieger algebras, graph C*-algebras or higher rank graph C*-algebras, many of which play important roles in the classification programme for C*-algebras. There is also an interesting link to group theory because Thompson's group V (see [4]) and many of its generalizations and variations [6, 21, 1] can be described as topological full groups of corresponding topological groupoids. In the case of V this observation goes back to [14].

While general structural properties [11, 12, 13, 17, 10] and rigidity results have been developed [19, 12], and several deep results have been established for particular examples of topological full groups [7, 8, 16, 20, 22], it would be desirable to create a dictionary between dynamical properties and invariants of topological groupoids on the one hand and group-theoretic properties and invariants of topological full groups on the other hand. This would allow us to study topological full groups – which are very interesting but in many aspects still remain mysterious – through the underlying topological groupoids which are often much more accessible. In my talk, I described recent work which develops this programme in the context of homological invariants by establishing a link between groupoid homology and group homology of topological full groups.

For the particular example class of Thompson's group V and its generalizations, the study of homological invariants and properties has a long history [3, 2]. It was shown in [2] that V is rationally acyclic. Only recently it was established in [22] that V is even integrally acyclic. The new approach in [22] also allows for many more homology computations for Higman-Thompson groups. However, for other classes of topological full groups, very little is known about homological invariants.

Inspired by [22], we have developed an approach to homological invariants of topological full groups in [9]. Let us now formulate our main results. Let G be a topological groupoid, i.e., a topological space which is at the same time a small category with invertible morphisms, such that all operations (range, source, multiplication and inversion maps) are continuous. We always assume the unit space $G^{(0)}$ consisting of the objects of G to be locally compact and Hausdorff. In addition, suppose that G is ample, in the sense that it has a basis for its topology given by compact open bisections. If $G^{(0)}$ is compact, then the topological full group $\boldsymbol{F}(G)$ is defined as the group of global compact open bisections. In the general case, $\boldsymbol{F}(G)$ is the inductive limit of topological full groups of restrictions of G to compact open subspaces of $G^{(0)}$. Given an ample groupoid G as above, we construct a small permutative category $\mathfrak{B}_G$ of compact open bisections of G. Let $\mathbb{K}(\mathfrak{B}_G)$ be the algebraic K-theory spectrum of $\mathfrak{B}_G$ and $\Omega^\infty\mathbb{K}(\mathfrak{B}_G)$ the associated infinite loop space.

Our first main result identifies reduced homology of $\mathbb{K}(\mathfrak{B}_G)$ with groupoid homology of G as introduced in [5] and studied in [11].

Theorem 1. *Let G be an ample groupoid with locally compact Hausdorff unit space $G^{(0)}$. Then we have*

$$\tilde{H}_*(\mathbb{K}(\mathfrak{B}_G)) \cong H_*(G).$$

For the second main result, we need the assumption that G is minimal, i.e., every G-orbit is dense in $G^{(0)}$. We also require G to have comparison, which roughly means that G-invariant measures on $G^{(0)}$ control when one compact open subspace of $G^{(0)}$ can be transported into another by compact open bisections of G. In this setting, we can identify group homology of the topological full group $\boldsymbol{F}(G)$ with the homology of $\Omega_0^\infty \mathbb{K}(\mathfrak{B}_G)$, the connected component of the base point in $\Omega^\infty \mathbb{K}(\mathfrak{B}_G)$.

Theorem 2. *Let G be an ample groupoid, with locally compact Hausdorff unit space $G^{(0)}$ without isolated points. Assume that G is minimal and has comparison. Then we have*

$$H_*(\boldsymbol{F}(G)) \cong H_*(\Omega_0^\infty \mathbb{K}(\mathfrak{B}_G)).$$

Among other things, our results lead to

- general vanishing and acyclicity results, explaining and generalizing the result that V is acyclic in [22],
- a complete description of rational group homology for large classes of topological full groups,
- a proof of Matui's AH-conjecture for minimal, ample groupoids satisfying comparison whose unit spaces have no isolated points.

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Amenability notions and Fell bundles

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(joint work with Alcides Buss)

There are, by now, numerous examples of interesting C^*-algebras arising from *dynamical systems* (see [1, 2, 5, 6, 9] and references therein). Classically speaking, a *dynamical system* [1] is given by an action of a (discrete) group Γ on a locally compact Hausdorff space X, that is, a collection of homeomorphisms $\alpha_\gamma\colon X \to X$ of X such that $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. More generally, the action α may, instead, be on a non-commutative C^*-algebra A [2], in which case $\alpha_\gamma\colon A \to A$ is a collection of *-isomorphisms. Even more generally, the action may be *twisted* which, in this setting, means that α_g may not be a *-isomorphism, but given by a Hilbert A-A-bimodule A_γ, and the same kind of composition relations hold. In the latter case the bimodules $(A_\gamma)_{\gamma\in\Gamma}$ form a *Fell bundle of A over Γ*, which is usually denoted by $(A_\gamma)_{\gamma\in\Gamma}$ (cf. [5, 6, 11]).

Given one of the above actions or Fell bundles, one can construct certain *crossed product C^*-algebras*, usually denoted by $C_0(X) \rtimes_i \Gamma$, $A \rtimes_i \Gamma$ or $C_i^*((A_\gamma)_{\gamma\in\Gamma})$, where $i \in \{\max, \mathrm{red}\}$ (cf. [2, 5, 6, 11]). These C^*-algebras have been known to exist for a long time now, and their construction is based on (or, rather, inspired by) the construction of the group measure construction from von Neumann algebras. The main interest of these C^*-algebras cannot be overstated, and lies in, mainly, two facts. On one hand, these algebras form an already very large class [4, 9], and hence any understanding of their structure is interesting in its own right. On the other hand, they implement the action *spatially* (cf. [2, 4, 5, 11]). For instance, the reduced *cross-sectional* C^*-algebra of a Fell bundle $(A_\gamma)_{\gamma\in\Gamma}$, denoted by $C^*_{\mathrm{red}}((A_\gamma)_{\gamma\in\Gamma})$, contains copies of the Banach spaces A_γ spanning a dense subalgebra and such that $A_{\gamma_1} \cdot A_{\gamma_2} = A_{\gamma_1\gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$. In other words, the Fell bundle $(A_\gamma)_{\gamma\in\Gamma}$ yields a Γ-*grading* when considered in $C^*_{\mathrm{red}}((A_\gamma)_{\gamma\in\Gamma})$.

In this talk we discussed and generalized the construction of cross-sectional C^*-algebras to Fell bundles over *inverse semigroups* (cf. [3, 4, 11]), and gave a sufficient condition to ensure that the algebra $C^*_{\mathrm{red}}((A_s)_{s\in S})$ is *nuclear* (cf. [4] or the discussion below). Thereby, we generalize the main results of [1, 5, 6, 7, 8, 11].

Inverse semigroups were introduced independently by Wagner [12] and Preston [10] in the 1950's. Recall that a semigroup S is said to be *inverse* if for every $s \in S$ there is a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. As groups model the global bijections/symmetries of a space, inverse semigroups model those that may only be partial. It is well known that inverse semigroups are closely related to *étale groupoids*, although that is not a point of view we will follow in this text. Moreover, one can substitute Γ by an inverse semigroup S in the above paragraph, and end up with an action of an inverse semigroup, or a Fell bundle $(A_s)_{s\in S}$. This kind of Fell bundle actions were introduced in the unpublished paper [11]. Throughout the rest of the text, S will be a (discrete) inverse semigroup with a *unit*, i.e. some (unique) $1 \in S$ such that $s \cdot 1 = 1 \cdot s = s$ for all $s \in S$.

