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Description of Generalized Isomonodromic Deformations of Rank-Two Linear Differential Equations Using Apparent Singularities

by

Arata Komyo

Abstract

In this paper, we consider the generalized isomonodromic deformations of rank-two irregular connections on the Riemann sphere. We introduce Darboux coordinates on the parameter space of a family of rank-two irregular connections by apparent singularities. Using the Darboux coordinates, we describe the generalized isomonodromic deformations as Hamiltonian systems.

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§1. Introduction

For connections on the trivial bundle on $\mathbb{P}^1$, the regular singular isomonodromic deformation is the Schlesinger equation, and the unramified irregular singular generalized isomonodromic deformation is the Jimbo–Miwa–Ueno equation which is completely given in [14, 12, 13]. Bertola–Mo and Bremer–Sage have generalized the Jimbo–Miwa–Ueno equation (see [1, 3, 4]) and Boalch [2] has given the symplectic geometry of the Jimbo–Miwa–Ueno equation. That is, the Jimbo–Miwa–Ueno equations are equivalent to a flat symplectic Ehresmann connection on a certain symplectic fiber bundle. The fibers of the symplectic fiber bundle are certain moduli spaces of meromorphic connections over $\mathbb{P}^1$. In this paper, we consider the generalized isomonodromic deformation from this point of view.

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A. Komyo: Department of Material Science, Graduate School of Science, University of Hyogo, 2167 Shosha, Himeji, Hyogo 671-2280, Japan;
e-mail: akomyo@sci.u-hyogo.ac.jp

As in [14] the monodromy data for certain families of irregular singular differential equations involve the asymptotic behavior of solutions along Stokes sectors at each singular point. Here we impose that the singularities of irregular singular differential equations satisfy some generic condition. More precisely, these singularities are regular or unramified irregular. If we have such a generic family of irregular singular differential equations, then we can locally define the monodromy map (in other words, Riemann–Hilbert map) from the space of parameters of this family to the moduli space of irregular monodromy representations (for details, for example, see [2, 11]). The fibers of the monodromy map are a foliation of the space of parameters of the family (see [2, 11, 17]). The foliation is called the (*generalized*) *isomonodromic foliation*. (The corresponding vector field is called the (*generalized*) *isomonodromic deformations*). On the other hand, there exists another approach to generalized isomonodromic deformations. As in [2, Appx.], a submanifold $\mathcal{L}$ in the space of parameters of the family is contained in a leaf of this foliation if and only if the family of connections corresponding to $\mathcal{L}$ is integrable.

When irregular singular differential equations have special singularities (so-called (generic) ramified irregular singularities), the formulation of the Riemann–Hilbert map is still not clear. But by using the integrable condition (which is the second point of view), we can define the generalized isomonodromic deformation for such a special family of irregular singular differential equations. Moreover, this generalized isomonodromic deformation is integrable. So we have the generalized isomonodromic foliation (see [1, 4, 10]).

In this paper, we consider generalized isomonodromic deformations only from the viewpoint of the integrability condition. More specifically, we construct a *horizontal lift* of the family of connections as in [11, Thm. 6.2] and [10, Sect. 9]. Here, the horizontal lift is a first-order infinitesimal extension of the relative connection with an integrability condition.

Let D be the effective divisor on $\mathbb{P}^1$ defined as

$$D = \sum_{i=1}^{\nu} n_i \cdot t_i + n_\infty \cdot \infty \quad \text{and} \quad n := \deg(D) = \sum_{i=1}^{\nu} n_i + n_\infty.$$

Let E be a rank-two vector bundle on $\mathbb{P}^1$. Let $\nabla \colon E \to E \otimes \Omega^1_{\mathbb{P}^1}(D)$ be a connection on E with the polar divisor D. We call such pairs (E, ∇) *connections*. Remark that we can shift the degree of the vector bundle E by arbitrary integers by applying some natural operations (twisting by rank-one meromorphic connections and birational bundle modifications, called *canonical transformations*, *elementary transformations*, or *Hecke modifications*). In this paper, we assume that the degree of the vector bundle is 1. If $H^1(\mathbb{P}^1, E^\vee) = 0$, then we have $E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Here, $E^\vee$ is the dual of the vector bundle E. If there exists a family of connections of degree 1, then there exists a Zariski open subset of the parameter space such that this open subset parametrizes connections with the bundle type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. For connections with the bundle type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, we can define *apparent singularities* of the connections and also we can define dual parameters for the apparent singularities. By the apparent singularities and dual parameters, we may give a map from a moduli space of connections with the bundle type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ to $\mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. On the other hand, Diarra–Loray [5, Sect. 6] gave global normal forms of the connections with the bundle type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$, whose connection matrices are companion matrices. By this normal form, we may construct a family of connections with bundle type $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ parametrized by a Zariski open subset of $\mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. By this family, we have a map from the Zariski open subset of $\mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$ to the moduli space of connections. Finally, we have a birational correspondence between the moduli space of connections and $\mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. Our point of view is that this birational correspondence gives coordinates on (a Zariski open set of) the moduli space of connections. In this paper, we consider the generalized isomonodromic deformations (integrable deformations) of connections. We may regard the generalized isomonodromic deformations as vector fields on the moduli space of connections. The main purpose of this paper is to give an explicit description of generalized isomonodromic deformations by using these coordinates. Here, the eigenvalues of the leading coefficients of the Laurent expansions of the connections at each irregular singular point are not necessarily distinct. (If any leading coefficients have distinct eigenvalues respectively, then the generalized isomonodromic deformations of this family of connections corresponds to the Jimbo–Miwa–Ueno equations). That is, we will consider not only *unramified irregular singular points* (Definition 2.5 below) but also *ramified irregular singular points* (Definition 2.6 below).

There exist many studies on Hamiltonians of the Jimbo–Miwa–Ueno equation ([7, 8, 20, 21, 19, 22]). The main subject of this paper is to give explicit descriptions of the symplectic structure and the Hamiltonians of the generalized isomonodromic deformations by using apparent singularities. For the regular singular isomonodromic deformations, Dubrovin–Mazzocco [6] have introduced isomonodromic Darboux coordinates on the moduli space of Fuchsian systems, which are connections on the trivial bundle over $\mathbb{P}^1$. They have described the isomonodromic deformations of Fuchsian systems as Hamiltonian systems by using the isomonodromic Darboux coordinates. Roughly speaking, we extend their argument for the regular singular case to the irregular singular (rank-two) case. In our calculation, Krichever's formula of the symplectic form [16, Sect. 5] is used as in [6]. On the other hand, Kimura [15] has studied the degeneration of the two-dimensional Garnier systems. By the confluence procedure, Hamiltonian systems of (generalized) isomonodromic deformations of certain rank-two linear differential equations are described explicitly. We try to compare our Hamiltonian systems and Kimura's Hamiltonian systems by an example. Our Hamiltonian systems are not necessarily commuting Hamiltonian systems in [6, Def. 3.5].

§1.1. Space of deformation parameters

Now we describe the *space of deformation parameters* for our generalized isomonodromic deformations. Put $I \coloneqq \{1, 2, \dots, \nu, \infty\}$, $t_1 \coloneqq 0$, $t_2 \coloneqq 1$, and $t_\infty \coloneqq \infty \in \mathbb{P}^1$. We take a decomposition $I = I_{\mathrm{reg}} \cup I_{\mathrm{un}} \cup I_{\mathrm{ra}}$ such that I_{reg}, I_{un}, and I_{ra} are disjoint from each other. We assume that $n_i = 1$ for $i \in I_{\mathrm{reg}}$ and $n_i > 1$ for $i \in I_{\mathrm{un}} \cup I_{\mathrm{ra}}$. We set

$$T_{\boldsymbol{t}} \coloneqq \left\{ (t_3, \dots, t_\nu) \in \mathbb{C}^{\nu-2} \;\middle|\; \begin{array}{l} t_i \neq t_j \ (i \neq j) \text{ and} \\ t_i \notin \{0,1\} \ (i = 3, \dots, \nu) \end{array} \right\}.$$

Moreover, put

$$T_{\boldsymbol{\theta}}^{\mathrm{res}} \coloneqq \left\{ \boldsymbol{\theta}_0 \in \mathbb{C}^{2(\nu+1)} \;\middle|\; \begin{array}{l} \sum_{i \in I_{\mathrm{reg}} \cup I_{\mathrm{un}}} (\theta^+_{n_i-1,t_i} + \theta^-_{n_i-1,t_i}) \\ \qquad + \sum_{i \in I_{\mathrm{ra}}} (\theta_{2n_i-2,t_i} - \frac{1}{2}) = -1, \\ \theta^+_{0,t_i} - \theta^-_{0,t_i} \notin \mathbb{Z} \text{ for } i \in I_{\mathrm{reg}} \end{array} \right\},$$

$$T_{\boldsymbol{\theta}} \coloneqq \left\{ \boldsymbol{\theta} \in \prod_{i \in I_{\mathrm{un}}} \mathbb{C}^{2(n_i-1)} \times \prod_{i \in I_{\mathrm{ra}}} \mathbb{C}^{2n_i-2} \;\middle|\; \begin{array}{l} \theta^+_{0,t_i} - \theta^-_{0,t_i} \neq 0 \text{ for } i \in I_{\mathrm{un}}, \\ \theta_{1,t_i} \neq 0 \text{ for } i \in I_{\mathrm{ra}} \end{array} \right\}.$$

Here we set

$$\boldsymbol{\theta}_0 \coloneqq ((\theta^+_{n_i-1,t_i}, \theta^-_{n_i-1,t_i})_{i \in I_{\mathrm{reg}} \cup I_{\mathrm{un}}}, (\theta_{2n_i-2,t_i})_{i \in I_{\mathrm{ra}}})$$

and we denote by $\boldsymbol{\theta} = (\boldsymbol{\theta}_{\mathrm{un}}, \boldsymbol{\theta}_{\mathrm{ra}})$ an element of $T_{\boldsymbol{\theta}}$ where

$$\begin{aligned} \boldsymbol{\theta}_{\mathrm{un}} &= ((\theta^+_{0,t_i}, \theta^-_{0,t_i}), \dots, (\theta^+_{n_i-2,t_i}, \theta^-_{n_i-2,t_i}))_{i \in I_{\mathrm{un}}}, \\ \boldsymbol{\theta}_{\mathrm{ra}} &= (\theta_{0,t_i}, \dots, \theta_{2n_i-3,t_i})_{i \in I_{\mathrm{ra}}}. \end{aligned}$$

The relation in the definition of $T_{\boldsymbol{\theta}}^{\mathrm{res}}$ is called the *Fuchs relation*. Fix a tuple of complex numbers $\boldsymbol{t}_{\mathrm{ra}} = (t_i)_{i \in \{3,4,\dots,\nu\} \cap I_{\mathrm{ra}}}$, where $t_i \neq t_j$ $(i \neq j)$ and $t_i \notin \{0,1\}$. We denote the fiber of $\boldsymbol{t}_{\mathrm{ra}}$ under the projection

$$T_{\boldsymbol{t}} \longrightarrow \prod_{i \in \{3,4,\dots,\nu\} \cap I_{\mathrm{ra}}} \mathbb{C}.$$

by $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$.

Definition 1.1. We define the *space of deformation parameters* as $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$.

Hence we consider the positions of the points t_i for $i \in I_{\mathrm{reg}} \cup I_{\mathrm{un}}$ as deformation parameters. On the other hand, we do not consider the positions of the points t_i for $i \in I_{\mathrm{ra}}$ as deformation parameters, since the integrable deformations whose deformation parameters are the positions of the ramified irregular points are more complicated.

§1.2. Symplectic fiber bundle

Next we define an algebraic variety over the space of deformation parameters $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ such that this algebraic variety parametrizes connections and there exists a symplectic form on each fiber. This algebraic variety is considered as the *phase space* of our generalized isomonodromic deformations. We set

$$\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} := \left\{ \begin{array}{l} (\{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\},(t_3,\dots,t_\nu)) \\ \in \mathrm{Sym}^{(n-3)}(\mathbb{C}^2) \times (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \end{array} \middle| \begin{array}{l} q_i \neq q_j \ (i \neq j) \text{ and} \\ q_j \notin \{0,1,t_3,\dots,t_\nu,\infty\} \\ (j=1,\dots,n-3) \end{array} \right\}.$$

If we take a point $\boldsymbol{t}_0 = (t_3,\dots,t_\nu)$ of $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$, we put

$$\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} := \left\{ \begin{array}{l} \{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\} \\ \in \mathrm{Sym}^{(n-3)}(\mathbb{C}^2) \end{array} \middle| \begin{array}{l} q_i \neq q_j \ (i \neq j) \text{ and } q_j \notin \{0,1,\infty\} \cup \boldsymbol{t}_0 \\ (j=1,\dots,n-3) \end{array} \right\}.$$

Definition 1.2. We define a symplectic fiber bundle $\pi_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}$ as the natural projection

$$\pi_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0} \colon \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \longrightarrow (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}. \tag{1.1}$$

Here, the symplectic structure on the fiber $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \times \{\boldsymbol{\theta}\}$ of $(\boldsymbol{t}_0,\boldsymbol{\theta}) \in (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ is defined by

$$\sum_{j=1}^{n-3} d\Big(\frac{p_j}{\prod_{i=1}^{\nu}(q_j-t_i)^{n_i}}\Big) \wedge dq_j. \tag{1.2}$$

Set $D(\boldsymbol{t}_0) := n_0 \cdot 0 + n_1 \cdot 1 + \sum_{i=3}^{\nu} n_i \cdot t_i + n_\infty \cdot \infty$ for $\boldsymbol{t}_0 = (t_3,\dots,t_\nu) \in (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$. We may check that the fiber $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \times \{\boldsymbol{\theta}\}$ is isomorphic to the moduli space $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. Here, $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ is the moduli space of $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-connections on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ such that the polar divisors of connections are $D(\boldsymbol{t}_0)$ and connections satisfy some generic conditions (see (2.12) below). The correspondence between $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \times \{\boldsymbol{\theta}\}$ and $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ is given by the theory of apparent singularities (Section 2.1 below) and construction of a family of connections parametrized by $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \times \{\boldsymbol{\theta}\}$ (Section 2.2 below). For the construction of a family, we will use Diarra–Loray's global normal form. We will call $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ an *extended moduli space of connections.*

§1.3. Main results

For the vector fields $\partial/\partial\theta^{\pm}_{l,t_i}$ ($i \in I_{\rm un}$, $l = 0,1,\ldots,n_i-2$), $\partial/\partial\theta_{l',t_i}$ ($i \in I_{\rm ra}$, $l' = 0,1,\ldots,2n_i-3$), and $\partial/\partial t_i$ ($i \in \{3,4\ldots,\nu\} \cap (I_{\rm reg} \cup I_{\rm un})$), we define the vector fields $\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}$, $\delta^{\rm IMD}_{\theta_{l',t_i}}$, and $\delta^{\rm IMD}_{t_i}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ by the integrable deformations of the family of connections parametrized by $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ (in Sections 3.3, 4.2, and 3.4). We define a 2-form $\widehat{\omega}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ such that the restriction of $\widehat{\omega}$ to each fiber of $\pi_{\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0}$ coincides with the symplectic form (1.2) and the interior products with the vector fields determined by the integrable deformations vanish:

$$\iota(\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}})\widehat{\omega} = \iota(\delta^{\rm IMD}_{\theta_{l',t_i}})\widehat{\omega} = \iota(\delta^{\rm IMD}_{t_i})\widehat{\omega} = 0.$$

We call the 2-form $\widehat{\omega}$ the *isomonodromy 2-form* as in [22]. The main result of this paper is an explicit description of the isomonodromy 2-form using apparent singularities. Our description of $\widehat{\omega}$ is

$$\begin{aligned}\widehat{\omega} = &\sum_{j=1}^{n-3} d\left(\frac{p_j}{\prod_{i=1}^{\nu}(q_j-t_i)^{n_i}} - \sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}} - D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})\right)\wedge dq_j \\ &+ \sum_{i\in I_{\rm un}}\sum_{l=0}^{n_i-2}(dH_{\theta^+_{l,t_i}}\wedge d\theta^+_{l,t_i} + dH_{\theta^-_{l,t_i}}\wedge d\theta^-_{l,t_i}) \\ &+ \sum_{i\in I_{\rm ra}}\sum_{l'=0}^{2n_i-3} dH_{\theta_{l',t_i}}\wedge d\theta_{l',t_i} + \sum_{\substack{i\in\{3,4,\ldots,\nu\}\\ \cap(I_{\rm reg}\cup I_{\rm un})}} dH_{t_i}\wedge dt_i \\ &+ [\text{a section of } \pi^*_{\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0}(\Omega^2_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}}})\ ]\end{aligned}$$

(Theorems 3.14 and 4.10). Here, $D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})$ where $i \in I$ are defined in Lemma 2.9 as D_i. The Hamiltonians $H_{\theta^{\pm}_{l,t_i}}$, $H_{\theta_{l',t_i}}$, and H_{t_i} are defined in Definitions 3.11, 4.9, and 3.12, respectively. Roughly speaking, the Hamiltonians $H_{\theta^{\pm}_{l,t_i}}$ and $H_{\theta_{l',t_i}}$ appear in the holomorphic parts of the diagonalizations of connections at each singular point t_i. By this description of the isomonodromy 2-form, we obtain Hamiltonian descriptions of the vector fields determined by the integrable deformations:

$$\begin{aligned}\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}} &= \frac{\partial}{\partial\theta^{\pm}_{l,t_i}} - \sum_{j=1}^{n-3}\left(\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j} - \frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\right), \\ \delta^{\rm IMD}_{\theta_{l',t_i}} &= \frac{\partial}{\partial\theta_{l',t_i}} - \sum_{j=1}^{n-3}\left(\frac{\partial H_{\theta_{l',t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j} - \frac{\partial H_{\theta_{l',t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\right), \\ \delta^{\rm IMD}_{t_i} &= \frac{\partial}{\partial t_i} - \sum_{j=1}^{n-3}\left(\frac{\partial H_{t_i}}{\partial\eta_j}\frac{\partial}{\partial q_j} - \frac{\partial H_{t_i}}{\partial q_j}\frac{\partial}{\partial\eta_j}\right)\end{aligned}$$

(Corollaries 3.15 and 4.11). Here we put

$$\eta_j := \frac{p_j}{\prod_{i=1}^{\nu}(q_j - t_i)^{n_i}} - \sum_{i=1}^{\nu} \frac{D_i(q_j; \boldsymbol{t}, \boldsymbol{\theta})}{(q_j - t_i)^{n_i}} - D_\infty(q_j; \boldsymbol{t}, \boldsymbol{\theta}).$$

The organization of this paper is as follows. In Section 2 we recall the definition of the apparent singularities and Diarra–Loray's global normal form. In Section 3 we consider the integrable deformations of connections which have only regular singularities and unramified irregular singularities. We define a 2-form on the fiber $\mathcal{M}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}}$ by Krichever's formula [16, Sect. 5]. We show that this 2-form coincides with the symplectic form (1.2). Also by Krichever's formula, we define a 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. We show that this 2-form is the isomonodromy 2-form. By calculation of this 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by using Diarra–Loray's global normal form, we have an explicit formula of this 2-form. In Section 4 we extend the argument of Section 3 to the integrable deformations of connections which have ramified irregular singularities. In Section 5 we consider two examples. The first example is the case where $D = 2 \cdot 0 + 2 \cdot 1 + 2 \cdot \infty$. We assume that $0, 1, \infty \in \mathbb{P}^1$ are unramified irregular singular points. The dimension of the space of deformation parameters is 6 and the dimension of the fiber $\widehat{\mathcal{M}}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}}$ is 6. The second example is the case where $D = 5 \cdot \infty$. We assume that $\infty \in \mathbb{P}^1$ is a ramified irregular singular point. This example corresponds to Kimura's $H(9/2)$ in [15]. We consider the family of connections corresponding to Kimura's family $L(9/2; 2)$. We reproduce the Hamiltonian system $H(9/2)$.

