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Shifted Yangians and Polynomial *R*-Matrices

by

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Abstract

We study the category $\mathcal{O}^{\mathrm{sh}}$ of representations over a shifted Yangian. This category has a tensor product structure and contains distinguished modules, the positive prefundamental modules and the negative prefundamental modules. Motivated by the representation theory of the Borel subalgebra of a quantum affine algebra and by the relevance of quantum integrable systems in this context, we prove that tensor products of prefundamental modules with irreducible modules are either cyclic or cocyclic. This implies the existence and uniqueness of morphisms, the R-matrices, for such tensor products. We prove the R-matrices are polynomial in the spectral parameter, and we establish functional relations for the R-matrices. As applications, we prove the Jordan–Hölder property in the category $\mathcal{O}^{\mathrm{sh}}$. We also obtain a proof, uniform for any finite type, that any irreducible module factorizes through a truncated shifted Yangian.

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Contents

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§1. Introduction

Shifted Yangians, and their truncations, appeared for type A in the context of the representation theory of finite W-algebras in the work of Brundan–Kleshchev [9], then in the study of quantized affine Grassmannian slices by Kamnitzer–Webster–Weekes–Yacobi [50] for general types, and in the study of quantized Coulomb branches of three-dimensional $N = 4$ SUSY quiver gauge theories by Braverman–Finkelberg–Nakajima [7] for simply laced types and by Nakajima–Weekes [62] for non-simply-laced types.

Fix a finite-dimensional complex simple Lie algebra $\mathfrak{g}$. The shifted Yangians $Y_\mu(\mathfrak{g})$ can be seen as variations of the ordinary Yangian $Y(\mathfrak{g})$ in its Drinfeld presentation, but depending on a coweight μ in the coweight lattice, denoted by $\mathbf{P}^\vee$, of the underlying simple Lie algebra $\mathfrak{g}$. In the particular case $\mu = 0$, we recover the Yangian $Y_0(\mathfrak{g}) = Y(\mathfrak{g})$. The representations of Yangians, and their trigonometric analogs the quantum affine algebras $U_q(\hat{\mathfrak{g}})$, have been under intense study for several decades. The truncated shifted Yangians, certain remarkable quotient of shifted Yangians, depend on additional parameters, including a dominant coweight λ. These parameters λ and μ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}^\lambda_\mu$ (the usual slices when μ is dominant). These varieties are also Coulomb branches, symplectic dual to Nakajima quiver varieties, and the truncated shifted Yangians can be seen as quantizations of these symplectic varieties.

For simply laced types, representations of shifted Yangians and related Coulomb branches have been intensively studied in this context; see [9, 48, 49] and references therein. For non-simply-laced types, representations of quantizations of Coulomb branches have been studied by Nakajima–Weekes [62] by using the method originally developed in [61] for simply laced types (the reader may refer to the discussion in [40, Introduction]).

One crucial property of shifted Yangians is the existence [20] of a family of algebra homomorphisms indexed by a pair of coweights μ and ν,

$$\Delta_{\mu,\nu} \colon Y_{\mu+\nu}(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}).$$

This is analogous to the Drinfeld–Jimbo coproduct for ordinary Yangians. These coproducts $\Delta_{\mu,\nu}$ induce a tensor product structure on a category

$$\mathcal{O}^{\mathrm{sh}} = \bigoplus_{\mu \in \mathbf{P}^\vee} \mathcal{O}_\mu,$$

which is a sum of categories $\mathcal{O}_\mu$ of representations over the shifted Yangians $Y_\mu(\mathfrak{g})$ for various coweights μ.

By [9, 48, 49], an irreducible module in category $\mathcal{O}^{\mathrm{sh}}$ is determined by its *highest weight*, which is a tuple of ratios of monic polynomials in u, one ratio for each Dynkin node of $\mathfrak{g}$. There is a natural $\mathbb{C}$-action on the shifted Yangians by algebra automorphisms. Each module V induces on the same underlying vector space a family of module structures $V(a)$, for $a \in \mathbb{C}$ referred to as the *spectral parameter*, such that $V(0) = V$. If V is irreducible, then $V(a)$ remains irreducible and its highest weight is obtained from that of V by the substitution $u \mapsto u - a$.

In a seemingly different direction, the category $\mathcal{O}$ of representations of the Borel subalgebra $U_q(\hat{\mathfrak{b}})$ of a quantum affine algebra $U_q(\hat{\mathfrak{g}})$ was introduced and studied by Jimbo and the first author in [41]. One crucial point for the approach therein is an asymptotical procedure to construct certain remarkable simple representations, the prefundamental representations, as limits of finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. It was observed by the second author in [70] that the category $\mathcal{O}^{\mathrm{sh}}$ for shifted Yangians provides a Yangian counterpart of the prefundamental representations and their asymptotical procedure. The prefundamental representations play a similar role in category $\mathcal{O}^{\mathrm{sh}}$ as the fundamental representations do in the subcategory of finite-dimensional representations of the ordinary Yangian [14], hence the terminology.

The trigonometric analogs of shifted Yangians, namely the shifted quantum affine algebras, are other examples of shifted quantum groups. These algebras $U_q^{\mu}(\hat{\mathfrak{g}})$ were introduced by Finkelberg–Tsymbaliuk [21] as variations of the quantum affine algebras $U_q(\hat{\mathfrak{g}})$, for a quantization parameter $q \in \mathbb{C}^*$ which is not a root of unity. The shifted quantum affine algebras also admit remarkable truncations which are closely related to K-theoretical Coulomb branches. The approach to the representation theory of $U_q^{\mu}(\hat{\mathfrak{g}})$ developed by the first author in [40] is based on the relations with representations in the category $\mathcal{O}$ of the Borel algebra $U_q(\hat{\mathfrak{b}})$ and on associated quantum integrable systems. For instance, the study of R-matrices and transfer matrices of $U_q(\hat{\mathfrak{g}})$ allows one to give a proof, uniform for any finite type, that simple finite-dimensional representations of shifted quantum affine algebras $U_q^{\mu}(\hat{\mathfrak{g}})$ descend to a truncation.

In this spirit, it is natural to raise the question of the construction of R-matrices for representations of shifted quantum groups. It is the problem we address in this paper and from which we obtain several applications. The Drinfeld–Jimbo coproduct is only conjecturally known for shifted quantum affine algebras, that is why we work with shifted Yangians.

More precisely, we study morphisms in category $\mathcal{O}^{\mathrm{sh}}$ of the form

$$\check{R}_{V,W} \colon V \otimes W \longrightarrow W \otimes V$$

for a pair (V, W) of irreducible representations of shifted Yangians.

To state our main results in a neat way, call an irreducible module *positive* (respectively, *negative*) if its highest weight is a tuple of (respectively, inverses of) monic polynomials. When the total degree of these polynomials is 1, these are the prefundamental modules [70] mentioned above. Positive modules are one-dimensional, while negative modules are infinite-dimensional except in the trivial case.

Construction of R-matrices. Let P be a positive module and N be a negative module. Let V be an *arbitrary* irreducible module in category $\mathcal{O}^{\mathrm{sh}}$. As our first main result, we prove the following cyclicity and cocyclicity properties (Theorem 4.8):

(i) The modules $P \otimes V$ and $V \otimes N$ are generated by tensor products of highest weight vectors. The modules $V \otimes P$ and $N \otimes V$ are cogenerated by tensor products of highest weight vectors.

Here, a module is cogenerated by a vector if this vector is contained in all non-zero submodules. As a consequence, we obtain unique module morphisms sending a tensor product of highest weight vectors to the opposite tensor product,

$$\check{R}_{P,V}(a)\colon P(a) \otimes V \longrightarrow V \otimes P(a)$$

and

$$\check{R}_{V,N}(a)\colon V(a) \otimes N \longrightarrow N \otimes V(a).$$

Another consequence of the cyclicity property is that the R-matrices $\check{R}_{P,V}(a)$ and $\check{R}_{V,N}(a)$, viewed as vector-valued functions of a, are polynomial.

Our cyclicity property differs from the case of finite-dimensional irreducible representations of ordinary quantum affine algebras and Yangians where cyclicity holds true for *generic* spectral parameters. Indeed, in the non-shifted case the failure of cyclicity is controlled by the poles of normalized R-matrices viewed as rational functions [1, 25, 51, 37, 38, 30] of the spectral parameter, which are rarely polynomial.

Properties of R-matrices. The R-matrices being module morphisms, we are able to compute them for positive modules, much in the spirit of Jimbo [44]. It turns out that they are connected to the Gerasimov–Kharchev–Lebedev–Oblezin truncation series [32], GKLO series for short, which are certain generating series of the shifted Yangians appearing in the definition of truncated shifted Yangians [9, 50, 7]. Note that in the trigonometric case, the truncation series defining truncated shifted quantum affine algebras were related to limits of transfer matrices associated to positive prefundamental representations (the Q-operators) in [40].

Since a positive module P is one-dimensional, we view $\check{R}_{P,V}(a)$ as a linear operator on V. We establish the following property in the "positive case":

(ii) For P a positive prefundamental module, the vector-valued polynomial function $a \mapsto \check{R}_{P,V}(a)$ from $\mathbb{C}$ to $\mathrm{End}(V)$ satisfies an additive difference equation determined by the action of a GKLO series; see equation (5.6).

In the case of finite-dimensional representations of the ordinary Yangian, there is a general construction of R-matrices by solving additive difference equations [28, 29, 30]. Point (ii) can be seen as a reverse statement: first the R-matrices are shown to exist, and then we find difference equations for them.

In the "negative case", the following is the key technical result of this paper.

(iii) Let V be a fundamental module equipped with a weight basis, and view $\check{R}_{V,N}(a)$ as a matrix whose entries are vector-valued polynomial functions from $\mathbb{C}$ to $\mathrm{End}(N)$. Then the diagonal entry associated to the lowest weight basis vector of V is the action of a GKLO series; Equation (8.1).

Here, a fundamental module [14] is a finite-dimensional irreducible module over the ordinary Yangian whose associated Drinfeld polynomials are of total degree 1 (as mentioned above, it should not be confused with a prefundamental representation).

Application I: Truncation of irreducible modules. We obtain a proof, uniform for any finite type, that any irreducible module in $\mathcal{O}^{\mathrm{sh}}$ factorizes through a truncated shifted Yangian. For $\mathfrak{g}$ of simply laced type, this can be derived from [48, 49], and then extended to non-simply-laced types by [62, 61], where the classification for non-simply-laced truncated shifted Yangians is reduced to the known classification in simply laced types via geometric arguments. In the case of shifted quantum affine algebras, the result was established for finite-dimensional irreducible modules [40, Thm. 12.9] by a method involving transfer matrices of quantum integrable systems.

We prove furthermore that if $\mathfrak{g}$ is not of type E_8, then any highest ℓ-weight module in category $\mathcal{O}^{\mathrm{sh}}$ descends to a truncation, by realizing such a module as a quotient of the tensor product of a positive module with a negative module (Theorem 4.15).

Application II: Jordan–Hölder property. As another application, we prove that in category $\mathcal{O}^{\mathrm{sh}}$ the tensor product of two (and hence finitely many) irreducible modules admits a finite Jordan–Hölder filtration. In other words, the full subcategory of $\mathcal{O}^{\mathrm{sh}}$ consisting of modules with a finite Jordan–Hölder filtration is closed under tensor product. This seems surprising, at least to us, as the analogous category $\mathcal{O}$ for the Borel subalgebra [41] does not satisfy this property.

We expect the R-matrices introduced in this paper for shifted quantum groups will be further studied in the future, keeping in mind their importance for ordinary quantum groups. Also, an approach to R-matrices using algebraic versions of Maulik–Okounkov stable maps for the category $\mathcal{O}$ of Borel subalgebras was proposed in [39]. As this category $\mathcal{O}$ is closely related to the category $\mathcal{O}^{\mathrm{sh}}$ for shifted quantum affine algebras, we expect such algebraic stable maps also to exist for shifted quantum groups. We also expect more applications of these R-matrices for the representation theory of shifted quantum groups, in particular to get advances on the conjecture in [40] which states a parameterization of irreducible representations of non-simply-laced truncated shifted quantum affine algebras in terms of Langlands dual q-characters.

The paper is organized as follows.

In Section 2 we recall the basic properties of shifted Yangians and of the truncated shifted Yangians. We also give a first estimation of the coproduct (Lemma 2.5).

In Section 3 we recall basic properties of representations of shifted Yangians, including the existence of Verma modules, the parameterization of irreducible modules in category $\mathcal{O}^{\mathrm{sh}}$, finite-dimensional irreducible modules, q-characters.

In Section 4 we prove cyclicity and cocyclicity properties for tensor products of prefundamental representations in category $\mathcal{O}^{\mathrm{sh}}$ (Theorem 4.8), which motivate our definitions of Weyl modules and standard modules (Definition 4.10). We describe these modules when $\mathfrak{g}$ is not of type E_8 (Theorem 4.15).

In Section 5 we construct the R-matrices for suitable highest ℓ-weight modules and establish their first properties in Theorem 5.2 and Propositions 5.3 and 5.7. We also get several results on the eigenvalues of certain of these R-matrices (Proposition 5.8).

In Section 6 we focus on the case $\mathfrak{g} = \mathrm{sl}_2$, for which we prove the existence and uniqueness of factorization for all irreducible modules in category $\mathcal{O}^{\mathrm{sh}}$ into tensor products of prefundamental modules and Kirillov–Reshetikhin modules (Theorem 6.4).

In Section 7 we compute diagonal entries of certain remarkable R-matrices, by relating them to one-dimensional R-matrices (Proposition 7.2 and Theorem 7.4). The proof uses a refined estimation of the coproduct that we establish (Lemma 7.1).

In Section 8 we prove uniformly that any standard module (and so any irreducible module) factorizes through a truncated shifted Yangian (Theorem 8.4).

In Section 9 we establish the Jordan–Hölder property of the category $\mathcal{O}^{\mathrm{sh}}$ (Theorem 9.5). We also get a uniform proof that a truncated shifted Yangian has only a finite number of irreducible representations (Theorem 9.3).

§2. Shifted Yangians

In this section we recall the basic properties of shifted Yangians from [7, 20, 52]. We review their definition, their standard gradings, and their triangular decomposition. We recall the shift homomorphism and the coproduct for which we give an estimation (Lemma 2.5). We also discuss the particular case of the ordinary Yangian, as well as certain remarkable quotients, the truncated shifted Yangians.

§2.1. Definition and structure

Fix $\mathfrak{g}$ to be a complex finite-dimensional simple Lie algebra. Set $\mathbb{N} := \mathbb{Z}_{\geq 0}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $I := \{1, 2, \dots, r\}$ be the set of Dynkin nodes. The dual space $\mathfrak{h}^*$ admits a basis of *simple roots* $(\alpha_i)_{i\in I}$ and a non-degenerate symmetric bilinear form $(\,,\,)\colon \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$. For $i, j \in I$ set

$$c_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad d_{ij} := \frac{(\alpha_i, \alpha_j)}{2}, \quad d_i := d_{ii}.$$

We assume that the $d_i \in \mathbb{Z}_{>0}$ are coprime. For $i \in I$, the *fundamental weight* $\varpi_i \in \mathfrak{h}^*$, the *fundamental coweight* $\varpi_i^\vee \in \mathfrak{h}$, and the *simple coroot* $\alpha_i^\vee \in \mathfrak{h}$ are determined by the following equations for $j \in I$:

$$(\varpi_i, \alpha_j) = d_i \delta_{ij}, \quad \langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}, \quad \langle \alpha_i^\vee, \alpha_j \rangle = c_{ij},$$

where $\langle\,,\,\rangle\colon \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ denotes the evaluation map. We shall need the coweight lattice, the root lattice, and some of their subsets defined as follows:

$$\text{coweight lattice} \quad \mathbf{P}^\vee := \bigoplus_{i\in I} \mathbb{Z}\varpi_i^\vee \subset \mathfrak{h}, \quad \mathbf{Q}_+^\vee := \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee;$$

$$\text{root lattice} \quad \mathbf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*, \quad \mathbf{Q}_+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i, \quad \mathbf{Q}_- := -\mathbf{Q}_+.$$

A coweight means an element of the coweight lattice $\mathbf{P}^\vee$. It is dominant (respectively antidominant) if all the coefficients of $\varpi_i^\vee$ belong to $\mathbb{N}$ (respectively $-\mathbb{N}$). On the other hand, a weight means an element of the dual space $\mathfrak{h}^*$. The weight lattice does not play any role in this paper. We let ϖ denote fundamental weights, and reserve ω for highest ℓ-weight vectors.

For a coweight $\mu \in \mathbf{P}^\vee$, the shifted Yangian $Y_\mu(\mathfrak{g})$ is the algebra with generators

$$x_{i,n}^\pm, \quad \xi_{i,p} \quad \text{for } (i, n, p) \in I \times \mathbb{N} \times \mathbb{Z},$$

called Drinfeld generators, subject to the following relations:

$$[\xi_{i,p}, \xi_{j,q}] = 0, \quad [x^+_{i,m}, x^-_{j,n}] = \delta_{ij}\xi_{i,m+n}, \tag{2.1}$$

$$[\xi_{i,p+1}, x^\pm_{j,n}] - [\xi_{i,p}, x^\pm_{j,n+1}] = \pm d_{ij}(\xi_{i,p}x^\pm_{j,n} + x^\pm_{j,n}\xi_{i,p}), \tag{2.2}$$

$$[x^\pm_{i,m+1}, x^\pm_{j,n}] - [x^\pm_{i,m}, x^\pm_{j,n+1}] = \pm d_{ij}(x^\pm_{i,m}x^\pm_{j,n} + x^\pm_{j,n}x^\pm_{i,m}), \tag{2.3}$$

$$\mathrm{ad}^{1-c_{ij}}_{x^\pm_{i,0}}(x^\pm_{j,0}) = 0 \quad \text{if } i \neq j, \tag{2.4}$$

$$\xi_{i,-\langle\mu,\alpha_i\rangle-1} = 1, \quad \xi_{i,p} = 0 \text{ for } p < -\langle\mu,\alpha_i\rangle - 1. \tag{2.5}$$

Here, $\mathrm{ad}_x(y) \coloneqq xy - yx$. Define the generating series for $i \in I$:

$$x^\pm_i(u) \coloneqq \sum_{n\in\mathbb{N}} x^\pm_{i,n}u^{-n-1}, \quad \xi_i(u) \coloneqq \sum_{p\in\mathbb{Z}} \xi_{i,p}u^{-p-1} \in Y_\mu(\mathfrak{g})(\!(u^{-1})\!). \tag{2.6}$$

These are Laurent series in u^{-1}, with leading terms $x^\pm_{i,0}u^{-1}$ and $u^{\langle\mu,\alpha_i\rangle}$.

Remark 2.1. Our generators $x^+_{i,n}$, $x^-_{i,n}$, and $\xi_{i,p}$ correspond to $E^{(n+1)}_i$, $F^{(n+1)}_i$, and $H^{(n+1)}_i$ in [7, 20]. The zero-shifted Yangian $Y_0(\mathfrak{g})$ is the ordinary Yangian [17] with deformation parameter $\hbar = 1$, which will also be denoted by $Y(\mathfrak{g})$.

The algebra $Y_\mu(\mathfrak{g})$ is $\mathbf{Q}$-graded, called *weight grading*, by declaring the weights of the generators $x^+_{i,n}$, $x^-_{i,n}$, and $\xi_{i,p}$ to be α_i, $-\alpha_i$, and 0. For $\beta \in \mathbf{Q}$, let $Y_\mu(\mathfrak{g})_\beta$ denote the subspace of elements of weight β. Setting $p = -\langle\mu,\alpha_i\rangle - 1$ in equation (2.2) and noticing $\xi_{i,p} = 1$, one obtains the following Cartan relation:

$$[\xi_{i,-\langle\mu,\alpha_i\rangle}, x^\pm_{j,n}] = \pm(\alpha_i,\alpha_j)x^\pm_{j,n}. \tag{2.7}$$

So the weight grading is characterized alternatively: $x \in Y_\mu(\mathfrak{g})$ is of weight β if and only if $[\xi_{i,-\langle\mu,\alpha_i\rangle}, x] = (\alpha_i,\beta)x$ for all $i \in I$.

As in the case of the ordinary Yangian (see for example [28, §2.8]), for $a \in \mathbb{C}$ there is an algebra automorphism $\tau_a \colon Y_\mu(\mathfrak{g}) \to Y_\mu(\mathfrak{g})$ defined by

$$\tau_a \colon Y_\mu(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}), \quad x^\pm_j(u) \mapsto x^\pm_j(u-a), \quad \xi_j(u) \mapsto \xi_j(u-a). \tag{2.8}$$

Note that $\tau_a \circ \tau_b = \tau_{a+b}$ for $a, b \in \mathbb{C}$ and $\tau_0 = \mathrm{Id}$. Indeed, τ_a can be obtained from the evaluation at $z = a$ of the following algebra homomorphism:

$$\tau_z \colon Y_\mu(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes \mathbb{C}[z], \quad X_p \mapsto \sum_{n\in\mathbb{N}} \binom{p}{n} X_{p-n} \otimes z^n, \tag{2.9}$$

where $X \in \{x^\pm_i, \xi_i\}$ and $p \in \mathbb{Z}$. It is understood that $x^\pm_{i,k} = 0$ for $k < 0$.

For ζ, η antidominant coweights, the following map extends uniquely to an algebra morphism $\iota_{\mu,\zeta,\eta} \colon Y_\mu(\mathfrak{g}) \to Y_{\mu+\zeta+\eta}(\mathfrak{g})$, called the *shift homomorphism*:

$$x^+_{i,n} \mapsto x^+_{i,n-\langle\zeta,\alpha_i\rangle}, \quad x^-_{i,n} \mapsto x^-_{i,n-\langle\eta,\alpha_i\rangle}, \quad \xi_{i,p} \mapsto \xi_{i,p-\langle\zeta+\eta,\alpha_i\rangle}. \tag{2.10}$$

The shifted Yangian $Y_\mu(\mathfrak{g})$ admits a *triangular decomposition*. Let us define three subalgebras by generating subsets. The first $Y_\mu^<(\mathfrak{g})$ is generated by the $x_{i,n}^-$, the second $Y_\mu^>(\mathfrak{g})$ is generated by the $x_{i,n}^+$, and the third $Y_\mu^=(\mathfrak{g})$ is generated by the $\xi_{i,p}$. These subalgebras inherit from $Y_\mu(\mathfrak{g})$ the weight grading: the first two are graded by $\mathbf{Q}_\mp$, and the third by $\{0\}$. Set

$$Y_\mu^+(\mathfrak{g}) := Y_\mu^=(\mathfrak{g})Y_\mu^>(\mathfrak{g}) \quad \text{and} \quad Y_\mu^-(\mathfrak{g}) := Y_\mu^<(\mathfrak{g})Y_\mu^=(\mathfrak{g});$$

these are subalgebras. The following result is a consequence of the PBW basis theorem [20, Cor. 3.15].

Theorem 2.2 ([20]). *All shift homomorphisms are injective. The multiplication map $Y_\mu^<(\mathfrak{g}) \otimes Y_\mu^=(\mathfrak{g}) \otimes Y_\mu^>(\mathfrak{g}) \to Y_\mu(\mathfrak{g})$ is an isomorphism of vector spaces, and $Y_\mu^\pm(\mathfrak{g})$ is the algebra generated by $x_{i,n}^\pm$ and $\xi_{i,p}$ for $(i,n,p) \in I \times \mathbb{N} \times \mathbb{Z}$ subject to equation* (2.5)*, the first half of equation* (2.1)*, and the $\pm$ part of equations* (2.2)–(2.4).

It follows that the assignments $x_i^\pm(u) \mapsto x_i^\pm(u)$ and $\xi_i(u) \mapsto u^{-\langle \mu, \alpha_i \rangle}\xi_i(u)$ extend uniquely to four algebra isomorphisms

$$Y_0^>(\mathfrak{g}) \cong Y_\mu^>(\mathfrak{g}), \quad Y_0^<(\mathfrak{g}) \cong Y_\mu^<(\mathfrak{g}), \qquad Y_0^\pm(\mathfrak{g}) \cong Y_\mu^\pm(\mathfrak{g}). \tag{2.11}$$

The first two isomorphisms being independent of μ, we omit μ from $Y_\mu^>(\mathfrak{g})$ and $Y_\mu^<(\mathfrak{g})$ when no confusion arises.

The next property of shifted Yangians is the *coproduct* of Drinfeld–Jimbo, which plays a key role in our study of representations. We rephrase [20, Thms. 4.8, 4.12, Prop. 4.14]. While [20] considered simply laced types, the proofs apply in general as commented in [20, Rem. 3.2].

Theorem 2.3 ([20]). *There exists a unique family of algebra homomorphisms*

$$\Delta_{\mu,\nu} \colon Y_{\mu+\nu}(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g})$$

for all coweights μ, ν such that $\Delta_{0,0}$ is the coproduct of the ordinary Yangian and properties (i)–(ii) *hold true.*

(i) *For μ and ν antidominant, $i \in I$, $n < -\langle \mu, \alpha_i \rangle$, and $m < -\langle \nu, \alpha_i \rangle$,*

$$\Delta_{\mu,\nu}(x_{i,n}^+) = x_{i,n}^+ \otimes 1, \quad \Delta_{\mu,\nu}(x_{i,m}^-) = 1 \otimes x_{i,m}^-. \tag{2.12}$$

(ii) *If ζ and η are antidominant, then the following diagram commutes:*

$$\begin{array}{ccc} Y_{\mu+\nu}(\mathfrak{g}) & \xrightarrow{\Delta_{\mu,\nu}} & Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}) \\ \downarrow{\scriptstyle \iota_{\mu+\nu,\zeta,\eta}} & & \downarrow{\scriptstyle \iota_{\mu,\zeta,0}\otimes\iota_{\nu,0,\eta}} \\ Y_{\mu+\nu+\zeta+\eta}(\mathfrak{g}) & \xrightarrow{\Delta_{\mu+\zeta,\nu+\eta}} & Y_{\mu+\zeta}(\mathfrak{g}) \otimes Y_{\nu+\eta}(\mathfrak{g}). \end{array} \tag{2.13}$$

Furthermore, if ν is antidominant, then the following diagram commutes:

$$\begin{array}{ccc} Y_{\mu+\nu+\rho}(\mathfrak{g}) & \xrightarrow{\Delta_{\mu+\nu,\rho}} & Y_{\mu+\nu}(\mathfrak{g}) \otimes Y_\rho(\mathfrak{g}) \\ \downarrow{\scriptstyle \Delta_{\mu,\nu+\rho}} & & \downarrow{\scriptstyle \Delta_{\mu,\nu}\otimes\mathrm{Id}} \\ Y_\mu(\mathfrak{g}) \otimes Y_{\nu+\rho}(\mathfrak{g}) & \xrightarrow{\mathrm{Id}\otimes\Delta_{\nu,\rho}} & Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}) \otimes Y_\rho(\mathfrak{g}). \end{array} \tag{2.14}$$

§2.2. The ordinary Yangian

The ordinary Yangian $Y(\mathfrak{g})$ endowed with the coproduct $\Delta_{0,0} =: \Delta$ is a Hopf algebra, which contains the universal enveloping algebra $U(\mathfrak{g})$ as a Hopf subalgebra by identifying the $x_{i,0}^\pm$ with root vectors in the Lie algebra $\mathfrak{g}$ associated to the roots $\pm\alpha_i$ so that $\alpha_i^\vee = \frac{1}{d_i}\xi_{i,0} \in \mathfrak{h}$. Let R denote the set of positive roots of $\mathfrak{g}$. One can extend $x_{i,0}^\pm =: x_{\alpha_i}^\pm$ to root vectors $x_\gamma^\pm \in \mathfrak{g}_{\pm\gamma}$ for $\gamma \in R$ suitably normalized with respect to an invariant bilinear form of $\mathfrak{g}$. Then the coproduct is determined by [17] (see [33, §4.2] for a proof):

$$\Delta(\xi_{i,1}) = \xi_{i,1} \otimes 1 + 1 \otimes \xi_{i,1} + \xi_{i,0} \otimes \xi_{i,0} - \sum_{\gamma \in R} (\alpha_i, \gamma) x_\gamma^- \otimes x_\gamma^+. \tag{2.15}$$

For $(\gamma, n) \in R \times \mathbb{N}$, the root vector $x_\gamma^- \in \mathfrak{g}_{-\gamma}$ proportional to an iterated commutator of the $x_{i,0}^-$, we choose exactly one of the $x_{i,0}^-$ in the commutator and replace it with $x_{i,n}^-$. It depends on the choice of i and the position of $x_{i,0}^-$ in the commutator. We fix such a choice for all $(\gamma, n) \in R \times \mathbb{N}$, and let $x_{\gamma,n}^-$ denote the resulting element in $Y^<(\mathfrak{g})$, called a *PBW variable* as in [20, §3.12].

Remark 2.4. For μ a coweight let us identify the subalgebra $Y_\mu^<(\mathfrak{g})$ with $Y^<(\mathfrak{g})$ via (2.11) so that the PBW variables make sense for $Y_\mu^<(\mathfrak{g})$. By [20, §3.12], with respect to a total order on the set of PBW variables which is in natural bijection with $R \times \mathbb{N}$, the ordered monomials in the PBW variables form a basis of $Y_\mu^<(\mathfrak{g})$.

We shall need the current algebra $\mathfrak{g}[t]$; the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ with bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} \quad \text{for } x, y \in \mathfrak{g} \text{ and } m, n \in \mathbb{N}.$$

It is bigraded by $\mathbb{N} \times \mathbf{Q}$: the weight grading comes from the adjoint action of $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{g}[t]$; the $\mathbb{N}$-grading is defined by declaring $x \otimes t^m$, for $x \in \mathfrak{g}$ and $m \in \mathbb{N}$, to be of degree m. Its universal enveloping algebra $U(\mathfrak{g}[t])$ is bigraded by $\mathbb{N} \times \mathbf{Q}$.

The algebra $Y(\mathfrak{g})$ is $\mathbb{N}$-filtrated, by declaring $Y(\mathfrak{g})^{\leq n}$, for $n \in \mathbb{N}$, to be the linear subspace spanned by the monomials in the generators $x_{i,m}^{\pm}$, $\xi_{i,m}$ for which the sum of the indexes m is at most n. The $\mathbb{N}$-filtration on $Y(\mathfrak{g})$ is compatible with the weight grading, so that the associated grading $\mathrm{gr}_{\mathbb{N}} Y(\mathfrak{g})$ is an algebra bigraded by $\mathbb{N} \times \mathbf{Q}$. Here, by definition, $Y(\mathfrak{g})^{\leq -1} = \{0\}$ and

$$\mathrm{gr}_{\mathbb{N}} Y(\mathfrak{g}) := \bigoplus_{n \in \mathbb{N}} Y(\mathfrak{g})^{\leq n} / Y(\mathfrak{g})^{\leq n-1}.$$

We have an isomorphism of $(\mathbb{N}, \mathbf{Q})$-bigraded algebras

$$U(\mathfrak{g}[t]) \longrightarrow \mathrm{gr}_{\mathbb{N}} Y(\mathfrak{g}), \quad x_{\alpha_i}^{\pm} \otimes t^m \mapsto \overline{x_{i,m}^{\pm}}. \tag{2.16}$$

Here, $\overline{x_{i,m}^{\pm}}$ denote the images of $x_{i,m}^{\pm}$ under the projection

$$Y(\mathfrak{g})^{\leq m} \longrightarrow Y(\mathfrak{g})^{\leq m} / Y(\mathfrak{g})^{\leq m-1} \subset \mathrm{gr}_{\mathbb{N}} Y(\mathfrak{g}).$$

This isomorphism appeared in [54]. For a complete proof, see [22, Thm. B.2].

§2.3. First coproduct estimation

A compact formula for the coproduct of the Drinfeld generators is unknown beyond sl_2 in [20, §6.3]. Still, some partial information on the weight space projection of the coproduct is sufficient.