The construction of the *reduced cross-sectional C^*-algebra* $C^*_{\mathrm{red}}((A_s)_{s\in S})$ is technical [3, 4], and relies on the *weak conditional expectation* P. Consider the map $P\colon C^*_{\max}((A_s)_{s\in S}) \to A_1^{**}$ that, when restricted to the subspaces A_s, is defined by

$$P\colon A_s \to A_1^{**}, \ \ a \mapsto a \cdot 1_{s,1},$$

where $1_{s,1}$ is the unit of the multiplier algebra of the ideal $I_{s,1} := \langle A_e \mid e \leq s, 1\rangle$ when considered in A_1^{**} in the usual manner (see [3, 4] for a more detailed construction). Such a map P is a weak conditional expectation, and can be used to construct the reduced cross-sectional C^*-algebra of $(A_s)_{s\in S}$ as follows.

Definition 1. Let $(A_s)_{s\in S}$ be a Fell bundle over S.

(i) The *nucleus of P* is the ideal $\mathcal{N}_P := \{x \mid P(x^*x) = 0\} \subseteq \oplus_{s\in S} A_s$.
(ii) The *algebraic cross-sectional *-algebra* is $\mathbb{C}[(A_s)_{s\in S}] := \oplus_{s\in S} A_s/\mathcal{N}_P$.
(iii) The *reduced cross-sectional C^*-algebra* is the (necessarily unique) C^*-algebra $C^*_{\mathrm{red}}((A_s)_{s\in S})$ densely containing $\mathbb{C}[(A_s)_{s\in S}]$ and such that P induces a faithful map $C^*_{\mathrm{red}}((A_s)_{s\in S}) \to A_1^{**}$.

On the other hand, we may now study *nuclearity* (cf. [2]). This is a property that has been most fruitful in the last decades (see [4, 2, 9] and references therein), as it is a finite dimensional type of approximation for a given C^*-algebra. Recall (cf. [2, Theorem 3.8.7]) that a C^*-algebra is *nuclear* if for any C^*-algebra B there is only one C^*-norm in the algebraic tensor product $A \odot B$. This is equivalent to saying that the canonical map $A \otimes_{\max} B \twoheadrightarrow A \otimes_{\min} B$ is injective. In the talk we discussed the following condition on a Fell bundle $(A_s)_{s\in S}$.

Definition 2. The Fell bundle $(A_s)_{s\in S}$ has the *approximation property* (cf. [4, Definition 4.3]) if there are finitely supported sections $(\xi_i\colon S \to A_1)_{i\in I}$ such that

(i) $\sup_{i\in I} \| \sum_{p,t\in S} 1_{p,t}\xi_i(p)^*\xi_i(t)\| < \infty$; and
(ii) $\| \sum_{p,t\in S} 1_{p,st}\xi_i(p)^* a\xi_i(t) - a\| \to 0$ for every $s \in S$ and $a \in A_s$.

The above definition of approximation property generalizes the homonymous property for Fell bundles over groups introduced by Exel in [5] (see also [6] and [4, Section 4]). On the other hand, it is proved in [4, Theorem 4.16] that the above property also restricts to (topological) amenability[1] of the associated *groupoid of germs* in the case that $(A_s)_{s\in S}$ is induced by an action $\alpha\colon S \curvearrowright X$. The main interest of the approximation property above comes from the following.

Theorem 3. *Let $(A_s)_{s\in S}$ be a Fell bundle with the approximation property. The canonical map $\Lambda\colon C^*_{\max}((A_s)_{s\in S}) \twoheadrightarrow C^*_{\mathrm{red}}((A_s)_{s\in S})$ is injective, and hence a *-isomorphism. Moreover, if A_1 is nuclear, then $C^*_{\mathrm{red}}((A_s)_{s\in S})$ is nuclear as well.*

Sketch of proof. We refer the reader to [4, Theorems 6.2 and 6.7] for a complete proof. The main ideas behind the first statement are the following. First, one can use a version of "Fell's absorption principle" in order to prove that there is a faithful representation

$$\pi^\Lambda\colon C^*_{\mathrm{red}}\left((A_s)_{s\in S}\right) \hookrightarrow \mathcal{B}\left(\ell^2_\pi\left(S,H\right)\right)$$

where $\ell^2_\pi(S,H)$ is a certain Hilbert space of sections of some bundle, and $\pi = (\pi_s)_{s\in S}$ is any given representation of the bundle $(A_s)_{s\in S}$ (cf. [4, Corollary 3.23]). Secondly, in the case when the Fell bundle $(A_s)_{s\in S}$ has the approximation property we can construct maps "in the wrong direction" (cf. [4, Lemma 6.1]), that is

$$\Psi_i\colon C^*_{\mathrm{red}}\left((A_s)_{s\in S}\right) \to C^*_{\max}\left((A_s)_{s\in S}\right), \quad x \mapsto T^*_{\xi_i} x T_{\xi_i},$$

where

$$T_{\xi_i}\colon H \to \ell^2_\pi\left(S,H\right), \quad v \mapsto \sum_{t\in S} \pi_1\left(\xi_i\left(t\right)\right) v\delta_t.$$

It can then be proved that Ψ_i is a completely positive map such that $\Psi_i(\Lambda(x)) \to x$ for all $x \in C^*_{\max}((A_s)_{s\in S})$, which shows the quotient map $\Lambda\colon C^*_{\max}((A_s)_{s\in S}) \to C^*_{\mathrm{red}}((A_s)_{s\in S})$ is injective, finishing the proof of the first statement.

The second statement, i.e. the fact that $C^*_{\mathrm{red}}((A_s)_{s\in S})$ is nuclear if A_1 is nuclear, follows from the first and the (non-trivial) considerations that taking cross-sectional C^*-algebras is a well-behaved process with respect to taking tensor products. We refer to [4, Section 5] for details. □

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The local bisection hypothesis for twists over étale Hausdorff groupoids

Kathryn McCormick
(joint work with Becky Armstrong, Jonathan Brown, Gilles de Castro, Lisa Orloff Clark, Kristen Courtney, Ying-Fen Lin, Jacqui Ramagge, Benjamin Steinberg, Aiden Sims)

Let G be a locally compact Hausdorff étale (ample) groupoid, and let $\Sigma \xrightarrow{q} G$ be a (discrete, R-) twist over G. Then using this data, one can build the twisted reduced groupoid C*-algebra $C_r^*(\Sigma; G)$ [8, 10], or the twisted Steinberg algebra $A_R(\Sigma; G)$ in the ample groupoid, discrete twist setting [4, 3]. These algebras have obvious abelian subalgebras by restricting to the twist above the units, $C_r^*(q^{-1}(G^{(0)}); G^{(0)})$ and $A_R(q^{-1}(G^{(0)}); G^{(0)})$, respectively. The (twisted) groupoid C^*-algebra and the (twisted) Steinberg algebra share some similar properties, despite one being an analytic object and the other being a purely algebraic object. For example, they both have uniqueness theorems that give sufficient conditions for which a homomorphism from the algebra can be injective [6, 1, 7, 4]. In both the algebraic and the C*-algebraic setting, it has also been fruitful and useful to study the special case where the groupoid G is the groupoid for a directed graph or k-graph.

Building on ideas of Feldman and Moore, [8, 11, 9] show that given a Cartan pair (A, B) of C^*-algebras, one can build an effective locally compact Hausdorff étale groupoid G and twist $\Sigma \xrightarrow{q} G$ so that $A \simeq C_r^*(\Sigma; G)$ and $B \simeq C_r^*(q^{-1}(G^{(0)}); G^{(0)})$; and moreover, if one starts with such a Cartan pair coming from a twisted groupoid C^*-algebra, the twist Σ' that one builds is isomorphic to Σ. Let R be an indecomposable commutative unital ring. In [3], we show that given a discrete R-twist $\Sigma \xrightarrow{q} G$ that satisfies the *local bisection hypothesis*, then we can recover the twist as well. We will describe in more detail what the local bisection hypothesis is, and summarize how the hypothesis gets used to reconstruct the groupoid twist in the above result. We will also discuss how the obvious analogue of the local bisection hypothesis for C*-algebras is 'hiding in plain sight' in known C*-algebra work, but also ends up being equivalent to G being effective.

