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On a Degenerating Limit Theorem of DeMarco–Faber

by

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Abstract

One of our aims is to complement the proof of DeMarco–Faber's degenerating limit theorem for the family of the unique maximal entropy measures parametrized by a punctured open disk associated to a meromorphic family of rational functions on the complex projective line, degenerating at the puncture. This complementation is done by our main result, which rectifies a key computation in their argument. We also establish and use a direct and explicit translation from degenerating complex dynamics into quantized Berkovich dynamics, instead of using DeMarco–Faber's more conceptual transfer principle between those dynamics.

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§1. Introduction

Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value. The action of a rational function $h \in K(z)$ on $\mathbb{P}^1 = \mathbb{P}^1(K)$ extends continuously to that on the Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$, which is a compact augmentation of $\mathbb{P}^1$. If in addition $\deg h > 0$, then this extended action of h on P^1 is surjective, open, and fiber-discrete and preserves the type (among I, II, III, and IV) of each point in P^1, and the local degree function $\deg_{\cdot} h$ of h on $\mathbb{P}^1$ also extends upper semicontinuously to P^1 so that for every open subset V in P^1 and every component U of $h^{-1}(V)$, $V \ni \mathcal{S}' \mapsto \sum_{\mathcal{S} \in h^{-1}(\mathcal{S}') \cap U} \deg_{\mathcal{S}} h \equiv \deg(h\colon U \to V)$.

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The pushforward operator $h_*\colon C^0(\mathsf{P}^1)\to C^0(\mathsf{P}^1)$ is defined so that for every $\psi\in C^0(\mathsf{P}^1)$, $(h_*\psi)(\cdot):=\sum_{\mathcal{S}\in h^{-1}(\cdot)}(\deg_{\mathcal{S}}h)\psi(\mathcal{S})$ on P^1. The pullback operator h^* from the space $M(\mathsf{P}^1)$ of all Radon measures on P^1 to itself is defined by the transpose of h_*, so that for every $\nu\in M(\mathsf{P}^1)$,

$$h^*\nu=\int_{\mathsf{P}^1}\Bigl(\sum_{\mathcal{S}'\in h^{-1}(\mathcal{S})}(\deg_{\mathcal{S}'}h)\delta_{\mathcal{S}'}\Bigr)\nu(\mathcal{S})\quad\text{on }\mathsf{P}^1,\tag{1.1}$$

where for each point $\mathcal{S}\in\mathsf{P}^1$, $\delta_{\mathcal{S}}$ is the Dirac measure at $\mathcal{S}$ on P^1; in particular, $(h^*\delta_{\mathcal{S}})(\mathsf{P}^1)=\deg h$.

§1.1. Factorization on P^1 and quantization

We follow the presentation in [5, §4.2]. For each finite subset Γ consisting of type II points (e.g., a semistable vertex set) in P^1, the family

$$S(\Gamma):=\bigl\{\text{either a component of }\mathsf{P}^1\setminus\Gamma\text{ or a singleton }\{\mathcal{S}\}\text{ for some }\mathcal{S}\in\Gamma\bigr\}\subset 2^{\mathsf{P}^1}$$

is a partition of P^1; the measurable factor space $\mathsf{P}^1/S(\Gamma)=S(\Gamma)$ equipped with the σ-algebra $2^{S(\Gamma)}$ is regarded as the measurable space $(\mathsf{P}^1,2^{S(\Gamma)})$, also regarding $2^{S(\Gamma)}$ as a σ-subalgebra in the Borel σ-algebra on P^1.

Let $M(\Gamma)$ be the set of all complex measures ω on $\mathsf{P}^1/S(\Gamma)$. The measurable factor map

$$\pi_\Gamma=\pi_{\mathsf{P}^1,\Gamma}\colon\mathsf{P}^1\to\mathsf{P}^1/S(\Gamma)$$

induces the pullback operator $(\pi_\Gamma)^*$ from the space of measurable functions on $\mathsf{P}^1/S(\Gamma)$ to that of measurable functions on P^1 and, in turn, the transpose (projection operator) $(\pi_\Gamma)_*\colon M(\mathsf{P}^1)\to M(\Gamma)$ of $(\pi_\Gamma)^*$ (by restricting each element of $M(\mathsf{P}^1)$ to $2^{S(\Gamma)}$), so in particular that for every $\nu\in M(\mathsf{P}^1)$,

$$((\pi_\Gamma)_*\nu)(\{U\})=\nu(U)\quad\text{for any }U\in S(\Gamma).\tag{1.2}$$

Set $M^1(\mathsf{P}^1):=\{\omega\in M(\mathsf{P}^1):\omega\ge 0\text{ and }\omega(\mathsf{P}^1)=1\}$ and $M^1(\Gamma):=\{\omega\in M(\Gamma):\omega\ge 0\text{ and }\omega(\mathsf{P}^1/S(\Gamma))=1\}$, so that $(\pi_\Gamma)_*(M^1(\mathsf{P}^1))\subset M^1(\Gamma)$. Also set

$$M^1(\Gamma)^\dagger:=\bigl\{\omega\in M^1(\Gamma):\omega(\{\mathcal{S}\})=0\text{ for every }\mathcal{S}\in\Gamma\bigr\}.$$

For any finite subsets Γ and Γ', $\Gamma\subset\Gamma'$, both consisting of type II points, the measurable factor map

$$\pi_{\Gamma',\Gamma}\colon\mathsf{P}^1/S(\Gamma')\to\mathsf{P}^1/S(\Gamma)$$

induces the pullback operator $(\pi_{\Gamma',\Gamma})^*$ from the space of measurable functions on $\mathsf{P}^1/S(\Gamma)$ to that of measurable functions on $\mathsf{P}^1/S(\Gamma')$ (so that $\pi_\Gamma^*=(\pi_{\Gamma'})^*(\pi_{\Gamma',\Gamma})^*$)

and, in turn, the transpose (or projection operator) $(\pi_{\Gamma',\Gamma})_*\colon M(\Gamma') \to M(\Gamma)$ of $(\pi_{\Gamma',\Gamma})^*$, so in particular that for every $\omega \in M(\Gamma')$,

$$((\pi_{\Gamma',\Gamma})_*\omega)(\{U\}) = \omega(\{V \in S(\Gamma') : V \subset U\}) \quad \text{for any } U \in S(\Gamma), \tag{1.3}$$

and that $(\pi_{\Gamma',\Gamma})_*(\pi_{\Gamma'})_* = (\pi_\Gamma)_*$. Then $(\pi_{\Gamma',\Gamma})_*(M^1(\Gamma')^\dagger) \subset M^1(\Gamma)^\dagger$.

Let us denote by $\mathcal{S}_G$ the Gauss (or canonical) point in P^1, which is a type II point (see Section 2.1). For a rational function $h \in K(z)$ on $\mathbb{P}^1$ of degree > 0, noting that $h(\mathcal{S}_G)$ is also a type II point and setting

$$\Gamma_G := \{\mathcal{S}_G\} \quad \text{and} \quad \Gamma_h := \{\mathcal{S}_G, h(\mathcal{S}_G)\},$$

the quantized pullback operator $h^*_G\colon M(\Gamma_h) \to M(\Gamma_G)$ is induced from the pullback operator h^* in (1.1); for every $\omega \in M(\Gamma_h)$, the measure $h^*_G\omega \in M(\Gamma_G)$ in particular satisfies

$$(h^*_G\omega)(\{U\}) = \int_{\mathsf{P}^1/S(\Gamma_h)} m_{V,U}(h)\omega(V) \quad \text{for any } U \in S(\Gamma_G),$$

where the quantized local degree $m_{V,U}(h)$ of h with respect to each pair $(U, V) \in S(\Gamma_G) \times S(\Gamma_h)$ is induced from the local degree function $\deg_{\cdot} h$ on P^1 so that, fixing any $\mathcal{S}' \in V$,

$$m_{V,U}(h) = \begin{cases} (h^*\delta_{\mathcal{S}'})(U) & \text{if } U \in S(\Gamma_G) \setminus \{\{\mathcal{S}_G\}\} \text{ and } V \in S(\Gamma_h) \setminus \{\{h(\mathcal{S}_G)\}\}, \\ (h^*\delta_{\mathcal{S}'})(\{\mathcal{S}_G\}) & \text{if } U = \{\{\mathcal{S}_G\}\} \end{cases}$$

(the remaining case that $U \in S(\Gamma_G) \setminus \{\{\mathcal{S}_G\}\}$ and $V = \{\{h(\mathcal{S}_G)\}\}$ is more subtle) and that for every $V \in S(\Gamma_h)$, $\sum_{U \in S(\Gamma_G)} m_{V,U}(h) = \deg h$. In particular,

$$(h^*_G\omega)(S(\Gamma_G)) = (\deg h) \cdot \omega(S(\Gamma_h)) \quad \text{for every } \omega \in M(\Gamma_h), \quad \text{and}$$
$$((\deg h)^{-1} h^*_G)(M^1(\Gamma_h)^\dagger) \subset M^1(\Gamma_G)^\dagger$$

(see Section 2.5 for more details, including the precise definitions of $m_{V,U}(h)$ and h^*_G).

§1.2. The f-balanced measures on P^1 and the maximal-ramification locus of f in P^1

From now on, let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of $\deg f =: d > 1$.

The equilibrium (or canonical) measure ν_f of f on P^1 is the weak limit

$$\nu_f := \lim_{n\to\infty} \frac{(f^n)^*\delta_{\mathcal{S}}}{d^n} \quad \text{in } M(\mathsf{P}^1) \text{ for any } \mathcal{S} \in \mathsf{P}^1 \setminus E(f) \tag{1.4}$$

(see [10] for the details), and is the unique $\nu \in M^1(\mathsf{P}^1)$ not only having the f-balanced property

$$f^*\nu = (\deg f) \cdot \nu \quad \text{on } \mathsf{P}^1,$$

but also satisfying the vanishing condition $\nu(E(f)) = 0$. Here, the (classical) exceptional set $E(f) := \{a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(a) < +\infty\}$ of f is the union of all (superattracting) cycles of f in $\mathbb{P}^1$ totally invariant under f (so is at most countable).

The ramification locus $\mathsf{R}(f) := \{\mathcal{S} \in \mathsf{P}^1 : \deg_{\mathcal{S}} f > 1\}$ of f contains the (classical) critical set $\mathrm{Crit}(f) := \{c \in \mathbb{P}^1 : f'(c) = 0\}$ of f, and the maximal-ramification locus

$$\mathsf{R}_{\max}(f) := \{\mathcal{S} \in \mathsf{P}^1 : \deg_{\mathcal{S}}(f) = d\} (\subset \mathsf{R}(f))$$

of f contains $E(f)\,(\subset \mathrm{Crit}(f))$; since $\mathsf{R}_{\max}(f)$ is connected (Faber [7, Thm. 8.2]), for every $c \in \mathsf{R}_{\max}(f) \cap \mathbb{P}^1$, $\mathsf{R}_{\max}(f)$ near c contains a closed interval $[c, \mathcal{S}]$ (see Section 2.1) in P^1 for some $\mathcal{S} \in \mathsf{P}^1 \setminus \{c\}$.

Definition 1.1 (Tame maximal-ramification). For each $c \in \mathsf{R}_{\max}(f) \cap \mathbb{P}^1$, we say f is tamely maximally ramified near c if $\mathsf{R}_{\max}(f)$ near c is a closed interval $[c, \mathcal{S}]$ in P^1 for some $\mathcal{S} \in \mathsf{P}^1 \setminus \{c\}$.

Fact 1.2 (Consequence of Faber [7, Cor. 6.6]). The function f is tamely maximally ramified at every $c \in \mathsf{R}_{\max}(f) \cap \mathbb{P}^1$ if the residue characteristic of K is either $= 0$ or $> d\,(= \deg f)$ (e.g., when $K = \mathbb{L}$ as in Section 1.4 below).

We note that when $\mathrm{char}\, K = 0$,

$$E(f) = \{a \in \mathbb{P}^1 : f^{-2}(a) = \{a\}\} \quad \text{and} \quad \# E(f) \le \#(\mathsf{R}_{\max}(f) \cap \mathbb{P}^1) \le 2. \tag{1.5}$$

The Berkovich Julia set $\mathsf{J}(f) := \mathrm{supp}\,\nu_f$ of f is in $\mathsf{P}^1 \setminus E(f)$ (by (1.4)); both $\mathsf{J}(f)$ and $E(f)$ are f-completely invariant. Any $\nu \in M^1(\mathsf{P}^1)$ (only) having the above f-balanced property on P^1 is written as

$$\nu = \nu(\mathsf{J}(f)) \cdot \nu_f + \sum_{\substack{\mathcal{E} \subset E(f): \\ \text{a cycle of } f}} \nu(\mathcal{E}) \cdot \frac{\sum_{a \in \mathcal{E}} \delta_a}{\# \mathcal{E}} \quad \text{on } \mathsf{P}^1$$

(by (1.4) and the countability of $E(f)$). For every $n \in \mathbb{N}$, we also have $\nu_{f^n} = \nu_f$ in $M^1(\mathsf{P}^1)$ (so $\mathsf{J}(f^n) = \mathsf{J}(f)$) and $E(f^n) = E(f)$.

Recall that for any $\mathcal{S} \in \mathsf{H}^1 := \mathsf{P}^1 \setminus \mathbb{P}^1$,

$$\begin{aligned} f^{-1}(\mathcal{S}) \neq \{\mathcal{S}\} \Leftrightarrow \nu_f(\{\mathcal{S}\}) < 1 \Leftrightarrow \mathrm{supp}(\nu_f) \neq \{\mathcal{S}\} \\ \Leftrightarrow \nu_f(\{\mathcal{S}\}) = 0 \Leftrightarrow \nu_f(\{f(\mathcal{S})\}) = 0 \end{aligned} \tag{1.6}$$

(see e.g., [2, Cor. 10.33]), so in particular, $f^{-1}(\mathcal{S}) \neq \{\mathcal{S}\}$ if and only if $f^{-n}(\mathcal{S}) \neq \{\mathcal{S}\}$ for every $n \in \mathbb{N}$. For every $\nu \in M^1(\mathsf{P}^1)$ having the f-balanced property on P^1 and every finite subset Γ in P^1 consisting of type II points, we have

$$((\pi_\Gamma)_*\nu)(S(\Gamma) \setminus F) = 0 \quad \text{for some countable subset } F \text{ in } S(\Gamma) \tag{1.7}$$

(by (1.4)) and

$$(\pi_\Gamma)_*\nu \in M^1(\Gamma)^\dagger \quad \text{if in addition } f^{-1}(\mathcal{S}) \neq \{\mathcal{S}\} \text{ for every } \mathcal{S} \in \Gamma. \tag{1.8}$$

§1.3. Main result: The projections of the f-balanced measures on P^1 to $\mathsf{P}^1/S(\Gamma_G)$

Recall that $d \coloneqq \deg f > 1$ and that $\Gamma_G \coloneqq \{\mathcal{S}_G\}$, and for each $n \in \mathbb{N}$, set

$$\Gamma_n \coloneqq \Gamma_{f^n} = \{\mathcal{S}_G, f^n(\mathcal{S}_G)\}.$$

Let us say $\omega \in M^1(\Gamma_f)$ has the quantized f-balanced property if

$$f_G^*\omega = d \cdot (\pi_{\Gamma_f,\Gamma_G})_*\omega \quad \text{in } M^1(\Gamma_G). \tag{1.9}$$

Set $\Delta_f \subset M^1(\Gamma_G)$ (resp. $\Delta_f^\dagger \subset M^1(\Gamma_G)^\dagger$) as

$$\begin{aligned} &\Delta_f \text{ (resp. } \Delta_f^\dagger) \\ &\coloneqq \Big\{\omega \in M^1(\Gamma_G) : \text{for (any) } n \gg 1, \text{ there is } \omega_n \in M^1(\Gamma_n) \text{ (resp. } \omega_n \in M^1(\Gamma_n)^\dagger) \\ &\qquad \text{such that } \omega_n(S(\Gamma_n) \setminus F) = 0 \text{ for some countable subset } F \text{ in } S(\Gamma_n) \text{ and} \\ &\qquad\qquad \text{that } d^{-n}((f^n)_G)^*\omega_n = \omega = (\pi_{\Gamma_n,\Gamma_G})_*\omega_n \text{ in } M^1(\Gamma_G)\Big\}; \end{aligned} \tag{1.10}$$

for a subtlety on the first vanishing assumption on each ω_n, see Remark 5.3.

Our principal result is the following computations of Δ_f and $\Delta_f^\dagger$ when $\operatorname{char} K = 0$, which in particular rectifies [5, Thm. 4.10, Cor. 4.13]; the assumption on the period of each $a \in E(f)$ is for simplicity, and f^2 always satisfies this condition, and the tame maximal-ramification condition for f near a in the case (ii) to obtain (1.11) below always holds when $K = \mathbb{L}$ as in Section 1.4.

Theorem A. *Let K be an algebraically closed field of characteristic 0 that is complete with respect to a non-trivial and non-archimedean absolute value, let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and suppose that $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ and that $f(a) = a$ (or equivalently $f^{-1}(a) = \{a\}$) for any $a \in E(f)$. Then one and only one of the following cases* (i) *and* (ii) *occurs:*

(i) $\Delta_f = \Delta_f^\dagger = \{(\pi_{\Gamma_G})_*\nu_f\}$;

(ii) *there is a (unique)* $a \in E(f)$ *such that* $\lim_{n\to\infty} f^n(\mathcal{S}_G) = a$ *and that* $f^n(\mathcal{S}_G)$ *is in the interval* $(\mathcal{S}_G, a]$ *in* P^1 *for* $n \gg 1$, *and then* $\deg_{f^n(\mathcal{S}_G)}(f) \equiv d$ *for* $n \gg 1$, *and* $\{(\pi_{\Gamma_G})_*\nu_f\} \subsetneq \{(\pi_{\Gamma_G})_*\delta_a, (\pi_{\Gamma_G})_*\nu_f\} \subset \Delta_f^\dagger$.

In the case (ii), *if in addition* f *is tamely maximally ramified near* a, *then*

(1.11)
$$\Delta_f = \Big\{\omega \in M^1(\Gamma_G) : \text{satisfying} \begin{cases} \omega(\{U_{\vec{v}}\}) = s\nu_f(U_{\vec{v}}) \text{ for every } \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}, \\ \omega(\{\{\mathcal{S}_G\}\}) = s', \text{ and} \\ \omega(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) = (s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s)) - s' \end{cases}$$
$$\text{for some } s \in [0,1] \text{ and some } s' \in \big[0, \min\{s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}), (1-s)(1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}))\}\big]\Big\},$$

which in particular yields

$$\Delta_f^\dagger = \big\{s \cdot (\pi_{\Gamma_G})_*\nu_f + (1-s) \cdot (\pi_{\Gamma_G})_*\delta_a : s \in [0,1]\big\},$$

and moreover, the three statements

- $\deg_{f^n(\mathcal{S}_G)}(f) \equiv d$ *(i.e.,* $f^n(\mathcal{S}_G) \in \mathsf{R}_{\max}(f)$*) for any* $n \in \mathbb{N} \cup \{0\}$,
- $\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) = 0$, *and*
- $\Delta_f = \Delta_f^\dagger$

are equivalent.

