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A Note on Injective Factors with Trivial Bicentralizer

by

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Abstract

We give an alternative proof that an injective factor on a Hilbert space with trivial bicentralizer is an infinite tensor product of factors of finte type I (ITPFI factor). Our proof is given in parallel with each type of factor and it is based on the strategy of Haagerup. As a consequence, the uniqueness theorem of injective factors, except for type III_0, follows from Araki–Woods' result.

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§1. Introduction

In Connes' fundamental works in operator algebras, it is proved that injective factors on a separable Hilbert space are hyperfinite. In [Co1], injective factors of types II_1, II_∞ and III_λ $(\lambda \neq 1)$ are classified. The remaining problem of the uniqueness of the injective type III_1 factor is solved by Haagerup [Ha2] by proving the so-called bicentralizer problem in [Co2].

Haagerup [Ha1] also gives another proof of the first result mentioned above without the automorphism group machinery of Connes. In Haagerup's proof, semidiscreteness rather than injectivity is applied. Popa [Po] gives a third approach to this result in the case of type II_1.

Alternative proofs of the uniqueness of injective factors of types II and III are also given by Haagerup [Ha1, Ha3, Ha4]. For each case, similar techniques are

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applied. One of the important notions is the trivial bicentralizer. It is essential in the case of type III_1, but the equivalent condition to a trivial bicentralizer is more important than the original definition in [Ha4]. This condition is similar to the Dixmier property. This property is applied in the case of type II_1 [Ha1], and its relative version is applied in the case of type III_λ [Ha3]. Another important notion is almost unitary equivalence in a Hilbert bimodule.

Moreover, in comparison with his papers [Ha1, Ha3, Ha4], the uniqueness of the injective type II_1 factor follows from Marray–von Neumann's fundamental result of the uniqueness of the hyperfinite type II_1 factor. In the case of type III_λ, it is directly proved that an injective type III_λ factor is isomorphic to the Powers factor R_λ. In the case of type III_1, by using Connes–Woods' characterization of ITPFI factors in [CW], it is proved that an injective type III_1 factor is ITPFI. Therefore, the uniqueness of the injective type III_1 factor follows from Araki–Woods' result [AW]. These are proved by similar arguments, but they are dependent on the choice of type of a given injective factor.

In this note, we give an alternative proof that an injective factor on a Hilbert space with trivial bicentralizer is ITPFI. Our proof is given in parallel with each type of factor and it is based on the strategy of Haagerup. One of our purposes is to unify his proof. Here we remark that there exists an injective type III_0 factor which is not ITPFI. However, the assumption of a trivial bicentralizer in the above assertion excludes the case of type III_0 from consideration. Namely, every type III_0 factor has a non-trivial bicentralizer. This fact may be folklore among specialists, but we do not find it in the literature. Hence we also give its proof in this note. To do so, we define the bicentralizer for a general weight by using the free ultrafilter. This is inspired by Houdayer–Isono's paper [HI]. The starting point is the semidiscreteness, which is equivalent to the injectivity by the work of Wassermann [Was]. To achieve the above assertion, we need to generalize Haagerup's works. One aim is to obtain an approximate factorization related to the modular automorphism from the semidiscreteness. In the case of type III, it relies on the uniqueness of the injective type II_1 factor [Ha3, Ha4]. However, we independently give such an approximate factorization for an arbitrary injective von Neumann algebra by combining a number of techniques in [Ha1]. The other aim is the almost unitary equivalence in Hilbert bimodules established in [Ha3], which is a generalization of [Ha1]. However, as in the case of [Ha4], we require a further generalization. Finally, based on Haagerup's approach, we give a proof of the above assertion by using Connes–Woods' characterization of ITPFI factors.

§2. Semidiscreteness with the modular automorphisms

Let M be a von Neumann algebra. We denote by $\mathcal{U}(M)$ the unitary group of M. For an fn (faithful normal) state φ, we denote by Δ_φ (resp. J_φ) the modular operator (resp. the modular conjugation operator) associated with φ. We put

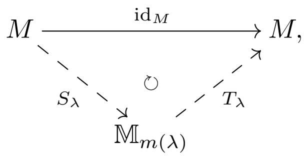
$$\|x\|_\varphi := \varphi(x^*x)^{1/2} \quad \text{and} \quad \|x\|_\varphi^\sharp := \varphi\Big(\frac{x^*x + xx^*}{2}\Big)^{1/2} \quad \text{for } x \in M.$$

We denote by $L^2(M, \varphi)$ the standard form for M with the cyclic unit vector ξ_φ, which becomes a normal M–M bimodule, where the left and right actions are given by

$$a\xi x := aJ_\varphi x^* J_\varphi \xi \quad \text{for } a, x \in M \text{ and } \xi \in L^2(M, \varphi).$$

The centralizer of φ is denoted by M_φ. For $m \in \mathbb{N}$, we denote by tr_m the normalized tracial state and by Tr_m the canonical trace with $\mathrm{Tr}_m(1) = m$ on the $m \times m$ matrix algebra $\mathbb{M}_m$.

If M is semidiscrete, then the identity map id_M on M has an approximate factorization through matrix algebras $\mathbb{M}_{m(\lambda)}$,



where (S_λ) and (T_λ) are nets of ucp (unital completely positive) maps. The purpose of this section is to show that for a given fn state φ on M and a positive number $\delta > 0$, one can choose an approximation factorization such that, moreover,

$$\varphi \circ T_\lambda = \psi_\lambda, \quad \psi_\lambda \circ S_\lambda = \varphi$$

and

$$\|\sigma_t^\varphi \circ T_\lambda - T_\lambda \circ \sigma_t^{\psi_\lambda}\| \le \delta|t| \quad \text{for } t \in \mathbb{R},$$

where (ψ_λ) is a net of fn states on $\mathbb{M}_{m(\lambda)}$.

To do so, we will prepare some lemmas, which are essentially proved in [Ha1, Ha3, Ha4]. The first lemma is given in [Ha1].

Lemma 2.1 ([Ha1, Lem. 3.1]). *Let τ be a tracial state on M, and $T\colon \mathbb{M}_m \to M$ be a faithful ucp map. Put*

$$\psi(x) := \tau \circ T(x) \quad \textit{for } x \in \mathbb{M}_m,$$

and let $h \in \mathbb{M}_m$ be the positive element for which

$$\psi(x) = \mathrm{tr}_m(xh) \quad \textit{for } x \in \mathbb{M}_m.$$

Then there exists a unique ucp map $S\colon M \to \mathbb{M}_m$ *such that* $\psi \circ S = \tau$ *and*

$$\mathrm{tr}_m(x^* h^{1/2} S(y) h^{1/2}) = \tau(T(x)^* y) \quad \text{for } x \in \mathbb{M}_m,\ y \in M.$$

Moreover,

$$\|T(x)\|_\tau^2 \le \mathrm{tr}_m(x^* h^{1/2} x h^{1/2}) \quad \text{for } x \in \mathbb{M}_m.$$

The second lemma is nothing but [Ha1, Lem. 3.2]. However, we necessarily sketch a proof to use it in Remark 2.3.

Lemma 2.2 ([Ha1, Lem. 3.2]). *Let* ψ *be a state on* $\mathbb{M}_m$ *of the form*

$$\psi(x) := \mathrm{tr}_m(xh) \quad \text{for } x \in \mathbb{M}_m,$$

where h *has strictly positive rational eigenvalues. Then there exist ucp maps* $S\colon \mathbb{M}_m \to \mathbb{M}_p$ *and* $T\colon \mathbb{M}_p \to \mathbb{M}_m$ *such that* $\mathrm{tr}_p \circ S = \psi$, $\psi \circ T = \mathrm{tr}_p$ *and*

$$\|x - T \circ S(x)\|_\psi^\sharp \le \|h^{1/2} x - x h^{1/2}\|_2 \quad \text{for } x \in \mathbb{M}_m.$$

Proof. We may assume that h is an $m \times m$ diagonal matrix with strictly positive rational eigenvalues $\lambda_1, \dots, \lambda_m$. Choose positive integers $p_1, \dots, p_m$, p such that

$$\frac{\lambda_i}{m} = \frac{p_i}{p} \quad \text{for } 1 \le i \le m.$$

Note that

$$\sum_{i=1}^m p_i = p.$$

For $1 \le i, j \le m$, we define the $p_i \times p_j$ matrix F_{ij} as

$$[F_{ij}]_{kl} := \delta_{kl} \quad \text{for } 1 \le k \le p_i,\ 1 \le l \le p_j,$$

and the $p \times p$ matrix f_{ij} with block matrix as

$$[f_{ij}]_{kl} = \delta_{ik}\delta_{jl} F_{ij} \quad \text{for } 1 \le k, l \le m.$$

Let (e_{ij}) be the matrix units for $\mathbb{M}_m$. We define the ucp map $S\colon \mathbb{M}_m \to \mathbb{M}_p$ as

$$S\Big(\sum_{i,j=1}^m x_{ij} e_{ij}\Big) := \sum_{i,j=1}^m x_{ij} f_{ij}.$$

Then we have

$$\mathrm{tr}_p \circ S(x) = \psi(x) \quad \text{for } x \in \mathbb{M}_m.$$

Moreover, S is faithful. By Lemma 2.1, there exists a unique ucp map $T\colon \mathbb{M}_p \to \mathbb{M}_m$ such that

$$\mathrm{tr}_m(x^* h^{1/2} T(y) h^{1/2}) = \mathrm{tr}_p(S(x)^* y) \quad \text{for } x \in \mathbb{M}_m,\ y \in \mathbb{M}_p.$$

In particular, we have

$$\psi\circ T(y)=\operatorname{tr}_p(y)\quad\text{for } y\in\mathbb{M}_p.$$

Moreover, for $1\le k,l\le m$, we have

$$\operatorname{tr}_m(e_{kl}^*h^{1/2}T(y)h^{1/2})=\operatorname{tr}_p(S(e_{kl})^*y)=\operatorname{tr}_p(f_{kl}^*y)\quad\text{for } y\in\mathbb{M}_p.$$

Hence the (k,l)th element of the $m\times m$ matrix $h^{1/2}T(y)h^{1/2}$ is $m\operatorname{tr}_p(f_{kl}^*y)$. This implies that the (k,l)th element of the $m\times m$ matrix $T(y)$ is

$$m\lambda_k^{-1/2}\lambda_l^{-1/2}\operatorname{tr}_p(f_{kl}^*y)=pp_k^{-1/2}p_l^{-1/2}\operatorname{tr}_p(f_{kl}^*y).$$

Note that the (i,j)th element of the $m\times m$ matrix $T\circ S(e_{ij})=T(f_{ij})$ is

$$(p_ip_j)^{-1/2}\min\{p_i,p_j\}$$

and all other elements of the matrix are zero. Hence

$$T\circ S(e_{ij})=(p_ip_j)^{-1/2}\min\{p_i,p_j\}e_{ij}.$$

Therefore we can obtain

$$(\|x-T\circ S(x)\|_\psi^\sharp)^2\le\|h^{1/2}x-xh^{1/2}\|_2^2.$$ □

Remark 2.3. Let k be an $m\times m$ diagonal matrix with eigenvalues $\mu_1,\dots,\mu_m$. Let p be the positive integer and T the ucp map as in the proof of Lemma 2.2. We define a $p\times p$ diagonal matrix $\bar{k}$ as

$$\bar{k}:=\sum_{i=1}^m\mu_iF_{ii}\in\mathbb{M}_p.$$

Then we have

$$kT(y)=T(\bar{k}y)\quad\text{and}\quad T(y)k=T(y\bar{k})\quad\text{for } y\in\mathbb{M}_p.$$

The third lemma is also given in [Ha1] in the case where φ is tracial.

Lemma 2.4 (Cf. [Ha1, Lem. 3.3]). *Let $T\colon\mathbb{M}_m\to M$ be a ucp map. For $\varepsilon>0$, there exists a ucp map $T'\colon\mathbb{M}_m\to M$ such that $\|T-T'\|<\varepsilon$ and*

$$\varphi\circ T'(x)=\operatorname{tr}_m(xh')\quad\textit{for } x\in\mathbb{M}_m,$$

where $h'\in\mathbb{M}_m^+$ has strictly positive rational eigenvalues.

Now we prove the main theorem in this section.