The local bisection hypothesis was introduced in 2010 by Steinberg [12] in the setting of graded ample Hausdorff groupoids. A discrete R-twist $\Sigma \xrightarrow{q} G$ satisfies the local bisection hypothesis exactly when for every (algebraic) normaliser n of $A_R(q^{-1}(G^{(0)}); G^{(0)})$, the open support $q(\mathrm{supp}(n))$ is a bisection of G. This should be compared to the "no nontrivial units" hypothesis on group rings, see [13]. For example, if R is a reduced and indecomposable commutative ring, and there is a dense subset of $G^{(0)}$ so that the isotropy over each $x \in G^{(0)}$ satisfies the unique product property, then the twist satisfies the local bisection hypothesis [3, 9.3]. Another example of a setting in which the local bisection hypothesis applies is when Γ is a countable higher rank graph that is row-finite with no sources and there is a normalised $R^\times$-valued cocycle on Γ [3, 9.4]. Note that the twist satisfying the local bisection hypothesis is more general than the groupoid G being effective.

One easy consequence of the local bisection hypothesis is that the canonical conditional expectation $P : A = A_R(\Sigma; G) \to B = A_R(q^{-1}(G^{(0)}); G^{(0)})$ is implemented by idempotents. That is, we have that for every normaliser n, there exists an idempotent e of the subalgebra such that $P(n) = en = ne$. Moreover, if (A, B) are an algebraic quasi-Cartan pair, then any conditional expectation implemented by idempotents can be also realized as restriction to $G^{(0)}$ and for any normaliser n, $q(\mathrm{supp}_g(n))$ is a bisection.

Moreover, the local bisection hypothesis allows us to connect the general setting of an algebraic quasi-Cartan pair with the concrete setting of a twisted Steinberg algebra. If we use the data (A, B) of an algebraic quasi-Cartan pair to build an ample groupoid G and discrete twist Σ as in [3] such that $A \simeq A_R(\Sigma; G)$ and $B \simeq A_R(q^{-1}(G^{(0)}); G^{(0)})$, then the twist Σ that we built will satisfy the local bisection hypothesis [3, Cor. 6.7]. Finally, if we start with a twisted Steinberg algebra $A_R(\Sigma; G)$ and build the associated twist $\Sigma' \xrightarrow{q'} G'$, then $\Sigma \xrightarrow{q} G$ satisfies the local bisection hypothesis if and only if the induced map between Σ and Σ' is a topological groupoid isomorphism. In summary, the local bisection hypothesis is the necessary and sufficient condition to recover the original groupoid Σ from our construction.

On the other hand, we can ask what the local bisection hypothesis might look like in the C*-algebraic setting, which is the question we pursue in [2]. Given a twist $\Sigma \xrightarrow{q} G$, we say the twist satisfies the *C*-algebraic local bisection hypothesis* if for every (C*-algebraic) normaliser n of $C^*_r(q^{-1}(G^{(0)}); G^{(0)})$, the open support $q(\mathrm{supp}(n))$ is a bisection of G. It is known ([11, 4.8(ii)], [5]) that if G is effective, then for any normaliser n of $C^*_r(q^{-1}(G^{(0)}), G^{(0)})$, $q(\mathrm{supp}(n))$ is a bisection. In [2] we show that the following are equivalent for a locally compact Hausdorff étale groupoid G:

(1) G is effective
(2) For any twist $\Sigma \xrightarrow{q} G$, $(C^*_r(\Sigma; G), C^*_r(q^{-1}(G^{(0)}); G^{(0)}))$ is a Cartan pair
(3) For any normaliser n of $C^*_r(q^{-1}(G^{(0)}); G^{(0)})$, $q(\mathrm{supp}(n))$ is an open bisection of G

Thus, for twisted groupoid C*-algebras, the analogue of the local bisection hypothesis forces the stronger condition of having a Cartan pair.

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A bicategorical perspective on generalized graph algebras and Leavitt path algebras and their underlying groupoids

Ralf Meyer

(joint work with Celso Antunes, Suliman Albandik, Joanna Ko, Fabian Rodatz, Muhammad Taufik bin Mohd Yusof)

Graph C*-algebras and Leavitt path algebras have been generalized in several ways. A common feature of these generalizations is that they are groupoid C*-algebras of suitable groupoids. This talk offers a bicategorical framework to study their analogues for topological, self-similar, and higher-rank graphs, where any combination of the three adjectives is possible. The underlying groupoid, its Steinberg algebra and its C*-algebra, that is, the generalized graph C*-algebra, are all limits in suitable bicategories of groupoids, rings, or C*-algebras.

A bicategory is like a category, but it has another layer of structure, namely, 2-arrows between arrows. These come with two compositions, subject to a lengthy list of conditions. Classic examples are the bicategory of categories, functors, and natural transformations or the bicategory of rings, bimodules, and bimodule maps. The limit construction in a bicategory is even more interesting than in an ordinary category because the commuting triangles that are used to define cones now commute only up to a 2-arrow, that is, conditions are replaced by extra data. As a result, the limit gets bigger than the data that is used to define it.

The simplest diagrams in a bicategory are homomorphisms from the monoid $(\mathbb{N}, +)$ into it. Such a homomorphism is produced by an object with an endomorphism. In the C*-correspondence bicategory, this gives a single C*-correspondence, that is, a Hilbert A-module with a nondegenerate left A-action by adjointable operators. In case the left action in the C*-correspondence is by compact operators, the limit of the resulting diagram is exactly its Cuntz-Pimsner algebra (see [2]). More generally, a diagram in the correspondence bicategory defined over a monoid P is the same as an essential product system over P, and if the left actions on all C*-correspondences in the diagram are by compact operators, then the limit is the Cuntz-Pimsner C*-algebra of the product system (see [2]). In particular, the C*-algebras of (self-similar and/or topological) graphs are defined as Katsura's variant of the Cuntz-Pimsner algebras of suitable C*-correspondences. However, if the left action in this C*-correspondence is not by an injective homomorphism to the compact operators, then the result above does not apply to Katsura's variant. Higher-Rank graph C*-algebras arise in a similar way, for homomorphisms defined on the monoid $(\mathbb{N}^k, +)$.

In his thesis [1], this limit construction in the world of C*-algebras is lifted to a suitable bicategory of groupoid correspondences. This is defined so that it comes with a homomorphism to the bicategory of C*-correspondences. More precisely, the relevant groupoids are étale, locally compact, possibly non-Hausdorff, and the homomorphism from groupoid to C*-correspondences takes a groupoid to its groupoid C*-algebra. See also [4] for more details. Albandik's main result is that for diagrams over a monoid P that satisfy suitable Ore conditions, the C*-algebra of the limit of the diagram in the bicategory of groupoid correspondences is isomorphic to the Cuntz–Pimsner algebra of the corresponding product system over P. A self-correspondence from a group to itself is the same as a self-similarity, and self-similar graphs also give natural examples of self-correspondences. In fact, it is useful to slightly generalise self-similar graphs and use groupoid correspondences on discrete groupoids instead: I expect these to have very much the same properties. A self-correspondence on an étale groupoid may then be interpreted as a topological self-similar graph.

A limit in a bicategory is well defined only up to equivalence, which is the same as Morita equivalence for the bicategory of groupoid correspondences. The particular groupoid constructed by Albandik may be characterised differently, using a universal property that specifies its actions on topological spaces. This universal property is used in [7] to define a groupoid model for any diagram of groupoid correspondences. It is shown in [6] that any diagram has a groupoid model, and that this is again locally compact if the diagram consists of proper correspondences.

More recently, I have lifted these results about C*-algebras to the bicategory of rings and bimodules. If the ring is unital and the bimodule is finitely generated and projective as a right module, then the limit turns out to be its algebraic Cuntz-Pimsner algebra, as defined by Carlsen and Ortega in [5]. This is shown in unpublished work by me, available upon request. The homomorphism from groupoid correspondences to C*-algebras has an algebraic counterpart, which gives the Steinberg algebra of an ample groupoid. This homomorphism is defined in the recent Master's Thesis by Fabian Rodatz. For certain diagrams of ample groupoid correspondences, he also proves that the Steinberg algebra of the groupoid model is a limit in the bicategory of rings and bimodules. My doctoral student bin Mohd Yusof is generalizing this theory to nonunital rings and non-proper groupoid correspondences. The paradigm for this work is a characterisation of relative Cuntz–Pimsner algebras of C*-correspondences in [8].