Let us define the *height function* $h\colon \mathbf{Q}_+ \to \mathbb{N}$ to be the additive function such that $h(\alpha_i) = 1$ for $i \in I$. In the following, when we write $h(\beta)$ or speak of the height of a weight β, it is understood that $\beta \in \mathbf{Q}_+$.

We shall need the notion of the *principal part*. For V a vector space, let $V[\![u, u^{-1}]\!]$ denote the space of formal power series with coefficients in V, and $V[\![u^{-1}]\!]$ its subspace of power series in u^{-1}. The principal part of a formal power series $f(u)$ in $V[\![u, u^{-1}]\!]$, denoted by $\langle f(u) \rangle_+$, is a power series in $V[\![u^{-1}]\!]$ defined by

$$\left\langle \sum_{p \in \mathbb{Z}} f_p u^{-p-1} \right\rangle_+ := \sum_{p \in \mathbb{N}} f_p u^{-p-1}.$$

This was denoted by $\underline{f(u)}$ in [48, Lem. 5.13]. As an example,

$$\left\langle \frac{g(u)}{u-a} \right\rangle_+ = \frac{g(a)}{u-a} \quad \text{for } g(u) \in \mathbb{C}[u] \text{ and } a \in \mathbb{C}.$$

As another example, the shift homomorphism of (2.10) acts on $x_i^{\pm}(u)$ as

$$x_i^+(u) \mapsto \langle u^{-\langle \zeta, \alpha_i \rangle} x_i^+(u) \rangle_+, \quad x_i^-(u) \mapsto \langle u^{-\langle \eta, \alpha_i \rangle} x_i^-(u) \rangle_+.$$

Multiplication endows $V[\![u,u^{-1}]\!]$ with a module structure over the polynomial algebra $\mathbb{C}[u]$. It is compatible with taking the principal part:

$$\langle g\langle hf\rangle_+\rangle_+ = \langle ghf\rangle_+ \quad \text{for } g,h\in\mathbb{C}[u] \text{ and } f\in V[\![u,u^{-1}]\!]. \tag{2.17}$$

Lemma 2.5. *For all coweights μ and ν, the coproduct $\Delta_{\mu,\nu}$ satisfies*

$$\Delta_{\mu,\nu}(x_i^+(u)) \equiv x_i^+(u)\otimes 1 + \langle \xi_i(u)\otimes x_i^+(u)\rangle_+ \operatorname{mod.} \sum_{h(\beta)>0} Y_\mu^-(\mathfrak{g})_{-\beta}\otimes Y_\nu^+(\mathfrak{g})_{\beta+\alpha_i},$$

$$\Delta_{\mu,\nu}(x_i^-(u)) \equiv 1\otimes x_i^-(u) + \langle x_i^-(u)\otimes \xi_i(u)\rangle_+ \operatorname{mod.} \sum_{h(\beta)>0} Y_\mu^-(\mathfrak{g})_{-\beta-\alpha_i}\otimes Y_\nu^+(\mathfrak{g})_{\beta},$$

$$\Delta_{\mu,\nu}(\xi_i(u)) \equiv \xi_i(u)\otimes \xi_i(u) \operatorname{mod.} \sum_{h(\beta)>0} Y_\mu^-(\mathfrak{g})_{-\beta}\otimes Y_\nu^+(\mathfrak{g})_{\beta},$$

$$\Delta_{\mu,\nu}(\xi_{i,-\langle\mu+\nu,\alpha_i\rangle}) = \xi_{i,-\langle\mu,\alpha_i\rangle}\otimes 1 + 1\otimes \xi_{i,-\langle\nu,\alpha_i\rangle}.$$

In the first three relations, the notation "mod. V" for $V\subset Y_\mu^-(\mathfrak{g})\otimes Y_\nu^+(\mathfrak{g})$ should be understood as "modulo $V(\!(u^{-1})\!)$".

Proof of Lemma 2.5. The case $\mu=\nu=0$ follows from [52, Lem. 1]; while the statement of [52] is weaker, its proof works in our situation; see also [14, Prop. 2.8] and [31, Prop. 2.9]. For arbitrary coweights we use the zigzag arguments as in the proof of [20, Thm. 4.12]. We shall treat only the first assertion, as the other three are parallel. Our goal is to prove the following relations, denoted by $P(n,\mu,\nu)$, for $n\in\mathbb{N}$:

$$\Delta_{\mu,\nu}(x_{i,n}^+) \equiv x_{i,n}^+\otimes 1 + \sum_{m\geq 0} \xi_{i,n-1-m}\otimes x_{i,m}^+ \operatorname{mod.} \sum_{h(\beta)>0} Y_\mu^-(\mathfrak{g})_{-\beta}\otimes Y_\nu^+(\mathfrak{g})_{\beta+\alpha_i}.$$

For ζ, η antidominant coweights, applying $\iota_{\mu,\zeta,0}\otimes\iota_{\nu,0,\eta}$ to $\Delta_{\mu,\nu}(x_{i,n}^+)$, from the commutative diagram (2.13) and injectivity of shift homomorphisms we deduce

$$P(n,\mu,\nu) \Longleftrightarrow P(n-\langle\zeta,\alpha_i\rangle, \mu+\zeta, \nu+\eta).$$

Let us choose ζ and η such that $\mu+\zeta$ and $\nu+\eta$ are antidominant.

If $n\geq -\langle\mu,\alpha_i\rangle$, then the above equivalence applied to $(n+\langle\mu,\alpha_i\rangle,0,0)$ and the antidominant coweights $\mu+\zeta,\nu+\eta$ gives

$$P(n+\langle\mu,\alpha_i\rangle,0,0) \Longleftrightarrow P(n-\langle\zeta,\alpha_i\rangle,\mu+\zeta,\nu+\eta).$$

So $P(n,\mu,\nu)\Leftrightarrow P(n+\langle\mu,\alpha_i\rangle,0,0)$, and the latter is true by [52].

If $n< -\langle\mu,\alpha_i\rangle$, then $n-\langle\zeta,\alpha_i\rangle < -\langle\mu+\zeta,\alpha_i\rangle$ and by equation (2.12),

$$\Delta_{\mu+\zeta,\nu+\eta}(x_{i,n-\langle\zeta,\alpha_i\rangle}^+) = x_{i,n-\langle\zeta,\alpha_i\rangle}^+\otimes 1.$$

In the commutative diagram (2.13) put $x^+_{i,n}$ at the top-left corner. Then the element at the bottom-right corner is $x^+_{i,n-\langle\zeta,\alpha_i\rangle}\otimes 1$. From the injectivity of the vertical maps we obtain that the element at the top-right corner is $x^+_{i,n}\otimes 1$, namely, $\Delta_{\mu,\nu}(x^+_{i,n}) = x^+_{i,n}\otimes 1$. For $m \geq 0$, we have $\xi_{i,n-m-1} = 0$ in $Y_\mu(\mathfrak{g})$ because

$$n-m-1 < -\langle\mu,\alpha_i\rangle - m - 1 \leq -\langle\mu,\alpha_i\rangle - 1.$$

So the summation $\sum_m$ in $P(n,\mu,\nu)$ vanishes. This proves $P(n,\mu,\nu)$, with the second summation $\sum_\beta$ being zero. □

The zigzag arguments will reappear in this paper at the level of representations.

§2.4. Truncated shifted Yangians

These are quotients of shifted Yangians appearing first in type A [9] as finite W-algebras, in the dominant case [50] as quantizations of slices in affine Grassmannians, and in the most general case [7] as quantized Coulomb branches. Their definition involves the notion of ℓ-weight.

To motivate equation (2.18) below, let μ be a coweight and V be a $Y_\mu(\mathfrak{g})$-module. Then the actions of the $\xi_{i,p}$ on V mutually commute. Suppose that $0 \neq v \in V$ is a common eigenvector with $e_{i,p}$ being the eigenvalue of $\xi_{i,p}$. We have

$$\xi_i(u)v = e_i(u)v, \quad e_i(u) := \sum_{p\in\mathbb{Z}} e_{i,p}u^{-p-1}.$$

From equation (2.5) we see that $e_i(u)$ is a Laurent series in u^{-1} whose leading term is fixed to be $u^{\langle\mu,\alpha_i\rangle}$. So the *coweight* μ can be recovered from the I-tuple of Laurent series $(e_i(u))_{i\in I}$. The actions of the $\xi_{i,-\langle\mu,\alpha_i\rangle}$ on v are encoded in the *weight* $\beta \in \mathfrak{h}^*$ defined below in the same way as equation (2.7):

$$\xi_{i,-\langle\mu,\alpha_i\rangle}v = (\alpha_i,\beta)v \quad \text{where } \beta := \sum_{i\in I} e_{i,-\langle\mu,\alpha_i\rangle}\frac{1}{d_i}\varpi_i.$$

In notation of Section 3.1, the vector v is of ℓ-weight $(e_i(u))_{i\in I}$ and weight β.

Consider the multiplicative group $\mathbb{C}((u^{-1}))^\times$ of the field $\mathbb{C}((u^{-1}))$ of Laurent series in u^{-1}. The set of ℓ-weights, denoted by $\mathcal{L}$, is the subset of the I-fold product group $\prod_{i\in I}\mathbb{C}((u^{-1}))^\times$ consisting of I-tuples of Laurent series in u^{-1} whose leading terms are of the form u^k for $k\in\mathbb{Z}$; it is clearly a subgroup. For $\mathbf{e}\in\mathcal{L}$ and $i\in I$, let $\mathbf{e}_i(u)$ be the ith component of $\mathbf{e}$, and let $\mathbf{e}_{i,p}\in\mathbb{C}$, for $p\in\mathbb{Z}$, be the coefficient of u^{-p-1} in $\mathbf{e}_i(u)$. By definition there exists a unique $k_i\in\mathbb{Z}$ such that

$$\mathbf{e}_i(u) = \sum_{p\in\mathbb{Z}} \mathbf{e}_{i,p}u^{-p-1} \quad \text{with } \mathbf{e}_{i,p} = 0 \text{ for } p < -k_i - 1 \text{ and } \mathbf{e}_{i,-k_i-1} = 1.$$

Define the *weight* and the *coweight* of $\mathbf{e}$ by

$$\varpi(\mathbf{e}) \coloneqq \sum_{i\in I} \frac{\mathbf{e}_{i,-k_i}}{d_i}\varpi_i \in \mathfrak{h}^*, \quad \varpi^\vee(\mathbf{e}) \coloneqq \sum_{i\in I} k_i\varpi_i^\vee \in \mathbf{P}^\vee. \tag{2.18}$$

This defines two morphisms of abelian groups $\varpi\colon \mathcal{L} \to \mathfrak{h}^*$ and $\varpi^\vee\colon \mathcal{L} \to \mathbf{P}^\vee$.

Definition 2.6. A pair $(\mu, \mathbf{r})$ of coweight μ and ℓ-weight $\mathbf{r}$ are *truncatable* if

$$\varpi^\vee(\mathbf{r}) - \mu \in \mathbf{Q}_+^\vee. \tag{2.19}$$

In this situation, let $m_i \in \mathbb{N}$, for $i \in I$, be the coefficient of $\alpha_i^\vee$ in $\varpi^\vee(\mathbf{r})-\mu$. We have the Gerasimov–Kharchev–Lebedev–Oblezin (GKLO for short) series $A_i(u)$, a $Y_\mu^=(\mathfrak{g})$-valued Laurent series in u^{-1} of leading term u^{m_i} for $i \in I$, uniquely determined by the following equations [32, Lem. 2.1 with $\iota\hbar = 1$] (see also [50, §4A] and [7, eq. (B.14)]):

$$\begin{aligned}\xi_i(u) &= \frac{\mathbf{r}_i(u)}{A_i(u)A_i(u-d_i)} \prod_{j:c_{ji}<0} \prod_{t=1}^{-c_{ji}} A_j(u-d_{ij}-td_j) \\ &= \frac{\mathbf{r}_i(u)}{A_i(u)A_i(u-d_i)} \prod_{j:c_{ji}=-1} A_j\Big(u-\frac{1}{2}d_j\Big) \prod_{j:c_{ji}=-2} A_j(u)A_j(u-1) \\ &\quad\times \prod_{j:c_{ji}=-3} A_j\Big(u+\frac{1}{2}\Big)A_j\Big(u-\frac{1}{2}\Big)A_j\Big(u-\frac{3}{2}\Big).\end{aligned} \tag{2.20}$$

The second equation comes from the fact that $c_{ji} = -1$ implies $d_{ij} = -\frac{1}{2}d_j$, while $c_{ji} < -1$ implies $d_j = 1$ and $d_{ij} = \frac{1}{2}c_{ji}$.

Lemma 2.7 ([32]). *Let $(\mu, \mathbf{r})$ be truncatable. In the shifted Yangian $Y_\mu(\mathfrak{g})$ we have*

$$A_i(u)x_{j,n}^- A_i(u)^{-1} = x_{j,n}^- + d_i\delta_{ij}\sum_{k\geq 0} x_{i,n+k}^- u^{-k-1}.$$

Proof. Write $A_i(u) = \sum_p a_{i,p}u^{-p-1}$. The following relation is a consequence of [32, eqs. (2.12),(2.14)]; see [49, Def. 4.1] in simply laced types:

$$[a_{i,p+1}, x_{j,n}^-] - [a_{i,p}, x_{j,n+1}^-] = \delta_{ij}d_i x_{i,n}^- a_{i,p}.$$

Since $a_{i,p} = 0$ for $p \ll 0$, the above relation can be rewritten as

$$[a_{i,p}, x_{j,n}^-] = d_i\delta_{ij}\sum_{k\geq 0} x_{i,n+k}^- a_{i,p-k-1}.$$

Multiplying the above equality by u^{-p-1} and summing over $p \in \mathbb{Z}$, we get

$$A_i(u)x_{j,n}^- - x_{j,n}^- A_i(u) = d_i\delta_{ij}\sum_{k\geq 0} x_{i,n+k}^- u^{-k-1}A_i(u).$$

Right multiplying by $A_i(u)^{-1}$ gives the desired identity from the lemma. □

Definition 2.8 ([50, 7]). For $(\mu, \mathbf{r}) \in \mathbf{P}^\vee \times \mathcal{L}$ a truncatable pair, the *truncated shifted Yangian* $Y_\mu^{\mathbf{r}}(\mathfrak{g})$ is the algebra defined as the quotient of the shifted Yangian $Y_\mu(\mathfrak{g})$ by the two-sided ideal generated by the coefficients of $\langle A_i(u)\rangle_+$ for $i \in I$.

Remark 2.9. Assume each $\mathbf{r}_i(u)$ is a monic polynomial of u. Our algebra $Y_\mu^{\mathbf{r}}(\mathfrak{g})$ and series $A_i(u)$ correspond to $\widetilde{Y}_\mu^\nu(\mathbf{r})$ and $u^{m_i}A_i(u)$ in [47, §3.4] with $\nu = \varpi^\vee(\mathbf{r})$. The original truncated shifted Yangian, denoted by $Y_\mu^\nu(\mathbf{r})$, is defined to be the image of $Y_\mu^{\mathbf{r}}(\mathfrak{g})$ under the so-called GKLO representation by difference operators; see [50, Thm. 4.5] and [7, Thm. B.15]. Conjecturally [50], the quotient map is an isomorphism, and a proof in type A is available in [46, Thm. 1.6] and [47, Thm. A.5]. The reason why we drop the polynomiality of $\mathbf{r}$ will be given in Section 8.

§3. Representations of shifted Yangians

We recall basic properties of representations of shifted Yangians: Verma modules, classification of irreducible modules in category $\mathscr{O}^{\mathrm{sh}}$, q-characters, finite-dimensional irreducible modules, and prefundamental modules.

Most of the definitions and results in this section are well known, and were also known for the representation theory of three classes of algebras: the ordinary Yangian [17, 14, 52, 28]; the upper Borel subalgebra of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ for $q \in \mathbb{C}^\times$ generic [41, 24, 39], which we refer to as a *Borel algebra*; the shifted quantum affine algebras recently developed in [21, 40]. Their proofs work for shifted Yangians as well, because the algebraic structures are common for these quantum groups.

§3.1. Verma modules

We begin with some general remarks on the notions of weights and ℓ-weights for modules over shifted Yangians. Fix μ a coweight. Let V be a module over $Y_\mu(\mathfrak{g})$. For $\beta \in \mathfrak{h}^*$ and $\mathbf{f} \in \mathcal{L}$, define

$$V_\beta := \{v \in V \mid \forall\, i \in I,\ \xi_{i,-\langle \mu,\alpha_i\rangle} v = (\alpha_i, \beta) v\},$$
$$V_{\mathbf{f}} := \{v \in V \mid \forall\, (i,p) \in I \times \mathbb{Z},\ \exists\, m \in \mathbb{N} \text{ such that } (\xi_{i,p} - \mathbf{f}_{i,p})^m v = 0\}.$$

If V_β is non-zero, then it is called a *weight space* of weight β, and a non-zero vector $v \in V_\beta$ is called a *weight vector* of weight $\mathrm{wt}(v) := \beta$. If $V_{\mathbf{f}}$ is non-zero, then it is an *ℓ-weight space* of ℓ-weight $\mathbf{f}$, and similar conventions for ℓ-weight vector and $\mathrm{wt}_\ell(v)$. Let $\mathrm{wt}(V) \subseteq \mathfrak{h}^*$ be the set of weights of V, and $\mathrm{wt}_\ell(V) \subseteq \mathcal{L}$ the set of ℓ-weights. By equations (2.5) and (2.7), for $\mathbf{f} \in \mathrm{wt}_\ell(V)$, $\beta \in \mathrm{wt}(V)$, and $\alpha \in \mathbf{Q}$, we have

$$\mu = \varpi^\vee(\mathbf{f}), \quad Y_\mu(\mathfrak{g})_\alpha V_\beta \subseteq V_{\alpha+\beta}.$$

Remark 3.1. While the automorphism τ_a from (2.8) preserves the weight grading on $Y_\mu(\mathfrak{g})$, this is not the case for modules. Define the weight $\tilde{\mu} \in \mathfrak{h}^*$ associated to μ and, by abuse of language, the group automorphism $\tau_a \colon \mathcal{L} \to \mathcal{L}$ by

$$\tilde{\mu} := \sum_{i\in I} \langle \mu, \alpha_i \rangle \frac{1}{d_i} \varpi_i, \quad \tau_a(\mathbf{f}) := (\mathbf{f}_i(u-a))_{i\in I} \text{ for } \mathbf{f} \in \mathcal{L}.$$

For V a $Y_\mu(\mathfrak{g})$-module, the pullback module $\tau_a^* V$ is denoted by $V(a)$, with a referred to as the *spectral parameter*. We have

$$V_\beta = V(a)_{\beta - a\tilde{\mu}} \text{ for } \beta \in \mathrm{wt}(V), \quad V_{\mathbf{f}} = V(a)_{\tau_a(\mathbf{f})} \text{ for } \mathbf{f} \in \mathrm{wt}_\ell(V).$$

Call V a *weight module* if it is a direct sum of weight spaces. In such a module, any ℓ-weight space $V_{\mathbf{f}}$ is contained in the weight space $V_{\varpi(\mathbf{f})}$. We shall say that V is weight graded by a subset $X \subset \mathfrak{h}^*$ if V is a weight module and $\mathrm{wt}(V) \subset X$.

Call V *top graded* if there exists $\lambda \in \mathfrak{h}^*$ such that V is weight graded by $\lambda + \mathbf{Q}_-$ and V_λ is one-dimensional. Clearly λ is unique and V_λ equals an ℓ-weight space $V_{\mathbf{e}}$ for a unique $\mathbf{e} \in \mathcal{L}$. We refer to λ, V_λ, $\mathbf{e}$, and $V_{\mathbf{e}}$ as the *top weight*, *top weight space*, *top ℓ-weight*, and *top ℓ-weight space*.

Let $\mathbf{e} \in \mathcal{L}$ be of coweight μ. The *Verma module* $M(\mathbf{e})$ is the $Y_\mu(\mathfrak{g})$-module defined by parabolic induction [48, §3.3]

$$M(\mathbf{e}) := Y_\mu(\mathfrak{g}) \otimes_{Y_\mu^+(\mathfrak{g})} \mathbb{C}.$$

Here $\mathbb{C} = \mathbb{C}1$ is viewed as a $Y_\mu^+(\mathfrak{g})$ by setting $x_i^+(u)1 = 0$ and $\xi_i(u)1 = \mathbf{e}_i(u)1$. The vector $\omega_{\mathbf{e}} := 1 \otimes 1 \in M(\mathbf{e})$ is of weight $\varpi(\mathbf{e})$ and ℓ-weight $\mathbf{e}$. From the triangular decomposition of Theorem 2.2 and the weight grading on $Y_\mu^<(\mathfrak{g})$ we obtain that $M(\mathbf{e})$ is top graded with $\mathbf{e}$ being the top ℓ-weight. Moreover, the linear map $Y_\mu^<(\mathfrak{g}) \to M(\mathbf{e})$ sending $x \in Y_\mu^<(\mathfrak{g})$ to $x\omega_{\mathbf{e}}$ is bijective.

By standard argument, the Verma module has a unique maximal submodule, the quotient by which is irreducible and denoted by $L(\mathbf{e})$. By abuse of language, we still let $\omega_{\mathbf{e}} \in L(\mathbf{e})$ denote the image of $\omega_{\mathbf{e}} \in M(\mathbf{e})$ under the quotient.

Let V be a $Y_\mu(\mathfrak{g})$-module and let v be a non-zero vector of V. Call v a *vector of highest ℓ-weight* $\mathbf{e}$ if there exists a module morphism $M(\mathbf{e}) \to V$ sending $\omega_{\mathbf{e}}$ to v. Namely, $\xi_i(u)v = \mathbf{e}_i(u)v$ and $x_i^+(u)v = 0$ for $i \in I$.

Definition 3.2. Call V a *module of highest ℓ-weight* $\mathbf{e}$ if there exists a non-zero surjective module morphism $M(\mathbf{e}) \to V$.

Equivalently, V is generated by a vector v of highest ℓ-weight $\mathbf{e}$. It follows that V is top graded with $\mathbf{e}$ being the top ℓ-weight. In particular, v is unique up to homothety, and there is a unique surjective module morphism $V \to L(\mathbf{e})$ sending v to $\omega_{\mathbf{e}}$.

Recall the coproduct for μ, ν coweights,

$$\Delta_{\mu,\nu}\colon Y_{\mu+\nu}(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}),$$

from Theorem 2.3. If W and V are modules over $Y_\mu(\mathfrak{g})$ and $Y_\nu(\mathfrak{g})$ respectively, then their tensor product $W \otimes V$ is naturally a module over $Y_{\mu+\nu}(\mathfrak{g})$. Since the coproduct respects the weight grading, we have

$$W_\alpha \otimes V_\beta \subset (W \otimes V)_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathfrak{h}^*.$$

So, a tensor product of weight modules is still weight graded.

Example 3.3. Let $\mathbf{e}, \mathbf{f} \in \mathcal{L}$. Consider the tensor product module $M(\mathbf{e}) \otimes M(\mathbf{f})$. From the coproduct of $\xi_i(u)$ and $x_i^+(u)$ in Lemma 2.5 we see that $\omega_{\mathbf{e}} \otimes \omega_{\mathbf{f}}$ is of highest ℓ-weight $\mathbf{ef}$. This implies that $L(\mathbf{ef})$ is a subquotient of $L(\mathbf{e}) \otimes L(\mathbf{f})$.

Lowest ℓ-weight vectors/modules can be defined by replacing $x_i^+(u)$ with $x_i^-(u)$.

Example 3.4. Let V be a $Y_\nu(\mathfrak{g})$-module containing a lowest ℓ-weight vector v_- and let W be a $Y_\mu(\mathfrak{g})$-module containing a highest ℓ-weight vector ω. Then for $j \in I$, $v \in V$, and $w \in W$, we have the following relations in the module $V \otimes W$ based on the coproduct estimation of Lemma 2.5:

$$\xi_j(u)(v_- \otimes w) = \xi_j(u)v_- \otimes \xi_j(u)w, \quad x_j^-(u)(v_- \otimes w) = v_- \otimes x_j^-(u)w,$$
$$x_j^-(u)(v \otimes \omega) = v \otimes x_j^-(u)\omega + \langle x_j^-(u)v \otimes \xi_j(u)\omega \rangle_+.$$

In particular, if w is an ℓ-weight vector, then so is $v_- \otimes w$.

§3.2. Finite-dimensional irreducible modules

In this subsection we recall the classification of finite-dimensional irreducible modules over shifted Yangians from [48, Thm. 3.5]. The result was proved in simply laced types by reduction to sl_2 and applying [9, §7.2], so it works in general types. See [40, Thm. 6.4] for a similar classification for shifted quantum affine algebras.

Example 3.5 ([70, Rem. 24]). Let $(i, a) \in I \times \mathbb{C}$. The *positive prefundamental module* $L_{i,a}^+$ is the one-dimensional $Y_{\varpi_i^\vee}(\mathfrak{g})$-module of highest ℓ-weight (our sign convention is opposite to [70, eq. (10)] and agrees with [41, Def. 3.7])

$$\Psi_{i,a} := (\underbrace{1, \ldots, 1}_{i-1}, u - a, \underbrace{1, \ldots, 1}_{r-i}) \quad \text{prefundamental weight.} \tag{3.1}$$

Our terminology follows [24, Def. 3.4]. In the framework of representations of the Borel algebra [41], the positive prefundamental module is an infinite-dimensional irreducible module whose ℓ-weights are rather simple, and it has important applications in quantum integrable systems (construction of Baxter's Q-operators [2] as transfer matrices of this modules, polynomiality of Q-operators). In another framework of representations of shifted quantum affine algebras, which is closer to our situation, the positive prefundamental modules are one-dimensional [40, Exa. 4.12].

For $(i, a) \in I \times \mathbb{C}$ define the *fundamental ℓ-weight* by

$$Y_{i,a} \coloneqq \frac{\Psi_{i,a-\frac{1}{2}d_i}}{\Psi_{i,a+\frac{1}{2}d_i}} \in \mathcal{L}. \tag{3.2}$$

In the notation of [14, §2.13], $L(Y_{i,a})$ is the finite-dimensional irreducible module over $Y(\mathfrak{g})$ with Drinfeld polynomials $P_i^+(u) = u - a - \frac{1}{2}d_i$ and $P_j^+(u) = 1$ for $j \neq i$. This justifies its name *fundamental module.*

Theorem 3.6 ([9, 48]). *For $\mathbf{e} \in \mathcal{L}$, the irreducible module $L(\mathbf{e})$ is finite-dimensional if and only if $\mathbf{e}$ is a monomial of the $\Psi_{i,a}$ and $Y_{i,a}$ for $i \in I$ and $a \in \mathbb{C}$. Furthermore, all finite-dimensional irreducible modules over shifted Yangians arise in this way.*

Example 3.7. Fix $(i, a) \in I \times \mathbb{C}$. Let $N_{i,a}$ be the irreducible module of highest ℓ-weight

$$Y_{i,a-\frac{1}{2}d_i} \prod_{j: c_{ij}<0} \Psi_{j,a-d_{ij}}.$$

It is realized on the vector space $\mathbb{C}^2$ with basis (e_1, e_2) such that the only non-zero actions of the generating series on the basis are

$$\xi_j(u)e_1 = e_1 \begin{cases} \frac{u-a+d_{ij}}{u-a} & \text{if } c_{ij} \geq 0, \\ u-a+d_{ij} & \text{if } c_{ij} < 0, \end{cases} \qquad \xi_j(u)e_2 = e_2 \begin{cases} \frac{u-a-d_{ij}}{u-a} & \text{if } c_{ij} \geq 0, \\ u-a-d_{ij} & \text{if } c_{ij} < 0, \end{cases}$$

$$x_i^+(u)e_2 = \frac{1}{u-a} e_1, \quad x_i^-(u)e_1 = \frac{d_i}{u-a} e_2.$$

Over the Borel algebra there is an infinite-dimensional irreducible module of similar highest ℓ-weight [43, §6.1.3], denoted by $N_{i,a}^+$ in [18, eq. (6.2)], which gives rise to cluster mutations [42, 43] and three-term Baxter TQ relations for transfer matrices [18, Prop. 6.8]. Over shifted quantum affine algebras, the irreducible module is two-dimensional [40, Exa. 6.6].

The ratio of the ℓ-weights of $N_{i,a}$ is a *generalized simple root*:

$$A_{i,a} \coloneqq \prod_{j \in I} \frac{\Psi_{j,a-d_{ij}}}{\Psi_{j,a+d_{ij}}} \in \mathcal{L}. \tag{3.3}$$

Notice that the $A_{i,a}$ for $(i,a) \in I \times \mathbb{C}$ generate a free abelian subgroup of $\mathcal{L}$. Originally, generalized simple roots were defined in [26, eqs. (3.11), (4.8)] as certain evaluations of the universal R-matrix of $U_q(\hat{\mathfrak{g}})$, and they were linked to ℓ-weights therein. Similar formulas hold [40, §5.5, Thm. 6.1] for shifted quantum affine algebras.

A finite-dimensional irreducible $Y(\mathfrak{g})$-module is necessarily weight graded, as an integrable $\mathfrak{g}$-module, and it is both of highest ℓ-weight and of lowest ℓ-weight.

Theorem 3.8 ([64, 63, 34]). *Let U and V be finite-dimensional irreducible $Y(\mathfrak{g})$-modules generated by highest ℓ-weight vectors ω_1 and ω_2 respectively. Let $a, b \in \mathbb{C}$.*

(i) *There exist a tensor product of fundamental modules T and an injective morphism from V to T whose image contains a tensor product of highest ℓ-weight vectors as well as a tensor product of lowest ℓ-weight vectors.*

(ii) *There exists a finite subset X of $\mathbb{C}$ such that the module $U(a) \otimes V(b)$ is irreducible if $a - b \notin X$.*

(iii) *The assignment $\omega_1 \otimes \omega_2 \mapsto \omega_2 \otimes \omega_1$ extends uniquely to a linear map*

$$\check{R}_{U,V}(u) \colon U \otimes V \longrightarrow V \otimes U \otimes \mathbb{C}(u)$$

such that the evaluation at $u = a - b$ of the vector-valued rational function is a module morphism from $U(a) \otimes V(b)$ to $V(b) \otimes U(a)$, if $a - b$ is not a pole.

We refer to [34, Thm. 3.10] for a proof of the theorem and for a discussion of relevant results for the quantum affine algebra. Part (i) is a weaker version of the main results of [64, 63]: such a tensor product can be chosen to have a unique irreducible submodule (of cohighest ℓ-weight in the sense of Definition 4.4). The vector-valued rational function $\check{R}_{U,V}(u)$ in part (iii) is called a *normalized R-matrix.* It is rarely polynomial, contrary to our R-matrices constructed later in Section 5.

As in [31, §3.5], set $\kappa := \frac{1}{2}\max(d_i \colon i \in I)h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. One has the involution $i \mapsto \bar{i}$ of the set I of Dynkin nodes of $\mathfrak{g}$ induced by $w_0(\alpha_i) = -\alpha_{\bar{i}}$, where w_0 is the longest element of the Weyl group of $\mathfrak{g}$. Define

$$V_i := L(Y_{\bar{i}, \frac{1}{2}d_i - \kappa}) \quad \text{for } i \in I. \tag{3.4}$$

Lemma 3.9 ([14, Prop. 3.2]). *For $i \in I$, the lowest ℓ-weight of V_i is $Y^{-1}_{i,\frac{1}{2}d_i}$.*

The ordinary Yangian $Y(\mathfrak{g})$ is a Hopf algebra with antipode S. For V a $Y(\mathfrak{g})$-module, its *Hopf dual* is the $Y(\mathfrak{g})$-module structure on the linear dual V^* defined by

$$(af)(v) = f(S(a)v) \quad \text{for } a \in Y(\mathfrak{g}),\ f \in V^* \text{ and } v \in V.$$

By the coproduct estimation of Lemma 2.5, the dual $L(Y_{i,a})^*$ of a fundamental module $L(Y_{i,a})$ is of lowest ℓ-weight $Y_{i,a}^{-1}$. The above lemma implies that $L(Y_{i,a})^*$ is the fundamental module $L(Y_{\bar{\imath},a-\kappa})$; see [25, Cor. 6.10] for similar arguments.