In the proof of Theorem A, we will also point out that for some f (indeed $f(z) = z^2 + t^{-1}z \in (\mathcal{O}(\mathbb{D})[t^{-1}])[z]$ $(\subset \mathbb{L}[z])$ and its iterations), we have the proper inclusion $\Delta_f^\dagger \subsetneq \Delta_f$.

§1.4. Application: The degenerating weak limit for the maximal entropy measures on $\mathbb{P}^1(\mathbb{C})$

We call an element $f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ of degree say $d \in \mathbb{N} \cup \{0\}$ a meromorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$ (of degree d and parametrized by

$$\mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$$

if for every $t \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$, the specialization f_t of f at t is a rational function on $\mathbb{P}^1(\mathbb{C})$ of degree d. Let us denote by $\mathbb{L}$ the (algebraically closed and complete) valued field of formal Puiseux series/$\mathbb{C}$ around $t = 0$,[1] i.e., the completion of the field $\overline{\mathbb{C}((t))}$ of Puiseux series/$\mathbb{C}$ around $t = 0$ valuated by their vanishing orders at $t = 0$. Noting that $\mathcal{O}(\mathbb{D})[t^{-1}]$ is a subring of the field $\mathbb{C}((t))$ of Laurent series/$\mathbb{C}$

[1]The terminology "formal Puiseux series" might be informal. The field $\mathbb{L}$ is known as the Levi-Civita field.

around $t = 0$, we also regard f as an element of $\mathbb{L}(z)$. If in addition $d > 1$, then for every $t \in \mathbb{D}^*$, there is the equilibrium (or canonical, and indeed the unique maximal entropy) measure μ_{f_t} of f_t on $\mathbb{P}^1(\mathbb{C})$ (see Fact 3.2). As already seen in Section 1.2, there is also the equilibrium (or canonical) measure ν_f of the $f \in \mathbb{L}(z)$ of degree $d > 1$ on $\mathsf{P}^1(\mathbb{L})$.

If in addition $\nu_f(\{\mathcal{S}_G\}) = 0$ or equivalently $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ in $\mathsf{P}^1(\mathbb{L})$ (mentioned in (1.6); see also another equivalent condition (2.2) below), then recalling that $\Gamma_G \coloneqq \{\mathcal{S}_G\}$ as in Section 1.1 and noting that

$$S(\Gamma_G) \setminus \{\{\mathcal{S}_G\}\} = T_{\mathcal{S}_G}(\mathsf{P}^1(\mathbb{L})) \cong \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}),$$

where $k_{\mathbb{L}}$ ($= \mathbb{C}$ as fields) is the residue field of $\mathbb{L}$ and where the bijection between the tangent (or directions) space $T_{\mathcal{S}_G}(\mathsf{P}^1(\mathbb{L}))$ of $\mathsf{P}^1(\mathbb{L})$ at $\mathcal{S}_G$ and $\mathbb{P}^1(k_{\mathbb{L}})$ is given by $\overrightarrow{\mathcal{S}_G a} \leftrightarrow \tilde{a}$ for each $a \in \mathbb{P}^1(\mathbb{L})$ (see Section 2.2 for the reduction $\tilde{a} \in \mathbb{P}^1(k_{\mathbb{L}})$ of a), the projection $(\pi_{\Gamma_G})_*\nu_f \in M^1(\Gamma_G)^\dagger$ of $\nu_f \in M_1(\mathsf{P}^1(\mathbb{L}))$ is also regarded as a purely atomic probability measure on $\mathbb{P}^1(\mathbb{C})$ (by (1.8)).

Using Theorem A and by some new arguments relating the absolute value on $\mathbb{L}$, which is an extension of the trivial (so non-archimedean) absolute value on $\mathbb{C} = k_{\mathbb{L}}$, with the (archimedean and non-trivial) Euclidean absolute value on $\mathbb{C}$, we complement the proof of the following degenerating limit theorem of DeMarco–Faber.

Theorem B ([5, Thm. B]). *For every meromorphic family*

$$f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)\ (\subset \mathbb{L}(z))$$

of rational functions on $\mathbb{P}^1(\mathbb{C})$ *of degree* > 1, *if* $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ *in* $\mathsf{P}^1(\mathbb{L})$, *then*

$$\lim_{t \to 0} \mu_{f_t} = (\pi_{\Gamma_G})_*\nu_f \quad \textit{weakly on } \mathbb{P}^1(\mathbb{C}). \tag{1.12}$$

We dispense with the intermediate “target bimeromorphically modified surface dynamics” part in the (conceptual) “transfer principle” from degenerating complex dynamics to quantized Berkovich dynamics in [5, Proof of Theorem B], and give and use a more direct and explicit translation from degenerating complex dynamics into quantized Berkovich dynamics (see Definition 4.3 and Proposition 4.4). We hope our argument could also be helpful for a further investigation of degenerating complex dynamics (see, e.g., [9, 6]).

Organization of the paper

In Sections 2 and 3, we recall some notions and facts from non-archimedean dynamics on P^1 and also recall some details on DeMarco–Faber’s degenerating balanced property for degenerating weak limit points of the maximal entropy measures on $\mathbb{P}^1(\mathbb{C})$, respectively. Section 4 is one of the main parts in this paper, as mentioned in the above paragraph. Theorem A is shown in Section 5, and our proof of Theorem B is given in Section 6. In Section 7, a specific example, which motivated our computation of Δ_f (and $\Delta_f^\dagger$) in Theorem A, is discussed. In Section 8, we further develop our direct translation from degenerating complex dynamics into quantized Berkovich dynamics, for completeness.

§2. Background from Berkovich dynamics

Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$.

§2.1. Berkovich projective line

We call $B(a,r) := \{z \in K : |z-a| \le r\}$ for some $a \in K$ and some $r \in \mathbb{R}_{\ge 0}$ a K-closed disk; for any K-closed disks B, B', if $B \cap B' \neq \emptyset$, then either $B \subset B'$ or $B \supset B'$. The Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ over K is a compact, uniquely arcwise connected, locally arcwise connected, and Hausdorff topological space; as sets,

$$\begin{aligned} \mathsf{P}^1 &= \mathbb{P}^1 \cup \mathsf{H}^1 = \mathbb{P}^1 \cup \mathsf{H}^1_{\mathrm{II}} \cup \mathsf{H}^1_{\mathrm{III}} \cup \mathsf{H}^1_{\mathrm{IV}} \quad \text{(the disjoint unions)},\\ \mathbb{P}^1 &= \mathbb{P}^1(K) = K \cup \{\infty\} \cong \big\{\{a\} = B(a,0) : a \in K\big\} \cup \{\{\infty\}\},\\ \mathsf{H}^1_{\mathrm{II}} &\cong \big\{B(a,r) : a \in K, r \in |K^*|\big\}, \quad \text{and}\\ \mathsf{H}^1_{\mathrm{III}} &\cong \big\{B(a,r) : a \in K, r \in \mathbb{R}_{>0} \setminus |K^*|\big\}. \end{aligned}$$

More precisely, each element of P^1 is regarded as either the cofinal equivalence class of a decreasing (i.e., non-increasing and nesting) sequence of K-closed disks or $\infty \in \mathbb{P}^1$. The inclusion relation $\subset$ among K-closed disks canonically extends to an ordering $\preceq$ on P^1, so that ∞ is the maximum element in $(\mathsf{P}^1, \preceq)$, and the diameter function $\mathrm{diam}_{|\cdot|}$ for K-closed disks also extends upper semicontinuously to P^1, so that $\mathrm{diam}_{|\cdot|}(\infty) = +\infty$. For $\mathcal{S}_1, \mathcal{S}_2 \in \mathsf{P}^1$, if $\mathcal{S}_1 \preceq \mathcal{S}_2$, then we set $[\mathcal{S}_1, \mathcal{S}_2] = [\mathcal{S}_2, \mathcal{S}_1] := \{\mathcal{S} \in \mathsf{P}^1 : \mathcal{S}_1 \preceq \mathcal{S} \preceq \mathcal{S}_2\}$, and in general there is the minimum element $\mathcal{S}'$ in $\{\mathcal{S} \in \mathsf{P}^1 : \mathcal{S}_1 \preceq \mathcal{S}$ and $\mathcal{S}_2 \preceq \mathcal{S}\}$ and we set

$$[\mathcal{S}_1, \mathcal{S}_2] = [\mathcal{S}_2, \mathcal{S}_1] := [\mathcal{S}_1, \mathcal{S}'] \cup [\mathcal{S}', \mathcal{S}_2];$$

we also set $(\mathcal{S}_1, \mathcal{S}_2] := [\mathcal{S}_1, \mathcal{S}_2] \setminus \{\mathcal{S}_1\}$. Those (closed) intervals $[\mathcal{S}, \mathcal{S}']$ in P^1 equip P^1 with a (profinite) tree structure in the sense of Jonsson [12, §2].

For every $\mathcal{S} \in \mathsf{P}^1$, the tangent (or direction) space $T_{\mathcal{S}}\mathsf{P}^1$ of P^1 at $\mathcal{S}$ is

$$T_{\mathcal{S}}\mathsf{P}^1 := \big\{\vec{v} = \overrightarrow{\mathcal{S}\mathcal{S}'} : \text{the germ of a non-empty left-half-open interval } (\mathcal{S}, \mathcal{S}']\big\};$$

then $\#T_{\mathcal{S}}\mathsf{P}^1 = 1$ if and only if $\mathcal{S} \in \mathbb{P}^1 \cup \mathsf{H}^1_{\mathrm{IV}}$, $\#T_{\mathcal{S}}\mathsf{P}^1 = 2$ if and only if $\mathcal{S} \in \mathsf{H}^1_{\mathrm{III}}$, and $T_{\mathcal{S}}\mathsf{P}^1 \cong \mathbb{P}^1(k)$ if and only if $\mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}$ (see (2.1) and Facts 2.3, 2.6 below). Identifying each $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$ with

$$U_{\vec{v}} = U_{\mathcal{S},\vec{v}} := \left\{\mathcal{S}' \in \mathsf{P}^1 \setminus \{\mathcal{S}\} : \overrightarrow{\mathcal{S}\mathcal{S}'} = \vec{v}\right\} \subset 2^{\mathsf{P}^1},$$

the collection $(U_{\mathcal{S},\vec{v}})_{\mathcal{S}\in\mathsf{P}^1,\vec{v}\in T_{\mathcal{S}}\mathsf{P}^1}$ is a quasi open basis of the (Gel'fand, weak, pointwise, or observer) topology on P^1(, and both $\mathbb{P}^1$ and $\mathsf{H}^1_{\mathrm{II}}$ are dense in P^1), and for every $\mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}$, we identify $T_{\mathcal{S}}\mathsf{P}^1$ with $S(\{\mathcal{S}\}) \setminus \{\{\mathcal{S}\}\}$ by the canonical bijection

$$T_{\mathcal{S}}\mathsf{P}^1 \ni \vec{v} \leftrightarrow U_{\vec{v}} \in S(\{\mathcal{S}\}) \setminus \{\{\mathcal{S}\}\}.$$

The Gauss (or canonical) point $\mathcal{S}_G \in \mathsf{H}^1_{\mathrm{II}}$ is represented by (the constant sequence of) the K-closed unit disk, that is, the ring $\mathcal{O}_K = B(0,1)$ of K-integers; the unique maximal ideal in $\mathcal{O}_K$ is $\mathcal{M}_K := \{z \in K : |z| < 1\}$, and

$$k = k_K := \mathcal{O}_K/\mathcal{M}_K$$

is the residue field of K, which is still algebraically closed under the standing assumption on K. The residue characteristic of K is $\operatorname{char} k$.

The reduction $\tilde{a} \in \mathbb{P}^1(k)$ of a point $a \in \mathbb{P}^1(K)$ is defined by the point $\widetilde{a_1}/\widetilde{a_0} \in \mathbb{P}^1(k)$, where $a_1, a_0 \in K$ are chosen so that $a = a_1/a_0$ (regarding $1/0 = \infty \in \mathbb{P}^1$) and that $\max\{|a_0|, |a_1|\} = 1$ (so $\widetilde{\infty} = \infty \in \mathbb{P}^1(k) = k \cup \{\infty\}$). There is also a canonical bijection

$$T_{\mathcal{S}_G}\mathsf{P}^1 \ni \overrightarrow{\mathcal{S}_G a} \leftrightarrow \tilde{a} \in \mathbb{P}^1(k). \tag{2.1}$$

For more details on (dynamics on) P^1, see e.g., the books [2, 3] and the survey article [12].

§2.2. Dynamics on P^1 and their reductions

For every $h \in K(z)$, writing

$$h(z) = \frac{P(z)}{Q(z)}, \quad P(z) = \sum_{j=0}^{\deg h} a_j z^j \in K[z], \quad \text{and} \quad Q(z) = \sum_{\ell=0}^{\deg h} b_\ell z^\ell \in K[z],$$

this h is regarded as the point $[b_0 : \cdots : b_{\deg h} : a_0 : \cdots : a_{\deg h}] \in \mathbb{P}^{2(\deg h)+1}(K)$. Then, choosing P, Q so that

$$\max\bigl\{|b_0|, \ldots, |b_{\deg h}|, |a_0|, \ldots, |a_{\deg h}|\bigr\} = 1,$$

we obtain the point $\tilde{h} = [\widetilde{b_0} : \cdots : \widetilde{b_{\deg h}} : \widetilde{a_0} : \cdots : \widetilde{a_{\deg h}}] \in \mathbb{P}^{2(\deg h)+1}(k)$; this point $\tilde{h} \in \mathbb{P}^{2(\deg h)+1}(k)$ is formally written as

$$\tilde{h} = H_{\tilde{h}}\phi_{\tilde{h}},$$

where we set $\widetilde{P}(\zeta) := \sum_{j=0}^{\deg h} \widetilde{a_j}\zeta^j \in k[\zeta]$, $\widetilde{Q}(z) := \sum_{\ell=0}^{\deg h} \widetilde{b_\ell}\zeta^\ell \in k[\zeta]$,

$$H_{\tilde{h}}(X_0, X_1) := \mathrm{GCD}\bigl(X_0^{\deg h}\widetilde{Q}(X_1/X_0), X_0^{\deg h}\widetilde{P}(X_1/X_0)\bigr) \in \bigcup_{\ell=0}^{\deg h} k[X_0, X_1]_\ell \setminus \{0\},$$

$$\text{and} \quad \phi_{\tilde{h}}(\zeta) := \frac{\widetilde{P}(\zeta)/H_{\tilde{h}}(1,\zeta)}{\widetilde{Q}(\zeta)/H_{\tilde{h}}(1,\zeta)} \in k(\zeta)$$

($H_{\tilde{h}}$ is unique up to multiplication in k^*). The rational function $\phi_{\tilde{h}} \in k(\zeta)$ on $\mathbb{P}^1(k)$ is called the reduction of h, the degree of which equals $\deg h - \deg H_{\tilde{h}}$.

Notation 2.1. When $\deg H_{\tilde{h}} > 0$, we denote by $[H_{\tilde{h}} = 0]$ the effective divisor on $\mathbb{P}^1(k)$ defined by the zeros of $H_{\tilde{h}}$ on $\mathbb{P}^1(k)$ taking into account their multiplicities, so that $\deg[H_{\tilde{h}} = 0] = \deg H_{\tilde{h}}$. When $\deg H_{\tilde{h}} = 0$, we set $[H_{\tilde{h}} = 0] := 0$ on $\mathbb{P}^1(k)$ by convention.

The action on $\mathbb{P}^1$ of $h \in K(z)$ extends continuously to that on P^1, and if in addition $\deg h > 0$, then this extended action is surjective, open, and fiber-discrete, and preserves $\mathbb{P}^1$, $\mathsf{H}^1_{\mathrm{II}}$, $\mathsf{H}^1_{\mathrm{III}}$, and $\mathsf{H}^1_{\mathrm{IV}}$, as already mentioned in Section 1. Then

$$h^{-1}(\mathcal{S}_G) = \{\mathcal{S}_G\} \quad \Leftrightarrow \quad \tilde{h} = \phi_{\tilde{h}} \quad \Leftrightarrow \quad \deg H_{\tilde{h}} = 0. \tag{2.2}$$

Fact 2.2 (Rivera-Letelier [15]; see also [2, Cor. 9.27]). We have $\deg(\phi_{\tilde{h}}) > 0$ if and only if $h(\mathcal{S}_G) = \mathcal{S}_G$. Moreover,

$$\phi_{\tilde{h}} \equiv \tilde{z} \text{ for some } z \in \mathbb{P}^1 \quad \Rightarrow \quad \overrightarrow{\mathcal{S}_G h(\mathcal{S}_G)} = \overrightarrow{\mathcal{S}_G z}. \tag{2.3}$$

Fact 2.3. The group $\mathrm{PGL}(2, K)$ of Möbius transformations on $\mathbb{P}^1$ acts transitively on $\mathsf{H}^1_{\mathrm{II}}$, and $\mathrm{PGL}(2, \mathcal{O}_K)$ is the stabilizer subgroup of $\mathcal{S}_G$ in $\mathrm{PGL}(2, K)$.

From now on, suppose that $\deg h > 0$.

§2.3. The tangent maps and the directional/surplus local degrees of rational functions

For the details on this and the next subsections, see Rivera-Letelier [16, 15]; see also Jonsson [12, §4.5] for an algebraic treatment.

For every $\mathcal{S} \in \mathsf{P}^1$, the tangent map $h_* = (h_*)_{\mathcal{S}} \colon T_{\mathcal{S}}\mathsf{P}^1 \to T_{h(\mathcal{S})}\mathsf{P}^1$ of h at $\mathcal{S}$ is defined so that for every $\vec{v} = \overrightarrow{\mathcal{S}\mathcal{S}'} \in T_{\mathcal{S}}\mathsf{P}^1$, if $\mathcal{S}'$ is close enough to $\mathcal{S}$, then h maps the interval $[\mathcal{S}, \mathcal{S}']$ onto the interval $[h(\mathcal{S}), h(\mathcal{S}')]$ homeomorphically, and

$$h_*(\vec{v}) = \overrightarrow{h(\mathcal{S})h(\mathcal{S}')}.$$

Moreover, for every $\mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}$ and every $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$, there is the directional local degree $m_{\vec{v}}(h) \in \mathbb{N}$ (indeed $\in \{1, \ldots, \deg_{\mathcal{S}}(h)\}$) of h on $U_{\vec{v}}$ such that choosing any

$A, B \in \mathrm{PGL}(2, K)$ satisfying $B^{-1}(\mathcal{S}) = A(h(\mathcal{S})) = \mathcal{S}_G$ (so $\deg(\widetilde{A \circ h \circ B}) > 0$ by Fact 2.2) and writing $(B^{-1})_*(\vec{v}) = \overrightarrow{\mathcal{S}_G z}$ and $A_*(h_*(\vec{v})) = \overrightarrow{\mathcal{S}_G w}$ by some $z, w \in \mathbb{P}^1$, we have

$$\phi_{\widetilde{A \circ h \circ B}}(\tilde{z}) = \widetilde{w} \quad \text{and} \tag{2.4}$$

$$m_{\vec{v}}(h) = \deg_{\tilde{z}}(\phi_{\widetilde{A \circ h \circ B}}). \tag{2.5}$$

For every $\mathcal{S} \in \mathsf{P}^1 \setminus \mathsf{H}^1_{\mathrm{II}}$ and every $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$, we set $m_{\vec{v}}(h) := \deg_{\mathcal{S}}(h)$.