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On automorphisms of Leavitt path algebras and applications

Tran Giang Nam

In the last years, I studied automorphisms of a Leavitt path algebra $L_K(E)$ associated to a directed graph E with coefficients in a field K via Cuntz's seminal paper [6]. In [7] Kuroda and I constructed a new class of automorphisms of a Leavitt path algebra $L_K(E)$ being analogous to the Anick automorphisms of a free associative algebra $K\langle x_1, x_2, \ldots, x_n\rangle$. The Anick automorphism has been shown by U. U. Umirbaev [9] to be a wild automorphism when $n = 3$ and K is a field of characteristic 0. Using these Anick type automorphisms of Leavitt path algebras, we constructed a new class of simple modules over the Leavitt path algebra $L_K(R_n)$, where R_n is the rose with n petals, that is, the directed graph with one vertex and n distinct edges. There are some by now well-known classes of simple modules over a Leavitt path algebra established in [5, 1, 2], called *Chen modules*. One instance of these modules occurs when considering a closed simple path c on

E and an irreducible polynomial $f \in K[x]$. The corresponding simple module is denoted by S_c^f in [2]. The new simple modules are obtained from the Chen simple modules S_c^f by twisting them by Anick type automorphisms of the Leavitt path algebra $L_K(E)$. The main result of my paper together with Kuroda [7] is that for some specific choices of closed simple paths on R_n, and some specific choices of Anick type automorphisms of $L_K(R_n)$, the new simple modules obtained using the above procedure are pairwise non-isomorphic, and also are not isomorphic to any other Chen simple module. In [8], Srivastava, Vien and I provided an additional class of simple modules over $L_K(R_n)$ from Chen modules $V_{[p]}$ associated with an infinite path in R_n by twisting them by automorphisms associated with invertible matrices $P \in GL_n(K)$.

One of the most frequently used tools to construct new examples of algebras is twisting the multiplicative structure of original algebra. Classic examples of algebras constructed by twisting multiplicative structure include skew polynomial rings and skew group rings. A notion of twist of a graded algebra A was introduced by Artin, Tate, and Van den Bergh in [3] as a deformation of the original graded product of A with the help of a graded automorphism of A. Let σ be an automorphism of the graded algebra $A = \oplus A_n$. Define a new multiplication $\star$ on the underlying graded K-module $\oplus A_n$ by $a \star b = a\sigma^n(b)$ where a and b are homogeneous elements in $A = \oplus A_n$ and $deg(a) = n$. The new graded algebra with the same underlying graded K-module $\oplus A_n$ and the new graded product $\star$ is called the twist of A and is denoted as A^σ. This notion of twist of a graded algebra was later generalized by Zhang in [10], where he introduced the concept of twisting of graded product with the help of a twisting system. Let $\tau = \{\tau_n \mid n \in \mathbb{Z}\}$ be a set of graded K-linear automorphisms of $A = \oplus A_n$. Then τ is called a twisting system if $\tau_n(y\tau_m(z)) = \tau_n(y)\tau_{n+m}(z)$ for all $n, m, l \in \mathbb{Z}$ and $y \in A_m$, $z \in A_l$. For example, if σ is a graded algebra automorphism of A, then $\tau = \{\sigma^n \mid n \in \mathbb{Z}\}$ is a twisting system. Thus, the twist of a graded algebra in the sense of Artin-Tate-Van den Bergh can be viewed as a special case of the twist introduced by Zhang. Such a twist of a graded algebra is now known as Zhang twist. Zhang twist of a graded algebra has played a vital role in the interaction of noncommutative algebra with noncommutative projective geometry. One of the main features of the study of Zhang twist of a graded algebra is that if an algebra B is isomorphic to the Zhang twist of an algebra A, then A is graded Morita equivalent to B. As a consequence it follows that the noncommutative projective schemes associated with A and B are equivalent ([4]).

In [8] Srivastava, Vien and I initiated the study of Zhang twist of Leavitt path algebras with a larger goal to develop the geometric theory of Leavitt path algebras. We twisted the multiplicative structure of Leavitt path algebras with the help of specially graded automorphisms ϕ. In a rather surprising result we obtained that the Leavitt path algebra $L_K(E)$ of an arbitrary graph E is always a subalgebra of the Zhang twist $L_K(E)^\phi$. We also characterized Leavitt path algebras $L_K(R_n)$ of the rose graph R_n with n petals that are rigid to Zhang twist in the sense that

$L_K(R_n)$ turns out to be isomorphic to its Zhang twist with respect to these graded automorphisms.

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Modeling groupoid algebras using left cancellative small categories

Enrique Pardo

(joint work with Eduard Ortega)

In [14], Spielberg described a new method of defining C^*-algebras associated to oriented combinatorial data, generalizing the construction of algebras from directed graphs, higher-rank graphs, and (quasi-)ordered groups. To this end, he introduced *categories of paths* –i.e. cancellative small categories with no (nontrivial) inverses– as a generalization of higher rank graphs, as well as ordered groups. The idea is to start with a suitable combinatorial object and define a C^*-algebra directly from what might be termed the generalized symbolic dynamics that it induces. Associated to the underlying symbolic dynamics, he presents a natural Deaconu-Renault étale groupoid derived from this structure. The construction also gives rise to a presentation by generators and relations, tightly related to the groupoid presentation. In [15] he showed that most of the results hold when relaxing the conditions, so that right cancellation or having no (nontrivial) inverses are taken out of the picture.

In [11], we studied Spielberg's construction, using a groupoid approach based in the Exel's tight groupoid construction [5], showing that the tight groupoid for these inverse semigroups coincide with Spielberg's groupoid [13]. With this tool at hand, following the approach of [2], we were able to characterize simplicity for the algebras associated to finitely aligned left cancellative small categories, and in particular in the case of Exel-Pardo systems [8], by using an inverse semigroup approach [7]. Finally, we gave, under mild and necessary hypotheses, a characterization of amenability for such a groupoid.

Therefore, it becames important to understand the internal structure of the left cancellative small category to check the desired properties of the associated groupoid, and hence of its associated (C^*-)algebra. A classical idea is to decompose our complex object in different simple pieces with well-behaved relations between them. This was well studied in [10], where it was proved that categories with length functions on $\mathbb{N}^k$ with certain decomposition properties can be written as the Zappa-Szép product of the groupoid of invertible elements of the category and a higher-rank graph subcategory generated by a transversal of generators of maximal right ideals. Zappa-Szép products of left cancellative small categories and groups were studied by Bédos, Kaliszewski, Quigg and Spielberg in [1], where they studied the representation theory for the Spielberg algebras of the new left cancellative small category associated to this construction.

In [12], we extended the scope of [1] to actions of groupoids, To this end, we defined groupoid actions on a left cancellative small category and their Zappa-Szép products (inspired in the construction in [9]), and we showed that Zappa-Szép products appear naturally in the context of left cancellative small categories with length functions. Finally, we extended the results of [11, Sections 7 & 8] to determine the essential properties of the tight groupoid associated to Zappa-Szép products of groupoid actions on a left cancellative small category.

In the talk, we summarized these work as follows: First, we explained why Spielberg's algebras associated to left cancellative small categories provide a different approach for studying algebras of second countable, ample groupoids; in particular, we explained how Donsig et al. [3, 4] and Exel [6] allowed us to connect ample groupoids with tight groupoids of inverse semigroups. Once this was done, we recalled some known results on small categories, and we defined length functions and factorization properties needed in the sequel. Also, we defined groupoid actions on left cancellative small categories, as well as the Zappa-Szép products of certain groupoid actions on left cancellative small categories; in particular, we showed that left cancellative small categories with nice length functions can be described as Zappa-Szép products of the action of their groupoid of invertible elements on certain nice subcategories. After recalling the basics on (topological) groupoids and their algebras, we carefully explained the construction the Exel's tight groupoid of an inverse semigroup. Finally, we analyzed the structure of the tight groupoid associated to Zappa-Szép products of groupoid actions to left cancellative small categories, extending Exel's construction.