§2. Normal forms for rank-two linear irregular differential equations

In the first part of this section, we will give a correspondence between the moduli space of connections and $\mathcal{M}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. First, we recall the theory of apparent singularities in Section 2.1. This theory gives a map from the moduli space of connections to $\mathcal{M}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}}$. Second, we recall Diarra–Loray's global normal form in Section 2.2. This normal form gives a map from $\mathcal{M}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}}$ to the moduli space of connections. In the second part of this section (Sections 2.5 and 2.6), we will consider infinitesimal deformations of connections and define horizontal lifts of connections. If we construct a horizontal lift of a connection, then we have an integrable deformation of a connection. After Section 2, we will discuss construction of horizontal lifts. In the third part of this section (Section 2.7), we will discuss local solutions of the differential equations with respect to the connections at the apparent singularities. We will use these solutions for the definition of the 2-form ω on $\mathcal{M}_{\boldsymbol{t}_0, \boldsymbol{t}_{\mathrm{ra}}}$ (in Section 3.1 below).

We take a natural affine open covering $\{U_0, U_\infty\}$ of $\mathbb{P}^1$. Denote by x a coordinate on U_0 and by w a coordinate on U_∞. That is, $w = x^{-1}$ on $U_0 \cap U_\infty$. Let ∞ be the point $w = 0$ on U_∞. Set $E_k := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k)$. Here we define the vector bundle E_k by two trivializations $\{\varphi^{(k)}_{U_0} : E_k|_{U_0} \xrightarrow{\cong} \mathcal{O}^{\oplus 2}_{U_0}, \varphi^{(k)}_{U_\infty} : E_k|_{U_\infty} \xrightarrow{\cong} \mathcal{O}^{\oplus 2}_{U_\infty}\}$ such that

$$\begin{array}{ccc} E_k|_{U_0\cap U_\infty} & \xrightarrow{\varphi^{(k)}_{U_\infty}} & \mathcal{O}^{\oplus 2}_{U_0\cap U_\infty} \\ \Big\downarrow{=} & & \Big\downarrow{G_k} \\ E_k|_{U_0\cap U_\infty} & \xrightarrow{\varphi^{(k)}_{U_0}} & \mathcal{O}^{\oplus 2}_{U_0\cap U_\infty}, \end{array} \tag{2.1}$$

where $G_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x^k \end{smallmatrix}\right)$. Fix a tuple of complex numbers $\boldsymbol{t}_{\mathrm{ra}} = (t_i)_{i\in\{3,4,\ldots,\nu\}\cap I_{\mathrm{ra}}}$, where $t_i \neq t_j$ $(i \neq j)$ and $t_i \notin \{0, 1\}$.

§2.1. Apparent singularities

Take $\boldsymbol{t}_0 = (t_i)_{i\in\{3,4,\ldots,\nu\}} \in (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$ and set $D = n_1 \cdot 0 + n_2 \cdot 1 + n_3 \cdot t_3 + \cdots + n_\nu \cdot t_\nu + n_\infty \cdot \infty$. For a connection $(E_1, \nabla \colon E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D))$, we define the *apparent singularities of* (E_1, ∇) as follows. Consider the sequence of maps

$$\mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\subset} E_1 \xrightarrow{\nabla} E_1 \otimes \Omega^1_{\mathbb{P}^1}(D) \xrightarrow{\text{quotient}} (E_1/\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \Omega^1_{\mathbb{P}^1}(D) \cong \mathcal{O}_{\mathbb{P}^1}(n-2).$$

This composition is an $\mathcal{O}_{\mathbb{P}^1}$-morphism, and we denote it by $\varphi_\nabla \colon \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(n-2)$. We assume that the subbundle $\mathcal{O}_{\mathbb{P}^1}(1) \subset E_1$ is not ∇-invariant. Then φ_∇ is not the zero morphism. The $\mathcal{O}_{\mathbb{P}^1}$-morphism φ_∇ has $n-3$ zeros counted with multiplicity.

Definition 2.1. We define *apparent singularities of* (E_1, ∇) as

$$\mathrm{div}(\varphi_\nabla) \in |\mathcal{O}_{\mathbb{P}^1}(n-3)| \cong \mathrm{Sym}^{(n-3)}(\mathbb{P}^1).$$

By the trivialization $\varphi^{(1)}_{U_0} \colon E_1|_{U_0} \xrightarrow{\cong} \mathcal{O}^{\oplus 2}_{U_0}$, we have the description

$$\nabla|_{U_0} = d + \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \frac{dx}{P(x)},$$

where $P(x) := \prod_{i=1}^{\nu}(x - t_i)^{n_i}$ and A, B, C, D are polynomials such that $\deg(A) \leq n-2$, $\deg(B) \leq n-3$, $\deg(C) \leq n-1$, $\deg(D) \leq n-2$. Then the apparent singularities of (E_1, ∇) are zeros of the polynomial $B(x)$.

Assume that the apparent singularities of (E_1, ∇) consist of distinct points and all of them are distinct from the poles $t_1, \ldots, t_\nu, \infty$ of the connection ∇. We can define a birational bundle transformation

$$\phi_\nabla := \mathrm{id} \oplus \varphi_\nabla \colon \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \dashrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)$$

and consider the pushed-forward connection $(\phi_\nabla)_*\nabla$ on $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n-2)$. Then we have a transformation of a connection with bundle type $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)$:

$$(E_1,\nabla)\longmapsto(E_{n-2},(\phi_\nabla)_*\nabla). \tag{2.2}$$

The connection $(\phi_\nabla)_*\nabla$ has simple poles $q_1,\dots,q_{n-3}$ with residual eigenvalues 0 and -1 at each pole. Let D_{App} be the effective divisor $q_1+\cdots+q_{n-3}$. We may decompose $(\phi_\nabla)_*\nabla$ as

$$\begin{pmatrix}\nabla_{11} & \Phi_{12}\\ \Phi_{21} & \nabla_{22}\end{pmatrix}, \tag{2.3}$$

where $\nabla_{11}\colon\mathcal{O}_{\mathbb{P}^1}\to\mathcal{O}_{\mathbb{P}^1}\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}})$ and $\nabla_{22}\colon\mathcal{O}_{\mathbb{P}^1}(n-2)\to\mathcal{O}_{\mathbb{P}^1}(n-2)\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}})$ are connections. Moreover, $\Phi_{12}\colon\mathcal{O}_{\mathbb{P}^1}(n-2)\to\mathcal{O}_{\mathbb{P}^1}\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}})$ and $\Phi_{21}\colon\mathcal{O}_{\mathbb{P}^1}\to\mathcal{O}_{\mathbb{P}^1}(n-2)\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}})$ are $\mathcal{O}_{\mathbb{P}^1}$-morphisms. Since the birational bundle transformation ϕ_∇ is given by $\left(\begin{smallmatrix}1&0\\0&B(x)\end{smallmatrix}\right)$, the connection (2.3) has the description

$$\begin{pmatrix}\nabla_{11} & \Phi_{12}\\ \Phi_{21} & \nabla_{22}\end{pmatrix}\Bigg|_{U_0}=\begin{pmatrix}d+\frac{A(x)\,dx}{P(x)} & \frac{dx}{P(x)}\\ \frac{C(x)B(x)\,dx}{P(x)} & d+\frac{D(x)\,dx}{P(x)}-\sum_{j=1}^{n-3}\frac{dx}{x-q_j}\end{pmatrix}.$$

By an automorphism of the bundle E_{n-2}, we may normalize the connection $(\phi_\nabla)_*\nabla$ so that the normalized connection has the following conditions (for details, see [5, Prop. 3]):

- the connection ∇_{11} is the trivial connection, and
- the $\mathcal{O}_{\mathbb{P}^1}$-morphism Φ_{12} corresponds to the section

$$\frac{dx}{\prod_{i=1}^{\nu}(x-t_i)^{n_i}}\ (\text{on } U_0)\quad\text{and}\quad\frac{-w^{-n+2}\,dw}{w^2\prod_{i=1}^{\nu}(1/w-t_i)^{n_i}}\ (\text{on } U_\infty) \tag{2.4}$$

under the isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(n-2),\mathcal{O}_{\mathbb{P}^1}\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}}))\\ &\qquad\cong H^0(\mathcal{O}_{\mathbb{P}^1}(-n+2)\otimes\Omega^1_{\mathbb{P}^1}(D+D_{\mathrm{App}})).\end{aligned}$$

Automorphisms of E_{n-2} preserving these conditions of $(\phi_\nabla)_*\nabla$ are just scalars (see [5, Sect. 3]). For each $j=1,2,\dots,n-3$, the 0-eigendirection of the residue matrix of the normalized connection at q_j corresponds to a point $\boldsymbol{p}_j\in\mathbb{P}(E_{n-2})|_{q_j}\cong\mathbb{P}^1$. Here, this identification is given by the trivialization $\varphi^{(n-2)}_{U_0}$.

Definition 2.2. Since the (-1)-eigendirection is contained in the second factor of $E_{n-2}=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n-2)$, we have $p_j\in\mathbb{C}$ such that $\boldsymbol{p}_j=[1:p_j]$. We call p_j a *dual parameter with respect to an apparent singularity* q_j.

§2.2. Global normal form for rank-two linear irregular differential equations

In the previous section, we assign a point on $\mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$ to a connection $E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D)$. Conversely, take a point $\{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\}$ on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. Then we may construct a connection $E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D)$ such that $q_1 + \cdots + q_{n-3}$ is the apparent singularity and p_j is the dual parameter of q_j, $j = 1,2,\dots,n-3$ (Proposition 2.4 below). Now we discuss this construction. We define an effective divisor D_{App} on $\mathbb{P}^1$ as $D_{\mathrm{App}} = q_1 + \cdots + q_{n-3}$.

Definition 2.3. For the point on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$, let

$$\nabla^{(n-2)}_{\mathrm{DL}} \colon E_{n-2} \longrightarrow E_{n-2} \otimes \Omega^1_{\mathbb{P}^1}(D + D_{\mathrm{App}})$$

be a connection with the following connection matrix on U_0:

$$\Omega^{(n-2)} = \begin{pmatrix} 0 & \frac{1}{P(x)} \\ c_0(x) & d_0(x) \end{pmatrix} dx. \tag{2.5}$$

Here we put $P(x) \coloneqq \prod_{i=1}^{\nu}(x-t_i)^{n_i}$,

$$\begin{aligned} c_0(x) &\coloneqq \sum_{i=1}^{\nu} \frac{C_i(x)}{(x-t_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{p_j}{x-q_j} + \widetilde{C}(x) + x^{n-3} C_\infty(x), \\ d_0(x) &\coloneqq \sum_{i=1}^{\nu} \frac{D_i(x)}{(x-t_i)^{n_i}} + \sum_{j=1}^{n-3} \frac{-1}{x-q_j} + D_\infty(x), \end{aligned} \tag{2.6}$$

where C_i, D_i $(i = 1,\dots,\nu)$, C_∞, D_∞, and $\widetilde{C}$ are polynomials in x such that

- $\deg(C_i), \deg(D_i) \leq n_i - 1$ for $i = 1,\dots,\nu$,
- $\deg(C_\infty) \leq n_\infty - 1$, $\deg(D_\infty) \leq n_\infty - 2$,
- $\deg(\widetilde{C}) \leq n - 4$.

We assume that $q_1,\dots,q_{n-3}$ are apparent singularities, that is, the elementary transformation

$$((\tilde{\Phi}_{q_j})^{-1} d\tilde{\Phi}_{q_j} + (\tilde{\Phi}_{q_j})^{-1}\Omega^{(n-2)}\tilde{\Phi}_{q_j}), \quad \text{where } \tilde{\Phi}_{q_j} \coloneqq \begin{pmatrix} 1 & 0 \\ p_j & x - q_j \end{pmatrix},$$

of $\Omega^{(n-2)}$ by $\tilde{\Phi}_{q_j}$ has no pole at q_j. We call such a connection $\nabla^{(n-2)}_{\mathrm{DL}}$ *Diarra–Loray's global normal form.*

The corresponding connection matrix $\Omega_\infty^{(n-2)}$ on U_∞ of $\nabla_{\mathrm{DL}}^{(n-2)}$ is

$$\begin{aligned}\Omega_\infty^{(n-2)} &= G_{n-2}^{-1}\, dG_{n-2} + G_{n-2}^{-1}\Omega^{(n-2)}G_{n-2}\\ &= \begin{pmatrix} 0 & 0 \\ 0 & -n+2\end{pmatrix}\frac{dw}{w} + \begin{pmatrix} 0 & \frac{1}{w^{n-2}P(1/w)} \\ w^{n-2}c_0(1/w) & d_0(1/w)\end{pmatrix}\frac{-dw}{w^2}.\end{aligned}$$

We may check that $\Omega_\infty^{(n-2)}$ has a pole of order n_∞ at ∞. We decompose the connection $\nabla_{\mathrm{DL}}^{(n-2)}$ as in (2.3). Since the $(1,1)$-entry of $\Omega^{(n-2)}$ is zero and the $(1,2)$-entry of $\Omega^{(n-2)}$ is $\frac{dx}{P(x)}$, the connection ∇_{11} is the trivial connection and the $\mathcal{O}_{\mathbb{P}^1}$-morphism Φ_{12} corresponds to the section (2.4). The vector $(1,p_j)$ is a 0-eigenvector of the residue matrix of $\Omega^{(n-2)}$ at q_j.

Now we consider a transformation of the connection $\nabla_{\mathrm{DL}}^{(n-2)}$ on E_{n-2} into a connection on E_1. Set $Q_1(x)=\prod_{j=1}^{n-3}(x-q_j)$. Let $Q_2(x)$ be the unique polynomial of degree $n-4$ such that $Q_2(q_j)=p_j$ for $j=1,2,\dots,n-3$. Set

$$\begin{aligned}\widetilde{G} &:= \begin{pmatrix} 1 & 0 \\ Q_2(x) & Q_1(x)\end{pmatrix} : \mathcal{O}_{U_0}^{\oplus 2} \dashrightarrow \mathcal{O}_{U_0}^{\oplus 2},\\ \widetilde{G}_\infty &:= G_{n-2}^{-1}\begin{pmatrix} 1 & 0 \\ Q_2(1/w) & Q_1(1/w)\end{pmatrix} G_1 \colon \mathcal{O}_{U_\infty}^{\oplus 2} \dashrightarrow \mathcal{O}_{U_\infty}^{\oplus 2}.\end{aligned} \tag{2.7}$$

Let $\Omega^{(1)}$ be the transformation of $\Omega^{(n-2)}$ by $\widetilde{G}$:

$$\begin{aligned}\Omega^{(1)} &= \widetilde{G}^{-1}\, d\widetilde{G} + \widetilde{G}^{-1}\Omega^{(n-2)}\widetilde{G}\\ &= \begin{pmatrix} \frac{Q_2(x)}{P(x)} & \frac{Q_1(x)}{P(x)} \\ \frac{c_0(x)+Q_2(x)d_0(x)+(Q_2(x))'}{Q_1(x)} - \frac{(Q_2(x))^2}{P(x)Q_1(x)} & d_0(x) + \frac{(Q_1(x))'}{Q_1(x)} - \frac{Q_2(x)}{P(x)}\end{pmatrix} dx.\end{aligned} \tag{2.8}$$

Proposition 2.4. *We set*

$$\nabla_{\mathrm{DL}}^{(1)} := \begin{cases} d+\Omega^{(1)} & \text{on } U_0,\\ d+G_1^{-1}\, dG_1 + G_1^{-1}\Omega^{(1)}G_1 & \text{on } U_\infty.\end{cases}$$

Then $\nabla_{\mathrm{DL}}^{(1)}$ is a connection

$$\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow (\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))\otimes\Omega^1_{\mathbb{P}^1}(D).$$

That is, the pole divisor of $\nabla_{\mathrm{DL}}^{(1)}$ is D. Moreover, the apparent singularity of $\nabla_{\mathrm{DL}}^{(1)}$ is $q_1+\cdots+q_{n-3}$ and the dual parameter with respect to q_j is p_j.

Proof. First we will show that $\nabla_{\mathrm{DL}}^{(1)}$ has no pole at $q_1,\dots,q_{n-3}$ by induction. Let $s\in\{1,\dots,n-3\}$. We define $Q_1^{(s)}(x)=\prod_{j=1}^{s}(x-q_j)$ and $Q_2^{(s)}(x)$ is the unique polynomial of degree $s-1$ such that $Q_2^{(s)}(q_j) = p_j$ for $j = 1, 2, \dots, s$. Set

$$\widetilde{G}^{(s)} := \begin{pmatrix} 1 & 0 \\ Q_2^{(s)}(x) & Q_1^{(s)}(x) \end{pmatrix}.$$

Assume that $d + (\widetilde{G}^{(s)})^{-1}\, d\widetilde{G}^{(s)} + (\widetilde{G}^{(s)})^{-1}\Omega^{(n-2)}\widetilde{G}^{(s)}$ has no pole at $q_1, q_2, \dots, q_s$. We will show that $d + (\widetilde{G}^{(s+1)})^{-1}\, d\widetilde{G}^{(s+1)} + (\widetilde{G}^{(s+1)})^{-1}\Omega^{(n-2)}\widetilde{G}^{(s+1)}$ has no pole at $q_1, q_2, \dots, q_{s+1}$. We may check the equalities

$$\begin{aligned}
&(\widetilde{G}^{(s)})^{-1}\, d\widetilde{G}^{(s)} + (\widetilde{G}^{(s)})^{-1}\Omega^{(n-2)}\widetilde{G}^{(s)} \\
&= \begin{pmatrix} \frac{Q_2^{(s)}(x)}{P(x)} & \frac{Q_1^{(s)}(x)}{P(x)} \\ \frac{c_0(x)+Q_2^{(s)}(x)d_0(x)+(Q_2^{(s)}(x))'}{Q_1^{(s)}(x)} - \frac{(Q_2^{(s)}(x))^2}{P(x)Q_1^{(s)}(x)} & d_0(x) + \frac{(Q_1^{(s)}(x))'}{Q_1^{(s)}(x)} - \frac{Q_2^{(s)}(x)}{P(x)} \end{pmatrix} dx \\
&= \begin{pmatrix} 0 & 0 \\ \frac{p_{s+1}-Q_2^{(s)}(q_{s+1})}{Q_1^{(s)}(q_{s+1})} & -1 \end{pmatrix} \frac{dx}{x-q_{s+1}} + [\text{holomorphic parts}].
\end{aligned}$$

Here, the last equality is the expansion at $x = q_{s+1}$. Since q_{s+1} is an apparent singularity, we can transform the connection $d + (\widetilde{G}^{(s)})^{-1}\, d\widetilde{G}^{(s)} + (\widetilde{G}^{(s)})^{-1}\Omega^{(n-2)}\widetilde{G}^{(s)}$ into a connection which is holomorphic at q_{s+1} by the matrix

$$\begin{pmatrix} 1 & 0 \\ \frac{p_{s+1}-Q_2^{(s)}(q_{s+1})}{Q_1^{(s)}(q_{s+1})} & x - q_{s+1} \end{pmatrix}.$$

We may check that

$$\widetilde{G}_0^{(s+1)} = \widetilde{G}_0^{(s)} \begin{pmatrix} 1 & 0 \\ \frac{p_{s+1}-Q_2^{(s)}(q_{s+1})}{Q_1^{(s)}(q_{s+1})} & x - q_{s+1} \end{pmatrix}.$$

Then we have that $d + (\widetilde{G}_0^{(s+1)})^{-1}\, d\widetilde{G}_0^{(s+1)} + (\widetilde{G}_0^{(s+1)})^{-1}\Omega^{(n-2)}\widetilde{G}_0^{(s+1)}$ has no pole at $q_1, q_2, \dots, q_{s+1}$. So $\nabla_{\mathrm{DL}}^{(1)}$ has no pole at $q_1, \dots, q_{n-3}$ by induction. Since $\widetilde{G}_\infty$ is holomorphic at $w = 0$ and the determinant of $\widetilde{G}_\infty$ does not vanish at $w = 0$,

$$d + G_1^{-1}\, dG_1 + G_1^{-1}\Omega^{(1)}G_1 = d + \widetilde{G}_\infty^{-1}\, d\widetilde{G}_\infty + \widetilde{G}_\infty^{-1}\Omega_\infty^{(n-2)}\widetilde{G}_\infty$$

has a pole of order n_∞ at ∞. Then the polar divisor of $\nabla_{\mathrm{DL}}^{(1)}$ is D.