Theorem 2.5 (Cf. [Ha4, Thm. 3.1]). *If M is injective, then for any $u_1, \dots, u_n \in \mathcal{U}(M)$, any $\varepsilon > 0$ and $\delta > 0$, there exist a ucp map $T\colon \mathbb{M}_m \to M$ and $v_1, \dots, v_n \in \mathcal{U}(\mathbb{M}_m)$ such that $\psi = \varphi \circ T$ is an fn state on $\mathbb{M}_m$, and*

$$\|\sigma_t^\varphi \circ T - T \circ \sigma_t^\psi\| \le \delta|t| \quad \text{for } t \in \mathbb{R},$$
$$\|T(v_k) - u_k\|_\varphi < \varepsilon \qquad \text{for } 1 \le k \le n.$$

Proof. *Step* 0. The first part is the same as in [Ha4, Lem. 3.4]. Let $N := M \rtimes_{\sigma^\varphi} \mathbb{R}$. We denote by $\lambda^\varphi(t)$ the implementing unitary for σ_t^φ, and by θ^φ the dual action of σ^φ. Then there exists an fns (faithful normal semifinite) operator-valued weight $P\colon N^+ \to \widehat{M}^+$, which is given by

$$P(y) := \int_{\mathbb{R}} \theta_s^\varphi(y) ds \quad \text{for } y \in N^+.$$

Let $\widetilde{\varphi} := \varphi \circ P$ be the dual weight of φ. Recall that N has an fns trace τ such that

$$\tau \circ \theta_s^\varphi = e^{-s}\tau \quad \text{for } s \in \mathbb{R}.$$

Let a be the positive self-adjoint operator affiliated with $N_{\widetilde{\varphi}}$ such that $\exp(ita) = \lambda^\varphi(t)$ for $t \in \mathbb{R}$. Then τ is given by

$$\tau = \widetilde{\varphi}(e^{-a} \cdot\ ).$$

Put $e_\alpha := 1_{[0,\alpha]}(a)$ for $\alpha > 0$. Thanks to [Ha4, Lem. 3.4], we have $P(e_\alpha) = \alpha 1$. Hence $\widetilde{\varphi}(e_\alpha) = \alpha$ and $\tau(e_\alpha) = 1 - e^{-\alpha} < \infty$. By using [Ha4, Lem. 3.4] again, we obtain

$$\lim_{\alpha \to \infty} \left\| \frac{1}{\alpha} P(e_\alpha x e_\alpha) - x \right\|_\varphi = 0 \quad \text{for } x \in M.$$

Let $u_1, \dots, u_n \in \mathcal{U}(M)$, $\varepsilon > 0$ and $\delta > 0$ be given. Take $\varepsilon' > 0$ such that

$$(2\varepsilon')^{1/2} + \varepsilon' < \varepsilon.$$

Then we choose $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that $1 > \varepsilon_3$ and

$$8\varepsilon_3^{1/2} + \varepsilon_2 + \varepsilon_1 + \varepsilon_0 < \varepsilon'.$$

Take $\alpha > 0$ such that

$$\left\| \frac{1}{\alpha} P(e_\alpha u_k e_\alpha) - u_k \right\|_\varphi < \varepsilon_0 \quad \text{for } 1 \le k \le n.$$

We define the ucp map $T_0 := \alpha^{-1} P|_{e_\alpha N e_\alpha}\colon e_\alpha N e_\alpha \to M$ and the fn state φ_0 on $e_\alpha N e_\alpha$ as

$$\varphi_0 := \varphi \circ T_0 = \frac{1}{\alpha} \varphi \circ P(e_\alpha \cdot e_\alpha) = \frac{1}{\alpha} \widetilde{\varphi}(e_\alpha \cdot e_\alpha).$$

Set $x_k := e_\alpha u_k e_\alpha \in e_\alpha N e_\alpha$ for $1 \le k \le n$. Then $\|x_k\| \le 1$ and

$$\|T_0(x_k) - u_k\|_\varphi < \varepsilon_0 \quad \text{for } 1 \le k \le n.$$

Moreover, we have

$$\sigma_t^\varphi \circ T_0 = T_0 \circ \sigma_t^{\varphi_0} \quad \text{for } t \in \mathbb{R}.$$

We define the fn tracial state τ_α on $e_\alpha N e_\alpha$ as

$$\tau_\alpha := \frac{1}{1-e^{-\alpha}} \widetilde{\varphi}(e^{-a} e_\alpha \cdot\ ).$$

Set

$$h_0 := \frac{d\varphi_0}{d\tau_\alpha} = \frac{1-e^{-\alpha}}{\alpha} e^a e_\alpha$$

and $\varphi_0 = \tau_\alpha(h_0 \cdot\ )$. Note that

$$\mathrm{sp}(h_0) \subset [c^{-1}, c] \quad \text{for some } c > 1.$$

Step 1. The second part is similar to [Ha4, Thm. 3.1]. Take $\delta' > 0$ with $\delta' < \varepsilon_1$ such that if positive elements a, b with $\mathrm{sp}(a), \mathrm{sp}(b) \subset [c^{-1}, c]$ and $\|a - b\| < \delta'$, then $\|\log(a) - \log(b)\| < \delta/2$.

Choose $(2c)^{-1} > \delta_1 > 0$ such that $3c^2\delta_1 < \delta'$. Then take $\lambda \in \mathbb{Q}$ such that $0 < \lambda < 1$, $\lambda^{-1} - 1 < \delta_1$. Then $1 - \lambda = \lambda(\lambda^{-1} - 1) < \delta_1$. Set

$$J := \max\{j \in \mathbb{N} \mid \lambda^{-j} \le c\}.$$

Since $\lambda^{-J} \le c$ and $c < \lambda^{-(J+1)}$, we have $c^{-1} \le \lambda^J$ and $\lambda^{J+1} < c^{-1}$. By using the spectral decomposition of h_0, we can choose projections

$$e_{J+1}, e_J, \dots, e_1, e_0, e_{-1}, \dots, e_{-J}$$

with $\sum_{-J \le j \le J+1} e_j = 1$ such that

$$h_0' := \sum_{-J \le j \le J+1} \lambda^j e_j \le h_0 \quad \text{and} \quad \|h_0 - h_0'\| < c\delta_1.$$

Put $C := \tau_\alpha(h_0') \le 1$ and $h_1 := C^{-1} h_0'$. Since $1 - C = \tau_\alpha(h_0 - h_0') \le c\delta_1$, we have

$$1 \le C^{-1} \le (1 - c\delta_1)^{-1}.$$

Then

$$\begin{aligned} \|h_0 - h_1\| &\le \|h_0 - h_0'\| + \|h_0' - h_1\| \\ &\le c\delta_1 + (C^{-1} - 1)\|h_0'\| \\ &\le (1 + 2c)c\delta_1 < 3c^2\delta_1 < \delta'. \end{aligned}$$

Hence we have

$$\|\log(h_0) - \log(h_1)\| < \frac{\delta}{2}.$$

Put the fn state $\varphi_1 := \tau_\alpha(h_1 \cdot\ )$ on $e_\alpha N e_\alpha$. We define a cp map $T_0'\colon e_\alpha N e_\alpha \to M$ as

$$T_0'(x) := T_0(b^{1/2} x b^{1/2}) \quad \text{for } x \in e_\alpha N e_\alpha,$$

where $b := h_0^{-1} h_0' \leq 1$. Then $T_0'(1) = T_0(b) \leq 1$ and

$$\varphi \circ T_0'(x) = \varphi_0(b^{1/2} x b^{1/2}) = \tau_\alpha(h_0 b^{1/2} x b^{1/2}) = \tau_\alpha(h_0' x) \quad \text{for } x \in e_\alpha N e_\alpha.$$

Next we define a cp map $T_1\colon e_\alpha N e_\alpha \to M$ as

$$T_1(x) := T_0'(x) + \frac{\tau_\alpha((h_1 - h_0')x)}{\tau_\alpha(h_1 - h_0')}(1 - T_0'(1)) \quad \text{for } x \in e_\alpha N e_\alpha.$$

Then $T_1(1) = 1$ and

$$\varphi \circ T_1(x) = \tau_\alpha(h_1 x) = \varphi_1(x) \quad \text{for } x \in e_\alpha N e_\alpha.$$

Moreover,

$$\begin{aligned}
\|T_0(x) - T_1(x)\| &\leq \|x - b^{1/2} x b^{1/2}\| + \|1 - b\| \, \|x\| \\
&= \frac{1}{2}\|(1 + b^{1/2})x(1 - b^{1/2}) + (1 - b^{1/2})x(1 + b^{1/2})\| + \|1 - b\| \, \|x\| \\
&\leq (\|1 + b^{1/2}\| \, \|1 - b^{1/2}\| + \|1 - b\|)\|x\| \\
&\leq 3c^2 \delta_1 \|x\|.
\end{aligned}$$

Hence

$$\|T_0 - T_1\| \leq 3c^2\delta_1 < \delta' < \varepsilon_1.$$

Therefore,

$$\|T_1(x_k) - u_k\|_\varphi \leq \|T_1(x_k) - T_0(x_k)\|_\varphi + \|T_0(x_k) - u_k\|_\varphi < \varepsilon_1 + \varepsilon_0 \quad \text{for } 1 \leq k \leq n.$$

Since $\sigma_t^\varphi \circ T_1 = T_1 \circ \sigma_t^{\varphi_0}$ and

$$\|h_0^{it} - h_1^{it}\| \leq \|\log(h_0) - \log(h_1)\| \, |t| \leq \frac{\delta}{2}|t|,$$

we have

$$\begin{aligned}
\|\sigma_t^\varphi \circ T_1(x) - T_1 \circ \sigma_t^{\varphi_1}(x)\| &= \|T_1(\sigma_t^{\varphi_0}(x) - \sigma_t^{\varphi_1}(x))\| \\
&\leq \|\sigma_t^{\varphi_0}(x) - \sigma_t^{\varphi_1}(x)\| \\
&= \|h_0^{it} x h_0^{-it} - h_1^{it} x h_1^{-it}\| \\
&\leq \delta|t| \, \|x\|.
\end{aligned}$$

Hence

$$\|\sigma_t^\varphi \circ T_1 - T_1 \circ \sigma_t^{\varphi_1}\| \leq \delta|t| \quad \text{for } t \in \mathbb{R}.$$

Step 2. The next part is similar to [Ha3, Lem. 5.3]. For $j \in \mathbb{N}$, we define the linear map

$$E_j \colon e_\alpha N e_\alpha \to N_j := \{x \in e_\alpha N e_\alpha \mid \sigma_t^{\varphi_1}(x) = \lambda^{ijt}x,\ t \in \mathbb{R}\}$$

as

$$E_j(x) := \frac{1}{t_0}\int_0^{t_0} \lambda^{-ijt}\sigma_t^{\varphi_1}(x)\,dt \quad \text{for } x \in e_\alpha N e_\alpha,$$

where $t_0 := -2\pi/\log\lambda$. For $q \in \mathbb{N}$, we define the ucp map γ_q on $e_\alpha N e_\alpha$ as

$$\gamma_q(x) := \sum_{j=-q+1}^{q-1}\Big(1 - \frac{|j|}{q}\Big)E_j(x) \quad \text{for } x \in e_\alpha N e_\alpha.$$

By [Ha3, Lem. 5.2], $\varphi_1 \circ \gamma_q = \varphi_1$ and $\|\gamma_q(x) - x\|_{\varphi_1} \to 0$ $(q \to \infty)$ for $x \in e_\alpha N e_\alpha$. Choose $q \in \mathbb{N}$ such that

$$\|\gamma_q(x_k) - x_k\|_{\varphi_1} < \varepsilon_2 \quad \text{for } 1 \le k \le n.$$

Let (e_{rs}) be the matrix units for $\mathbb{M}_q$. We define the fn state ψ_λ on $\mathbb{M}_q$ as

$$\psi_\lambda := \mathrm{tr}_q(h_\lambda\,\cdot\,),$$

where

$$h_\lambda := \sum_{r=1}^{q}\lambda_r e_{rr} \quad \text{and} \quad \lambda_r := q\Big(\sum_{s=1}^{q}\lambda^s\Big)^{-1}\lambda^r.$$

Note that

$$\sigma_t^{\psi_\lambda}(e_{rs}) = h_\lambda^{it}(e_{rs})h_\lambda^{-it} = \lambda^{i(r-s)t}e_{rs}.$$

In particular, $\sigma_{t_0}^{\psi_\lambda} = \mathrm{id}$. Put the fn state $\chi := \varphi_1 \otimes \psi_\lambda$ on $e_\alpha N e_\alpha \otimes \mathbb{M}_q$. Since

$$\sigma_t^{\chi}(x \otimes e_{rs}) = \lambda^{i(r-s)t}\sigma_t^{\varphi_1}(x) \otimes e_{rs},$$

the centralizer $N_\chi := (e_\alpha N e_\alpha \otimes \mathbb{M}_q)_\chi$ is given by

$$N_\chi = \Big\{\textstyle\sum_{r,s=1}^{q} x_{rs} \otimes e_{rs} \in e_\alpha N e_\alpha \otimes \mathbb{M}_q \;\Big|\; x_{rs} \in N_{s-r}\Big\}.$$

We define ucp maps $S_2 \colon e_\alpha N e_\alpha \to N_\chi$ and $T_2 \colon N_\chi \to e_\alpha N e_\alpha$ as

$$S_2(x) := \sum_{r,s=1}^{q} E_{s-r}(x) \otimes e_{rs},$$

$$T_2\Big(\sum_{r,s=1}^{q} x_{rs} \otimes e_{rs}\Big) := \frac{1}{q}\sum_{r,s=1}^{q} x_{rs}.$$

Since $T_2 \circ S_2 = \gamma_q$, we have

$$\|T_2 \circ S_2(x_k) - x_k\|_{\varphi_1} < \varepsilon_2 \quad \text{for } 1 \le k \le n.$$

Put the state $\varphi_2 := \varphi_1 \circ T_2$ on N_χ. Then $\varphi_2 \circ S_2 = \varphi_1 \circ \gamma_q = \varphi_1$. For $x_{rs} \in N_{s-r}$, we have

$$\varphi_2\Big(\sum_{r,s=1}^{q} x_{rs} \otimes e_{rs}\Big) = \frac{1}{q}\sum_{r=1}^{q} \varphi_1(x_{rr}).$$

Hence

$$\varphi_2(y) = (\varphi_1 \otimes \mathrm{tr}_q)(y) \quad \text{for } y \in N_\chi \subset e_\alpha N e_\alpha \otimes \mathbb{M}_q.$$

Namely φ_2 is the restriction of $\varphi_1 \otimes \mathrm{tr}_q$ on N_χ. Hence we have

$$\sigma_t^{\varphi_2}(y) = (\sigma_t^{\varphi_1} \otimes \mathrm{id})(y) \quad \text{for } y \in N_\chi.$$

By definition, we have

$$\sigma_t^{\varphi_2} \circ S_2 = S_2 \circ \sigma_t^{\varphi_1} \quad \text{and} \quad \sigma_t^{\varphi_1} \circ T_2 = T_2 \circ \sigma_t^{\varphi_2}.$$