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Classification of graph C^*-algebras

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(joint work with Søren Eilers)

A graph C^*-algebra is a C^*-algebra defined by generators and relations governed by an underlying directed graph. The class of graph C^*-algebras is a tractable class of C^*-algebras due to the combinatorial tools from the directed graph that defines a graph C^*-algebra. Structural properties of a graph C^*-algebra such as its ideal lattice, whether it is unital, stably finite, purely infinite, and its K-theory are obtained through the underlying directed graph. The authors with Restorff and Sørensen [10] exploited these combinatorial tools to obtain a *geometric classification* of unital graph C^*-algebras. We proved that the equivalence relation induced on all vertex-finite directed graphs (but possibly countably infinitely many edges) by stable isomorphism of the associated graph C^*-algebras is the coarsest equivalence relation containing a number of moves on the graphs. These moves are all local in nature – affecting only the graph in a small neighborhood of the vertex to which it is applied – and the basic moves have their origin in symbolic dynamics, viz. *in-splitting* and *out-splitting* as defined and studied by Williams ([17]) as a way of characterizing conjugacy among shifts of finite type.

Recently, in a movement pioneered by Kengo Matsumoto, it has been discovered that for finite essential graphs – the so-called Cuntz-Krieger algebras – have rigidity properties when they are considered not as C^*-algebras alone, but as C^*-algebras with additional natural structures. More precisely, the shifts of finite type associated to finite essential graphs are remembered by the Cuntz-Krieger algebras at varying level of precision depending upon how much structure is considered. The canonical objects associated to the Cuntz-Krieger algebra $\mathcal{O}_A$ or its stabilization $\mathcal{O}_A \otimes \mathbb{K}$ are the diagonal, the gauge action, the stabilized diagonal, or the stabilized gauge action (denoted by $\mathcal{D}_A$, γ^A, $\mathcal{D}_A \otimes c_0$ and $\gamma^A \otimes \mathrm{id}$, respectively). Table 1 lists a collection of such results presently known, showing that standard dynamical notions such as conjugacy and flow equivalence are indeed rigidly remembered by the operator algebras, and providing motivation for the study of new concepts of sameness of such dynamical systems.

Notion	*Data*	*Which SFTs?*	*Ref.*
Flow equivalence	$(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0)$	Irreducible	[16]
		All	[7]
Conjugacy	$(\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0, \gamma^A \otimes \mathrm{id})$	All	[8]
Shift equivalence	$(\mathcal{O}_A \otimes \mathbb{K}, \gamma^A \otimes \mathrm{id})$	Primitive	[5]
		Irreducible	[12]
Continuous orbit equivalence	$(\mathcal{O}_A, \mathcal{D}_A)$	Irreducible	[14]
		No isolated points	[4]
		All	[1] [9]
Eventual conjugacy	$(\mathcal{O}_A, \mathcal{D}_A, \gamma^A)$	Irreducible	[15]
		All	[8]

Table 1. Assorted rigidity results

To systematically address questions of this nature, in [11], we give the following definition. Let E and F be vertex-finite directed graphs. With $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \{0, 1\}$ we say that E and F are xyz-equivalent when there exists a $*$-isomorphism $\phi\colon C^*(E) \otimes \mathbb{K} \to C^*(F) \otimes \mathbb{K}$ which additionally satisfies

- $\phi(1_{C^*(E)} \otimes e_{1,1}) = 1_{C^*(F)} \otimes e_{1,1}$ when $\mathsf{x} = 1$
- $\phi \circ (\gamma^E_z \otimes \mathrm{id}_{\mathbb{K}}) = (\gamma^F_z \otimes \mathrm{id}_{\mathbb{K}}) \circ \phi$ when $\mathsf{y} = 1$
- $\phi(\mathcal{D}_E \otimes c_0) = \mathcal{D}_F \otimes c_0$ when $\mathsf{z} = 1$.

The goal of our project is to generate each xyz-equivalence relation by a finite collection of graph moves similar to [10]. We have defined a collection of moves

$$\texttt{(O)}, \texttt{(I+)}, \texttt{(I-)}, \texttt{(R+)}, \texttt{(S)}, \texttt{(C+)}, \texttt{(P+)}, \texttt{(K+)}$$

and prove the appropriate xyz-equivalence for each move except (K+). More precisely,

(1) E and F are 111-equivalent when E and F are move equivalent via Move (O) or Move (I+),
(2) E and F are 011-equivalent when E and F are move equivalent via Move (I-),
(3) E and F are 101-equivalent when E and F are move equivalent via Move (R+),
(4) E and F are 001-equivalent when E and F are move equivalent via Move (S), and
(5) E and F are 100-equivalent when E and F are move equivalent via Move (C+) or Move (P+).

Invariance of (K+) is the analytic analog to [13, Hazrat's Conjecture] which conjectures that E and F should be 110-equivalent. At the moment, we are only able to show that E and F are 100-equivalent when E and F are move equivalent via Move (K+). This fact was independently proved by Ara, Hazrat, and Li [3] for finite graphs.

For the class of countable vertex-finite graphs, we conjecture that

$$(0) \quad \overline{\underline{000}} = \langle \texttt{(O)}, \texttt{(I-)}, \texttt{(R+)}, \texttt{(S)}, \texttt{(C+)}, \texttt{(P+)}, \texttt{(K+)} \rangle$$
$$(1) \quad \overline{\underline{001}} = \langle \texttt{(O)}, \texttt{(I-)}, \texttt{(R+)}, \texttt{(S)} \rangle$$
$$(2) \quad \overline{\underline{010}} = \langle \texttt{(O)}, \texttt{(I-)}, \texttt{(K+)} \rangle$$
$$(3) \quad \overline{\underline{011}} = \langle \texttt{(O)}, \texttt{(I-)} \rangle$$
$$(4) \quad \overline{\underline{100}} = \langle \texttt{(O)}, \texttt{(I+)}, \texttt{(R+)}, \texttt{(C+)}, \texttt{(P+)}, \texttt{(K+)} \rangle$$
$$(5) \quad \overline{\underline{101}} = \langle \texttt{(O)}, \texttt{(I+)}, \texttt{(R+)} \rangle$$
$$(6) \quad \overline{\underline{110}} = \langle \texttt{(O)}, \texttt{(I+)}, \texttt{(K+)} \rangle$$
$$(7) \quad \overline{\underline{111}} = \langle \texttt{(O)}, \texttt{(I+)} \rangle.$$

We use the notation $\overline{\underline{\mathsf{xyz}}}$ to refer to the equivalence relation among vertex-finite graphs defined by $(E, F) \in \overline{\underline{\mathsf{xyz}}}$ if and only if E and F are xyz-equivalent. We use $\langle \cdot \rangle$ to denote the smallest equivalence relation generated by a collection of relations.

The geometric classification of the authors with Restorff and Sørensen [10] proves Conjecture (0). Conjecture (4) was completed by the authors with Arklint in [2]. Brix in [6] proved that Conjecture (7) is true for the class of finite directed graphs with no sinks. In [11], we proved these conjectures are true for several subclasses of vertex-finite directed graphs. One such example is that

$$\overline{\underline{010}} = \langle \texttt{(O)}, \texttt{(I-)}, \texttt{(K+)} \rangle = \langle \texttt{(O)}, \texttt{(I-)} \rangle = \overline{\underline{011}}$$

for acyclic directed graphs.