By (2.8), the $(1,2)$-term of $\Omega^{(1)}$ is $\frac{Q_1(x)}{P(x)}$. The apparent singularities of $\nabla_{\mathrm{DL}}^{(1)}$ are the zeros of $\frac{Q_1(x)}{P(x)}$. Then the apparent singularities of $\nabla_{\mathrm{DL}}^{(1)}$ are $q_1 + \dots + q_{n-3}$, since the birational bundle transformation ϕ_∇ is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & Q_1(x) \end{smallmatrix}\right)$. Moreover,

$$\begin{pmatrix} 1 & 0 \\ Q_2(x) & Q_1(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Q_2(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q_1(x) \end{pmatrix}.$$

Here, $\begin{pmatrix} 1 & 0 \\ Q_2(x) & 1 \end{pmatrix}$ is an automorphism of E_{n-2}. Since the vector $(1, p_j)$ is a 0-eigenvector of the residue matrix of $\Omega^{(n-2)}$ at q_j, the dual parameter with respect to q_j is p_j. □

§2.3. Local formal data

We put $x_{t_i} = (x - t_i)$ for $i = 1, \dots, \nu$ and $x_\infty = w$. Put $I := \{1, 2, \dots, \nu, \infty\}$, $t_1 := 0$, $t_2 := 1$, and $t_\infty := \infty \in \mathbb{P}^1$. We take a decomposition $I = I_{\mathrm{reg}} \cup I_{\mathrm{un}} \cup I_{\mathrm{ra}}$ such that I_{reg}, I_{un}, and I_{ra} are disjoint from each other. We assume that $n_i = 1$ for $i \in I_{\mathrm{reg}}$ and $n_i > 1$ for $i \in I_{\mathrm{un}} \cup I_{\mathrm{ra}}$.

Let ∇ be a connection on E_1:

$$\nabla \colon E_1 \longrightarrow E_1 \otimes \Omega^1_{\mathbb{P}^1}(D),$$

where $D = \sum_{i \in I} n_i \cdot t_i$. For each $i \in I$, we take an affine open subset $U_i \subset \mathbb{P}^1$ such that $t_i \in U_i$. We take a trivialization $E_1|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus 2}$ and choose the coordinate x_{t_i} on U_i such that the point t_i is defined by $x_{t_i} = 0$. Let Ω be the connection matrix of ∇ associated to this trivialization. We may describe Ω as

$$\Omega = \Omega_{t_i}(0) \frac{dx_{t_i}}{x_{t_i}^{n_i}} + [\text{higher-order terms}], \quad \Omega_{t_i}(0) \in \mathfrak{gl}(2, \mathbb{C})$$

for each $i \in I$.

Definition 2.5. We say t_i is an *unramified irregular singular point of* ∇ if $n_i > 1$ and $\Omega_{t_i}(0)$ has distinct eigenvalues.

Let $(\mathrm{Det}(E_1), \mathrm{Tr}(\nabla))$ be the determinant bundle of E_1 with the induced connection $\mathrm{Tr}(\nabla)$, that is, $\mathrm{Tr}(\nabla) = \nabla \wedge \mathrm{id} + \mathrm{id} \wedge \nabla$. We consider the trivialization $\mathrm{Det}(E_1)|_{U_i} \cong \mathcal{O}_{U_i}$ induced by the trivialization of $E_1|_{U_i}$. Let $\alpha' \in \mathcal{O}_{U_i}\, dx_{t_i}/x_{t_i}^{n_i}$ be the connection matrix of $\mathrm{Tr}(\nabla)|_{U_i}$ associated to this trivialization (if necessary, U_i shrinks). We consider the tensor product $(\mathcal{O}_{U_i}^{\oplus 2}, d + \Omega) \otimes (\mathcal{O}_{U_i}, d - \frac{1}{2}\alpha')$. Let $N(x_{t_i})\, dx_{t_i}/x_{t_i}^{n_i}$ be the connection matrix of this tensor product. Remark that $N(x_{t_i}) \in \mathrm{End}(\mathcal{O}_{U_i}^{\oplus 2})$.

Definition 2.6. We say t_i is a *ramified irregular singular point of* ∇ if $n_i > 1$, $N(0)$ is a nonzero nilpotent matrix, and $N(x_{t_i})^2 \not\equiv 0 \pmod{x_{t_i}^2}$.

We assume that

- the differences of the eigenvalues of $\Omega_{t_i}(0)$ are not integers for any $i \in I_{\mathrm{reg}}$,
- t_i are unramified irregular singular points for any $i \in I_{\mathrm{un}}$, and
- t_i are ramified irregular singular points for any $i \in I_{\mathrm{ra}}$.

Lemma 2.7 (For example [5, Props. 9 and 10]). *Let Ω be the connection matrix which satisfies the assumption above.*

(1) *If $i \in I_{\mathrm{un}}$, then there exists a matrix $M \in \mathrm{GL}_2(\mathbb{C}[\![x_{t_i}]\!])$ such that*

$$M^{-1}\,dM + M^{-1}\Omega M = \frac{\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix}}{x_{t_i}^{n_i}}\,dx_{t_i} + \cdots + \frac{\begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix}}{x_{t_i}}\,dx_{t_i}.$$

We call the tuple $((\theta^+_{0,t_i},\theta^-_{0,t_i}),\dots,(\theta^+_{n_i-1,t_i},\theta^-_{n_i-1,t_i}))$ the local formal data of ∇ at t_i.

(2) *If $i \in I_{\mathrm{ra}}$, then there exists a matrix $M \in \mathrm{GL}_2(\mathbb{C}[\![x_{t_i}]\!])$ such that*

$$M^{-1}\,dM + M^{-1}\Omega M = \begin{pmatrix} \alpha_i & \beta_i \\ x_{t_i}\beta_i & \alpha_i - \frac{dx_{t_i}}{2x_{t_i}} \end{pmatrix},$$

where

$$\begin{cases} \alpha_i := \dfrac{\theta_{0,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \dfrac{\theta_{2l,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \dfrac{\theta_{2n_i-2,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}}, \\ \beta_i := \dfrac{\theta_{1,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \dfrac{\theta_{2l+1,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \dfrac{\theta_{2n_i-3,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^2}. \end{cases}$$

We call the tuple $(\theta_{0,t_i},\dots,\theta_{2n_i-2,t_i})$ the local formal data of ∇ at t_i.

If we define ζ_i as $x_{t_i} = \zeta_i^2$ and put

$$M_{\zeta_i} := \begin{pmatrix} 1 & 1 \\ \zeta_i & -\zeta_i \end{pmatrix}, \tag{2.9}$$

then we have the following diagonalization:

$$\begin{aligned} &M_{\zeta_i}^{-1}\,dM_{\zeta_i} + M_{\zeta_i}^{-1}\begin{pmatrix} \alpha_i & \beta_i \\ x_{t_i}\beta_i & \alpha_i - \frac{dx_{t_i}}{2x_{t_i}} \end{pmatrix} M_{\zeta_i} \\ &= \sum_{l=0,1,\dots,n_i-1} \begin{pmatrix} \frac{\theta_{2l,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-1}} & 0 \\ 0 & \frac{\theta_{2l,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-1}} \end{pmatrix} + \sum_{l=0,1,\dots,n_i-2} \begin{pmatrix} \frac{\theta_{2l+1,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-2}} & 0 \\ 0 & -\frac{\theta_{2l+1,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-2}} \end{pmatrix}. \end{aligned}$$

Definition 2.8. Let $(\boldsymbol{\theta},\boldsymbol{\theta}_0) \in T_{\boldsymbol{\theta}} \times T^{\mathrm{res}}_{\boldsymbol{\theta}}$.

- Let $\nabla\colon E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D)$ be a connection. If the tuple of the local formal data of ∇ is $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$, we call this connection a $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-*connection on* E_1.
- We say that Diarra–Loray's normal form $\nabla^{(n-2)}_{\mathrm{DL}}\colon E_{n-2} \to E_{n-2} \otimes \Omega^1_{\mathbb{P}^1}(D + D_{\mathrm{App}})$ is a $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-*connection* if the corresponding $\nabla^{(1)}_{\mathrm{DL}}\colon E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D)$ is a $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-connection on E_1.

Lemma 2.9 ([5, Lems. 16 and 19]). *Let* $(\{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\},(t_3,\dots,t_\nu)) \in \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}$ *and* $(\boldsymbol{\theta},\boldsymbol{\theta}_0)\in T_{\boldsymbol{\theta}}\times T^{\mathrm{res}}_{\boldsymbol{\theta}}$. *Set* $D=n_1\cdot 0+n_2\cdot 1+\sum_{i=3}^{\nu} n_i\cdot t_i+n_\infty\cdot\infty$. *There exists a unique tuple of the polynomials* $((C_i,D_i)_{i\in I},\widetilde{C})$ *in* (2.6) *such that*

- *the polar divisor of* $\nabla^{(n-2)}_{\mathrm{DL}}$ *is* $D+q_1+\cdots+q_{n-3}$,
- $q_1,\dots,q_{n-3}$ *are apparent singularities,*
- *the dual parameter with respect to* q_j *is* p_j $(j=1,2,\dots,n-3)$,
- $\nabla^{(n-2)}_{\mathrm{DL}}$ *is a* $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$*-connection.*

Let $((C_i,D_i)_{i\in I},\widetilde{C})$ be the tuple of the polynomials in Lemma 2.9. The polynomials C_i and D_i $(i\in I)$ have simple descriptions. Now we give explicit descriptions of C_i and D_i $(i\in I)$. For $i\in I_{\mathrm{reg}}\cup I_{\mathrm{un}}$, we define a polynomial $\Theta_i^{\pm}$ in x as

$$\sum_{l=0}^{n_i-1}\frac{\theta^{\pm}_{l,t_i}}{(x-t_i)^{n_i-l}}=\frac{\Theta_i^{\pm}}{(x-t_i)^{n_i}}.$$

For $i\in I_{\mathrm{ra}}$, we define polynomials A_i and B_i in x as

$$\sum_{l=0}^{n_i-1}\frac{\theta_{2l,t_i}}{2(x-t_i)^{n_i-l}}=\frac{A_i}{(x-t_i)^{n_i}}\quad\text{and}\quad\sum_{l=0}^{n_i-2}\frac{\theta_{2l+1,t_i}}{2(x-t_i)^{n_i-l}}=\frac{B_i}{(x-t_i)^{n_i}},$$

respectively. For $i\in I_{\mathrm{reg}}\cup I_{\mathrm{un}}$, the polynomials C_i and D_i have the description

$$\begin{cases} C_i=-\left(\Theta_i^+\Theta_i^-\prod_{j\neq i}(x-t_j)^{n_j}\right)\bmod (x-t_i)^{n_i},\\ D_i=\Theta_i^++\Theta_i^-. \end{cases}\tag{2.10}$$

For $i\in I_{\mathrm{ra}}$, the polynomials C_i and D_i have the description

$$\begin{cases} C_i=-\Bigg(\left(A_i^2-\dfrac{(x-t_i)^{n_i-1}}{2}A_i-(x-t_i)B_i^2\right)\\ \qquad\cdot\prod_{j\neq i}(x-t_j)^{n_j}\Bigg)\bmod (x-t_i)^{n_i},\\ D_i=2A_i-\dfrac{(x-t_i)^{n_i-1}}{2}. \end{cases}\tag{2.11}$$

§2.4. Family of connections

Let $(\boldsymbol{\theta},\boldsymbol{\theta}_0)\in T_{\boldsymbol{\theta}}\times T^{\mathrm{res}}_{\boldsymbol{\theta}}$ and $\boldsymbol{t}_0=(t_3,\dots,t_\nu)\in(T_{\boldsymbol{t}})_{\boldsymbol{t}_0}$. Set $D=n_1\cdot 0+n_2\cdot 1+n_3\cdot t_3+\cdots+n_\nu\cdot t_\nu+n_\infty\cdot\infty$. Let $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ be the moduli space of $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-connections satisfying some generic conditions:

$$\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} := \left\{ (E,\nabla) \;\middle|\; \begin{array}{l} E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \text{ and} \\ \nabla \text{ is a } (\boldsymbol{\theta},\boldsymbol{\theta}_0)\text{-connection such that} \\ \mathcal{O}_{\mathbb{P}^1}(1) \subset E_1 \text{ is not } \nabla\text{-invariant,} \\ D_{\mathrm{App}} \text{ is reduced, and} \\ D_{\mathrm{App}} \text{ has disjoint support with } D \end{array} \right\} \Big/ \sim . \tag{2.12}$$

Here, $(E,\nabla) \sim (E',\nabla')$ means that there exists an isomorphism $\varphi\colon E \to E'$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & E \times \Omega^1_{\mathbb{P}^1}(D) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi\otimes \mathrm{id}} \\ E' & \xrightarrow{\nabla'} & E' \times \Omega^1_{\mathbb{P}^1}(D). \end{array}$$

By taking apparent singularities and the dual parameters from a connection $(E_1,\nabla) \in \mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$, we may define a map

$$\begin{aligned} \mathrm{App}\colon \mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} &\longrightarrow \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathbb{C}^2), \\ (E,\nabla) &\longmapsto \{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\}. \end{aligned}$$

Now we construct an inverse map of App as follows. Let d be the relative exterior derivative of $\mathbb{P}^1 \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \to \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. By Definition 2.3 and Lemma 2.9, we may construct an algebraic family

$$\widetilde{\nabla}^{(n-2)}_{\mathrm{DL}} = \begin{cases} d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} & \text{on } U_0 \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \\ d + G_{n-2}^{-1}\, dG_{n-2} + G_{n-2}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} G_{n-2} & \text{on } U_\infty \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \end{cases}$$

of $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-connections on E_{n-2} parametrized by $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. We set

$$\Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} = \widetilde{G}^{-1}\, d\widetilde{G} + \widetilde{G}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\widetilde{G}.$$

Then we have an algebraic family

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL}} = \begin{cases} d + \Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} & \text{on } U_0 \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \\ d + G_1^{-1}\, dG_1 + G_1^{-1}\Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} G_1 & \text{on } U_\infty \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \end{cases}$$

of $(\boldsymbol{\theta},\boldsymbol{\theta}_0)$-connections $E_1 \to E_1 \otimes \Omega^1_{\mathbb{P}^1}(D)$ parametrized by $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ by Proposition 2.4. The algebraic family $\widetilde{\nabla}^{(1)}_{\mathrm{DL}}$ parametrized by $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ gives the inverse map of App:

$$\begin{aligned} \mathrm{App}^{-1}\colon \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} &\longrightarrow \mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}, \\ \boldsymbol{p} = \{(q_1,p_1),\dots,(q_{n-3},p_{n-3})\} &\longmapsto (E_1, \widetilde{\nabla}^{(1)}_{\mathrm{DL}}|_{\mathbb{P}^1\times\{\boldsymbol{p}\}}). \end{aligned}$$

Next we consider the extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ of $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. We set $D(\boldsymbol{t}_0) := n_1 \cdot 0 + n_2 \cdot 1 + \sum_{i=3}^{\nu} n_i \cdot t_i + n_\infty \cdot \infty$ for $\boldsymbol{t}_0 = (t_3, \ldots, t_\nu) \in (T_{\boldsymbol{t}})_{\boldsymbol{t}_0}$. This extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ is defined by

$$\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} := \left\{ (E, \nabla, \boldsymbol{t}_0, \boldsymbol{\theta}) \;\middle|\; \begin{array}{l} \boldsymbol{t}_0 \in (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}},\ \boldsymbol{\theta} \in T_{\boldsymbol{\theta}},\ E \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \text{ and} \\ \nabla \text{ is a } (\boldsymbol{\theta},\boldsymbol{\theta}_0)\text{-connection} \\ \text{such that the polar divisor of } \nabla \text{ is } D(\boldsymbol{t}_0), \\ \mathcal{O}_{\mathbb{P}^1}(1) \subset E_1 \text{ is not } \nabla\text{-invariant}, \\ D_{\mathrm{App}} \text{ is reduced, and} \\ D_{\mathrm{App}} \text{ has disjoint support with } D(\boldsymbol{t}_0) \end{array} \right\} \Big/ \sim .$$

Here, $(E, \nabla, \boldsymbol{t}_0, \boldsymbol{\theta}) \sim (E', \nabla', \boldsymbol{t}'_0, \boldsymbol{\theta}')$ means that $\boldsymbol{t}_0 = \boldsymbol{t}'_0$, $\boldsymbol{\theta} = \boldsymbol{\theta}'$, and there exists an isomorphism $\varphi\colon E \to E'$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\ \nabla\ } & E \times \Omega^1_{\mathbb{P}^1}(D) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi\otimes \mathrm{id}} \\ E' & \xrightarrow{\ \nabla'\ } & E' \times \Omega^1_{\mathbb{P}^1}(D). \end{array}$$

By taking apparent singularities, the dual parameters, the position of singular points, and the local formal data from a connection $(E_1, \nabla) \in \widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$, we may define a map

$$\begin{aligned} \widehat{\mathrm{App}}\colon \widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} &\longrightarrow \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \subset (\mathrm{Sym}^{(n-3)}(\mathbb{C}^2) \times T_{\boldsymbol{t}}) \times T_{\boldsymbol{\theta}}, \\ (E, \nabla, \boldsymbol{t}_0, \boldsymbol{\theta}) &\longmapsto ((\{(q_1,p_1), \ldots, (q_{n-3}, p_{n-3})\}, \boldsymbol{t}_0), \boldsymbol{\theta}). \end{aligned} \tag{2.13}$$

Now we may also construct an inverse map of $\widehat{\mathrm{App}}$ as follows. Here, let d be the relative exterior derivative of $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \to \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. By Definition 2.3 and Lemma 2.9, we may construct an algebraic family

$$\widetilde{\nabla}^{(n-2)}_{\mathrm{DL,ext}} = \begin{cases} d + \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \\ d + G_{n-2}^{-1}\, dG_{n-2} + G_{n-2}^{-1} \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} G_{n-2} & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \end{cases}$$

of connections on E_{n-2} parametrized by $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. We set

$$\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} = \widetilde{G}^{-1}\, d\widetilde{G} + \widetilde{G}^{-1} \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} \widetilde{G}.$$