Fact 2.4 (Decomposition of the local degree [16, Prop. 3.5]). For every $\mathcal{S} \in \mathsf{P}^1$, also using the notation in the above paragraph if $\mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}$, we have

$$(1 \le)\ \deg_{\mathcal{S}}(h) = \sum_{\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1 : h_*(\vec{v}) = \vec{w}} m_{\vec{v}}(h)\ (= \deg(\phi_{\widetilde{A \circ h \circ B}}) \text{ if } \mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}) \quad \text{for any } \vec{w} \in T_{h(\mathcal{S})}\mathsf{P}^1; \tag{2.6}$$

in particular, $h_* \colon T_{\mathcal{S}}\mathsf{P}^1 \to T_{h(\mathcal{S})}\mathsf{P}^1$ is surjective.

Fact 2.5 (Non-archimedean argument principle [15, Lem. 2.1]). For every $\mathcal{S} \in \mathsf{P}^1$ and every $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$, there is the surplus local degree $s_{\vec{v}}(h) \in \{0, 1, \ldots, \deg_{\mathcal{S}}(h)\}$ of h on $U_{\vec{v}}$ such that for every $\mathcal{S}' \in \mathsf{P}^1 \setminus \{h(\mathcal{S})\}$,

$$(h^*\delta_{\mathcal{S}'})(U_{\vec{v}}) = \begin{cases} m_{\vec{v}}(h) + s_{\vec{v}}(h) & \text{if } U_{h_*(\vec{v})} \ni \mathcal{S}', \\ s_{\vec{v}}(h) & \text{otherwise;} \end{cases} \tag{2.7}$$

moreover, $h(U_{\vec{v}})$ is either P^1 or $U_{h_*(\vec{v})}$, the latter of which is the case if and only if $s_{\vec{v}}(h) = 0$. For every $\mathcal{S} \in \mathsf{P}^1$, $s_{\vec{v}}(h) > 0$ for at most finitely many $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$, and then

$$\sum_{\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1} s_{\vec{v}}(h) = \deg h - \deg_{\mathcal{S}}(h) \tag{2.8}$$

since fixing any $\mathcal{S}' \in \mathsf{P}^1 \setminus \{h(\mathcal{S})\}$, we have

$$\begin{aligned} \deg h &= (h^*\delta_{\mathcal{S}'})(\mathsf{P}^1) = (h^*\delta_{\mathcal{S}'})(\mathsf{P}^1 \setminus \{\mathcal{S}\}) \\ &= \sum_{\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1 : h_*(\vec{v}) = \overrightarrow{\mathcal{S}\mathcal{S}'}} m_{\vec{v}}(h) + \sum_{\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1} s_{\vec{v}}(h) = \deg_{\mathcal{S}}(h) + \sum_{\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1} s_{\vec{v}}(h). \end{aligned}$$

Fact 2.6. In the case that $h \in \mathrm{PGL}(2, K)$, the tangent map $h_* \colon T_{\mathcal{S}}\mathsf{P}^1 \to T_{h(\mathcal{S})}\mathsf{P}^1$ is bijective, and for every $\mathcal{S} \in \mathsf{P}^1$ and every $\vec{v} \in T_{\mathcal{S}}\mathsf{P}^1$, $h(U_{\vec{v}}) = U_{h_*(\vec{v})}$.

Fact 2.7 (Faber [7, Lem. 3.17]). For every $\mathcal{S}\in\mathsf{H}^1_{\mathrm{II}}$ and every $\vec{v}\in T_{\mathcal{S}}\mathsf{P}^1$, choosing any such $A,B\in\mathrm{PGL}(2,K)$ that $B^{-1}(\mathcal{S})=A(h(\mathcal{S}))=\mathcal{S}_G$ and any such $z\in\mathbb{P}^1$ that $(B^{-1})_*(\vec{v})=\overrightarrow{\mathcal{S}_G z}$ (as in the paragraph before Fact 2.4), we have

$$s_{\vec{v}}(h)\begin{cases}=\mathrm{ord}_{\zeta=\tilde{z}}\big[H_{\widetilde{A\circ h\circ B}}=0\big] & \text{if } \deg H_{\widetilde{A\circ h\circ B}}>0,\\ \equiv 0 & \text{otherwise.}\end{cases} \tag{2.9}$$

§2.4. The hyperbolic metric ρ on H^1 and the piecewise affine action of h on (H^1,ρ)

The hyperbolic metric ρ on H^1, which is defined so that

$$\rho(\mathcal{S}_1,\mathcal{S}_2)=\log\Big(\frac{\mathrm{diam}_{|\cdot|}\mathcal{S}_2}{\mathrm{diam}_{|\cdot|}\mathcal{S}_1}\Big)\quad\text{if } \mathcal{S}_1\preceq\mathcal{S}_2,$$

would be used at some part in the proof of Theorem A. The topology on (H^1,ρ) is finer than the relative topology on H^1 from P^1.

Fact 2.8 ([16, Prop. 3.5]). For every $\mathcal{S}\in\mathsf{P}^1$ and every $\vec{v}=\overrightarrow{\mathcal{S}\mathcal{S}'}\in T_{\mathcal{S}}\mathsf{P}^1$, if $\mathcal{S}'$ is close enough to $\mathcal{S}$, then for every $\mathcal{S}''\in(\mathcal{S},\mathcal{S}']$,

$$\rho(h(\mathcal{S}''),h(\mathcal{S}'))=m_{\vec{v}}(h)\cdot\rho(\mathcal{S}'',\mathcal{S}'), \tag{2.10}$$

which still holds for $\mathcal{S}''\in[\mathcal{S},\mathcal{S}']$ if $\mathcal{S}\in\mathsf{H}^1$.

§2.5. Quantized local degrees and quantized pullbacks

Let us precisely define the quantized local degree $m_{V,U}(h)$, mentioned in Section 1.1, in terms of the (directional/surplus) local degrees of h, and then also (re)define the quantized pullback operator $h^*_G\colon M(\Gamma_h)\to M(\Gamma_G)$. Recall

$$\Gamma_G:=\{\mathcal{S}_G\}\quad\text{and}\quad\Gamma_h:=\{\mathcal{S}_G,h(\mathcal{S}_G)\}\quad\text{in } \mathsf{H}^1_{\mathrm{II}}.$$

Definition 2.9 (Quantized local degree). For every $U_{\vec{v}}\in S(\Gamma_G)\setminus\{\{\mathcal{S}_G\}\}=T_{\mathcal{S}_G}\mathsf{P}^1$ and every $V\in S(\Gamma_h)$, set

$$\begin{aligned} m_{V,U_{\vec{v}}}(h) &:= \begin{cases} m_{\vec{v}}(h)+s_{\vec{v}}(h) & \text{if } V\subset U_{h_*(\vec{v})},\\ s_{\vec{v}}(h) & \text{if } V\cap U_{h_*(\vec{v})}=\emptyset,\end{cases}\\ &\underset{(2.7)}{=} (h^*\delta_{\mathcal{S}'})(U_{\vec{v}})\quad\text{for any } \mathcal{S}'\in V \text{ if } V\in S(\Gamma_h)\setminus\{\{h(\mathcal{S}_G)\}\},\end{aligned}$$

and for every $V \in S(\Gamma_h)$, set

$$m_{V,\{\mathcal{S}_G\}}(h) := \begin{cases} \deg_{\mathcal{S}_G}(h) & \text{if } V = \{h(\mathcal{S}_G)\}, \\ 0 & \text{if } V \in S(\Gamma_h) \setminus \{\{h(\mathcal{S}_G)\}\}, \end{cases}$$
$$\underset{(1.1)}{=} (h^*\delta_{\mathcal{S}'})(\{\mathcal{S}_G\}) \quad \text{for any } \mathcal{S}' \in V.$$

Fact 2.10. The fundamental equality

$$\sum_{U \in S(\Gamma_G)} m_{V,U}(h) = \deg h \quad \text{for any } V \in S(\Gamma_h) \tag{2.11}$$

holds; indeed, for every $V \in S(\Gamma_h) \setminus \{\{h(\mathcal{S}_G)\}\}$, there is a unique $\vec{w} \in T_{h(\mathcal{S}_G)}\mathsf{P}^1$ satisfying $V \subset U_{\vec{w}}$, and then

$$\sum_{U \in S(\Gamma_G)} m_{V,U}(h) = \sum_{\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1 : h_*(\vec{v}) = \vec{w}} m_{\vec{v}}(h) + \sum_{\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1} s_{\vec{v}}(h) + 0$$
$$\underset{(2.6)\&(2.8)}{=} \deg_{\mathcal{S}_G}(h) + (\deg h - \deg_{\mathcal{S}_G}(h)) = \deg h,$$

and similarly,

$$\sum_{U \in S(\Gamma_G)} m_{\{h(\mathcal{S}_G)\},U}(h) = \sum_{\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1} s_{\vec{v}}(h) + \deg_{\mathcal{S}_G}(h)$$
$$\underset{(2.8)}{=} (\deg h - \deg_{\mathcal{S}_G}(h)) + \deg_{\mathcal{S}_G}(h) = \deg h.$$

The quantized pushforward operator $h_{G,*}$ from the space of measurable functions on $\mathsf{P}^1/S(\Gamma_G)$ to that of measurable functions on $\mathsf{P}^1/S(\Gamma_h)$ is defined so that for every measurable function ψ on $\mathsf{P}^1/S(\Gamma_G)$, the measurable function $h_{G,*}\psi$ on $\mathsf{P}^1/S(\Gamma_h)$ satisfies

$$(h_{G,*}\psi)(V) = \sum_{U \in S(\Gamma_h)} m_{V,U}(h)\psi(U) \quad \text{for any } V \in S(\Gamma_h) \quad \text{or equivalently}$$
$$(\pi_{\Gamma_h})^*(h_{G,*}\psi) \equiv \sum_{U \in S(\Gamma_G)} m_{V,U}(h) \cdot ((\pi_{\Gamma_G})^*\psi)|U \quad \text{on each } V \in S(\Gamma_h),$$

so, in particular,

$$(\pi_{\Gamma_h})^*(h_{G,*}\psi) = \sum_{\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1} (h^*\delta_.)(U_{\vec{v}}) \cdot ((\pi_{\Gamma_G})^*\psi)|U_{\vec{v}} \quad \text{on } \mathsf{P}^1 \setminus \{h(\mathcal{S}_G)\}. \tag{2.12}$$

The quantized pullback operator $h_G^*\colon M(\Gamma_h) \to M(\Gamma_G)$ is the transpose of this quantized pushforward operator $h_{G,*}$ so, in particular, for every $\omega \in M(\Gamma_h)$, the measure $h_G^*\omega \in M(\Gamma_G)$ satisfies

$$
\begin{aligned}
(h_G^*\omega)(\{U\}) &= \langle 1_{\{U\}}, h_G^*\omega\rangle = \langle h_{G,*}(1_{\{U\}}), \omega\rangle \\
&= \int_{\mathsf{P}^1/S(\Gamma_h)} \Bigl(\sum_{W \in S(\Gamma_G)} m_{V,W}(h) \cdot 1_{\{U\}}(W) \Bigr) \omega(V) \\
&= \int_{\mathsf{P}^1/S(\Gamma_h)} m_{V,U}(h)\omega(V) \quad \text{for any } U \in S(\Gamma_G). \qquad (2.13)
\end{aligned}
$$

§3. Degenerating balanced property for degenerating weak limit points of the maximal entropy measures on $\mathbb{P}^1(\mathbb{C})$

We follow the presentation in [5, §2.1–§2.4].

Fixing $r \in (0,1)$ (e.g., $r = e^{-1}$) once and for all, the field $\mathbb{C}((t))$ of Laurent series around $t = 0$ over $\mathbb{C}$ is equipped with the non-trivial and non-archimedean absolute value

$$
|x|_r = r^{\min\{n \in \mathbb{Z}:\ a_n \neq 0\}} \qquad (3.1)
$$

for $x(t) = \sum_{n\in\mathbb{Z}} a_n t^n \in \mathbb{C}((t))$ (under the convention that $\min \emptyset = +\infty$ and $r^{+\infty} = 0$), which extends the trivial absolute value on $\mathbb{C}$ to $\mathbb{C}((t))$.

An algebraic closure $\overline{\mathbb{C}((t))}$ of $\mathbb{C}((t))$ is the field of Puiseux series around $t = 0$ over $\mathbb{C}$, $|\cdot|_r$ extends to $\overline{\mathbb{C}((t))}$ as an absolute value, and the completion $\mathbb{L}$ of $\overline{\mathbb{C}((t))}$ is the field of formal Puiseux series around $t = 0$ over $\mathbb{C}$ and is still algebraically closed. We note that $\mathcal{O}(\mathbb{D})[t^{-1}] \subset \mathbb{C}((t))$,

$$
\begin{aligned}
\mathbb{C} \subset \mathcal{O}(\mathbb{D}) \subset \mathcal{O}_{\mathbb{C}((t))} &= \Bigl\{ \sum_{n\in\mathbb{Z}} a_n t^n \in \mathbb{C}((t)) : a_n = 0 \text{ if } n < 0 \Bigr\} = \mathbb{C}[[t]], \\
\mathcal{M}_{\mathbb{C}((t))} &= t \cdot \mathcal{O}_{\mathbb{C}((t))}, \\
k_{\mathbb{L}} &= k_{\mathbb{C}((t))} = \mathbb{C} \ \text{(as fields)}, \quad \text{and} \\
T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) &\cong \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}) \quad \text{(the bijection is the canonical one in (2.1)).}
\end{aligned}
$$

Notation 3.1. Let $M(\mathbb{P}^1(\mathbb{C}))$ be the space of all complex Radon measures on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. The pullback of each $\mu \in M(\mathbb{P}^1(\mathbb{C}))$ under a rational function $R \in \mathbb{C}(z)$ on $\mathbb{P}^1(\mathbb{C})$ of degree > 0 is $R^*\mu := \int_{\mathbb{P}^1(\mathbb{C})} (\sum_{w \in R^{-1}(z)} (\deg_w R)\delta_w)\mu(z)$ on $\mathbb{P}^1(\mathbb{C})$, where for each $z \in \mathbb{P}^1(\mathbb{C})$, δ_z is the Dirac measure at z on $\mathbb{P}^1(\mathbb{C})$; if R is constant, then $R^*\mu := 0$ by convention. Also set

$$
\begin{aligned}
M^1(\mathbb{P}^1(\mathbb{C})) &:= \bigl\{ \mu \in M(\mathbb{P}^1(\mathbb{C})) : \mu \geq 0 \text{ and } \mu(\mathbb{P}^1(\mathbb{C})) = 1 \bigr\} \quad \text{and} \\
M^1(\mathbb{P}^1(\mathbb{C}))^\dagger &:= \bigl\{ \mu \in M^1(\mathbb{P}^1(\mathbb{C})) : \mu \text{ is purely atomic} \bigr\}.
\end{aligned}
$$

Fact 3.2 (Maximal entropy measure on $\mathbb{P}^1(\mathbb{C})$ [4, 14, 11]). For a rational function $R \in \mathbb{C}(z)$ on $\mathbb{P}^1(\mathbb{C})$ of degree > 1, the equilibrium (or canonical, and indeed the unique maximal entropy) measure μ_R of R on $\mathbb{P}^1(\mathbb{C})$ is the unique $\mu \in M^1(\mathbb{P}^1(\mathbb{C}))$ satisfying $R^*\mu = (\deg R)\mu$ on $\mathbb{P}^1(\mathbb{C})$ and $\mu(E(R)) = 0$, where $E(R) := \{a \in \mathbb{P}^1(\mathbb{C})\colon \#\bigcup_{n\in\mathbb{N}} R^{-n}(a) < +\infty\}$. Then, for every $n \in \mathbb{N}$, $\mu_{R^n} = \mu_R$ on $\mathbb{P}^1(\mathbb{C})$ and $E(R^n) = E(R)$. The measure μ_R is $\mathrm{PGL}(2,\mathbb{C})$-equivariant in that for every Möbius transformation $M \in \mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$, $\mu_{M\circ R\circ M^{-1}} = M_*\mu_R$ on $\mathbb{P}^1(\mathbb{C})$.

When $R \in \mathbb{C}[z]$ or equivalently $R(\infty) = \infty \in E(R)$, μ_R is supported by $\partial(K_R)$, where the filled-in Julia set $K_R := \{z \in \mathbb{C} : \limsup_{n\to\infty} |R^n(z)| < +\infty\}$ of R is a compact subset in $\mathbb{C}$.

Let $h \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)\ (\subset \mathbb{L}(z))$ be a meromorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$, and let us regard $\tilde{h} = H_{\tilde{h}}\phi_{\tilde{h}} \in \mathbb{P}^{2(\deg h)+1}(k_{\mathbb{L}})$ as a point in $\mathbb{P}^{2(\deg h)+1}(\mathbb{C})$, $\phi_{\tilde{h}}$ as a rational function on $\mathbb{P}^1(\mathbb{C})$ of degree $\deg h - \deg H_{\tilde{h}}$, and the effective divisor $[H_{\tilde{h}} = 0]$ on $\mathbb{P}^1(k_{\mathbb{L}})$ as that on $\mathbb{P}^1(\mathbb{C})$ and in turn also as the Radon measure $\sum_{z\in\mathbb{P}^1(\mathbb{C})}(\mathrm{ord}_z[H_{\tilde{h}} = 0])\delta_z$ on $\mathbb{P}^1(\mathbb{C})$, under $k_{\mathbb{L}} = \mathbb{C}$ as fields. Then

$$\lim_{t\to 0} h_t = \phi_{\tilde{h}} \quad \text{locally uniformly on } \mathbb{P}^1(\mathbb{C}) \setminus (\mathrm{supp}[H_{\tilde{h}} = 0]). \tag{3.2}$$

Definition 3.3. For every $\mu \in M^1(\mathbb{P}^1(\mathbb{C}))$, the (possibly degenerating) pullback $\tilde{h}^*\mu \in M(\mathbb{P}^1(\mathbb{C}))$ of μ under $\tilde{h}$ is defined by

$$\tilde{h}^*\mu := (\phi_{\tilde{h}})^*\mu + [H_{\tilde{h}} = 0] \quad \text{on } \mathbb{P}^1(\mathbb{C}), \tag{3.3}$$

still satisfying $(\tilde{h}^*\mu)(\mathbb{P}^1(\mathbb{C})) = \deg h$.