§3.3. Category $\mathcal{O}^{\mathrm{sh}}$ and rationality

In this subsection we study a category of representations of shifted Yangians, which appeared in [49, §5] in simply laced types.

For μ a coweight, define $\mathcal{O}_\mu$ to be the full subcategory of the category of $Y_\mu(\mathfrak{g})$-modules. An object of $\mathcal{O}_\mu$ is a $Y_\mu(\mathfrak{g})$-module V such that

(O1) it is a direct sum of finite-dimensional weight spaces;

(O2) there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathfrak{h}^*$ such that

$$\mathrm{wt}(V) \subseteq \bigcup_{j=1}^{n} (\lambda_j + \mathbf{Q}_-).$$

Remark 3.10. Assume μ is dominant. The quotient of $Y_\mu(\mathfrak{g})$ by the ideal generated by the $[x, y]$ for $x, y \in Y_\mu(\mathfrak{g})$ is isomorphic to the polynomial algebra in finitely many variables $\xi_{i,p}$ for $i \in I$ and $-\langle \mu, \alpha_i \rangle \leq p < 0$. A finite-dimensional module over the quotient algebra is in category $\mathcal{O}_\mu$ if and only if it is semisimple. If $\mu \neq 0$, then the quotient algebra, as a polynomial algebra in at least one variable, admits finite-dimensional modules which are non-semisimple and therefore do not belong to category $\mathcal{O}_\mu$; see [9, §5.1] for similar arguments. If $\mu = 0$, then a finite-dimensional $Y(\mathfrak{g})$-module is necessarily in category $\mathcal{O}_0$ viewed as an integrable $\mathfrak{g}$-module.

The category $\mathcal{O}_\mu$ is abelian. Let us describe its irreducible objects. The following rationality is well known for quantum affine algebras [36] and Yangians [28].

Lemma 3.11. *Let V be a $Y_\mu(\mathfrak{g})$-module which is a direct sum of finite-dimensional weight spaces. The generating series $x_i^\pm(u)$ and $\xi_i(u)$ restricted to each weight space of V are rational in the sense that they are expansions at ∞ of rational functions of u with values in finite-dimensional vector spaces.*

Proof. The rationality of the Laurent series $\xi_i(u)$ and $x_i^\pm(u)$ is proved in the same way as [36, Prop. 3.8], [28, Prop. 3.6(i)]: first, one shows explicitly the rationality of the $x_i^\pm(u)$, which implies that of $\langle \xi_i(u) \rangle_+$; then $\xi_i(u)$ is $\langle \xi_i(u) \rangle_+$ plus a polynomial of u. □

Define $\mathcal{R}$ to be the subgroup of $\mathcal{L}$ generated by the $\Psi_{i,a}$. An element $\mathbf{e} \in \mathcal{L}$ belongs to $\mathcal{R}$ if and only if all the components $\mathbf{e}_i(u)$ are ratios of monic polynomials of u. Let $\mathcal{R}_\mu$ be the set of $\mathbf{e} \in \mathcal{R}$ of coweight μ.

Theorem 3.12. *For μ a coweight, the $L(\mathbf{e})$ for $\mathbf{e} \in \mathcal{R}_\mu$ form the set of mutually non-isomorphic irreducible modules in category $\mathcal{O}_\mu$.*

Proof. Standard arguments based on the triangular decomposition and rationality of Lemma 3.11 show that any irreducible module in category $\mathcal{O}_\mu$ is of the form $L(\mathbf{e})$ for $\mathbf{e} \in \mathcal{R}_\mu$. It suffices to prove that $L(\mathbf{e})$ is in category $\mathcal{O}_\mu$ for $\mathbf{e} \in \mathcal{R}_\mu$. Repeatedly using equation (2.3) as in [15, §5, Proof of (b)], we are reduced to showing that for fixed $i \in I$ the vectors $x_{i,n}^- \omega_{\mathbf{e}}$ with $n \in \mathbb{N}$ span a finite-dimensional subspace of $L(\mathbf{e})$. Write

$$\mathbf{e}_i(u) = \frac{P(u)}{Q(u)} \quad \text{with } P(u) \text{ and } Q(u) \text{ monic polynomials.}$$

It suffices to prove the recurrence relation $\langle Q(u) x_i^-(u) \omega_{\mathbf{e}} \rangle_+ = 0$. Indeed,

$$\begin{aligned} x_{j,m}^+ x_i^-(u) &= x_i^-(u) x_{j,m}^+ + \delta_{ij} \sum_{n \geq 0} \xi_{i,m+n} u^{-n-1} \\ &= x_i^-(u) x_{j,m}^+ + \delta_{ij} \langle u^m \xi_i(u) \rangle_+, \\ x_{j,m}^+ \langle Q(u) x_i^-(u) \omega_{\mathbf{e}} \rangle_+ &= \langle Q(u) x_{j,m}^+ x_i^-(u) \omega_{\mathbf{e}} \rangle_+ = \delta_{ij} \langle Q(u) \langle u^m \xi_i(u) \omega_{\mathbf{e}} \rangle_+ \rangle_+ \\ &= \delta_{ij} \langle u^m Q(u) \xi_i(u) \omega_{\mathbf{e}} \rangle_+ \\ &= \delta_{ij} \left\langle u^m Q(u) \frac{P(u)}{Q(u)} \omega_{\mathbf{e}} \right\rangle_+ = 0. \end{aligned}$$

The power series $\langle Q(u) x_i^-(u) \omega_{\mathbf{e}} \rangle_+$ is annihilated by all the $x_{j,m}^+$. If it is non-zero, then by applying the triangular decomposition to its coefficients we obtain a non-zero submodule of $L(\mathbf{e})$ weight graded by $\varpi(\mathbf{e}) - \alpha_i + \mathbf{Q}_-$, contradicting the irreducibility of $L(\mathbf{e})$. ☐

We define the *completed Grothendieck group* $K_0(\mathcal{O}_\mu)$ as in [43, §3.2]: its elements are formal sums $\sum_{\mathbf{e} \in \mathcal{R}_\mu} n_{\mathbf{e}} [L(\mathbf{e})]$ of the symbols $[L(\mathbf{e})]$, for $\mathbf{e} \in \mathcal{R}_\mu$ and $n_{\mathbf{e}} \in \mathbb{Z}$, such that the direct sum of $Y_\mu(\mathfrak{g})$-modules $\bigoplus_{\mathbf{e} \in \mathcal{R}_\mu} L(\mathbf{e})^{\oplus |n_{\mathbf{e}}|}$ is in category $\mathcal{O}_\mu$; addition is the usual one of formal sums. Let V be in category $\mathcal{O}_\mu$. As in the case of Kac–Moody algebras [45, §9.3], for $\mathbf{e} \in \mathcal{R}_\mu$ the multiplicity $m_{L(\mathbf{e}),V} \in \mathbb{N}$ of the irreducible module $L(\mathbf{e})$ in V makes sense, and we get a well-defined isomorphism class of V,

$$[V] \coloneqq \sum_{\mathbf{e} \in \mathcal{R}_\mu} m_{L(\mathbf{e}),V} [L(\mathbf{e})] \in K_0(\mathcal{O}_\mu).$$

The coproduct $\Delta_{\mu,\nu}$ of Theorem 2.3 induces a functor

$$\mathcal{O}_\mu \times \mathcal{O}_\nu \longrightarrow \mathcal{O}_{\mu+\nu}, \quad (W, V) \mapsto W \otimes V.$$

Define the direct sum of abelian categories and its Grothendieck group

$$\mathcal{O}^{\mathrm{sh}} := \bigoplus_{\mu \in \mathbf{P}^\vee} \mathcal{O}_\mu, \quad K_0(\mathcal{O}^{\mathrm{sh}}) := \bigoplus_{\mu \in \mathbf{P}^\vee} K_0(\mathcal{O}_\mu).$$

Then the above functor extends to a tensor product functor

$$\otimes \colon \mathcal{O}^{\mathrm{sh}} \times \mathcal{O}^{\mathrm{sh}} \longrightarrow \mathcal{O}^{\mathrm{sh}}.$$

The exactness of the tensor product induces a group homomorphism

$$K_0(\mathcal{O}^{\mathrm{sh}}) \times K_0(\mathcal{O}^{\mathrm{sh}}) \longrightarrow K_0(\mathcal{O}^{\mathrm{sh}}), \quad ([W], [V]) \mapsto [W \otimes V].$$

Remark 3.13. Let $\mathcal{O}^{\mathrm{sh}}_-$ denote the direct sum of the categories $\mathcal{O}_\mu$ for μ anti-dominant. Then the commutative diagram (2.14) implies that $(\mathcal{O}^{\mathrm{sh}}_-, \otimes)$ is a monoidal category with trivial associators. It is unclear to us whether category $(\mathcal{O}^{\mathrm{sh}}, \otimes)$ is monoidal because the coproducts fail to be coassociative for general coweights [20, Rem. 4.15].

If V is in category $\mathcal{O}_\mu$, then each weight space V_β is a direct sum of ℓ-weight spaces and each ℓ-weight belongs to $\mathcal{R}_\mu$ by Lemma 3.11. Following Knight [52], we define the *q-character* of V to be (we adopt the terminology of [26])

$$\chi_{\mathrm{q}}(V) := \sum_{\mathbf{f} \in \mathrm{wt}_\ell(V)} \dim(V_{\mathbf{f}}) \mathbf{f} \in \mathcal{E}_\ell.$$

The target $\mathcal{E}_\ell$ is the set of formal sums $\sum_{\mathbf{f} \in \mathcal{R}} n_{\mathbf{f}} \mathbf{f}$ of $\mathbf{f} \in \mathcal{R}$ with integer coefficients $n_{\mathbf{f}}$ subject to the following conditions [41, §3.4]:

(E1) for each $\beta \in \mathfrak{h}^*$ the set $\{\mathbf{f} \in \mathcal{R} \mid n_{\mathbf{f}} \neq 0, \ \varpi(\mathbf{f}) = \beta\}$ is finite;

(E2) there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathfrak{h}^*$ such that

$$\varpi(\mathbf{f}) \in \bigcup_{j=1}^m (\lambda_j + \mathbf{Q}_-) \quad \text{if } n_{\mathbf{f}} \neq 0.$$

It is a ring: addition is the usual one of formal sums; multiplication is induced by that of $\mathcal{R}$. One views $\mathcal{E}_\ell$ as a completion of the group ring $\mathbb{Z}[\mathcal{R}]$.

Since χ_{q} respects exact sequences, the assignment $[V] \mapsto \chi_{\mathrm{q}}(V)$ extends uniquely to a group homomorphism

$$\chi_{\mathrm{q}} \colon K_0(\mathcal{O}^{\mathrm{sh}}) \longrightarrow \mathcal{E}_\ell$$

called the q-character map. The next result is proved in the same way as [52, Thm. 2] and [26, Prop. 1], based on the coproduct estimation of Lemma 2.5.

Theorem 3.14 ([52, 26]). *The q-character map is an injective group homomorphism. Furthermore, for W and V in category $\mathcal{O}^{\mathrm{sh}}$, we have*

$$\chi_{\mathrm{q}}(W \otimes V) = \chi_{\mathrm{q}}(W)\chi_{\mathrm{q}}(V).$$

As an important consequence, the Grothendieck group $K_0(\mathcal{O}^{\mathrm{sh}})$ endowed with the multiplication is a commutative ring: the associativity follows from that of the target ring $\mathcal{E}_\ell$, and so does the commutativity as in [41, Rem. 3.13]. In the case of shifted quantum affine algebras, the ring structure of the Grothendieck group is given by the fusion product of highest ℓ-weight modules [40, Thm. 5.4].

For V a top-graded module in category $\mathcal{O}^{\mathrm{sh}}$, we define its *normalized q-character* by

$$\widetilde{\chi_{\mathrm{q}}}(V) \coloneqq \chi_{\mathrm{q}}(V) \times \mathbf{e}^{-1} \in \mathcal{E}_\ell,$$

where $\mathbf{e}$ is the top ℓ-weight of V. In Example 3.7 we have $\widetilde{\chi_{\mathrm{q}}}(N_{i,a}) = 1 + A_{i,a}^{-1}$. A tensor product of top-graded modules is still top graded, and the normalized q-characters are multiplicative with respect to tensor product as in Theorem 3.14.

We shall also need the notion of character, which is defined in a standard way. Let $\mathcal{E}$ denote the set of formal sums $\sum_{\lambda\in\mathfrak{h}^*} n_\lambda e^\lambda$ of the symbols e^λ with integer coefficients n_λ under the following condition: there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathfrak{h}^*$ such that $n_\lambda \neq 0$ implies $\lambda \in \bigcup_{j=1}^m (\lambda_j + \mathbf{Q}_-)$. This is again a ring: addition is the usual one of formal sums; multiplication is induced by $e^\lambda e^\mu = e^{\lambda+\mu}$ for $\lambda, \mu \in \mathfrak{h}^*$. In particular, the weight map $\varpi\colon \mathcal{L} \to \mathfrak{h}^*$ induces a ring morphism

$$\varpi\colon \mathcal{E}_\ell \longrightarrow \mathcal{E}, \quad \sum_{\mathbf{f}\in\mathcal{R}} n_{\mathbf{f}}\mathbf{f} \mapsto \sum_{\mathbf{f}\in\mathcal{R}} n_{\mathbf{f}} e^{\varpi(\mathbf{f})}.$$

The character of a module V in category $\mathcal{O}^{\mathrm{sh}}$ is defined as

$$\chi(V) \coloneqq \varpi(\chi_{\mathrm{q}}(V)) = \sum_{\lambda\in\mathrm{wt}(V)} \dim(V_\lambda)e^\lambda \in \mathcal{E}.$$

In Example 3.5, we have $\chi(L_{i,a}^+) = e^{-ad_i^{-1}\varpi_i}$ by equation (2.18).

For $(i,a,k) \in I \times \mathbb{C} \times \mathbb{N}$, the Kirillov–Reshetikhin (KR for short) module $W_{k,a}^{(i)}$ is the finite-dimensional irreducible $Y(\mathfrak{g})$-module of highest ℓ-weight

$$\frac{\Psi_{i,a-kd_i}}{\Psi_{i,a}} = Y_{i,a-\frac{1}{2}d_i} Y_{i,a-\frac{3}{2}d_i} \cdots Y_{i,a-\frac{2k-1}{2}d_i}.$$

Following [24, Def. 3.4], define the negative prefundamental module $L_{i,a}^-$ to be $L(\Psi_{i,a}^{-1})$ in category $\mathcal{O}_{-\varpi_i^\vee}$ for $(i,a) \in I \times \mathbb{C}$. As in the case of the Borel algebra [41], it can be realized as a limit of KR modules [70].

Proposition 3.15 ([70]). *Fix $(i,a) \in I \times \mathbb{C}$. As the integer $k \in \mathbb{N}$ tends to infinity, the normalized q-character of the KR module $W_{k,a}^{(i)}$ converges to the normalized q-character of $L_{i,a}^-$ as a power series in $\mathbb{N}[\![A_{j,b}^{-1}]\!]_{j \in I, b \in \mathbb{C}}$.*

Proof. In [70, Prop. 23] we have constructed a module L in category $\mathcal{O}_{-\varpi_i^\vee}$ with q-character $\Psi_{i,a}^{-1} \lim_{k\to\infty} \widetilde{\chi_q}(W_{k,a}^{(i)})$. In particular, $\Psi_{i,a}^{-1}$ is a highest ℓ-weight of L and $L_{i,a}^-$ is an irreducible subquotient of L. It suffices to show that $\Psi_{i,a}\chi_q(L)$ is bounded above by $\widetilde{\chi_q}(L_{i,a}^-)$, so that $L_{i,a}^- \cong L$. Since the former is the limit of $\widetilde{\chi_q}(W_{k,a}^{(i)})$, we are led to prove that $\widetilde{\chi_q}(W_{k,a}^{(i)})$ is bounded above by $\widetilde{\chi_q}(L_{i,a}^-)$ for $k \in \mathbb{N}$. This follows by viewing $W_{k,a}^{(i)}$ as an irreducible subquotient of $L_{i,a-kd_i}^+ \otimes L_{i,a}^-$ and taking normalized q-characters. (Since $L_{i,a-kd_i}^+$ is one-dimensional, its normalized q-character is 1.) □

The character of a negative prefundamental module has a fermionic form [41, Thm. 6.4]. We shall need its product form, conjectured in [57] and partly proved recently in [53]. While [53] is about KR-modules over $U_q(\hat{\mathfrak{g}})$, its main result holds true in the Yangian case by the functor of [28] relating finite-dimensional modules over $U_q(\hat{\mathfrak{g}})$ and $Y(\mathfrak{g})$. Recall that R is the set of positive roots of $\mathfrak{g}$. For γ a positive root and for $i \in I$, by definition $\langle \varpi_i^\vee, \gamma \rangle$ is the coefficient of α_i in γ.

Theorem 3.16 ([53]). *Assume $\mathfrak{g}$ is not of type E_8. For $(i,a) \in I \times \mathbb{C}$ we have*

$$\chi(L_{i,a}^-) = e^{ad_i^{-1}\varpi_i} \prod_{\gamma \in R} \Big(\frac{1}{1-e^{-\gamma}}\Big)^{\langle \varpi_i^\vee, \gamma\rangle}.$$

§3.4. Examples in the sl_2-case

For the simple Lie algebra sl_2, we omit the Dynkin node 1 everywhere: $x_n^\pm = x_{1,n}^\pm$ and $\xi_p = \xi_{1,p}$ as generators; $N_a = N_{1,a}$ and $L_a^\pm = L_{1,a}^\pm$ as modules: $\Psi_a = u - a$ and $A_a = \frac{u-a+1}{u-a-1}$ as ℓ-weights. We identify the coweight lattice with $\mathbb{Z}$, so that 1 is the fundamental coweight and 2 is the simple coroot. Similarly, the set of weights is $\mathbb{C}$, so that 1 is the fundamental weight and 2 the simple root.

Example 3.17 ([13, Prop. 2.6]). Let $a, b \in \mathbb{C}$. On the vector space with basis $(v_i)_{i\in\mathbb{N}}$ there is a $Y(\mathrm{sl}_2)$-module structure, denoted by $\mathscr{L}_b^a$:

$$x^+(u)v_i = \frac{1}{u-b+i-1}v_{i-1}, \quad x^-(u)v_i = \frac{(b-a-i)(i+1)}{u-b+i}v_{i+1},$$
$$\xi(u)v_i = \frac{(u-b-1)(u-a)}{(u-b+i-1)(u-b+i)}v_i.$$

Its normalized q-character is

$$\widetilde{\chi_q}(\mathscr{L}_b^a) = 1 + A_b^{-1} + A_b^{-1}A_{b-1}^{-1} + A_b^{-1}A_{b-1}^{-1}A_{b-2}^{-1} + \cdots.$$

The vector v_0 generates an irreducible submodule, denoted by L_b^a, of highest ℓ-weight $\frac{u-a}{u-b}$. We have $\mathscr{L}_b^a = L_b^a$ if and only if $b-a \notin \mathbb{N}$. When $b-a \in \mathbb{N}$,

$$\widetilde{\chi_{\mathrm{q}}}(L_b^a) = 1 + A_b^{-1} + A_b^{-1}A_{b-1}^{-1} + A_b^{-1}A_{b-1}^{-1}A_{b-2}^{-1} + \cdots + A_b^{-1}A_{b-1}^{-1}\cdots A_{a+1}^{-1}.$$

Let us define $\Delta_b^a := \{k \in \mathbb{N} \mid k < b-a\}$ if $b-a \in \mathbb{N}$, and $\Delta_b^a := \mathbb{N}$ otherwise. Then $k \in \Delta_b^a$ if and only if A_{b-k}^{-1} is a factor of an ℓ-weight in $\widetilde{\chi_{\mathrm{q}}}(L_b^a)$.

For $a, b \in \mathbb{C}$, the $Y(\mathrm{sl}_2)$-module $\mathscr{L}_b^a$ can be obtained as the pullback by an evaluation morphism $Y(\mathrm{sl}_2) \to U(\mathrm{sl}_2)$ of an sl_2-module M of cohighest weight $b-a$; more precisely, M is the graded Hopf dual of the Verma module of lowest weight $a-b$.

In the special case $a = b-1$, comparing with Example 3.7 we get a module isomorphism $N_b \cong L_b^{b-1}$ sending e_1 to v_0 and e_2 to v_1.

We recall the following result of Tarasov [65, 66] on $Y(\mathrm{sl}_2)$-modules with detailed proof in [55, Prop. 3.6]. It was stated for the larger Yangian $Y(\mathrm{gl}_2)$ which contains $Y(\mathrm{sl}_2)$ as a Hopf subalgebra. Irreducible highest ℓ-weight modules over $Y(\mathrm{gl}_2)$ remain irreducible when restricted to $Y(\mathrm{sl}_2)$.

Theorem 3.18 ([65, 66, 55]). *The $Y(\mathrm{sl}_2)$-module $L_{b_1}^{a_1} \otimes L_{b_2}^{a_2} \otimes \cdots \otimes L_{b_n}^{a_n}$ is irreducible if and only if $b_i - a_j \notin \Delta_{b_i}^{a_i} \cap \Delta_{b_j}^{a_j}$ for any $1 \leq i, j \leq n$.*

The tensor product factorization in category $\mathcal{O}_0$ of $Y(\mathrm{sl}_2)$-modules is not unique:

$$L_0^9 \otimes L_2^3 \cong L\Big(\frac{(u-9)(u-3)}{u(u-2)}\Big) \cong L_0^3 \otimes L_2^9.$$

A similar example appeared for $U_q(\widehat{\mathrm{sl}}_2)$ in [57, Rem. 4.3]. The non-uniqueness issue will be resolved in the larger category $\mathcal{O}^{\mathrm{sh}}$; see Theorem 6.4.

Example 3.19. [70, Prop. 23] In Example 3.17, fix b, divide the right-hand sides of $x^-(u)v_i$ and $\xi(u)v_i$ by $b-a$, and take the limit as a goes to infinity. In this way, we obtain the negative prefundamental module L_b^- over $Y_{-1}(\mathrm{sl}_2)$:

$$x^+(u)v_i = \frac{1}{u-b+i-1}v_{i-1}, \quad x^-(u)v_i = \frac{i+1}{u-b+i}v_{i+1},$$
$$\xi(u)v_i = \frac{u-b-1}{(u-b+i-1)(u-b+i)}v_i.$$

§4. Tensor products of prefundamental modules

In this section we study two distinguished families of irreducible modules in category $\mathcal{O}^{\mathrm{sh}}$, the one-dimensional positive prefundamental modules, and the infinite-dimensional negative prefundamental modules. We prove cyclicity and cocyclicity properties for tensor products of these modules (Theorem 4.8), which motivate our definitions of Weyl modules and standard modules (Definition 4.10). In the end we identify these two modules when $\mathfrak{g}$ is not of type E_8 (Theorem 4.15).

§4.1. One-dimensional modules

Let $\mathcal{D}$ be the submonoid of $\mathcal{R}$ generated by the $\Psi_{i,a}$ for $(i,a) \in I \times \mathbb{C}$. This is indeed the classifying set for one-dimensional modules in category $\mathcal{O}^{\mathrm{sh}}$ in the following sense.

Lemma 4.1. *Let* $\mathbf{e} \in \mathcal{L}$. *Then* $\dim L(\mathbf{e}) = 1$ *if and only if* $\mathbf{e} \in \mathcal{D}$.

Proof. One-dimensional $Y_\mu(\mathfrak{g})$-modules are necessarily irreducible in category $\mathcal{O}_\mu$, and they factorize through the quotient of $Y_\mu(\mathfrak{g})$ by the ideal generated by the $[x,y]$ for $x,y \in Y_\mu(\mathfrak{g})$. Since such an ideal contains $\xi_{i,n}$ for $(i,n) \in I \times \mathbb{N}$, in the quotient each $\xi_i(u)$ is a monic polynomial. □

The category $\mathcal{O}^{\mathrm{sh}}$ does not admit a non-trivial invertible object: if D and E are modules such that $D \otimes E \cong L(1)$, then both D and E are isomorphic to $L(1)$. In the case of the Borel algebra [41] or shifted quantum affine algebras [40], there are infinitely many invertible objects, the one-dimensional weight modules.

By definition (compare with [24, Thm. 4.1])

$$\chi_{\mathrm{q}}(\mathbf{s}) = \mathbf{s} \quad \text{for } \mathbf{s} \in \mathcal{D}.$$

The generalized Baxter relations for representations of the Borel algebra [24, Thm. 4.8] and its proof hold true in category $\mathcal{O}^{\mathrm{sh}}$. Recall from Lemma 3.11 that the q-character of a finite-dimensional module in category $\mathcal{O}^{\mathrm{sh}}$ lies in $\mathbb{Z}[\mathcal{R}]$, which is the ring of Laurent polynomials in the $\Psi_{i,a}$.

Corollary 4.2. *Let* V *be a finite-dimensional module in category* $\mathcal{O}^{\mathrm{sh}}$. *Replace in* $\chi_{\mathrm{q}}(V)$ *each variable* $\Psi_{i,a}$ *by* $[L^+_{i,a}]$ *and* $\chi_{\mathrm{q}}(V)$ *by* $[V]$. *Then multiplying by denominators, we get a relation in the Grothendieck ring of* $\mathcal{O}^{\mathrm{sh}}$.

Let us apply the corollary to the module $N_{i,a}$ of Example 3.7:

$$\chi_{\mathrm{q}}(N_{i,a}) = \frac{\Psi_{i,a-d_i}}{\Psi_{i,a}} \prod_{j: c_{ij}<0} \Psi_{j,a-d_{ij}} + \frac{\Psi_{i,a+d_i}}{\Psi_{i,a}} \prod_{j: c_{ij}<0} \Psi_{j,a+d_{ij}},$$

$$[L^+_{i,a}][N_{i,a}] = \prod_{j: c_{ij}\neq 0} [L^+_{j,a-d_{ij}}] + \prod_{j: c_{ij}\neq 0} [L^+_{j,a+d_{ij}}].$$

For $\mathbf{s} \in \mathcal{D}$ of coweight μ, let $\rho_{\mathbf{s}} \colon Y_\mu(\mathfrak{g}) \to \mathbb{C}$ denote the representation of the one-dimensional module $L(\mathbf{s})$. Let ν be another coweight. Define the algebra morphisms $\iota_1^{\mathbf{s}}$ and $\iota_2^{\mathbf{s}}$, both from $Y_{\mu+\nu}(\mathfrak{g})$ to $Y_\nu(\mathfrak{g})$, as follows (we omit the dependence of these morphisms on ν which will always be clear from the context):

$$\iota_1^{\mathbf{s}} := (\rho_{\mathbf{s}} \otimes 1)\Delta_{\mu,\nu}, \quad \iota_2^{\mathbf{s}} := (1 \otimes \rho_{\mathbf{s}})\Delta_{\nu,\mu}.$$

Then for V a $Y_\nu(\mathfrak{g})$-module, the tensor product modules $L(\mathbf{s}) \otimes V$ and $V \otimes L(\mathbf{s})$ are pullbacks of V by $\iota_1^{\mathbf{s}}$ and $\iota_2^{\mathbf{s}}$ respectively. Based on Lemma 2.5 we have the following precise formulas for the algebra morphisms:

$$\begin{aligned} &\iota_1^{\mathbf{s}}\colon\ x_i^+(u) \mapsto \langle \mathbf{s}_i(u) x_i^+(u)\rangle_+, \quad x_i^-(u) \mapsto x_i^-(u), \quad \xi_i(u) \mapsto \mathbf{s}_i(u)\xi_i(u), \\ &\iota_2^{\mathbf{s}}\colon\ x_i^+(u) \mapsto x_i^+(u), \quad x_i^-(u) \mapsto \langle \mathbf{s}_i(u) x_i^-(u)\rangle_+, \quad \xi_i(u) \mapsto \mathbf{s}_i(u)\xi_i(u). \end{aligned} \tag{4.1}$$

Remark 4.3. Let $(\nu, \mathbf{r}) \in \mathbf{P}^\vee \times \mathcal{L}$ be truncatable and let $\mathbf{s} \in \mathcal{D}$ be of coweight μ. Then a $Y_\nu(\mathfrak{g})$-module V factorizes through the truncated shifted Yangian $Y_\nu^{\mathbf{r}}(\mathfrak{g})$ if and only if $L(\mathbf{s}) \otimes V$ factorizes through $Y_{\nu+\mu}^{\mathbf{rs}}(\mathfrak{g})$. Indeed, the uniqueness of factorization in equation (2.20) shows that $\iota_1^{\mathbf{s}}(A_i(u)) = A_i(u)$ for $i \in I$, where the first GKLO series is taken in the shifted Yangian $Y_{\nu+\mu}(\mathfrak{g})$ with respect to the truncatable pair $(\nu + \mu, \mathbf{rs})$ and the second in $Y_\nu(\mathfrak{g})$ with respect to $(\nu, \mathbf{r})$.

§4.2. Cyclicity and cocyclicity

The main result of this subsection is the cyclicity and cocyclicity properties of tensor product modules. By cyclicity we mean the module is generated by a highest ℓ-weight vector. Let us explain cocyclicity.

Definition 4.4. Call a $Y_\mu(\mathfrak{g})$-module V *of cohighest ℓ-weight* if it is top graded and its top weight space is contained in all non-zero submodules of V.

It follows that the submodule of V generated by the top weight space is isomorphic to $L(\mathbf{e})$, where $\mathbf{e} \in \mathcal{L}$ is the top ℓ-weight of V. We shall also say that V is cogenerated by a vector of highest ℓ-weight $\mathbf{e}$.

Remark 4.5. Suppose that V is a module of highest ℓ-weight $\mathbf{e}$ and W of cohighest ℓ-weight $\mathbf{e}$. Then there exists a non-zero module morphism $V \to W$ which factorizes through $L(\mathbf{e})$. Such a map is unique up to homothety. It is surjective if and only if W is irreducible, and injective if and only if V is irreducible.

Lemma 4.6. *Let V be a top-graded $Y_\mu(\mathfrak{g})$-module. Then V is of cohighest ℓ-weight if and only if its top weight space equals the subspace of vectors in V annihilated by the $x_{i,n}^+$ for all $(i, n) \in I \times \mathbb{N}$.*

Proof. Let $\lambda \in \mathfrak{h}^*$ be the top weight of V. Let $S \subset V$ be the subspace of vectors annihilated by all the $x_{i,n}^+$. Then $V_\lambda \subset S$ because $\lambda + \alpha_i$ is not a weight of V by assumption and $x_{i,n}^+ V_\lambda \subset V_{\lambda+\alpha_i}$. Moreover, S is weight graded.

Assume V is of cohighest ℓ-weight. Let $\beta \in \lambda + \mathbf{Q}_-$ be any weight of S. The non-zero submodule $S' = Y_\mu(\mathfrak{g})S_\beta$ contains V_λ. In particular, $\lambda \in \mathrm{wt}(S')$. Applying the triangular decomposition to S_β gives $\mathrm{wt}(S') \subset \beta + \mathbf{Q}_-$ and $\lambda \in \beta + \mathbf{Q}_-$. So $\beta = \lambda$. This proves $\mathrm{wt}(S) = \{\lambda\}$ and $S = V_\lambda$.

Assume $S = V_\lambda$. Let T be a non-zero submodule of V. Since T is weight graded by $\lambda + \mathbf{Q}_-$, there exists $\beta \in \mathrm{wt}(T)$, such that $\beta + \alpha_i \notin \mathrm{wt}(T)$ for all $i \in I$. This implies $x_{i,n}^+ T_\beta = \{0\}$ for all $(i,n) \in I \times \mathbb{N}$ and therefore $\{0\} \neq T_\beta \subset S = V_\lambda$. Since V_λ is one-dimensional, $V_\lambda = T_\beta \subset T$. $\square$

Recall from (2.9) the algebra homomorphism for μ a coweight,

$$\tau_z \colon Y_\mu(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes \mathbb{C}[z].$$

Let V be a $Y_\nu(\mathfrak{g})$-module and W be a $Y_\mu(\mathfrak{g})$-module. The vector space $W \otimes V \otimes \mathbb{C}[z]$ is a module over the tensor product algebra $Y_{\mu+\nu}(\mathfrak{g}) \otimes \mathbb{C}[z]$: the tensor factor $\mathbb{C}[z]$ acts by polynomial multiplication; the tensor factor $Y_{\mu+\nu}(\mathfrak{g})$ acts by

$$(1 \otimes \tau_z)\Delta_{\mu,\nu} \colon Y_{\mu+\nu}(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}) \longrightarrow Y_\mu(\mathfrak{g}) \otimes Y_\nu(\mathfrak{g}) \otimes \mathbb{C}[z].$$

Similarly, the $Y_{\mu+\nu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $V \otimes \mathbb{C}[z] \otimes W$ is defined using $(\tau_z \otimes 1)\Delta_{\nu,\mu}$.