Let τ_χ be the restriction of χ on N_χ, which is tracial. Then

$$\tau_\chi(y) = (\varphi_1 \otimes \psi_\lambda)(y) = (\varphi_1 \otimes \mathrm{tr}_q)((1 \otimes h_\lambda)y) \quad \text{for } y \in N_\chi.$$

Since $1 \otimes h_\lambda \in N_\chi$, we have

$$\frac{d\varphi_2}{d\tau_\chi} = 1 \otimes h_\lambda^{-1}$$

and

$$\sigma_t^{\varphi_2}(y) = \Big(\frac{d\varphi_2}{d\tau_\chi}\Big)^{it} y \Big(\frac{d\varphi_2}{d\tau_\chi}\Big)^{-it} = (1 \otimes h_\lambda^{-it})y(1 \otimes h_\lambda^{it}) \quad \text{for } y \in N_\chi,\, t \in \mathbb{R}.$$

Step 3. In this step we use the semidiscreteness of M. Since $\|S_2(x_k)\| \leq 1$ for $1 \leq k \leq n$, we have

$$S_2(x_k) = \frac{1}{2}(w_{k1} + w_{k2}) + \frac{i}{2}(w_{k3} + w_{k4})$$

for some unitaries w_{kl}. Put

$$w_{kl} = \sum_{r,s} w_{rs}^{(kl)} \otimes e_{rs} \in N_\chi \subset e_\alpha N e_\alpha \otimes \mathbb{M}_q \quad \text{for } 1 \leq k \leq n,\, 1 \leq l \leq 4.$$

Set $c_\lambda := \max\{1, \|1 \otimes h_\lambda^{-1}\|\} \geq 1$. Since $e_\alpha N e_\alpha$ is semidiscrete, we can take ucp maps $S_3\colon e_\alpha N e_\alpha \to \mathbb{M}_p$ and $T_3\colon \mathbb{M}_p \to e_\alpha N e_\alpha$ such that

$$\|T_3 \circ S_3(w_{rs}^{(kl)}) - w_{rs}^{(kl)}\|_{\varphi_1} < \frac{\varepsilon_3}{c_\lambda \sqrt{q}} \quad \text{for } 1 \leq k \leq n,\, 1 \leq l \leq 4,\, 1 \leq r,s \leq q.$$

Here, by using Lemma 2.4, we may also assume that T_3 satisfies

$$\varphi_1 \circ T_3(x) = \mathrm{tr}_p(h_3' x) \quad \text{for } x \in \mathbb{M}_p,$$

where $h_3' \in \mathbb{M}_p^+$ has strictly positive rational eigenvalues. Let E_χ be the χ-invariant conditional expectation from $e_\alpha N e_\alpha \otimes \mathbb{M}_q$ onto N_χ, which is given by

$$E_\chi\left(\sum_{r,s=1}^{q} x_{rs} \otimes e_{rs}\right) = \sum_{r,s=1}^{q} E_{s-r}(x_{rs}) \otimes e_{rs}.$$

Set $S_3^{(q)} := S_3 \otimes \mathrm{id}_{\mathbb{M}_q}$ and $T_3^{(q)} := T_3 \otimes \mathrm{id}_{\mathbb{M}_q}$. Then we have

$$\|E_\chi \circ T_3^{(q)} \circ S_3^{(q)}(w_{kl}) - w_{kl}\|_{\tau_\chi} < \frac{\varepsilon_3}{c_\lambda}$$

and

$$\|T_3^{(q)} \circ S_3^{(q)}(S_2(x_k)) - S_2(x_k)\|_{\varphi_1 \otimes \mathrm{tr}_q} < 2\varepsilon_3.$$

Step 4. Let $h_3 := h_3' \otimes h_\lambda \in \mathbb{M}_p \otimes \mathbb{M}_q = \mathbb{M}_{pq}$ be the diagonal matrix with strictly positive rational eigenvalues. We define

$$\varphi_3 := \tau_\chi \circ E_\chi \circ T_3^{(q)}.$$

Then

$$\varphi_3 = \mathrm{tr}_{pq}(h_3 \cdot\ ).$$

By Lemma 2.2, there exist ucp maps $S_4 \colon \mathbb{M}_{pq} \to \mathbb{M}_m$ and $T_4 \colon \mathbb{M}_m \to \mathbb{M}_{pq}$ such that $\varphi_3 \circ T_4 = \mathrm{tr}_m$, $\mathrm{tr}_m \circ S_4 = \varphi_3$ and

$$\|T_4 \circ S_4(y) - y\|_{\varphi_3}^\sharp \le \|y h_3^{1/2} - h_3^{1/2} y\|_2 \quad \text{for } y \in \mathbb{M}_{pq}.$$

Set $y_{kl} := S_3^{(q)}(w_{kl}) \in \mathbb{M}_{pq}$. By Lemma 2.1, we have

$$\begin{aligned}
\mathrm{tr}_{pq}(y_{kl}^* h_3^{1/2} y_{kl} h_3^{1/2}) &\ge \|E_\chi \circ T_3^{(q)}(y_{kl})\|_{\tau_\chi}^2 \\
&\ge (\|w_{kl}\|_{\tau_\chi} - \|E_\chi \circ T_3^{(q)} \circ S_3^{(q)}(w_{kl}) - w_{kl}\|_{\tau_\chi})^2 \\
&> \left(1 - \frac{\varepsilon_3}{c_\lambda}\right)^2 > 1 - \frac{2\varepsilon_3}{c_\lambda}.
\end{aligned}$$

Hence

$$\|h_3^{1/2} y_{kl} - y_{kl} h_3^{1/2}\|_2^2 \le 2 - 2\,\mathrm{tr}_{pq}(y_{kl}^* h_3^{1/2} y_{kl} h_3^{1/2}) < \frac{4\varepsilon_3}{c_\lambda}.$$

Therefore,

$$\|T_4 \circ S_4(y_{kl}) - y_{kl}\|_{\varphi_3}^\sharp < \frac{2\varepsilon_3^{1/2}}{\sqrt{c_\lambda}} \quad \text{for } 1 \le k \le n,\ 1 \le l \le 4.$$

Set

$$y_k := \frac{1}{2}(y_{k1} + y_{k2}) + \frac{i}{2}(y_{k3} + y_{k4}) = S_3^{(q)} \circ S_2(x_k) \in \mathbb{M}_{pq} \quad \text{for } 1 \le k \le n.$$

Then

$$\|T_4 \circ S_4(y_k) - y_k\|_{\varphi_3}^\sharp \le \frac{4\varepsilon_3^{1/2}}{\sqrt{c_\lambda}} \quad \text{for } 1 \le k \le n.$$

Step 5. Set $k_\lambda := 1 \otimes h_\lambda^{-1} \in \mathbb{M}_p \otimes \mathbb{M}_q = \mathbb{M}_{pq}$. The $m \times m$ diagonal matrix $\bar{k}_\lambda$ is defined in Remark 2.3. We define an fn state ψ on $\mathbb{M}_m$ as

$$\psi := \mathrm{tr}_m(\bar{k}_\lambda \,\cdot\,).$$

Now we define a ucp map $T\colon \mathbb{M}_m \to M$ as

$$T := T_1 \circ T_2 \circ E_\chi \circ T_3^{(q)} \circ T_4.$$

Then, by Remark 2.3, for $z \in \mathbb{M}_m$, we have

$$\begin{aligned}\varphi \circ T(z) &= \varphi \circ T_1 \circ T_2 \circ E_\chi \circ T_3^{(q)} \circ T_4(z)\\ &= \varphi_1 \circ T_2 \circ E_\chi \circ T_3^{(q)} \circ T_4(z)\\ &= \varphi_2 \circ E_\chi \circ T_3^{(q)} \circ T_4(z)\\ &= \tau_\chi(1 \otimes h_\lambda^{-1}(E_\chi \circ T_3^{(q)} \circ T_4)(z))\\ &= \tau_\chi \circ E_\chi \circ T_3^{(q)}(k_\lambda T_4(z))\\ &= \tau_\chi \circ E_\chi \circ T_3^{(q)} \circ T_4(\bar{k}_\lambda z)\\ &= \varphi_3 \circ T_4(\bar{k}_\lambda z)\\ &= \mathrm{tr}_m(\bar{k}_\lambda z) = \psi(z).\end{aligned}$$

Hence $\psi = \varphi \circ T$ and

$$\sigma_t^\psi(z) = \bar{k}_\lambda^{it} z \bar{k}_\lambda^{-it} \quad \text{for } z \in \mathbb{M}_m.$$

By Remark 2.3, for $z \in \mathbb{M}_m$ we have

$$\begin{aligned}\sigma_t^{\varphi_2} \circ E_\chi \circ T_3^{(q)} \circ T_4(z) &= 1 \otimes h_\lambda^{-it}(E_\chi \circ T_3^{(q)} \circ T_4(z)) 1 \otimes h_\lambda^{it}\\ &= E_\chi \circ T_3^{(q)}(k_\lambda^{it} T_4(z) k_\lambda^{-it})\\ &= E_\chi \circ T_3^{(q)} \circ T_4(\bar{k}_\lambda^{it} z \bar{k}_\lambda^{-it})\\ &= E_\chi \circ T_3^{(q)} \circ T_4 \circ \sigma_t^\psi(z).\end{aligned}$$

Therefore we have

$$\|\sigma_t^\varphi \circ T - T \circ \sigma_t^\psi\| \le \delta|t| \quad \text{for } t \in \mathbb{R}.$$

Put $z_k := S_4(y_k) = S_4 \circ S_3^{(q)} \circ S_2(x_k) \in \mathbb{M}_m$ for $1 \le k \le n$. By the Kadison–Schwarz inequality we have

$$\begin{aligned}\|E_\chi \circ T_3^{(q)} \circ T_4(z_k) - E_\chi \circ T_3^{(q)}(y_k)\|_{\varphi_2}^2 &\le \|k_\lambda\| \, \|T_4(z_k) - y_k\|_{\varphi_3}^2\\ &\le 2c_\lambda \|T_4(z_k) - y_k\|_{\varphi_3}^{\sharp 2}\\ &\le 32\varepsilon_3.\end{aligned}$$

Hence

$$
\begin{aligned}
&\|E_\chi \circ T_3^{(q)} \circ T_4(z_k) - S_2(x_k)\|_{\varphi_2} \\
&\quad \le \|E_\chi \circ T_3^{(q)} \circ T_4(z_k) - E_\chi \circ T_3^{(q)}(y_k)\|_{\varphi_2} + \|E_\chi \circ T_3^{(q)}(y_k) - E_\chi(S_2(x_k))\|_{\varphi_2} \\
&\quad \le \sqrt{32}\varepsilon_3^{1/2} + \|T_3^{(q)} \circ S_3^{(q)}(S_2(x_k)) - S_2(x_k)\|_{\varphi_1 \otimes \mathrm{tr}_q} \\
&\quad \le 6\varepsilon_3^{1/2} + 2\varepsilon_3 \le 8\varepsilon_3^{1/2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|T_1 \circ T_2(E_\chi \circ T_3^{(q)} \circ T_4(z_k) - S_2(x_k))\|_\varphi^2 &\le \|E_\chi \circ T_3^{(q)} \circ T_4(z_k) - S_2(x_k)\|_{\varphi_2}^2 \\
&< 64\varepsilon_3.
\end{aligned}
$$

Similarly, we have

$$
\|T_1(T_2 \circ S_2(x_k) - x_k)\|_\varphi^2 \le \|T_2 \circ S_2(x_k) - x_k\|_{\varphi_1}^2 < \varepsilon_2^2.
$$

Therefore,

$$
\begin{aligned}
\|T(z_k) - u_k\|_\varphi &\le \|T_1 \circ T_2(E_\chi \circ T_3^{(q)} \circ T_4(z_k) - S_2(x_k))\|_\varphi \\
&\quad + \|T_1(T_2 \circ S_2(x_k) - x_k)\|_\varphi + \|T_1(x_k) - u_k\|_\varphi \\
&< 8\varepsilon_3^{1/2} + \varepsilon_2 + \varepsilon_1 + \varepsilon_0 < \varepsilon'.
\end{aligned}
$$

By the polar decomposition, we obtain unitaries v_k in $\mathbb{M}_m$ such that

$$
z_k = v_k|z_k| \quad \text{for } 1 \le k \le n.
$$

Then

$$
\|z_k\|_\psi^2 \ge \|T(z_k)\|_\varphi^2 > 1 - 2\varepsilon'.
$$

Since $\|z_k\| \le 1$ and $|z_k|^2 + (1 - |z_k|)^2 \le 1$, we have

$$
\begin{aligned}
\|v_k - z_k\|_\psi^2 &= \|1 - |z_k|\,\|_\psi^2 \\
&\le 1 - \|\,|z_k|\,\|_\psi^2 \\
&< 2\varepsilon'.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|T(v_k) - u_k\|_\varphi &\le \|T(v_k - z_k)\|_\varphi + \|T(z_k) - u_k\|_\varphi \\
&\le \|v_k - z_k\|_\psi + \varepsilon' \\
&< (2\varepsilon')^{1/2} + \varepsilon' < \varepsilon.
\end{aligned}
$$

□

Remark 2.6. In Theorem 2.5 we obtain the ucp map $T\colon \mathbb{M}_m \to M$ with $\psi = \varphi \circ T$ such that h_ψ has strictly positive rational eigenvalues, where $\psi = \mathrm{tr}_m(h_\psi \cdot\,)$.