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C^*-algebraic representations of suspensions of graphs

Aidan Sims

Let $E = (E^0, E^1, r, s)$ be a directed graph; so E^0 is a countable set whose elements are the vertices of the graph, E^1 is a countable set whose elements are the edges, and $r, s : E^1 \to E^0$ indicate the directions of the edges: the edge e points from the vertex $s(e)$ to the vertex $r(e)$. For simplicity, we will discuss only the situation where E is finite, in the sense that both E^0 and E^1 are finite, and has neither sources or sinks in the sense that $r, s : E^1 \to E^0$ are both surjective. The (one-sided) infinite-path space E^∞ of E is the space of words $e_0e_1e_2\dots$ in the alphabet E^1 with the property that $r(e_i) = s(e_{i+1})$ for all i, given the topology inherited as a subspace of $\prod_{i=1}^{\infty} E^1$. The map $\sigma : E^\infty \to E^\infty$ that deletes the first edge in an infinite path is a local homeomorphism called the *shift map*, and the pair (E^∞, σ) is called the (one-sided) edge-shift of E. A well-known result from symbolic dynamics

[8, Theorem 2.3.2] says that every one-sided subshift of finite type is conjugate to an edge-shift.

In the world of two-sided shift spaces, an important invariant of a symbolic-dynamical system (X, σ) is its flow space $M(\sigma) := (X \times \mathbb{R})/\langle (x, t+1) \sim (\sigma(x), t)\rangle$—that is, the mapping torus of the shift map. Two shifts are flow-equivalent if there is a homeomorphism between their flow spaces that preserves orientation of the $\mathbb{R}$ action and preserves $\mathbb{R}$-orbits. Let A_E denote the adjacency matrix of the graph E. Work of Parry and Sullivan [11], Bowen and Franks [1] and of Franks [5] showed that the pair $(\operatorname{coker}(I - A_E), \operatorname{sign}(\det(I - A_E)))$, now called the Bowen–Franks invariant, is a complete invariant for flow-equivalence of irreducible shifts of finite type. Subsequently Cuntz and Krieger showed [3, 2] that the Cuntz–Krieger algebra of the matrix A_E—or equivalently the graph C^*-algebra $C^*(E)$ [4, 7]—remembers $\operatorname{coker}(I - A_E)$ as its K_0-group. Matsumoto and Matui [9] later showed that $C^*(E)$ together with its natural Cartan subalgebra remembers the whole Bowen–Franks invariant. The resulting interaction between symbolic dynamics and C^*-algebras via graph algebras has been enormously productive.

In this talk I discussed the problem of encoding the flow-space $M(\sigma)$ directly in a C^*-algebra. The starting point is to build from the graph E a family of quivers $\mathcal{S}^l E$ indexed by positive real numbers l as follows. The space $\mathcal{S}^l E^0$ of vertices is independent of l and is the topological realisation of E: the space

$$\big((E^1 \times [0,1]) \cup E^0\big)/\langle (e, 0) \sim s(e) \text{ and } (e, 1) \sim r(e)\rangle.$$

Roughly speaking, the space $\mathcal{S}^l E^1$ of edges consists of continuous paths of length l in this vertex space that respect the orientation of edges. When $l = 1$, the infinite-path space $\mathcal{S}^1 E^\infty$ with its natural shift map is exactly the flow-space of E.

Unfortunately, $\mathcal{S}^l E$ is neither a topological graph in the sense of Katsura [6] nor a topological quiver in the sense of Muhly and Tomforde [10] because neither the range nor the source map from $\mathcal{S}^l E_1$ to $\mathcal{S}^l E^0$ is open; and for the same reason its natural groupoid model does not admit a Haar system in the sense of Renault [12]. But one can associate a C^*-algebra to $\mathcal{S}^l E$ "by hand:" for each $v \in \mathcal{S}^l E^0$, write $\mathcal{S}^l E^* v$ for the (discrete) space of finite paths in $\mathcal{S}^l E$ that terminate at v, and let $\mathcal{H}_v := \ell^2(\mathcal{S}^l E^* v)$; then there are natural maps $\pi_v : C(\mathcal{S}^l E^0) \to \mathcal{B}(\mathcal{H}_v)$ and $\psi_v : C(\mathcal{S}^l E^1) \to \mathcal{B}(\mathcal{H}_v)$ that, for $l = 1$, amount to a summand in the standard path-space representation of E or of its dual $\widehat{E}$. Composing with the quotient map $q_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{Q}(\mathcal{H}_v)$ onto the Calkin algebra gives maps $\tilde{\pi}_v : C(\mathcal{S}^l E^0) \to \mathcal{Q}(\mathcal{H}_v)$ and $\tilde{\psi}_v : C(\mathcal{S}^l E^1) \to \mathcal{Q}(\mathcal{H}_v)$ whose images, when $l = 1$, contain a Cuntz-Krieger E-family and generate a representation of $C^*(E)$. We define

$$C^*(\mathcal{S}^l E) := C^*\Big(\bigoplus_v \tilde{\pi}_v(C(\mathcal{S}^l E^0)), \bigoplus_v \tilde{\psi}_v(C(\mathcal{S}^l E^1))\Big).$$

The main results that I discussed in the talk appear in [13], and concern the structure of $C^*(\mathcal{S}^l E)$. When E is a simple cycle of length n, the C^*-algebra $C^*(\mathcal{S}^l E)$ is isomorphic to the rotation algebra $A_{l/n}$. When $l = 1$, $C^*(\mathcal{S}^l E)$ turns out to be isomorphic to $C([0,1]) \otimes C^*(E)$, and so homotopy equivalent to $C^*(E)$.

When $l = m/n$ is rational, we obtain a similar homotopy result: if $D_n E^m$ is the graph obtained by first *delaying* E by inserting $n-1$ vertices along each edge of E, and then by taking the mth higher-power graph of $D_n E$ (the graph whose vertices are those of $D_n E$ but whose edges are paths of length m in $D_n E$), then $C^*(\mathcal{S}^{m/n}E)$ is homotopy equivalent to $C^*(D_n E^m)$. Since the edge shift of $D_n E^m$ can naturally be regarded as the $\frac{m}{n}$th higher-power shift of the edge-shift of E, this suggests that for irrational l, the C^*-algebra $C^*(\mathcal{S}^l E)$ may be a natural candidate for a C^*-algebraic representation of the lth higher-power shift of the graph E.

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On von Neumann Regularity of Ample Groupoid Algebras

Benjamin Steinberg
(joint work with Daniel van Wyk)

The notion of a regular ring was introduced by von Neumann in the 1930s. A ring R is *regular* if, for all $a \in R$, there exists $b \in R$ with $aba = a$. The class of regular rings enjoys good closure properties and can be characterized elegantly as the rings for which all modules are flat. The notion has since become fundamental in ring theory [3].

Commutative regular rings are rings of compactly supported global sections of sheaves of fields over a locally compact and totally disconnected Hausdorff space. In particular, the ring of locally constant functions with values in a field on a locally compact and totally disconnected Hausdorff space is regular. Connell characterized in the 1960s [2] which group algebras are regular: the group must be locally finite and the characteristic of the field cannot divide the order of any element of the group. Attempts were made to generalize this result to inverse semigroups. The best result to date is due to Okniński [5]. He showed that if K is a field of characteristic 0 and S is an inverse semigroup, then KS is regular if and only if S is locally finite. His proof was analytic and took advantage of the embedding of $\mathbb{C}S$ into the Banach algebra $\ell_1 S$. Abrams and Rangaswamy characterized the Leavitt path algebras which are regular as those coming from acyclic graphs [1], which is quite restrictive.

One can generalize regularity to group-graded rings. A Γ-graded ring $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is *graded regular* if, for each homogeneous element $a \in R_\gamma$, there is $b \in R$ (necessarily in $R_{\gamma^{-1}}$) with $aba = a$. Hazrat proved every Leavitt path algebra is graded regular with respect to its natural $\mathbb{Z}$-grading [4].

Ample groupoid algebras (aka Steinberg algebras) include algebras of locally constant functions on locally compact and totally disconnected Hausdorff spaces, group algebras, inverse semigroup algebras and Leavitt path algebras as special cases. Our main result [6] is that if K is a field of characteristic 0 and $\mathcal{G}$ is an ample groupoid, then $K\mathcal{G}$ is regular if and only if $\mathcal{G}$ is a directed union of quasicompact open subgroupoids. This, in turn, is equivalent to the inverse semigroup of compact open bisections being locally finite. The two key ingredients of the proof are the embedding of $\mathbb{C}\mathcal{G}$ into $C^*_r(\mathcal{G})$, which plays the role of the embedding of $\mathbb{C}S$ into $\ell_1 S$ in Okniński's proof, and a classical result of Birkhoff from universal algebra on the local finiteness of algebras belonging to the variety generated by a finite collection of finite algebras. This generalizes the previous results in characteristic 0. We also have positive results [6] in characteristic $p > 0$, that were not discussed in the talk.