Remark 4.7. For $x \in Y_{\mu+\nu}(\mathfrak{g})$ and $w \otimes v \in W \otimes V$, the action of x on $w \otimes v$ in the $Y_{\mu+\nu}(\mathfrak{g})$-module $W \otimes V(a)$ is the evaluation at $z = a$ of the vector-valued polynomial $x(w \otimes v)$ computed in the $Y_{\mu+\nu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $W \otimes V \otimes \mathbb{C}[z]$. A similar statement holds for the $Y_{\mu+\nu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $V \otimes \mathbb{C}[z] \otimes W$.

Theorem 4.8. *Let $\mathbf{s} \in \mathcal{D}$ be of coweight μ and $\mathbf{e} \in \mathcal{L}$ be of coweight ν.*

(i) *If V is a $Y_\nu(\mathfrak{g})$-module of cohighest ℓ-weight, then so are the $Y_{\nu+\mu}(\mathfrak{g})$-module $V \otimes L(\mathbf{s})$ and the $Y_{\nu-\mu}(\mathfrak{g})$-module $L(\mathbf{s}^{-1}) \otimes V$.*

(ii) *The assignment $\omega_{\mathbf{s}} \otimes \omega_{\mathbf{e}} \mapsto \omega_{\mathbf{se}}$ extends uniquely to a module isomorphism*

$$L(\mathbf{s}) \otimes M(\mathbf{e}) \cong M(\mathbf{se}).$$

(iii) *The $Y_{\nu-\mu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $M(\mathbf{e}) \otimes \mathbb{C}[z] \otimes L(\mathbf{s}^{-1})$ is generated by $\omega_{\mathbf{e}} \otimes \omega_{\mathbf{s}^{-1}}$.*

Therefore, if a $Y_\nu(\mathfrak{g})$-module V is of highest ℓ-weight, then so are the $Y_{\nu+\mu}(\mathfrak{g})$-module $L(\mathbf{s}) \otimes V$ and the $Y_{\nu-\mu}(\mathfrak{g})$-module $V \otimes L(\mathbf{s}^{-1})$.

Proof. Assume (ii)–(iii). If V is of highest ℓ-weight, then V is a quotient of a Verma module $M(\mathbf{e})$. So $L(\mathbf{s}) \otimes V$ is a quotient of $L(\mathbf{s}) \otimes M(\mathbf{e})$, which is another Verma module $M(\mathbf{se})$ by (ii). Therefore $L(\mathbf{s}) \otimes V$ is of highest ℓ-weight. Similarly, by evaluating the $Y_{\nu-\mu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $M(\mathbf{e}) \otimes \mathbb{C}[z] \otimes L(\mathbf{s}^{-1})$ at $z = 0$, which makes sense because of Remark 4.7, we obtain from (iii) that the $Y_{\nu-\mu}(\mathfrak{g})$-module $M(\mathbf{e})\otimes L(\mathbf{s}^{-1})$ is of highest ℓ-weight, so is its quotient $V\otimes L(\mathbf{s}^{-1})$.

We shall prove (i)–(iii) for $L(\mathbf{s})$ and $L(\mathbf{s}^{-1})$ separately.

First half of part (i). Let λ be the top weight of V. The tensor product $V\otimes L(\mathbf{s})$ is top graded with $V_\lambda\otimes\omega_{\mathbf{s}}$ being the top weight space. From the formula $\iota_2^{\mathbf{s}}(x_i^+(u)) = x_i^+(u)$ of equation (4.1) we get $x(v\otimes\omega_{\mathbf{s}}) = xv\otimes\omega_{\mathbf{s}}$ for $v\in V$ and $x\in Y^>(\mathfrak{g})\cong Y^>_{\nu+\mu}(\mathfrak{g})\cong Y^>_\nu(\mathfrak{g})$. By Lemma 4.6, the module V is of cohighest ℓ-weight if and only if the module $V\otimes L(\mathbf{s})$ is of cohighest ℓ-weight.

Part (ii). By Example 3.3, $\omega_{\mathbf{s}}\otimes\omega_{\mathbf{e}}\in L(\mathbf{s})\otimes M(\mathbf{e})$ is a vector of highest ℓ-weight $\mathbf{se}$. This induces a module morphism $F\colon M(\mathbf{se})\to L(\mathbf{s})\otimes M(\mathbf{e})$ sending $\omega_{\mathbf{se}}$ to $\omega_{\mathbf{s}}\otimes\omega_{\mathbf{e}}$. From the formula $\iota_1^{\mathbf{s}}(x_i^-(u)) = x_i^-(u)$ of equation (4.1) we get $F(x\omega_{\mathbf{se}}) = \omega_{\mathbf{s}}\otimes x\omega_{\mathbf{e}}$ for $x\in Y^<(\mathfrak{g})\cong Y^<_{\nu+\mu}(\mathfrak{g})\cong Y^<_\nu(\mathfrak{g})$. Identifying the underlying space of Verma modules with $Y^<(\mathfrak{g})$, we see that F is an isomorphism.

From now on fix $W := L(\mathbf{s}^{-1})$ and $\omega := \omega_{\mathbf{s}^{-1}}$. For $i\in I$ setting $P(u)=1$ and $Q(u)=\mathbf{s}_i(u)\in\mathbb{C}[u]$ in the proof of Theorem 3.12, we get

$$\langle \mathbf{s}_i(u)x_i^-(u)\omega\rangle_+ = 0 \quad \text{for } i\in I. \tag{4.2}$$

Applying $x_i^+(u)x_{i,n}^- = x_{i,n}^-x_i^+(u) + \langle u^n\xi_i(u)\rangle_+$ to the highest ℓ-weight vector ω gives

$$\langle \mathbf{s}_i(u)x_i^+(u)x_{i,n}^-\omega\rangle_+ = 0 \quad \text{for } (i,n)\in I\times\mathbb{N}. \tag{4.3}$$

Indeed, the term $x_{i,n}^-x_i^+(u)$ annihilates ω and

$$\langle \mathbf{s}_i(u)\langle u^n\xi_i(u)\omega\rangle_+\rangle_+ = \langle u^n\mathbf{s}_i(u)\mathbf{s}_i(u)^{-1}\omega\rangle_+ = \langle u^n\rangle_+\omega = 0.$$

Part (iii). The Verma module $M(\mathbf{e})$ is $\mathbb{N}$-graded $M(\mathbf{e}) = \bigoplus_{n\in\mathbb{N}} M(\mathbf{e})_n$ by declaring $M(\mathbf{e})_n$ to be the subspace spanned by the weight vectors $x_{i_1,m_1}^-x_{i_2,m_2}^-\cdots x_{i_n,m_n}^-\omega_{\mathbf{e}}$, where $(i_k,m_k)\in I\times\mathbb{N}$ for $1\le k\le n$. Similarly, the $\mathbb{N}$-grading on the highest ℓ-weight module W is defined. Let S be the $Y_{\nu-\mu}(\mathfrak{g})\otimes\mathbb{C}[z]$-submodule of $M(\mathbf{e})\otimes\mathbb{C}[z]\otimes W$ generated by $\omega_{\mathbf{e}}\otimes\omega$. It suffices to show that $M(\mathbf{e})\otimes W\subset S$.

Step 1. Prove $M(\mathbf{e})_n\otimes\omega\subset S$ by induction on $n\in\mathbb{N}$. The initial case $n=0$ is trivial because $M(\mathbf{e})_0 = \mathbb{C}\omega_{\mathbf{e}}$ and $\omega_{\mathbf{e}}\otimes\omega\in S$ by definition. Let $n>0$ and $v\in M(\mathbf{e})_n$. By linearity one may assume $v = x_{i,m}^-v'$ for certain $v'\in M(\mathbf{e})_{n-1}$ and $(i,m)\in I\times\mathbb{N}$. By the induction hypothesis we have $v'\otimes\omega\in S$.

In the module $M(\mathbf{e})\otimes\mathbb{C}[z]\otimes W$ we have by Example 3.4 and equations (2.8)–(2.9),

$$x_i^-(u)(v'\otimes\omega) = x_i^-(u-z)v'\otimes\mathbf{s}_i(u)^{-1}\omega + v'\otimes x_i^-(u)\omega.$$

Here one views $x_i^-(u-z)$ as the Laurent series $\sum_{k\geq 0}\tau_z(x_{i,k}^-)u^{-k-1}$ whose coefficients belong to $Y_\nu(\mathfrak{g})\otimes\mathbb{C}[z]$ and act as linear maps $M(\mathbf{e})\to M(\mathbf{e})\otimes\mathbb{C}[z]$. The principal part at the right-hand side is unnecessary because $\mathbf{s}_i(u)^{-1}x_i^-(u-z)$ is a power series of u^{-1}. Multiply the above equation by the polynomial $\mathbf{s}_i(u)$ and then take the principal part. We obtain from equation (4.2) that

$$\langle\mathbf{s}_i(u)x_i^-(u)\rangle_+(v'\otimes\omega)=x_i^-(u-z)v'\otimes\omega\in S[\![u^{-1}]\!]. \tag{4.4}$$

For $p\in\mathbb{N}$, let C_p be the coefficient of u^{-p-1} at the right-hand side. We have

$$C_p=\sum_{k=0}^{p}\binom{p}{k}x_{i,p-k}^-v'\otimes z^k\otimes w\in S.$$

It follows from the Newton formula $\tau_{-a}\tau_a=\mathrm{Id}$ that

$$v\otimes w=x_{i,m}^-v'\otimes w=\sum_{k=0}^{m}\binom{m}{k}(-z)^{m-k}C_k\in S.$$

Therefore $M(\mathbf{e})_n\otimes\omega\subset S$ for all $n\in\mathbb{N}$ and $M(\mathbf{e})\otimes\omega\subset S$.

Step 2. Prove $M(\mathbf{e})\otimes W_n\subset S$ by induction on $n\in\mathbb{N}$. The initial case $n=0$ follows from Step 1 because $W_0=\mathbb{C}\omega$. Let $n>0$ and $w\in W_n$. By linearity assume $w=x_{i,m}^-w'$ for a certain weight vector $w'\in W_{n-1}$ and $(i,m)\in I\times\mathbb{N}$. By the induction hypothesis we have $v\otimes w'\in S$. In the module $M(\mathbf{e})\otimes\mathbb{C}[z]\otimes W$, take an arbitrary vector $v\in M(\mathbf{e})$ and apply $x_{i,m}^-\in Y_{\nu-\mu}^{<}(\mathfrak{g})$ to $v\otimes w'$. From the coproduct formula $\Delta_{\nu,-\mu}(x_{i,m}^-)$ of Lemma 2.5 we get

$$\begin{aligned}S\ni x_{i,m}^-(v\otimes w')&=(\tau_z\otimes 1)\Delta_{\nu,-\mu}(x_{i,m}^-)(v\otimes\omega)\\&\equiv v\otimes x_{i,m}^-w'\ \mathrm{mod.}\sum_{w''\in W:\mathrm{wt}(w'')-\mathrm{wt}(w')\in\mathbf{Q}_+}M(\mathbf{e})\otimes\mathbb{C}[z]\otimes w''.\end{aligned}$$

Since $\mathrm{wt}(w)=\mathrm{wt}(w')-\alpha_i$, any w'' in the summation belongs to $W_{n'}$ for a certain $0\leq n'<n$. The induction hypothesis applied to w'', together with the fact that S is stable by $\mathbb{C}[z]$, gives $M(\mathbf{e})\otimes\mathbb{C}[z]\otimes w''\subset S$. So $v\otimes w\in S$ for all $v\in M(\mathbf{e})$ and $w\in W_n$. Therefore, $M(\mathbf{e})\otimes W_n\subset S$ for all $n\in\mathbb{N}$ and $M(\mathbf{e})\otimes W\subset S$.

Second half of part (i). As in the first half of part (i), it suffices to show that if $g\in W\otimes V$ is annihilated by the $x_i^+(u)$, then $g\in\omega\otimes V_\lambda$. Assume $0\neq g$ is a weight vector of weight β. Choose a weight basis $\mathcal{B}_V$ of V and write $g=\sum_{v\in\mathcal{B}_V}g_v\otimes v$. Then each $g_v\in W$ is a weight vector and $g_v\neq 0$ only if $\mathrm{wt}(g_v)+\mathrm{wt}(v)=\beta$. Moreover, $g_v=0$ for all but finitely many v. Choose $\gamma\in\mathrm{wt}(V)$ such that

(A) there exists $v_1\in\mathcal{B}_V$ of weight γ such that $g_{v_1}\neq 0$;

(B) if $g_v\neq 0$ and $\mathrm{wt}(v)\neq\gamma$ then $\mathrm{wt}(v)\notin\gamma+\mathbf{Q}_-$.

Let $i \in I$. By Lemma 2.5, $x_i^+(u)(g_v \otimes v)$ is $x_i^+(u)g_v \otimes v$ plus a linear combination of vectors in $W \otimes v'$, where $v' \in \mathcal{B}_V$ satisfies $\mathrm{wt}(v') \in \mathrm{wt}(v) + \alpha_i + \mathbf{Q}_+$. We get from assumption (B) that the component of $W_{\beta-\gamma+\alpha_i} \otimes V_\gamma$ in $x_i^+(u)g = 0$ is

$$\sum_{v \in \mathcal{B}_V : \mathrm{wt}(v)=\gamma} x_i^+(u)g_v \otimes v = 0.$$

Since the second tensor factors are linearly independent, for each v in the summation, we have $x_i^+(u)g_v = 0$ for $i \in I$. Since W is cogenerated by the highest ℓ-weight vector ω, there exists $c_v \in \mathbb{C}$ with $g_v = c_v\omega$. By assumption (A), $c_{v_1} \neq 0$ and so $\beta - \gamma = \mathrm{wt}(g_{v_1}) = \mathrm{wt}(\omega)$.

Next we consider the component $W_{\mathrm{wt}(\omega)} \otimes V_{\gamma+\alpha_i}$ of in $x_i^+(u)g$. This comes from two parts by the coproduct estimation of Lemma 2.5 and assumption (B):

$$0 = \sum_{v \in \mathcal{B}_V : \mathrm{wt}(v)=\gamma} \xi_i(u)c_v\omega \otimes x_i^+(u)v + \sum_{v' \in \mathcal{B}_V : \mathrm{wt}(v')=\gamma+\alpha_i} x_i^+(u)g_{v'} \otimes v'.$$

In the first summation, $\xi_i(u)\omega = \mathbf{s}_i(u)^{-1}\omega$. Multiply the above equality by $\mathbf{s}_i(u)$ and take the principal part. We obtain

$$0 = \omega \otimes x_i^+(u) \sum_{v \in \mathcal{B}_V : \mathrm{wt}(v)=\gamma} c_v v + \sum_{v' \in \mathcal{B}_V : \mathrm{wt}(v')=\gamma+\alpha_i} \langle \mathbf{s}_i(u)x_i^+(u)g_{v'}\rangle_+ \otimes v'.$$

For each v' in the second summation, $\mathrm{wt}(g_{v'}) = \beta - \gamma - \alpha_i = \mathrm{wt}(\omega) - \alpha_i$. So $g_{v'}$ is a linear combination of the $x_{i,n}^-\omega$ for $n \in \mathbb{N}$ and $\langle \mathbf{s}_i(u)x_i^+(u)g_{v'}\rangle_+ = 0$ by equation (4.3). The vector $g' := \sum_{v \in \mathcal{B}_V : \mathrm{wt}(v)=\gamma} c_v v \in V_\gamma$ is annihilated by all the $x_i^+(u)$. Since V is of cohighest ℓ-weight and since $c_{v_1} \neq 0$, we obtain $0 \neq g' \in V_\lambda$ and so $\gamma = \lambda$. In particular, V is weight graded by $\gamma + \mathbf{Q}_-$. Assumption (B) forces $g_v = 0$ if $\mathrm{wt}(v) \neq \gamma$. So $g = \omega \otimes g' \in \omega \otimes V_\lambda$. □

Part (i) of Theorem 4.8 was known for $V \otimes L$ where V is an irreducible $U_q(\hat{\mathfrak{g}})$-module in category $\mathcal{O}$ of [35, §4.3] and L is a tensor product of positive prefundamental modules over the Borel algebra [18, Lem. 5.7]. Part (iii) can be seen as an integral version of cyclicity results in the fusion constructions, over the field $\mathbb{C}(z)$, of representations of current algebras [19, Prop. 1.1] and quantum affinizations [36, Thm. 6.2]. In the non-shifted case, the field $\mathbb{C}(z)$ is necessary because cyclicity holds true only for generic spectral parameters; see [1, 25, 67, 51, 10] for $U_q(\hat{\mathfrak{g}})$ and [64, 63, 31] for $Y(\mathfrak{g})$.

Corollary 4.9. *Let $\mathbf{r}, \mathbf{s} \in \mathcal{D}$. The assignment $\omega_{\mathbf{r}^{-1}} \otimes \omega_{\mathbf{s}^{-1}} \mapsto \omega_{\mathbf{r}^{-1}\mathbf{s}^{-1}}$ extends uniquely to a module isomorphism*

$$L(\mathbf{r}^{-1}) \otimes L(\mathbf{s}^{-1}) \cong L(\mathbf{r}^{-1}\mathbf{s}^{-1}).$$

Proof. It suffices to prove the irreducibility of the tensor product $L(\mathbf{r}^{-1}) \otimes L(\mathbf{s}^{-1})$. By Theorem 4.8, the tensor product is at the same time of highest ℓ-weight and of cohighest ℓ-weight. So it must be irreducible. □

For the Borel algebra and shifted quantum affine algebras, a tensor product of negative prefundamental modules is shown to be irreducible by realizing it as a limit of an inductive system of irreducible tensor products of KR modules over $U_q(\hat{\mathfrak{g}})$; see [24, Thm. 4.11], [43, Thm. 7.6], and [40, Thm. 5.5]. A similar limit procedure was carried out in [5] with KR-modules replaced by finite-dimensional standard modules [58, §13.2], resulting in modules outside the category $\mathcal{O}$ for the Borel algebra [41].

§4.3. Weyl modules and standard modules

We introduce two families of highest ℓ-weight modules in category $\mathcal{O}^{\mathrm{sh}}$ based on the properties of tensor product modules in the previous subsection. Their definitions resemble those in the category of finite-dimensional modules over the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ [16, 58, 67].

Definition 4.10. For $\mathbf{r}, \mathbf{s} \in \mathcal{D}$, define the *standard module* $\mathcal{W}(\mathbf{r}, \mathbf{s})$ to be the tensor product of irreducible modules $L(\mathbf{r}) \otimes L(\mathbf{s}^{-1})$. Define the *Weyl module* $W(\mathbf{r}, \mathbf{s})$ to be the quotient of the Verma module $M(\mathbf{s}^{-1}\mathbf{r})$ by the relations

$$\langle \mathbf{s}_i(u) x_i^-(u) \rangle_+ \omega_{\mathbf{s}^{-1}\mathbf{r}} = 0 \quad \text{for } i \in I.$$

For $\mathbf{r} = \Psi_{i_1,a_1}\Psi_{i_2,a_2}\cdots\Psi_{i_M,a_M}$ and $\mathbf{s} = \Psi_{j_1,b_1}\Psi_{j_2,b_2}\cdots\Psi_{j_N,b_N}$ we have the following factorization of the standard module (one need not care about non-associativity because each tensor product in the parentheses is irreducible)

$$\mathcal{W}(\mathbf{r}, \mathbf{s}) \cong (L^+_{i_1,a_1} \otimes L^+_{i_2,a_2} \otimes \cdots \otimes L^+_{i_M,a_M}) \otimes (L^-_{j_1,b_1} \otimes L^-_{j_2,b_2} \otimes \cdots \otimes L^-_{j_N,b_N}).$$

This resembles the case of finite-dimensional standard modules over $U_q(\hat{\mathfrak{g}})$ in [67, Cor. 7.17] and [59, Cor. 6.13]. It implies the following equation in $K_0(\mathcal{O}^{\mathrm{sh}})$:

$$[\mathcal{W}(\mathbf{r}, \mathbf{s}) \otimes \mathcal{W}(\mathbf{m}, \mathbf{n})] = [\mathcal{W}(\mathbf{rm}, \mathbf{sn})] \quad \text{for } \mathbf{r}, \mathbf{s}, \mathbf{m}, \mathbf{n} \in \mathcal{D}. \tag{4.5}$$

Our definition of a Weyl module is similar to the one in the categories of finite-dimensional modules over $U_q(\hat{\mathfrak{g}})$ [16, §4] and over the quantum affine superalgebra $U_q(\widehat{\mathrm{sl}}(m,n))$ [69, §4.1]. The difference from [16] is that apart from the highest ℓ-weight $\mathbf{s}^{-1}\mathbf{r}$ we have to introduce a new parameter $\mathbf{s}$. This is because in category $\mathcal{O}^{\mathrm{sh}}$ and the related category for the quantum affine superalgebra among the highest ℓ-weight modules of a fixed highest ℓ-weight, there is no universal one.

Remark 4.11. For dominantly shifted Yangians of type A, there is another definition of a standard module in [9, eq. (7.1)] as an ordered tensor product of irreducible modules. These modules are parameterized by their highest ℓ-weight. We do not know whether they are particular cases of our standard modules.

Proposition 4.12. *Let* $\mathbf{m}, \mathbf{r}, \mathbf{s} \in \mathcal{D}$.

(i) *There exists a unique surjective module morphism* $W(\mathbf{r}, \mathbf{s}) \to \mathcal{W}(\mathbf{r}, \mathbf{s})$ *sending* $\omega_{\mathbf{s}^{-1}\mathbf{r}}$ *to* $\omega_{\mathbf{r}} \otimes \omega_{\mathbf{s}^{-1}}$.

(ii) *The map* $\omega_{\mathbf{m}} \otimes \omega_{\mathbf{s}^{-1}\mathbf{r}} \mapsto \omega_{\mathbf{s}^{-1}\mathbf{mr}}$ *extends uniquely to a module isomorphism*

$$L(\mathbf{m}) \otimes W(\mathbf{r}, \mathbf{s}) \longrightarrow W(\mathbf{mr}, \mathbf{s}).$$

(iii) *Weyl modules are in category* $\mathcal{O}^{\mathrm{sh}}$. *A highest* ℓ*-weight module in category* $\mathcal{O}^{\mathrm{sh}}$ *is necessarily a quotient of a Weyl module.*

Proof. Part (i). By Theorem 4.8, the module $\mathcal{W}(\mathbf{r}, \mathbf{s})$ is of highest ℓ-weight $\mathbf{s}^{-1}\mathbf{r}$. Combining equation (4.2) in $L(\mathbf{s}^{-1})$ with equation (4.1), we get for $j \in I$,

$$\langle \mathbf{s}_j(u) x_j^-(u)\rangle_+(\omega_{\mathbf{r}} \otimes \omega_{\mathbf{s}^{-1}}) = \omega_{\mathbf{r}} \otimes \langle \mathbf{s}_j(u) x_j^-(u)\rangle_+ \omega_{\mathbf{s}^{-1}} = 0.$$

All the defining relations of the Weyl module $W(\mathbf{r}, \mathbf{s})$ are realized in $\mathcal{W}(\mathbf{r}, \mathbf{s})$.

Part (ii). Let $\mu = \sum_{j\in I} k_j \varpi_j^\vee$ be the coweight of $\mathbf{s}^{-1}\mathbf{r}$. Set $\tilde{x}_i^-(u) := \langle \mathbf{s}_i(u) x_i^-(u)\rangle_+$. Let $V(\mathbf{r}, \mathbf{s})$ be the subspace of the Verma module $M(\mathbf{s}^{-1}\mathbf{r})$ spanned by the coefficients of $\tilde{x}_i^-(u)\omega_{\mathbf{s}^{-1}\mathbf{r}}$ for $i \in I$, and let $K(\mathbf{r}, \mathbf{s})$ be the submodule generated by this subspace. Then $W(\mathbf{r}, \mathbf{s}) = M(\mathbf{s}^{-1}\mathbf{r})/K(\mathbf{r}, \mathbf{s})$. For $(j, m) \in I \times \mathbb{N}$, as in the proof of Theorem 3.12 we have $x_{j,m}^+ V(\mathbf{r}, \mathbf{s}) = \{0\}$. Next we prove by induction on $p \geq -k_j - 1$ that $\xi_{j,p} V(\mathbf{r}, \mathbf{s}) \subset V(\mathbf{r}, \mathbf{s})$. The initial case is trivial because $\xi_{j,-k_j-1} = 1$. Applying the following relation to the highest ℓ-weight vector $\omega_{\mathbf{s}^{-1}\mathbf{r}}$, which is a common eigenvector of the $\xi_{j,q}$ for $q \in \mathbb{Z}$,

$$\begin{aligned}\xi_{j,p+1}\tilde{x}_i^-(u) = \tilde{x}_i^-(u)\xi_{j,p+1} + \xi_{j,p}\langle u\tilde{x}_i^-(u)\rangle_+ \\ - \langle u\tilde{x}_i^-(u)\rangle_+\xi_{j,p} - d_{ij}\xi_{j,p}\tilde{x}_i^-(u) - d_{ij}\tilde{x}_i^-(u)\xi_{j,p},\end{aligned}$$

we derive the case of $p+1$ from the case of p. From the triangular decomposition of Theorem 2.2 we get $K(\mathbf{r}, \mathbf{s}) = Y^<(\mathfrak{g})V(\mathbf{r}, \mathbf{s})$.

Theorem 4.8(ii) affords a module isomorphism $F\colon M(\mathbf{s}^{-1}\mathbf{mr}) \to L(\mathbf{m}) \otimes M(\mathbf{s}^{-1}\mathbf{r})$ which sends $x\omega_{\mathbf{s}^{-1}\mathbf{mr}}$ to $\omega_{\mathbf{m}} \otimes x\omega_{\mathbf{s}^{-1}\mathbf{r}}$ for $x \in Y^<(\mathfrak{g})$. In particular, F maps the subspace $V(\mathbf{mr}, \mathbf{s})$ of the Verma module $M(\mathbf{s}^{-1}\mathbf{mr})$ onto $\omega_{\mathbf{m}} \otimes V(\mathbf{r}, \mathbf{s})$. Furthermore,

$$F(K(\mathbf{mr}, \mathbf{s})) = F(Y^<(\mathfrak{g})V(\mathbf{mr}, \mathbf{s})) = \omega_{\mathbf{m}} \otimes Y^<(\mathfrak{g})V(\mathbf{r}, \mathbf{s}) = \omega_{\mathbf{m}} \otimes K(\mathbf{r}, \mathbf{s}).$$

This induces the desired module isomorphism $W(\mathbf{mr}, \mathbf{s}) \cong L(\mathbf{m}) \otimes W(\mathbf{r}, \mathbf{s})$.

Part (iii). Let W be a Weyl module generated by a highest ℓ-weight vector ω, so that $\mathrm{wt}(W) \subset \mathrm{wt}(\omega) + \mathbf{Q}_-$. By definition, $W_{\mathrm{wt}(\omega)-\alpha_i}$ is finite-dimensional for $i \in I$. One can copy [15, §5, Proof of (b)] to show that all weight spaces of W are finite-dimensional. Therefore W is in category $\mathcal{O}^{\mathrm{sh}}$.

Let V be a module in category $\mathcal{O}^{\mathrm{sh}}$ generated by a highest ℓ-weight vector v of ℓ-weight $\mathbf{n}$. Fix $i \in I$. The $x_{i,m}^- v$ for $m \in \mathbb{N}$ span a finite-dimensional weight space. We get a monic polynomial $Q_i(u) = \sum_{k=0}^n c_k u^k$ with $\sum_{k=0}^n c_k x_{i,k}^- v = 0$. Applying $x_{i,m}^+$ for $m \in \mathbb{N}$ to this equality we get that $Q_i(u)\mathbf{n}_i(u)$ is a monic polynomial of u. Apply $\xi_{i,-k_i+1} - \frac{1}{2}\xi_{i,-k_i}^2$ repeatedly to the equality. From

$$\left[\xi_{i,-\langle \mu,\alpha_i\rangle+1} - \frac{1}{2}\xi_{i,-\langle \mu,\alpha_i\rangle}^2, x_{i,m}^-\right] = -2d_i x_{i,m+1}^-,$$

we get $\sum_{k=0}^n c_k x_{i,k+m}^- v = 0$ for all $m \in \mathbb{N}$, namely, $\langle Q_i(u) x_i^-(u)\rangle_+ v = 0$. Let us set $\mathbf{s} := (Q_i(u))_{i\in I}$ and $\mathbf{r} := \mathbf{ns}$. Then $\mathbf{r}, \mathbf{s} \in \mathcal{D}$ and V is a quotient of $W(\mathbf{r},\mathbf{s})$. □

In the rest of this subsection we show that Weyl modules are standard modules when $\mathfrak{g}$ is not of type E_8. Recall from Section 2.2 the root vectors $x_\gamma^\pm \in \mathfrak{g}_{\pm\gamma}$ and the PBW variables $x_{\gamma,n}^- \in Y(\mathfrak{g})^{\leq n}$ for $(\gamma, n) \in R \times \mathbb{N}$. Identify the associated grading $\mathrm{gr}_{\mathbb{N}} Y(\mathfrak{g})$ with $U(\mathfrak{g}[t])$ via the isomorphism of (2.16). Then for $i \in I$ and $n \in \mathbb{N}$,

$$x_{i,n}^- + Y(\mathfrak{g})^{\leq n-1} = x_{\alpha_i,n}^- + Y(\mathfrak{g})^{\leq n-1} = x_{\alpha_i}^- \otimes t^n \quad \text{and} \quad d_i\alpha_i^\vee = \xi_{i,0}.$$

In general, $x_{\gamma,n}^- + Y(\mathfrak{g})^{\leq n-1}$ is proportional to $x_\gamma^- \otimes t^n \in \mathfrak{g}[t]$ for $\gamma \in R$.

Let $\mathbf{s} \in \mathcal{D}$. Consider the Weyl module $W(\mathbf{s},\mathbf{s})$ over $Y(\mathfrak{g})$. Its zero weight space $W(\mathbf{s},\mathbf{s})_0$ is spanned by a highest ℓ-weight vector ω. The $\mathbb{N}$-filtration of $Y(\mathfrak{g})$ descends to the module $W(\mathbf{s},\mathbf{s})$ by setting

$$W(\mathbf{s},\mathbf{s})^{\leq m} := Y(\mathfrak{g})^{\leq m}\omega \quad \text{for } m \in \mathbb{N}.$$

The associated grading $\mathrm{gr}_{\mathbb{N}} W(\mathbf{s},\mathbf{s})$ is then naturally an $\mathbb{N}$-graded $U(\mathfrak{g}[t])$-module. This is referred to as the *classical limit* of $W(\mathbf{s},\mathbf{s})$, denoted by $\overline{W(\mathbf{s},\mathbf{s})}$.