Moreover, in Theorem 2.5 we assume M is a finite von Neumann algebra with an fn tracial state $\varphi = \tau$. Then we can choose the ucp map T satisfying $\tau \circ T = \mathrm{tr}_m$. This fact is exactly [Ha1, Lem. 3.4]. Indeed, in the proof of Theorem 2.5, we omit Steps 0, 1 and 2. In Step 3, we set $M = e_\alpha N e_\alpha = N_\chi$, $\tau = \varphi_1$, $u_k = S_2(x_k)$ and $q = 1$. Then we obtain ucp maps $S_2\colon M \to \mathbb{M}_p$ and $T_2\colon \mathbb{M}_p \to M$ such that

$$\|T_2 \circ S_2(u_k) - u_k\|_\tau < \varepsilon_3 \quad \text{for } 1 \le k \le n.$$

In Step 4, we set $\varphi_3 = \tau \circ T_2$ and $y_k = S_2(u_k)$. Then we have ucp maps $S_3\colon \mathbb{M}_p \to \mathbb{M}_m$ and $T_3\colon \mathbb{M}_m \to \mathbb{M}_p$ such that

$$\|T_3 \circ S_3(y_k) - y_k\|_{\varphi_3} < 2\varepsilon_3^{1/2} \quad \text{for } 1 \le k \le n.$$

In Step 5, if we define $T := T_2 \circ T_3$ and $z_k = S_3(y_k)$, then we have $\tau \circ T = \mathrm{tr}_m$ and

$$\|T(z_k) - u_k\|_\tau \le \varepsilon_3 + 2\varepsilon_3^{1/2} \quad \text{for } 1 \le k \le n.$$

Next we consider the case where M is a type III_λ factor with $0 < \lambda < 1$ and an fn state φ on M satisfies $\sigma^\varphi_{t_0} = \mathrm{id}$ with $t_0 = -2\pi/\log\lambda$. Then we can choose the ucp map $T\colon \mathbb{M}_m \to M$ with $\psi = \varphi \circ T = \mathrm{tr}_m(h_\psi \cdot\,)$ such that

$$\lambda_1/\lambda_2 \in \{\lambda^n \mid n \in \mathbb{Z}\} \quad \text{for } \lambda_1, \lambda_2 \in \mathrm{sp}(h_\psi).$$

This fact is weaker than [Ha3, Thm. 3.4], but it is sufficient for our purpose. By identifying $\widehat{\mathbb{Z}}$ with $\mathbb{R}/t_0\mathbb{Z}$,

$$N_0 = M \rtimes_{\sigma^\varphi} (\mathbb{R}/t_0\mathbb{Z})$$

is generated by $\pi^\varphi_0(x)$ and $\lambda^\varphi_0(t)$, where

$$(\pi^\varphi_0(x)\xi)(s) = \sigma^\varphi_{-s}(x)\xi(s),$$
$$(\lambda^\varphi_0(t)\xi)(s) = \xi(s-t)$$

for $\xi \in L^2(\mathbb{R}/t_0\mathbb{Z}, H_\varphi)$. By [HS, Prop. 5.6], we have $N \simeq N_0 \otimes L^\infty(0,\gamma_0)$ by identifying

$$\pi^\varphi(x) = \pi^\varphi_0(x) \otimes 1,$$
$$\lambda^\varphi(t) = \lambda^\varphi_0(t) \otimes m(e^{it}),$$

where $\gamma_0 = -\log\lambda$ and $m(e^{it})$ is the multiplication operator

$$(m(e^{it})\xi)(\gamma) = e^{it\gamma}\xi(\gamma) \quad \text{for } \xi \in L^2(0,\gamma_0).$$

We denote the canonical traces by τ_0 and τ, the dual weights by $\widetilde{\varphi}_0$ and $\widetilde{\varphi}$, on N_0 and N, respectively. Let $h_\varphi = d\widetilde{\varphi}_0/d\tau_0$ and $k_\varphi = d\widetilde{\varphi}/d\tau$. Then

$$k_\varphi = h_\varphi \otimes m(e^\gamma).$$

Note that $\lambda_0^\varphi(t+t_0) = \lambda_0^\varphi(t)$ for $t \in \mathbb{R}$ and $h_\varphi^{it} = \lambda_0^\varphi(t)$. Hence $\mathrm{sp}(h_\varphi) = \{\lambda^n\}_{n\in\mathbb{Z}} \cup \{0\}$. Therefore we have

$$h_0 = \frac{1-e^{-\alpha}}{\alpha}(h_\varphi \otimes m(e^\gamma))e_\alpha.$$

In Step 1, if $\lambda \notin \mathbb{Q}$, then we can take $\mu \in \mathbb{Q}$ such that μ is sufficiently close to λ and define

$$h_1 := \frac{1-e^{-\alpha}}{\alpha}(h'_\varphi \otimes m(e^\gamma))e_\alpha$$

satisfying $\|h_0 - h_1\| < \delta'$ and $\mathrm{sp}(h'_\varphi) \subset \{\mu^n\}_{n\in\mathbb{Z}} \cup \{0\}$. Therefore, by the proof of Theorem 2.5, we have the ucp map $T\colon \mathbb{M}_m \to M$ with $\psi = \varphi \circ T$ such that $\psi = \mathrm{tr}_m(\bar{k}_\mu\, \cdot\,)$. Then, by small perturbation of T, we can obtain T' such that $\psi = \varphi \circ T' = \mathrm{tr}_m(\bar{k}_\lambda\, \cdot\,)$.

§3. The bicentralizer of a type III$_0$ factor

Let M be a von Neumann algebra. We denote by $\mathcal{W}(M)$ and $\mathcal{W}_0(M)$ the set of ns (normal semifinite) weights and fns (faithful normal semifinite) weights on M, respectively. For $\varphi \in \mathcal{W}(M)$, we define

$$\mathfrak{n}_\varphi := \{x \in M \mid \varphi(x^*x) < \infty\}$$

and $\mathfrak{m}_\varphi := \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$. Let $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$.

Definition 3.1 ([HI, Def. 3.1]). For an fn state φ on M, we define the *asymptotic centralizer* and *ω-asymptotic centralizer* of φ as

$$\mathrm{AC}(M,\varphi) := \big\{(x_n)_n \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\infty} \|x_n\varphi - \varphi x_n\| = 0\big\},$$
$$\mathrm{AC}_\omega(M,\varphi) := \big\{(x_n)_n \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\omega} \|x_n\varphi - \varphi x_n\| = 0\big\},$$

respectively. We also define the *bicentralizer* and *ω-bicentralizer* of φ as

$$B(M,\varphi) := \big\{a \in M \mid \lim_{n\to\infty} \|ax_n - x_n a\|_\varphi = 0 \ \text{ for } (x_n)_n \in \mathrm{AC}_\omega(M,\varphi)\big\},$$
$$B_\omega(M,\varphi) := \big\{a \in M \mid \lim_{n\to\omega} \|ax_n - x_n a\|_\varphi = 0 \ \text{ for } (x_n)_n \in \mathrm{AC}_\omega(M,\varphi)\big\},$$

respectively.

We define

$$\mathcal{I}_\omega(M) := \big\{(x_n)_n \in \ell^\infty(M) \mid x_n \to 0 \ *\text{-strongly as } n \to \omega\big\},$$
$$\mathcal{M}^\omega(M) := \big\{(x_n)_n \in \ell^\infty(M) \mid (x_n)_n \mathcal{I}_\omega(M) \subset \mathcal{I}_\omega(M) \text{ and } \mathcal{I}_\omega(M)(x_n)_n \subset \mathcal{I}_\omega(M)\big\}.$$

Then the *multiplier algebra* $\mathcal{M}^\omega(M)$ is a C*-algebra and $\mathcal{I}_\omega(M) \subset \mathcal{M}^\omega(M)$ is a norm closed two-sided ideal. Following [Oc], we define the *ultrapower von Neumann algebra* $M^\omega := \mathcal{M}^\omega(M)/\mathcal{I}_\omega(M)$, which is indeed well known to be a von Neumann algebra.

Definition 3.2 ([AH, Def. 4.25]). For $\varphi \in \mathcal{W}(M)$, we define $\varphi^\omega \in \mathcal{W}(M)$ as

$$\varphi^\omega := \varphi \circ E,$$

where

$$E\colon M^\omega \ni (x_n)^\omega \mapsto \operatorname*{wot-lim}_{n\to\omega} x_n \in M$$

is the canonical fn conditional expectation and where wot-lim is the limit with respect to the weak operator topology. If φ is faithful, then so is φ^ω.

The following fact induces us to define the bicentralizer of a general weight.

Proposition 3.3 ([HI, Props. 3.2, 3.3]). *For an fn state φ on M, we have*

$$B(M,\varphi) = B_\omega(M,\varphi) = [(M^\omega)_{\varphi^\omega}]' \cap M.$$

In particular, the bicentralizer of φ does not depend on the choice of a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$.

Definition 3.4. For any $\varphi \in \mathcal{W}_0(M)$, we define the ω-*bicentralizer* of φ as

$$B_\omega(M,\varphi) := [(M^\omega)_{\varphi^\omega}]' \cap M.$$

Remark 3.5. Let $\varphi \in \mathcal{W}_0(M)$. Does $B_\omega(M,\varphi)$ depend on the choice of a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$? We give a partial answer to this question at the end of this section.

Lemma 3.6. *If $\varphi \in \mathcal{W}_0(M)$ is lacunary, then we have*

$$B_\omega(M,\varphi) \supset \mathcal{Z}(M_\varphi).$$

Proof. Since $(M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega$ by [AH, Prop. 4.27], we have

$$B_\omega(M,\varphi) = [(M^\omega)_{\varphi^\omega}]' \cap M = [(M_\varphi)^\omega]' \cap M \supset \mathcal{Z}(M_\varphi). \qquad \square$$

Remark 3.7. If M is a type III_λ factor with $0 \leq \lambda < 1$ and $\varphi \in \mathcal{W}_0(M)$ is lacunary, then by [Tak, Lem. XII.4.7] we have

$$B_\omega(M,\varphi) = [(M^\omega)_{\varphi^\omega}]' \cap M = [(M_\varphi)^\omega]' \cap M \subset M_\varphi' \cap M = \mathcal{Z}(M_\varphi),$$

and therefore

$$B_\omega(M,\varphi) = \mathcal{Z}(M_\varphi).$$

The following arguments are based on the work of Connes–Takesaki [CT]. From now on, we assume that M is a σ-finite type III_λ factor for $0 \leq \lambda < 1$ and $\varphi \in \mathcal{W}_0(M)$ is a lacunary weight of infinite multiplicity. Then M_φ is a type II_∞ von Neumann algebra, and there exists a unitary $U \in M$ such that

$$UM_\varphi U^* = M_\varphi, \quad \varphi \circ \mathrm{Ad}(U) \leq \lambda_0 \varphi, \;\; 0 < \lambda_0 < 1;$$

$$M \simeq M_\varphi \rtimes_\theta \mathbb{Z}, \quad \theta = \mathrm{Ad}(U)|_{M_\varphi}.$$

Moreover, $\tau \coloneqq \varphi|_{M_\varphi}$ is an fns trace on M_φ. For $m \in \mathbb{Z}$, there exists a non-singular positive self-adjoint operator ρ_m affiliated to $\mathcal{Z}(M_\varphi)$ such that

(1) $\varphi \circ \theta^m = \varphi_{\rho_m}$,

(2) $\rho_{m+n} = \rho_m \theta^{-m}(\rho_n)$,

(3) $\sigma_t^\varphi(U^m) = U^m \rho_m^{it}$,

(4) $\rho_m \leq \rho_1 < 1$ for $m > 0$,

(5) $\rho_m \geq \rho_{-1} > 1$ for $m < 0$.

We simply write $\rho \coloneqq \rho_1$.

Remark 3.8. Let $u, v \in M$ be unitaries and $x \in M$ with the polar decomposition $x = w|x|$. If $ux = xv$, then $uw = wv$, $u(ww^*) = (ww^*)u$ and $v(w^*w) = (w^*w)v$.

Lemma 3.9 (Cf. [CT, §3.2, Lem. 2.6]). *Suppose that $\psi_j = \varphi_{h_j}$ with $h_j \in M_\varphi^+$ satisfying*

$$\rho s(h_j) \leq h_j < 1 \quad \textit{for } j = 1, 2.$$

It $\psi_2^\omega = (\psi_1^\omega)_u$ for a partial isometry $u \in M^\omega$ with $uu^* = s(\psi_1^\omega)$ and $u^*u = s(\psi_2^\omega)$, then $u \in (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega$.

Proof. The proof is similar to that of [Tak, Lem. XII.4.14]. Set

$$k_j \coloneqq \rho(1 - s(h_j)) + h_j \quad \text{for } j = 1, 2.$$

Then $\rho \leq k_j < 1$. Note that

$$s(\psi_j^\omega) = s(\psi_j) = s(h_j) \in M_\varphi$$

by [Tak, Lem. XII.4.13].