Then we have an algebraic family

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}} = \begin{cases} d + \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \\ d + G_1^{-1}\, dG_1 + G_1^{-1} \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} G_1 & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \end{cases} \tag{2.14}$$

of connections on E_1 parametrized by $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by Proposition 2.4. Let $\tilde{\boldsymbol{t}}_0 = (\tilde{t}_1, \ldots, \tilde{t}_\nu, \tilde{t}_\infty)$ be a family of $(\nu+1)$-points on $\mathbb{P}^1$ parametrized by $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$ and $\tilde{\boldsymbol{\theta}}$ be a family of tuples of complex numbers parametrized by $T_{\boldsymbol{\theta}}$. We denote by the same characters $\tilde{\boldsymbol{t}}_0$ and $\tilde{\boldsymbol{\theta}}$ the pull-backs of $\tilde{\boldsymbol{t}}_0$ and $\tilde{\boldsymbol{\theta}}$ under the compositions

$$\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \xrightarrow{\mathrm{id}\times\pi_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}} (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \xrightarrow{\text{projection}} (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}},$$
$$\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \xrightarrow{\mathrm{id}\times\pi_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}} (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \xrightarrow{\text{projection}} T_{\boldsymbol{\theta}},$$

respectively. The algebraic family $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ parametrized by $\mathcal{M}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ gives the inverse map of $\widehat{\mathrm{App}}$:

$$\widehat{\mathrm{App}}^{-1} : \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \longrightarrow \widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)},$$
$$\hat{\boldsymbol{p}} = ((\{(q_1,p_1),\ldots,(q_{n-3},p_{n-3})\},\boldsymbol{t}_0),\boldsymbol{\theta}) \longmapsto (E_1, \widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}|_{\mathbb{P}^1\times\{\hat{\boldsymbol{p}}\}}, \tilde{\boldsymbol{t}}_0|_{\hat{\boldsymbol{p}}}, \tilde{\boldsymbol{\theta}}|_{\hat{\boldsymbol{p}}}).$$

§2.5. Infinitesimal deformations of connections

Let U be an open subset of $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$. Let δ be a vector field on U. By the vector field δ, we have a map

$$\mathbb{P}^1 \times \mathrm{Spec}\,\mathcal{O}_U[\varepsilon] \longrightarrow \mathbb{P}^1 \times U, \tag{2.15}$$

where $\varepsilon^2 = 0$. We take the pull-back of the family $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL}}|_{\mathbb{P}^1\times U}$ under the map (2.15). We denote the expansion of this pull-back of $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL}}|_{\mathbb{P}^1\times U}$ with respect to ε by

$$\begin{cases} d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} + \varepsilon\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) & \text{on } U_0 \times \mathrm{Spec}\,\mathcal{O}_U[\varepsilon], \\ d + G_{n-2}^{-1}\,dG_{n-2} + G_{n-2}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}G_{n-2} & \\ \qquad + \varepsilon G_{n-2}^{-1}\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})G_{n-2} & \text{on } U_\infty \times \mathrm{Spec}\,\mathcal{O}_U[\varepsilon]. \end{cases}$$

We also denote the expansion of the pull-back of the family $\widetilde{\nabla}^{(1)}_{\mathrm{DL}}|_{\mathbb{P}^1\times U}$ under the map (2.15) by

$$\begin{cases} d + \Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} + \varepsilon\delta(\Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) & \text{on } U_0 \times \mathrm{Spec}\,\mathcal{O}_U[\varepsilon], \\ d + G_1^{-1}\,dG_1 + G_1^{-1}\Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}G_1 + \varepsilon G_1^{-1}\delta(\Omega^{(1)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})G_1 & \text{on } U_\infty \times \mathrm{Spec}\,\mathcal{O}_U[\varepsilon]. \end{cases}$$

Let $\widehat{U}$ be an open subset of $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. Let $\hat{\delta}$ be a vector field on $\widehat{U}$. By the vector field $\hat{\delta}$, we have a map

$$\mathbb{P}^1 \times \mathrm{Spec}\,\mathcal{O}_{\widehat{U}}[\varepsilon] \longrightarrow \mathbb{P}^1 \times \widehat{U}, \tag{2.16}$$

where $\varepsilon^2 = 0$. We take the pull-back of the family $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL,ext}}|_{\mathbb{P}^1\times\widehat{U}}$ under the map (2.16). We denote the expansion of this pull-back of $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL,ext}}|_{\mathbb{P}^1\times\widehat{U}}$ with respect to

ε by

$$\begin{cases} d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \text{on } U_0\times \operatorname{Spec}\mathcal{O}_{\widehat{U}}[\varepsilon] \\ d+G_{n-2}^{-1}\,dG_{n-2}+G_{n-2}^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_{n-2}, & \\ \qquad +\varepsilon G_{n-2}^{-1}\delta(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_{n-2} & \text{on } U_\infty\times \operatorname{Spec}\mathcal{O}_{\widehat{U}}[\varepsilon]. \end{cases}$$

Here, this $\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ means the pull-back of $\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ on $U_0\times\widehat{U}$ by the trivial projection $U_0\times\operatorname{Spec}\mathcal{O}_{\widehat{U}}[\varepsilon]\to U_0\times\widehat{U}$. Remark that there is a difference between this $\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ and the pull-back of $\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ on $U_0\times\widehat{U}$ by (2.16). The ε-part $\delta(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})$ adjusts this difference. We also denote the expansion of the pull-back of the family $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}|_{\mathbb{P}^1\times\widehat{U}}$ under the map (2.16) by

$$\begin{cases} d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \text{on } U_0\times \operatorname{Spec}\mathcal{O}_{\widetilde{U}}[\varepsilon], \\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1+\varepsilon G_1^{-1}\delta(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1 & \text{on } U_\infty\times \operatorname{Spec}\mathcal{O}_{\widetilde{U}}[\varepsilon]. \end{cases}$$

§2.6. Horizontal lifts of a family of connections

Let $\widehat{E}_1$ and $\widehat{E}_{n-2}$ be the pull-backs of E_1 and E_{n-2}, respectively, under the projection $\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\to\mathbb{P}^1$. Set $D(\tilde{\boldsymbol{t}}_0):=\sum_{i=1}^{\nu}n_i\cdot\tilde{t}_i+n_\infty\cdot\tilde{t}_\infty$, which is a Cartier divisor on $\mathbb{P}^1\times\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$, which is flat over $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. Let

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}=\begin{cases} d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}), \\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}), \end{cases}$$

be the family (2.14). This family is a relative connection on $\widehat{E}_1$:

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}\colon \widehat{E}_1\longrightarrow\widehat{E}_1\otimes\Omega^1_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})/\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}}(D(\tilde{\boldsymbol{t}}_0)).$$

We will consider an infinitesimal deformation of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$, which means an "integrable deformation".

Let $\widehat{E}_1^{\varepsilon}$ be the pull-back of $\widehat{E}_1$ under the trivial projection

$$\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\longrightarrow\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}). \tag{2.17}$$

Let δ_{time} be a vector field on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. The vector field δ_{time} gives a map

$$\pi_{\delta_{\mathrm{time}}}\colon((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\longrightarrow(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}.$$

Set

$$D_{\mathrm{red}}(\tilde{\boldsymbol{t}}_0):=\sum_{i\in I}\tilde{t}_i.$$

We consider $D_{\mathrm{red}}(\tilde{\boldsymbol{t}}_0)$ as a Cartier divisor on $\mathbb{P}^1\times((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})$. We denote by $D(\tilde{\boldsymbol{t}}_0)_\varepsilon$ and $D_{\mathrm{red}}(\tilde{\boldsymbol{t}}_0)_\varepsilon$ the pull-backs of $D(\tilde{\boldsymbol{t}}_0)$ and $D_{\mathrm{red}}(\tilde{\boldsymbol{t}}_0)$, respectively, under the composition

$$\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \xrightarrow{\mathrm{id}\times\pi_{\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0}\times\mathrm{id}} \mathbb{P}^1 \times ((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$$

$$\xrightarrow{\mathrm{id}\times\pi_{\delta_{\rm time}}} \mathbb{P}^1 \times (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}. \tag{2.18}$$

Take local defining equations $\tilde{x}_{t_i}$ of the Cartier divisor $D_{\rm red}(\tilde{\boldsymbol{t}}_0)_\varepsilon$. Let $\widetilde{\Omega}^1_{\delta_{\rm time}}$ be a coherent subsheaf of $\Omega^1_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]/\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}}}(D(\tilde{\boldsymbol{t}}_0)_\varepsilon)$ which is locally defined by

$$\widetilde{\Omega}^1_{\delta_{\rm time}} = \mathcal{O}_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]} \frac{d\tilde{x}_{t_i}}{\tilde{x}_{t_i}^{n_i}} + \mathcal{O}_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})} \frac{d\varepsilon}{\tilde{x}_{t_i}^{n_i-1}}. \tag{2.19}$$

Let $\nabla_{\delta_{\rm time}} \colon \widehat{E}_1^\varepsilon \to \widehat{E}_1^\varepsilon \otimes \widetilde{\Omega}^1_{\delta_{\rm time}}$ be a morphism with the Leibniz rule. That is, $\nabla_{\delta_{\rm time}}(fa) = a\otimes\hat{d}f + f\nabla_{\delta_{\rm time}}(a)$ for $f \in \mathcal{O}_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]}$ and $a \in \widehat{E}_1^\varepsilon$. Here, $\hat{d}$ is the relative exterior derivative of $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$. We denote the expansion of the morphism $\nabla_{\delta_{\rm time}}$ with respect to ε by

$$\nabla_{\delta_{\rm time}} = \begin{cases} \hat{d} + \widehat{\Omega}^{(1)}_{\delta_{\rm time}} + \varepsilon\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}}) + \Upsilon_{\delta_{\rm time}}\, d\varepsilon & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon], \\ \hat{d} + G_1^{-1}\, dG_1 + G_1^{-1}\widehat{\Omega}^{(1)}_{\delta_{\rm time}} G_1 & \\ \qquad + \varepsilon G_1^{-1}\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}}) G_1 & \\ \qquad + G_1^{-1}\Upsilon_{\delta_{\rm time}} G_1\, d\varepsilon & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]. \end{cases}$$

Here, the connection matrices are decomposed into dx_{t_i}-terms and $d\varepsilon$-terms. Remark that dx_{t_i} is the pull-back of dx_{t_i} under the morphism $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \mathbb{P}^1 \times (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ defined by $\pi_{\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0}$ and the trivial projection. On the other hand, $d\tilde{x}_{t_i}$ is the pull-back of dx_{t_i} under the morphism $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \mathbb{P}^1 \times (T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ defined in (2.18). Moreover, remark that $\widehat{\Omega}^{(1)}_{\delta_{\rm time}}$ has a pole on the divisor $D(\tilde{\boldsymbol{t}}_0) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$, which is different from the divisor $D(\tilde{\boldsymbol{t}}_0)_\varepsilon$. So $\widehat{\Omega}^{(1)}_{\delta_{\rm time}}$ does not belong to $\mathcal{E}nd(\widehat{E}_1^\varepsilon) \otimes \widetilde{\Omega}^1_{\delta_{\rm time}}$. The ε-term $\varepsilon\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}}) + \Upsilon_{\delta_{\rm time}}\, d\varepsilon$ adjusts the condition that the image of $\nabla_{\delta_{\rm time}}$ is contained in $\widehat{E}_1^\varepsilon \otimes \widetilde{\Omega}^1_{\delta_{\rm time}}$.

Let $\overline{\nabla}_{\delta_{\rm time}}$ be the relative connection induced by $\nabla_{\delta_{\rm time}}$:

$$\overline{\nabla}_{\delta_{\rm time}} \colon \widehat{E}_1^\varepsilon \longrightarrow \widehat{E}_1^\varepsilon \otimes \Omega^1_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]/(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]}(D(\tilde{\boldsymbol{t}}_0)_\varepsilon).$$

That is,

$$\overline{\nabla}_{\delta_{\rm time}} = \begin{cases} d + \widehat{\Omega}^{(1)}_{\delta_{\rm time}} + \varepsilon\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}}) & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon], \\ d + G_1^{-1}\, dG_1 + G_1^{-1}\widehat{\Omega}^{(1)}_{\delta_{\rm time}} G_1 & \\ \qquad + \varepsilon G_1^{-1}\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}}) G_1 & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]. \end{cases}$$

We consider $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}$ as a matrix with values in $\Omega^1_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})}(D(\tilde{\boldsymbol{t}}_0))$. We take a pull-back of the matrix $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}$ by the trivial projection

$$U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}}).$$

This pull-back induces a matrix with values in

$$\Omega^1_{\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]/(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}})}(D(\tilde{\boldsymbol{t}}_0)\times\operatorname{Spec}\mathbb{C}[\varepsilon]).$$

We also denote this induced matrix by $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}$.

Definition 2.10. We say $\nabla_{\delta_{\rm time}}$ is a *horizontal lift of* $\widetilde{\nabla}^{(1)}_{\rm DL,ext}$ if $\nabla_{\delta_{\rm time}}$ satisfies $\widehat{\Omega}^{(1)}_{\delta_{\rm time}}=\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}$ and the integrable condition

$$\delta(\widehat{\Omega}^{(1)}_{\delta_{\rm time}})\wedge d\varepsilon=d\Upsilon_{\delta_{\rm time}}\wedge d\varepsilon+[\widehat{\Omega}^{(1)}_{\delta_{\rm time}},\Upsilon_{\delta_{\rm time}}]\wedge d\varepsilon.$$

If $\nabla_{\delta_{\rm time}}$ is a horizontal lift of $\widetilde{\nabla}^{(1)}_{\rm DL,ext}$, the relative connection $\overline{\nabla}_{\delta_{\rm time}}$ means an integrable deformation of $\widetilde{\nabla}^{(1)}_{\rm DL,ext}$.

The construction of horizontal lifts of $\widetilde{\nabla}^{(1)}_{\rm DL,ext}$ is discussed in Sections 3.3, 3.4, and 4.2.

§2.7. Solutions of $d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}=0$ at the apparent singularities

Since q_j $(j=1,2,\dots,n-3)$ are apparent singularities, then we have the following lemma:

Lemma 2.11. *For each* $j\in\{1,\dots,n-3\}$*, the equation* $(d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\Psi=0$ *has a solution* $\psi_{q_j}=\Phi_{q_j}\Xi_{q_j}(x)\Lambda_{q_j}(x)$ *at* q_j*. Here,*

$$\Phi_{q_j}:=\begin{pmatrix}1&0\\p_j&1\end{pmatrix},\qquad\Lambda_{q_j}(x):=\begin{pmatrix}1&0\\0&x-q_j\end{pmatrix},$$
$$\Xi_{q_j}(x):=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{s=1}^{\infty}\begin{pmatrix}(\xi^{q_j}_s)_{11}&(\xi^{q_j}_s)_{12}\\(\xi^{q_j}_s)_{21}&(\xi^{q_j}_s)_{22}\end{pmatrix}(x-q_j)^s, \tag{2.20}$$

where $(\xi^{q_j}_1)_{11}=-\frac{p_j}{P(q_j)}$, $(\xi^{q_j}_1)_{12}=-\frac{1}{2P(q_j)}$, $(\xi^{q_j}_1)_{21}=0$, *and*

$$(\xi^{q_j}_1)_{22}=\frac{p_j}{P(q_j)}-\sum_{i=1}^{\nu}\frac{D_i(q_j)}{(q_j-t_i)^{n_i}}+\sum_k\frac{1}{q_j-q_k}-D_\infty(q_j).$$

The solution ψ_{q_j} *has converging entries.*

Proof. The connection matrix $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ has the following description at q_j by (2.5):

$$\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} = \begin{pmatrix} 0 & 0 \\ \frac{p_j}{x-q_j} & \frac{-1}{x-q_j} \end{pmatrix} dx + \begin{pmatrix} 0 & \frac{1}{P(q_j)} \\ c^{(0)}_{q_j} & d^{(0)}_{q_j} \end{pmatrix} dx + O(x-q_j).$$

Here we set

$$c^{(0)}_{q_j} := \sum_{i=1}^{\nu} \frac{C_i(q_j)}{(q_j-t_i)^{n_i}} + \sum_{k\neq j} \frac{p_j}{q_j-q_k} + \widetilde{C}(q_j) + q_j^{n-3}C_\infty(q_j),$$

$$d^{(0)}_{q_j} := \sum_{i=1}^{\nu} \frac{D_i(q_j)}{(q_j-t_i)^{n_i}} + \sum_{k\neq j} \frac{-1}{q_j-q_k} + D_\infty(q_j).$$

Let Φ_{q_j} be the matrix in (2.20). We may check the equality

$$\Phi_{q_j}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi_{q_j} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{x-q_j} \end{pmatrix} dx + \begin{pmatrix} \frac{p_j}{P(q_j)} & \frac{1}{P(q_j)} \\ 0 & d^{(0)}_{q_j} - \frac{p_j}{P(q_j)} \end{pmatrix} dx + O(x-q_j).$$

Here, the $(2,1)$-entry of this constant term is zero, since q_j is an apparent singular point. Since $\Phi_{q_j}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi_{q_j}$ has simple pole at q_j and q_j is an apparent singular point, there exists a convergent power series $\Xi_{q_j}(x) = \mathrm{id} + \Xi^{(1)}_{q_j}(x-q_j) + \cdots$ such that

$$\begin{aligned} &(\Phi_{q_j}\Xi_{q_j}(x))^{-1}d(\Phi_{q_j}\Xi_{q_j}(x)) + (\Phi_{q_j}\Xi_{q_j}(x))^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\Phi_{q_j}\Xi_{q_j}(x)) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \frac{dx}{x-q_j}. \end{aligned} \tag{2.21}$$

We calculate the left-hand side of (2.21). The constant term of this left-hand side is

$$\Xi^{(1)}_{q_j} + \left[\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \Xi^{(1)}_{q_j} \right] + \begin{pmatrix} \frac{p_j}{P(q_j)} & \frac{1}{P(q_j)} \\ 0 & d^{(0)}_{q_j} - \frac{p_j}{P(q_j)} \end{pmatrix}.$$

This matrix is a zero matrix. So we may check that $\Xi^{(1)}_{q_j}$ is determined as

$$\Xi^{(1)}_{q_j} = \begin{pmatrix} -\frac{p_j}{P(q_j)} & -\frac{1}{2P(q_j)} \\ (\xi_1^{q_j})_{21} & -d^{(0)}_{q_j} + \frac{p_j}{P(q_j)} \end{pmatrix}.$$

We may determine $(\xi_1^{q_j})_{21}$ freely. Here we set $(\xi_1^{q_j})_{21} = 0$. □

§3. Unramified irregular singularities

In this section we assume that $I_{\mathrm{ra}} = \emptyset$. In Section 3.1, we define a 2-form on the fiber $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ by Krichever's formula [16, Sect. 5]. Remark that $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ is isomorphic to the moduli space $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. We show that this 2-form coincides with the symplectic form (1.2). In Sections 3.3 and 3.4, we will construct horizontal lifts of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$. Let $\partial/\partial\theta^{\pm}_{l,t_i}$ ($i \in I_{\mathrm{un}}$ and $l = 0,1,\dots,n_i-2$) and $\partial/\partial t_i$ ($i = 3,4\dots,\nu$) be the vector fields on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. By the construction of the horizontal lifts, we have the vector fields $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ and $\delta^{\mathrm{IMD}}_{t_i}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ determined by the integrable deformations with respect to $\partial/\partial\theta^{\pm}_{l,t_i}$ and $\partial/\partial t_i$, respectively. Remark that $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ is isomorphic to the extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. In Section 3.5 we define a 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by Krichever's formula. We show that this 2-form is the isomonodromy 2-form. In Section 3.6 we calculate this 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by using Diarra–Loray's global normal form. Then we obtain an explicit formula of this 2-form.