Lemma 4.13. *Let $\mathbf{s} \in \mathcal{D}$. As a module over $\mathfrak{h} \subset \mathfrak{g}[t]$, the classical limit $\overline{W(\mathbf{s},\mathbf{s})}$ is semisimple and has character equal to $\chi(W(\mathbf{s},\mathbf{s}))$. Moreover, the $\mathfrak{g}[t]$-module $\overline{W(\mathbf{s},\mathbf{s})}$ is generated by ω and satisfies the relations*

$$(x_{\alpha_i}^+ \otimes t^n)\omega = 0 = (\alpha_i^\vee \otimes t^n)\omega = (x_{\alpha_i}^- \otimes t^{\langle \varpi^\vee(\mathbf{s}),\alpha_i\rangle})\omega \quad \textit{for } (i,n) \in I \times \mathbb{N}.$$

Proof. The first part comes from the compatibility of $\mathbb{N}$-filtration and weight grading on $Y(\mathfrak{g})$. For the second part, fix $i \in I$ and set $N := \langle \varpi^\vee(\mathbf{s}), \alpha_i\rangle$. By definition,

$\mathbf{s}_i(u) = \sum_{k=0}^{N} c_k u^k \in \mathbb{C}[u]$ with $c_N = 1$. In the Weyl module $W(\mathbf{s}, \mathbf{s})$ we have

$$x_{i,N}^- \omega = -\sum_{k=0}^{N-1} c_k x_{i,k}^- \omega \in W(\mathbf{s}, \mathbf{s})^{\leq N-1}.$$

In the associated grading, the left-hand side becomes $(x_{\alpha_i}^- \otimes t^N)\omega$ and is of degree N, while the right-hand side is of degree $N-1$. So both sides vanish and $(x_{\alpha_i}^- \otimes t^N)\omega = 0$ in the classical limit. The first two relations are proved in the same way. □

Let $\mathfrak{a}_{\mathbf{s}}$ be the Lie subalgebra of $\mathfrak{g}[t]$ generated by the elements $x_{\alpha_i}^+ \otimes t^n$, $\alpha_i^\vee \otimes t^n$, and $x_{\alpha_i}^- \otimes t^{\langle \varpi^\vee(\mathbf{s}), \alpha_i \rangle}$ for $(i, n) \in I \times \mathbb{N}$. Define the $U(\mathfrak{g}[t])$-module

$$P_{\mathbf{s}} \coloneqq U(\mathfrak{g}[t]) \otimes_{U(\mathfrak{a}_{\mathbf{s}})} \mathbb{C},$$

where $\mathfrak{a}_{\mathbf{s}}$ acts on $\mathbb{C}$ as zero. Lemma 4.13 shows that $\overline{W(\mathbf{s}, \mathbf{s})}$ is a quotient of $P_{\mathbf{s}}$.

Lemma 4.14. *The action of $\mathfrak{h} \subset \mathfrak{g}[t]$ on $P_{\mathbf{s}}$ is semisimple with character*

$$\chi(P_{\mathbf{s}}) = \prod_{\gamma \in R} \Big(\frac{1}{1 - e^{-\gamma}} \Big)^{\langle \varpi^\vee(\mathbf{s}), \gamma \rangle}.$$

Proof. We claim that the subalgebra $\mathfrak{a}_{\mathbf{s}}$ of $\mathfrak{g}[t]$ is spanned by

$$\text{(B)} \qquad x_\gamma^+ \otimes t^n, \quad \alpha_i^\vee \otimes t^n, \quad x_\gamma^- \otimes t^m,$$

where $(i, \gamma, m, n) \in I \times R \times \mathbb{N}^2$ and $m \geq \langle \varpi^\vee(\mathbf{s}), \gamma \rangle$. Assume the claim. Take V to be the subspace of $\mathfrak{g}[t]$ with basis $x_\gamma^- \otimes t^k$, where $(\gamma, k) \in R \times \mathbb{N}$ and $k < \langle \varpi^\vee(\mathbf{s}), \gamma \rangle$. Then $\mathfrak{g}[t] = \mathfrak{a}_{\mathbf{s}} \oplus V$ is a direct sum of semisimple $\mathfrak{h}$-modules. By the PBW theorem for $U(\mathfrak{g}[t])$, the $\mathfrak{h}$-module $P_{\mathbf{s}}$ is isomorphic to the symmetric algebra $\mathrm{Sym}(V)$ whose $\mathfrak{h}$-module structure is induced from that of V. Each basis vector $x_\gamma^- \otimes t^k$ of V is of weight $-\gamma$ and gives rise to a factor $\frac{1}{1-e^{-\gamma}}$ in the character $\chi(\mathrm{Sym}(V)) = \chi(P_{\mathbf{s}})$. Taking their products gives the desired product character formula.

First we show that each vector in (B) belongs to $\mathfrak{a}_{\mathbf{s}}$. This is clear for the first two families of vectors because $x_\gamma^+ \otimes t^n$ is proportional to a commutator of the $x_{\alpha_i}^+ \otimes t^k$ for $(i, k) \in I \times \mathbb{N}$. For the third family, one may assume $m = \langle \varpi^\vee(\mathbf{s}), \gamma \rangle$, as the other cases can be deduced from adjoint actions of the $\alpha_i^\vee \otimes t$. Write $\gamma = \sum_{i \in I} m_i \alpha_i$. Then x_γ^- is proportional to a commutator of the $x_{\alpha_i}^-$, where each $x_{\alpha_i}^-$ appears exactly m_i times. Replacing each $x_{\alpha_i}^-$ in the commutator formula of x_γ^- with $x_{\alpha_i}^- \otimes t^{\langle \varpi^\vee(\mathbf{s}), \alpha_i \rangle}$, we obtain $x_\gamma^- \otimes t^m$ in the Lie subalgebra $\mathfrak{a}_{\mathbf{s}}$.

It remains to show that the subspace of $\mathfrak{g}[t]$ spanned by (B) is a Lie subalgebra. The only non-trivial case is to check that the following vector belongs to this subspace for $(\delta,\gamma,n,m)\in R^2\times\mathbb{N}^2$ with $m\geq\langle\varpi^\vee(\mathbf{s}),\gamma\rangle$:

$$[x_\delta^+\otimes t^n, x_\gamma^-\otimes t^m]=[x_\delta^+,x_\gamma^-]\otimes t^{m+n}.$$

If $\gamma-\delta\notin R$, then $[x_\delta^+,x_\gamma^-]$ is spanned by the x_β^+ and $\alpha_i^\vee$ for $(i,\beta)\in I\times R$, so we get the first two families of vectors in (B). If $\gamma-\delta\in R$, then $[x_\delta^+,x_\gamma^-]\in\mathbb{C}x_{\gamma-\delta}^-$ and

$$m+n\geq m\geq\langle\varpi^\vee(\mathbf{s}),\gamma\rangle>\langle\varpi^\vee(\mathbf{s}),\gamma-\delta\rangle.$$

So the commutator does belong to the third family of vectors in (B). □

Theorem 4.15. *Assume that $\mathfrak{g}$ is not of type E_8. Then for $\mathbf{r},\mathbf{s}\in\mathcal{D}$, the surjective morphism $W(\mathbf{r},\mathbf{s})\to\mathcal{W}(\mathbf{r},\mathbf{s})$ of Proposition 4.12(i) is an isomorphism. Furthermore, the ordered monomials in the $x_{\gamma,n}^-$ for $(\gamma,n)\in R\times\mathbb{N}$ and $n<\langle\varpi^\vee(\mathbf{s}),\gamma\rangle$ applied to $\omega_{\mathbf{s}^{-1}\mathbf{r}}$ form a basis of the Weyl module $W(\mathbf{r},\mathbf{s})$.*

Proof. By Proposition 4.12(i), $\chi(\mathcal{W}(\mathbf{s},\mathbf{s}))$ is bounded above by $\chi(W(\mathbf{s},\mathbf{s}))$. Lemma 4.14 implies that $\chi(W(\mathbf{s},\mathbf{s}))=\chi(\overline{W(\mathbf{s},\mathbf{s})})$ is bounded above by $\chi(P_\mathbf{s})$. In view of Corollary 4.9 and Theorem 3.16, if we write $\mathbf{s}=\Psi_{i_1,a_1}\Psi_{i_2,a_2}\cdots\Psi_{i_N,a_N}$, then

$$\chi(\mathcal{W}(\mathbf{s},\mathbf{s}))=\prod_{k=1}^N\chi(\mathcal{W}(\Psi_{i_k,a_k},\Psi_{i_k,a_k}))=\prod_{k=1}^N\prod_{\gamma\in R}\Big(\frac{1}{1-e^{-\gamma}}\Big)^{\langle\varpi_{i_k}^\vee,\gamma\rangle}=\chi(P_\mathbf{s}).$$

So all the characters are equal and the quotient map $P_\mathbf{s}\to\overline{W(\mathbf{s},\mathbf{s})}$ is bijective. From the tensor product factorizations of Proposition 4.12(ii) we get

$$\widetilde{\chi_q}(W(\mathbf{r},\mathbf{s}))=\widetilde{\chi_q}(W(\mathbf{s},\mathbf{s}))=\widetilde{\chi_q}(\mathcal{W}(\mathbf{s},\mathbf{s}))=\widetilde{\chi_q}(L(\mathbf{s}^{-1}))=\widetilde{\chi_q}(\mathcal{W}(\mathbf{r},\mathbf{s})).$$

This implies $W(\mathbf{r},\mathbf{s})\cong\mathcal{W}(\mathbf{r},\mathbf{s})$.

For the second part, the case $\mathbf{r}=\mathbf{s}$ follows from the PBW basis of the classical limit $\overline{W(\mathbf{s},\mathbf{s})}\cong P_\mathbf{s}$ obtained in the proof of Lemma 4.14. The rest follows from the fact that the isomorphism $L(\mathbf{m})\otimes W(\mathbf{n},\mathbf{s})\cong W(\mathbf{mn},\mathbf{s})$ of Proposition 4.12(ii) for $\mathbf{m},\mathbf{n}\in\mathcal{D}$ identifies the actions of $Y^<(\mathfrak{g})$ on the Weyl modules $W(\mathbf{n},\mathbf{s})$ and $W(\mathbf{mn},\mathbf{s})$. □

Remark 4.16. Our proof of Theorem 4.15 in category $\mathcal{O}^{\mathrm{sh}}$ is simpler than the finite-dimensional case [12, Thm. 7.5]. The reason is that we take the quotient of $U(\mathfrak{g}[t])$ by a left ideal generated by elements of the Lie algebra $\mathfrak{g}[t]$. In the finite-dimensional case, depending on a dominant integral weight $\sum_{i\in I}k_i\varpi_i$, the left ideal of $U(\mathfrak{g}[t])$ to define the classical limit is generated by [16, §2]:

$$x_{\alpha_i}^+\otimes t^n,\quad \alpha_i^\vee\otimes t^n-\delta_{n0}k_i,\quad (x_{\alpha_i}^-\otimes 1)^{k_i+1}\quad\text{for } (i,n)\in I\times\mathbb{N}.$$

The third family of generators is not in $\mathfrak{g}[t]$. This makes it highly non-trivial to find a basis for the quotient, even in type A [11].

§5. Properties of R-matrices

In this section we study R-matrices, which are module morphisms

$$V \otimes W \longrightarrow W \otimes V,$$

where V and W are suitably chosen highest ℓ-weight modules. First we establish properties of the R-matrices: existence, uniqueness, factorization, and polynomiality (Theorem 5.2, Propositions 5.3 and 5.7). We also compute the eigenvalues of certain of these R-matrices in Proposition 5.8.

Definition 5.1. A module in category $\mathcal{O}^{\mathrm{sh}}$ is called *negative* if it is irreducible and its highest ℓ-weight is of the form $\mathbf{s}^{-1}$ with $\mathbf{s} \in \mathcal{D}$.

Theorem 5.2. *Let V be a $Y_\nu(\mathfrak{g})$-module generated by a highest ℓ-weight vector v_0 and W be a $Y_\mu(\mathfrak{g})$-module generated by a highest ℓ-weight vector ω. Then the assignment $v_0 \otimes \omega \mapsto \omega \otimes v_0$ extends uniquely to a $Y_{\mu+\nu}(\mathfrak{g})$-module morphism*

$$\check{R}_{V,W} \colon V \otimes W \longrightarrow W \otimes V$$

under one of the following conditions:

(i) *The module V is irreducible, and W is negative.*

(ii) *The module V is one-dimensional, and W is either a Verma module, or a highest ℓ-weight irreducible module, or a Weyl module.*

Proof. *Part* (i). By Theorem 4.8, $V \otimes W$ is of highest ℓ-weight, and $W \otimes V$ is of cohighest ℓ-weight. Their highest ℓ-weights coincide by Example 3.3. The existence and uniqueness of $\check{R}_{V,W}$ follow from Remark 4.5.

Part (ii). The same arguments as above work when $\dim V = 1$ and W is irreducible.

Suppose $V = L(\mathbf{s})$ and $W = M(\mathbf{e})$ with $\mathbf{s} \in \mathcal{D}$ and $\mathbf{e} \in \mathcal{L}$. Then $V \otimes W$ is isomorphic to the Verma module $M(\mathbf{se})$ by Theorem 4.8(ii). Since $\omega \otimes v_0 \in W \otimes V$ is of highest ℓ-weight $\mathbf{se}$, the existence and uniqueness of $\check{R}_{V,W}$ follow.

Suppose $V = L(\mathbf{m})$ and $W = W(\mathbf{r}, \mathbf{s})$ with $\mathbf{m}, \mathbf{r}, \mathbf{s} \in \mathcal{D}$. Then $V \otimes W$ is isomorphic to the Weyl module $W(\mathbf{mr}, \mathbf{s})$ by Proposition 4.12(ii). The existence and uniqueness of $\check{R}_{V,W}$ follow if the highest ℓ-weight vector $\omega \otimes v_0 \in W \otimes V$ satisfies the defining relations of the Weyl module $W(\mathbf{mr}, \mathbf{s})$. We have $\langle \mathbf{s}_i(u) x_i^-(u) \omega \rangle_+ = 0$ in the Weyl module $W(\mathbf{r}, \mathbf{s}) = W$ and $x_i^-(u)(\omega \otimes v_0) = \langle \mathbf{m}_i(u) x_i^-(u) \omega \rangle_+ \otimes v_0$ in the tensor product module $W \otimes L(\mathbf{m}) = W \otimes V$ by equation (4.1). So

$$\begin{aligned} \langle \mathbf{s}_i(u) x_i^-(u)(\omega \otimes v_0) \rangle_+ &= \langle \mathbf{s}_i(u) \langle \mathbf{m}_i(u) x_i^-(u) \omega \rangle_+ \rangle_+ \otimes v_0 \\ &= \langle \mathbf{m}_i(u) \langle \mathbf{s}_i(u) x_i^-(u) \omega \rangle_+ \rangle_+ \otimes v_0 = 0. \end{aligned}$$

The second equality used equation (2.17) twice. □

The morphism $\check{R}_{V,W}$ is independent of the choice of the highest ℓ-weight vectors v_0 and ω because both of them span a one-dimensional weight space. It is normalized as in Theorem 3.8.

As a first application, we consider the situation of Theorem 5.2(i). For $a \in \mathbb{C}$ there exists a unique $Y_{\mu+\nu}(\mathfrak{g})$-module morphism

$$\check{R}_{V,W}(a) \colon V(a) \otimes W \longrightarrow W \otimes V(a), \quad v_0 \otimes \omega \mapsto \omega \otimes v_0.$$

Proposition 5.3. *Let V be a highest ℓ-weight irreducible $Y_\nu(\mathfrak{g})$-module and let W be a negative module. Then there exists a unique linear map*

$$\check{R}_{V,W}(u) \colon V \otimes W \longrightarrow W \otimes V \otimes \mathbb{C}[u]$$

whose evaluation at $u = a$ is $\check{R}_{V,W}(a)$ for any $a \in \mathbb{C}$.

(i) *Assume ν is antidominant. For $\mathbf{r}, \mathbf{s} \in \mathcal{D}$, we have*

$$\check{R}_{V,L(\mathbf{r}^{-1}\mathbf{s}^{-1})} = (1 \otimes \check{R}_{V,L(\mathbf{s}^{-1})})(\check{R}_{V,L(\mathbf{r}^{-1})} \otimes 1),$$

where we have identified $L(\mathbf{r}^{-1}) \otimes L(\mathbf{s}^{-1})$ with $L(\mathbf{r}^{-1}\mathbf{s}^{-1})$ as in Corollary 4.9.

(ii) *Let U and V be finite-dimensional irreducible $Y(\mathfrak{g})$-modules. Then we have the following* quantum Yang–Baxter equation*:*

$$\check{R}^{23}_{U,V}(u-v)\check{R}^{12}_{U,W}(u)\check{R}^{23}_{V,W}(v) = \check{R}^{12}_{V,W}(v)\check{R}^{23}_{U,W}(u)\check{R}^{12}_{U,V}(u-v). \tag{5.1}$$

It is an equality of linear maps from $U \otimes V \otimes W$ to $W \otimes V \otimes U \otimes \mathbb{C}(u,v)$, where $R^{23} = 1 \otimes R$, $R^{12} = R \otimes 1$, and $\check{R}_{U,V}(u-v)$ is the normalized R-matrix of Theorem 3.8.

Proof. Suppose that the negative module W is defined over the shifted Yangian $Y_\mu(\mathfrak{g})$ and fix highest ℓ-weight vectors ω in W and v_0 in V. We need to show that for any vectors $v \in V$ and $w \in W$, the function $a \mapsto \check{R}_{V,W}(a)(v \otimes w)$ is polynomial, in the sense that it is the evaluation at $z = a$ of polynomial of z taking values in $W \otimes V$. By Remark 4.7, the $Y_{\mu+\nu}(\mathfrak{g})$-module $V(a) \otimes W$ is the evaluation at $z = a$ of the $Y_{\mu+\nu}(\mathfrak{g}) \otimes \mathbb{C}[z]$-module $V \otimes \mathbb{C}[z] \otimes W$. By Theorem 4.8(iii), there exists a polynomial $\sum_{s=0}^N X_s z^s$ with coefficients $X_s \in Y_{\mu+\nu}(\mathfrak{g})$ such that in the module $V(a) \otimes W$,

$$v \otimes w = \sum_{s=0}^N a^s X_s (v_0 \otimes \omega) \in V(a) \otimes W.$$

Applying $\check{R}_{V,W}(a)$ to the relation yields

$$\check{R}_{V,W}(a)(v \otimes w) = \sum_{s=0}^N a^s X_s (\omega \otimes v_0) \in W \otimes V(a).$$

Conclude from the polynomial actions of the X_s on $W \otimes V(a)$.

Part (i). Both sides are module morphisms because all the tensor factors V, $L(\mathbf{r}^{-1})$, and $L(\mathbf{s}^{-1})$ are modules over antidominantly shifted Yangians and belong to the monoidal category $\mathcal{O}_-^{\mathrm{sh}}$ of Remark 3.13. Both sides send $v_0 \otimes \omega_{\mathbf{r}^{-1}} \otimes \omega_{\mathbf{s}^{-1}}$ to $\omega_{\mathbf{r}^{-1}} \otimes \omega_{\mathbf{s}^{-1}} \otimes v_0$. They have to be equal by uniqueness of the R-matrix in Theorem 5.2.

Part (ii). Let ω_1, ω_2, and ω_3 denote highest ℓ-weight vectors of U, V, and W. By the rationality of $\check{R}_{U,V}(u)$ and polynomiality of $\check{R}_{U,W}(u)$ and $\check{R}_{V,W}(u)$, it suffices to prove equation (5.1) evaluated at $u = a$ and $v = b$ for $a, b \in \mathbb{C}$ satisfying the conditions of Theorem 3.8(ii)–(iii). Both sides are module morphisms from $U(a) \otimes V(b) \otimes W$ to $W \otimes V(b) \otimes U(a)$ since the tensor factors are modules over antidominantly shifted Yangians and belong to the monoidal category $\mathcal{O}_-^{\mathrm{sh}}$ of Remark 3.13. The source module is generated by $\omega_1 \otimes \omega_2 \otimes \omega_3$ by Theorem 4.8 and the irreducibility of $U(a) \otimes V(b)$. Both sides send the generator to $\omega_3 \otimes \omega_2 \otimes \omega_1$, so they must coincide. □

For v and w weight vectors in the modules V and W respectively, the vector $v \otimes w$ in the module $V(a) \otimes W$ is of weight $\mathrm{wt}(v) + \mathrm{wt}(w) - a\tilde{\nu}$ by Remark 3.1; recall that V is defined over $Y_\nu(\mathfrak{g})$. So $\check{R}_{V,W}(a)(v \otimes w)$ belongs to the weight space of the module $W \otimes V(a)$ of the same weight, which by Remark 3.1 coincides with the finite-dimensional weight space $(W \otimes V)_{\mathrm{wt}(v)+\mathrm{wt}(w)}$ of the module $W \otimes V$. It follows that $\check{R}_{V,W}(u)$ restricts to polynomials taking values in finite-dimensional vector spaces:

$$\check{R}_{V,W}(u) \in \mathrm{Hom}((V \otimes W)_\gamma, (W \otimes V)_\gamma) \otimes \mathbb{C}[u] \quad \text{for } \gamma \in \mathrm{wt}(V \otimes W).$$

Definition 5.4. In the situation of Theorem 5.2(i), the *highest diagonal entry* of $\check{R}_{V,W}(u)$ is the linear operator $s_{V,W}(u) \in \mathrm{Hom}(V, V \otimes \mathbb{C}[u])$ such that for $v \in V$,

$$\check{R}_{V,W}(u)(v \otimes \omega) \equiv \omega \otimes s_{V,W}(u)v \mod \sum_{\mathrm{wt}(\omega) \neq \gamma \in \mathrm{wt}(W)} W_\gamma \otimes V \otimes \mathbb{C}[u]. \tag{5.2}$$

If we assume furthermore that V is finite-dimensional with v_- being a lowest ℓ-weight vector, then the *lowest diagonal entry* of $\check{R}_{V,W}(u)$ is defined to be the linear operator $t_{V,W}(u) \in \mathrm{Hom}(W, W \otimes \mathbb{C}[u])$ such that for $w \in W$,

$$\check{R}_{V,W}(u)(v_- \otimes w) \equiv t_{V,W}(u)w \otimes v_- \mod \sum_{\mathrm{wt}(v_-) \neq \gamma \in \mathrm{wt}(V)} W \otimes V_\gamma \otimes \mathbb{C}[u]. \tag{5.3}$$

Define the polynomial $\lambda_{V,W}(u) \in \mathbb{C}[u]$ to be the eigenvalue of $t_{V,W}(u)$ associated to ω.

The above definition makes sense because both of the vectors ω and v_- span a one-dimensional weight space.

Remark 5.5. Let W be a highest ℓ-weight module over the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and V be a lowest ℓ-weight module over the Borel subalgebra. We specialize the universal R-matrix $\mathcal{R}(z)$ of $U_q(\hat{\mathfrak{g}})$ to obtain $\check{R}_{V,W}(z)\colon V\otimes W \to (W\otimes V)[\![z]\!]$. Then we attach the highest diagonal entry $s_{V,W}(z)\colon V\to V[\![z]\!]$ to a highest ℓ-weight vector ω of W and the lowest diagonal entry $t_{V,W}(z)\colon W\to W[\![z]\!]$ to a lowest ℓ-weight vector v_- of V as above. The proofs of [25, Lem. 2.6] and [24, Prop. 5.5] indicate that

$$s_{V,W}(z) = \mathcal{R}^0(z)\mathcal{R}^\infty|_{V\otimes\omega}, \quad t_{V,W}(z) = \mathcal{R}^0(z)\mathcal{R}^\infty|_{v_-\otimes W}.$$

Here the abelian part $\mathcal{R}^0(z)$ and the Cartan part $\mathcal{R}^\infty$ of $\mathcal{R}(z)$ are written in terms of Drinfeld–Cartan generators. Therefore, both diagonal entries are determined by the action of the Drinfeld–Cartan generators. Our computation of the diagonal entries in the Yangian situation (Proposition 7.2 and Theorem 7.4) is guided by this principle, although we do not have a universal R-matrix for shifted Yangians.

Example 5.6. Fix $i\in I$. Let $V=N_{i,0}$ and v_0 be the highest ℓ-weight vector e_1 in Example 3.7. Write $\mathbf{s}_i(u)=\sum_{k=0}^m c_k u^k$ with $c_m=1$. Apply $\check{R}_{V,W}(a)$ to equation (4.4) with $v'=e_1$ and $z=a$ and then use Example 3.4 to compute $x_i^-(u)(\omega\otimes e_1)$. We obtain

$$\begin{aligned}
\frac{d_i}{u-a}\check{R}_{V,W}(a)(e_2\otimes\omega) &= \check{R}_{V,W}(a)(x_i^-(u)e_1\otimes\omega) = \langle \mathbf{s}_i(u)x_i^-(u)(\omega\otimes e_1)\rangle_+ \\
&= \langle \mathbf{s}_i(u)x_i^-(u)\omega\otimes\xi_i(u)e_1\rangle_+ + \langle \mathbf{s}_i(u)\omega\otimes x_i^-(u)e_1\rangle_+ \\
&= \Big\langle \mathbf{s}_i(u)\frac{u-a+d_i}{u-a}x_i^-(u)\Big\rangle_+\omega\otimes e_1 + \Big\langle \mathbf{s}_i(u)\frac{d_i}{u-a}\Big\rangle_+\omega\otimes e_2, \\
\check{R}_{V,W}(a)(e_2\otimes\omega) &= \mathbf{s}_i(a)\omega\otimes e_2 + \sum_{n=0}^{m-1}\sum_{k=n+1}^{m} c_k a^{k-n-1}x_{i,n}^-\omega\otimes e_1.
\end{aligned}$$

Since e_2 is a lowest ℓ-weight vector, $\lambda_{V,W}(u)=\mathbf{s}_i(u)$.

Next consider Theorem 5.2(ii). Let $\mathbf{s}\in\mathcal{D}$ denote the highest ℓ-weight of $V=\mathbb{C}v_0$. We have a unique linear operator $R_{\mathbf{s}}^W$ on W such that $R_{\mathbf{s}}^W(\omega)=\omega$ and

$$\check{R}_{V,W}(v_0\otimes w) = R_{\mathbf{s}}^W(w)\otimes v_0 \quad \text{for } w\in W.$$

Proposition 5.7. *Let W be either a Verma module, or its irreducible quotient, or a Weyl module, of highest ℓ-weight* $\mathbf{e}$.

(i) *We have $R_{\mathbf{r}}^W\circ R_{\mathbf{s}}^W = R_{\mathbf{rs}}^W$ for $\mathbf{r},\mathbf{s}\in\mathcal{D}$. In particular, the $R_{\mathbf{s}}^W$ for $\mathbf{s}\in\mathcal{D}$ form a commuting family of linear endomorphisms on W.*

(ii) *For $i\in I$, there exists a unique linear map*

$$R_i^W(u)\colon W \longrightarrow W\otimes\mathbb{C}[u]$$

whose evaluation at $u = a$, for $a \in \mathbb{C}$, is $R^W_{\Psi_{i,a}}$. Furthermore, for $\beta \in \varpi(\mathbf{e}) + \mathbf{Q}_-$, restricted to the weight space W_β, the operator $R^W_i(-u)$ is an $\mathrm{End}(W_\beta)$*-valued monic polynomial of degree $\langle \varpi_i^\vee, \varpi(\mathbf{e}) - \beta\rangle$.*

Proof. Suppose $W = M(\mathbf{e})$ is a Verma module and write $\omega = \omega_{\mathbf{e}}$.

Part (i). Comparing the actions of the shifted Yangian on the two tensor products, we see that the linear map $R^W_{\mathbf{s}} : W \to W$ for $\mathbf{s} \in \mathcal{D}$ is uniquely characterized by the equations $R^W_{\mathbf{s}}(\omega) = \omega$ and $R^W_{\mathbf{s}} \circ \iota_1^{\mathbf{s}} = \iota_2^{\mathbf{s}} \circ R^W_{\mathbf{s}}$, namely,

$$R^W_{\mathbf{s}}(\omega) = \omega, \qquad R^W_{\mathbf{s}} x_i^-(u) = \langle \mathbf{s}_i(u) x_i^-(u)\rangle_+ R^W_{\mathbf{s}}, \tag{5.4}$$

$$R^W_{\mathbf{s}} \xi_i(u) = \xi_i(u) R^W_{\mathbf{s}}, \quad R^W_{\mathbf{s}} \langle \mathbf{s}_i(u) x_i^+(u)\rangle_+ = x_i^+(u) R^W_{\mathbf{s}}. \tag{5.5}$$

Equation (5.4) already determines $R^W_{\mathbf{s}}$, because W is obtained from ω by repeatedly applying the $x^-_{i,n}$. We need to check equation (5.4) for $R^W_{\mathbf{r}} \circ R^W_{\mathbf{s}}$ with $\mathbf{s}$ replaced by $\mathbf{rs}$. The first half is evident. For the second half,

$$\begin{aligned} R^W_{\mathbf{r}} \circ R^W_{\mathbf{s}} x_i^-(u) &= R^W_{\mathbf{r}} \langle \mathbf{s}_i(u) x_i^-(u)\rangle_+ R^W_{\mathbf{s}} \\ &= \langle \mathbf{s}_i(u) \langle \mathbf{r}_i(u) x_i^-(u)\rangle_+\rangle_+ R^W_{\mathbf{r}} \circ R^W_{\mathbf{s}} \\ &= \langle \mathbf{s}_i(u)\mathbf{r}_i(u) x_i^-(u)\rangle_+ R^W_{\mathbf{r}} \circ R^W_{\mathbf{s}}. \end{aligned}$$

The second half of part (i) follows from the commutativity of $\mathcal{D}$.

Part (ii). For $(j, n) \in I \times \mathbb{N}$ we have by equation (5.4),

$$R^W_{\Psi_{i,-a}} x^-_{i,n} = (x^-_{i,n+1} + a x^-_{i,n}) R^W_{\Psi_{i,-a}}, \quad R^W_{\Psi_{i,-a}} x^-_{j,n} = x^-_{j,n} R^W_{\Psi_{i,a}} \ \text{ if } j \neq i.$$

Write $\varpi(\mathbf{e}) - \beta = \sum_{j\in I} h_j \alpha_j$ so that $h_j = \langle \varpi_j^\vee, \varpi(\mathbf{e}) - \beta\rangle$. By the triangular decomposition, W_β is spanned by the vectors of the form $x^-_{j_1,n_1} \cdots x^-_{j_K,n_K}\omega$, where each $j \in I$ appears exactly h_j times. Applying $R^W_{\Psi_{i,-a}}$ to such a vector gives

$$R^W_{\Psi_{i,-a}}(x^-_{j_1,n_1} \cdots x^-_{j_K,n_K}\omega) = \prod_{s=1}^K (\delta_{j_s i} x^-_{i,n_s+1} + a^{\delta_{j_s i}} x^-_{j_s,n_s}) \times \omega.$$

The right-hand side is the evaluation at $u = a$ of an W_β-valued polynomial whose dominant term is $u^{h_i} x^-_{j_1,n_1} \cdots x^-_{j_K,n_K}\omega$. We have therefore proved for each $v \in W_\beta$ the existence of h_i vectors $v_0, v_1, \dots, v_{h_i-1} \in W_\beta$ such that

$$R^W_{\Psi_{i,-a}}(v) = a^{h_i} v + a^{h_i-1} v_{h_i-1} + \cdots + a v_1 + v_0 \quad \text{for } a \in \mathbb{C}.$$

By a standard argument of the Vandermonde determinant, each $v \mapsto v_s$ defines a linear operator Q_s on W_β for $0 \le s < h_i$. The $\mathrm{End}(W_\beta)$-valued monic polynomial

$$u^{h_i}\mathrm{Id} + u^{h_i-1} Q_{h_i-1} + \cdots + u Q_1 + Q_0$$

defines the restriction of $R^W_i(-u)$ to W_β.