By [Tak, Lem. XII.4.3], we have

$$\begin{aligned} uk_2^{it} &= us(h_2)k_2^{it} = uh_2^{it} = u(D\psi_2^\omega : D\varphi^\omega)_t = uu^*(D\psi_1^\omega : D\varphi^\omega)_t\sigma_t^{\varphi^\omega}(u) \\ &= s(h_1)h_1^{it}\sigma_t^{\varphi^\omega}(u) = k_1^{it}s(h_1)\sigma_t^{\varphi^\omega}(u) = k_1^{it}\sigma_t^{\varphi^\omega}(u). \end{aligned}$$

Therefore we obtain

$$uk_2^{it} = k_1^{it}\sigma_t^{\varphi^\omega}(u) \quad \text{for } t \in \mathbb{R}.$$

By [AH, Prop. 6.23], M^ω is canonically isomorphic to $(M_\varphi)^\omega \rtimes_{\theta^\omega} \mathbb{Z}$. Hence we choose a sequence $(x^{(m)})$ in $(M_\varphi)^\omega = (M^\omega)_{\varphi^\omega}$ such that $u = \sum_{m\in\mathbb{Z}} x^{(m)}U^m$ in M^ω. Now we have

$$uk_2^{it} = \sum_{m\in\mathbb{Z}} x^{(m)}U^m k_2^{it} = \sum_{m\in\mathbb{Z}} x^{(m)}\theta^m(k_2^{it})U^m$$

and

$$k_1^{it}\sigma_t^{\varphi^\omega}(u) = k_1^{it}\sum_{m\in\mathbb{Z}} \sigma_t^{\varphi^\omega}(x^{(m)}U^m) = \sum_{m\in\mathbb{Z}} k_1^{it}x^{(m)}U^m\rho_m^{it} = \sum_{m\in\mathbb{Z}} k_1^{it}x^{(m)}\theta^m(\rho_m^{it})U^m.$$

By the uniqueness of the expansion, we have

$$k_1^{it}x^{(m)}\theta^m(\rho_m^{it}) = x^{(m)}\theta^m(k_2^{it}).$$

Hence

$$k_1^{it}x^{(m)} = x^{(m)}\theta^m(k_2^{it}\rho_m^{-it}).$$

For each $m \in \mathbb{Z}$, by Remark 3.8 we may and do assume that $w := x^{(m)}$ is a partial isometry in $(M_\varphi)^\omega$ such that w^*w commutes with $\theta^m(k_2^{it}\rho_m^{-it})$ and ww^* commutes with k_1^{it}. Then

$$k_1^{it}ww^* = w\theta^m(k_2^{it}\rho_m^{-it})w^*.$$

If $m > 0$, then $\rho_m \le \rho \le k_2$. Hence $H := \theta^m(k_2\rho_m^{-1}) \ge 1$ and $0 \le K := k_1 < 1$. Then the functions

$$\{z \in \mathbb{C} \mid \mathrm{Im}(z) \ge 0\} \ni z \mapsto wH^{iz}w^*$$

and

$$\{z \in \mathbb{C} \mid \mathrm{Im}(z) \le 0\} \ni z \mapsto K^{iz}ww^*$$

are analytic and bounded, and $H^{it}ww^* = wK^{it}w^*$ for $t \in \mathbb{R}$. Hence the function $z \mapsto K^{iz}ww^*$ can be extended to a bounded entire function, which must be constant by Liouville's theorem. Therefore, $ww^* = ww^*k_1$ and $ww^*(1-k_1) = 0$. Since $1 - k_1$ is non-singular, we have $ww^* = 0$. Similarly, we can show the case $m < 0$. □

Lemma 3.10 (Cf. [Tak, Lem. XII.4.14]). *Let $\psi \in \mathcal{W}_0(M)$. If $\psi = \varphi_h$ for some $h \in M_\varphi^+$ with $\rho s(h) \leq h < 1$, then*

$$(M^\omega)_{\psi^\omega} \subset (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega.$$

Proof. If $u \in (M^\omega)_{\psi^\omega}$ is a partial isometry such that $uu^* = s(\psi^\omega) = u^*u$, then $\psi^\omega = \psi^\omega_u$. By Lemma 3.9, we have $u \in (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega$. □

Theorem 3.11. *If M is a type III_0 factor with separable prequel, then*

$$B_\omega(M, \psi) \neq \mathbb{C}1$$

for any $\psi \in \mathcal{W}_0(M)$. In particular,

$$B(M, \psi) \neq \mathbb{C}1$$

for any fn state ψ on M.

Proof. By [Tak, Thm. XII.4.10], there exists $h \in M_\varphi^+$ satisfying $\rho s(h) \leq h < 1$ such that $\psi \sim \varphi_h$, i.e., there exists an isometry $u \in M$ such that $1 = s(\psi) = u^*u$, $s(\varphi_h) = uu^*$ and $\psi(x) = \varphi_h(uxu^*)$ for $x \in M$.

By Lemma 3.10 we have

$$(M^\omega)_{\varphi_h^\omega} \subset (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega.$$

If $a \in M_{\psi^\omega}$ satisfies $uau^* = 0$, then

$$0 = \varphi_h^\omega(ua^*u^*uau^*) = \psi^\omega(a^*a).$$

Since ψ^ω is faithful, we have $a = 0$. Hence the adjoint map $\mathrm{Ad}(u)\colon M_{\psi^\omega} \to M_{\varphi_h^\omega}$ is an injective normal $*$-homomorphism. Since

$$u(M^\omega)_{\psi^\omega}u^* \subset (M^\omega)_{\varphi_h^\omega} \subset (M^\omega)_{\varphi^\omega}$$

we have

$$B_\omega(M, \varphi) = [(M^\omega)_{\varphi^\omega}]' \cap M \subset [(M^\omega)_{\varphi_h^\omega}]' \cap M \subset [u(M^\omega)_{\psi^\omega}u^*]' \cap M.$$

By Lemma 3.6 we have

$$\mathcal{Z}(M_\varphi) \subset B_\omega(M, \varphi) \subset [u(M^\omega)_{\psi^\omega}u^*]' \cap M.$$

Let $a \in \mathcal{Z}(M_\varphi)$. For $x \in (M^\omega)_{\psi^\omega}$, since u is an isometry, we have

$$u^*aux = u^*a(uxu^*)u = u^*(uxu^*)au = xu^*au.$$

Hence $u^*au \in B_\omega(M, \psi)$.

Now suppose that $u^*au \in \mathbb{C}1$ for any $a \in \mathcal{Z}(M_\varphi)$, i.e., $u^*au = \gamma 1$ for some $\gamma \in \mathbb{C}$. Recall that $\mathcal{Z}(M_\varphi) = M_\varphi' \cap M$ by [Tak, Cor. XII.4.17]. Then $ae = \gamma e$, because $e := uu^* = s(h) \in M_\varphi$. Since $axe = xae = \gamma xe$ for any $x \in M_\varphi$, we have $ac(e) = \gamma c(e)$, where $c(e)$ is the central support of e in M_φ. Therefore we have $\mathcal{Z}(M_\varphi)c(e) = \mathbb{C}c(e)$, which contradicts the fact that $\mathcal{Z}(M_\varphi)$ is non-atomic by [Tak, Cor. XII.3.15]. Hence it follows that $B_\omega(M, \psi) \neq \mathbb{C}1$. □

Next we discuss a von Neumann algebra M with trivial bicentralizer, except for type III_1.

Proposition 3.12. *Let ψ be an fn state on M with $B(M, \psi) = \mathbb{C}1$.*

(1) *If M is a semifinite von Neumann algebra, then M is finite and ψ is tracial.*

(2) *If M is a type III_λ factor with $0 < \lambda < 1$, then $\sigma_{t_0}^\psi = \mathrm{id}$, where $t_0 = -2\pi/\log\lambda$.*

Proof. (1) Assume M is a semifinite von Neumann algebra with an fns trace τ. Then there exists a non-singular positive self-adjoint operator h affiliated with M_ψ such that $\psi = \tau_h$ by [Tak, Thm. VIII.3.14]. Hence

$$\sigma_t^\psi(x) = h^{it}xh^{-it} \quad \text{for } t \in \mathbb{R},\ x \in M.$$

For $(x_n)^\omega \in (M^\omega)_{\psi^\omega}$, thanks to [AH, Thm. 4.1], we have

$$h^{it}(x_n)^\omega h^{-it} = (\sigma_t^\psi(x_n))^\omega = \sigma_t^{\psi^\omega}((x_n)^\omega) = (x_n)^\omega.$$

Hence

$$h^{it} \in [(M^\omega)_{\psi^\omega}]' \cap M = B(M, \psi) = \mathbb{C}1.$$

Therefore we have $\sigma_t^\psi = \mathrm{id}$, which means that ψ is tracial.

(2) We assume that M is a type III_λ factor with $0 < \lambda < 1$. Let $\varphi \in \mathcal{W}_0(M)$ be a lacunary weight with infinite multiplicity. By [Tak, Cor. XII.4.10], there exist $h \in M_\varphi^+$ and an isometry $u \in M$ such that $1 = s(\psi) = u^*u$, $s(\varphi_h) = uu^*$ and $\psi(x) = \varphi_h(uxu^*)$ for $x \in M$. By Lemma 3.10 we have

$$(M^\omega)_{\varphi_h^\omega} \subset (M^\omega)_{\varphi^\omega} = (M_\varphi)^\omega.$$

By the proof of Theorem 3.11,

$$u(M^\omega)_{\psi^\omega}u^* \subset (M^\omega)_{\varphi_h^\omega}.$$

Hence we obtain

$$h \in [(M^\omega)_{\varphi_h^\omega}]' \cap M \subset [u(M^\omega)_{\psi^\omega}u^*]' \cap M.$$

Then

$$u^*hu \in [(M^\omega)_{\psi^\omega}]' \cap M = B_\omega(M, \psi) = \mathbb{C}1.$$

Therefore, $u^*hu = \gamma 1$ for some constant γ. Since $h = uu^*huu^* = \gamma uu* = \gamma s(h)$, for $x \in M$ we have

$$\psi(x) = \varphi_h(uxu^*) = \varphi(huxu^*) = \varphi(uu^*huxu^*) = \gamma\varphi(uxu^*).$$

By [Tak, Lem. XII.4.3],

$$(D\psi \colon D\varphi)_t = u^*(D\gamma\varphi \colon D\varphi)_t \sigma_t^\varphi(u) = \gamma^{it} u^* \sigma_t^\varphi(u) =: u_t.$$

Then

$$\sigma_t^\psi(x) = u_t \sigma_t^\varphi(x) u_t^* = u^* \sigma_t^\varphi(u) \sigma_t^\varphi(x) \sigma_t^\varphi(u^*) u = u^* \sigma_t^\varphi(uxu^*) u.$$

Therefore,

$$\sigma_{t_0}^\psi(x) = u^* \sigma_{t_0}^\varphi(uxu^*) u = u^*(uxu^*)u = x.$$

□

Finally we discuss the problem in Remark 3.5. Recall that $\varphi \in \mathcal{W}_0(M)$ is *strictly semifinite* if its restriction to M_φ is also semifinite.

Proposition 3.13. *If $\varphi \in \mathcal{W}_0(M)$ is strictly semifinite, then $B_\omega(M, \varphi)$ does not depend on the choice of a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. We claim that $a \in B_\omega(M, \varphi) = [(M^\omega)_{\varphi^\omega}]' \cap M$ if and only if

$$a \in \bigcap_{\substack{e \in M_\varphi \\ \varphi(e) < \infty}} [(M_e^\omega)_{\varphi_e^\omega}]' \cap M.$$

Assume that $a \in M$ commutes with any element in $(M_e^\omega)_{\varphi_e^\omega}$ for any projection $e \in M_\varphi$ with $\varphi(e) < \infty$. Since φ is strictly semifinite, there exists an orthogonal family $(e_k)_{k \in I}$ of projections in M_φ with sum 1 such that $\varphi(e_k) < \infty$. Put $p_F := \sum_{k \in F} e_k$ for a finite subset $F \Subset I$. Then $M_\varphi \ni p_F \nearrow 1$ and $\varphi(p_F) < \infty$. Let $x \in (M^\omega)_{\varphi^\omega}$. Then $p_F x p_F \in (M_{p_F}^\omega)_{\varphi_{p_F}^\omega}$. Hence

$$p_F(ax)p_F = a(p_F x p_F) = (p_F x p_F)a = p_F(xa)p_F$$

and so $ax = xa$, namely $a \in B_\omega(M, \varphi)$.

Conversely, let $a \in B_\omega(M, \varphi)$. Let $e \in M_\varphi$ be a projection with $\varphi(e) < \infty$. Since $(M_e^\omega)_{\varphi_e^\omega} \subset (M^\omega)_{\varphi^\omega}$, we have $ax = xa$ for $x \in (M_e^\omega)_{\varphi_e^\omega}$.

Moreover,

$$[(M_e^\omega)_{\varphi_e^\omega}]' \cap M = ([(M_e^\omega)_{\varphi_e^\omega}]' \cap M_e) \oplus M_{e^\perp}.$$

Indeed, let $a \in [(M_e^\omega)_{\varphi_e^\omega}]' \cap M$. Since $e \in (M_e^\omega)_{\varphi_e^\omega}$, we have

$$a = eae + eae^\perp \in ([(M_e^\omega)_{\varphi_e^\omega}]' \cap M_e) \oplus M_{e^\perp}.$$

Conversely, let $a = a_1 + a_2 \in ([(M_e^\omega)_{\varphi_e^\omega}]' \cap M_e) \oplus M_{e^\perp}$. For any $x \in (M_e^\omega)_{\varphi_e^\omega}$, we have $xa_2 = a_2 x = 0$. Hence $ax = xa$, and so $a \in [(M_e^\omega)_{\varphi_e^\omega}]' \cap M$.

By Proposition 3.3, we have

$$[(M_e^\omega)_{\varphi_e^\omega}]' \cap M_e = B_\omega(M_e, \varphi_e^\omega) = B(M_e, \varphi_e^\omega).$$

Therefore,

$$B_\omega(M, \varphi) = \bigcap_{\substack{e \in M_\varphi \\ \varphi(e) < \infty}} B(M_e, \varphi_e^\omega) \oplus M_{e^\perp},$$

which means independence in the choice of a free ultrafilter ω. □

§4. Almost unitary equivalence

In this section we generalize the notion of δ-relatedness for two n-tuples of unit vectors in a Hilbert bimodule. In [Ha4, Rem. 2.9], this is stated, but there is no proof, and so we give details.

Throughout this section, M is a von Neumann algebra, and H is a Hilbert M-bimodule, i.e., H is a Hilbert space with left and right actions

$$(x, \xi) \mapsto x\xi, \quad (x, \xi) \mapsto \xi x$$

such that the above maps are bilinear and $(x\xi)y = x(\xi y)$ for $x, y \in M$, $\xi \in H$. Moreover,

$$x \mapsto L_x, \quad x \mapsto R_x$$

are a normal $*$-homomorphism and $*$-antihomomorphism, respectively, where $L_x \xi := x\xi$ and $R_x \xi := \xi x$ for $x \in M$, $\xi \in H$.