Recall Fact 2.2. The following target rescaling theorem is a special case of [13, Lem. 3.7] (see also [5, Lem. 2.1]).

Theorem 3.4. *For every meromorphic family $f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ $(\subset \mathbb{L}(z))$ of rational functions on $\mathbb{P}^1(\mathbb{C})$ of degree > 1, there is a meromorphic family $A \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ such that $(A\circ f)(\mathcal{S}_G) = \mathcal{S}_G$ in $\mathsf{P}^1(\mathbb{L})$. Such a family A is unique up to a postcomposition to A of any meromorphic family $B \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ satisfying $\widetilde{B} = \phi_{\widetilde{B}} \in \mathrm{PGL}(2,\mathbb{C})$.*

Also recall (2.2). The degenerating f-balanced property of the pair $\mu = (\mu_C, \mu_E)$ (the former half in (3.4)) is a consequence of (3.2) and the complex argument principle. The proof of the purely atomicness of μ (the latter half in (3.4)) is more involved.

Theorem 3.5 (Consequence of [5, Thms. 2.4 and A]). *Let*

$$f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)\,(\subset \mathbb{L}(z))$$

be a meromorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$ *of degree* $d > 1$ *satisfying* $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ *in* $\mathsf{P}^1(\mathbb{L})$, *let* $A \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ *be a meromorphic family of Möbius transformations on* $\mathbb{P}^1(\mathbb{C})$ *such that* $(A \circ f)(\mathcal{S}_G) = \mathcal{S}_G$, *and let*

$$\mu_C = \lim_{j\to\infty} \mu_{f_{t_j}}, \quad \mu_E = \lim_{j\to\infty} (A_{t_j})_* \mu_{f_{t_j}} \in M^1(\mathbb{P}^1(\mathbb{C}))$$

be weak limit points on $\mathbb{P}^1(\mathbb{C})$ *as* $t \to 0$ *of the families* $(\mu_{f_t})_{t\in\mathbb{D}^*}$ *and* $((A_t)_*\mu_{f_t})_{t\in\mathbb{D}^*}$ *of the unique maximal entropy measures* μ_{f_t} *and* $(A_t)_*\mu_{f_t} = \mu_{A_t\circ f_t\circ A_t^{-1}}$ *on* $\mathbb{P}^1(\mathbb{C})$ *of* f_t *and of* $A_t \circ f_t \circ A_t^{-1}$, *respectively, for some sequence* $(t = t_j)$ *in* $\mathbb{D}^*$ *tending to* 0 *as* $j \to \infty$. *Then*

$$(\widetilde{A \circ f})^* \mu_E = d \cdot \mu_C \text{ on } \mathbb{P}^1(\mathbb{C}) \quad \text{and} \quad \mu := (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2. \tag{3.4}$$

§4. A direct translation

Pick a meromorphic family $f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)\,(\subset \mathbb{L}(z))$ of rational functions on $\mathbb{P}^1(\mathbb{C})$ of degree $d > 1$, and suppose that $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ in $\mathsf{P}^1(\mathbb{L})$. Choose a meromorphic family $A \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ such that $(A \circ f)(\mathcal{S}_G) = \mathcal{S}_G$ (by Theorem 3.4). Also recall

$$\Gamma_G := \{\mathcal{S}_G\} \quad \text{and} \quad \Gamma_f := \{\mathcal{S}_G, f(\mathcal{S}_G)\} \quad \text{in } \mathsf{H}^1_{\mathrm{II}}(\mathbb{L}).$$

From Fact 2.2 and (2.2), the following five statements

$$\Gamma_G = \Gamma_f, \quad f(\mathcal{S}_G) = \mathcal{S}_G, \quad \deg(\phi_{\tilde{f}}) > 0, \quad \text{and moreover,}$$
$$A(\mathcal{S}_G) = \mathcal{S}_G \quad \text{and}$$
$$\tilde{A} = \phi_{\tilde{A}} \in \mathrm{PGL}(2, k_{\mathbb{L}}) = \mathrm{PGL}(2, \mathbb{C}) \text{ (under } k_{\mathbb{L}} = \mathbb{C} \text{ as fields, here and below)}$$

are equivalent. Alternatively, when $\Gamma_G \neq \Gamma_f$, there are $h_A, a_A \in \mathbb{P}^1(\mathbb{C})$ such that

$$\begin{aligned} &\mathrm{supp}[H_{\tilde{A}} = 0] = \{h_A\} \quad \text{in } \mathbb{P}^1(\mathbb{C}), \quad \phi_{\tilde{A}} \equiv a_A \quad \text{on } \mathbb{P}^1(\mathbb{C}), \\ &\text{and moreover} \quad \phi_{\tilde{f}} \equiv h_A \quad \text{on } \mathbb{P}^1(\mathbb{C}) \quad \text{(by (3.2) and Fact 2.2).} \end{aligned} \tag{4.1}$$

We note that

$$T_{f(\mathcal{S}_G)}\mathsf{P}^1(\mathbb{L}) \underset{(A^{-1})_*}{\overset{\cong}{\leftarrow}} T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) \underset{(2.1)}{\cong} \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}),$$

also recalling Fact 2.6.

Lemma 4.1. *When $\Gamma_f \neq \Gamma_G$, we have*

$$(A^{-1})_*(\overrightarrow{\mathcal{S}_G A(\mathcal{S}_G)}) = \overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G}. \tag{4.2}$$

Proof. If $(A^{-1})_*(\vec{v}) = \overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G}$ $(= \overrightarrow{A^{-1}(\mathcal{S}_G)\mathcal{S}_G})$ for some (indeed unique) $\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L})$, then we have $\mathcal{S}_G \in U_{(A^{-1})_*(\vec{v})}$, which yields $A(\mathcal{S}_G) \in A(U_{(A^{-1})_*(\vec{v})}) = U_{A_*(A^{-1})_*(\vec{v})} = U_{\vec{v}}$ (using Fact 2.6), and in turn $\vec{v} = \overrightarrow{\mathcal{S}_G A(\mathcal{S}_G)}$. □

Lemma 4.2. *When $\Gamma_f \neq \Gamma_G$, for any $\tilde{x}, \tilde{y} \in \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C})$ (and any representatives $x, y \in \mathbb{P}^1(\mathbb{L})$ of $\tilde{x}$, $\tilde{y}$, respectively), we have*

$$\begin{cases} \overrightarrow{\mathcal{S}_G x} = \overrightarrow{\mathcal{S}_G f(\mathcal{S}_G)} \text{ in } T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) & \Leftrightarrow \ \tilde{x} = h_A \text{ in } \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}), \\ (A^{-1})_*(\overrightarrow{\mathcal{S}_G y}) = \overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G} \text{ in } T_{f(\mathcal{S}_G)}\mathsf{P}^1(\mathbb{L}) & \Leftrightarrow \ \tilde{y} = a_A \text{ in } \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}). \end{cases} \tag{4.3}$$

Proof. The former assertion is by $\phi_{\tilde{f}} \equiv h_A$ on $\mathbb{P}^1(\mathbb{C})$ (in (4.1)) and (2.3). On the other hand, by (4.2), we have

$$(A^{-1})_*(\overrightarrow{\mathcal{S}_G y}) = \overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G} \ \Leftrightarrow \ \overrightarrow{\mathcal{S}_G y} \left(= A_*(\overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G})\right) = \overrightarrow{\mathcal{S}_G A(\mathcal{S}_G)},$$

so the latter assertion holds by $\phi_{\tilde{A}} \equiv a_A$ on $\mathbb{P}^1(\mathbb{C})$ (in (4.1)) and (2.3). □

Definition 4.3 (Admissibility of μ and construction of the measure ω_μ). For every $\mu = (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C})))^2$ satisfying the following admissibility

$$\begin{cases} \tilde{A}^*\mu_E = \mu_C \ \text{ on } \mathbb{P}^1(\mathbb{C}) & \text{when } \Gamma_f = \Gamma_G \, (\Leftrightarrow \tilde{A} = \phi_{\tilde{A}} \Leftrightarrow A(\mathcal{S}_G) = \mathcal{S}_G), \\ \mu_C(\{h_A\}) + \mu_E(\{a_A\}) \geq 1 & \text{when } \Gamma_f \neq \Gamma_G \end{cases} \tag{4.4}$$

(for A), there is a unique probability measure

$$\omega_\mu \in M^1(\Gamma_f) \quad (\text{and indeed } \omega_\mu \in M^1(\Gamma_f)^\dagger \text{ if } \mu \in (M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2)$$

on $\mathsf{P}^1/S(\Gamma_f) = S(\Gamma_f)$ such that, writing $\mu_C = \nu_C + \tilde{\nu}_C$ (resp. $\mu_E = \nu_E + \tilde{\nu}_E$) in $M(\mathbb{P}^1)$ where ν_C (resp. ν_E) has no atoms on $\mathbb{P}^1(\mathbb{C})$ and $\tilde{\nu}_C = \mu_C - \nu_C$ (resp. $\tilde{\nu}_E = \mu_E - \nu_E$) is purely atomic, when $\Gamma_f = \Gamma_G$,

$$\begin{cases} \omega_\mu(\{\{\mathcal{S}_G\}\}) = \nu_E(\mathbb{P}^1(\mathbb{C})) \left(= \nu_C(\mathbb{P}^1(\mathbb{C}))\right) \quad \text{and} \\ \omega_\mu(\{U_{(A^{-1})_*(\overrightarrow{\mathcal{S}_G y})}\}) = \mu_E(\{\tilde{y}\}) \quad \text{for every } \tilde{y} \in \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}) \\ \qquad \left(\underset{(4.4)\&(2.4)}{\Leftrightarrow} \ \omega_\mu(\{U_{\overrightarrow{\mathcal{S}_G y}}\}) = \mu_C(\{\tilde{y}\}) \quad \text{for every } \tilde{y} \in \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C})\right) \end{cases}$$

and, when $\Gamma_f \neq \Gamma_G$(, noting also Lemma 4.2),

$$
\begin{cases}
\omega_\mu(\{\{\mathcal{S}_G\}\}) = \nu_C(\mathbb{P}^1(\mathbb{C})), \\
\omega_\mu(\{U_{\overrightarrow{\mathcal{S}_G\tilde{x}}}\}) = \mu_C(\{\tilde{x}\}) \text{ for every } \tilde{x} \in \mathbb{P}^1(\mathbb{C}) \setminus \{h_A\}, \\
\omega_\mu(\{\{f(\mathcal{S}_G)\}\}) = \nu_E(\mathbb{P}^1(\mathbb{C})), \\
\omega_\mu(\{U_{(A^{-1})_*(\overrightarrow{\mathcal{S}_G\tilde{y}})}\}) = \mu_E(\{\tilde{y}\}) \text{ for every } \tilde{y} \in \mathbb{P}^1(\mathbb{C}) \setminus \{a_A\}, \quad \text{and} \\
\omega_\mu(\{U_{\overrightarrow{\mathcal{S}_G f(\mathcal{S}_G)}} \cap U_{\overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G}}\}) = \mu_C(\{h_A\}) + \mu_E(\{a_A\}) - 1 \, (\underset{(4.4)}{\geq} 0).
\end{cases}
\tag{4.5}
$$

For every $\mu = (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C})))^2$ satisfying the admissibility (4.4) (for A), we note that

$$\omega_\mu(S(\Gamma_f) \setminus F) = 0 \quad \text{for some countable subset } F \text{ in } S(\Gamma_f),$$

and also have

$$
\begin{aligned}
\omega_\mu \in M^1(\Gamma_f)^\dagger &\Rightarrow (\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu \in M^1(\Gamma_G)^\dagger \\
&\Rightarrow \mu_C = (\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu \quad \text{in } M^1(\mathbb{P}^1(\mathbb{C}))^\dagger = M^1(\Gamma_G)^\dagger
\end{aligned}
\tag{4.6}
$$

identifying $M^1(\Gamma_G)^\dagger$ with $M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$ under the bijection

$$S(\Gamma_G) \setminus \{\mathcal{S}_G\} = T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) \cong \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}).$$

The following direct translation from degenerating complex dynamics into quantized Berkovich dynamics is based on the above explicit definition of ω_μ and bypasses a correspondence between semistable models of $\mathsf{P}^1(\mathbb{L})$ and semistable vertex sets in $\mathsf{P}^1(\mathbb{L})$ from rigid analytic geometry (see, e.g., [1]), which is used in [5]. See Section 8 for a complement of this proposition.

Proposition 4.4 (Direct translation, cf. [5, Prop. 5.1(1)]). *For every ordered pair $\mu = (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C})))^2$ satisfying the admissibility (4.4) (for A), we have*

$$
\begin{aligned}
&(\widetilde{A \circ f})^*\mu_E = d \cdot \mu_C \quad \textit{in } M(\mathbb{P}^1(\mathbb{C})) \\
&\qquad \Rightarrow \quad f_G^*\omega_\mu = d \cdot (\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu \quad \textit{in } M(\Gamma_G).
\end{aligned}
\tag{4.7}
$$

Proof. Pick an ordered pair $\mu = (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C})))^2$ satisfying the admissibility (4.4) (for A), and write $\mu_C = \nu_C + \tilde{\nu}_C$, $\mu_E = \nu_E + \tilde{\nu}_E$ as in Definition 4.3.

(a-1). When $\Gamma_f \neq \Gamma_G$, for every $\tilde{x} \in \mathbb{P}^1(\mathbb{C}) = \mathbb{P}^1(k_{\mathbb{L}}) \cong T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) = S(\Gamma_G) \setminus \{\mathcal{S}_G\}$ (and every representative $x \in \mathbb{P}^1(\mathbb{L})$ of $\tilde{x}$), recalling Definitions 2.9 and 4.3, we compute both

$$
\begin{aligned}
&(f_G^*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\})\\
&\underset{(2.13)}{=} \int_{\mathsf{P}^1/S(\Gamma_f)} m_{V,U_{\overrightarrow{\mathcal{S}_G x}}}(f)\omega_\mu(V)\\
&= s_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot 1 + m_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot\omega_\mu\big(\{V\in S(\Gamma_f): V\subset U_{f_*(\overrightarrow{\mathcal{S}_G x})}\}\big)\\
&= s_{\overrightarrow{\mathcal{S}_G x}}(f) + m_{\overrightarrow{\mathcal{S}_G x}}(f)\times\\
&\quad\times\begin{cases}
1-\omega_\mu\big(\{U_{\vec{w}}\in S(\Gamma_f):\vec{w}\in(T_{f(\mathcal{S}_G)}\mathsf{P}^1(\mathbb{L}))\setminus\{f_*(\overrightarrow{\mathcal{S}_G x})\}\}\cup\{\{f(\mathcal{S}_G)\}\}\big)\\
\qquad = 1-\omega_\mu\big(\{U_{(A^{-1})_*(\overrightarrow{\mathcal{S}_G y})}: y\in\mathbb{P}^1(\mathbb{L}) \text{ satisfying } \tilde{y}\in\mathbb{P}^1(k_{\mathbb{L}})\setminus\{a_A\}\}\big)\\
\qquad\quad -\omega_\mu(\{\{f(\mathcal{S}_G)\}\})\\
\qquad = \mu_E(\{a_A\}) \quad \text{if } f_*(\overrightarrow{\mathcal{S}_G x}) = \overrightarrow{f(\mathcal{S}_G)\mathcal{S}_G},\\
\omega_\mu\big(\{U_{f_*(\overrightarrow{\mathcal{S}_G x})}\}\big) = \omega_\mu\big(\{U_{(A^{-1})_*(\overrightarrow{\mathcal{S}_G y})}\}\big)\\
\qquad \text{for any such } y\in\mathbb{P}^1(\mathbb{L}) \text{ that } f_*(\overrightarrow{\mathcal{S}_G x}) = (A^{-1})_*(\overrightarrow{\mathcal{S}_G y}) \quad \text{otherwise}
\end{cases}\\
&\underset{(4.3)}{=} s_{\overrightarrow{\mathcal{S}_G x}}(f) + m_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot\mu_E(\{\tilde{y}\}) \text{ for any such } y\in\mathbb{P}^1(\mathbb{L})\\
&\qquad\qquad \text{that } (A\circ f)_*(\overrightarrow{\mathcal{S}_G x}) = \overrightarrow{\mathcal{S}_G y}\ (\Leftrightarrow f_*(\overrightarrow{\mathcal{S}_G x}) = (A^{-1})_*(\overrightarrow{\mathcal{S}_G y}))\\
&\underset{(2.9),(2.5),\&(2.4)}{=} \mathrm{ord}_{\tilde{x}}[H_{\widetilde{A\circ f}}=0] + (\deg_{\tilde{x}}(\phi_{\widetilde{A\circ f}}))\cdot\mu_E(\{\phi_{\widetilde{A\circ f}}(\tilde{x})\})\\
&\underset{(3.3)}{=} ((\widetilde{A\circ f})^*\mu_E)(\{\tilde{x}\})
\end{aligned}
$$

and

$$
\begin{aligned}
&((\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\})\\
&\quad\underset{(1.3)}{=} \omega_\mu\big(\{V\in S(\Gamma_f): V\subset U_{\overrightarrow{\mathcal{S}_G x}}\}\big)\\
&\quad= \begin{cases}
1-\omega_\mu\big(\{U_{\vec{v}}\in S(\Gamma_f):\vec{v}\in(T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}))\setminus\{\overrightarrow{\mathcal{S}_G x}\}\}\cup\{\{\mathcal{S}_G\}\}\big)\\
\qquad = 1-\mu_C(\mathbb{P}^1(\mathbb{C})\setminus\{h_A\}) = \mu_C(\{h_A\}) \text{ if } \overrightarrow{\mathcal{S}_G x} = \overrightarrow{\mathcal{S}_G f(\mathcal{S}_G)},\\
\omega_\mu(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) \text{ otherwise}
\end{cases}\\
&\quad\underset{(4.3)}{=} \mu_C(\{\tilde{x}\}).
\end{aligned}
$$

Hence, if $(\widetilde{A\circ f})^*\mu_E = d\cdot\mu_C$ on $\mathbb{P}^1(\mathbb{C})$, then for the x, we have the equality $(f_G^*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) = (d\cdot(\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\})$.