We consider the leading coefficient of $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ at t_i:

$$\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} = \begin{pmatrix} 0 & \frac{1}{\prod_{j\neq i}(t_i-t_j)^{n_j}} \\ \theta^{+}_{0,t_i}\theta^{-}_{0,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j} & \theta^{+}_{0,t_i}+\theta^{-}_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + [\text{higher-order terms}].$$

Remark that this leading coefficient at t_i is independent of $\{(q_j,p_j)\}_{j=1,2,\dots,n-3}$. We fix $\Phi_i \in \mathrm{GL}(2,\mathbb{C})$ so that

$$\Phi_i^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi_i = \begin{pmatrix} \theta^{+}_{0,t_i} & 0 \\ 0 & \theta^{-}_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + [\text{higher-order terms}].$$

We call the matrix Φ_i a *compatible framing at* t_i. If we have another Φ'_i such that the leading coefficient matrix of $(\Phi'_i)^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi'_i$ is diagonal as above, then there exists a diagonal matrix C_{t_i} such that $\Phi'_i = \Phi_i C_{t_i}$, since $\theta^{+}_{0,t_i} - \theta^{-}_{0,t_i} \neq 0$.

Lemma 3.1 (For example [16, Lem. 3.1]). *Assume that* $\theta^{+}_{0,t_i} - \theta^{-}_{0,t_i} \neq 0$ *if* $n_i > 1$ *and* $\theta^{+}_{0,t_i} - \theta^{-}_{0,t_i} \notin \mathbb{Z}$ *if* $n_i = 1$. *For a compatible framing* Φ_i, *there exist unique*

- $\theta^{\pm}_{l,t_i} \in \Gamma(\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \mathcal{O}_{\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}})$ *(*$l \geq n_i$ *and* $i \in I$*),*
- $\sum_{s=1}^{\infty}(\xi^{(i)}_s)_{12}x^s_{t_i} \in \Gamma(\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \mathcal{O}_{\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}})[\![x_{t_i}]\!]$*, and*
- $\sum_{s=1}^{\infty}(\xi^{(i)}_s)_{21}x^s_{t_i} \in \Gamma(\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \mathcal{O}_{\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}})[\![x_{t_i}]\!]$

such that $\psi_i := \Phi_i\Xi_i(x_{t_i})\exp(-\Lambda_i(x_{t_i}))$ *satisfies the equation* $(d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_i = 0$ *formally at* t_i. *Here we put*

$$\Lambda_i(x_{t_i}) := \begin{pmatrix} \hat{\lambda}^{+}_i(x_{t_i}) & 0 \\ 0 & \hat{\lambda}^{-}_i(x_{t_i}) \end{pmatrix},$$

$$\Xi_i(x_{t_i}) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{s=1}^{\infty} \begin{pmatrix} 0 & (\xi^{(i)}_s)_{12} \\ (\xi^{(i)}_s)_{21} & 0 \end{pmatrix} x^s_{t_i},$$

where $\hat{\lambda}_i^{\pm}(x_{t_i}) := \sum_{l=0}^{\infty} \theta^{\pm}_{l,t_i} \int x_{t_i}^{-n_i+l}\, dx_{t_i}$. *That is,* ψ_i *is a formal fundamental matrix solution at* t_i.

Remark 3.2. By the equations in (2.10), we have that the polynomials C_i and D_i $(i = 1, 2, \dots, \nu, \infty)$ in (2.5) are independent of the parameters $\{(q_j, p_j)\}_{j=1,2,\dots,n-3}$ of $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. If we take a compatible framing Φ_i so that Φ_i is independent of $\{(q_j, p_j)\}_{j=1,2,\dots,n-3}$, then the coefficients of the formal power series $\Phi_i \Xi_i(x_{t_i})$ up to the $(x - t_i)^{n_i-1}$-term are independent of $\{(q_j, p_j)\}_{j=1,2,\dots,n-3}$. This independency is the assumption of Lemma 3.13 (below). We will use this fact for the calculation of Hamiltonians and the isomonodromy 2-form.

§3.1. Symplectic structure

Definition 3.3 ([16, Sect. 5] and [6, form. (3.16), p. 306]). Let δ_1 and δ_2 be vector fields on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathbb{C}^2)$, which is isomorphic to the moduli space $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. We fix a formal fundamental matrix solution ψ_i of $(d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_i = 0$ at $x = t_i$ as in Lemma 3.1. Moreover, we fix a fundamental matrix solution ψ_{q_j} of $(d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_{q_j} = 0$ at $x = q_j$ as in Lemma 2.11. We set

$$\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1} := \delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_i)\psi_i^{-1} - \delta_1(\psi_i)\psi_i^{-1}\delta_2(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})$$

and

$$\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_{q_j})\psi_{q_j}^{-1} := \delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_{q_j})\psi_{q_j}^{-1} - \delta_1(\psi_{q_j})\psi_{q_j}^{-1}\delta_2(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}).$$

We define a 2-form ω on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ as

$$\begin{aligned}\omega(\delta_1, \delta_2) := {}& \frac{1}{2}\sum_{i\in I} \mathrm{res}_{x=t_i} \mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1}) \\ &+ \frac{1}{2}\sum_{j=1}^{n-3} \mathrm{res}_{x=q_j} \mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_{q_j})\psi_{q_j}^{-1}),\end{aligned} \tag{3.1}$$

where $I := \{1, 2, \dots, \nu, \infty\}$.

In [16, Sect. 5], it is discussed that this definition is well defined. We recall this argument in [16, Sect. 5]. First, we show that the right-hand side of (3.1) is independent of the choice of ψ_{q_j}. If we have another solution ψ'_{q_j}, then we have matrix C_{q_j} such that this matrix is independent of parameters on $\mathbb{P}^1$ and $\psi'_{q_j} = \psi_{q_j} C_{q_j}$. By the Leibniz rule, we have

$$\begin{aligned}\mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi'_{q_j})(\psi'_{q_j})^{-1}) &= \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_{q_j}C_{q_j})(\psi_{q_j}C_{q_j})^{-1}) \\ &= \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_{q_j})\psi_{q_j}^{-1}) \\ &\quad + \mathrm{Tr}(\psi_{q_j}^{-1}\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_{q_j}\delta_2(C_{q_j})C_{q_j}^{-1}).\end{aligned}$$

We take variations of both sides of the equation $d\psi_{q_j} = -\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\psi_{q_j}$ with respect to δ_1. Then we have the equalities

$$\begin{aligned}\psi_{q_j}^{-1}\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_{q_j} &= -\psi_{q_j}^{-1}d(\delta_1(\psi_{q_j})) - \psi_{q_j}^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\delta_1(\psi_{q_j})\\ &= -\psi_{q_j}^{-1}d(\delta_1(\psi_{q_j})) + \psi_{q_j}^{-1}d(\psi_{q_j})\psi_{q_j}^{-1}\delta_1(\psi_{q_j})\\ &= -d(\psi_{q_j}^{-1}\delta_1(\psi_{q_j})).\end{aligned} \tag{3.2}$$

Here, the second equality is given by $d\psi_{q_j} = -\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\psi_{q_j}$. So we have

$$\begin{aligned}\mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi'_{q_j})(\psi'_{q_j})^{-1}) &= \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_{q_j})\psi_{q_j}^{-1})\\ &\quad - \mathrm{Tr}(d(\psi_{q_j}^{-1}\delta_1(\psi_{q_j}))\delta_2(C_{q_j})C_{q_j}^{-1}).\end{aligned}$$

Since the solution ψ_{q_j} is holomorphic at q_j and C_{q_j} is independent of parameters on $\mathbb{P}^1$, the residue of the second term on the right-hand side is zero. This fact means that the right-hand side of (3.1) is independent of the choice of ψ_{q_j}.

We may check that the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1})$ at $\tilde{t}_i$ is well defined as follows. We have the equality

$$\begin{aligned}\delta(\psi_i)\psi_i^{-1} &= \delta\big(\Phi_i\Xi_i(x_{t_i})\exp(-\Lambda_i(x_{t_i}))\big)\big(\Phi_i\Xi_i(x_{t_i})\exp(-\Lambda_i(x_{t_i}))\big)^{-1}\\ &= \delta(\Phi_i\Xi_i(x_{t_i}))(\Phi_i\Xi_i(x_{t_i}))^{-1}\\ &\quad - (\Phi_i\Xi_i(x_{t_i}))\delta(\Lambda_i(x_{t_i}))(\Phi_i\Xi_i(x_{t_i}))^{-1}.\end{aligned} \tag{3.3}$$

Since $\theta^{\pm}_{n_i-1,t_i}$ is constant on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$, $\delta(\theta^{\pm}_{n_i-1,t_i}) = 0$. Then $\delta(\theta^{\pm}_{n_i-1,t_i}\int x_{t_i}^{-1}\,dx_{t_i}) = \delta(c)$. Here, c is an integration constant. If we fix integration constants on $\Lambda_i(x_{t_i})$, then we can take the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})$ at $\tilde{t}_i$. We may check that $\mathrm{res}_{x=t_i}\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1})$ is independent of the choice of the integration constant as follows. We take other integration constants and a formal solution ψ'_i is given for the integration constants. There exists a diagonal matrix C_{t_i} such that $\psi'_i = \psi_i C_{t_i}$ and C_{t_i} is independent of parameters on $\mathbb{P}^1$. By the same argument as above, we have the equality

$$\begin{aligned}&\mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi'_i)(\psi'_i)^{-1})\\ &= \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_i)\psi_i^{-1}) - \mathrm{Tr}\big(d(\psi_i^{-1}\delta_1(\psi_i))\delta_2(C_{t_i})C_{t_i}^{-1}\big).\end{aligned} \tag{3.4}$$

Since $\Lambda_i(x_{t_i})$ and C_{t_i} are diagonal, we have

$$\exp(-\Lambda_i(x_{t_i}))\delta_2(C_{t_i})C_{t_i}^{-1}\exp(-\Lambda_i(x_{t_i}))^{-1} = \delta_2(C_{t_i})C_{t_i}^{-1}.$$

We calculate the second term on the left-hand side of (3.4):

$$\begin{aligned}
&\mathrm{Tr}\big(d(\psi_i^{-1}\delta_1(\psi_i))\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\quad= \mathrm{Tr}\big(d\big(\exp(-\Lambda_i(x_{t_i}))^{-1}(\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i}))\exp(-\Lambda_i(x_{t_i}))\big)\\
&\qquad\quad \times \delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\qquad + \mathrm{Tr}\Big(d\big(\exp(-\Lambda_i(x_{t_i}))^{-1}\delta_1\big(\exp(-\Lambda_i(x_{t_i}))\big)\big)\delta_2(C_{t_i})C_{t_i}^{-1}\Big)\\
&\quad= \mathrm{Tr}\big(d\Lambda_i(x_{t_i})(\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i}))\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\qquad - \mathrm{Tr}\big((\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i}))d\Lambda_i(x_{t_i})\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\qquad + \mathrm{Tr}\big(d\big((\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i}))\big)\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\qquad + \mathrm{Tr}\big(d\big(\delta_1(-\Lambda_i(x_{t_i}))\big)\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\quad= \mathrm{Tr}\big(d\big((\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i}))\big)\delta_2(C_{t_i})C_{t_i}^{-1}\big)\\
&\qquad + \mathrm{Tr}\big(d\big(\delta_1(-\Lambda_i(x_{t_i}))\big)\delta_2(C_{t_i})C_{t_i}^{-1}\big).
\end{aligned} \tag{3.5}$$

The residue parts of $d((\Phi_i\Xi_i(x_{t_i}))^{-1}\delta_1(\Phi_i\Xi_i(x_{t_i})))$ and $d(\delta_1(-\Lambda_i(x_{t_i})))$ vanish. Since $\delta_2(C_{t_i})C_{t_i}^{-1}$ is independent of parameters on $\mathbb{P}^1$, the residues of the formal meromorphic differentials of the last line of (3.5) at t_i are zero. Then we have that $\mathrm{res}_{x=t_i}\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})$ is independent of the choice of the integration constant. Finally, the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})$ at $\tilde{t}_i$ is well defined.

Next we show that the right-hand side of (3.1) is independent of the choice of a formal solution ψ_i. Let $C_{t_i}(x_{t_i})$ be the following diagonal matrix:

$$\begin{pmatrix} c_{t_i,11}(x_{t_i}) & 0 \\ 0 & c_{t_i,22}(x_{t_i})\end{pmatrix} = \begin{pmatrix} c^{(0)}_{t_i,11} & 0 \\ 0 & c^{(0)}_{t_i,22}\end{pmatrix} + \begin{pmatrix} c^{(1)}_{t_i,11} & 0 \\ 0 & c^{(1)}_{t_i,22}\end{pmatrix} x_{t_i} + \begin{pmatrix} c^{(2)}_{t_i,11} & 0 \\ 0 & c^{(2)}_{t_i,22}\end{pmatrix} x_{t_i}^2 + \cdots .$$

We define $\Xi'(x_{t_i})$ and $\Lambda'_i(x_{t_i})$ by

$$\begin{aligned}
&\Xi'(x_{t_i}) = \Xi(x_{t_i})C_{t_i}(x_{t_i}),\\
&\Lambda'_i(x_{t_i}) = \Lambda_i(x_{t_i}) + \begin{pmatrix} \int c_{t_i,11}(x_{t_i})^{-1}\, d(c_{t_i,11}(x_{t_i})) & 0 \\ 0 & \int c_{t_i,22}(x_{t_i})^{-1}\, d(c_{t_i,22}(x_{t_i}))\end{pmatrix}.
\end{aligned}$$

Then we have another formal fundamental matrix solution $\psi'_i = \Phi_i\Xi'(x_{t_i})\times\exp(-\Lambda'_i(x_{t_i}))$. There exists a diagonal matrix C_{t_i} such that $\psi'_i = \psi_i C_{t_i}$ and C_{t_i} is independent of parameters on $\mathbb{P}^1$. By the same argument as above, we have

$$\mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi'_i)(\psi'_i)^{-1}) = \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_i)\psi_i^{-1}).$$

Then we obtain that the right-hand side of (3.1) is independent of the choice of ψ_i.

Theorem 3.4. *Let ω be the 2-form on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\rm ra}}$ defined by (3.1) in Definition 3.3. The 2-form ω coincides with*

$$\sum_{j=1}^{n-3} d\Big(\frac{p_j}{P(q_j)}\Big) \wedge dq_j.$$

Proof. Recall that $\omega(\delta_1,\delta_2)$ is

$$\begin{aligned}&\frac{1}{2}\sum_{i\in I} \operatorname{res}_{x=t_i} \operatorname{Tr}\big(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1}\big)\\ &\qquad + \frac{1}{2}\sum_{j=1}^{n-3} \operatorname{res}_{x=q_j} \operatorname{Tr}\big(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_{q_j})\psi_{q_j}^{-1}\big).\end{aligned}$$

We calculate the residue of $\operatorname{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_{q_j})\psi_{q_j}^{-1})$ at $x=q_j$. For this purpose, first, we calculate $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})$ around $x=q_j$ as follows. The connection matrix $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ has the following description at q_j by (2.5):

$$\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} = \begin{pmatrix} 0 & 0 \\ \frac{p_j}{x-q_j} & \frac{-1}{x-q_j}\end{pmatrix} dx + \begin{pmatrix} 0 & b_0' \\ c_0' & d_0'\end{pmatrix} dx. \tag{3.6}$$

Here, b_0', c_0', and d_0' are holomorphic at $x=q_j$. Since t_i and $\theta^{\pm}_{l,t_i}$ $(i\in I,\ 0\le l\le n_i-1)$ are constants on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\rm ra}}$, we have $\delta(t_i)=0$ and $\delta(\theta^{\pm}_{l,t_i})=0$. By $\delta(t_i)=0$ we have $\delta(b_0')=0$. By (2.10), we have $\delta(D_i)=0$ for $i=1,2,\dots,\nu,\infty$. We take the variation $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})$ of $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ associated to δ:

$$\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) = \begin{pmatrix} 0 & 0 \\ \delta(c_0) & \delta(d_0)\end{pmatrix} dx,$$

where

$$\begin{aligned}\delta(c_0) &= \frac{p_j\delta(q_j)}{(x-q_j)^2} + \frac{\delta(p_j)}{(x-q_j)} + O(x-q_j)^0,\\ \delta(d_0) &= -\frac{\delta(q_j)}{(x-q_j)^2} - \sum_{k\neq j}\frac{\delta(q_k)}{(q_j-q_k)^2} + O(x-q_j).\end{aligned}$$

Second, we consider $\delta(\psi_{q_j})\psi_{q_j}^{-1}$. We have

$$\begin{aligned}\delta(\psi_{q_j})\psi_{q_j}^{-1} &= \delta(\Phi_{q_j}\Xi_{q_j}(x))(\Phi_{q_j}\Xi_{q_j}(x))^{-1}\\ &\qquad + (\Phi_{q_j}\Xi_{q_j}(x))\begin{pmatrix} 0 & 0 \\ 0 & \frac{-\delta(q_j)}{x-q_j}\end{pmatrix}(\Phi_{q_j}\Xi_{q_j}(x))^{-1}.\end{aligned} \tag{3.7}$$

By using Lemma 2.11, we have the equality

$$\begin{aligned}\Phi_{q_j}\Xi_{q_j}(x) = &\begin{pmatrix} 1 & 0 \\ p_j & 1 \end{pmatrix} \\ &+ \begin{pmatrix} -\frac{p_j}{P(q_j)} & -\frac{1}{2P(q_j)} \\ -\frac{p_j^2}{P(q_j)} & \frac{p_j}{2P(q_j)} - \sum_{i=1}^{\nu} \frac{D_i(q_j)}{(q_j-t_i)^{n_i}} + \sum_{k\neq j} \frac{1}{q_j-q_k} - D_\infty(q_j) \end{pmatrix}(x-q_j) \\ &+ O(x-q_j)^2.\end{aligned}$$

By this description of $\Phi_{q_j}\Xi_{q_j}(x)$, we may check that the constant term of the expansion of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ at q_j has the description

$$\begin{pmatrix} 0 & 0 \\ \delta(p_j) & 0 \end{pmatrix} - \delta(q_j) \begin{pmatrix} -\frac{p_j}{P(q_j)} & -\frac{1}{2P(q_j)} \\ -\frac{p_j^2}{P(q_j)} & * \end{pmatrix},$$

and the coefficient of the $(x-q_j)$-term of the expansion of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ has the description

$$\begin{pmatrix} * & 0 \\ * & \frac{\delta(p_j)}{2P(q_j)} + \sum_{k\neq j} \frac{\delta(q_k)}{(q_j-q_k)^2} \end{pmatrix} - \delta(q_j) \begin{pmatrix} * & * \\ * & * \end{pmatrix}.$$

Here we put the entries having $\delta(q_j)$ together in the second matrices. Moreover, we may check that the constant term of the expansion of $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ at q_j is $\left(\begin{smallmatrix} 1 & 0 \\ -p_j & 1 \end{smallmatrix}\right)$ and the coefficient of the $(x-q_j)$-term of the expansion of $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ has the description

$$-\begin{pmatrix} -\frac{p_j}{2P(q_j)} & -\frac{1}{2P(q_j)} \\ * & \frac{p_j}{P(q_j)} - \sum_{i=1}^{\nu} \frac{D_i(q_j)}{(q_j-t_i)^{n_i}} + \sum_{k\neq j} \frac{1}{q_j-q_k} - D_\infty(q_j) \end{pmatrix}.$$