Definition 4.1 (Cf. [Ha3, Def. 2.1]). Two n-tuples $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ of unit vectors in H are called *almost δ-related* if, for any $\varepsilon > 0$, there exist $a_1, \dots, a_p \in M$ such that for $1 \le k \le n$,

(a1) $\|(1 - \sum_{j=1}^p a_j^* a_j)\xi_k\| < \varepsilon$, $\|(1 - \sum_{j=1}^p a_j^* a_j)\eta_k\| < \varepsilon$;

(a2) $\|\xi_k(1 - \sum_{j=1}^p a_j^* a_j)\| < \varepsilon$, $\|\eta_k(1 - \sum_{j=1}^p a_j^* a_j)\| < \varepsilon$;

(b1) $\|(1 - \sum_{j=1}^p a_j a_j^*)\xi_k\| < \varepsilon$, $\|(1 - \sum_{j=1}^p a_j a_j^*)\eta_k\| < \varepsilon$;

(b2) $\|\xi_k(1 - \sum_{j=1}^p a_j a_j^*)\| < \varepsilon$, $\|\eta_k(1 - \sum_{j=1}^p a_j a_j^*)\| < \varepsilon$;

(c) $\sum_{j=1}^p \|a_j \xi_k - \eta_k a_j\|^2 < \delta$.

Remark 4.2. In this case, we can easily check that

(d) $\sum_{j=1}^p \|a_j^* \eta_k - \xi_k a_j^*\|^2 < 2\delta$ for $1 \le k \le n$.

Indeed, for $\delta/4 > \varepsilon > 0$, we take $a_1, \dots, a_p \in M$ satisfying (a1), (a2), (b1), (b2) and (c). Then we have

$$\begin{aligned}\sum_{j=1}^p \|a_j^*\eta_k - \xi_k a_j^*\|^2 &= \sum_{j=1}^p \|a_j^*\eta_k\|^2 + \|\xi_k a_j^*\|^2 - 2\operatorname{Re}\langle a_j^*\eta_k, \xi_k a_j^*\rangle \\ &= \left\langle \left(\sum_{j=1}^p a_j a_j^* - 1\right)\eta_k, \eta_k \right\rangle + \left\langle \eta_k, \eta_k\left(1 - \sum_{j=1}^p a_j a_j^*\right)\right\rangle \\ &\quad + \left\langle \eta_k, \eta_k \sum_{j=1}^p a_j a_j^*\right\rangle + \left\langle \xi_k, \xi_k\left(\sum_{j=1}^p a_j^* a_j - 1\right)\right\rangle \\ &\quad + \left\langle \left(1 - \sum_{j=1}^p a_j^* a_j\right)\xi_k, \xi_k\right\rangle + \left\langle \sum_{j=1}^p a_j^* a_j \xi_k, \xi_k\right\rangle \\ &\quad - 2\operatorname{Re}\langle a_j\xi_k, \eta_k a_j\rangle \\ &< 4\varepsilon + \sum_{j=1}^p \|a_j\xi_k - \eta_k a_j\|^2 < 2\delta.\end{aligned}$$

Remark 4.3. If two n-tuples $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ are almost δ-related, then for each $\varepsilon > 0$, we can choose $a_1, \dots, a_p$ such that for $1 \le k \le n$,

(a′1) $\|(1 - \sum_{j=1}^p a_j^* a_j)\xi_k\| < 2\varepsilon$, $\|(1 - \sum_{j=1}^p a_j^* a_j)\eta_k\| < 2\varepsilon$;

(a′2) $\|\xi_k(1 - \sum_{j=1}^p a_j^* a_j)\| < 2\varepsilon$, $\|\eta_k(1 - \sum_{j=1}^p a_j^* a_j)\| < 2\varepsilon$;

(b′1) $\|(1 - \sum_{j=1}^p a_j a_j^*)\xi_k\| < 2\varepsilon$, $\|(1 - \sum_{j=1}^p a_j a_j^*)\eta_k\| < 2\varepsilon$;

(b′2) $\|\xi_k(1 - \sum_{j=1}^p a_j a_j^*)\| < 2\varepsilon$, $\|\eta_k(1 - \sum_{j=1}^p a_j a_j^*)\| < 2\varepsilon$;

(c′) $\sum_{j=1}^p \|a_j\xi_k - \eta_k a_j\|^2 < \delta$;

(d′) $\sum_{j=1}^p \|a_j^*\eta_k - \xi_k a_j^*\|^2 < 2\delta$;

(e′) $\|(\sum_{j=1}^p a_j^* a_j)\xi_k\| \le 1$, $\|(\sum_{j=1}^p a_j^* a_j)\eta_k\| \le 1$;

(f′) $\|(\sum_{j=1}^p a_j a_j^*)\xi_k\| \le 1$, $\|(\sum_{j=1}^p a_j a_j^*)\eta_k\| \le 1$.

Indeed, we take $a_1, \dots, a_p$ satisfying (a1), (a2), (b1), (b2), (c) and (d) by Remark 4.2. Then it is easy to check that operators

$$a_j' := \left(\frac{1}{1+\varepsilon}\right)^{1/2} a_j \quad \text{for } j = 1, \dots, p$$

satisfy the above properties for sufficiently small $\varepsilon > 0$.

Lemma 4.4. *Let ξ, η be two almost δ-related unit vectors in a Hilbert M-bimodule. Then there exist $b_1, \dots, b_p \in M$ such that*

(1) $|\langle (1 - \sum_{j=1}^p b_j^* b_j)\xi, \xi\rangle| < 4\delta^{1/2}$;

(2) $|\langle (1 - \sum_{j=1}^p b_j b_j^*)\eta, \eta\rangle| < 4\delta^{1/2}$;

(3) $\sum_{j=1}^p \|b_j\xi - \eta b_j\|^2 < 4\delta$;

(4) $\sum_{j=1}^p \|b_j^*\eta - \xi b_j^*\|^2 < 4\delta$;

(5) $\sum_{j=1}^p b_j^* b_j \leq 1$;

(6) $\sum_{j=1}^p b_j b_j^* \leq 1$.

Proof. Choose $0 < \varepsilon < 1$ such that $5\varepsilon \leq 2\delta^{1/2}$ and $4\varepsilon^{1/2} \leq \delta^{1/2}$. By Remark 4.3, there exist $a_1, \dots, a_p$ of M satisfying

(a′1) $\|(1 - \sum_{j=1}^p a_j^* a_j)\xi\| < \varepsilon^2$, $\|(1 - \sum_{j=1}^p a_j^* a_j)\eta\| < \varepsilon^2$;

(a′2) $\|\xi(1 - \sum_{j=1}^p a_j^* a_j)\| < \varepsilon^2$, $\|\eta(1 - \sum_{j=1}^p a_j^* a_j)\| < \varepsilon^2$;

(b′1) $\|(1 - \sum_{j=1}^p a_j a_j^*)\xi\| < \varepsilon^2$, $\|(1 - \sum_{j=1}^p a_j a_j^*)\eta\| < \varepsilon^2$;

(b′2) $\|\xi(1 - \sum_{j=1}^p a_j a_j^*)\| < \varepsilon^2$, $\|\eta(1 - \sum_{j=1}^p a_j a_j^*)\| < \varepsilon^2$;

(c′) $\sum_{j=1}^p \|a_j\xi - \eta a_j\|^2 < \delta$;

(d′) $\sum_{j=1}^p \|a_j^*\eta - \xi a_j^*\|^2 < 2\delta$;

(e′) $\|(\sum_{j=1}^p a_j^* a_j)\xi\| \leq 1$, $\|(\sum_{j=1}^p a_j^* a_j)\eta\| \leq 1$;

(f′) $\|(\sum_{j=1}^p a_j a_j^*)\xi\| \leq 1$, $\|(\sum_{j=1}^p a_j a_j^*)\eta\| \leq 1$.

Then we define cp maps S and T on M as

$$S(x) := \sum_{j=1}^p a_j^* x a_j, \quad T(x) := \sum_{j=1}^p a_j x a_j^* \quad \text{for } x \in M.$$

We define $e := 1_{[1-\varepsilon, 1+\varepsilon]}(S(1))$ and $f := 1_{[1-\varepsilon, 1+\varepsilon]}(T(1))$. Since $(S(1) - 1)^2 \geq \varepsilon^2(1 - e)$, we have

$$\varepsilon^2\|(1-e)\xi\|^2 = \varepsilon^2\langle (1-e)\xi, \xi\rangle \leq \|(S(1) - 1)\xi\|^2 < \varepsilon^4.$$

Hence

$$\|(1-e)\xi\| \leq \varepsilon.$$

Similarly, we have

$$\|(1-e)\eta\| \leq \varepsilon, \quad \|\xi(1-e)\| \leq \varepsilon, \quad \|\eta(1-e)\| \leq \varepsilon.$$

We also obtain

$$\|(1-f)\xi\| \le \varepsilon, \quad \|(1-f)\eta\| \le \varepsilon, \quad \|\xi(1-f)\| \le \varepsilon, \quad \|\eta(1-f)\| \le \varepsilon.$$

Next we define cp maps S' and T' on M as

$$S'(x) := \frac{1}{1+\varepsilon} eS(x)e, \quad T'(x) := \frac{1}{1+\varepsilon} fT(x)f \quad \text{for } x \in M.$$

Then $S'(1) \le 1$ and $T'(1) \le 1$. In particular, S' and T' are contractive.

Now we define

$$b_j := \frac{1}{\sqrt{1+\varepsilon}} f a_j e \quad \text{for } 1 \le j \le p.$$

Then

$$\sum_{j=1}^p b_j^* b_j = \frac{1}{1+\varepsilon} \sum_{j=1}^p e a_j^* f a_j e = S'(f) \le 1.$$

Similarly, we have $\sum_{j=1}^p b_j b_j^* \le 1$. Thus we obtain (5) and (6).

We will check (1). Since ξ and η are unit vectors, we have

$$\begin{aligned}
\left|\left\langle\left(1-\sum_{j=1}^p b_j^* b_j\right)\xi, \xi\right\rangle\right| &= \left|\langle \eta, \eta\rangle - \frac{1}{1+\varepsilon}\sum_{j=1}^p \langle e a_j^* f a_j e\xi, \xi\rangle\right| \\
&\le \varepsilon + \left|\left\langle \eta\left(1-\sum_{j=1}^p a_j a_j^*\right), \eta\right\rangle\right| + \left|\left\langle (1-f)\eta, \eta\sum_{j=1}^p a_j a_j^*\right\rangle\right| \\
&\quad + \left|\sum_{j=1}^p \langle f\eta a_j, (\eta a_j - a_j\xi)\rangle\right| + \left|\sum_{j=1}^p \langle f(\eta a_j - a_j\xi), a_j\xi\rangle\right| \\
&\quad + \left|\sum_{j=1}^p \langle f a_j \xi, a_j(1-e)\xi\rangle\right| + \left|\sum_{j=1}^p \langle f a_j(1-e)\xi, a_j e\xi\rangle\right| \\
&\le \varepsilon + \left\|\eta\left(1-\sum_{j=1}^p a_j a_j^*\right)\right\| \|\eta\| + \|(1-f)\eta\| \left\|\eta\sum_{j=1}^p a_j a_j^*\right\| \\
&\quad + \left(\sum_{j=1}^p \|\eta a_j\|^2\right)^{1/2}\left(\sum_{j=1}^p \|\eta a_j - a_j\xi\|^2\right)^{1/2} \\
&\quad + \left(\sum_{j=1}^p \|\eta a_j - a_j\xi\|^2\right)^{1/2}\left(\sum_{j=1}^p \|a_j\xi\|^2\right)^{1/2} \\
&\quad + \left(\sum_{j=1}^p \|a_j\xi\|^2\right)^{1/2}\left(\sum_{j=1}^p \|a_j(1-e)\xi\|^2\right)^{1/2} \\
&\quad + \left(\sum_{j=1}^p \|a_j(1-e)\xi\|^2\right)^{1/2}\left(\sum_{j=1}^p \|a_j e\xi\|^2\right)^{1/2}.
\end{aligned}$$

By (e′), (f′), we have $\sum_{j=1}^p \|\eta a_j\|^2 \le 1$, and $\sum_{j=1}^p \|a_j\xi\|^2 \le 1$. Since the projection e commutes with $\sum_{j=1}^p a_j^* a_j$, we have

$$\begin{aligned}\sum_{j=1}^p \|a_j(1-e)\xi\|^2 &= \left\langle \sum_{j=1}^p a_j^* a_j (1-e)\xi, (1-e)\xi \right\rangle \\ &= \left\langle \sum_{j=1}^p a_j^* a_j \xi, (1-e)\xi \right\rangle \le \|(1-e)\xi\| \le \varepsilon\end{aligned}$$

and

$$\sum_{j=1}^p \|a_j e\xi\|^2 = \left\langle \sum_{j=1}^p a_j^* a_j e\xi, e\xi \right\rangle = \left\langle \sum_{j=1}^p a_j^* a_j \xi, e\xi \right\rangle \le 1.$$

Therefore,

$$\left| \left\langle \left(1 - \sum_{j=1}^p b_j^* b_j \right) \xi, \xi \right\rangle \right| \le 5\varepsilon + 2\delta^{1/2} \le 4\delta^{1/2}.$$

Similarly, we have (2).