Suppose W is either the irreducible quotient of $M(\mathbf{e})$, or a Weyl module. Let $\pi\colon M(\mathbf{e}) \to W$ denote the quotient map. The diagram

$$\begin{array}{ccc} L(\mathbf{s}) \otimes M(\mathbf{e}) & \xrightarrow{\check{R}_{L(\mathbf{s}),M(\mathbf{e})}} & M(\mathbf{e}) \otimes L(\mathbf{s}) \\ \downarrow{\scriptstyle 1\otimes\pi} & & \downarrow{\scriptstyle \pi\otimes 1} \\ L(\mathbf{s}) \otimes W & \xrightarrow{\check{R}_{L(\mathbf{s}),W}} & W \otimes L(\mathbf{s}) \end{array}$$

is commutative because the top-left module is generated by $\omega_{\mathbf{s}} \otimes \omega_{\mathbf{e}}$, which is sent to $\omega_{\mathbf{e}} \otimes \omega_{\mathbf{s}}$ by both paths. So (i)–(ii) for the Verma module descend to W. ☐

In the rest of this section, we compute the eigenvalues of $R_i^W(u)$ for W a Weyl module or its irreducible quotient. From equation (5.5) we get $R_i^W(u)\xi_{j,p} = \xi_{j,p}R_i^W(u)$. So $R_i^W(u)$ restricts to an $\mathrm{End}(W_{\mathbf{f}})$-valued polynomial for each ℓ-weight $\mathbf{f}$ of W.

Proposition 5.8. *Let $(\mu, \mathbf{r}) \in \mathbf{P}^\vee \times \mathcal{L}$ be a truncatable pair and $A_i(u)$ for $i \in I$ be the GKLO series in $Y_\mu(\mathfrak{g})$. Let W be either a Weyl module or an irreducible module, generated by a vector ω of highest ℓ-weight $\mathbf{e} \in \mathcal{R}_\mu$. Let $g_i(u) \in \mathbb{C}(\!(u^{-1})\!)^\times$ be the eigenvalue of $A_i(u)$ associated to ω and normalize $\bar{A}_i(u) := g_i(u)^{-1}A_i(u)$.*

(i) *We have an additive difference equation*

$$R_i^W(u + d_i) = R_i^W(u)\bar{A}_i(u) \in \mathrm{Hom}(W, W \otimes \mathbb{C}[u]). \tag{5.6}$$

(ii) *Each ℓ-weight $\mathbf{f}$ of W has a unique decomposition*

$$\mathbf{f} = \mathbf{e} \prod_{j\in I} \prod_{s=1}^{h_j} A_{j,a_{j,s}}^{-1},$$

where $h_j = \langle \varpi_j^\vee, \varpi(\mathbf{f}^{-1}\mathbf{e})\rangle$ and $a_{j,s} \in \mathbb{C}$ for $1 \le s \le h_j$. Furthermore, both of the operators $R_i^W(u)$ and $A_i(u)$ acting on $W_{\mathbf{f}}$ have a unique eigenvalue, respectively

$$\prod_{s=1}^{h_i}(a_{i,s} - u) \quad \textit{and} \quad g_i(u)\prod_{s=1}^{h_i} \frac{u - a_{i,s} + d_i}{u - a_{i,s}}.$$

As a consequence, the normalized q-character of an arbitrary highest ℓ-weight module in category $\mathcal{O}^{\mathrm{sh}}$ is a power series in $\mathbb{N}[\![A_{i,a}^{-1}]\!]_{i\in I, a\in\mathbb{C}}$ with leading term 1.

Proof. *Part* (i). The left-hand side of equation (5.6) sends W to $W \otimes \mathbb{C}[u]$ by Proposition 5.7, and the right-hand side sends each finite-dimensional weight space W_β of W to $W_\beta \otimes \mathbb{C}(\!(u^{-1})\!)$. Since W is obtained from ω by repeatedly applying the $x^-_{j,n}$ for $j \in I$ and $n \in \mathbb{N}$, and since both sides send ω to ω, it suffices to show that both sides have the same commutation relations with the $x^-_{j,n}$.

By Lemma 2.7 and equation (5.4), if $j \neq i$, then $R^W_i(u)$, $A_i(u)$, and both sides of equation (5.6) commute with $x^-_{j,n}$. If $j = i$, then

$$\begin{aligned} R^W_i(u+d_i)x^-_{i,n} &= (x^-_{i,n+1} - ux^-_{i,n} - d_i x^-_{i,n})R^W_i(u+d_i), \\ R^W_i(u)\bar{A}_i(u)x^-_{i,n} &= R^W_i(u)\Big(x^-_{i,n} + d_i\sum_{k\geq 0} x^-_{i,n+k}u^{-k-1}\Big)\bar{A}_i(u) \\ &= \Big(x^-_{i,n+1} - ux^-_{i,n} + d_i\sum_{k\geq 0}(x^-_{i,n+k+1} - ux^-_{i,n+k})u^{-k-1}\Big)R^W_i(u)\bar{A}_i(u) \\ &= (x^-_{i,n+1} - ux^-_{i,n} - d_i x^-_{i,n})R^W_i(u)\bar{A}_i(u). \end{aligned}$$

This proves equation (5.6).

Part (ii). The finite-dimensional ℓ-weight space $W_{\mathbf{f}}$ admits mutually commuting actions of the series $R^W_j(u)$, $\xi_j(u)$, $\bar{A}_i(u)$ for $j \in I$. One can choose a basis B of $W_{\mathbf{f}}$ with respect to which the matrices of these series are upper triangular. Fix a basis vector $b \in B$. For $X(u)$ any of these series, let $[X(u)]_b$ denote the bth diagonal of the matrix of $X(u)$. Then $[\xi_j(u)]_b = \mathbf{f}_j(u)$ by definition of the ℓ-weight space.

By Proposition 5.7, the linear map $R^W_j(-u)\colon W_{\varpi(\mathbf{f})} \to W_{\varpi(\mathbf{f})} \otimes \mathbb{C}[u]$, viewed as an $\mathrm{End}(W_{\varpi(\mathbf{f})})$-valued polynomial in u, is monic of degree h_j. Its restriction to the ℓ-weight space $W_{\mathbf{f}}$ as an $\mathrm{End}(W_{\mathbf{f}})$-valued polynomial is also monic of degree h_j, so is its diagonal entry $[R^W_j(-u)]_b \in \mathbb{C}[u]$ associated to the basis vector b. Let $-a_{j,s}$ for $1 \leq s \leq h_j$ denote the roots of the eigenvalue, which may depend on b. Then $[\bar{A}_j(u)]_b$ can be computed from the difference equation (5.6):

$$\begin{aligned} [R^W_j(u)]_b &= \prod_{s=1}^{h_j}(a_{j,s} - u), \\ [\bar{A}_j(u)]_b &= \frac{[R^W_j(u+d_j)]_b}{[R^W_j(u)]_b} = \prod_{s=1}^{h_j}\frac{u - a_{j,s} + d_j}{u - a_{j,s}}. \end{aligned}$$

By definition, $\bar{A}_i(u)$ is the normalization of $A_i(u)$ by its eigenvalue associated to ω. Applying (2.20) to ω and then to $b \in B$ we get

$$\begin{aligned} \frac{\mathbf{f}_i(u)}{\mathbf{e}_i(u)} &= \frac{1}{[\bar{A}_i(u)]_b[\bar{A}_i(u-d_i)]_b}\prod_{j:c_{ji}<0}\prod_{t=1}^{-c_{ji}}[\bar{A}_j(u - d_{ij} - td_j)]_b \\ &= \prod_{s=1}^{h_i}\frac{(u-a_{i,s})(u-a_{i,s}-d_i)}{(u-a_{i,s}+d_i)(u-a_{i,s})} \times \prod_{j:c_{ji}<0}\prod_{s'=1}^{h_j}\prod_{t=1}^{-c_{ji}}\frac{u - d_{ij} - td_j - a_{j,s'} + d_j}{u - d_{ij} - td_j - a_{j,s'}} \end{aligned}$$

$$= \prod_{s=1}^{h_i} \frac{u - a_{i,s} - d_i}{u - a_{i,s} + d_i} \times \prod_{j : c_{ji} < 0} \prod_{s'=1}^{h_j} \frac{u - d_{ij} - a_{j,s'}}{u - d_{ij} + c_{ji} d_j - a_{j,s'}}$$

$$= \prod_{j \in I} \prod_{s=1}^{h_j} \frac{u - a_{j,s} - d_{ij}}{u - a_{j,s} + d_{ij}} \quad (\text{because } c_{ji} d_j = 2 d_{ij}).$$

From equation (3.3) we get $\mathbf{f} = \mathbf{e} \prod_{j \in I} \prod_{s=1}^{h_j} A_{j,a_{j,s}}^{-1}$. Since the generalized simple roots generate a free abelian subgroup of $\mathcal{R}$, the $a_{i,s}$ for $i \in I$ and $1 \le s \le h_i$ are uniquely determined by $\mathbf{f}^{-1}\mathbf{e}$ and they are independent of the basis vector $b \in B$. This shows that both $R_i^W(u)$ and $A_i(u)$ have a single eigenvalue of the desired form. □

Remark 5.9. Let us compare with R-matrices of ordinary Yangians and quantum affine algebras:

(i) Recall from (2.11) that $Y_\mu^-(\mathfrak{g}) \cong Y_0^-(\mathfrak{g})$. As in [27, §2.6], there are *shift operators* σ_i for $i \in I$, which are algebra endomorphisms defined by

$$\sigma_i \colon Y_\mu^-(\mathfrak{g}) \longrightarrow Y_\mu^-(\mathfrak{g}), \quad \xi_{j,p} \mapsto \xi_{j,p}, \quad x_{j,n}^- \mapsto x_{j,n+\delta_{ij}}^-.$$

It follows from Lemma 2.7 and equation (5.4) that

$$A_i(u) x_{j,n}^- = \frac{u - \sigma_i + d_i \delta_{ij}}{u - \sigma_i} x_{j,n}^- A_i(u), \tag{5.7}$$

$$R_i^W(u) x_{j,n}^- = (-u + \sigma_i)^{\delta_{ij}} x_{j,n}^- R_i^W(u). \tag{5.8}$$

This also illustrates the additive difference equation (5.6).

(ii) In the Borel situation, there is a universal solution to equation (5.8), denoted by $T_i(z)$ in [24, Prop. 5.5], whose action on a $U_q(\hat{\mathfrak{g}})$-module W is the lowest diagonal entry of the specialization $\mathcal{R}(z)|_{L \otimes W}$, where L is a lowest ℓ-weight module over the Borel subalgebra obtained as the graded Hopf dual of a negative prefundamental module; see [24, §3.4, §7.2] and Remark 5.5. The polynomiality of $T_i(z)$ follows from the stronger one for the transfer matrix; see [24, Thms. 5.9, 5.17]. Our difference equation (5.6) corresponds to [40, eqs. (9.18), (10.20)] for shifted quantum affine algebras.

(iii) The ordinary Yangian $Y(\mathfrak{g})$ processes the abelian part $\mathcal{R}^0(u)$ of the universal R-matrix. While it is divergent as a formal infinite product, its specialization on a tensor product of finite-dimensional $Y(\mathfrak{g})$-modules makes sense by viewing it as a solution to a difference equation [29, §5.8]. Proposition 5.8 is close to this approach. We expect that a suitably shifted version of $\mathcal{R}^0(u)$ can be specialized to a tensor product of modules in category $\mathcal{O}^{\mathrm{sh}}$ and that it recovers the highest/lowest diagonal entries as in the case of quantum affine algebras in Remark 5.5.

Corollary 5.10. *Let*

$$\mathbf{s} = \Psi_{i_1,a_1}\Psi_{i_2,a_2}\cdots\Psi_{i_N,a_N} \in \mathcal{D}$$

and W be an irreducible module in category $\mathcal{O}^{\mathrm{sh}}$. The module $L(\mathbf{s})\otimes W$ is irreducible if and only if, for all $1 \le s \le N$ and $\mathbf{f} \in \mathrm{wt}_\ell(W)$, we must have $A^{-1}_{i_s,a_s}\mathbf{f} \notin \mathrm{wt}_\ell(W)$.

Proof. From the proof of Theorem 5.2 we get a module morphism $\check{R}_{L(\mathbf{s}),W}$ from the highest ℓ-weight module $L(\mathbf{s})\otimes W$ to the cohighest ℓ-weight module $W\otimes L(\mathbf{s})$. It is injective if and only if $L(\mathbf{s})\otimes W$ is irreducible. We have

$$R^W_{\mathbf{s}} = R^W_{i_1}(a_1)R^W_{i_2}(a_2)\cdots R^W_{i_N}(a_N),$$

which is a product of mutually commuting operators on W. The linear operator $R^W_{\mathbf{s}}$ is injective if and only if 0 is not an eigenvalue of any of the operators $R^W_{i_s}(a_s)$. The rest follows from Proposition 5.8(ii). □

The "if" part of the corollary was known [18, Lem. 5.9] for $L\otimes V$, where L is a tensor product of positive prefundamental modules over the Borel algebra and V is an irreducible $U_q(\hat{\mathfrak{g}})$-module in category $\mathcal{O}$ of [35, §4.3].

Example 5.11. Let $W = N_{i,a}$ as in Example 3.7 and write $L^+_{i,a} = \mathbb{C}\mathbf{1}$. Then e_1 and e_2 are eigenvectors of $R^W_i(u)$ of eigenvalues 1 and $a-u$ respectively. Consider the module morphism $\check{R}_{L^+_{i,a},W}$ from $L^+_{i,a}\otimes W$ to $W\otimes L^+_{i,a}$: its image is spanned by the vector $e_1\otimes\mathbf{1}$ of ℓ-weight $\prod_{j:c_{ij}\neq 0}\Psi_{j,a-d_{ij}}$; its kernel is spanned by $\mathbf{1}\otimes e_2$ of ℓ-weight $\prod_{j:c_{ij}\neq 0}\Psi_{j,a+d_{ij}}$. We obtain a short exact sequence of modules in category $\mathcal{O}^{\mathrm{sh}}$:

$$0 \longrightarrow \bigotimes_{j:c_{ij}\neq 0} L^+_{i,a+d_{ij}} \longrightarrow L^+_{i,a}\otimes N_{i,a} \longrightarrow \bigotimes_{j:c_{ij}\neq 0} L^+_{j,a-d_{ij}} \longrightarrow 0.$$

A similar short exact sequence appeared in the category $\mathcal{O}$ of the Borel algebra [39, Thm. 5.16], whose proof also made use of R-matrices.

§6. Tensor product factorization in the sl_2-case

In this section $\mathfrak{g}$ is fixed to be sl_2. We prove existence and uniqueness of factorization for *all* irreducible modules in category $\mathcal{O}^{\mathrm{sh}}$ into tensor products of prefundamental modules and KR modules (Theorem 6.4). This result will be crucial in the proof of the Jordan–Hölder property in Section 9.

In our situation, $\mathcal{R}$ is the subgroup of the multiplicative group of the field $\mathbb{C}(u)$ generated by the $u-a$ for $a\in\mathbb{C}$. From Section 3.4 recall the subset $\Delta^a_b \subset \mathbb{N}$ and the irreducible modules L^a_b and $L^\pm_a$ for $a,b\in\mathbb{C}$.

Definition 6.1. Let $\mathbf{e} \in \mathcal{R}$. A *standard factorization* of $\mathbf{e}$ is

$$\mathbf{e} = \prod_{r=1}^{m}(u - x_r) \times \prod_{s=1}^{n} \frac{u - y_s}{u - z_s} \times \prod_{t=1}^{k} \frac{1}{u - w_t},$$

where for $1 \leq r \leq m$, $1 \leq s$, $l \leq n$, and $1 \leq t \leq k$,

$$0 \neq z_s - y_s \in \mathbb{N}, \quad z_s - y_l \notin \Delta_{z_s}^{y_s} \cap \Delta_{z_l}^{y_l},$$
$$z_s - x_r \notin \Delta_{z_s}^{y_s}, \quad w_t - x_r \notin \mathbb{N}, \quad w_t - y_s \notin \Delta_{z_s}^{y_s}.$$

For example, $(u-3)(u-9) \times \frac{u-5}{u-6} \times \frac{1}{u}\frac{1}{u-2}$ is a standard factorization, while this is false for $(u-5)(u-9) \times \frac{u-3}{u-6} \times \frac{1}{u}\frac{1}{u-2}$.

Proposition 6.2. *Consider the following factorization and tensor product:*

$$\text{(F)}: \quad \mathbf{e} = \prod_{r=1}^{m}(u - x_r) \times \prod_{s=1}^{n} \frac{u - y_s}{u - z_s} \times \prod_{t=1}^{k} \frac{1}{u - w_t},$$
$$T = (L_{x_1}^{+} \otimes \cdots \otimes L_{x_m}^{+}) \otimes (L_{z_1}^{y_1} \otimes \cdots \otimes L_{z_n}^{y_n}) \otimes (L_{w_1}^{-} \otimes \cdots \otimes L_{w_k}^{-}).$$

Suppose $0 \neq z_s - y_s \in \mathbb{N}$ *for all* $1 \leq s \leq n$*. Then* (F) *is a standard factorization of* $\mathbf{e}$ *if and only if* T *is an irreducible module isomorphic to* $L(\mathbf{e})$*.*

Proof. Note that the irreducibility of T would force it to be isomorphic to $L(\mathbf{e})$, by comparing highest ℓ-weights. Write $T = T_1 \otimes T_2$, where T_1 denotes the tensor product of the first m factors, and T_2 the remaining part. First, notice that T_1, being one-dimensional, is isomorphic to $L((u - x_1)\cdots(u - x_m))$.

For $1 \leq t \leq k$ let us choose $w_t' \in \mathbb{C}$ in such a way that $w_t' - w_l \notin \mathbb{Z}$ and $w_t' - z_s \notin \mathbb{Z}$ for all $1 \leq l \leq k$ and $1 \leq s \leq n$. Set $\mathbf{s} := (u - w_1')\cdots(u - w_k')$.

Claim 1. The module T_2 is irreducible if and only if $T_2 \otimes L(\mathbf{s})$ is irreducible. The $\Leftarrow$ part is trivial since tensor product is exact. For the $\Rightarrow$ part, assume the irreducibility of T_2. By Corollary 5.10, it suffices to prove that none of the $A_{w_t'}^{-1}$ for $1 \leq t \leq k$ appears as a factor of any ℓ-weight in the normalized q-character of T_2:

$$\widetilde{\chi_q}(T_2) = \prod_{s=1}^{n} \widetilde{\chi_q}(L_{z_s}^{y_s}) \prod_{l=1}^{k} \widetilde{\chi_q}(L_{w_l}^{-}),$$

where $\widetilde{\chi_q}(L_{z_s}^{y_s})$ only admits the $A_{z_s - c}^{-1}$ for $c \in \Delta_{z_s}^{y_s}$ as factors, and $\widetilde{\chi_q}(L_{w_l}^{-})$ only admits the $A_{w_l - c}^{-1}$ for $c \in \mathbb{N}$ as factors. The assumptions $w_l - w_t' \notin \mathbb{Z}$ and $z_s - w_t' \notin \mathbb{Z}$ guarantee the condition of Corollary 5.10, hence the irreducibility of $T_2 \otimes L(\mathbf{s})$.

Claim 2. The module $T_2 \otimes L(\mathbf{s})$ is irreducible if and only if the tensor product

$$T_2' := (L_{z_1}^{y_1} \otimes \cdots \otimes L_{z_n}^{y_n}) \otimes (L_{w_1}^{w_1'} \otimes \cdots \otimes L_{w_k}^{w_k'})$$

is irreducible. This is because the two modules have the same q-character.

Applying Theorem 3.18 to T_2', in view of our choice of the additional parameters w_t', we get that T_2 is irreducible if and only if

$$z_l - y_s \notin \Delta_{z_s}^{y_s} \cap \Delta_{z_l}^{y_l}, \quad w_t - y_s \notin \Delta_{z_s}^{y_s} \quad \text{for } 1 \le s, l \le n,\ 1 \le t \le k.$$

Note that the irreducibility of T is equivalent to the irreducibility of the module T_2 and the tensor product $L((u-x_1)\cdots(u-x_m)) \otimes T_2$. The latter is again in the situation of Corollary 5.10. It is irreducible if and only if for $1 \le r \le m$, $1 \le s \le n$, and $1 \le t \le k$,

- $A_{x_r}^{-1}$ does not appear in $\widetilde{\chi_{\mathrm{q}}}(L_{z_s}^{y_s})$, which means $z_s - x_r \notin \Delta_{z_s}^{y_s}$;
- $A_{x_r}^{-1}$ does not appear in $\widetilde{\chi_{\mathrm{q}}}(L_{w_t}^{-})$, which means $w_t - x_r \notin \mathbb{N}$.

Therefore, T is irreducible if and only if all the conditions from Definition 6.1 on the x_r, y_s, z_s, w_t are satisfied, meaning that (F) is a standard factorization. □

If $k \le m$ in the standard factorization then the irreducibility of the tensor product follows from [9, Thm. 7.7], by first identifying $L_{x_t}^{+} \otimes L_{w_t}^{-}$ with $L_{w_t}^{x_t}$ for $1 \le t \le k$ (so that there is no negative prefundamental module), and then replacing L_b^a and L_b^+ with the modules $L(\binom{a}{b})$ and $L(_b)$ respectively in [9, Thm. 7.7].

If $\mathbf{e}$ is a product of the $\frac{u-a+1}{u-a}$ for $a \in \mathbb{C}$, then a standard factorization is equivalent to writing a finite set of complex numbers with multiplicities as a union of pairwise non-interacting strings [13, Prop. 3.5].

Lemma 6.3. *Let $\mathbf{e} \in \mathcal{R}$. Standard factorizations of $\mathbf{e}$ as in Definition 6.1 exist, and they are unique in the sense that the two polynomials $(u-x_1)\cdots(u-x_m)$ and $(u-w_1)\cdots(u-w_k)$, and the pairs (y_s, z_s) for $1 \le s \le n$ up to $\mathfrak{S}_n$-permutations are completely determined by $\mathbf{e}$.*

Proof. We begin with some easy observations.

Observation 1. A standard factorization gives rise to the reduced form of the rational function $\mathbf{e}$: its numerator and denominator as monic polynomials are

$$(u-x_1)\cdots(u-x_m)(u-y_1)\cdots(u-y_n), \quad (u-z_1)\cdots(u-z_n)(u-w_1)\cdots(u-w_n).$$

This is because the two polynomials are coprime by Definition 6.1.

Observation 2. If for all zero a and pole b of $\mathbf{e}$ we have $b-a \notin \mathbb{N}$, then the reduced form of $\mathbf{e}$ is the unique standard factorization.

Observation 3. In Definition 6.1, deleting a factor of the form $u-x_r$, $\frac{u-y_s}{u-z_s}$, or $\frac{1}{u-w_t}$, one gets another standard factorization.

We prove the existence and uniqueness of standard factorization by induction on the number, denoted by $d(\mathbf{e})$, of zeros and poles of $\mathbf{e}$ counted with multiplicities.

Namely, $d(\mathbf{e})$ is the degree of the numerator plus that of the denominator. The initial case $d(\mathbf{e}) = 0$ is trivial since $\mathbf{e} = 1$.

Suppose $d(\mathbf{e}) > 0$. If $\mathbf{e}$ satisfies the hypothesis of Observation 2, then we conclude. Assume that there exist a zero y_0 and a pole z_0 of $\mathbf{e}$ such that $z_0 - y_0 \in \mathbb{N}$. Since there are finitely many such pairs, we assume further that

(H1) If y is a zero of $\mathbf{e}$ and z a pole, then $z - y \in \mathbb{N}$ implies $z - y \geq z_0 - y_0$.

Set $\mathbf{f} := \frac{u-z_0}{u-y_0}\mathbf{e}$. Since $\frac{u-y_0}{u-z_0}$ appears in the reduced form of $\mathbf{e}$, it cancels with the factor $\frac{u-z_0}{u-y_0}$ and we get $d(\mathbf{f}) = d(\mathbf{e}) - 2$. By the induction hypothesis, the existence and uniqueness of standard factorization hold for $\mathbf{f}$. Fix such a factorization:

$$\text{(F1)} \qquad \mathbf{f} = \prod_{r=1}^{m}(u - x_r) \times \prod_{s=1}^{n} \frac{u - y_s}{u - z_s} \times \prod_{t=1}^{k} \frac{1}{u - w_t}.$$

Step 1: Existence. We show that the following is a standard factorization:

$$\text{(F2)} \qquad \mathbf{e} = \prod_{r=1}^{m}(u - x_r) \times \prod_{s=0}^{n} \frac{u - y_s}{u - z_s} \times \prod_{t=1}^{k} \frac{1}{u - w_t}.$$

In view of Definition 6.1, it suffices to show that none of the following complex numbers belongs to $\Delta_{z_0}^{y_0}$ for $1 \leq r \leq m$, $1 \leq s \leq n$, and $1 \leq t \leq k$:

$$z_0 - x_r, \quad z_0 - y_s, \quad z_s - y_0, \quad w_t - y_0.$$

Let us prove it for the first number, the other three being parallel. Applying Observation 1 to the standard factorization of $\mathbf{e}$, and noting that $d(\mathbf{e}) = d(\mathbf{f}) + 2$, we see that the above factorization of $\mathbf{e}$ is reduced. In particular, x_r is a zero of $\mathbf{e}$. If $z_0 - x_r \in \Delta_{z_0}^{y_0}$, then by definition of $\Delta_{z_0}^{y_0} \subset \mathbb{N}$ we have $z_0 - x_r < z_0 - y_0$, in contradiction with hypothesis (H1) which forces $z_0 - x_r \geq z_0 - y_0$.

Step 2: Uniqueness. Let the following be a standard factorization:

$$\text{(F3)} \qquad \mathbf{e} = \prod_{r=1}^{m'}(u - x'_r) \times \prod_{s=0}^{n'} \frac{u - y'_s}{u - z'_s} \times \prod_{t=1}^{k'} \frac{1}{u - w'_t}.$$

We claim that $(y_0, z_0) = (y'_l, z'_l)$ for certain $0 \leq l \leq n'$. When this is the case, by Observation 3 we have another standard factorization of $\mathbf{f}$:

$$\text{(F4)} \qquad \mathbf{f} = \prod_{r=1}^{m'}(u - x'_r) \times \prod_{0 \leq s \leq n', s \neq l'} \frac{u - y'_s}{u - z'_s} \times \prod_{t=1}^{k'} \frac{1}{u - w'_t}.$$

Applying the induction hypothesis to (F1) and (F4), we get

$$\prod_{r=1}^{m'}(u - x'_r) = \prod_{r=1}^{m}(u - x_r), \quad \prod_{t=1}^{k}(u - w_t) = \prod_{t=1}^{k'}(u - w'_t),$$

and the pairs (y'_s, z'_s) for $0 \le s \le n'$, $s \ne l'$ are in one-to-one correspondence with the (y_s, z_s) for $1 \le s \le n$. In other words, the standard factorizations (F2) and (F3) are the same after permutation.

To prove the claim, notice by Observation 1 that

$$y_0 \in \{x'_1, \dots, x'_{m'}, y'_0, y'_1, \dots, y'_{n'}\}, \quad z_0 \in \{z'_0, z'_1, \dots, z'_{n'}, w'_1, \dots, w'_{k'}\}.$$

Applying (H1) to the zero y'_s and pole z'_s we get the following:

(H2) For $0 \le s \le n'$ we have $z_0 - y_0 \le z'_s - y'_s$.

In the standard factorization (F3), we have $w'_t - x'_r \notin \mathbb{N}$. So $z_0 - y_0 \in \mathbb{N}$ forces $(y_0, z_0) \ne (x'_r, w'_t)$ for $1 \le r \le m'$ and $1 \le t \le k'$. There remain three cases.

Case 1. We have $(y_0, z_0) = (x'_r, z'_s)$ for certain $1 \le r \le m'$ and $0 \le s \le n'$. Again in the standard factorization (F3) we have $z'_s - x'_r \notin \Delta^{y'_s}_{z'_s}$. Together with the assumption $z'_s - x'_r = z_0 - y_0 > 0$ and the definition of $\Delta^{y'_s}_{z'_s}$ we have $z'_s - x'_r = z_0 - y_0 \ge z'_s - y'_s$. In view of (H2), equality holds. Now $z_0 = z'_s$ forces $y_0 = y'_s$.

Case 2. We have $(y_0, z_0) = (y'_s, w'_t)$ for certain $0 \le s \le n'$ and $1 \le t \le k'$. Similar arguments to Case 1 show $(y_0, z_0) = (y'_s, z'_s)$.

Case 3. We have $(y_0, z_0) = (y'_s, z'_l)$ for certain $0 \le s, l \le n'$. From the condition $z_0 - y_0 = z'_l - y'_s \notin \Delta^{y'_s}_{z'_s} \cap \Delta^{y'_l}_{z'_l}$ imposed by (F3) we get either $z_0 - y_0 \ge z'_s - y'_s$ or $z_0 - y_0 \ge z'_l - y'_l$. In both situations, equality holds by (H2). Applying $y_0 = y'_s$ to the first situation and $z_0 = z'_l$ to the second situation, we get either $(y_0, z_0) = (y'_s, z'_s)$ or $(y_0, z_0) = (y'_l, z'_l)$. □

We point out that the existence arguments closely follow [55, Prop. 3.6]. It is the uniqueness that is the key point of Lemma 6.3.

Proposition 6.2 together with Lemma 6.3 implies the following.

Theorem 6.4. *Each simple module in category $\mathcal{O}^{\mathrm{sh}}$ factorizes uniquely as a tensor product of prefundamental modules $L_a^{\pm}$ ($a \in \mathbb{C}$) and of Kirillov–Reshetikhin modules L_b^a ($a, b \in \mathbb{C}$, $0 < a - b \in \mathbb{N}$).*

Example 6.5. Let us revisit the example after Theorem 3.18,

$$L_0^9 \otimes L_2^3 \cong L_2^9 \otimes L_0^3 \cong L_9^+ \otimes L_3^+ \otimes L_0^- \otimes L_2^-.$$

In the subcategory $\mathcal{O}_0$ of $\mathcal{O}^{\mathrm{sh}}$, the irreducible module L_b^a for $b - a \notin \mathbb{N}$ is prime in the sense that if $L_b^a \cong V \otimes W$ in category $\mathcal{O}_0$ then either V or W is the one-dimensional trivial module. The first isomorphism forms two non-equivalent factorizations into primes of the same irreducible module. The issue of non-uniqueness is resolved in category $\mathcal{O}^{\mathrm{sh}}$ by further factorizing $L_b^a \cong L_a^+ \otimes L_b^-$.

§7. Computation of diagonal entries

In this section we compute the diagonal entries introduced in Definition 5.4 for the R-matrix $\check{R}_{V,W}(u)$, where V is a finite-dimensional irreducible module and W is a negative module (Proposition 7.2 and Theorem 7.4). A technical point in the proofs is a refined estimation of the coproduct that we establish in Lemma 7.1.

§7.1. Second coproduct estimation

As a preparatory step, we refine the coproduct estimation of Lemma 2.5 for the Drinfeld–Cartan series $\xi_i(u)$. In the ordinary Yangian $Y(\mathfrak{g})$ the following relations hold:

$$[x_j^+(u), x_{j,0}^-] = [x_{j,0}^+, x_j^-(u)] = \langle \xi_j(u)\rangle_+ = \xi_j(u) - 1, \tag{7.1}$$

$$[\xi_i(u), x_{j,0}^-] = -2d_{ij}\xi_i(u)x_j^-(u-d_{ij}) = -2d_{ij}x_j^-(u+d_{ij})\xi_i(u). \tag{7.2}$$

The first relation follows from (2.1). The second is obtained by taking specializations $v = u + d_{ij}$ and $v = u - d_{ij}$ of the relation [28, §2.4]

$$(u - v + d_{ij})\xi_i(u)x_j^-(v) - (u - v - d_{ij})x_j^-(v)\xi_i(u) = -[\xi_i(u), x_{j,0}^-].$$

Lemma 7.1. *For all coweights μ and ν, the coproduct $\Delta_{\mu,\nu}$ satisfies*

$$\Delta_{\mu,\nu}(\xi_i(u)) \equiv \xi_i(u)\otimes\xi_i(u) - \sum_{j\in I} 2d_{ij}x_j^-(u+d_{ij})\xi_i(u)\otimes\xi_i(u)x_j^+(u+d_{ij})$$
$$\mod. \sum_{h(\beta)\geq 2} Y_\mu^-(\mathfrak{g})_{-\beta}\otimes Y_\nu^+(\mathfrak{g})_\beta.$$

Proof. One adapts the zigzag arguments of [20, Thm. 4.12] to reduce to the case $\mu = \nu = 0$, as in the proof of Lemma 2.5. For $\alpha\in\mathbf{Q}$, let π_α denote the projection of $Y(\mathfrak{g})$ onto the weight space $Y(\mathfrak{g})_\alpha$. It suffices to prove for $i, j\in I$,

$$(\pi_{-\alpha_j}\otimes\pi_{\alpha_j})\circ\Delta(\xi_i(u)) = -2d_{ij}x_j^-(u+d_{ij})\xi_i(u)\otimes\xi_i(u)x_j^+(u+d_{ij}). \tag{7.3}$$

The strategy is to produce a system of linear equations which will have a unique solution, given by both sides of this equation. Let

$$A(u) := \sum_{p\geq -1} A_p u^{-p-1}$$

denote the power series on the left-hand side. We view the coefficients A_p as elements in $Y(\mathfrak{g})_{-\alpha_j}\otimes Y(\mathfrak{g})_{\alpha_j}$. From equation (2.15) we get $A_{-1} = A_0 = 0$.