By the calculation of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ and $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$, we may show that $\delta(\Phi_{q_j}\Xi_{q_j}(x))(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ is

$$\begin{pmatrix} * & \frac{\delta(q_j)}{2P(q_j)} \\ * & * \end{pmatrix} + \begin{pmatrix} * & f_{12}^{(1)}\delta(q_j) \\ * & \frac{\delta(p_j)}{P(q_j)} + \sum_k \frac{\delta(q_k)}{(q_j-q_k)^2} - f_{22}^{(1)}\delta(q_j) \end{pmatrix}(x-q_j) + O(x-q_j)^2, \tag{3.8}$$

where $f_{12}^{(1)}$ and $f_{22}^{(1)}$ are rational functions on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. We consider the second term of (3.7). We may show that

$$\begin{aligned}&(\Phi_{q_j}\Xi_{q_j}(x)) \begin{pmatrix} 0 & 0 \\ 0 & \frac{-\delta(q_j)}{x-q_j} \end{pmatrix} (\Phi_{q_j}\Xi_{q_j}(x))^{-1} \\ &= \frac{\begin{pmatrix} 0 & 0 \\ * & -\delta(q_j) \end{pmatrix}}{x-q_j} + \begin{pmatrix} * & \frac{\delta(q_j)}{2P(q_j)} \\ * & * \end{pmatrix} + \begin{pmatrix} * & g_{12}^{(1)}\delta(q_j) \\ * & g_{22}^{(1)}\delta(q_j) \end{pmatrix}(x-q_j) + O(x-q_j)^2,\end{aligned} \tag{3.9}$$

where $g_{12}^{(1)}$ and $g_{22}^{(1)}$ are rational functions on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. By (3.8) and (3.9), we have

$$\delta(\psi_{q_j})\psi_{q_j}^{-1} = \frac{\begin{pmatrix} 0 & 0 \\ * & -\delta(q_j) \end{pmatrix}}{x - q_j} + \begin{pmatrix} * & \frac{\delta(q_j)}{P(q_j)} \\ * & * \end{pmatrix}$$
$$+ \begin{pmatrix} * & (g_{12}^{(1)} + f_{12}^{(1)})\delta(q_j) \\ * \frac{\delta(p_j)}{P(q_j)} + \sum_k \frac{\delta(q_k)}{(q_j - q_k)^2} & + (g_{22}^{(1)} - f_{22}^{(1)})\delta(q_j) \end{pmatrix}(x - q_j) + O(x - q_j)^2.$$

By (3.6) and this equality, we have

$$\begin{aligned} &\mathrm{res}_{x=q_j} \mathrm{Tr}(\delta_1(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\delta_2(\psi_{q_j})\psi_{q_j}^{-1}) \\ &\quad = \frac{\delta_1(p_j)\delta_2(q_j)}{P(q_j)} - \frac{\delta_1(q_j)\delta_2(p_j)}{P(q_j)} + \sum_{k\neq j} \frac{\delta_1(q_k)\delta_2(q_j)}{(q_j - q_k)^2} - \sum_{k\neq j} \frac{\delta_1(q_j)\delta_2(q_k)}{(q_j - q_k)^2} \\ &\qquad + (p_j(g_{12}^{(1)} + f_{12}^{(1)}) - (g_{22}^{(1)} - f_{22}^{(1)}))\delta_1(q_j)\delta_2(q_j). \end{aligned}$$

Since $\sum_{j=1}^{n-3}\sum_{k\neq j} \frac{\delta_1(q_k)\delta_2(q_j)-\delta_1(q_j)\delta_2(q_k)}{(q_j-q_k)^2} = 0$, we have

$$\sum_{j=1}^{n-3} \mathrm{res}_{x=q_j} \mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_{q_j})\psi_{q_j}^{-1}) = 2\sum_{j=1}^{n-3}\Big(\frac{\delta_1(p_j)\delta_2(q_j)}{P(q_j)} - \frac{2\delta_2(p_j)\delta_1(q_j)}{P(q_j)}\Big).$$

Next we calculate the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1})$ at $x = t_i$. First we consider the expansion of $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})$ at $x = t_i$. Since $\delta(C_i) = \delta(D_i) = 0$ for $i = 1, 2, \ldots, \nu, \infty$, we have $\delta(c_0) = O(x^0_{t_i})$ and $\delta(d_0) = O(x^0_{t_i})$. Second we consider $\delta(\psi_i)\psi_i^{-1}$. By Lemma 3.1 we have

$$\begin{aligned} \delta(\psi_i)\psi_i^{-1} &= \delta(\Phi_i\Xi_i(x_{t_i}))(\Phi_i\Xi_i(x_{t_i}))^{-1} \\ &\quad + (\Phi_i\Xi_i(x_{t_i}))\begin{pmatrix} -\delta(\hat{\lambda}_i^+(x_{t_i})) & 0 \\ 0 & -\delta(\hat{\lambda}_i^-(x_{t_i})) \end{pmatrix}(\Phi_i\Xi_i(x))^{-1}. \end{aligned}$$

Since $\delta(\hat{\lambda}_i^\pm(x_{t_i})) = O(x_{t_i})$, we have that the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}) \wedge \delta(\psi_i)\psi_i^{-1})$ at t_i is zero. Then we obtain

$$\omega(\delta_1, \delta_2) = \sum_{j=1}^{n-3}\Big(\frac{\delta_1(p_j)\delta_2(q_j)}{P(q_j)} - \frac{\delta_2(p_j)\delta_1(q_j)}{P(q_j)}\Big),$$

which means that ω coincides with $\sum_{j=1}^{n-3} d(\frac{p_j}{P(q_j)}) \wedge dq_j$. □

§3.2. Note on the relation to the symplectic structure of the coadjoint orbits

We apply the argument in [6, proof of Thm. 3.3] for our ω. Let $d + \Omega^0$ be a connection on $E_1 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, whose polar divisor is D. Remark that the connection $d+\Omega^0$ is related to a connection on $E_{n-2}=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(n-2)$ via the transformation (2.2). Let t be a component of the divisor D. Choosing a formal coordinate x_t near t and a trivialization of E on the formal neighborhood of t, we describe ∇ near t by

$$d+\Omega_0^0\frac{dx_t}{x_t^{n_t}}+[\text{higher-order terms}],\quad \Omega_0^0\in\mathfrak{gl}(2,\mathbb{C}).$$

Let ψ be a formal solution at t, that is, $(d+\Omega^0)\psi=0$. For $j=1,2$, let $\delta_j(\Omega^0)$ and $\delta_j(\psi)$ be the variations of Ω^0 and ψ, respectively. Here, δ_j $(j=1,2)$ mean vector fields on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$.

We define G_{n_t} as $G_{n_t}=\mathrm{GL}(2,\mathbb{C}[x_t]/(x_t^{n_t}))$. Let $\mathfrak{g}_{n_t}$ be the Lie algebra of G_{n_t} and $\mathfrak{g}_{n_t}^*$ be the dual of $\mathfrak{g}_{n_t}$. We define $\Omega^0_{\leq n_t-1}$ and $U^{(j)}_{\leq n_t-1}$ $(j=1,2)$ as $\Omega^0=(\Omega^0_{\leq n_t-1})x_t^{-n_t}+O(x_t^0)$ and $\delta_j(\psi)\psi^{-1}=U^{(j)}_{\leq n_t-1}+O(x_t^{n_t})$, respectively. We identify $\Omega^0_{\leq n_t-1}$ and $U^{(j)}_{\leq n_t-1}$ as elements of $\mathfrak{g}^*_{n_t}$ by the pairing $\langle X,Y\rangle=\sum_{k=0}^{n_t-1}(X_kY_{n_t-1-k})$, where $X=X_0+X_1x_t+\cdots+X_{n_t-1}x_t^{n_t-1}\in\mathfrak{g}_{n_t}$ and $Y=Y_0+Y_1x_t+\cdots+Y_{n_t-1}x_t^{n_t-1}\in\mathfrak{g}_{n_t}$. Since $\delta_j(\Omega^0)=-[\Omega^0,\delta_j(\psi)\psi^{-1}]-\frac{d}{dx_t}(\delta_j(\psi)\psi^{-1})$ for $j=1,2$, we have

$$\delta_j(\Omega^0_{\leq n_t-1})=-[\Omega^0_{\leq n_t-1},U^{(j)}_{\leq n_t-1}]. \tag{3.10}$$

By this equality, we have the following equality:

$$\begin{aligned}&\frac{1}{2}\mathrm{res}_{x_t=0}\,\mathrm{Tr}(\delta_1(\Omega^0)\delta_2(\psi)(\psi)^{-1}-\delta_1(\psi)(\psi)^{-1}\delta_2(\Omega^0))\\&\quad=-\mathrm{Tr}\langle\Omega^0_{\leq n_t-1},[U^{(1)}_{\leq n_t-1},U^{(2)}_{\leq n_t-1}]\rangle.\end{aligned}\tag{3.11}$$

If we consider the elementary transformation (in other words, the Hecke modification), we have a connection on the rank-two trivial bundle from the connection $d+\Omega^0$ on E_1. By equalities (3.10) and (3.11), we have a relation between ω and the symplectic form on the product of the coadjoint orbits of G_{n_t} for each component t of D (see [2, Prop. 2.1]).

§3.3. Integrable deformations associated to $T_{\boldsymbol{\theta}}$

First we fix $i\in I$ and $l\in\{0,1,\dots,n_i-2\}$. Let $\widehat{E}_1$ be the pull-back of E_1 under the projection $\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\to\mathbb{P}^1$. Let

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}=\begin{cases}d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}),\\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}),\end{cases}$$

be the family (2.14) of connections on $\widehat{E}_1$. Let $\theta^{\pm}_{l,t_i}$ be the natural coordinate of $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ and $\partial/\partial\theta^{\pm}_{l,t_i}$ be the vector field on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ associated to $\theta^{\pm}_{l,t_i}$. We will construct a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial\theta^{\pm}_{l,t_i}$.

We consider diagonalizations of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ until some degree term at each $\tilde{t}_{i'}$ ($i' \in I$). By using the explicit form of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$, we take a family of compatible framings of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ at $\tilde{t}_{i'}$ for each $i' \in I$. We denote this family of compatible framings at $\tilde{t}_{i'}$, for each $i' \in I$, by $\Phi_{i'}$. Let $\Xi_{i'}(x_{t_{i'}})$ be the formal transformation of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ at $\tilde{t}_{i'}$ with respect to $\Phi_{i'}$ appearing in Lemma 3.1. Let $\widetilde{G}$ be the matrix defined in (2.7). We write the formal expansion of $\widetilde{G}^{-1}\Phi_{i'}\Xi_{i'}(x_{t_{i'}})$ at $x_{t_{i'}}=0$ by

$$\widetilde{G}^{-1}\Phi_{i'}\Xi_{i'}(x_{t_{i'}}) = P_{i',0} + P_{i',1}x_{t_{i'}} + P_{i',2}x_{t_{i'}}^2 + \cdots . \tag{3.12}$$

Set

$$\begin{aligned} P_{i'} &\coloneqq P_{i',0} + P_{i',1}x_{t_{i'}} + P_{i',2}x_{t_{i'}}^2 + \cdots + P_{i',2n_{i'}-1}x_{t_{i'}}^{2n_{i'}-1} \quad (\text{for } i' \in I),\\ P_{\nu+1} &\coloneqq \mathrm{id}. \end{aligned}$$

We take an affine open covering $\{\widehat{U}_{i'}\}_{i'\in I\cup\{\nu+1\}}$ of $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})$ such that

- for $i' \in I$, we have $\tilde{t}_{i'} \subset \widehat{U}_{i'}$, $\tilde{t}_j \cap \widehat{U}_{i'} = \emptyset$ (for any $j \neq i'$, $j \in I$), and $\sum_{s=1}^{2n_{i'}-1} P_{i',s}x_{t_{i'}}^s$ is invertible on each point of $\widehat{U}_{i'}$,
- for $i' = \nu+1$, $\tilde{t}_j \cap \widehat{U}_{i'} = \emptyset$ (for any $j \in I$).

Set $\widehat{U}_{i'_1,i'_2} \coloneqq \widehat{U}_{i'_1} \cap \widehat{U}_{i'_2}$.

Now we define new trivializations $\{(\widehat{U}_{i'},\widehat{\varphi}_{i'})\}_{i'\in I\cup\{\nu+1\}}$ of $\widehat{E}_1$. Also, we denote by $(U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \varphi^{(1)}_{U_0})$ and $(U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \varphi^{(1)}_{U_\infty})$ the trivializations of $\widehat{E}_1$ induced by (2.1). We define $\widehat{\varphi}_{i'}$ (for $i' \in (I\cup\{\nu+1\})\setminus\{\infty\}$) by the composition

$$\widehat{\varphi}_{i'} \colon \widehat{E}_1|_{\widehat{U}_{i'}} \xrightarrow{\varphi^{(1)}_{U_0}|_{\widehat{U}_{i'}}} \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'}} \xrightarrow{P_{i'}^{-1}} \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'}}.$$

We define $\widehat{\varphi}_\infty$ by the composition

$$\widehat{\varphi}_\infty \colon \widehat{E}_1|_{\widehat{U}_\infty} \xrightarrow{\varphi^{(1)}_{U_\infty}|_{\widehat{U}_\infty}} \mathcal{O}^{\oplus 2}_{\widehat{U}_\infty} \xrightarrow{P_\infty^{-1}} \mathcal{O}^{\oplus 2}_{\widehat{U}_\infty}.$$

Then we have new trivializations $\{(\widehat{U}_{i'},\widehat{\varphi}_{i'})\}_{i'\in I\cup\{\nu+1\}}$ of $\widehat{E}_1$. Let $\widehat{\Omega}_{i'}$ be the connection matrix of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ under the new trivialization $\widehat{\varphi}_{i'}$:

$$\begin{aligned} \widehat{\Omega}_{i'} &= P_{i'}^{-1}\,dP_{i'} + P_{i'}^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{U}_{i'}}P_{i'} \quad \text{for } i' \in (I\cup\{\nu+1\})\setminus\{\infty\},\\ \widehat{\Omega}_\infty &= (G_1P_\infty)^{-1}\,d(G_1P_\infty) + (G_1P_\infty)^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{U}_\infty}(G_1P_\infty). \end{aligned} \tag{3.13}$$

Remark that $\widehat{\Omega}_{i'}$ is diagonal until the $x_{t_{i'}}^{n_{i'}-1}$-term for each $i' \in I$.

Now we construct an integrable deformation of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$. For the fixed $i \in I$ and l $(0 \le l \le n_i - 2)$, we define matrices $B_{\theta^{\pm}_{l,t_i}}(x_{t_i})$ by

$$B_{\theta^{+}_{l,t_i}}(x_{t_i}) := \frac{\begin{pmatrix} \frac{\delta(\theta^{+}_{l,t_i})}{-n_i+l+1} & 0 \\ 0 & 0 \end{pmatrix}}{x_{t_i}^{n_i-l-1}} \quad \text{and} \quad B_{\theta^{-}_{l,t_i}}(x_{t_i}) := \frac{\begin{pmatrix} 0 & 0 \\ 0 & \frac{\delta(\theta^{-}_{l,t_i})}{-n_i+l+1} \end{pmatrix}}{x_{t_i}^{n_i-l-1}}.$$

For each $i' \in I \cup \{\nu+1\}$, we set $(\widehat{E}_1)_{i',\varepsilon} = \widehat{E}_1|_{\widehat{U}_{i'}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$, $\widehat{U}^{\varepsilon}_{i'} = \widehat{U}_{i'} \times \operatorname{Spec}\mathbb{C}[\varepsilon]$, and $\widehat{U}^{\varepsilon}_{i'_1,i'_2} = \widehat{U}^{\varepsilon}_{i'_1} \cap \widehat{U}^{\varepsilon}_{i'_2}$. We define matrices P^{ε}_i and $P^{\varepsilon}_{i'}$ by

$$P^{\varepsilon}_i = P_i(\mathrm{id} + \varepsilon B_{\theta^{\pm}_{l,t_i}}(x_{t_i})) \quad \text{and} \quad P^{\varepsilon}_{i'} = P_{i'} \otimes \mathrm{id} \tag{3.14}$$

(where $i' \in (I \setminus \{i\}) \cup \{\nu+1\}$), respectively. In the argument below, we will replace P^{ε}_{∞} with $G_1 P^{\varepsilon}_{\infty}$. The matrices give isomorphisms

$$\mathcal{O}^{\oplus 2}_{\widehat{U}_{i'_1,i'_2}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] \xrightarrow{P^{\varepsilon}_{i'_1}} \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'_1,i'_2}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$$

for each $i'_1, i'_2 \in I \cup \{\nu+1\}$. First we define a vector bundle $(\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}$ on $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$ by gluing $\{(\widehat{E}_1)_{i',\varepsilon}\}_{i' \in I \cup \{\nu+1\}}$ as follows: we glue $(\widehat{E}_1)_{i'_1,\varepsilon}$ and $(\widehat{E}_1)_{i'_2,\varepsilon}$ $(i'_1, i'_2 \in I \cup \{\nu+1\})$ by the composition

$$(\widehat{E}_1)_{i'_1,\varepsilon}|_{\widehat{U}^{\varepsilon}_{i'_1,i'_2}} \xrightarrow{\widehat{\varphi}_{i'_1}|_{\widehat{U}_{i'_1,i'_2}} \otimes 1} \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'_1,i'_2}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] \xrightarrow{(P^{\varepsilon}_{i'_2})^{-1} P^{\varepsilon}_{i'_1}} \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'_1,i'_2}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] \xrightarrow{\widehat{\varphi}^{-1}_{i'_2}|_{\widehat{U}_{i'_1,i'_2}} \otimes 1} (\widehat{E}_1)_{i'_2,\varepsilon}|_{\widehat{U}^{\varepsilon}_{i'_1,i'_2}}.$$

By construction, we have $(\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}} \otimes \mathbb{C}[\varepsilon]/(\varepsilon) = \widehat{E}_1$. Second we define a morphism

$$\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}} : (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}} \longrightarrow (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}} \otimes \widetilde{\Omega}^1_{\partial/\partial\theta^{\pm}_{l,t_i}}$$

with the Leibniz rule. Here, $\widetilde{\Omega}^1_{\partial/\partial\theta^{\pm}_{l,t_i}}$ is the coherent subsheaf (2.19). We define $\nabla^{\varepsilon}_{i'}$ $(i' \in I \cup \{\nu+1\})$ as

$$\begin{cases} \nabla^{\varepsilon}_{i'} = \hat{d} + \widehat{\Omega}_{i'} \quad \text{for } i' \in (I \setminus \{i\}) \cup \{\nu+1\}, \\ \nabla^{\varepsilon}_{i} = \hat{d} + \widehat{\Omega}_i + \varepsilon\Big(\dfrac{\partial}{\partial x_{t_i}}(B_{\theta^{\pm}_{l,t_i}})\, dx_{t_i} + [\widehat{\Omega}_i, B_{\theta^{\pm}_{l,t_i}}]\Big) + B_{\theta^{\pm}_{l,t_i}}\, d\varepsilon. \end{cases} \tag{3.15}$$

We can consider $\nabla^{\varepsilon}_{i'}$ $(i' \in I \cup \{\nu+1\})$ as a morphism

$$\nabla^{\varepsilon}_{i'} : (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}|_{\widehat{U}^{\varepsilon}_{i'}} \longrightarrow (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}|_{\widehat{U}^{\varepsilon}_{i'}} \otimes \widetilde{\Omega}^1_{\partial/\partial\theta^{\pm}_{l,t_i}}|_{\widehat{U}^{\varepsilon}_{i'}}$$

by using the trivialization

$$\widehat{\varphi}_{i'} \otimes 1 \colon (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}|_{\widehat{U}^{\varepsilon}_{i'}} = \widehat{E}_1|_{\widehat{U}_{i'}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] \to \mathcal{O}^{\oplus 2}_{\widehat{U}_{i'}} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon].$$

We may glue $\{\nabla^{\varepsilon}_{i'}\}_{i' \in I \cup \{\nu+1\}}$. Finally, we obtain $\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$ by this gluing. Since $\widehat{\Omega}_i$ and $B_{\theta^{\pm}_{l,t_i}}$ are diagonal until the $x_{t_i}^{n_i-1}$-terms, the negative parts of the relative connections $\overline{\nabla}^{\varepsilon}_{\partial/\partial\theta^{+}_{l,t_i}}$ and $\overline{\nabla}^{\varepsilon}_{\partial/\partial\theta^{-}_{l,t_i}}$ along the divisor $\tilde{t}_i$ are

$$\begin{pmatrix} \theta^{+}_{0,t_i} & 0 \\ 0 & \theta^{-}_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^{+}_{l,t_i} + \varepsilon\delta(\theta^{+}_{l,t_i}) & 0 \\ 0 & \theta^{-}_{l,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \begin{pmatrix} \theta^{+}_{n_i-1,t_i} & 0 \\ 0 & \theta^{-}_{n_i-1,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}},$$

$$\begin{pmatrix} \theta^{+}_{0,t_i} & 0 \\ 0 & \theta^{-}_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^{+}_{l,t_i} & 0 \\ 0 & \theta^{-}_{l,t_i} + \varepsilon\delta(\theta^{-}_{l,t_i}) \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \begin{pmatrix} \theta^{+}_{n_i-1,t_i} & 0 \\ 0 & \theta^{-}_{n_i-1,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}},$$

respectively.