(a-2). Moreover, we also compute both

$$(f_G^*\omega_\mu)(\{\{\mathcal{S}_G\}\}) \underset{(2.13)}{=} \int_{\mathsf{P}^1/S(\Gamma_f)} m_{V,\{\mathcal{S}_G\}}(f)\omega_\mu(V) = \deg_{\mathcal{S}_G}(f)\cdot\omega_\mu(\{\{f(\mathcal{S}_G)\}\})$$
$$\underset{(2.6)}{=} \deg(\phi_{\widetilde{A\circ f}})\cdot\nu_E(\mathbb{P}^1(\mathbb{C})) = ((\widetilde{A\circ f})^*\mu_E)(\mathbb{P}^1(\mathbb{C})\setminus F_1)$$

and

$$((\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{\{\mathcal{S}_G\}\}) \underset{(1.3)}{=} \omega_\mu(\{\{\mathcal{S}_G\}\}) = \nu_C(\mathbb{P}^1(\mathbb{C})) = \mu_C(\mathbb{P}^1(\mathbb{C})\setminus F_2),$$

where F_1, F_2 are any sufficiently large countable subsets in $\mathbb{P}^1(\mathbb{C})$.

Hence, if $(\widetilde{A\circ f})^*\mu_E = d\cdot\mu_C$ on $\mathbb{P}^1(\mathbb{C})$, then we also have $(f_G^*\omega_\mu)(\{\{\mathcal{S}_G\}\}) = (d\cdot(\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{\{\mathcal{S}_G\}\})$. Now the proof is complete in this case.

(b-1). When $\Gamma_f=\Gamma_G$, for every $\tilde{x}\in\mathbb{P}^1(\mathbb{C})=\mathbb{P}^1(k_{\mathbb{L}})\cong T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L})=S(\Gamma_G)\setminus\{\mathcal{S}_G\}$, similarly to (a-1), we compute both

$$\begin{aligned}
&(f_G^*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\})\\
&\underset{(2.13)}{=} \int_{\mathsf{P}^1/S(\Gamma_G)} m_{V,U_{\overrightarrow{\mathcal{S}_G x}}}(f)\omega_\mu(V) = s_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot 1 + m_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot\omega_\mu(\{U_{f_*(\overrightarrow{\mathcal{S}_G x})}\})\\
&= s_{\overrightarrow{\mathcal{S}_G x}}(f) + m_{\overrightarrow{\mathcal{S}_G x}}(f)\cdot\mu_E(\{\tilde{y}\})\\
&\qquad \text{for any such } y\in\mathbb{P}^1(\mathbb{L}) \text{ that } f_*(\overrightarrow{\mathcal{S}_G x}) = (A^{-1})_*(\overrightarrow{\mathcal{S}_G y})\\
&\underset{(2.9),(2.5),\,\&(2.4)}{=} \operatorname{ord}_{\tilde{x}}[H_{\widetilde{A\circ f}}=0] + (\deg_{\tilde{x}}(\phi_{\widetilde{A\circ f}}))\cdot\mu_E(\{\phi_{\widetilde{A\circ f}}(\tilde{x})\})\\
&\underset{(3.3)}{=} ((\widetilde{A\circ f})^*\mu_E)(\{\tilde{x}\})
\end{aligned}$$

and

$$((\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) \underset{(1.3)}{=} \omega_\mu(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) = \mu_C(\{\tilde{x}\}).$$

Hence, if $(\widetilde{A\circ f})^*\mu_E = d\cdot\mu_C$ on $\mathbb{P}^1(\mathbb{C})$, then we have the equality $(f_G^*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) = (d\cdot(\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{U_{\overrightarrow{\mathcal{S}_G x}}\})$.

(b-2). Similarly to (a-2), we also compute both

$$(f_G^*\omega_\mu)(\{\{\mathcal{S}_G\}\}) \underset{(2.13)}{=} \int_{\mathsf{P}^1/S(\Gamma_G)} m_{V,\{\mathcal{S}_G\}}(f)\omega_\mu(V) = \deg_{\mathcal{S}_G}(f)\cdot\omega_\mu(\{\{\mathcal{S}_G\}\})$$
$$\underset{(2.6)}{=} \deg(\phi_{\widetilde{A\circ f}})\cdot\nu_E(\mathbb{P}^1(\mathbb{C})) = ((\widetilde{A\circ f})^*\mu_E)(\mathbb{P}^1(\mathbb{C})\setminus F_1)$$

and

$$((\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{\{\mathcal{S}_G\}\}) \underset{(1.3)}{=} \omega_\mu(\{\{\mathcal{S}_G\}\}) = \nu_C(\mathbb{P}^1(\mathbb{C})) = \mu_C(\mathbb{P}^1(\mathbb{C})\setminus F_2),$$

where F_1, F_2 are any sufficiently large countable subsets in $\mathbb{P}^1(\mathbb{C})$.

Hence, if $(\widetilde{A\circ f})^*\mu_E=d\cdot\mu_C$ on $\mathbb{P}^1(\mathbb{C})$, then we also have $(f_G^*\omega_\mu)(\{\{\mathcal{S}_G\}\})=(d\cdot(\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu)(\{\{\mathcal{S}_G\}\})$. Now the proof is also complete in this case. □

The following complements Theorem 3.5.

Proposition 4.5. *If $\mu_C=\lim_{j\to\infty}\mu_{f_{t_j}}$, $\mu_E=\lim_{j\to\infty}(A_{t_j})_*\mu_{f_{t_j}}$ are weak limit points on $\mathbb{P}^1(\mathbb{C})$ as $t\to 0$ of $(\mu_{f_t})_{t\in\mathbb{D}^*}$, $((A_t)_*\mu_{f_t})_{t\in\mathbb{D}^*}$, respectively, for some sequence $(t=t_j)$ in $\mathbb{D}^*$ tending to 0 as $j\to\infty$, then $\mu:=(\mu_C,\mu_E)\in(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2$ also satisfies the admissibility* (4.4) *(for A).*

Proof. When $\Gamma_f=\Gamma_G$ or equivalently $\tilde{A}=\phi_{\tilde{A}}$, by the uniform convergence (3.2) and $\mathrm{supp}[H_{\tilde{A}}=0]=\emptyset$, we have $\tilde{A}_*\mu_C=\mu_E$ on $\mathbb{P}^1(\mathbb{C})$, that is, the admissibility $\tilde{A}^*\mu_E=\mu_C$ on $\mathbb{P}^1(\mathbb{C})$ in this case holds.

When $\Gamma_f\neq\Gamma_G$, for $0<\varepsilon\ll 1$, by the outer regularity of μ_E, there is a continuous test function ψ on $\mathbb{P}^1(\mathbb{C})$ such that $\psi\geq 0$ on $\mathbb{P}^1(\mathbb{C})$, that $\psi\equiv 1$ on an open neighborhood of a_A, and that $\mu_E(\{a_A\})+\varepsilon/2>\int_{\mathbb{P}^1(\mathbb{C})}\psi\mu_E$. Then, for any continuous test function η on $\mathbb{P}^1(\mathbb{C})$ supported by $\mathbb{P}^1(\mathbb{C})\setminus\{h_A\}$ and satisfying $0\leq\eta\leq 1$ on $\mathbb{P}^1(\mathbb{C})$, we have

$$\begin{aligned}\mu_E(\{a_A\})+\varepsilon>\int_{\mathbb{P}^1(\mathbb{C})}\psi((A_{t_j})_*\mu_{f_{t_j}})&=\int_{\mathbb{P}^1(\mathbb{C})}(\psi\circ A_{t_j})\mu_{f_{t_j}}\\&\geq\int_{\mathrm{supp}\,\eta}(\psi\circ A_{t_j})\cdot\eta\mu_{f_{t_j}}\quad\text{for } j\gg 1.\end{aligned}$$

Then, by the uniform convergence (3.2) and the first item in (4.1), we even have $\mu_E(\{a_A\})+\varepsilon>\int_{\mathrm{supp}\,\eta}1\cdot\eta\mu_{f_{t_j}}=\int_{\mathbb{P}^1(\mathbb{C})}\eta\mu_{f_{t_j}}$ for $j\gg 1$, so that $\mu_E(\{a_A\})+\varepsilon\geq\int_{\mathbb{P}^1(\mathbb{C})}\eta\mu_C$ making $j\to\infty$. Hence, by the inner regularity of μ_C, we have $\mu_E(\{a_A\})+2\varepsilon\geq\mu_C(\mathbb{P}^1(\mathbb{C})\setminus\{h_A\})$, and in turn $\mu_E(\{a_A\})\geq\mu_C(\mathbb{P}^1(\mathbb{C})\setminus\{h_A\})$, that is, the admissibility $\mu_C(\{h_A\})+\mu_E(\{a_A\})\geq(\mu_C(\mathbb{P}^1(\mathbb{C}))=)\ 1$ in this case also holds. □

§5. Proof of Theorem A

Let K be an algebraically closed field that is complete with respect to a non-trivial and non-archimedean absolute value $|\cdot|$, and let $f\in K(z)$ be a rational function on $\mathbb{P}^1$ of $\deg f=:d>1$. Recall that

$$\Gamma_G:=\{\mathcal{S}_G\}\quad\text{and}\quad\Gamma_n:=\Gamma_{f^n}:=\{\mathcal{S}_G,f^n(\mathcal{S}_G)\}\quad\text{in }\mathsf{H}^1_{\mathrm{II}}$$

for each $n\in\mathbb{N}$, and the definitions of Δ_f, $\Delta_f^\dagger$ in Section 1.3.

Lemma 5.1 (Cf. [5, Lem. 4.4]). *For every $\nu\in M^1(\mathsf{P}^1)$, if ν has the f-balanced property $f^*\nu=d\cdot\nu$ on P^1 and satisfies $\nu(\{f(\mathcal{S}_G)\})=0$, then for every $n\in\mathbb{N}$,*

$(\pi_{\Gamma_n})_*\nu \in M^1(\Gamma_n)$ has the quantized f^n-balanced property (see (1.9)), and if in addition $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$, then $(\pi_{\Gamma_G})_*\nu \in \Delta_f^\dagger$.

Proof. Under the assumption on ν, for every $U \in S(\Gamma_G) \setminus \{\{\mathcal{S}_G\}\}$, we compute

$$\begin{aligned}(f_G^*((\pi_{\Gamma_f})_*\nu))(\{U\}) &= \langle (\pi_{\Gamma_f})^*(f_{G,*}1_{\{U\}}), \nu\rangle \\ &\underset{(2.12)}{=} \int_{\mathsf{P}^1\setminus\{f(\mathcal{S}_G)\}} ((f^*\delta_\cdot)(U))\nu = \langle (f^*\delta_\cdot)(U), \nu\rangle = \langle 1_U, f^*\nu\rangle \\ &= \langle (\pi_{\Gamma_G})^* 1_{\{U\}}, d\cdot\nu\rangle = d\cdot((\pi_{\Gamma_G})_*\nu)(\{U\}) \\ &= d\cdot\big((\pi_{\Gamma_f,\Gamma_G})_*((\pi_{\Gamma_f})_*\nu)\big)(\{U\}),\end{aligned}$$

so that also recalling (1.7), $(\pi_{\Gamma_f})_*\nu \in M^1(\Gamma_f)$ has the quantized f-balanced property (1.9). On the other hand, for any $n \in \mathbb{N}$, we have $(f^n)^*\nu = d^n \cdot \nu$ on P^1, and in turn

$$\begin{aligned}0 &= d^{n-1}\cdot(\deg_{\mathcal{S}_G} f)\cdot\nu(\{f(\mathcal{S}_G)\}) = d^{n-1}\cdot(f^*\nu)(\{\mathcal{S}_G\}) \\ &= ((f^n)^*\nu)(\{\mathcal{S}_G\}) = \deg_{\mathcal{S}_G}(f^n)\cdot\nu(\{f^n(\mathcal{S}_G)\}) \geq \nu(\{f^n(\mathcal{S}_G)\})\,(\geq 0),\end{aligned}$$

so $\nu(\{f^n(\mathcal{S}_G)\}) = 0$. Hence the former assertion holds, and so does the latter by (1.6), (1.7), (1.8)(, and $(\pi_{\Gamma_G})_* = (\pi_{\Gamma_n,\Gamma_G})_*(\pi_{\Gamma_n})_*$). □

Proof of Theorem A. Suppose that $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$, which is equivalent to

$$\nu_f(\{f(\mathcal{S}_G)\}) = \nu_f(\{\mathcal{S}_G\}) = ((\pi_{\Gamma_G})_*\nu_f)(\{\{\mathcal{S}_G\}\}) = 0 \tag{5.1}$$

(by (1.6)). Then, by Lemma 5.1, we have $(\pi_{\Gamma_G})_*\nu_f \in \Delta_f^\dagger$. Suppose also that $\operatorname{char} K = 0$ (so $\#E(f) \leq 2$) and, in turn, that for any $a \in E(f)$, $f(a) = a$ or equivalently $f^{-1}(a) = \{a\}$. Then, for every $a \in E(f)$, by Lemma 5.1, we also have $(\pi_{\Gamma_G})_*\delta_a \in \Delta_f^\dagger$. Moreover, for every $a \in E(f)$, every $n \in \mathbb{N}$, and every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$, by Facts 2.5 and 2.4, we have

$$s_{\vec{v}}(f^n) = 0 \ \ (\Leftrightarrow f^n(U_{\vec{v}}) = U_{(f^n)_*\vec{v}}) \quad \text{and} \tag{5.2}$$

$$(f^n)_*(\vec{v}) \neq \overrightarrow{f^n(\mathcal{S}_G)a}, \tag{5.3}$$

and for every $a \in E(f)$ and every $n \in \mathbb{N}$, we also have

$$s_{\overrightarrow{\mathcal{S}_G a}}(f^n) = d^n - \deg_{\mathcal{S}_G}(f^n) \quad \text{(also using (2.8))} \quad \text{and} \tag{5.4}$$

$$(f^n)_*(\overrightarrow{\mathcal{S}_G a}) = \overrightarrow{f^n(\mathcal{S}_G)a} \quad \text{(also using Fact 2.4).} \tag{5.5}$$

(a). Let us see the former half in Theorem A. If, for any $\vec{v} \in T_{\mathcal{S}_G}\mathsf{P}^1$, we have

$$\limsup_{n\to\infty} \frac{s_{\vec{v}}(f^n)}{d^n} \geq \nu_f(U_{\vec{v}})\,(= ((\pi_{\Gamma_G})_*\nu_f)(\{U_{\vec{v}}\})), \tag{5.6}$$

then for every $\omega\in\Delta_f$ and $n\gg 1$, fixing $\omega_n\in M^1(\Gamma_n)$ such that $\omega_n(S(\Gamma_n)\setminus F)=0$ for some countable subset F in $S(\Gamma_n)$ and that $d^{-n}(f^n)^*_G\omega_n=\omega\,(=(\pi_{\Gamma_n,\Gamma_G})_*\omega_n)$ in $M^1(\Gamma_G)$, also recalling Definition 2.9, for every $\vec{v}\in T_{\mathcal{S}_G}\mathsf{P}^1$, we have

$$\omega(\{U_{\vec{v}}\})=\limsup_{n\to\infty}\frac{((f^n)^*_G\omega_n)(\{U_{\vec{v}}\})}{d^n}\underset{(2.13)}{\geq}\limsup_{n\to\infty}\Bigl(\frac{s_{\vec{v}}(f^n)}{d^n}\cdot\omega_n(\mathsf{P}^1/S(\Gamma_n))\Bigr)$$
$$=\limsup_{n\to\infty}\frac{s_{\vec{v}}(f^n)}{d^n}\geq((\pi_{\Gamma_G})_*\nu_f)(\{U_{\vec{v}}\}),$$

which with (5.1) and (1.7) yields $\omega=(\pi_{\Gamma_G})_*\nu_f$ in $M^1(\Gamma_G)$. Hence we have $\Delta_f=\Delta_f^\dagger=\{(\pi_{\Gamma_G})_*\nu_f\}$, i.e., the case (i) in Theorem A holds, under this "surplus equidistribution" assumption (5.6) (see [5, p. 27]).

(b.1). Alternatively, suppose that there is $\vec{u}\in T_{\mathcal{S}_G}\mathsf{P}^1$ not satisfying (5.6). Then, fixing any $\mathcal{S}\in\mathbb{P}^1\setminus E(f)\,(\subset\mathsf{P}^1\setminus(E(f)\cup\{f^n(\mathcal{S}_G):n\in\mathbb{N}\}))$, we have

$$\begin{aligned}\nu_f(U_{\vec{u}})&\leq\limsup_{n\to\infty}\frac{((f^n)^*\delta_{\mathcal{S}})(U_{\vec{u}})}{d^n}\\&\underset{(2.7)}{\leq}\limsup_{n\to\infty}\frac{m_{\vec{u}}(f^n)}{d^n}+\limsup_{n\to\infty}\frac{s_{\vec{u}}(f^n)}{d^n}<\limsup_{n\to\infty}\frac{m_{\vec{u}}(f^n)}{d^n}+\nu_f(U_{\vec{u}}),\end{aligned}$$

the first inequality in which is by the inner regularity of ν_f and (1.4)(, and the equality holds if $\mathcal{S}_G\in\mathsf{P}^1\setminus\mathsf{J}(f)$). Hence $0<\limsup_{n\to\infty}(m_{\vec{u}}(f^n)/d^n)=\prod_{j=0}^{\infty}(m_{(f^j)_*(\vec{u})}(f)/d)$, so that $m_{(f^n)_*(\vec{u})}(f)\equiv d\ (>1)$ for $n\gg 1$, and in turn, also recalling (2.6) (and the maximal-ramification locus $\mathsf{R}_{\max}(f)$ of f in Section 1.2), that

$$\deg_{f^n(\mathcal{S}_G)}(f)\equiv d,\ \text{ i.e., } f^n(\mathcal{S}_G)\in\mathsf{R}_{\max}(f),\quad\text{for } n\gg 1; \tag{5.7}$$

then $f^{n+1}(\mathcal{S}_G)\neq f^n(\mathcal{S}_G)$ for $n\gg 1$ under the assumption that $f^{-1}(\mathcal{S}_G)\neq\{\mathcal{S}_G\}$.

Also recall that $\mathsf{R}_{\max}(f)$ of f is connected in P^1. Hence, for $n\gg 1$, we have $f^{-1}([f^n(\mathcal{S}_G),f^{n+1}(\mathcal{S}_G)])=[f^{n-1}(\mathcal{S}_G),f^n(\mathcal{S}_G)]\subset\mathsf{R}_{\max}(f)$, and then f restricts to a homeomorphism from $[f^{n-1}(\mathcal{S}_G),f^n(\mathcal{S}_G)]$ onto $[f^n(\mathcal{S}_G),f^{n+1}(\mathcal{S}_G)]$ and, recalling (2.6), we also have $\mathcal{S}\mapsto m_{\overrightarrow{\mathcal{S}f^n(\mathcal{S}_G)}}(f)=\deg_{\mathcal{S}}(f)\equiv d\ (>1)$ on $[f^{n-1}(\mathcal{S}_G),f^n(\mathcal{S}_G)]$. Then, for any $m\geq n\gg 1$, $\rho(f^m(\mathcal{S}_G),f^{m+1}(\mathcal{S}_G))=d^{m-n}\cdot\rho(f^n(\mathcal{S}_G),f^{n+1}(\mathcal{S}_G))$ by (2.10). Consequently, also by the upper semicontinuity of $\deg_{\cdot}(f)$ on P^1, there is $a\in\mathbb{P}^1$ such that

$$\bigl\{f^n(a):n\in\mathbb{N}\cup\{0\}\bigr\}\subset(\mathbb{P}^1\cap\mathsf{R}_{\max}(f))\cap\bigcap_{N\in\mathbb{N}}\overline{\{f^n(\mathcal{S}_G):n\geq N\}},$$

which with $\#(\mathbb{P}^1\cap\mathsf{R}_{\max}(f))\leq 2$ (mentioned in (1.5)) still implies

$$a\in E(f).$$

Under the assumption that $f(a)=a$ (or equivalently $f^{-1}(a)=\{a\}$ so $f'(a)=0$), we conclude that $\lim_{n\to\infty} f^n(\mathcal{S}_G)=a$ (and $\mathcal{S}_G\in\mathsf{P}^1\setminus\mathsf{J}(f)$) and, moreover, that $f^n(\mathcal{S}_G)\in(\mathcal{S}_G,a]$ for $n\gg 1$, using [8, Thm. F] and (2.10) (see [5, p. 25]) (or now assuming that f is tamely maximally ramified near this $a\in E(f)\subset\mathsf{R}_{\max}(f)\cap\mathbb{P}^1$, for simplicity).