Next, applying Δ to the second formula of $[\xi_i(u), x_{j,0}^-]$ from equation (7.2), taking into account $\Delta(x_{j,0}^-) = 1\otimes x_{j,0}^- + x_{j,0}^-\otimes 1$ and Lemma 2.5,

$$\Delta(x_j^-(u)) \equiv 1\otimes x_j^-(u) + x_j^-(u)\otimes\xi_j(u) \mod. \sum_{h(\beta)>0} Y(\mathfrak{g})_{-\beta-\alpha_j}\otimes Y(\mathfrak{g})_\beta,$$

after projection onto the weight space of bi-weight $(-\alpha_j, 0)$ we obtain

$$[\xi_i(u) \otimes \xi_i(u), x_{j,0}^- \otimes 1] + [A(u), 1 \otimes x_{j,0}^-]$$
$$= -2d_{ij}(1 \otimes x_j^-(u + d_{ij}))A(u) - 2d_{ij}x_j^-(u + d_{ij})\xi_i(u) \otimes \xi_j(u + d_{ij})\xi_i(u).$$

Making use of equations (7.1)–(7.2) we simplify the equality as follows:

$$\begin{aligned} &[A(u), 1 \otimes x_{j,0}^-] + (2d_{ij} \otimes x_j^-(u + d_{ij}))A(u) \\ &\quad = -2d_{ij}x_j^-(u + d_{ij})\xi_i(u) \otimes \langle \xi_j(u + d_{ij})\rangle_+ \xi_i(u). \end{aligned} \tag{7.4}$$

Equation (7.4) forms a linear system whose unknown variables are the A_p for $p \geq -1$. It expresses $[A_p, 1 \otimes x_{j,0}^-]$ in terms of the A_m for $m < p$. Therefore, the system has a unique solution provided that the following linear map is injective:

$$Y(\mathfrak{g})_{-\alpha_j} \otimes Y(\mathfrak{g})_{\alpha_j} \longrightarrow Y(\mathfrak{g})_{-\alpha_j} \otimes Y(\mathfrak{g})_0, \quad a \mapsto [a, 1 \otimes x_{j,0}^-].$$

This map is the restriction of $-\mathrm{Id} \otimes \mathrm{ad}_{x_{j,0}^-}$. It suffices to establish the injectivity of $\mathrm{ad}_{x_{j,0}^-}$ restricted to $Y(\mathfrak{g})_{\alpha_j}$. Note that $x_{j,0}^-$, $\frac{1}{d_j}\xi_{j,0}$, $\frac{1}{d_j}x_{j,0}^+$ span a sub-Lie algebra of $Y(\mathfrak{g})$ isomorphic to sl_2. The adjoint action of sl_2 on $Y(\mathfrak{g})$ is integrable by the Serre relation (2.4). If $w \in \ker(\mathrm{ad}_{x_{j,0}^-}) \cap Y(\mathfrak{g})_{\alpha_j}$ is non-zero, then w is a vector of lowest weight $\frac{1}{d_j}(\alpha_j, \alpha_j) = 2$, contradicting the integrable representation theory of sl_2.

It remains to show that the right-hand side is a solution to equation (7.4). This follows from equations (7.1)–(7.2) and commutativity of the $\xi_i(u)$. □

When $\mathfrak{g} = \mathrm{sl}_2$, Lemma 7.1 agrees with the term $k = 1$ of the coproduct formula of $\Delta(h(u))$ in [56, Def. 2.24] and [20, eq. (6.9)].

§7.2. Highest diagonal entry

Let V be an irreducible module in category $\mathcal{O}^{\mathrm{sh}}$ and W be a negative module as in Definition 5.4. We identify the highest diagonal entry $s_{V,W}(u)$ with R-matrices of Proposition 5.7. When V is finite-dimensional, this leads to a formula for the polynomial $\lambda_{V,W}(u)$ in terms of ℓ-weights of V and W.

Proposition 7.2. *Let $\mathbf{r} = \Psi_{i_1,a_1}\Psi_{i_2,a_2}\cdots\Psi_{i_N,a_N} \in \mathcal{D}$ and V be a highest ℓ-weight irreducible module over $Y_\nu(\mathfrak{g})$. Then the highest diagonal entry of $\check{R}_{V,L(\mathbf{r}^{-1})}(u)$ is*

$$s_{V,L(\mathbf{r}^{-1})}(u) = R_{i_1}^V(a_1 - u)R_{i_2}^V(a_2 - u)\cdots R_{i_N}^V(a_N - u).$$

Proof. Write $W = L(\mathbf{r}^{-1})$ and $\omega = \omega_{\mathbf{r}^{-1}}$. Choose a highest ℓ-weight vector v_0 of V. Since both sides are linear maps from V to $V \otimes \mathbb{C}[u]$, it suffices to prove the equality specialized at an arbitrary complex number $a \in \mathbb{C}$.

Set $v = v_0$ in equation (5.2). Since $\check{R}_{V,W}(a)$ sends $v_0 \otimes \omega$ to $\omega \otimes v_0$, we get

$$s_{V,W}(a)v_0 = v_0.$$

Next, for $i \in I$ and $v' \in V$, from equation (4.4) we obtain the following relation in the module $V(a) \otimes W$:

$$x_i^-(u)v' \otimes \omega = \langle \mathbf{r}_i(u)x_i^-(u)\rangle_+(v' \otimes \omega).$$

Applying the module morphism $\check{R}_{V,W}(a)\colon V(a) \otimes W \to W \otimes V(a)$ gives

$$\check{R}_{V,W}(a)(x_i^-(u)v' \otimes \omega) = \langle \mathbf{r}_i(u)x_i^-(u)\rangle_+\check{R}_{V,W}(a)(v' \otimes \omega).$$

We compute the components of $\omega \otimes V(a)$ at both sides. By definition, the left-hand side is $\omega \otimes s_{V,W}(a)x_i^-(u)v'$. On the right-hand side, $\check{R}_{V,W}(a)(v' \otimes \omega)$ is $\omega \otimes s_{V,W}(a)v'$ plus a linear combination of vectors in $W_\gamma \otimes V(a)$, where $\gamma \in \mathrm{wt}(\omega) + \mathbf{Q}_-$ and $\gamma \neq \mathrm{wt}(\omega)$. By Lemma 2.5, the coproduct of $x_i^-(u)$ is $1 \otimes x_i^-(u)$ plus a linear combination of tensor products of elements of shifted Yangians such that the weight of each first tensor factor is in $\mathbf{Q}_- \setminus \{0\}$. For weight reasons the desired component is

$$\omega \otimes \langle \mathbf{r}_i(u)x_i^-(u)\rangle_+ s_{V,W}(a)v'.$$

So we have the following equality in the module $V(a)$:

$$s_{V,W}(a)x_i^-(u)v' = \langle \mathbf{r}_i(u)x_i^-(u)\rangle_+ s_{V,W}(a)v'.$$

In the module V the equality becomes

$$s_{V,W}(a)x_i^-(u-a)v' = \langle \mathbf{r}_i(u)x_i^-(u-a)\rangle_+ s_{V,W}(a)v'.$$

Replacing u by $u + a$, we obtain the following commutation relation in the module V:

$$s_{V,W}(a)x_i^-(u) = \langle \mathbf{r}_i(u+a)x_i^-(u)\rangle_+ s_{V,W}(a) \quad \text{for } i \in I.$$

Combining with $s_{V,W}(a)v_0 = v_0$ we obtain all the defining properties of the operator $R^V_{\tau_{-a}(\mathbf{r})}$ in equation (5.4). Here we recall from Remark 3.1 the one-parameter family of group automorphisms $\tau_b\colon \mathcal{L} \to \mathcal{L}$ for $b \in \mathbb{C}$. By uniqueness, $s_{V,W}(a) = R^V_{\tau_{-a}(\mathbf{r})}$ and

$$s_{V,W}(a) = R^V_{\Psi_{i_1,a_1-a}\Psi_{i_2,a_2-a}\cdots\Psi_{i_N,a_N-a}} = R^V_{i_1}(a_1-a)R^V_{i_2}(a_2-a)\cdots R^V_{i_N}(a_N-a).$$

This is exactly the equality of the proposition evaluated at $u = a$. □

Proposition 7.2 implies that $R_i^V(-u)$ is the highest diagonal entry of $\check{R}_{V,L^-_{i,0}}(u)$. This is dual to the Borel situation of Remark 5.9(ii).

Theorem 7.3. *Let* $\mathbf{s} \in \mathcal{D}$ *and* V *be a finite-dimensional irreducible module in category* $\mathcal{O}^{\mathrm{sh}}$. *Write the ratio of the highest* ℓ*-weight to the lowest* ℓ*-weight of* V *as a monomial of the* $A_{j,b}$*, and replace each* $A_{j,b}$ *with the polynomial* $\mathbf{s}_j(u+b)$. *Then we obtain* $\lambda_{V,L(\mathbf{s}^{-1})}(u)$. *In particular, the polynomial* $\lambda_{V,L(\mathbf{s}^{-1})}(u) \in \mathbb{C}[u]$ *is monic.*

Proof. Setting v in equation (5.2) to be a lowest ℓ-weight vector v_- of V, and $w = \omega_{\mathbf{s}^{-1}}$ in equation (5.3), we see that $\lambda_{V,L(\mathbf{s}^{-1})}(u)$ is also the eigenvalue of the highest diagonal entry $s_{V,L(\mathbf{s}^{-1})}(u)\colon V \to V \otimes \mathbb{C}[u]$ associated to the lowest ℓ-weight vector v_- of V.

Let $\prod_{i\in I}\prod_{t=1}^{h_i} A_{i,a_{i,t}}$ be the ratio of highest ℓ-weight to lowest ℓ-weight of V. By Proposition 7.2, each factor $u-b$ of the polynomial $\mathbf{s}_i(u)$ gives rise to a factor $R_i^V(b-u)$ of $s_{V,L(\mathbf{s}^{-1})}(u)$. The eigenvalue of $R_i^V(b-u)$ associated to v_- is $\prod_{t=1}^{h_i}(a_{i,t}-b+u)$ by Proposition 5.8(ii). Each component $\mathbf{s}_i(u)$ of $\mathbf{s}$ gives rise to a factor $\prod_{t=1}^{h_i}\mathbf{s}_i(u+a_{i,t})$ of the eigenvalue of $s_{V,L(\mathbf{s}^{-1})}(u)$. After taking the product over all $i \in I$, we get

$$\lambda_{V,L(\mathbf{s}^{-1})}(u) = \prod_{i\in I}\prod_{t=1}^{h_i}\mathbf{s}_i(u+a_{i,t}). \qquad \square$$

§7.3. Lowest diagonal entry

We express the lowest diagonal entry $t_{V,W}(u)$ from Definition 5.4 in terms of one-dimensional R-matrices from Proposition 5.7, assuming that V is a finite-dimensional irreducible module over the ordinary Yangian. The idea is to reduce to the case of a fundamental module by a fusion construction of R-matrices, which appeared for example in [37, Cor. 5.5].

Theorem 7.4. *Let* W *be a negative module and* V *be a finite-dimensional irreducible* $Y(\mathfrak{g})$*-module whose lowest* ℓ*-weight is*

$$Y^{-1}_{i_1,a_1+\frac{1}{2}d_{i_1}} Y^{-1}_{i_2,a_2+\frac{1}{2}d_{i_2}} \cdots Y^{-1}_{i_m,a_m+\frac{1}{2}d_{i_m}}.$$

Then we have the following equality in $\mathrm{Hom}(W, W\otimes\mathbb{C}[u])$*:*

$$t_{V,W}(u)\prod_{s=1}^{m} R^W_{i_s}(u+a_s) = \lambda_{V,W}(u)\prod_{s=1}^{m} R^W_{i_s}(u+a_s+d_{i_s}). \tag{7.5}$$

Moreover, $t_{V,W}(u)$ *is an* $\mathrm{End}(W)$*-valued monic polynomial of degree* $\deg\lambda_{V,W}(u)$.

Proof. Let ω denote a highest ℓ-weight vector of the module W defined over $Y_\mu(\mathfrak{g})$. Since W is an irreducible module in category $\mathcal{O}_\mu$, by Proposition 5.8(ii) there exists a countable subset $\Gamma \subset \mathbb{C}$ such that the normalized q-character of W is a power series in the $A_{j,b}^{-1}$ with $(j,b) \in I \times \Gamma$. Notably, $a \in \mathbb{C} \setminus \Gamma$ and $\mathbf{f} \in \mathrm{wt}_\ell(W)$ imply $A_{i,a}^{-1}\mathbf{f} \notin \mathrm{wt}_\ell(W)$.

Restricted to each weight space W_β of W, by Propositions 5.3 and 5.7, $R_i^W(-u)$ is a monic polynomial of u and $t_{V,W}(u)$ is a polynomial. Equation (7.5) implies that the degree of $t_{V,W}(u)|_{W_\beta}$ is $\deg \lambda_{V,W}(u)$, which is independent of the weight β. This proves the second part of the theorem assuming equation (7.5). By polynomiality, it suffices to prove equation (7.5) evaluated at $u = a$ for $a \in \mathbb{C}$ such that $a + a_s \notin \Gamma$ for all $1 \leq s \leq m$.

Step 1: Reduction to the fundamental case. For $1 \leq s \leq m$, let v_s and v_-^s denote a highest ℓ-weight vector and a lowest ℓ-weight vector of the fundamental module V_{i_s} of equation (3.4). By Theorem 3.8(i), one may assume, after a permutation of the pairs (i_s, a_s), that V is the irreducible submodule of the tensor product

$$T \coloneqq V_{i_1}(a_1) \otimes V_{i_2}(a_2) \otimes \cdots \otimes V_{i_m}(a_m)$$

generated by $v_0 \coloneqq v_1 \otimes v_2 \otimes \cdots \otimes v_m$ and $v_- \coloneqq v_-^1 \otimes v_-^2 \otimes \cdots \otimes v_-^m$. In particular, v_0 and v_- are the highest ℓ-weight vector and lowest ℓ-weight vector of V.

From the fact that τ_b is a Hopf algebra automorphism of $Y(\mathfrak{g})$ and from the equation $\tau_b\tau_c = \tau_{b+c}$, for $b, c \in \mathbb{C}$, we get an identification of modules

$$T(a) = V_{i_1}(a_1 + a) \otimes V_{i_2}(a_2 + a) \otimes \cdots \otimes V_{i_m}(a_m + a).$$

Consider the composite map

$$R(a) = \prod_{s=1}^{m} (1^{\otimes s-1} \otimes \check{R}_{V_{i_s},W}(a + a_s) \otimes 1^{\otimes m-s})\colon T(a) \otimes W \longrightarrow W \otimes T(a).$$

Since all modules are defined over antidominantly shifted Yangians, it follows from the monoidality of category $\mathcal{O}_-^{\mathrm{sh}}$ in Remark 3.13 that $R(a)$ is a module morphism from $T(a) \otimes W$ to $W \otimes T(a)$. It restricts to a module morphism from $V(a) \otimes W$ to $W \otimes V(a)$ because the former is generated by the highest ℓ-weight vector $v_0 \otimes \omega$ by Theorem 4.8(iii) and $R(a)$ sends $v_0 \otimes \omega$ to $\omega \otimes v_0$ by definition of the $\check{R}_{V_{i_s},W}(a+a_s)$. From uniqueness of the R-matrix in Theorem 5.2, we obtain

$$R(a)|_{V(a)\otimes W} = \check{R}_{V,W}(a).$$

For $w \in W$, by definition $t_{V,W}(a)w \otimes v_-$ is the projection to $W \otimes v_-^1 \otimes v_-^2 \otimes \cdots \otimes v_-^m$ of the vector $R(a)(v_-^1 \otimes v_-^2 \otimes \cdots \otimes v_-^m \otimes w)$, which by the factorization of $R(a)$ is

$$t_{V_{i_1},W}(a + a_1) t_{V_{i_2},W}(a + a_2) \cdots t_{V_{i_m},W}(a + a_m) w \otimes v_-^1 \otimes v_-^2 \otimes \cdots \otimes v_-^m.$$

We obtain therefore a factorization

$$t_{V,W}(a) = t_{V_{i_1},W}(a + a_1) t_{V_{i_2},W}(a + a_2) \cdots t_{V_{i_m},W}(a + a_m) \in \mathrm{End}(W).$$

We are reduced to computing the lowest diagonal entry $t_{V_i,W}(u)$ for a fixed $i \in I$. Assume from now on that $m = 1$, $(i_1, a_1) = (i, 0)$, and $a \in \mathbb{C} \setminus \Gamma$.

By Lemma 3.9, the finite-dimensional $Y(\mathfrak{g})$-module V contains a $\mathfrak{g}$-submodule of lowest weight $-\varpi_i$ and $V_{-\varpi_i} = \mathbb{C}v_-$. By Weyl group symmetry, $v_+ := x_{i,0}^+ v_-$ spans the weight space $V_{\alpha_i - \varpi_i}$. Complete $v_\pm$ to a weight basis $\mathcal{B}_V$ of V. If $v \in \mathcal{B}_V \setminus \{v_-, v_+\}$, then $0 \neq \mathrm{wt}(v) - \alpha_i + \varpi_i \in \mathbf{Q}_+$. Since V is defined over the ordinary Yangian, by Remark 3.1 it has the same weight grading as $V(a)$. We have the following relations in the module $V(a)$ similar to Example 3.7:

$$
\begin{aligned}
x_j^+(u)v_- &= \frac{\delta_{ij}}{u-a}v_+, \quad & \xi_j(u)v_+ &= \frac{u-a-d_i\delta_{ij}}{u-a}\frac{u-a+d_{ij}}{u-a-d_{ij}}v_+, \\
x_j^-(u)v_+ &= \frac{d_i\delta_{ij}}{u-a}v_-, \quad & \xi_j(u)v_- &= \frac{u-a-d_i\delta_{ij}}{u-a}v_-.
\end{aligned} \tag{7.6}
$$

From the paragraph above Definition 5.4, we get a family of linear operators $f_v(u)\colon W \to W \otimes \mathbb{C}[u]$ for $v \in \mathcal{B}_V$ such that for $a \in \mathbb{C}$ and $w \in W$ we have

$$\check{R}_{V,W}(a)(v_- \otimes w) = \sum_{v \in \mathcal{B}_V} f_v(a)w \otimes v.$$

In particular, $f_{v_-}(u) = t_{V,W}(u)$ by equation (5.2). To simplify notation, write

$$
\begin{gathered}
R_a := \check{R}_{V,W}(a), \quad \lambda_a := \lambda_{V,W}(a), \quad C_a := f_{v_+}(a), \quad D_a := f_{v_-}(a), \\
\widetilde{C}_a := R^W_{\Psi_{i,a}} C_a, \quad \widetilde{D}_a := R^W_{\Psi_{i,a}} D_a, \quad \widetilde{E}_a := \widetilde{C}_a + x_{i,0}^- \widetilde{D}_a.
\end{gathered}
$$

Our goal is to show that for $a \in \mathbb{C} \setminus \Gamma$ we have $\widetilde{D}_a = \lambda_a R^W_{\Psi_{i,a+d_i}}$ as linear operators on W. By polynomiality, this would imply that $R_i^W(u)t_{V,W}(u) = \lambda_{V,W}(u)R_i^W(u+d_i)$. From Proposition 5.7 we get that the operators $t_{V,W}(u)$, $R_i^W(u)$ for $i \in I$ acting on W mutually commute, so that the order of the products in equation (7.5) does not matter.

Step 2: Projection formulas. . The $Y(\mathfrak{g})$-modules V and $V(a)$ have the same weight grading. With respect to the weight basis $\mathcal{B}_V$ of $V(a)$, let

$$\phi\colon W \otimes V(a) \longrightarrow W \quad \text{and} \quad \psi\colon W \otimes V(a) \longrightarrow W$$

denote linear maps which send $\sum_{v \in \mathcal{B}_V} g_v \otimes v$ to g_{v_-} and to g_{v_+} respectively. We apply these maps to the following relations in $W \otimes V(a)$ for $j \in I$ and $w \in W$:

$$R_a x_j^-(u)(v_- \otimes w) = x_j^-(u)R_a(v_- \otimes w), \quad R_a \xi_j(u)(v_- \otimes w) = \xi_j(u)R_a(v_- \otimes w).$$

By Example 3.4, the left-hand sides of the above equations are $R_a(v_- \otimes x_j^-(u)w)$ and $R_a(\xi_j(u)v_- \otimes \xi_j(u)w)$ respectively. Based on the coproduct estimations of

$\Delta_{\mu,0}(x_j^-(u))$ from Lemma 2.5 and of $\Delta_{\mu,0}(\xi_j(u))$ from Lemma 7.1, we have

$$\begin{aligned}
\phi(R_a x_j^-(u)(v_-\otimes w)) &= \phi(R_a(v_-\otimes x_j^-(u)w)) = D_a x_j^-(u)w,\\
\phi(x_j^-(u)R_a(v_-\otimes w)) &= \phi\Big(x_j^-(u)\Big(\sum_{v\in\mathcal{B}_V} f_v(a)w\otimes v\Big)\Big)\\
&= \phi(x_j^-(u)D_a(w)\otimes \xi_j(u)v_-) + \phi(C_a(w)\otimes x_j^-(u)v_-)\\
&= \frac{u-a-d_i\delta_{ij}}{u-a}x_j^-(u)D_a(w) + \frac{d_i\delta_{ij}}{u-a}C_a(w),\\
\phi(R_a\xi_j(u)(v_-\otimes w)) &= \phi\big(R_a(\xi_j(u)v_-\otimes \xi_j(u)w)\big) = \frac{u-a-d_i\delta_{ij}}{u-a}D_a\xi_j(u)w,\\
\phi(\xi_j(u)R_a(v_-\otimes w)) &= \phi\Big(\xi_j(u)\Big(\sum_{v\in\mathcal{B}_V} f_v(a)w\otimes v\Big)\Big)\\
&= \phi(\xi_j(u)D_a(w)\otimes\xi_j(u)v_-) = \frac{u-a-d_i\delta_{ij}}{u-a}\xi_j(u)D_a(w),\\
\psi(R_a\xi_j(u)(v_-\otimes w)) &= \psi\big(R_a(\xi_j(u)v_-\otimes \xi_j(u)w)\big) = \frac{u-a-d_i\delta_{ij}}{u-a}C_a\xi_j(u)w,\\
\psi(\xi_j(u)R_a(v_-\otimes w)) &= \psi\Big(\xi_j(u)\Big(\sum_{v\in\mathcal{B}_V} f_v(a)w\otimes v\Big)\Big)\\
&= \psi(\xi_j(u)C_a(w)\otimes \xi_j(u)v_+)\\
&\quad - 2d_{ij}\psi(x_i^-(u+d_{ij})\xi_j(u)D_a(w)\otimes\xi_j(u)x_i^+(u+d_{ij})v_-)\\
&= \frac{u-a-d_i\delta_{ij}}{u-a}\frac{u-a+d_{ij}}{u-a-d_{ij}}\xi_j(u)C_a(w)\\
&\quad - \frac{u-a-d_i\delta_{ij}}{u-a}\frac{2d_{ij}}{u-a-d_{ij}}x_i^-(u+d_{ij})\xi_j(u)D_a(w).
\end{aligned}$$

Step 3: Commutation relations. It follows from the projection formulas that

$$D_a x_j^-(u) = \frac{u-a-d_i\delta_{ij}}{u-a}x_j^-(u)D_a + \frac{d_i\delta_{ij}}{u-a}C_a, \tag{7.7}$$

$$D_a\xi_j(u) = \xi_j(u)D_a, \tag{7.8}$$

$$C_a\xi_j(u) = \frac{u-a+d_{ij}}{u-a-d_{ij}}\xi_j(u)C_a - \frac{2d_{ij}}{u-a-d_{ij}}x_i^-(u+d_{ij})\xi_j(u)D_a. \tag{7.9}$$

Notice that $x_i^-(u+d_{ij})\xi_j(u) = \xi_j(u)x_i^-(u-d_{ij})$ as a rewriting of the second half of equation (7.2). Left multiplying equation (7.9) by $\xi_j(u)^{-1}$, we obtain

$$\xi_j(u)^{-1}C_a\xi_j(u) = \frac{u-a+d_{ij}}{u-a-d_{ij}}C_a - \frac{2d_{ij}}{u-a-d_{ij}}x_i^-(u-d_{ij})D_a.$$

Left multiplying the above equation by $R^W_{\Psi_{i,a}}$, which commutes with $\xi_j(u)$, we get

$$\begin{aligned}\xi_j(u)^{-1}\widetilde{C}_a\xi_j(u) &= \frac{u-a+d_{ij}}{u-a-d_{ij}}\widetilde{C}_a - \frac{2d_{ij}}{u-a-d_{ij}}\langle (u-a-d_{ij})x_i^-(u-d_{ij})\rangle_+\widetilde{D}_a \\ &= \frac{u-a+d_{ij}}{u-a-d_{ij}}(\widetilde{C}_a + x_{i,0}^-\widetilde{D}_a) - 2d_{ij}x_i^-(u-d_{ij})\widetilde{D}_a - x_{i,0}^-\widetilde{D}_a.\end{aligned}$$

Combining with equation (7.8) and the relation $\xi_j(u)^{-1}x_{i,0}^-\xi_j(u) = x_{i,0}^- + 2d_{ij}x_i^-(u-d_{ij})$, which rewrites the first half of equation (7.2), we obtain

$$\xi_j(u)^{-1}\widetilde{E}_a\xi_j(u) = \frac{u-a+d_{ij}}{u-a-d_{ij}}\widetilde{E}_a. \tag{7.10}$$

Recall that $a \in \mathbb{C}\setminus\Gamma$. Let $w \in W$ be a vector of ℓ-weight $\mathbf{f}$. If the vector $\widetilde{E}_a(w)$ is non-zero, then it is of ℓ-weight $A_{i,a}^{-1}\mathbf{f}$ by equations (7.10) and (3.3), contradicting our choice of Γ. So $\widetilde{E}_a(w) = 0$ for all ℓ-weight vectors $w \in W$ and $\widetilde{E}_a = 0$.

Let us multiply equation (7.7) by $R^W_{\Psi_{i,a}}$. Making use of the defining properties equation (5.4), we recover all the defining properties of $\lambda_a R^W_{\Psi_{i,a+d_i}}$:

$$\begin{gathered}\widetilde{D}_a(\omega) = \lambda_a\omega, \quad \widetilde{D}_a x_j^-(u) = x_j^-(u)\widetilde{D}_a \quad \text{for } j \neq i, \\ \widetilde{D}_a x_i^-(u) = \langle (u-a-d_i)x_i^-(u)\rangle_+\widetilde{D}_a + \frac{d_i}{u-a}\widetilde{E}_a = \langle (u-a-d_i)x_i^-(u)\rangle_+\widetilde{D}_a.\end{gathered}$$

This proves the desired identity $\widetilde{D}_a = \lambda_a R^W_{\Psi_{i,a+d_i}}$ for $a \in \mathbb{C}\setminus\Gamma$. □

Example 7.5. Let $\mathfrak{g} = \mathrm{sl}_2$. Consider $\check{R}_{V,W}(a)$ with $V = N_0$ and $W = L_0^-$. Example 5.6 gives $\lambda_{V,W}(u) = u$. Furthermore, v_i is an eigenvector of $R_1^W(u)$ of eigenvalue $-u(-u-1)(-u-2)\cdots(-u-i+1)$ by Proposition 5.8(ii) and Example 3.19. From equation (7.5) and its proof we obtain that

$$\begin{aligned}D_a(v_i) &= a\frac{R_1^W(a+1)}{R_1^W(a)}v_i = a\frac{(a+1)(a+2)\cdots(a+i)}{a(a+1)\cdots(a+i-1)}v_i = (a+i)v_i, \\ C_a(v_i) &= -R_1^W(a)^{-1}x_0^-R_1^W(a)t^W_{e_2,e_2}(a)v_i = (i+1)v_{i+1}.\end{aligned}$$

This gives

$$\check{R}_{V,W}(a)(e_2\otimes v_i) = (a+i)v_i\otimes e_2 + (i+1)v_{i+1}\otimes e_1.$$

Apply x_0^+ to the equality and notice that $\Delta_{0,-1}(x_0^+) = x_0^+\otimes 1 + 1\otimes x_0^+$ and $\Delta_{-1,0}(x_0^+) = x_0^+\otimes 1$:

$$\begin{aligned}x_0^+(e_2\otimes v_i) &= x_0^+e_2\otimes v_i + e_2\otimes x_0^+v_i = e_1\otimes v_i + e_2\otimes v_{i-1}, \\ x_0^+(v_i\otimes e_2) &= x_0^+v_i\otimes e_2 = v_{i-1}\otimes e_2, \\ x_0^+(v_{i+1}\otimes e_1) &= x_0^+v_{i+1}\otimes e_1 = v_i\otimes e_1.\end{aligned}$$

We obtain from the commutativity of $\check{R}_{V,W}(a)$ with x_0^+ that

$$\check{R}_{V,W}(a)(v_i \otimes e_1 + e_2 \otimes v_{i-1}) = (a+i)v_{i-1} \otimes e_2 + (i+1)v_i \otimes e_1,$$
$$\check{R}_{V,W}(a)(v_i \otimes e_1) = e_1 \otimes v_i + e_2 \otimes v_{i-1}.$$

With respect to the basis (e_1, e_2) of V we get

$$\check{R}_{V,W}(u) = \begin{pmatrix} 1 & \mathbf{a}^+ \\ \mathbf{a}^- & u + \mathbf{a}^+\mathbf{a}^- \end{pmatrix} \quad \text{where } \begin{cases} \mathbf{a}^+(v_i) = (i+1)v_{i+1}, \\ \mathbf{a}^-(v_i) = v_{i-1}. \end{cases}$$

This is a monodromy matrix of Baxter's Q-operator for $Y(\mathrm{gl}_2)$; see [4, eq. (3.38)]. The Yang–Baxter equation [4, eq. (3.1)] is a particular case of equation (5.1).

In Definition 5.4, we think of $\check{R}_{V,W}(u)$ as a monodromy matrix. Taking a suitable trace over V gives a transfer matrix acting on W, and $t_{V,W}(u)$ is a leading term of the transfer matrix. We expect equation (7.5) to be a leading term of generalized Baxter relations for transfer matrices [24, Thm. 5.11].

§8. Truncations of standard modules

In this section, we prove that any standard module (and so any irreducible module) in category $\mathcal{O}^{\mathrm{sh}}$ factorizes through a truncated shifted Yangian (Theorem 8.4). Our proof is uniform for all finite types (see the introduction for a discussion of known results).

From equation (3.4) and the paragraph above recall the involution $i \mapsto \bar{\imath}$ on I, the rational number κ, and the fundamental module V_i. From Definition 5.4 and Theorem 7.3 recall the monic polynomial $\lambda_{V,W}(u)$.