Let $\widehat{E}^{\varepsilon}_1$ be the pull-back of $\widehat{E}_1$ under the projection (2.17). We consider a short exact sequence

$$0 \longrightarrow \varepsilon\,\mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1) \longrightarrow \mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1) \longrightarrow \mathcal{E}\mathrm{nd}(\widehat{E}_1) \longrightarrow 0.$$

Note that

$$\varepsilon\,\mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1) \cong (\varepsilon) \otimes \mathcal{E}\mathrm{nd}(\widehat{E}_1).$$

Since the bundle type is $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, we have $R^1\pi_*((\varepsilon) \otimes \mathcal{E}\mathrm{nd}(\widehat{E}_1)) = 0$, which means the rigidity of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Here, π is the projection $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \to \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. So we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow \pi_*\big(\varepsilon\,\mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1)\big) &\longrightarrow \pi_*\big(\mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1)\big) \\ &\longrightarrow \pi_*(\mathcal{E}\mathrm{nd}(\widehat{E}_1)\big) \longrightarrow 0. \end{aligned}$$

Since $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ is affine, we have that

$$\begin{aligned} \mathrm{Hom}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1) &= \Gamma\big(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}, \pi_*\big(\mathcal{H}\mathrm{om}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1)\big)\big) \\ &\longrightarrow \Gamma\big(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}, \pi_*(\mathcal{E}\mathrm{nd}(\widehat{E}_1))\big) = \mathrm{End}(\widehat{E}_1) \end{aligned}$$

is surjective. Then we have a lift $\varphi_{\Upsilon} \in \mathrm{Hom}((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \widehat{E}^{\varepsilon}_1)$ of $\mathrm{id} \in \mathrm{End}(\widehat{E}_1)$. This lift φ_{Υ} is an isomorphism

$$\varphi_{\Upsilon} \colon (\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}} \xrightarrow{\cong} \widehat{E}^{\varepsilon}_1.$$

We consider a pair $(\widehat{E}^{\varepsilon}_1, (\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}})$ induced by $((\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}, \nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}})$. By construction, the pair $(\widehat{E}^{\varepsilon}_1, (\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}})$ is a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial\theta^{\pm}_{l,t_i}$. Let $\overline{(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}}$ be the relative connection induced by $(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$. Since φ_{Υ} is holomorphic and invertible along the pole divisor of $\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$, the local formal data of $\overline{(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}}$ is the same as in $\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$. The family of connections $(\widehat{E}^{\varepsilon}_1, \overline{(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}})$ parametrized by $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]$ gives a map from the base space $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]$ to the moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. By taking the composition with $\widehat{\mathrm{App}}$ defined in (2.13), we have a map

$$(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\longrightarrow \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}. \tag{3.16}$$

Definition 3.5. Then we may define the vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ associated to the map (3.16). We denote this vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ by $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$.

Let $f^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\colon (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ be the map induced by the vector field $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$. We have $\widehat{E}^{\varepsilon}_1=(\mathrm{id}\times f^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})^*\widehat{E}_1$. We denote the pull-back of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ under the map $\mathrm{id}\times f^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ by

$$\begin{cases} d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon],\\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \\ \qquad +\varepsilon G_1^{-1}\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1 & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\,. \end{cases} \tag{3.17}$$

Since $(\widehat{E}^{\varepsilon}_1,(\mathrm{id}\times f^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})^*\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}})$ is isomorphic to $(\widehat{E}^{\varepsilon}_1,\overline{(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}})$, we have a lift of $(\mathrm{id}\times f^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})^*\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$:

$$\begin{cases} \hat{d}+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \\ \qquad +\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\,d\varepsilon & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\\ \hat{d}+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \\ \qquad +\varepsilon G_1^{-1}\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1 & \\ \qquad +G_1^{-1}\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}G_1\,d\varepsilon & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]\,, \end{cases} \tag{3.18}$$

which is a morphism $\widehat{E}^{\varepsilon}_1\to\widehat{E}^{\varepsilon}_1\otimes\widetilde{\Omega}^1_{\partial/\partial\theta^{\pm}_{l,t_i}}$ with the Leibniz rule. Then, since $(\widehat{E}^{\varepsilon}_1,\overline{(\varphi_{\Upsilon}^{-1})^*\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}})$ is relativization of the horizontal lift, we have the equality

$$\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})=d\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}+[\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)},\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}], \tag{3.19}$$

which means the integrable condition.

§3.4. Integrable deformations associated to $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}$

First we fix $i \in \{3, 4, \dots, \nu\}$. Let

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}} = \begin{cases} d + \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}}, \boldsymbol{\theta}_0)} & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \\ d + G_1^{-1}\, dG_1 + G_1^{-1} \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}}, \boldsymbol{\theta}_0)} G_1 & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}), \end{cases}$$

be the family (2.14). Let t_i be the natural coordinate of $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ and $\partial/\partial t_i$ be the vector field on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ associated to t_i. We will construct a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial t_i$.

For the fixed index i, we define a matrix $B_{t_i}(x)$ by

$$B_{t_i}(x) \coloneqq - \sum_{l=0}^{n_i - 1} \begin{pmatrix} \theta^+_{l,t_i} & 0 \\ 0 & \theta^-_{l,t_i} \end{pmatrix} \frac{\hat{\delta}(t_i)}{(x - t_i)^{n_i - l}}.$$

We define a vector bundle $(\widehat{E}_1)^{\varepsilon}_{t_i}$ on $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec} \mathbb{C}[\varepsilon]$ by the same argument as in the construction of $(\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}$. That is, we replace $B_{\theta^{\pm}_{l,t_i}}(x_{t_i})$ in (3.14) with $B_{t_i}(x)$. We define a morphism

$$\nabla^{\varepsilon}_{\partial/\partial t_i} \colon (\widehat{E}_1)^{\varepsilon}_{t_i} \longrightarrow (\widehat{E}_1)^{\varepsilon}_{t_i} \otimes \widetilde{\Omega}^1_{\partial/\partial t_i}$$

by gluing the connections

$$\begin{cases} \nabla^{\varepsilon}_{i'} = \hat{d} + \widehat{\Omega}_{i'} & \text{for } i' \in (I \setminus \{i\}) \cup \{\nu + 1\}, \\ \nabla^{\varepsilon}_{i} = \hat{d} + \widehat{\Omega}_{i} + \varepsilon \Big(\dfrac{\partial}{\partial x_{t_i}} (B_{t_i})\, dx_{t_i} + [\widehat{\Omega}_i, B_{t_i}] \Big) + B_{t_i}\, d\varepsilon, \end{cases}$$

as in the construction of $\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$ in the previous section. Here, $\widehat{\Omega}_{i'}$ ($i' \in I \cup \{\nu+1\}$) is defined in (3.13). Now we check that the connection matrix of ∇^{ε}_{i} is a section of $\widetilde{\Omega}^1_{\partial/\partial t_i}$ defined in (2.19). We set $\tilde{x}_{t_i} \coloneqq x - (t_i + \varepsilon \hat{\delta}(t_i)) = x_{t_i} - \varepsilon \hat{\delta}(t_i)$. Since $\varepsilon^2 = 0$ and $\varepsilon\, d\varepsilon = 0$, we may check the equalities

$$\begin{aligned} &\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{\hat{d}\tilde{x}_{t_i}}{\tilde{x}_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{\hat{d}\tilde{x}_{t_i}}{\tilde{x}_{t_i}} \\ &\quad = \begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{(x_{t_i} - \varepsilon\hat{\delta}(t_i))^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i} - \varepsilon\hat{\delta}(t_i)} \\ &\qquad - \left(\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{\hat{\delta}(t_i)\, d\varepsilon}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{\hat{\delta}(t_i)\, d\varepsilon}{x_{t_i}} \right) \end{aligned}$$

$$= \begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}}$$

$$- \varepsilon\hat{\delta}(t_i)\frac{\partial}{\partial x_{t_i}}\left(\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{1}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{1}{x_{t_i}}\right) dx_{t_i}$$

$$+ B_{t_i}\, d\varepsilon$$

$$= \begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}} + \varepsilon\frac{\partial}{\partial x_{t_i}}(B_{t_i})\, dx_{t_i}$$

$$+ B_{t_i}\, d\varepsilon.$$

Moreover, $\widehat{\Omega}_i$ and B_{t_i} are diagonal until the $x_{t_i}^{n_i-1}$-terms. So we have that

$$\widehat{\Omega}_i + \varepsilon\Big(\frac{\partial}{\partial x_{t_i}}(B_{t_i})\, dx_{t_i} + [\widehat{\Omega}_i, B_{t_i}]\Big) + B_{t_i}\, d\varepsilon$$

is a section of $\widetilde{\Omega}^1_{\partial/\partial t_i}$.

As in the previous section, $\widehat{E}_1^{\varepsilon} \cong (\widehat{E}_1)^{\varepsilon}_{t_i}$. If we consider the pull-back of $\nabla^{\varepsilon}_{\partial/\partial t_i}$ under this isomorphism, then we have a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial t_i$. If we take a relativization of this horizontal lift, we have a family of connections parametrized by $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$. This family gives a map from the base space $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$ to the moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. By taking composition with $\widehat{\mathrm{App}}$ defined in (2.13), we have a map

$$(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \longrightarrow \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}. \tag{3.20}$$

Definition 3.6. Then we may define the vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ associated to the map (3.20). We denote this vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by $\delta^{\mathrm{IMD}}_{t_i}$.

Let

$$f^{\mathrm{IMD}}_{t_i} \colon (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$$

be the map induced by the vector field $\delta^{\mathrm{IMD}}_{t_i}$. We have $\widehat{E}_1^{\varepsilon} = (\mathrm{id} \times f^{\mathrm{IMD}}_{t_i})^* \widehat{E}_1$. We denote the pull-back of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ under the map $\mathrm{id} \times f^{\mathrm{IMD}}_{t_i}$ by

$$\begin{cases} d + \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} + \varepsilon\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon], \\ d + G_1^{-1}\, dG_1 + G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} G_1 & \\ \qquad +\varepsilon G_1^{-1}\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) G_1 & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]. \end{cases} \tag{3.21}$$

As in the previous section, we have a lift of $(\mathrm{id} \times f^{\mathrm{IMD}}_{t_i})^* \widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$:

$$\begin{cases} \hat{d} + \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} + \varepsilon \delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \\ \qquad + \Upsilon^{\mathrm{IMD}}_{t_i}\, d\varepsilon & \text{on } U_0 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec} \mathbb{C}[\varepsilon], \\ \hat{d} + G_1^{-1}\, dG_1 + G_1^{-1} \widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} G_1 \\ \qquad + \varepsilon G_1^{-1} \delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) G_1 \\ \qquad + G_1^{-1} \Upsilon^{\mathrm{IMD}}_{t_i} G_1\, d\varepsilon & \text{on } U_\infty \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec} \mathbb{C}[\varepsilon], \end{cases} \tag{3.22}$$

which is a morphism $\widehat{E}^{\varepsilon}_1 \to \widehat{E}^{\varepsilon}_1 \otimes \widetilde{\Omega}^1_{\partial/\partial t_i}$ with the Leibniz rule and the equality

$$\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) = d\Upsilon^{\mathrm{IMD}}_{t_i} + [\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}, \Upsilon^{\mathrm{IMD}}_{t_i}], \tag{3.23}$$

which means the integrable condition.

§3.5. Isomonodromy 2-form

Definition 3.7. Let $\hat{\delta}_1$ and $\hat{\delta}_2$ be vector fields on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$, which is isomorphic to the extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. We fix a formal fundamental matrix solution ψ_i of $(d + \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\psi_i = 0$ at $x = t_i$ as in Lemma 3.1. We take a fundamental matrix solution ψ_{q_j} of $(d + \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\psi_{q_j} = 0$ at $x = q_j$ as in Lemma 2.11. We define a 2-form $\widehat{\omega}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ as

$$\begin{aligned} \widehat{\omega}(\hat{\delta}_1, \hat{\delta}_2) &:= \frac{1}{2} \sum_{i \in I} \operatorname{res}_{x=t_i} \operatorname{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_i)\psi_i^{-1}) \\ &\quad + \frac{1}{2} \sum_{j=1}^{n-3} \operatorname{res}_{x=q_j} \operatorname{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}), \end{aligned} \tag{3.24}$$

where $I := \{1, 2, \ldots, \nu, \infty\}$. Here we set $\hat{\delta}(A) \wedge \hat{\delta}(\psi)\psi^{-1} := \hat{\delta}_1(A)\hat{\delta}_2(\psi)(\psi)^{-1} - \hat{\delta}_1(\psi)(\psi)^{-1}\hat{\delta}_2(A)$.

Since $\theta^{\pm}_{n_i-1,t_i}$ is constant on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$, $\hat{\delta}(\theta^{\pm}_{n_i-1,t_i}) = 0$, we have

$$\hat{\delta}\left(\theta^{\pm}_{n_i-1,t_i} \int x_{t_i}^{-1}\, dx_{t_i}\right) = \frac{-\theta^{\pm}_{n_i-1,t_i}\hat{\delta}(t_i)}{x - t_i} + \hat{\delta}(c).$$

Here, c is an integration constant. By the same argument as in Section 3.1, we have that the residue of $\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_i)\psi_i^{-1}$ at $\tilde{t}_i$ is well defined. By the same argument as in Section 3.1, we may check that the right-hand side of (3.24) is independent of the choices of ψ_{q_j} and ψ_i.

We will show a transformation formula (Lemma 3.8 below). We will use this transformation formula for calculation of $\widehat{\omega}(\hat{\delta}_1, \hat{\delta}_2)$. We show this transformation formula for general situations. Let C be a smooth projective curve over $\mathbb{C}$ and let M be an algebraic variety over $\mathbb{C}$. Let U be an analytic open subset of C. Let x be a parameter on U. Let $d + A\,dx$ be a family of connections on $\mathcal{O}_U^{\oplus 2}$ parameterized by M. Assume we can take a (formal) fundamental matrix solution ψ of $(d + A\,dx)\psi = 0$, that is, there exists $\psi \in \mathcal{E}\mathrm{nd}(\mathcal{O}_M^{\oplus 2}) \otimes \widehat{\mathcal{O}}_{U,0}$ such that $d\psi + A\psi\,dx = 0$. Here, d means the relative exterior derivative on the projection $U \times M \to M$.

Lemma 3.8. *Let δ_1 and δ_2 be vector fields on M. Let g be a family of matrices parameterized by M such that the entries of the matrix g_m for $m \in M$ are meromorphic functions on U. We assume that we can define g^{-1}, which is the family of matrices parameterized by M such that for $m \in M$, $g_m(g^{-1})_m = \mathrm{id}$ and the entries of $(g^{-1})_m$ are meromorphic functions on U. Set $A' := g^{-1}\,dg + g^{-1}Ag$ and $\psi' := g^{-1}\psi$. Moreover, set $u^{(l)} := \delta_l(g)g^{-1}$ and $\tilde{u}^{(l)} := g^{-1}\delta_l(g)$ for $l \in \{1,2\}$. Then we have the equality*

$$\begin{aligned}
&\mathrm{Tr}(\delta(A') \wedge \delta(\psi')(\psi')^{-1}) - \mathrm{Tr}(\delta(A) \wedge \delta(\psi)\psi^{-1}) \\
&\quad = -\mathrm{Tr}(\delta_1(A')\tilde{u}^{(2)} - \tilde{u}^{(1)}\delta_2(A')) - \mathrm{Tr}(\delta_1(A)u^{(2)} - u^{(1)}\delta_2(A)) \\
&\qquad + \mathrm{Tr}\big(d(\psi^{-1}u^{(1)}\delta_2(\psi) - \psi^{-1}u^{(2)}\delta_1(\psi))\big).
\end{aligned} \tag{3.25}$$

Proof. Since $\psi' = g^{-1}\psi$, we have the equalities

$$\begin{aligned}
\mathrm{Tr}(\delta(A') \wedge \delta(\psi')(\psi')^{-1}) &= \mathrm{Tr}(\delta(A') \wedge \delta(g^{-1}\psi)\psi^{-1}g) \\
&= \mathrm{Tr}\big(\delta(A') \wedge (-g^{-1}\delta(g)g^{-1}\psi\psi^{-1}g + g^{-1}\delta(\psi)\psi^{-1}g)\big) \\
&= \mathrm{Tr}\big(\delta(A') \wedge (-g^{-1}\delta(g) + g^{-1}\delta(\psi)\psi^{-1}g)\big).
\end{aligned} \tag{3.26}$$

We calculate $\mathrm{Tr}(\delta(A') \wedge (g^{-1}\delta(\psi)\psi^{-1}g))$ as

$$\begin{aligned}
&\mathrm{Tr}\big(\delta(A') \wedge (g^{-1}\delta(\psi)\psi^{-1}g)\big) \\
&\quad = \mathrm{Tr}\big(\delta(g^{-1}dg + g^{-1}Ag) \wedge (g^{-1}\delta(\psi)\psi^{-1}g)\big) \\
&\quad = \mathrm{Tr}\big((-g^{-1}\delta(g)g^{-1}dg + g^{-1}\delta(dg)) \wedge (g^{-1}\delta(\psi)\psi^{-1}g)\big) \\
&\qquad + \mathrm{Tr}\big((-g^{-1}\delta(g)g^{-1}Ag + g^{-1}\delta(A)g + g^{-1}A\delta(g)) \wedge (g^{-1}\delta(\psi)\psi^{-1}g)\big) \\
&\quad = \mathrm{Tr}\big(d(\delta(g)g^{-1}) \wedge (\delta(\psi)\psi^{-1})\big) - \mathrm{Tr}\big((\delta(g)g^{-1}A) \wedge (\delta(\psi)\psi^{-1})\big) \\
&\qquad + \mathrm{Tr}\big(\delta(A) \wedge (\delta(\psi)\psi^{-1})\big) + \mathrm{Tr}\big((A\delta(g)g^{-1}) \wedge (\delta(\psi)\psi^{-1})\big).
\end{aligned} \tag{3.27}$$