Our second main result [6] says that if $\mathcal{G}$ is an ample groupoid with a locally constant 1-cocycle $c\colon \mathcal{G} \to \Gamma$ to a discrete group Γ, then the algebra $K\mathcal{G}$ is graded regular with respect to the Γ-grading induced by c if and only if the homogeneous component of the identity e, which is the groupoid algebra $Kc^{-1}(e)$, is regular. This allows us to apply the results discussed above to graded regularity. In particular, we recover Hazrat's theorem [4], as well as extend it to higher rank graphs.

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Stein's groups as topological full groups

Owen Tanner

Topological full groups are a way to build simple groups with finiteness properties from ample groupoids. More specifically, I focus on effective ample groupoids with a compact unit space. The topological full group $\mathsf{F}(\mathcal{G})$ is the unital subgroup of the inverse semigroup of compact open bisections. We can think of our topological full group as a group of homeomorphisms of the unit space, since our groupoid is effective.

There are two fundamental background results about topological full groups.

Theorem 1 (Matui's Isomorphism Theorem [2])**.** *Let $\mathcal{G}_1, \mathcal{G}_2$ be minimal, effective, Cantor groupoids. TFAE:*

- $\mathcal{G}_1 \cong \mathcal{G}_2$ *as topological groupoids.*
- $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$ *as abstract groups.*

Theorem 2 (Nekrashevych's Theorem [1])**.** *Let $\mathcal{G}$ be an ample effective Cantor groupoid that is minimal and expansive. Then, the derived subgroup of $\mathsf{F}(\mathcal{G})$ is simple and finitely generated.*

Recently, I have become interested in a class of topological full groups called Stein's groups, which are certain generalisations of Thompson's group V [3].

Definition 3. Let Λ be a subgroup of the positive real numbers with respect to multiplication, and let ℓ be a positive real number. Then Stein's group $V(\Lambda, \ell)$ is the group of piecewise linear, right continious bijections of $[0, \ell]$ with finitely many slopes, all in Λ, and finitely many nondifferentiable points, all in $\mathbb{Z}[\Lambda]$.

I was able to prove the following in my preprint on the ArXiV [4]. The first nontrivial thing to show is that these admit a nice groupoid model. The groupoid model is a certain full corner in Patersons universal groupoid of the cancellative semigroup formed of the semidirect product of $\mathbb{Z}[\Lambda] \cap [0, +\infty)$ by Λ where the action is by multiplication. Next I show via Nekrashevych's theorem:

Theorem 4. *$D(V(\Lambda, \ell))$ is finitely generated iff Λ is finitely generated.*

In this talk, I make use of Matui's isomorphism theorem and the diversity of the groupoid models to explain 3 interesting cases in greater detail:

- The Higman-Thompson groups $V_{k,r} \cong V(\langle k \rangle, r)$ where k, r are natural numbers, are associated to certain graph groupoids. Namely, the graph groupoid of the rose with k petals and a *stem* of length $r - 1$. It is mentioned that via this connection, and via Leavitt path algebras, Pardo completed the classification of these groups [5].
- In the case where Λ is generated by $k > 1$ integers, we can associate $V(\Lambda, 1)$ to a certain class of single-vertex k-graphs. This allows us to show that these examples of Stein's groups are always rationally acyclic, appealing to the homology framework of Li. An open question is mentioned– the classification of these groups is open.

- The case where Λ is generated by an irrational number, which is interesting from the perspective of homology because it is not always a rationally acyclic, or even virtually simple group.

We hope to inspire interest in this class of groups, and the connection via groupoids to combinatorial *-algebras.

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Functoriality for étale groupoid algebras

Jonathan Taylor

With the exception of some particular classes, continuous functors between groupoids do not generally lift to homomorphisms of their (C^*-)algebras. For (discrete) groups, group homomorphisms lift covariantly to homomorphisms of their associated algebras, whereas continuous maps between topological spaces lift contravariantly via pullback to their algebras of continuous functions. Viewing groupoids as a hybrid of these two concepts with morphisms that generalise both these listed above, any way of lifting morphisms to homomorphisms of the groupoid algebras needs to be simultaneously covariant along source/range fibres in the groupoid and contravariant along the unit space. The two main classes of groupoid functor that satisfy this are open embeddings (where one may use the continuous partial inverse to reverse the direction of arrows on the unit space), and fibrewise-bijective continuous functors (allowing one to reverse the direction of arrows along the fibres).

Buneci and Stachura [1] introduced a morphism between groupoids as an alternative to considering functors. A morphism from a groupoid G to a groupoid H consists of a left action of G on H that commutes with the right multiplication of H on itself. Buneci and Stachura showed that such morphisms induce *-homomorphisms from the C^*-algebra of G to the multiplier algebra of the C^*-algebra of H via a convolution formula using the action multiplication in place of the multiplication in H. We call such morphisms *actors*. Buneci and Stachura showed that actors lift functorially to *-homomorphisms of the groupoid C^*-algebras, providing at least one answer to the question of functoriality for groupoid C^*-algebras. Moreover, the *-homomorphisms induced by open embeddings and fibrewise bijective continuous functors arise as special cases of actors, unifying two of the main approaches to constructing *-homomorphisms between groupoid C^*-algebras.

When considering étale groupoids and their C^*-algebras, we gain some extra structure. The algebra of continuous functions on the unit space of an étale groupoid embeds into the groupoid C^*-algebra, and there is a canonical conditional expectation from the groupoid C^*-algebra to this subalgebra of functions. One may then consider *-homomorphisms that also preserve this structure, that is, *-homomorphisms between étale groupoid C^*-algebras that intertwine the conditional expectations. An actor that is *free*, that is, an actor where only units act trivially, will induce such a *-homomorphism. Moreover, if the domain groupoid for the actor is Hausdorff, then the induced *-homomorphism intertwines expectations if and only if the actor is free. If the groupoids concerned are effective, and so lift to Cartan pairs of C^*-algebras with the canonical subalgebras of functions on the unit spaces, then any non-degenerate *-homomorphism between these two groupoid C^*-algebras that preserves the Cartan structure then arises from an actor of the underlying groupoids. This leads to an equivalence of categories between effective étale groupoids and the C^*-pairs they induce (cf. [3], [4]). By extending the construction of actors to twists over effective groupoids, this equivalence can be extended one between the category of twist over effective groupoids and all Cartan pairs.

In his seminal paper [2], Li demonstrated that the inductive limit of Cartan pairs is again a Cartan pair. In particular, inductive limits of these groupoid C^*-algebras are again groupoid C^*-algebras. The main methodology used to show this involves reconstructing *-homomorphisms between Cartan pairs as a combination of those arising from open inclusions and proper fibrewise-bijective functors between the underlying groupoids, with an intermediate groupoid filling in part of the resulting zig-zag. By viewing the *-homomorphism as arising from an actor between the underlying groupoids, the intermediate groupoid defined by Li can be identified with the transformation groupoid associated to the actor, and the construction of the inductive limit groupoid can be framed in this language too. Using this heuristic, given an inductive system of free actors, one may construct the inductive limit groupoid mimicking the construction of Li for Cartan pairs. This inductive limit groupoid is the colimit object for this diagram in the category of groupoids with actors. Moreover, the functor to C^*-algebras will lift this colimit groupoid to the colimit of the inductive system of C^*-algebras induced by the system of actors.