Definition 8.1. Let $\mathbf{r} \mapsto \tilde{\mathbf{r}}$ denote the group automorphism on $\mathcal{R}$ which sends each generator $\Psi_{i,a}$ to $\Psi_{\bar{\imath},a+\kappa}$. For $\mathbf{s} \in \mathcal{D}$, define the polynomial ℓ-weight $(g_i^{\mathbf{s}}(u))_{i\in I} \in \mathcal{D}$ and the rational ℓ-weight $\bar{\mathbf{s}} = (\bar{\mathbf{s}}_i(u))_{i\in I} \in \mathcal{R}$ as follows:

$$g_i^{\mathbf{s}}(u) := \lambda_{V_i, L(\mathbf{s}^{-1})}(u),$$
$$\bar{\mathbf{s}}_i(u) := \frac{g_i^{\mathbf{s}}(u) g_i^{\mathbf{s}}(u-d_i)}{\mathbf{s}_i(u)} \prod_{j: c_{ji}<0} \prod_{t=1}^{-c_{ji}} \frac{1}{g_j^{\mathbf{s}}(u - d_{ij} - t d_j)}.$$

Example 8.2. We describe the map $\mathbf{s} \mapsto \bar{\mathbf{s}}$ of Definition 8.1 for $\mathfrak{g}$ of type B_2. Let α_1 be the long root and α_2 the short root so that $d_1 = 2$, $d_2 = 1$, and $d_{12} = -1$. The dual Coxeter number is 3 and so $\kappa = 3$. The Dynkin diagram automorphism $i \mapsto \bar{\imath}$ is the identity. We have $V_1 = L(Y_{1,-2})$ and $V_2 = L(Y_{2,-\frac{5}{2}})$. The ratios of highest to lowest ℓ-weights for V_1 and V_2 are given by

$$A_{1,0}A_{1,-1}A_{2,0}A_{2,-1}, \quad A_{1,-1}A_{2,0}A_{2,-2}.$$

We obtain from Theorem 7.3 and Definition 8.1 that for $\mathbf{s} \in \mathcal{D}$,

$$g_1^{\mathbf{s}}(u) = \mathbf{s}_1(u-1)\mathbf{s}_1(u)\mathbf{s}_2(u-1)\mathbf{s}_2(u), \quad g_2^{\mathbf{s}}(u) = \mathbf{s}_1(u-1)\mathbf{s}_2(u-2)\mathbf{s}_2(u),$$
$$\bar{\mathbf{s}}_1(u) = \frac{g_1^{\mathbf{s}}(u)g_1^{\mathbf{s}}(u-2)}{\mathbf{s}_1(u)g_2^{\mathbf{s}}(u)g_2^{\mathbf{s}}(u-1)} = \mathbf{s}_1(u-3), \quad \bar{\mathbf{s}}_2(u) = \frac{g_2^{\mathbf{s}}(u)g_2^{\mathbf{s}}(u-1)}{\mathbf{s}_2(u)g_1^{\mathbf{s}}(u-1)} = \mathbf{s}_2(u-3).$$

Therefore, in type B_2 we have $\bar{\mathbf{s}} = \tilde{\mathbf{s}} \in \mathcal{D}$.

Example 8.3. Assume $\mathfrak{g}$ is of type G_2. Let α_1 be the long root and α_2 the short root, so that $d_1 = 3$, $d_2 = 1$, and $d_{12} = -\frac{3}{2}$. The dual Coxeter number is 4 and so $\kappa = 6$. The Dynkin diagram automorphism is the identity. We have $V_1 = L(Y_{1,-\frac{9}{2}})$ and $V_2 = L(Y_{2,-\frac{11}{2}})$. The ratios of highest to lowest ℓ-weights for V_1 and V_2 are given by

$$A_{1,0}A_{1,-1}A_{1,-2}A_{1,-3}A_{2,\frac{1}{2}}A_{2,-\frac{1}{2}}A_{2,-\frac{3}{2}}^2A_{2,-\frac{5}{2}}A_{2,-\frac{7}{2}},$$
$$A_{1,-\frac{3}{2}}A_{1,-\frac{7}{2}}A_{2,0}A_{2,-2}A_{2,-3}A_{2,-5}.$$

As in the previous example, we have for $\mathbf{s} \in \mathcal{D}$,

$$\begin{aligned}
g_1^{\mathbf{s}}(u) &= \mathbf{s}_1(u)\mathbf{s}_1(u-1)\mathbf{s}_1(u-2)\mathbf{s}_1(u-3) \\
&\quad \times \mathbf{s}_2\Big(u+\frac{1}{2}\Big)\mathbf{s}_2\Big(u-\frac{1}{2}\Big)\mathbf{s}_2\Big(u-\frac{3}{2}\Big)^2\mathbf{s}_2\Big(u-\frac{5}{2}\Big)\mathbf{s}_2\Big(u-\frac{7}{2}\Big), \\
g_2^{\mathbf{s}}(u) &= \mathbf{s}_1\Big(u-\frac{3}{2}\Big)\mathbf{s}_1\Big(u-\frac{7}{2}\Big)\mathbf{s}_2(u)\mathbf{s}_2(u-2)\mathbf{s}_2(u-3)\mathbf{s}_2(u-5), \\
\bar{\mathbf{s}}_1(u) &= \frac{g_1^{\mathbf{s}}(u)g_1^{\mathbf{s}}(u-3)}{\mathbf{s}_1(u)g_2^{\mathbf{s}}(u+\frac{1}{2})g_2^{\mathbf{s}}(u-\frac{1}{2})g_2^{\mathbf{s}}(u-\frac{3}{2})} = \mathbf{s}_1(u-6), \\
\bar{\mathbf{s}}_2(u) &= \frac{g_2^{\mathbf{s}}(u)g_2^{\mathbf{s}}(u-1)}{\mathbf{s}_2(u)g_1^{\mathbf{s}}(u-\frac{3}{2})} = \mathbf{s}_2(u-6).
\end{aligned}$$

Therefore, in type G_2 we have $\bar{\mathbf{s}} = \tilde{\mathbf{s}} \in \mathcal{D}$.

Theorem 8.4. *For $\mathbf{r}, \mathbf{s} \in \mathcal{D}$, the standard module $\mathcal{W}(\mathbf{r}, \mathbf{s})$ factorizes through the truncated shifted Yangian $Y_{\varpi^\vee(\mathbf{s}^{-1}\mathbf{r})}^{\mathbf{r}\bar{\mathbf{s}}}(\mathfrak{g})$. In particular, any irreducible module in category $\mathcal{O}^{\mathrm{sh}}$ factorizes through a truncated shifted Yangian.*

Proof. We have $\mathcal{W}(\mathbf{r}, \mathbf{s}) = L(\mathbf{r}) \otimes L(\mathbf{s}^{-1})$, with $L(\mathbf{r})$ being one-dimensional. For an irreducible module $L(\mathbf{e})$ in category $\mathcal{O}^{\mathrm{sh}}$, one can write $\mathbf{e} = \mathbf{n}^{-1}\mathbf{m}$ with $\mathbf{m}, \mathbf{n} \in \mathcal{D}$. Then Theorem 4.8 shows that $L(\mathbf{e})$ is a quotient of the standard module $\mathcal{W}(\mathbf{m}, \mathbf{n})$. By Remark 4.3, it suffices to show that $L(\mathbf{s}^{-1})$ factorizes through $Y_{-\varpi^\vee(\mathbf{s})}^{\bar{\mathbf{s}}}(\mathfrak{g})$.

Note that $(-\varpi^\vee(\mathbf{s}), \bar{\mathbf{s}}) \in \mathbf{P}^\vee \times \mathcal{R}$ is truncatable:

$$\varpi^\vee(\bar{\mathbf{s}}) + \varpi^\vee(\mathbf{s}) = \sum_{i \in I} \deg(g_i^{\mathbf{s}}(u))\alpha_i^\vee.$$

In the situation of Proposition 5.8 with $W = L(\mathbf{s}^{-1})$, we have $g_i(u) = g_i^{\mathbf{s}}(u)$ by Definition 8.1. Take $V = V_i$ in Theorem 7.4 and compare with equation (5.6). We get the following equality of $\mathrm{End}(L(\mathbf{s}^{-1}))$-valued Laurent series in u^{-1}:

$$A_i(u)|_{L(\mathbf{s}^{-1})} = t_{V_i, L(\mathbf{s}^{-1})}(u) \quad \text{for } i \in I. \tag{8.1}$$

The polynomiality of the lowest diagonal entries in Theorem 7.4 implies in the module $L(\mathbf{s}^{-1})$ the defining relation $\langle A_i(u)\rangle_+ = 0$ of the truncated shifted Yangian. $\square$

Remark 8.5. A similar identification of GKLO series with matrix entries as in equation (8.1) was given in [48, Cor. 5.9]. As commented in [48, Rem. 5.10], these should be specializations of RTT realizations of shifted Yangians [68]. Some particular cases of such a realization appeared in [8, 23]. Notice from equation (5.1) that our R-matrix $\check{R}_{V,W}(u)$ satisfies an RTT relation when V is a finite-dimensional irreducible module over the ordinary Yangian and W is a negative module.

Remark 8.6. There are other approaches to truncations of simple modules over shifted Yangians and shifted quantum affine algebras:

(i) Fix $\mathbf{r}, \mathbf{s} \in \mathcal{D}$ and set μ and ν to be the coweights of $\mathbf{s}^{-1}\mathbf{r}$ and $\mathbf{r}\tilde{\mathbf{s}}$ respectively. Recall from Remark 2.9 the quotient map $\widetilde{Y}_\mu^\nu(\mathbf{r}\tilde{\mathbf{s}}) \to Y_\mu^\nu(\mathbf{r}\tilde{\mathbf{s}})$. If $\mathfrak{g}$ is simply laced, then there is a classification of irreducible highest ℓ-weight modules for the original truncated shifted Yangian $Y_\mu^\nu(\mathbf{r}\tilde{\mathbf{s}})$ in terms of monomial crystals [48, 49], which translated into the language of q-characters by [60, Thm. 3.3] implies that the $Y_\mu(\mathfrak{g})$-module $\mathcal{W}(\mathbf{r}, \mathbf{s})$ factorizes through

$$Y_\mu(\mathfrak{g}) \longrightarrow Y_\mu^{\mathbf{r}\tilde{\mathbf{s}}}(\mathfrak{g}) = \widetilde{Y}_\mu^\nu(\mathbf{r}\tilde{\mathbf{s}}) \longrightarrow Y_\mu^\nu(\mathbf{r}\tilde{\mathbf{s}})$$

and is a module over $Y_\mu^{\mathbf{r}\tilde{\mathbf{s}}}(\mathfrak{g})$. Theorem 8.4 shows that $\mathcal{W}(\mathbf{r}, \mathbf{s})$ is a module over $Y_\mu^{\mathbf{r}\bar{\mathbf{s}}}(\mathfrak{g})$. So we expect that $\tilde{\mathbf{s}} = \bar{\mathbf{s}}$ for $\mathfrak{g}$ simply laced. By Examples 8.2–8.3, such an equality seems to hold for $\mathfrak{g}$ of arbitrary type. This would imply that $\bar{\mathbf{s}} \in \mathcal{D}$ in Definition 8.1.

(ii) For $\mathfrak{g}$ of non-simply-laced types, the classification of irreducible highest ℓ-weight modules over the original truncated shifted Yangians of Remark 2.9 can be reduced to the known classification in simply laced types [48, 49] via geometric arguments [62, 61]. Theorem 8.4 in non-simply-laced types follows from this analysis.

(iii) For shifted quantum affine algebras, the second part of Theorem 8.4 was known for finite-dimensional irreducible modules [40, Thm. 12.9] (see also the discussion in [40, Introduction]). More generally, a conjectural description of highest ℓ-weights of irreducible modules over truncated shifted quantum affine algebras in terms of Langlands dual q-characters was formulated in [40, Conj. 12.3].

§9. Jordan–Hölder property

In this section, as an application of our study of R-matrices in Sections 5–7, we establish a property in category $\mathcal{O}^{\mathrm{sh}}$: finite-length modules are stable under tensor product (Theorem 9.5). This is referred to as the Jordan–Hölder property.

We also obtain a uniform proof of the following known result (see Remark 9.4): a truncated Yangian has only a finite number of irreducible representations in the category $\mathcal{O}^{\mathrm{sh}}$ (Theorem 9.3).

The following result is well known in the non-shifted case. We omit its proof as it is a direct consequence of the definition of shifted Yangians.

Lemma 9.1. *Let $\mu = \sum_{j\in I} k_j \varpi_j^\vee$ be a coweight of $\mathfrak{g}$. Then for $i \in I$ there exists a unique algebra homomorphism $f_{\mu,i} \colon Y_{k_i}(\mathrm{sl}_2) \to Y_\mu(\mathfrak{g})$ such that*

$$x^+(u) \mapsto d_i^{-k_i} x_i^+(ud_i), \quad x^-(u) \mapsto d_i x_i^-(ud_i), \quad \xi(u) \mapsto d_i^{-k_i}\xi_i(ud_i).$$

For $f(u) \in \mathbb{C}(u)$ a rational function, by the denominator of $f(u)$ we mean the monic polynomial $q(u)$ of u of smallest degree such that $q(u)f(u) \in \mathbb{C}[u]$. The numerator of $f(u)$ is the denominator of $f(u)^{-1}$.

Lemma 9.2. *Let $\mathbf{e} \in \mathcal{R}$ and $(i, a, m) \in I \times \mathbb{C} \times \mathbb{Z}_{>0}$. Then $(u - a)^m$ divides the denominator of $\mathbf{e}_i(u)$ if and only if $A_{i,a}^{-m}\mathbf{e}$ is an ℓ-weight of $L(\mathbf{e})$.*

Proof. Write $\mu = \varpi^\vee(\mathbf{e}) = \sum_{j\in I} k_j\varpi_j^\vee$. Set $L := f_{\mu,i}(Y_{k_i}(\mathrm{sl}_2))\omega_{\mathbf{e}} \subset L(\mathbf{e})$; it is a $Y_{k_i}(\mathrm{sl}_2)$-module via the pullback by $f_{\mu,i}$. We claim that

(i) the $Y_{k_i}(\mathrm{sl}_2)$-module L is isomorphic to $L(d_i^{-k_i}\mathbf{e}_i(ud_i))$;

(ii) as subspaces of $L(\mathbf{e})$, the ℓ-weight space of the $Y_{k_i}(\mathrm{sl}_2)$-module L of ℓ-weight $d_i^{-k_i}\mathbf{e}_i(ud_i)A_{a_1}^{-1}A_{a_2}^{-1}\cdots A_{a_N}^{-1}$ is equal to the ℓ-weight space of the $Y_\mu(\mathfrak{g})$-module $L(\mathbf{e})$ of ℓ-weight $\mathbf{e}A_{i,a_1d_i}^{-1}A_{i,a_2d_i}^{-1}\cdots A_{i,a_Nd_i}^{-1}$.

Part (i) can be proved as in [14, Lem. 4.3] by restriction to diagram subalgebras.

For (ii), write $\varpi(\mathbf{e}) = \sum_{j\in I} m_j\varpi_j$. Then $\omega_{\mathbf{e}}$ is of weight m_i in the $Y_{k_i}(\mathrm{sl}_2)$-module L. From $L(\mathbf{e}) = Y_\mu^<(\mathfrak{g})\omega_{\mathbf{e}}$ and the $\mathbf{Q}_-$-grading on $Y_\mu^<(\mathfrak{g})$ we get an identification of weight spaces for $N \in \mathbb{N}$:

$$\begin{aligned} L_{m_i-2N} &= L(\mathbf{e})_{\varpi(\mathbf{e})-N\alpha_i} \\ &= \mathrm{Vect}(x_{i,m_1}^- x_{i,m_2}^- \cdots x_{i,m_N}^-\omega_{\mathbf{e}} \mid m_1, m_2, \ldots, m_N \in \mathbb{N}). \end{aligned}$$

Since a weight space is a direct sum of ℓ-weight spaces, it suffices to prove that the right-hand side of the equality of (ii) is contained in the left-hand side. This is obvious from $f_{\mu,i}(\xi(u)) = d_i^{-k_i}\xi_i(ud_i)$.

It follows that $A_{i,a}^{-m}\mathbf{e}$ is an ℓ-weight of $L(\mathbf{e})$ if and only if $d_i^{-k_i}\mathbf{e}_i(ud_i)A_{ad_i^{-1}}^{-1}$ is an ℓ-weight of the $Y_{k_i}(\mathrm{sl}_2)$-module $L(d_i^{-k_i}\mathbf{e}_i(ud_i))$. We are reduced to the case $\mathfrak{g} = \mathrm{sl}_2$. Then this follows from the tensor product factorization of Proposition 6.2 because, by Definition 6.1, the standard factorization of a rational function fixes its denominator and each factor $u - a$ in the denominator contributes to a factor A_a^{-1} of an ℓ-weight. □

For $(\mu, \mathbf{r}) \in \mathbf{P}^\vee \times \mathcal{R}$ a truncatable pair, define $\mathcal{R}_\mu^{\mathbf{r}}$ to be the set of $\mathbf{e} \in \mathcal{R}_\mu$ such that the $Y_\mu(\mathfrak{g})$-module $L(\mathbf{e})$ factorizes through the truncated shifted Yangian $Y_\mu^{\mathbf{r}}(\mathfrak{g})$.

Theorem 9.3. *The set $\mathcal{R}_\mu^{\mathbf{r}}$ is finite for any truncatable pair $(\mu, \mathbf{r}) \in \mathbf{P}^\vee \times \mathcal{R}$.*

Remark 9.4. When $\mathbf{r} \in \mathcal{D}$, there is a geometric proof of the finiteness, as explained in [48, Cor. 3.13], by viewing the truncated shifted Yangian $Y_\mu^{\mathbf{r}}(\mathfrak{g})$ as a quantization of a scheme supported on a generalized affine Grassmannian slice $\overline{\mathcal{W}}_\mu^\lambda$ with $\lambda := \varpi^\vee(\mathbf{r})$ (see [47, Prop. 4.10]), and then applying the general result [6, Prop. 5.1]. If $\mathbf{r} \notin \mathcal{D}$, choose $\mathbf{s} \in \mathcal{D}$ such that $\mathbf{rs} \in \mathcal{D}$. Remark 4.3 shows that $\mathcal{R}_\mu^{\mathbf{r}} \subset \mathbf{s}^{-1}\mathcal{R}_{\mu+\varpi^\vee(\mathbf{s})}^{\mathbf{rs}}$. The finiteness of $\mathcal{R}_\mu^{\mathbf{r}}$ follows from the known case. Our proof is algebraic and close to the situation of truncated shifted quantum affine algebras [40, Thm. 11.15].

Proof of Theorem 9.3. For $\mathbf{e} \in \mathcal{R}$ and $i \in I$, let $p_i^{\mathbf{e}}(u)$ and $q_i^{\mathbf{e}}(u)$ denote respectively the numerator and denominator of $\mathbf{e}_i(u)$. Recall from equation (2.19) the coefficient $m_i \in \mathbb{N}$ of $\alpha_i^\vee$ in $\varpi^\vee(\mathbf{r}) - \mu$.

Step 1. We shall give a necessary condition for $\mathbf{e} \in \mathcal{R}_\mu^{\mathbf{r}}$. As in Proposition 5.8, let $g_i(u) \in \mathbb{C}[u]$ be the eigenvalue of $A_i(u) \in Y_\mu^{\mathbf{r}}(\mathfrak{g})[u]$ associated to the eigenvector $\omega_{\mathbf{e}}$ of $L(\mathbf{e})$. Then $g_i(u)$ is a monic polynomial of degree m_i. We claim that $g_i(u)\mathbf{e}_i(u)$ is a polynomial, which by equation (2.20) is equivalent to divisibility of polynomials:

$$g_i(u)q_i^{\mathbf{r}}(u+d_i) \mid p_i^{\mathbf{r}}(u+d_i) \times \prod_{j: c_{ji}<0} \prod_{t=1}^{-c_{ji}} g_j(u+d_i-d_{ij}-td_j). \tag{9.1}$$

We need to prove $q_i^{\mathbf{e}}(u) \mid g_i(u)$. Namely, for any $(a, m) \in \mathbb{C} \times \mathbb{Z}_{>0}$ such that $(u-a)^m \mid q_i^{\mathbf{e}}(u)$, we must have $(u-a)^m \mid g_i(u)$. By Lemma 9.2, $\mathbf{e}A_{i,a}^{-m}$ is an ℓ-weight of $L(\mathbf{e})$. From Proposition 5.8(ii) we see that the eigenvalue of $A_i(u)$ on $L(\mathbf{e})_{\mathbf{e}A_{i,a}^{-m}}$ is $g_i(u)(\frac{u-a+d_i}{u-a})^m$, which must be a polynomial. Since $(u-a+d_i)^m$ and $(u-a)^m$ are coprime, we have $(u-a)^m \mid g_i(u)$, as desired.

Step 2. Introduce the following finite subsets of $\mathbb{C}$ for $s \in \mathbb{Z}_{>0}$:

$$D_{\mathfrak{g}} \coloneqq \{d_i - d_{ij} - td_j \mid i, j \in I,\ t \in \mathbb{Z},\ c_{ji} < 0,\ 1 \leq t \leq -c_{ji}\},$$
$$X_{\mathbf{r}}^0 \coloneqq \{a \in \mathbb{C} \mid p_i^{\mathbf{r}}(a + d_i) = 0 \text{ for certain } i \in I\},$$
$$X_{\mathbf{r}}^s \coloneqq \{a - c_1 - c_2 - \cdots - c_s \in \mathbb{C} \mid a \in X_{\mathbf{r}}^0,\ c_k \in D_{\mathfrak{g}} \text{ for } 1 \leq k \leq s\}.$$

Clearly, $D_{\mathfrak{g}}$ depends on $\mathfrak{g}$, and $X_{\mathbf{r}}^s$ on the triple $(\mathbf{r}, s, \mathfrak{g})$. Set $m \coloneqq \sum_{i \in I} m_i$ and $X \coloneqq \bigcup_{s=0}^{m-1} X_{\mathbf{r}}^s$. Then X is a finite set depending on the triple $(\mathbf{r}, \mu, \mathfrak{g})$.

Each $\mathbf{e} \in \mathcal{R}_{\mu}^{\mathbf{r}}$ is uniquely determined by the monic polynomials $g_i(u)$ for $i \in I$ from Step 1. Let us attach a quiver $\Gamma_{\mathbf{e}}$ to $\mathbf{e}$ as follows:

- The set of vertices is $V_{\mathbf{e}} \coloneqq \{(i, a) \in I \times \mathbb{C} \mid g_i(a) = 0\}$.
- Draw an arrow $(i, a) \to (j, b)$ if $c_{ji} < 0$ and there exists $1 \leq t \leq -c_{ji}$ such that $b = a + d_i - d_{ij} - td_j$ and $p_i^{\mathbf{r}}(a + d_i) \neq 0$. In particular, $b - a \in D_{\mathfrak{g}}$.

We prove that $V_{\mathbf{e}} \subset I \times X$. Since $g_i(u)$ is of fixed degree m_i, this will imply that there are finitely many choices of $g_i(u)$, from which comes the finiteness of $\mathcal{R}_{\mu}^{\mathbf{r}}$.

Let $V_{\mathbf{e}}^0$ be the set of sink vertices (namely, vertices with no outgoing arrows) of $\Gamma_{\mathbf{e}}$. If (i, a) is sink, by relation (9.1) we must have $p_i^{\mathbf{r}}(a + d_i) = 0$, since the product $\prod_j$ is non-zero. This means $V_{\mathbf{e}}^0 \subset I \times X_{\mathbf{r}}^0$.

For $s \in \mathbb{Z}_{>0}$, define $V_{\mathbf{e}}^s$ to be the set of vertices (i, a) of $\Gamma_{\mathbf{e}}$ which can be joint to a sink vertex with s arrows. Namely, $(i, a) \in V_{\mathbf{e}}^s$ if there exist s vertices

$$(i_1, a_1), (i_2, a_2), \ldots, (i_s, a_s)$$

such that (i_s, a_s) is sink and there are arrows

$$(i, a) \longrightarrow (i_1, a_1) \longrightarrow (i_2, a_2) \longrightarrow \cdots \longrightarrow (i_{s-1}, a_{s-1}) \longrightarrow (i_s, a_s).$$

From our definition of arrows it follows that $a_{k+1} - a_k \in D_{\mathfrak{g}},\ a_s \in X_{\mathbf{r}}^0$, and

$$a = a_s - (a_s - a_{s-1}) - \cdots - (a_2 - a_1) - (a_1 - a_0) \in X_{\mathbf{r}}^s.$$

This gives $V_{\mathbf{e}}^s \subset I \times X_{\mathbf{r}}^s$.

By Claim 1 below, the quiver $\Gamma_{\mathbf{e}}$ is acyclic and every vertex is connected to a sink vertex by at most $m - 1$ arrows. We obtain the desired relation

$$V_{\mathbf{e}} = \bigcup_{s=0}^{m-1} V_{\mathbf{e}}^s \subset \bigcup_{s=0}^{m-1} (I \times X_{\mathbf{r}}^s) = I \times X.$$

Claim 1. In the quiver $\Gamma_{\mathbf{e}}$ there does not exist any sequence of vertices (i_k, a_k) for $0 \leq k \leq m$ with arrows $(i_l, a_l) \to (i_{l+1}, a_{l+1})$ for $0 \leq l < m$.

Assume the contrary and fix such a sequence

$$S\colon \quad (i_0, a_0) \longrightarrow (i_1, a_1) \longrightarrow (i_2, a_2) \longrightarrow \cdots \longrightarrow (i_m, a_m).$$

Since $m+1 > \sum_{j\in I} m_j$, there must exist $j \in I$ which appears in the sequence $i_0 i_1 i_2 \cdots i_m$ at least $m_j + 1$ times. Each appearance of j, say $i_k = j$, gives a root a_k of $g_j(u)$. If $i_k = i_l$ and $k < l$ then necessarily $l - k > 1$ and from Claim 2 below we get $a_k \neq a_l$. So the polynomial $\prod_{0\le k\le m, i_k = j}(u - a_k)$ divides $g_j(u)$; the former is of degree at least $m_j + 1$, while the latter of degree m_j, contradiction.

Claim 2. In the sequence S, if $0 \le k < k+1 < l \le m$ then $a_l - a_k \in \frac{1}{2}\mathbb{Z}_{>0}$.

If $\mathfrak{g}$ is not of type G_2, then $c_{ji} \ge -2$ for all $i, j \in I$. By equation (2.20), we have $D_{\mathfrak{g}} \subset \frac{1}{2}\mathbb{N}$; moreover, if $0 \in D_{\mathfrak{g}}$, then it must arise from $0 = d_i - \frac{1}{2}d_j$ so that $d_i = 1$ and $d_j = 2$. This means that for an arrow $(i, a) \to (j, b)$ of the quiver $\Gamma_{\mathbf{e}}$, either $b - a \in \frac{1}{2}\mathbb{Z}_{>0}$, or $b = a$ and $(d_i, d_j) = (1, 2)$. Apply this to the sequence S:

$$a_1 - a_0, a_2 - a_1, \ldots, a_m - a_{m-1} \in \frac{1}{2}\mathbb{N}.$$

So $a_l - a_k \in \frac{1}{2}\mathbb{N}$. If $a_l = a_k$, then $a_k = a_{k+1} = a_{k+2}$. From $a_k = a_{k+1}$ we obtain $d_{i_{k+1}} = 2$, while from $a_{k+1} = a_{k+2}$ we obtain $d_{i_{k+1}} = 1$, contradiction.

If $\mathfrak{g}$ is of type G_2 with $d_1 = 3$ and $d_2 = 1$, we check Claim 2 directly. By equation (2.20), the arrows in the quiver $\Gamma_{\mathbf{e}}$ are of the form

$$(1, a) \longrightarrow \Big(2, a + \frac{3}{2}\Big), \quad (1, a) \longrightarrow \Big(2, a + \frac{5}{2}\Big),$$
$$(1, a) \longrightarrow \Big(2, a + \frac{7}{2}\Big), \quad (2, a) \longrightarrow \Big(1, a - \frac{1}{2}\Big).$$

The sequence $i_0 i_1 \cdots i_{m-1} i_m$ is alternating of the form $1212\cdots$ or $2121\cdots$. For an arrow $(1, a) \to (2, b)$, we have $b - a \ge \frac{3}{2}$, while for an arrow $(2, a) \to (1, b)$ we have $b - a = -\frac{1}{2}$. It follows that $a_l - a_k = \sum_{t=k}^{l-1}(a_{t+1} - a_t)$ as a half integer is at least S, where S is an alternating sum of $l - k > 1$ terms of the following form:

$$\frac{3}{2} - \frac{1}{2} + \frac{3}{2} - \frac{1}{2} + \cdots, \quad -\frac{1}{2} + \frac{3}{2} - \frac{1}{2} + \frac{3}{2} - \cdots.$$

Clearly $a_l - a_k \ge S > 0$. □

Define $\mathcal{O}_\mu^{\mathrm{fin}}$ to be the full subcategory of $\mathcal{O}_\mu$ consisting of modules with a finite Jordan–Hölder filtration. In other words, a module V in category $\mathcal{O}_\mu$ is in $\mathcal{O}_\mu^{\mathrm{fin}}$ if and only if V admits a finite number of irreducible subquotients, if and only if $[V]$ is a finite sum of irreducible isomorphism classes $[L(\mathbf{e})]$ for $\mathbf{e} \in \mathcal{R}_\mu$. Let $\mathcal{O}_{\mathrm{fin}}^{\mathrm{sh}}$ be the direct sum of the categories $\mathcal{O}_\mu^{\mathrm{fin}}$ over all coweights μ.

Theorem 9.5. *The category $\mathcal{O}_{\mathrm{fin}}^{\mathrm{sh}}$ is closed under tensor product.*

Proof. We need to show that the tensor product T of two arbitrary irreducible modules belongs to category $\mathcal{O}_{\mathrm{fin}}^{\mathrm{sh}}$, namely, T admits a finite number of irreducible subquotients. From Theorem 4.8 we see that T can be realized as a quotient of a tensor product T' of two standard modules. By equation (4.5) the module T' has the same isomorphism class as a third standard module $\mathcal{W}$. By Theorem 8.4, the standard module $\mathcal{W}$ factorizes through a truncated shifted Yangian $Y_\nu^{\mathbf{s}}(\mathfrak{g})$. Since $\mathcal{R}_\nu^{\mathbf{s}}$ is finite, $\mathcal{W}$ admits a finite number of irreducible subquotients, so do the tensor products T' and T. □

The Grothendieck group of category $K_0(\mathcal{O}^{\mathrm{sh}}_{\mathrm{fin}})$, which is the abelian subgroup of $K_0(\mathcal{O}^{\mathrm{sh}})$ freely generated by the $[L(\mathbf{e})]$ for $\mathbf{e} \in \mathcal{R}$, is a therefore a subring.

Remark 9.6. Let us make further comments on the Jordan–Hölder property:

(i) Assume $\mathfrak{g}$ is not of type E_8. Combining Proposition 4.12(iii) and Theorems 4.15 and 8.4, we obtain that any highest ℓ-weight module in category $\mathcal{O}^{\mathrm{sh}}$ factorizes through a truncated shifted Yangian and belongs to category $\mathcal{O}^{\mathrm{sh}}_{\mathrm{fin}}$.

(ii) The Jordan–Hölder property is known to be true [36, Thm. 5.27] for a certain category of integrable modules over a quantum affinization. It fails in the original category $\mathcal{O}$ of $\mathfrak{g}$-modules (counterexample: the tensor product of two irreducible Verma modules over sl_2 has infinitely many irreducible subquotients) and in the category $\mathcal{O}$ of modules over the Borel algebra [41] as observed in [43, Rem. 5.12].

(iii) For $\mathfrak{g} = \mathrm{sl}_{r+1}$ there is another proof of Theorem 9.5 by extending $Y(\mathrm{sl}_{r+1})$ to the Yangian $Y(\mathrm{gl}_{r+1})$ and using the big center of $Y(\mathrm{gl}_{r+1})$; see [9, Lems. 6.13, 7.16]. This is close to the proof of classical fact in the original category $\mathcal{O}$ of $\mathfrak{g}$-modules that each Verma module admits a finite Jordan–Hölder filtration.

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