Next we will check (3):

$$\begin{aligned}&\left(\sum_{j=1}^p \|b_j\xi - \eta b_j\|^2 \right)^{1/2} \\ &\le \left(\sum_{j=1}^p \|f a_j e\xi - \eta f a_j e\|^2 \right)^{1/2} \\ &= \left(\sum_{j=1}^p \|f a_j (e-1)\xi\|^2 \right)^{1/2} + \left(\sum_{j=1}^p \|f a_j \xi (1-e)\|^2 \right)^{1/2} \\ &\quad + \left(\sum_{j=1}^p \|f(a_j\xi - \eta a_j e)\|^2 \right)^{1/2} + \left(\sum_{j=1}^p \|(f-1)\eta a_j e\|^2 \right)^{1/2} \\ &\quad + \left(\sum_{j=1}^p \|\eta(1-f) a_j e\|^2 \right)^{1/2} \\ &\le (\langle S(f)(e-1)\xi, (e-1)\xi \rangle)^{1/2} + (\langle S(f)\xi(1-e), \xi(1-e) \rangle)^{1/2} \\ &\quad + \left(\sum_{j=1}^p \|a_j\xi - \eta a_j\|^2 \right)^{1/2} \\ &\quad + (\langle (f-1)\eta, (f-1)\eta T(e) \rangle)^{1/2} + (\langle \eta(1-f), \eta(1-f) T(e) \rangle)^{1/2} \\ &< 4\varepsilon^{1/2} + \delta^{1/2} \le 2\delta^{1/2}.\end{aligned}$$

Similarly, we can check (4). The proof is complete. □

Then we can show Theorem 4.6 by using following lemma, which is also proved by similar arguments to [Ha3, Lem. 2.5].

Lemma 4.5 (Cf. [Ha3, Lem. 2.5]). *Assume that $\delta > 0$ and $r \in \mathbb{N}$ satisfy*

$$\delta^{1/2} < \frac{1}{8r}.$$

Let ξ, η be two almost δ-related unit vectors in a Hilbert M-bimodule. Then there exist r operators $c_1, \dots, c_r \in M$ such that $\|c_j\| \leq 1$, $1 \leq j \leq r$ and

$$\left\| \left(\sum_{j=1}^{r} c_j^* c_j - 1 \right) \xi \right\|^2 < \frac{12}{r}, \quad \left\| \left(\sum_{j=1}^{r} c_j c_j^* - 1 \right) \eta \right\|^2 < \frac{12}{r},$$

$$\sum_{j=1}^{p} \|c_j \xi - \eta c_j\|^2 < 32\delta, \qquad \sum_{j=1}^{p} \|c_j^* \eta - \xi c_j^*\|^2 < 32\delta.$$

By similar arguments to [Ha3] with Lemma 4.5, we can prove almost unitary equivalence for two almost δ-related n-tuples in a Hilbert bimodule, which is a generalization of [Ha3, Thm. 2.3].

Theorem 4.6 (Cf. [Ha3, Thm. 2.3]). *For every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(n, \varepsilon) > 0$ such that for all von Neumann algebras M and all almost δ-related n-tuples $(\xi_1, \dots, \xi_n)$, $(\eta_1, \dots, \eta_n)$ of unit vectors in a Hilbert M-bimodule, there exists a unitary $u \in M$ such that*

$$\|u\xi_k - \eta_k u\| < \varepsilon \quad \textit{for } 1 \leq k \leq n.$$

§5. Γ-stable states

Definition 5.1 (Cf. [Ha4, Def. 4.1]). Let Γ be a multiplicative subgroup of $\mathbb{R}^+$. An fn state φ on a von Neumann algebra M is called Γ-*stable* if for every $n \in \mathbb{N}$, $0 < r \leq 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ with $1 = r\gamma_1 + \cdots + r\gamma_n$, there exist n partial isometries $v_1, \dots, v_n \in M$ and a projection $e \in M$ such that

$$\sum_{j=1}^{n} v_j v_j^* = 1, \quad \varphi(e) = r$$

and

$$e = v_j^* v_j, \quad \varphi v_j = \gamma_j v_j \varphi \quad \text{for } 1 \leq j \leq n.$$

Remark 5.2. If $\Gamma = \mathbb{Q}^+$, then $\mathbb{Q}$-stable states in [Ha4, Def. 4.1] are equivalent to our $\mathbb{Q}^+$-stable states. Indeed, an fn state φ is $\mathbb{Q}$-stable in the sense of [Ha4, Def. 4.1] if and only if, for $q_1, \dots, q_n \in \mathbb{Q}^+$ with $1 = q_1 + \cdots + q_n$, there exist n isometries $v_1, \dots, v_n \in M$ such that

$$\sum_{j=1}^{n} v_j v_j^* = 1 \quad \text{and} \quad \varphi v_j = q_j v_j \varphi,$$

because of [Ha4, Lem. 4.6]. Therefore, if φ is $\mathbb{Q}^+$-stable, then for $q_1, \dots, q_n \in \mathbb{Q}^+$ with $1 = q_1 + \cdots + q_n$, there are partial isometries $v_1, \dots, v_n \in M$ and a projection $e \in M$ such that

$$\sum_{j=1}^{n} v_j v_j^* = 1, \quad \varphi(e) = 1$$

and

$$e = v_j^* v_j, \quad \varphi v_j = q_j v_j \varphi \quad \text{for } 1 \leq j \leq n.$$

Since $\varphi(e) = 1$ and φ is faithful, we have $e = 1$. Hence $v_1, \dots, v_n$ are isometries, and thus φ is $\mathbb{Q}$-stable.

Conversely, let $0 < r \leq 1$ and $\gamma_1, \dots, \gamma_n \in \mathbb{Q}^+$ with $1 = r\gamma_1 + \cdots + r\gamma_n$. Then $r \in \mathbb{Q}^+$ and put $q_j := r\gamma_j \in \mathbb{Q}^+$ for $1 \leq j \leq n$. By using $\mathbb{Q}$-stability, there are isometries $v_1, \dots, v_n \in M$ such that

$$\sum_{j=1}^{n} v_j v_j^* = 1 \quad \text{and} \quad \varphi v_j = q_j v_j \varphi.$$

Moreover, there is an isometry w such that $\varphi w = rw\varphi$. We define partial isometries $w_j := v_j w^*$. Then $w_j^* w_j = ww^*$ and

$$\sum_{j=1}^{n} w_j w_j^* = \sum_{j=1}^{n} v_j v_j^* = 1.$$

Moreover,

$$\varphi w_j = \varphi v_j w^* = q_j r^{-1} v_j w^* \varphi = \gamma_j w_j \varphi$$

and

$$\varphi(w_j^* w_j) = \gamma_j^{-1} \varphi(w_j w_j^*) = \gamma_j^{-1} \varphi(v_j v_j^*) = \gamma_j^{-1} q_j \varphi(v_j^* v_j) = r.$$

Lemma 5.3 (Cf. [Ha4, Thm. 4.5]). *Let φ be a Γ-stable fn state on a von Neumann algebra M, and let $0 < r \leq 1$, $\gamma_1, \dots, \gamma_n \in \Gamma$ with $1 = r\gamma_1 + \cdots + r\gamma_n$. Then there exists a type I_n subfactor F of M such that $\sigma_t^\varphi(F) = F$ for $t \in \mathbb{R}$ and $\varphi|_F = \mathrm{Tr}_n(h\,\cdot\,)$, where*

$$h = \begin{bmatrix} r\gamma_1 & & \\ & \ddots & \\ & & r\gamma_n \end{bmatrix}.$$

Proof. There exist n partial isometries $v_1, \dots, v_n \in M$ and a projection $e \in M$ such that

$$\sum_{j=1}^{n} v_j v_j^* = 1, \quad \varphi(e) = r$$

and

$$e = v_j^* v_j, \quad \varphi v_j = \gamma_j v_j \varphi \quad \text{for } 1 \le j \le n.$$

Then $e_{jk} := v_j v_k^*$ for $1 \le j, k \le n$ give a system of matrix units. Moreover, we have

$$\varphi(e_{jk}) = \varphi(v_j v_k^*) = \gamma_j \varphi(v_k^* v_j) = \delta_{jk} r \gamma_j.$$

Since

$$\sigma_t^\varphi(e_{jk}) = \sigma_t^\varphi(v_j v_k^*) = \gamma_j^{it} \gamma_k^{-it} v_j v_k^* = \gamma_j^{it} \gamma_k^{-it} e_{jk},$$

we have $\sigma_t^\varphi(F) = F$ for $t \in \mathbb{R}$. □

Remark 5.4. If τ is a tracial fn state on a type II_1 factor, then it is easy to see that τ is $\{1\}$-stable.

If φ is an fn state on a type III_λ factor $(0 < \lambda < 1)$ for which $\sigma_{t_0}^\varphi = \mathrm{id}$, where $t_0 = -2\pi/\log\lambda$, then φ is $\{\lambda^m\}_{m\in\mathbb{Z}}$-stable. Indeed, let $0 < r \le 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$ with $1 = r\gamma_1 + \cdots + r\gamma_n$. Put $\lambda_j := r\gamma_j$. Since M_φ is a type II_1 factor, we can choose a projection e and mutually orthogonal projections $e_1, \dots, e_n$ in M_φ with sum 1 such that

$$\varphi(e) = r \quad \text{and} \quad \varphi(e_j) = \lambda_j \quad \text{for } 1 \le j \le n.$$

By using [Ha3, Lem. 4.2], there exist partial isometries $v_1, \dots, v_n$ in M such that

$$e = v_j^* v_j, \quad e_j = v_j v_j^* \quad \text{and} \quad \varphi v_j = \gamma_j v_j \varphi \quad \text{for } 1 \le j \le n.$$

In these cases, by Lemma 5.3, we obtain a finite-dimensional subfactor F of M such that

$$\sigma_t^\varphi(F) = F \quad \text{for } t \in \mathbb{R}.$$

Note that it is equivalent to

$$\varphi = \varphi|_F \otimes \varphi|_{F^c},$$

where $F^c := F' \cap M$. We expect that $\varphi|_{F^c}$ is also Γ-stable. If τ is tracial, then $\tau|_{F^c}$ is also tracial, and thus is $\{1\}$-stable. If M is a type III_λ factor with an fn state φ for which $\sigma_{t_0}^\varphi = \mathrm{id}$, then F^c is also a type III_λ factor and $\sigma_t^{\varphi|_{F^c}}$ is the restriction of σ_t^φ to F^c. Hence $\varphi|_{F^c}$ is also $\{\lambda^m\}_{m\in\mathbb{Z}}$-stable. In the case of type III_1 factors, it is proved in [Ha4, Thm. 4.5] that if φ is $\mathbb{Q}^+$-stable, then $\varphi|_{F^c}$ is also $\mathbb{Q}^+$-stable.

Lemma 5.5 (Cf. [Ha4, Lem. 4.6]). *If φ is Γ-stable, then for $\gamma \in \Gamma$, there exist $m \in \mathbb{N}$ and partial isometries $w_1, \dots, w_m \in M$ such that*

$$\sum_{j=1}^{m} w_j^* w_j = 1 \quad \textit{and} \quad \varphi w_j = \gamma w_j \varphi \quad \textit{for } 1 \leq j \leq m.$$

Proof. If $\gamma = 1 \in \Gamma$, then $m = 1$ and $w_1 = 1$. If $0 < \gamma < 1$, then set $r := \gamma$. By using Γ-stability for $1 = r\gamma^{-1}$, there exists a partial isometry $v \in M$ such that $vv^* = 1$, $\varphi v = \gamma^{-1} v \varphi$ and $\varphi(v^* v) = \gamma$. Then $w_1 := v^*$ satisfies the desired properties. If $1 < \gamma$, then there is $m \in \mathbb{N}$ such that

$$m\gamma^{-1} \geq 1.$$

Then take $0 < r \leq 1$ such that $r(m\gamma^{-1}) = 1$. By using Γ-stability, there exist partial isometries $v_1, \dots, v_m \in M$ such that

$$\sum_{j=1}^{m} v_j v_j^* = 1, \quad \varphi v_j = \gamma^{-1} v_j \varphi \quad \text{for } 1 \leq j \leq m.$$

Then $w_1 := v_1^*, \dots, w_m := v_m^*$ satisfy the desired properties. □

§6. Injective factors and ITPFI factors

Throughout this section, we assume that Γ is a multiplicative subgroup of $\mathbb{R}^+$, which is $\{1\}$, $\{\lambda^m\}_{m \in \mathbb{Z}}$ with $0 < \lambda < 1$ or $\mathbb{Q}^+$. We also assume that M is an injective factor M not of type I, with separable predual and φ is a Γ-stable fn state on M with $B(M, \varphi) = \mathbb{C}1$. If M is of type II_1, then φ is tracial with $\Gamma = \{1\}$, and if M is of type III_λ $(0 < \lambda < 1)$, then $\sigma_{t_0}^{\varphi} = \mathrm{id}$ with $\Gamma = \{\lambda^m\}_{m \in \mathbb{Z}}$, where $t_0 = -2\pi / \log \lambda$. If M is of type III_1, then we assume that φ is $\mathbb{Q}^+$-stable. We recall that every type III_1 factor with separable predual has a $\mathbb{Q}^+$-stable fn state by [Ha4, Thm. 4.2]. Moreover, every injective type III_1 with separable predual factor has trivial bicentralizer by [Ha2, Thm. 2.3].

We prove the main theorem in this section.