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Porcupine-quotient graphs, the fourth primary color, and some updates about the Graded Classification Conjecture

Lia Vaš

If E is a directed graph and K a field the Leavitt path algebra $L_K(E)$ is naturally graded by the group of integers. The lattice of graded $L_K(E)$-ideals corresponds to the lattice of pairs of certain sets of vertices called the admissible pairs. The ideal $I(H,S)$ corresponding to an admissible pair (H,S) is graded isomorphic to the Leavitt path algebra of a graph introduced in [2] which is called the porcupine graph. The porcupine graph resembles the older construction of a hedgehog graph ([1, Definitions 2.5.16 and 2.5.20]) except that the "spines" added to the "body" determined by $H \cup S$ are longer, so the name "porcupine" was chosen to reflect that. While the Leavitt path algebra of the hedgehog of (H,S) is isomorphic to $I(H,S)$, this isomorphism does not have to be graded. In contrast, the Leavitt path algebra of the porcupine of (H,S) is *graded* isomorphic to $I(H,S)$.

One can also define the quotient graph $E/(H,S)$ ([1, Definition 2.4.14]) in such a way that the quotient $L_K(E)/I(H,S)$ is graded isomorphic to the Leavitt path algebra of $E/(H,S)$. We present a construction from [3] which generalizes both the porcupine and the quotient graph constructions and enables one to represent the quotient of two graded ideals as the Leavitt path algebra of this newly defined graph. Specifically, if (H,S) and (G,T) are admissible pairs such that $(H,S) \leq (G,T)$ (in the sense which corresponds exactly to $I(H,S) \subseteq I(G,T)$), we define the porcupine-quotient graph $(G,T)/(H,S)$ and show that its Leavitt path algebra is graded isomorphic to the quotient $I(G,T)/I(H,S)$.

We also consider two pre-ordered monoids, M_E and M_E^Γ, originated in relation to some classification questions. The *graph monoid* M_E is isomorphic to the monoid $\mathcal{V}(L_K(E))$ of the isomorphism classes of finitely generated projective modules. The natural grading of a Leavitt path algebra induces an action of the infinite cyclic group $\Gamma = \langle t \rangle \cong \mathbb{Z}$ on the graded isomorphism classes of finitely generated graded projective $L_K(E)$-modules and there is a Γ-isomorphism of the monoid $\mathcal{V}^\Gamma(L_K(E))$ of such graded isomorphism classes and the monoid M_E^Γ, also known as the *talented monoid* or the *graph Γ-monoid*. In particular, the following lattices are isomorphic: the lattice of order-ideals of M_E, the lattice of Γ-order-ideals of M_E^Γ, the lattice of graded ideals of $L_K(E)$, and the lattice of admissible pairs of E. If $(G,T)/(H,S)$ is the porcupine-quotient graph, $M_{(G,T)/(H,S)}$ is isomorphic to the quotient of the order-ideals corresponding to (G,T) and (H,S) and $M^\Gamma_{(G,T)/(H,S)}$ is isomorphic to the quotient of the Γ-order-ideals corresponding to (G,T) and (H,S).

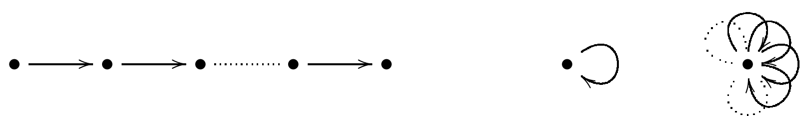
We say that $L_K(E)$ has a graded composition series if there is a finite and increasing chain of graded ideals, starting with the trivial ideal and ending with the improper ideal, such that the quotient of each two consecutive ideals is graded simple. Since a Leavitt path algebra is graded simple if and only if the underlying graph is cofinal, the porcupine-quotient construction enables us to relate the existence of a graded composition series of $L_K(E)$ with the existence of a finite and increasing chain of admissible pairs, starting with the trivial pair and ending with the improper pair, such that the porcupine-quotient of two consecutive pairs is cofinal. If such a chain exists, we say that E has a composition series. For any graph, the following conditions are equivalent.

(1) E has a composition series.
(2) $L_K(E)$ has a graded composition series.
(3) M_E has a composition series.
(4) M_E^Γ has a composition series.

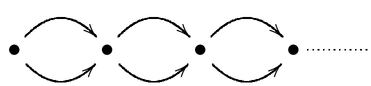
We characterize the existence of the above composition series by a set of conditions on E which can be directly checked and which produce a specific composition series. In order to obtain this result, we start by introducing a type of vertices which are "terminal" in the same sense as the vertices of any of the three types below.

(1) A *sink* is a vertex which emits no edges. A sink connects to no other vertex in the graph except, trivially, to itself.
(2) A *cycle without exits* is a cycle whose vertices emit only one edge to another vertex in the cycle. The vertices in such a cycle do not connect to any vertices outside of the cycle.
(3) An *extreme cycle* is a cycle such that the range of every exit from the cycle connects back to a vertex in the cycle. The vertices in such a cycle c connect only to the vertices on cycles in the same "cluster" as c.

The significance of these three groups of vertices lies in the fact that the Leavitt path algebra of a finite graph is *graded simple* exactly when there is a unique "cluster" of vertices of one of the three types above. Because of this, the three graphs below are the three quintessential examples of graphs with the above three types of vertices. The authors of [1] refer to the Leavitt path algebras of these three graphs as the *three primary colors* of Leavitt path algebras.



However, if the graph is not finite, its Leavitt path algebra can be graded simple without having exactly one cluster of the three types of vertices as above. For example, the Leavitt path algebras of the graph below is graded simple and the graph has neither cycles nor sinks.



We introduce terminal paths as the infinite paths whose vertices are terminal in the same sense as the above three types. According to this definition, every infinite path of the above graph is terminal. Then, we characterize graded simplicity of a Leavitt path algebra $L_K(E)$ by a set of conditions on E which are direct to check and which are given in terms of the existence of exactly one cluster of the four types of terminal vertices. The existence of the fourth type does not contradict the Trichotomy Principle ([1, Proposition 3.1.14]), but it refines it: it distinguishes between sinks and terminal paths.

Using the previous results, we present a set of conditions on E which are equivalent with E having a composition series. Such conditions are constructive in the following sense: given a graph, one can construct a chain of admissible pairs such that the porcupine-quotient graphs of two consecutive pairs are cofinal and check if such a chain terminates after finitely many steps. Informally, such a chain is obtained by iteratively cutting the terminal vertices (and their breaking sets if E is not row-finite). A direct corollary of this characterization is that every unital Leavitt path algebra has a graded composition series.

The last portion of the talk contained some new developments, soon to be submitted for publication, regarding the the Graded Classification Conjecture (GCC). The GCC states that the pointed K_0^{gr}-group is a complete invariant of the Leavitt path algebras of finite graphs when these algebras are considered with their natural grading by $\mathbb{Z}$. The conjecture has been shown to hold in some special cases including finite graphs such that each cycle has no exits (edges leaving the cycle). We prove that the conjecture holds for graphs with cycles which can have exits – we consider graphs with a composition series such that each composition factor is a cofinal graph with either a unique sink or a unique cycle without exits. This class of graphs coincides with the class of graphs with disjoint cycles, finitely many infinite emitters, sinks and cycles and such that every infinite path ends in a cycle. We say that a graph in this class is a composition S-NE graph (where S stands for sink and NE for no-exits). For the main result, we also require infinite emitters of a graph to emit only countably many edges. In particular, our result holds for finite graphs with disjoint cycles (the Toeplitz graph is such, for example) and we formulate it also for the graph C^*-algebras. As a consequence of this result, the Isomorphism Conjecture also holds for the class of graphs we consider.

For S-NE graphs we regard, we show that the two conditions from the GCC are equivalent also with a condition expressed only in terms of properties of graphs. In particular, we introduce an invariant we refer to as the S-NE-invariant. Featuring this invariant, our main result does more than prove the GCC for the class of graph we consider – it describes the graded (*-)isomorphism class of a graph from this class. Such description is relevant for the active program of classification of the graph C^*-algebras. The S-NE invariant and the methods of our proof indicate that the GCC should generally be considered also with the graph-theoretic condition and that using the length of a composition series for induction is promising when attempting to prove the GCC for other types of graphs.

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