Remark 5.2. Conversely, if there is such an $a\in E(f)$ that $\lim_{n\to\infty} f^n(\mathcal{S}_G)=a$ and that $f^n(\mathcal{S}_G)\in(\mathcal{S}_G,a]$ for $n\gg 1$, then (5.7) is the case (since there is $\mathcal{S}\in(a,\mathcal{S}_G]$ so close to a that $(a,\mathcal{S}]\subset\mathsf{R}_{\max}(f)$), and (5.7) together with (5.2) and (5.4) implies that the inequality (5.6) for this a does not hold for some $\vec{v}\in T_{\mathcal{S}_G}\mathsf{P}^1$.

(b.2). Once such an $a\in E(f)$ is at our disposal, noting that $f^{-1}(a)=\{a\}$, that $\lim_{n\to\infty} f^n(\mathcal{S}_G)=a$, and that $f^n(\mathcal{S}_G)\in(\mathcal{S}_G,a]$ for $n\gg 1$, we have

$$f(U_{\overrightarrow{f^n(\mathcal{S}_G)a}})=U_{\overrightarrow{f^{n+1}(\mathcal{S}_G)a}}\quad\text{for } n\gg 1 \tag{5.8}$$

(also by Fact 2.5 applied to $\overrightarrow{f^n(\mathcal{S}_G)a}\in T_{f^n(\mathcal{S}_G)}\mathsf{P}^1$) and have not only

$$\nu_f(U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}})=1\quad\text{for } n\gg 1 \tag{5.9}$$

but also $\mathcal{S}_G\in\mathsf{P}^1\setminus\mathsf{J}(f)$ (also since $a\in\mathsf{P}^1\setminus\mathsf{J}(f)$ and $f(\mathsf{J}(f))=\mathsf{J}(f)$). Hence fixing such $n_0\gg 1$ that $\deg_{\mathcal{S}_G}(f^n)/d^n$ is constant for $n\geq n_0$ (by (5.7)) and fixing any $\mathcal{S}\in\mathsf{P}^1\setminus E(f)$, for every $n\geq n_0$, we also have

$$\begin{aligned}
0&<\frac{\deg_{\mathcal{S}_G}(f^n)}{d^n}\Bigl(\underset{(5.4)}{=}\frac{d^n-s_{\overrightarrow{\mathcal{S}_Ga}}(f^n)}{d^n}=\\
&\underset{\substack{(5.5)\&(2.7)\\ \text{when } n\gg1}}{=}1-\frac{(f^n)^*\delta_{\mathcal{S}}}{d^n}(U_{\overrightarrow{\mathcal{S}_Ga}})\equiv 1-\limsup_{n\to\infty}\frac{(f^n)^*\delta_{\mathcal{S}}}{d^n}(U_{\overrightarrow{\mathcal{S}_Ga}})=\Bigr)\\
&\underset{\substack{(1.4)\&\\ \mathcal{S}_G\in\mathsf{P}^1\setminus\mathsf{J}(f)}}{\equiv}1-\nu_f(U_{\overrightarrow{\mathcal{S}_Ga}});
\end{aligned}\tag{5.10}$$

in particular, $\nu_f(U_{\overrightarrow{\mathcal{S}_Ga}})<1$, and in turn $(\pi_{\Gamma_G})_*\nu_f\neq(\pi_{\Gamma_G})_*\delta_a$.

Now the case (ii) in Theorem A holds under this "surplus *in*equidistribution" assumption, and the proof of the former half in Theorem A is complete.

Remark 5.3. In [5, §4.6], the condition $\mathsf{J}(f)\subset\mathsf{P}^1\setminus(U_{\overrightarrow{\mathcal{S}_Ga}}\cup\{\mathcal{S}_G\})$ was assumed with loss of some generality; under this condition, the vanishing assumption on each ω_n in the definition (1.10) of Δ_f does not matter (and did not appear in

[5, §4.6]). By (5.10) (and $\deg_{\cdot}(f) \in \{1,\dots,d\}$), the statement $\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) = 0$ ($\Leftarrow$ $\mathsf{J}(f) \subset \mathsf{P}^1 \setminus (U_{\overrightarrow{\mathcal{S}_G a}} \cup \{\mathcal{S}_G\})$) is equivalent to

$$\deg_{f^n(\mathcal{S}_G)}(f) \equiv d \quad \text{for any } n \in \mathbb{N} \cup \{0\}, \tag{5.7'}$$

and is indeed not always the case (as seen in Section 7 below).

(c.1). Let us show the latter half, i.e., the equality (1.11), in Theorem A. For $n \gg 1$, by (5.9), (5.2), the f^n-balanced property of ν_f on P^1, and Fact 2.5, for every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$, we have the equivalence

$$\nu_f(U_{\vec{v}}) > 0 \;\Leftrightarrow\; (f^n)_*(\vec{v}) = \overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G} \;\Leftrightarrow\; f^n(U_{\vec{v}}) = U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}} \tag{5.11}$$

(one of) which is the case for at least one $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$ since $\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) < 1$. Hence, for $n \gg 1$, using the f^n-balanced property of ν_f on P^1 again, for every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}}) > 0$, we have

(5.12)
$$\begin{aligned}(0 <)\, \nu_f(U_{\vec{v}}) &= \frac{(f^n)^*\nu_f}{d^n}(U_{\vec{v}}) = \frac{1}{d^n}\int_{f^n(U_{\vec{v}})} ((f^n)^*\delta_{\mathcal{S}})(U_{\vec{v}})\nu_f(\mathcal{S}) \\ &\underset{(5.11)\&(2.7)}{=} \frac{m_{\vec{v}}(f^n) + s_{\vec{v}}(f^n)}{d^n} \cdot \nu_f(U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}) \underset{(5.2)\&(5.9)}{=} \frac{m_{\vec{v}}(f^n)}{d^n}\end{aligned}$$

(and $m_{(f^n)_*\vec{v}}(f) \equiv d$). On the other hand, for $n \gg 1$, by (5.3), (5.5), (5.11), and Fact 2.4, we have

$$\begin{aligned}&\bigl\{(f^n)_*(\vec{v}) : \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\} \text{ satisfying } \nu_f(U_{\vec{v}}) = 0\bigr\} \\ &\quad = (T_{f^n(\mathcal{S}_G)}\mathsf{P}^1) \setminus \bigl\{\overrightarrow{f^n(\mathcal{S}_G)a}, \overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}\bigr\}.\end{aligned} \tag{5.13}$$

Now we assume that f is tamely maximally ramified near this $a \in E(f) \subset \mathsf{R}(f) \cap \mathbb{P}^1$. Then there is $\mathcal{S} \in (\mathcal{S}_G, a] \setminus \{a\}$ such that $\mathsf{R}_{\max}(f) \cap U_{\overrightarrow{\mathcal{S}a}} = (\mathcal{S}, a]$, and in turn for every $\mathcal{S}' \in (\mathcal{S}, a] \setminus \{a\}$ and every $\vec{w} = \overrightarrow{\mathcal{S}'\mathcal{S}''} \in (T_{\mathcal{S}'}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}'a}, \overrightarrow{\mathcal{S}'\mathcal{S}}\}$, diminishing $[\mathcal{S}', \mathcal{S}'']$ if necessary, we have $m_{\vec{w}}(f) = m_{\overrightarrow{\mathcal{S}''\mathcal{S}'}}(f) \le \deg_{\mathcal{S}''}(f) < d$ by (2.10) and (2.6). Hence, by (5.13), for $n \gg 1$, since $\lim_{n\to\infty} f^n(\mathcal{S}_G) = a$ and $f^n(\mathcal{S}_G) \in (\mathcal{S}_G, a]$, for every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}}) = 0$, we have

$$m_{(f^n)_*(\vec{v})}(f) \le d - 1. \tag{5.14}$$

(c.2). Pick $\omega \in \Delta_f$ and, for $n \gg 1$, fix $\omega_n \in M^1(\Gamma_n)$ satisfying $\omega_n(S(\Gamma_n) \setminus F) = 0$ for some countable subset F in $S(\Gamma_n)$ and $d^{-n}(f^n)^*_G\omega_n = \omega = (\pi_{\Gamma_n,\Gamma_G})_*\omega_n$

in $M^1(\Gamma_G)$. Then, by the latter equality $\omega = (\pi_{\Gamma_n,\Gamma_G})_*\omega_n$, ω also satisfies $\omega(S(\Gamma_G)\setminus F)=0$ for some countable subset F in $S(\Gamma_G)$.

Let us compute $\omega(\{U\})$ for each $U\in S(\Gamma_G)$. For $n\gg 1$, using the equality $d^{-n}(f^n)^*_G\omega_n=\omega$ (and recalling Definition 2.9), we have both

$$\omega(\{U_{\vec{v}}\}) \underset{(5.2)\&(2.13)}{=} \frac{m_{\vec{v}}(f^n)}{d^n}\cdot\omega_n\bigl(\{V\in S(\Gamma_n): V\subset U_{(f^n)_*(\vec{v})}\}\bigr) \quad \text{for any } \vec{v}\in (T_{\mathcal{S}_G}\mathsf{P}^1)\setminus\{\overrightarrow{\mathcal{S}_G a}\} \tag{5.15}$$

and

$$\omega_n(\{\{f^n(\mathcal{S}_G)\}\}) \underset{(2.13)}{=} \frac{d^n\cdot\omega(\{\{\mathcal{S}_G\}\})}{\deg_{\mathcal{S}_G}(f^n)} \underset{(5.10)}{=} \frac{\omega(\{\{\mathcal{S}_G\}\})}{1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})}. \tag{5.16}$$

Then, for $n\gg 1$, by (5.15), (5.11), and (5.12), we have $\omega_n(\{V\in S(\Gamma_n): V\subset U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}\}) = \omega(\{U_{\vec{v}}\})/\nu_f(U_{\vec{v}})$ for every $\vec{v}\in(T_{\mathcal{S}_G}\mathsf{P}^1)\setminus\{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}})>0$. Hence there exists a constant $s_\omega\in[0,1]$ such that for $n\gg1$,

$$\omega_n\bigl(\{V\in S(\Gamma_n): V\subset U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}\}\bigr)\equiv s_\omega \tag{5.17}$$

and that for every $\vec{v}\in(T_{\mathcal{S}_G}\mathsf{P}^1)\setminus\{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}})>0$,

$$\omega(\{U_{\vec{v}}\}) = s_\omega\nu_f(U_{\vec{v}}). \tag{5.18}$$

Moreover, for every $\vec{v}\in(T_{\mathcal{S}_G}\mathsf{P}^1)\setminus\{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}})=0$, we have

$$0\le\omega(\{U_{\vec{v}}\}) \underset{(5.15)}{\le} \frac{m_{\vec{v}}(f^n)}{d^n}\cdot 1 = \prod_{j=0}^{n-1}\frac{m_{(f^j)_*(\vec{v})}(f)}{d} \underset{(5.14)}{\to} 0 \quad \text{as } n\to\infty,$$

so we still have

$$\omega(\{U_{\vec{v}}\}) = 0 = s_\omega\nu_f(U_{\vec{v}}). \tag{5.19}$$

Now, for $n\gg1$, we also have

$$\begin{aligned}\omega(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) &= 1-\omega\bigl(\{U_{\vec{v}}\in S(\Gamma_G): \vec{v}\in(T_{\mathcal{S}_G}\mathsf{P}^1)\setminus\{\overrightarrow{\mathcal{S}_G a}\}\}\cup\{\{\mathcal{S}_G\}\}\bigr)\\ &\Bigl(\underset{\substack{(5.18),(5.19),\\ \&(1.7)}}{=} 1-s_\omega\nu_f(\mathsf{P}^1\setminus U_{\overrightarrow{\mathcal{S}_G a}})-\omega(\{\{\mathcal{S}_G\}\})\Bigr)\\ &= (s_\omega\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})+(1-s_\omega))-\omega(\{\{\mathcal{S}_G\}\}).\end{aligned} \tag{5.20}$$

(c.3). Let us also see the desired estimate on $\omega(\{\{\mathcal{S}_G\}\})$. For $n \gg 1$, recalling $f^n(\mathcal{S}_G) \in (\mathcal{S}_G, a]$, we compute

$$\begin{aligned} 0 &\le \omega_n\bigl(\{U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}} \cap U_{\overrightarrow{\mathcal{S}_G a}}\}\bigr) \\ &\bigl(= \omega_n\bigl(\{V \in S(\Gamma_n) : V \subset U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}\}\bigr) \\ &\quad - \omega_n\bigl(\{U_{\vec{v}} \in S(\Gamma_n) : \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}\} \cup \{\{\mathcal{S}_G\}\}\bigr) \\ &\underset{(5.17)\&(5.20)}{\equiv} s_\omega + \bigl((s_\omega \nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s_\omega)) - \omega(\{\{\mathcal{S}_G\}\}) - 1\bigr)\bigr) \\ &= s_\omega \nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) - \omega(\{\{\mathcal{S}_G\}\}), \end{aligned} \tag{5.21}$$

which yields the upper bound $\omega(\{\{\mathcal{S}_G\}\}) \le s_\omega \nu_f(U_{\overrightarrow{\mathcal{S}_G a}})$. Moreover, for $n \gg 1$, by (5.13), (1.7), (5.15), and (5.19), we have

$$\omega_n\bigl(\{U_{\vec{w}} \in S(\Gamma_n) : \vec{w} \in (T_{f^n(\mathcal{S}_G)}\mathsf{P}^1) \setminus \{\overrightarrow{f^n(\mathcal{S}_G)a}, \overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}\}\}\bigr) = 0, \tag{5.22}$$

and using the equality $(\pi_{\Gamma_n,\Gamma_G})_*\omega_n = \omega$ in $M^1(\Gamma_G)$ (and (1.3)), we also have

$$\begin{aligned} \omega_n\bigl(\{V \in S(\Gamma_n) : V \subset U_{\overrightarrow{\mathcal{S}_G a}}\}\bigr) &= \omega(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) \\ &\underset{(5.20)}{=} (s_\omega \nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s_\omega)) - \omega(\{\{\mathcal{S}_G\}\}). \end{aligned} \tag{5.23}$$

Then, for $n \gg 1$, we compute

$$\begin{aligned} 0 &\le \omega_n\bigl(\{U_{\overrightarrow{f^n(\mathcal{S}_G)a}}\}\bigr) \\ &= \omega_n\bigl(\{V \in S(\Gamma_n) : V \subset U_{\overrightarrow{\mathcal{S}_G a}}\}\bigr) - \omega_n\bigl(\{U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}} \cap U_{\overrightarrow{\mathcal{S}_G a}}\}\bigr) \\ &\quad - \omega_n\bigl(\{U_{\vec{w}} : \vec{w} \in (T_{f^n(\mathcal{S}_G)}\mathsf{P}^1) \setminus \{\overrightarrow{f^n(\mathcal{S}_G)a}, \overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}\}\}\bigr) - \omega_n(\{\{f^n(\mathcal{S}_G)\}\}) \\ &\underset{\substack{(5.23),(5.21),\\(5.22),\&(5.16)}}{=} (1-s_\omega) - \frac{\omega(\{\{\mathcal{S}_G\}\})}{1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})}, \end{aligned}$$

which yields the other upper bound $\omega(\{\{\mathcal{S}_G\}\}) \le (1-s_\omega)(1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}))$. Hence Δ_f is contained in the right-hand side in (1.11).