Theorem 6.1. *Let M be an injective factor M with separable predual and φ be a Γ-stable fn state on M with $B(M, \varphi) = \mathbb{C}1$. Then M is ITPFI.*

Lemma 6.2 (Cf. [Ha4, Lem. 5.4]). *Let φ be a Γ-stable fn state with $B(M, \varphi) = \mathbb{C}1$ on an injective factor M. Let $u_1, \dots, u_n \in \mathcal{U}(M)$ and $\delta > 0$. Then there exist a finite-dimensional σ^{φ}-invariant subfactor F of M and $v_1, \dots, v_n \in \mathcal{U}(F)$ satisfying the following: for $\varepsilon > 0$, there exist operators $b_1, \dots, b_p$ in M such that for $1 \leq k \leq n$,*

(a) $\|(1-\sum_{j=1}^{p} b_j^* b_j)u_k\xi_\varphi\| < \varepsilon$, $\|(1-\sum_{j=1}^{p} b_j^* b_j)v_k\xi_\varphi\| < \varepsilon$;

(b) $\|(1-\sum_{j=1}^{p} \varepsilon_{F,\varphi}(b_j b_j^*))u_k\xi_\varphi\| < \varepsilon$, $\|(1-\sum_{j=1}^{p} \varepsilon_{F,\varphi}(b_j b_j^*))v_k\xi_\varphi\| < \varepsilon$;

(c) $\sum_{j=1}^{p} \|b_j\xi_\varphi - \xi_\varphi b_j\|^2 < \delta$;

(d) $\sum_{j=1}^{p} \|b_j u_k - v_k b_j\|_\varphi^2 < \delta$.

Proof. By Theorem 2.5 and Remark 2.6, for $1 > \delta > 0$ there exists a ucp map $T\colon \mathbb{M}_m \to M$ and $v_1, \ldots, v_n \in \mathcal{U}(\mathbb{M}_m)$ such that an fn state $\psi := \varphi \circ T = \mathrm{tr}_m(h_\psi \cdot\,)$ on $\mathbb{M}_m$ satisfies

$$\|\sigma_t^\varphi \circ T - T \circ \sigma_t^\psi\| \le \delta|t| \qquad \text{for } t \in \mathbb{R},$$
$$\|T(v_k) - u_k\|_\varphi < \frac{\delta^{1/2}}{2} \qquad \text{for } 1 \le k \le n$$

and

$$\lambda_1/\lambda_2 \in \Gamma \quad \text{for } \lambda_1, \lambda_2 \in \mathrm{sp}(h_\psi).$$

Since φ is Γ-stable, as in the proof of [Ha4, Lem. 5.4], we may assume that $F := \mathbb{M}_m \subset M$, and $T\colon F \to M$ satisfies $\varphi \circ T = \varphi|_F$ and

$$\|\sigma_t^\varphi \circ T - T \circ \sigma_t^{\varphi|_F}\| \le \delta|t| \quad \text{for } t \in \mathbb{R}.$$

Set $\xi_k := u_k\xi_\varphi$ and $\eta_k := v_k\xi_\varphi$ for $0 \le k \le n$, where $u_0 = v_0 = 1$.

By [Ha1, Prop. 2.1], there exist $a_1, \ldots, a_p \in M$ such that

$$T(x) = \sum_{j=1}^{p} a_j^* x a_j \quad \text{for } x \in F.$$

Since T is unital, we have

$$\sum_{j=1}^{p} a_j^* a_j = 1.$$

If M is of type II_1, then operators $a_1, \ldots, a_p$ satisfy the desired properties by the proof of [Ha1, Prop. 5.2].

Next we assume that M is properly infinite. By [Ha1, Prop. 2.1], we can take a single operator $a \in M$ such that

$$T(x) = a^* x a \quad \text{for } x \in F.$$

Then we can find finitely many operators a_j satisfying the following conditions:

(a′) $\|(1-\sum_{j=-p}^{p} a_j^* a_j)\xi_k\| < \varepsilon/2$, $\|(1-\sum_{j=1}^{p} a_j^* a_j)\eta_k\| < \varepsilon/2$;

(b′) $\|(1-\sum_{j=-p}^{p} \lambda^{-j}\varepsilon_{F,\varphi}(a_j a_j^*))\xi_k\| < \varepsilon/2$, $\|(1-\sum_{j=-p}^{p} \lambda^{-j}\varepsilon_{F,\varphi}(a_j a_j^*))\eta_k\| < \varepsilon/2$;

(c′) $\sum_{j=-p}^{p} \|a_j\xi_\varphi - \lambda^{-j/2}\xi_\varphi a_j\|^2 < \delta$;

(d′) $\sum_{j=-p}^{p} \|a_j u_k - v_k a_j\|_\varphi^2 < \delta$.

If M is of type III_1 and φ is $\mathbb{Q}_+$-stable, then by the argument of [Ha4, Lem. 5.4], we can choose $\lambda \in \mathbb{Q}_+$ and operators a_j for $-p \le j \le p$ satisfy conditions (a′)–(d′).

If M is of type III_λ $(0 < \lambda < 1)$, then it suffices to set $a_j := \varepsilon_j(a)$, where ε_j is the projection of norm 1 of M onto

$$M_j = \{x \in M \mid \sigma_t^\varphi(x) = \lambda^{ijt}(x),\ t \in \mathbb{R}\}$$

given by

$$\varepsilon_j(x) := \frac{1}{t_0}\int_0^{t_0} \sigma_t^\varphi(x)\lambda^{-ijt}\,dt.$$

Indeed, it follows from similar arguments to [Ha4, Lem. 5.4]. We give a sketch proof below. Note that every $x \in M$ has a formal expansion

$$x \sim \sum_{j=-\infty}^{\infty} \varepsilon_j(x).$$

For $\xi \in L^2(M, \varphi)$, we have

$$\sum_{j=-\infty}^{\infty} \|\varepsilon_j(a)\xi\|^2 = \frac{1}{t_0}\int_0^{t_0} \|\sigma_t^\varphi(a)\xi\|^2\,dt = \|\xi\|^2.$$

Let $x \in F$. Since

$$\begin{aligned}
\varphi\big(\varepsilon_{F,\varphi}(\varepsilon_j(a)\varepsilon_j(a)^*)x\big) &= \varphi \circ \varepsilon_{F,\varphi}(\varepsilon_j(a)\varepsilon_j(a)^*x) \\
&= \varphi(\varepsilon_j(a)\varepsilon_j(a)^*x) \\
&= \varphi\big(\varepsilon_j(a)^* x \sigma_{-i}^\varphi(\varepsilon_j(a))\big) \\
&= \lambda^j \varphi(\varepsilon_j(a)^* x \varepsilon_j(a)),
\end{aligned}$$

we obtain

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \lambda^{-j}\varphi\big(\varepsilon_{F,\varphi}(\varepsilon_j(a)\varepsilon_j(a)^*)x\big) &= \sum_{j=-\infty}^{\infty} \varphi(\varepsilon_j(a)^* x \varepsilon_j(a)) \\
&= \sum_{j=-\infty}^{\infty} \langle x\varepsilon_j(a)\xi_\varphi, \varepsilon_j(a)\xi_\varphi\rangle \\
&= \frac{1}{t_0}\int_0^{t_0} \langle x\sigma_t^\varphi(a)\xi_\varphi, \sigma_t^\varphi(a)\xi_\varphi\rangle\,dt \\
&= \frac{1}{t_0}\int_0^{t_0} \varphi(\sigma_t^\varphi \circ T \circ \sigma_{-t}^\varphi(x))\,dt \\
&= \varphi(x).
\end{aligned}$$

Since $\xi_\varphi \varepsilon_j(a) = \lambda^{j/2}\varepsilon_j(a)\xi_\varphi$, we have

$$\sum_{j=-\infty}^{\infty} \|\varepsilon_j(a)\xi_\varphi - \lambda^{-j/2}\xi_\varphi\varepsilon_j(a)\|^2 = 0.$$

For $\xi, \eta \in L^2(M, \varphi)$, we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \langle x\varepsilon_j(a)\xi, \varepsilon_j(a)\eta\rangle &= \frac{1}{t_0}\int_0^{t_0} \langle x\sigma_t^\varphi(a)\xi, \sigma_t^\varphi(a)\eta\rangle \, dt \\ &= \frac{1}{t_0}\int_0^{t_0} \langle \sigma_t^\varphi \circ T \circ \sigma_{-t}^\varphi(x)\xi, \eta\rangle \, dt.\end{aligned}$$

Hence

$$\left\| T(x) - \sum_{j=-\infty}^{\infty} \varepsilon_j(a)^* x \varepsilon_j(a) \right\| \leq \delta\|x\|.$$

Therefore, it follows from the above arguments that, for sufficiently large integer $p > 0$, operators a_j for $-p \leq j \leq p$ satisfy conditions (a′)–(d′).

By Remark 5.4, $\varphi|_{F^c}$ is also Γ-stable. By Lemma 5.5, for each $-p \leq j \leq p$, there exist a finite set of operators $c_{j,1}, \dots, c_{j,p(j)}$ in F^c such that

$$\varphi c_{j,l} = \lambda^{-j} c_{j,l}\varphi \quad \text{and} \quad \sum_{l=1}^{p(j)} c_{j,l}^* c_{j,l} = 1.$$

Then operators $b_{j,l} \coloneqq c_{j,l}a_j$ for $-p \leq j \leq p$ and $l = 1, \dots, p(j)$ satisfy the desired properties as in the proof of [Ha4, Lem. 5.4]. □

Lemma 6.3 (Cf. [Ha4, Lem. 5.5]). *Let $\delta > 0$ and $u_1, \dots, u_n \in \mathcal{U}(M)$, Then there exist a finite-dimensional σ^φ-invariant subfactor F of M and unitaries $v_1, \dots, v_n \in \mathcal{U}(F)$ such that $(n+1)$-tuples of unit vectors $(\xi_\varphi, u_1\xi_\varphi, \dots, u_n\xi_\varphi)$ and $(\xi_\varphi, v_1\xi_\varphi, \dots, v_n\xi_\varphi)$ are almost δ-related.*

Proof. The proof is the same as [Ha4, Lem. 5.5]. So we only give a sketch of the proof. Let $\delta > 0$ and $u_1, \dots, u_n \in \mathcal{U}(M)$. Thanks to Lemma 6.2, there exist a finite-dimensional σ^φ-invariant subfactor F of M and $v_1, \dots, v_n \in \mathcal{U}(F)$ as in the statement of Lemma 6.2. For a given $\varepsilon > 0$, we can choose operators $b_1, \dots, b_p$ in M satisfying conditions (a)–(d) in Lemma 6.2.

Let $\delta' > 0$ be arbitrary. Set $b \coloneqq \sum_{j=1}^p b_j b_j^*$. Since $B(M, \varphi) = \mathbb{C}$, thanks to [Ha2] and [Ha4, Prop. 2.6] we have

$$\varepsilon_{F,\varphi}(b) \in \overline{\mathrm{conv}}\{wbw^* \mid w \in \mathcal{U}(F^c),\ \|w\xi_\varphi - \xi_\varphi w\| < \delta'\},$$

where the closure means the σ-strong operator topology. Hence there exist $w_1, \dots, w_q \in \mathcal{U}(F^c)$ and $\lambda_1, \dots, \lambda_q \in \mathbb{R}^+$ with $\sum_{l=1}^q \lambda_q = 1$ such that

$$\|w_l \xi_\varphi - \xi_\varphi w_l\| < \delta'$$

and

$$\left\| \left(1 - \sum_{l=1}^q \lambda_l w_l b w_l^*\right) u_k \xi_\varphi \right\| < \varepsilon \quad \text{and} \quad \left\| \left(1 - \sum_{l=1}^q \lambda_l w_l b w_l^*\right) v_k \xi_\varphi \right\| < \varepsilon.$$

Then one can easily check that operators

$$a_{j,l} := \lambda_l^{1/2} w_l b_j$$

for $j = 1, \dots, p$ and $l = 1, \dots, q$ have the desired properties for the almost δ-relatedness. $\square$

Now we prove our main theorem in this section.

Proof of Theorem 6.1. Let $u_1, \dots, u_n \in \mathcal{U}(M)$ and $\varepsilon > 0$. Then we take $\delta = \delta(n, \varepsilon/4) > 0$ with properties in Theorem 4.6. By Lemma 6.3, we choose a finite-dimensional σ^φ-invariant subfactor F of M and $v_1, \dots, v_n \in \mathcal{U}(F)$ such that $(n+1)$-tuples of unit vectors $(\xi_\varphi, u_1\xi_\varphi, \dots, u_n\xi_\varphi)$ and $(\xi_\varphi, v_1\xi_\varphi, \dots, v_n\xi_\varphi)$ are almost δ-related. Therefore, by Theorem 4.6, there exists $w \in \mathcal{U}(M)$ such that

$$\|w\xi_\varphi - \xi_\varphi w\| < \frac{\varepsilon}{4}$$

and

$$\|w u_k \xi_\varphi - v_k \xi_\varphi w\| < \frac{\varepsilon}{4} \quad \text{for } 1 \le k \le n.$$

Then

$$\begin{aligned} \|u_k - w^* v_k w\|_\varphi &= \|w^*(w u_k - v_k w)\xi_\varphi\| \\ &\le \|w u_k \xi_\varphi - v_k \xi_\varphi w\| + \|v_k(\xi_\varphi w - w\xi_\varphi)\| \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Set $F_0 := w^* F w$ and $w_k := w^* v_k w$ for $1 \le k \le n$. Then

$$\|u_k - w_k\|_\varphi < \frac{\varepsilon}{2} \quad \text{for } 1 \le k \le n.$$

Put $\varphi_0 := w^* \varphi w$. Then since F is σ^φ-invariant, F_0 is also σ^{φ_0}-invariant, i.e.,

$$\varphi_0 = \varphi_0|_{F_0} \otimes \varphi_0|_{F_0^c}.$$

Since $\xi_{\varphi_0} = w^*\xi_\varphi w$, we have

$$\begin{aligned}\|\varphi - \varphi_0\| &\leq \|\xi_\varphi - w^*\xi_\varphi w\| \, \|\xi_\varphi + w^*\xi_\varphi w\| \\ &\leq 2\|w\xi_\varphi - \xi_\varphi w\| \\ &< \varepsilon.\end{aligned}$$

Therefore, by [CW, Lem. 7.6], M is ITPFI. □

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