(c.4). Conversely, pick ω in the right-hand side in (1.11), so that for some $s \in [0,1]$ and some $s' \in [0, \min\{s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}), (1-s)(1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}))\}]$, we have

$$\begin{aligned} \omega(\{U_{\vec{v}}\}) &= s\nu_f(U_{\vec{v}}) \quad \text{for every } \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}, \\ \omega(\{\{\mathcal{S}_G\}\}) &= s', \quad \text{and} \\ \omega(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) &= (s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s)) - s'. \end{aligned}$$

For $n \gg 1$, recalling that $\lim_{n\to\infty} f^n(\mathcal{S}_G) = a$, that $f^n(\mathcal{S}_G) \in (\mathcal{S}_G, a]$, and that $\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) < 1$, there is $\omega_n \in M^1(\Gamma_n)$ such that

$$\begin{cases} \omega_n(\{\{\mathcal{S}_G\}\}) = s', \\ \omega_n(\{\{f^n(\mathcal{S}_G)\}\}) = \dfrac{s'}{1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})}, \\ \omega_n(\{U_{\vec{v}}\}) = \begin{cases} s\nu_f(U_{\vec{v}}) & \text{for every } \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}, \\ 0 & \text{for every } \vec{v} \in (T_{f^n(\mathcal{S}_G)}\mathsf{P}^1) \setminus \{\overrightarrow{f^n(\mathcal{S}_G)a}, \overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}\}, \end{cases} \\ \omega_n\bigl(\{U_{\overrightarrow{\mathcal{S}_G a}} \cap U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}\}\bigr) = s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) - s' \,(\ge 0), \quad \text{and} \\ \omega_n\bigl(\{U_{\overrightarrow{f^n(\mathcal{S}_G)a}}\}\bigr) = 1 - s - \dfrac{s'}{1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})} \,(\ge 0) \end{cases}$$

(indeed, $\omega_n \ge 0$ and $\omega_n(\mathsf{P}^1/S(\Gamma_n)) = 1 - s + s\nu_f(\mathsf{P}^1 \setminus \{\mathcal{S}_G\}) \underset{(5.1)}{=} 1$) and that $\omega_n(S(\Gamma_n) \setminus F) = 0$ for some countable subset F in $S(\Gamma_n)$ (by (1.7)). Then, for $n \gg 1$, we have $(\pi_{\Gamma_n,\Gamma_G})_*\omega_n = \omega$ in $M^1(\Gamma_G)$ (also by (1.3)). Moreover, for $n \gg 1$, recalling Definition 2.9,

(I) for every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}}) > 0$, we have

$$\begin{aligned} (d^{-n}(f^n)^*_G\omega_n)(\{U_{\vec{v}}\}) &\underset{\substack{(2.13),(5.2),\\ \&(5.11)}}{=} \frac{m_{\vec{v}}(f^n) \cdot \omega_n\bigl(\{V \in S(\Gamma_n) : V \subset U_{\overrightarrow{f^n(\mathcal{S}_G)\mathcal{S}_G}}\}\bigr)}{d^n} \\ &\underset{(5.12)\&(1.7)}{=} \nu_f(U_{\vec{v}}) \cdot s\nu_f(\mathsf{P}^1 \setminus \{\mathcal{S}_G\}) \\ &\underset{(5.1)}{=} s\nu_f(U_{\vec{v}}) = \omega(\{U_{\vec{v}}\}), \end{aligned}$$

(II) for every $\vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}$ satisfying $\nu_f(U_{\vec{v}}) = 0$,

$$\begin{aligned} (d^{-n}(f^n)^*_G\omega_n)(\{U_{\vec{v}}\}) &\underset{(2.13)\&(5.2)}{=} \frac{m_{\vec{v}}(f^n) \cdot \omega_n\bigl(\{U_{(f^n)_*(\vec{v})}\}\bigr)}{d^n} \\ &\underset{(5.13)}{=} 0 = s\nu_f(U_{\vec{v}}) = \omega(\{U_{\vec{v}}\}), \end{aligned}$$

(III) and we have

$$\begin{aligned} (d^{-n}(f^n)^*_G\omega_n)(\{\{\mathcal{S}_G\}\}) &\underset{(2.13)}{=} \frac{\deg_{\mathcal{S}_G}(f^n) \cdot \omega_n(\{\{f^n(\mathcal{S}_G)\}\})}{d^n} \\ &\underset{(5.10)}{=} (1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}})) \cdot \omega_n(\{\{f^n(\mathcal{S}_G)\}\}) \\ &= s' = \omega(\{\{\mathcal{S}_G\}\}), \end{aligned}$$

and then

$$(d^{-n}(f^n)^*_G\omega_n)(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) = 1 - (d^{-n}(f^n)^*_G\omega_n)(S(\Gamma_G) \setminus \{U_{\overrightarrow{\mathcal{S}_G a}}\}) \\ = 1 - \omega(S(\Gamma_G) \setminus \{U_{\overrightarrow{\mathcal{S}_G a}}\}) = \omega(\{U_{\overrightarrow{\mathcal{S}_G a}}\}).$$

Hence, for $n \gg 1$, we also have $d^{-n}(f^n)^*_G\omega_n = \omega$ in $M^1(\Gamma_G)$, and the right-hand side in (1.11) is contained in Δ_f.

(d). Once the equality (1.11) is at our disposal, the final assertion in case (ii) in Theorem A (under the assumption that f is tamely maximally ramified near a) is clear, also recalling Remark 5.3. Now the proof of Theorem A is complete. □

§6. Proof of Theorem B

We use the notation in Sections 3 and 4. Let

$$f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)(\subset \mathbb{L}(z))$$

be a meromorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$ of degree $d > 1$, and suppose that $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ in $\mathsf{P}^1(\mathbb{L})$. Then $f^{-n}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ for every $n \in \mathbb{N}$ (see Section 1.2). Recall that $\operatorname{char}\mathbb{L} = \operatorname{char} k_{\mathbb{L}} = \operatorname{char}\mathbb{C} = 0$ and that the absolute value $|\cdot|_r$ on $\mathbb{L}$ is (the extension of) (3.1), fixing $r \in (0,1)$ once and for all. Since $\nu_{f^2} = \nu_f$ on $\mathsf{P}^1(\mathbb{L})$, $\mu_{(f_t)^2} = \mu_{f_t}$ on $\mathbb{P}^1(\mathbb{C})$ for every $t \in \mathbb{D}^*$, $E(f^2) = E(f)$, and $\#E(f) \leq 2$, replacing f with f^2 if necessary, we can assume that $f(a) = a$ or equivalently $f^{-1}(a) = \{a\}$ for any $a \in E(f)$ with no loss of generality.

Recall that

$$\Gamma_G := \{\mathcal{S}_G\} \quad \text{and} \quad \Gamma_n = \Gamma_{f^n} := \{\mathcal{S}_G, f^n(\mathcal{S}_G)\} \quad \text{in } \mathsf{H}^1_{\mathrm{II}}(\mathbb{L})$$

for every $n \in \mathbb{N}$, and that $M^1(\Gamma_G)^\dagger$ is identified with $M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$ under the bijection $S(\Gamma_G) \setminus \{\mathcal{S}_G\} = T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) \cong \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C})$. For every $n \in \mathbb{N}$, pick a meromorphic family

$$A_n \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$$

of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ such that $(A_n \circ f^n)(\mathcal{S}_G) = \mathcal{S}_G$ in $\mathsf{P}^1(\mathbb{L})$ (by the existence part of Theorem 3.4).

Let

$$\mu_0 = \lim_{j\to\infty} \mu_{f_{t_j}}$$

be any weak limit point of $(\mu_{f_t})_{t\in\mathbb{D}^*}$ on $\mathbb{P}^1(\mathbb{C})$ as $t \to 0$, where the sequence (t_j) in $\mathbb{D}^*$ tends to 0 as $j \to \infty$. Then, taking a subsequence of (t_j) if necessary, for any $n \in \mathbb{N}$, there also exists the weak limit

$$\mu_E^{(n)} := \lim_{j\to\infty} ((A_n)_{t_j})_*\mu_{f_{t_j}} \quad \text{on } \mathbb{P}^1(\mathbb{C}).$$

For every $n \in \mathbb{N}$, by Theorem 3.5 and Proposition 4.5, the ordered pair

$$\mu^{(n)} := (\mu_0, \mu_E^{(n)}) \in (M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2$$

not only has the degenerating f^n-balanced property (the former half in (3.4)) but also satisfies the admissibility (4.4) (for A_n), and in turn also by Proposition 4.4, we have

$$\omega_0 := (\pi_{\Gamma_n,\Gamma_G})_*\omega_{\mu^{(n)}} \in \Delta_f^\dagger;$$

this measure ω_0 is indeed independent of $n \in \mathbb{N}$, and is identified with μ_0 under the identification of $M^1(\Gamma_G)^\dagger$ with $M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$ (by (4.6)).

Hence, in the case (i) in Theorem A, we have the desired $\mu_0\,(=\omega_0) = (\pi_{\Gamma_G})_*\nu_f$ in $M^1(\Gamma_G)^\dagger = M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$.

(a). Suppose now that the case (ii) in Theorem A occurs. Then there is $a = a(t) \in E(f)\,(\subset \mathbb{P}^1(\mathbb{L}))$ such that $\lim_{n\to\infty} f^n(\mathcal{S}_G) = a$ and that $f^n(\mathcal{S}_G) \in (\mathcal{S}_G, a]$ for $n \gg 1$, and then $\deg_{f^n(\mathcal{S}_G)}(f) \equiv d$ for $n \gg 1$; since $\nu_{f^n} = \nu_f$ on $\mathsf{P}^1(\mathbb{L})$ for every $n \in \mathbb{N}$, $\mu_{(f_t)^n} = \mu_{f_t}$ on $\mathbb{P}^1(\mathbb{C})$ for every $t \in \mathbb{D}^*$ and every $n \in \mathbb{N}$, and $E(f^n) = E(f)$ for every $n \in \mathbb{N}$, replacing f with f^ℓ for some $\ell \gg 1$ if necessary, we also assume that for every $n \in \mathbb{N}$, $f^n(\mathcal{S}_G) \in (\mathcal{S}_G, a]$ (so $\Gamma_n \neq \Gamma_G$), $\deg_{f^n(\mathcal{S}_G)}(f) \equiv d$, and both (5.8) and (5.10) hold, with no loss of generality.

(b). Set

$$B_1(z) := \begin{cases} \dfrac{1}{z-a} & \text{if } a \in \mathcal{O}_\mathbb{L}, \\ \dfrac{-z}{(z/a)-1} & \text{if } a \in \mathbb{L} \setminus \mathcal{O}_\mathbb{L}, \\ z & \text{if } a = \infty \in \mathbb{P}^1(\mathbb{L})\ (= \mathbb{L} \cup \{\infty\}) \end{cases} \in \mathrm{PGL}(2, \mathcal{O}_\mathbb{L}),$$

so that $B_1(a) = \infty$ and that $B_1(\mathcal{S}_G) = \mathcal{S}_G$ (or equivalently $\widetilde{B_1} = \phi_{\widetilde{B_1}} \in \mathrm{PGL}(2, k_\mathbb{L})$ $= \mathrm{PGL}(2,\mathbb{C})$, and then $\widetilde{B_1^{-1}} = \phi_{\widetilde{B_1^{-1}}} = \phi_{\widetilde{B_1}}^{-1} \in \mathrm{PGL}(2, k_\mathbb{L}) = \mathrm{PGL}(2,\mathbb{C})$), and set

$$f_{B_1} := B_1 \circ f \circ B_1{}^{-1} \in \mathbb{L}[z].$$

(c.1). Write $f_{B_1}(z) = \sum_{j=0}^d c_j(t) z^j \in \mathbb{L}[z]$ (so $c_d \in \mathbb{L} \setminus \{0\}$) and set

$$d_0 := \max\Big\{ j \in \{0,1,\ldots,d\} : |c_j|_r = \max_{i \in \{0,1,\ldots,d\}} |c_i|_r \Big\}.$$

Then, noting that $f_{B_1}(\mathcal{S}_G) \in (\mathcal{S}_G, \infty]$, we have $|c_{d_0}|_r > 1$, and $f_{B_1}(\mathcal{S}_G)$ is represented by (the constant sequence of) the $\mathbb{L}$-closed disk $B(0, |c_{d_0}|_r)$. Setting

$$B_2(z) := c_{d_0}^{-1} z \in \mathbb{L}[z] \cap \mathrm{PGL}(2, \mathbb{L}),$$

so that $(B_2\circ f_{B_1})(\mathcal{S}_G)=\mathcal{S}_G$, we have $\phi_{\widetilde{B_2\circ f_{B_1}}}(\zeta)=\sum_{j=0}^{d_0}\widetilde{(\frac{c_j}{c_{d_0}})}\cdot\zeta^j$,

$$\text{(6.1)}\quad d_0=\deg(\phi_{\widetilde{B_2\circ f_{B_1}}}) \underset{(2.6)}{=} \deg_{\mathcal{S}_G}(B_2\circ f_{B_1})$$
$$=\deg_{f(\mathcal{S}_G)}(B_2\circ B_1)\cdot\deg_{B_1^{-1}(\mathcal{S}_G)}(f)\cdot\deg_{\mathcal{S}_G}(B_1^{-1})=\deg_{\mathcal{S}_G}(f)\,(>0),$$

and $(H_{\widetilde{B_2\circ f_{B_1}}}(\zeta_0,\zeta_1)=\zeta_0^{d-d_0}$, so in particular)

$$\text{(6.2)}\qquad \operatorname{ord}_{\zeta=\infty}\big[H_{\widetilde{B_2\circ f_{B_1}}}=0\big]=d-d_0=d-\deg_{\mathcal{S}_G}(f).$$

(c.2). For each $j\in\{0,\dots,d\}$, set

$$C_j=C_j(t):=\frac{c_j}{c_{d_0}}\cdot c_{d_0}^{j-d_0}\in\mathbb{L},\quad\text{so that } C_{d_0}=1 \text{ and that } |C_j|_r<1 \text{ if } j<d_0,$$

and also set

$$f_{B_2B_1}(w):=(B_2\circ f_{B_1}\circ B_2^{-1})(w)=c_{d_0}^{d_0}\Bigg(w^{d_0}+\sum_{j\in\{0,1,\dots,d\}\setminus\{d_0\}}C_jw^j\Bigg)\in\mathbb{L}[z].$$

Then, using Fact 2.6 and (2.4) (for $B_2^{-1},B_2\in\mathrm{PGL}(2,\mathbb{L})$), we have

$$\text{(6.3)}\qquad f_{B_2B_1}(U_{\overrightarrow{\mathcal{S}_G\infty}})$$
$$\Big(=(B_2\circ f_{B_1})(B_2^{-1}(U_{\overrightarrow{\mathcal{S}_G\infty}}))=(B_2\circ f_{B_1})(U_{\overrightarrow{B_2^{-1}(\mathcal{S}_G)\infty}})$$
$$\underset{(B_2\circ f_{B_1})(\mathcal{S}_G)=\mathcal{S}_G}{=}(B_2\circ f_{B_1})(U_{\overrightarrow{f_{B_1}(\mathcal{S}_G)\infty}})=B_2(f_{B_1}(U_{\overrightarrow{f_{B_1}(\mathcal{S}_G)\infty}}))$$
$$\underset{\substack{(5.8)\text{ applied}\\ \text{to } n=1}}{=}B_2(U_{\overrightarrow{f_{B_1}^2(\mathcal{S}_G)\infty}})=U_{\overrightarrow{((B_2\circ f_{B_1}^2)(\mathcal{S}_G))\infty}}\Big)$$
$$\subsetneq\mathsf{P}^1(\mathbb{L}).$$

Claim 1. *Either $d_0=d$ or there is $j>d_0$ such that $|C_j|_r\ge 1$.*

Proof. Otherwise, $d_0<d$ and $|C_j|_r<1$ for every $j\in\{0,\dots,d\}\setminus\{d_0\}$. Then, since $|c_{d_0}^{d_0}|_r=|c_{d_0}|_r^{d_0}>1$, we have $H_{\widetilde{f_{B_2B_1}}}(\zeta_0,\zeta_1)=\zeta_0^{d-d_0}\zeta_1^{d_0}$ (and $\phi_{\widetilde{f_{B_2B_1}}}\equiv\infty\in\mathbb{P}^1(k_{\mathbb{L}})$), so that $\operatorname{ord}_{\zeta=\infty}[H_{\widetilde{f_{B_2B_1}}}=0]=d-d_0$. In particular, we must have

$$s_{\overrightarrow{\mathcal{S}_G\infty}}(f_{B_2B_1})=\operatorname{ord}_{\zeta=\infty}[H_{\widetilde{f_{B_2B_1}}}=0]=d-d_0>0$$

(by Fact 2.7), so $f_{B_2B_1}(U_{\overrightarrow{\mathcal{S}_G\infty}})=\mathsf{P}^1(\mathbb{L})$ (by Fact 2.5). This contradicts (6.3). □

(c.3). Since this $a\in E(f)$ is a fixed point in $\mathbb{P}^1(\mathbb{L})$ of $f\in(\mathcal{O}(\mathbb{D})[t^{-1}])(z)$, this $a=a(t)$ is indeed in $\mathbb{P}^1(\mathbb{K})$ over a finite algebraic field extension $\mathbb{K}$ of the quotient field of the domain $\mathcal{O}(\mathbb{D})[t^{-1}]$, that is, for $0 < s_0 \ll 1$, by the substitution/change of indeterminants $t = s^m$ for some $m \in \mathbb{N}$, we have

$$a = a(s^m) \in \mathbb{P}^1(\mathcal{O}(\mathbb{D}_{s_0})[s^{-1}])\,(\subset \mathbb{P}^1(\mathbb{L})),$$

where $\mathbb{D}_{s_0} := \{s \in \mathbb{C} : |s| < s_0\}$ (cf. [5, Proof of Corollary 5.3]). Then, decreasing $0 < s_0 \ll 1$ if necessary, we have not only $c_j(s^m), C_j(s^m) \in \mathcal{O}(\mathbb{D}_{s_0})[s^{-1}] \subset \mathbb{L}$ for every $j \in \{0, 1, \ldots, d\}$ but also $(B_1)_{s^m}, (B_2)_{s^m} \in \mathrm{PGL}(2, \mathcal{O}(\mathbb{D}_{s_0})[s^{-1}])\,(\subset \mathrm{PGL}(2, \mathbb{L})$ and indeed $(B_1)_{s^m} \in \mathrm{PGL}(2, \mathcal{O}_{\mathbb{L}}))$, and still $(B_1)_{s^m}(\mathcal{S}_G) = \mathcal{S}_G$ in $\mathsf{P}^1(\mathbb{L})$ or equivalently $\widetilde{(B_1)_{s^m}} = \phi_{\widetilde{(B_1)_{s^m}}}\ (= \phi_{\widetilde{B_1}} = \widetilde{B_1})$ in $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{PGL}(2, k_{\mathbb{L}})$.

Let us, for notational simplicity, denote by

$$A := A_1 = (A_1)_t \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$$

the meromorphic family A_1 (A_n for $n = 1$) of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$, and also by

$$\mu_E := \mu_E^{(1)} \in M^1(\mathbb{C})^\dagger \quad \text{and} \quad \mu := \mu^{(1)} = (\mu_0, \mu_E) \in \left(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger\right)^2$$

the probability measure $\mu_E^{(1)}$ and the ordered pair $\mu^{(1)}$, respectively. Set

$$D = D_s := (B_2 \circ B_1)_{s^m} \circ (A_{s^m})^{-1} \in \mathrm{PGL}(2, \mathcal{O}(\mathbb{D}_{s_0})[s^{-1}])\,(\subset \mathrm{PGL}(2, \mathbb{L})),$$

so that $\widetilde{D} = \phi_{\widetilde{D}}$ in $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{PGL}(2, k_{\mathbb{L}})$ (by the uniqueness part in Theorem 3.4) since $((B_2 \circ B_1)_{s^m} \circ f_{s^m})(\mathcal{S}_G) = (B_2 \circ f_{B_1})_{s^m}((B_1)_{s^m}(\mathcal{S}_G)) = (B_2 \circ f_{B_1})_{s^m}(\mathcal{S}_G) = \mathcal{S}_G = (A \circ f)_{s^m}(\mathcal{S}_G)$.

Claim 2. $\mathrm{supp}((\phi_{\widetilde{D}})_* \mu_E) \subset \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$.

Proof. Recall that $|\cdot|_r$ and $|\cdot|$ are the absolute values on $\mathbb{L}$ and on $\mathbb{C}$, respectively. For every $s \in \mathbb{D}_{s_0}^*$ and every $z \in \mathbb{C}$, we compute

$$(f_{B_1})_{s^m}(c_{d_0}(s^m)z) = \left(c_{d_0}(s^m)\right)^{d_0+1} z^{d_0} \cdot \left\{1 + \sum_{j \in \{0,1,\ldots,d\} \setminus \{d_0\}} C_j(s^m) z^{j-d_0}\right\}.$$

Let us see that for $\ell \gg 1$, if $0 < |s| \ll s_0$, then

$$\inf_{|z|=\ell} \left|1 + \sum_{j \in \{0,1,\ldots,d\} \setminus \{d_0\}} C_j(s^m) z^{j-d_0}\right| \geq \frac{1}{2}\,(> 0);$$

for, in the latter case in Claim 1, we set

$$d_1 := \max\left\{j \in \{d_0 + 1, \ldots, d\} : |C_j|_r = \max_{j > d_0} |C_j|_r\ \ (\geq 1)\right\},$$

so that ($d_1 > d_0$, that) $\limsup_{s \to 0} |C_{d_1}(s^m)| \in (0, +\infty]$ (since $|C_{d_1}(s^m)|_r = |C_{d_1}|_r^m \geq 1$), and that for every $j > d_0$, $C_j(s^m)/C_{d_1}(s^m) \in \mathcal{O}(\mathbb{D}_{s_0})$, which vanishes at $s=0$ if $j>d_1$ (since $|C_j(s^m)/C_{d_1}(s^m)|_r=|C_j|_r^m/|C_{d_1}|_r^m$ is ≤ 1 if $j>d_0$, and is <1 if $j>d_1$). Then, for $\ell\gg 1$ (so that the second and third inequalities below hold), if $0<|s|\ll s_0$ (so that the first and fourth ones below hold), then

$$\begin{aligned}
&\Big|\sum_{j<d_0} C_j(s^m)z^{j-d_0}\Big| \\
&\quad\bigg(\le \sum_{j<d_0}(0+1)\ell^{j-d_0}\quad (\text{by } |C_j(s^m)|_r=|C_j|_r^m<1 \text{ if } j<d_0) \\
&\quad\le d_0\quad (\text{noting that the sum above is over } j<d_0) \\
&\quad\le \Big(\min\Big\{1,2^{-1}\cdot \limsup_{s\to 0}|C_{d_1}(s^m)|\Big\}\Big) \\
&\quad\quad\times\Big(\ell^{d_1-d_0}\Big(1-\sum_{d_1>j>d_0}\Big(\Big|\frac{C_j(s^m)}{C_{d_1}(s^m)}\Big|\Big|_{s=0}+1\Big)\ell^{j-d_1}\Big)-1\Big)-\frac{3}{2} \\
&\quad\le |C_{d_1}(s^m)|\cdot\bigg(\ell^{d_1-d_0}-\sum_{d_1>j>d_0}\Big|\frac{C_j(s^m)}{C_{d_1}(s^m)}\Big|\ell^{j-d_0}-\sum_{j>d_1}\Big|\frac{C_j(s^m)}{C_{d_1}(s^m)}\Big|\ell^{j-d_0}\bigg)-\frac{3}{2}\bigg) \\
&\quad\le \Big|\sum_{j>d_0}C_j(s^m)z^{j-d_0}\Big|-\frac{3}{2}
\end{aligned}$$

on $\{z\in\mathbb{C}:|z|=\ell\}$, which yields the desired inequality in this case. Similarly, in the former case ($d_0=d$) in Claim 1, for $\ell\gg 1$ (so that the final inequality below holds), if $0<|s|\ll s_0$ (so that the second inequality below holds), then

$$\Big|\sum_{j<d_0}C_j(s^m)z^{j-d_0}\Big|\le\sum_{j<d_0}|C_j(s^m)|\ell^{j-d_0}\le\sum_{j<d_0}(0+1)\ell^{j-d_0}\le\frac{1}{2}$$

on $\{z\in\mathbb{C}:|z|=\ell\}$, which still yields the desired inequality in this case.

Hence, since $d_0\ge 1$ (in (6.1)) and $|c_{d_0}(s^m)|_r=|c_{d_0}|_r^m>1$, fixing $\ell_0\gg 1$, if $0<|s|\ll s_0$, then $(f_{B_1})_{s^m}(\{z\in\mathbb{C}:|z|=|c_{d_0}(s^m)|\ell_0\})\subset\{z\in\mathbb{C}:|z|\ge |c_{d_0}(s^m)|^{d_0+1}\ell_0^{d_0}/2\}\subset\{z\in\mathbb{C}:|z|\ge 2|c_{d_0}(s^m)|\ell_0\}$, which with the maximum modulus principle for holomorphic functions applied to $1/((f_{B_1})_{s^m}(1/w))$ near $w=0\in\mathbb{C}$ in turn yields

$$(f_{B_1})_{s^m}\big(\{z\in\mathbb{C}:|z|>|c_{d_0}(s^m)|\ell_0\}\big)\subset\{z\in\mathbb{C}:|z|>2|c_{d_0}(s^m)|\ell_0\},$$

so that $\mathrm{supp}(((B_1)_{s^m})_*(\mu_{f_{s^m}}))$ $(=\mathrm{supp}(\mu_{(f_{B_1})_{s^m}}))\subset\{z\in\mathbb{C}:|z|\le|c_{d_0}(s^m)|\ell_0\}$ (see Fact 3.2). Hence, for $0<|s|\ll s_0$, recalling that $(B_2)_{s^m}(z)=(c_{d_0}(s^m))^{-1}z$, we have

$$\mathrm{supp}((D_s)_*(A_{s^m})_*\mu_{f_{s^m}})\Big(=\mathrm{supp}\big(((B_2\circ B_1)_{s^m})_*\mu_{f_{s^m}}\big)\Big)\subset\big\{z\in\mathbb{C}:|z|\le\ell_0\big\}.$$

Recall that $\mu_E := \lim_{j\to\infty}(A_{t_j})_*\mu_{f_{t_j}}$ weakly on $\mathbb{P}^1(\mathbb{C})$, and pick a sequence (s_j) in $\mathbb{D}^*$ so that $t_j = s_j^m$ for every $j \in \mathbb{N}$. Then $\lim_{j\to\infty} D_{s_j} = \phi_{\widetilde{D}} (= \widetilde{D})$ uniformly on $\mathbb{P}^1(\mathbb{C})$ (by (3.2)). Now the above inclusion for $s = s_j$, $j \gg 1$, completes the proof of Claim 2, by making $j \to \infty$. $\square$

(d). Recalling that $\omega_0 \,(= (\pi_{\Gamma_f,\Gamma_G})_*\omega_\mu) \in \Delta_f^\dagger \,(\subset \Delta_f)$, there are $s \in [0,1]$ and $s' \in [0, \min\{s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}), (1-s)(1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}))\}]$ such that

$$\begin{cases} \omega_0(\{U_{\vec{v}}\}) = s\nu_f(U_{\vec{v}}) \text{ for every } \vec{v} \in (T_{\mathcal{S}_G}\mathsf{P}^1) \setminus \{\overrightarrow{\mathcal{S}_G a}\}, \\ \omega_0(\{\{\mathcal{S}_G\}\}) = s', \quad \text{and} \\ \omega_0(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) = (s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s)) - s', \end{cases}$$

using the computation (1.11) of Δ_f under the standing assumption that the case (ii) in Theorem A occurs and by char $k_{\mathbb{L}} = 0$. Since $\omega_0 \in \Delta_f^\dagger$, we first have $s' = 0$.

Recalling the identification $\omega_0 = \mu_0$ in $M^1(\Gamma_G)^\dagger = M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$ and the degenerating f-balanced property (the former half in (3.4)) of $\mu = (\mu_0, \mu_E)$, we compute

$$\begin{aligned}(s\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) + (1-s)) - s' &= \omega_0(\{U_{\overrightarrow{\mathcal{S}_G a}}\}) = \mu_0(\{\tilde{a}\}) \\ &= \frac{(\widetilde{A \circ f})^*\mu_E}{d}(\{\tilde{a}\}) = \frac{((\phi_{\widetilde{A\circ f}})^*\mu_E + [H_{\widetilde{A\circ f}} = 0])(\{\tilde{a}\})}{d}\end{aligned}$$

and, moreover, recalling that $\widetilde{D} = \phi_{\widetilde{D}}$, $\widetilde{B_1} = \phi_{\widetilde{B_1}} \in \mathrm{PGL}(2,\mathbb{C})$, that $a = B_1^{-1}(\infty)$, that $(B_2 \circ f_{B_1})(\infty) = \infty$, and that $\deg(\phi_{\widetilde{B_2\circ f_{B_1}}}) = d_0 > 0$ (in (6.1)) and using Claim 2, we compute

$$\begin{aligned}((\phi_{\widetilde{A\circ f}})^*\mu_E)(\{\tilde{a}\}) &= ((\phi_{(\widetilde{D^{-1}\circ B_2\circ B_1 \circ f \circ B_1^{-1}})})^*\mu_E)(\{\infty\}) \\ &= ((\phi_{\widetilde{B_2\circ f_{B_1}}})^*(\phi_{\widetilde{D}})_*\mu_E)(\{\infty\}) \\ &= (\deg_\infty(\phi_{\widetilde{B_2\circ f_{B_1}}}))\cdot((\phi_{\widetilde{D}})_*\mu_E)(\{\infty\}) = 0,\end{aligned}$$

and on the other hand, we compute

$$\begin{aligned}&\mathrm{ord}_{\zeta=\tilde{a}}[H_{\widetilde{A\circ f}} = 0] \left(\underset{(2.9)}{=} s_{\overrightarrow{\mathcal{S}_G a}}(f) \underset{(2.7)}{=} s_{(B_1)_*(\overrightarrow{\mathcal{S}_G a})}(f_{B_1}) \underset{(2.4)}{=} s_{\overrightarrow{\mathcal{S}_G \infty}}(f_{B_1}) = \right) \\ &\underset{(2.9)}{=} \mathrm{ord}_{\zeta=\infty}[H_{\widetilde{B_2\circ f_{B_1}}} = 0] \underset{(6.2)}{=} d - \deg_{\mathcal{S}_G}(f) \underset{\substack{(5.10)\\ \text{for } n=1}}{=} d\cdot\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}).\end{aligned}$$

Hence we also have $s' = (1-s)(1-\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}))$.

Consequently, we have not only $s' = 0$ but also $s = 1$ since $\nu_f(U_{\overrightarrow{\mathcal{S}_G a}}) < 1$ (which is a consequence of (5.10)) in the case (ii) in Theorem A. Then we still have the desired $\mu_0 = \omega_0 = (\pi_{\Gamma_G})_*\nu_f$ in $M^1(\mathbb{P}^1(\mathbb{C}))^\dagger = M^1(\Gamma_G)^\dagger$ (also by (1.2)).

Now the proof of Theorem B is complete. □

Remark 6.1. The arguments in steps (c.1), (c.2), and (c.3) in the proof of Theorem B relate the non-archimedean absolute value $|\cdot|_r$ on $\mathbb{L}$, which is an extension of the trivial absolute value on $\mathbb{C} = k_{\mathbb{L}}$, with the Euclidean absolute value $|\cdot|$ on $\mathbb{C}$ and complement [5, Proof of Theorem B]. The final assertion in [5, Cor. 5.3], which [5, Proof of Theorem B] is based on, was shown in [5] under the condition (5.7′) (see also Remark 5.3).

§7. Examples

Pick a meromorphic family

$$f(z) = z^2 + t^{-1}z \in (\mathcal{O}(\mathbb{D})[t^{-1}])[z]\,(\subset \mathbb{L}[z])$$

of quadratic polynomials on $\mathbb{P}^1(\mathbb{C})$. Then $f^{-1}(\infty) = \{\infty\} = E(f)$, and the case (ii) (for $a = \infty$) in Theorem A occurs (indeed, $(\mathcal{S}_G, a] \ni f^n(\mathcal{S}_G) = \mathcal{S}_{B(0,|t^{-2^n-1}|_r)} \to \infty$ as $n \to \infty$ since $\mathcal{S}_G$ is represented by (the constant sequence of) the $\mathbb{L}$-closed disk $\mathcal{O}_{\mathbb{L}} = B(0,1)$, $f(0) = 0$, $|f(1)|_r = |t^{-1}|_r$ (> 1), $|f(t^{-1})|_r = |t^{-2}|_r > |t^{-1}|_r$, and $|f(z)|_r = |z|_r^2$ on $\mathbb{L} \setminus B(0, |t^{-1}|_r)$; see (3.1) for the absolute value $|\cdot|_r$ on $\mathbb{L}$). Since $f'(z) = 2z + t^{-1} \in \mathbb{L}[z]$, the point $-t^{-1} + 1 \in U_{\overrightarrow{\mathcal{S}_G\infty}} \cap \mathbb{L}$ is a (classical) repelling fixed point of f (indeed $f(-t^{-1}+1) = -t^{-1}+1$ and $|f'(-t^{-1}+1)|_r = |t^{-1}|_r > 1$), which is in $\mathsf{J}(f) = \operatorname{supp}\nu_f$, so we in particular have $\nu_f(U_{\overrightarrow{\mathcal{S}_G\infty}}) > 0$. Hence (5.7′) in Remark 5.3 is not the case for this f.

§8. A complement of Proposition 4.4

Let us continue to use the notation in Sections 3 and 4. Let

$$f \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)\,(\subset \mathbb{L}(z))$$

be a meromorphic family of rational functions on $\mathbb{P}^1(\mathbb{C})$ of degree $d > 1$, and suppose that $f^{-1}(\mathcal{S}_G) \neq \{\mathcal{S}_G\}$ in $\mathsf{P}^1(\mathbb{L})$. Recall that $\Gamma_G := \{\mathcal{S}_G\}$ and $\Gamma_n := \Gamma_{f^n} := \{\mathcal{S}_G, f^n(\mathcal{S}_G)\}$ in $\mathsf{H}^1_{\mathrm{II}}(\mathbb{L})$ for every $n \in \mathbb{N}$ and that $M^1(\Gamma_G)^\dagger$ is identified with $M^1(\mathbb{P}^1(\mathbb{C}))^\dagger$ under the bijection $S(\Gamma_G)\setminus\{\mathcal{S}_G\} = T_{\mathcal{S}_G}\mathsf{P}^1(\mathbb{L}) \cong \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C})$.

For every $n \in \mathbb{N}$, pick a meromorphic family $A_n \in (\mathcal{O}(\mathbb{D})[t^{-1}])(z)$ of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ such that $(A_n \circ f^n)(\mathcal{S}_G) = \mathcal{S}_G$ in $\mathsf{P}^1(\mathbb{L})$ (by Theorem 3.4), and set

$$A := A_1.$$

We note that for any $\mu = (\mu_C, \mu_E) \in (M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2$ satisfying the admissibility (4.4) (for this A), we still have $\omega_\mu \in M^1(\Gamma_f)^\dagger$ (and $\omega_\mu(S(\Gamma_f) \setminus F) = 0$ for some countable subset F in $S(\Gamma_f)$).

Conversely, for every $\omega \in M^1(\Gamma_f)^\dagger$ satisfying $\omega(S(\Gamma_f) \setminus F) = 0$ for some countable subset F in $S(\Gamma_f)$, there is a unique ordered pair

$$\mu_\omega = (\mu_{\omega,C}, \mu_{\omega,E}) \in \left(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger\right)^2 = (M^1(\Gamma_G)^\dagger)^2$$

such that when $\Gamma_f = \Gamma_G$ ($\Leftrightarrow \tilde{A} = \phi_{\tilde{A}}$),

$$\begin{cases} \mu_{\omega,C} := (\pi_{\Gamma_f,\Gamma_G})_*\omega \in M^1(\Gamma_G)^\dagger = M^1(\mathbb{P}^1(\mathbb{C}))^\dagger, \\ \mu_{\omega,E} := \tilde{A}_*(\pi_{\Gamma_f,\Gamma_G})_*\omega = \tilde{A}_*\mu_{\omega,C} \in M^1(\Gamma_G)^\dagger = M^1(\mathbb{P}^1(\mathbb{C}))^\dagger \end{cases}$$

and that when $\Gamma_f \neq \Gamma_G$, noting that $\{f(\mathcal{S}_G)\} \subset \Gamma_f \subset \mathsf{H}^1_{\mathrm{II}}(\mathbb{L})$,

$$\begin{cases} \mu_{\omega,C}(\{\tilde{x}\}) := ((\pi_{\Gamma_f,\Gamma_G})_*\omega)(\{U_{\overrightarrow{\mathcal{S}_G x}}\}) & \text{for every } \tilde{x} \in \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}), \\ \mu_{\omega,E}(\{\tilde{y}\}) := ((\pi_{\Gamma_f,\{f(\mathcal{S}_G)\}})_*\omega)(\{U_{(A^{-1})_*(\overrightarrow{\mathcal{S}_G y})}\}) & \text{for every } \tilde{y} \in \mathbb{P}^1(k_{\mathbb{L}}) = \mathbb{P}^1(\mathbb{C}). \end{cases}$$

Then this ordered pair $\mu_\omega = (\mu_{\omega,C}, \mu_{\omega,E})$ satisfies the admissibility (4.4) (for A) (by Lemma 4.2 when $\Gamma_f \neq \Gamma_G$), and in turn we have both

$$\omega_{\mu_\omega} = \omega \text{ in } M^1(\Gamma_f)^\dagger \quad \text{and} \quad \mu_{\omega_\mu} = \mu \text{ in } \left(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger\right)^2, \tag{8.1}$$

that is, the map $(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger)^2 \ni \mu \mapsto \omega_\mu \in M^1(\Gamma_f)^\dagger$ is bijective.

We conclude with the following complement of Proposition 4.4.

Proposition 8.1 (Cf. [5, Prop. 5.1 and Thm. 5.2]). *There is the bijection*

$$\Big\{(\mu_C, \mu_E) \in \left(M^1(\mathbb{P}^1(\mathbb{C}))^\dagger\right)^2 : \textit{satisfying the admissibility (4.4) (for } A\textit{) and the degenerating } f\textit{-balanced property } (\widetilde{A \circ f})^*\mu_E = d \cdot \mu_C \textit{ in } M(\mathsf{P}^1)\Big\} \ni \mu$$
$$\mapsto \omega_\mu \in \Big\{\omega \in M^1(\Gamma_f)^\dagger : \textit{satisfying } \omega(S(\Gamma_f) \setminus F) = 0 \textit{ for some countable subset } F \textit{ in } S(\Gamma_f) \textit{ and } f_G^*\omega = d \cdot (\pi_{\Gamma_f,\Gamma_G})_*\omega \textit{ in } M(\Gamma_G)\Big\},$$

the inverse of which is given by the map $\omega \mapsto \mu_\omega$. This bijection induces the bijection

$$\Delta_0^\dagger \ni \mu_C \mapsto (\pi_{\Gamma_n,\Gamma_G})_*(\omega_{(\mu_C,\mu_E^{(n)})}) \in \Delta_f^\dagger,$$

where

$$\Delta_0^\dagger := \Big\{\mu_C \in M^1(\mathbb{P}^1(\mathbb{C}))^\dagger : \textit{for (any) } n \gg 1, \textit{ there is } \mu_E^{(n)} \in M^1(\mathbb{P}^1(\mathbb{C}))^\dagger \textit{ such that } (\widetilde{A_n \circ f^n})^*\mu_E^{(n)} = d \cdot \mu_C\Big\}.$$

Proof. The former assertion follows from (8.1) and the computations in (a-1) and (b-1) in the proof of Proposition 4.4. Then the latter assertion holds also by (4.6). ∎

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