By equalities (3.26) and (3.27), we have

$$\begin{aligned}
&\mathrm{Tr}(\delta(A') \wedge \delta(\psi')(\psi')^{-1}) + \mathrm{Tr}\big(\delta(A') \wedge (g^{-1}\delta(g))\big) - \mathrm{Tr}\big(\delta(A) \wedge (\delta(\psi)\psi^{-1})\big) \\
&\quad = \mathrm{Tr}\big(d(\delta(g)g^{-1}) \wedge (\delta(\psi)\psi^{-1})\big) \\
&\qquad - \mathrm{Tr}\big((\delta(g)g^{-1}A) \wedge (\delta(\psi)\psi^{-1})\big) + \mathrm{Tr}\big((A\delta(g)g^{-1}) \wedge (\delta(\psi)\psi^{-1})\big).
\end{aligned} \tag{3.28}$$

We calculate $\mathrm{Tr}(d(\psi^{-1}u^{(1)}\delta_2(\psi)-\psi^{-1}u^{(2)}\delta_1(\psi)))$ as

$$\begin{aligned}
&\mathrm{Tr}\big(d(\psi^{-1}u^{(1)}\delta_2(\psi)-\psi^{-1}u^{(2)}\delta_1(\psi))\big)\\
&\quad=\mathrm{Tr}(-\psi^{-1}d(\psi)\psi^{-1}u^{(1)}\delta_2(\psi)+\psi^{-1}d(u^{(1)})\delta_2(\psi)+\psi^{-1}u^{(1)}\delta_2(d\psi))\\
&\qquad-\mathrm{Tr}(-\psi^{-1}\,d(\psi)\psi^{-1}u^{(2)}\delta_1(\psi)+\psi^{-1}\,d(u^{(2)})\delta_1(\psi)+\psi^{-1}u^{(2)}\delta_1(d\psi))\\
&\quad=\mathrm{Tr}(Au^{(1)}\delta_2(\psi)\psi^{-1}+d(u^{(1)})\delta_2(\psi)\psi^{-1}-u^{(1)}\delta_2(A\psi)\psi^{-1})\\
&\qquad-\mathrm{Tr}(Au^{(2)}\delta_1(\psi)\psi^{-1}+d(u^{(2)})\delta_1(\psi)\psi^{-1}-u^{(2)}\delta_1(A\psi)\psi^{-1})\\
&\quad=\mathrm{Tr}\big((A\delta(g)g^{-1})\wedge(\delta(\psi)\psi^{-1})\big)+\mathrm{Tr}\big(d(\delta(g)g^{-1})\wedge(\delta(\psi)\psi^{-1})\big)\\
&\qquad-\mathrm{Tr}\big(u^{(1)}\delta_2(A)-u^{(2)}\delta_1(A)+u^{(1)}A\delta_2(\psi)\psi^{-1}-u^{(2)}A\delta_1(\psi)\psi^{-1}\big)\\
&\quad=\mathrm{Tr}\big((A\delta(g)g^{-1})\wedge(\delta(\psi)\psi^{-1})\big)+\mathrm{Tr}\big(d(\delta(g)g^{-1})\wedge(\delta(\psi)\psi^{-1})\big)\\
&\qquad+\mathrm{Tr}\big(\delta(A)\wedge(\delta(g)g^{-1})\big)-\mathrm{Tr}\big((\delta(g)g^{-1}A)\wedge(\delta(\psi)\psi^{-1})\big).
\end{aligned}\tag{3.29}$$

Here, the second equality follows from $d\psi=-A\psi$. Equality (3.25) follows from equalities (3.28) and (3.29). □

Proposition 3.9. *Let $\widetilde{G}$ and $\widetilde{G}_\infty$ be the matrices defined in (2.7). Set $\widetilde{\psi}_i:=\widetilde{G}^{-1}\psi_i$ for any $i\in I_{\mathrm{un}}\setminus\{\infty\}$ and $\widetilde{\psi}_\infty:=\widetilde{G}_\infty^{-1}\psi_\infty$. We have the equality*

$$\widehat{\omega}(\hat{\delta}_1,\hat{\delta}_2)=\frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\widetilde{\psi}_i)(\widetilde{\psi}_i)^{-1}).\tag{3.30}$$

Proof. By Proposition 2.4, we have $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}=\widetilde{G}^{-1}\,d\widetilde{G}+\widetilde{G}^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}\widetilde{G}$. Set $u^{(l)}_{\widetilde{G}}:=\delta_l(\widetilde{G})\widetilde{G}^{-1}$ and $\tilde{u}^{(l)}_{\widetilde{G}}:=\widetilde{G}^{-1}\delta_l(\widetilde{G})$ for $l\in\{1,2\}$. Set $\widetilde{\psi}_{q_j}:=\widetilde{G}^{-1}\psi_{q_j}$ for any $j\in\{1,2,\dots,n-3\}$. We calculate the difference between the right- and left-hand sides of (3.30). By Lemma 2.11, we have $\widetilde{\psi}_{q_j}=\widetilde{G}^{-1}\Phi_{q_j}\Xi_{q_j}(x)\Lambda_{q_j}(x)$. We calculate $\widetilde{G}^{-1}\Phi_{q_j}$ as

$$\widetilde{G}^{-1}\Phi_{q_j}=\begin{pmatrix}1&0\\-\frac{Q_2(x)}{Q_1(x)}&\frac{1}{Q_1(x)}\end{pmatrix}\begin{pmatrix}1&0\\p_j&1\end{pmatrix}=\begin{pmatrix}1&0\\\frac{-Q_2(x)+p_j}{Q_1(x)}&1\end{pmatrix}\begin{pmatrix}1&0\\0&\frac{1}{Q_1(x)}\end{pmatrix}.$$

Since $Q_2(q_j)=p_j$, we may remove a pole of $\frac{-Q_2(x)+p_j}{Q_1(x)}$ at q_j. By Lemma 2.11, we may check that the pole of

$$\begin{pmatrix}1&0\\0&\frac{1}{Q_1(x)}\end{pmatrix}\Xi_{q_j}(x)\Lambda_{q_j}(x)$$

at q_j is removable. So we have that $\mathrm{Tr}(\delta(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\delta(\widetilde{\psi}_{q_j})(\widetilde{\psi}_{q_j})^{-1})$ has no pole at q_j $(j=1,2,\dots,n-3)$. Then the difference between the right- and left-hand sides of (3.30) is equal to

$$\frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\big(\mathrm{Tr}(\delta(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\delta(\widetilde{\psi}_i)(\widetilde{\psi}_i)^{-1})-\mathrm{Tr}(\delta(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})\big)$$
$$+\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\big(\mathrm{Tr}(\delta(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\delta(\widetilde{\psi}_{q_j})(\widetilde{\psi}_{q_j})^{-1})$$
$$-\mathrm{Tr}(\delta(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\delta(\psi_{q_j})\psi_{q_j}^{-1})\big). \tag{3.31}$$

By equation (3.25), the difference (3.31) is equal to

$$-\frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\big(\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\tilde{u}^{(2)}_{\widetilde{G}}-\tilde{u}^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))$$
$$-\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})u^{(2)}_{\widetilde{G}}-u^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))\big)$$
$$-\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\big(\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\tilde{u}^{(2)}_{\widetilde{G}}-\tilde{u}^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))$$
$$-\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})u^{(2)}_{\widetilde{G}}-u^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))\big). \tag{3.32}$$

Here, note that the third term of the right-hand side of (3.25) is an exact form. Then the residue of this third term vanishes. We claim that (3.32) vanishes. We show this claim as follows. Set $u^{(l)}_{\widetilde{G},\infty}:=\delta_l(\widetilde{G}_\infty)\widetilde{G}_\infty^{-1}$ and $\tilde{u}^{(l)}_{\widetilde{G},\infty}:=\widetilde{G}_\infty^{-1}\delta_l(\widetilde{G}_\infty)$ for $l\in\{1,2\}$. Since $\widetilde{G}_\infty=G_{n-2}^{-1}\widetilde{G}G_1$, we have $\tilde{u}^{(l)}_{\widetilde{G},\infty}=G_1^{-1}\tilde{u}^{(l)}_{\widetilde{G}}G_1$ and $u^{(l)}_{\widetilde{G},\infty}=G_{n-2}^{-1}u^{(l)}_{\widetilde{G}}G_{n-2}$ for $l=1,2$. The meromorphic differential form

$$\begin{cases}\frac{1}{2}\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\tilde{u}^{(2)}_{\widetilde{G}}-\tilde{u}^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\ \quad -\frac{1}{2}\mathrm{Tr}(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})u^{(2)}_{\widetilde{G}}-u^{(1)}_{\widetilde{G}}\delta_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}), \\ \frac{1}{2}\mathrm{Tr}(G_1^{-1}\delta_1(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1\tilde{u}^{(2)}_{\widetilde{G},\infty} \\ \qquad -\tilde{u}^{(1)}_{\widetilde{G},\infty}G_1^{-1}\delta_2(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1) \\ \quad -\frac{1}{2}\mathrm{Tr}(G_{n-2}^{-1}\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_{n-2}u^{(2)}_{\widetilde{G},\infty} \\ \qquad -u^{(1)}_{\widetilde{G},\infty}G_{n-2}^{-1}\delta_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_{n-2}) & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}),\end{cases} \tag{3.33}$$

is a family of global meromorphic differential forms on $\mathbb{P}^1$ parametrized by $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. The differential forms have poles at only t_i and q_j ($i\in I$ and $j=1,2,\dots,n-3$). The sums of residues (3.32) are just the sums of all residues of the global meromorphic differential forms (3.33) on $\mathbb{P}^1$ parametrized by $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. By the residue theorem, we have that (3.32) is zero. Finally, we obtain equality (3.30). □

Theorem 3.10. *For the vector field* $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$, *we have* $\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}},\hat{\delta})=0$ *for any vector field* $\hat{\delta}\in\Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}}$. *Moreover, for the vector field* $\delta^{\mathrm{IMD}}_{t_i}$, *we have* $\widehat{\omega}(\delta^{\mathrm{IMD}}_{t_i},\hat{\delta})=0$ *for any vector field* $\hat{\delta}\in\Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}}$.

Proof. By equality (3.30), we have

$$
\begin{aligned}
\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}},\hat{\delta}) = \frac{1}{2}\sum_{i'\in I}\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(&\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\\
&-\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\big).
\end{aligned}
\tag{3.34}
$$

Here, $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})$ appears in the ε-term of the morphism (3.17).

Now we consider replacement of $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}$ in (3.34) for each $i'\in I$. We will show that we may replace $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}$ with $\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ as follows. Here, $\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ appeared in (3.18). We take an analytic open subset V of $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. We take an inverse image of V under the projection

$$p_{\mathbb{P}^1}\colon \mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\longrightarrow \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}.$$

Let $\widehat{\Delta}^{\mathrm{an}}_{i'}$ $(i'\in I)$ be an analytic open subset of the inverse image $p_{\mathbb{P}^1}^{-1}(V)$ such that $\tilde{t}_{i'}\cap p_{\mathbb{P}^1}^{-1}(V)\subset\widehat{\Delta}^{\mathrm{an}}_{i'}$ and the fibers of $p_{\mathbb{P}^1}|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}\colon\widehat{\Delta}^{\mathrm{an}}_{i'}\to V$ for each point of V are unit disks such that x_{t_i} gives a coordinate of the unit disks. Let U^{an}_t be an analytic open subset of $\mathbb{C}^1=\operatorname{Spec}\mathbb{C}[t]$ such that $0\in U^{\mathrm{an}}_t$ and U^{an}_t is small enough. We consider the restriction of (3.17) to $\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]$. This is a morphism

$$\widehat{E}^{\varepsilon}_1|_{\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]}\to(\widehat{E}^{\varepsilon}_1\otimes\widetilde{\Omega}^1_{\partial/\partial\theta^{\pm}_{l,t_i}})|_{\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]}.$$

Let $\widehat{E}^t_1$ be the pull-back of E_1 under the first projection

$$\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t\hookrightarrow\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times U^{\mathrm{an}}_t\to\mathbb{P}^1.$$

Let $D(\tilde{\boldsymbol{t}}_0)_\varepsilon$ be the pull-back of $D(\tilde{\boldsymbol{t}}_0)$ under the composition (2.18). We take a divisor $D(\tilde{\boldsymbol{t}}_0)_t$ on $\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t$ such that the pull-back of $D(\tilde{\boldsymbol{t}}_0)_t$ under the map $\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t$ is $D(\tilde{\boldsymbol{t}}_0)_\varepsilon|_{\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]}$. Here, this map $\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t$ is given by the substitution $t=\varepsilon$. We take a relative connection on $\widehat{E}^t_1$:

$$\widehat{E}^t_1\longrightarrow\widehat{E}^t_1\otimes\Omega^1_{\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t/\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}\times U^{\mathrm{an}}_t}(D(\tilde{\boldsymbol{t}}_0)_t)$$

such that the pull-back of this relative connection on $\widehat{E}^t_1$ under the map $\widehat{\Delta}^{\mathrm{an}}_{i'}\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to\widehat{\Delta}^{\mathrm{an}}_{i'}\times U^{\mathrm{an}}_t$ is just the restriction of (3.17). We denote the connection matrix of the relative connection by $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)$, where

$$\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},0)=\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{\Delta}^{\mathrm{an}}_{i'}},$$
$$\frac{\partial}{\partial t}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)|_{t=0}=\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}.$$

Let $\widehat{\Sigma}\subset\widehat{\Delta}^{\mathrm{an}}_{i'}$ be a family of sufficiently small sectors in $\widehat{\Delta}^{\mathrm{an}}_{i'}\to\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. We take a fundamental matrix solution $\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)$ on $\widehat{\Sigma}\times U^{\mathrm{an}}_t$ of the connection

$$d\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)=0$$

with uniform asymptotic relation

$$\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)\exp(\Lambda^-_{i'}(x_{t_{i'}},t))\sim\widehat{P}_{i'}(x_{t_{i'}},t)\quad(x_{t_{i'}}\to0,\ x_{t_{i'}}\in\widehat{\Sigma}).$$

Here we set

$$\Lambda^-_{i'}(x_{t_{i'}}):=\sum_{l=0}^{n_i-1}\begin{pmatrix}\theta^+_{l,t_{i'}}\int x_{t_{i'}}^{-n_{i'}+l}\,dx_{t_{i'}} & 0\\ 0 & \theta^-_{l,t_{i'}}\int x_{t_{i'}}^{-n_{i'}+l}\,dx_{t_{i'}}\end{pmatrix}$$

and we take

$$\widehat{P}_{i'}(x_{t_{i'}},t)=\widehat{P}_{i',0}(t)+\widehat{P}_{i',1}(t)x_{t_{i'}}+\cdots,$$
$$\Lambda^-_{i'}(x_{t_{i'}},t)=\sum_{l=0}^{n_i-1}\begin{pmatrix}\theta^+_{l,t_{i'}}(t)\int x_{t_{i'}}^{-n_{i'}+l}\,dx_{t_{i'}} & 0\\ 0 & \theta^-_{l,t_{i'}}(t)\int x_{t_{i'}}^{-n_{i'}+l}\,dx_{t_{i'}}\end{pmatrix},$$

so that the expansions of $\Lambda^-_{i'}(x_{t_{i'}},\varepsilon)$ and $\widehat{P}_{i'}(x_{t_{i'}},\varepsilon)$ with respect to ε are

$$\Lambda^-_{i'}(x_{t_{i'}},\varepsilon)=\Lambda^-_{i'}(x_{t_{i'}})+\varepsilon\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\Lambda^-_{i'}(x_{t_{i'}})),$$
$$\widehat{P}_{i'}(x_{t_{i'}},\varepsilon)=\widetilde{\psi}_{i'}\exp(\Lambda^-_{i'}(x_{t_{i'}}))+\varepsilon\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\big(\widetilde{\psi}_{i'}\exp(\Lambda^-_{i'}(x_{t_{i'}}))\big).$$

The uniform asymptotic relation means that

$$\lim_{\substack{x_{t_{i'}}\to0\\ x_{t_{i'}}\in\Gamma_{\widehat{\Sigma}}}}\frac{\|\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)\exp(\Lambda^-_{i'}(x_{t_{i'}},t))-\sum_{j=0}^N\widehat{P}_{i',j}x^j_{t_{i'}}\|}{|x_{t_{i'}}|^N}=0\quad\text{(uniformly)}$$

for any positive integer N. We may check that

$$\frac{\partial\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)}{\partial t}\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)^{-1}\Big|_{t=0}\sim\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}.\tag{3.35}$$

By the integrable condition (3.19), we may take a fundamental matrix solution $\Psi^{\mathrm{flat}}_{\widehat{\Sigma}}(x_{t_{i'}},t)$ on $\widehat{\Sigma}\times U^{\mathrm{an}}_t$ of the connection $d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)=0$ such that

$\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)$ satisfies $\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},0)=\Psi_{\widehat{\Sigma}}(x_{t_{i'}},0)$ and

$$\frac{\partial\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)}{\partial t}\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)^{-1}\Big|_{t=0}=\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}|_{\widehat{\Sigma}}.$$

There exists a matrix $C_{t_{i'}}(t)$ such that $C_{t_{i'}}(t)$ is independent of $x_{t_{i'}}$ and $\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)=\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)C_{t_{i'}}(t)$. We calculate $\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}|_{\widehat{\Sigma}}$ as

$$\begin{aligned}\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}|_{\widehat{\Sigma}}&=\frac{\partial\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)}{\partial t}\Psi^{\rm flat}_{\widehat{\Sigma}}(x_{t_{i'}},t)^{-1}\Big|_{t=0}\\&=\frac{\partial\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)}{\partial t}\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)^{-1}\Big|_{t=0}\\&\quad+\Psi_{\widehat{\Sigma}}(x_{t_{i'}},0)\Big(\frac{\partial C_{t_{i'}}(t)}{\partial t}C_{t_{i'}}(t)^{-1}\Big)\Big|_{t=0}\Psi_{\widehat{\Sigma}}(x_{t_{i'}},0)^{-1}.\end{aligned}\tag{3.36}$$

Set

$$\widetilde{C}_{t_{i'}}(x_{t_{i'}}):=\exp(-\Lambda^{-}_{i'}(x_{t_{i'}},t))\Big(\frac{\partial C_{t_{i'}}(t)}{\partial t}C_{t_{i'}}(t)^{-1}\Big)\exp(\Lambda^{-}_{i'}(x_{t_{i'}},t))\Big|_{t=0}.$$

By equalities (3.35) and (3.36) we have

$$\begin{aligned}\widetilde{C}_{t_{i'}}(x_{t_{i'}})\sim\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}\widehat{P}_{i'}(x_{t_{i'}},0)\\-\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\psi'_{i'})(\psi'_{i'})^{-1}\widehat{P}_{i'}(x_{t_{i'}},0).\end{aligned}$$

By this asymptotic relation, we have that $x_{t_{i'}}^{n_{i'}}\widetilde{C}_{t_{i'}}(x_{t_{i'}})$ is bounded on $\widehat{\Sigma}\times U^{\rm an}_{t}$. Then we may check that $(\frac{\partial C_{t_{i'}}(t)}{\partial t}C_{t_{i'}}(t)^{-1})|_{t=0}$ is a triangular matrix and that $\widetilde{C}_{t_{i'}}(x_{t_{i'}})\sim\widetilde{C}^{\rm diag}_{t_{i'}}$, where $\widetilde{C}^{\rm diag}_{t_{i'}}$ is a diagonal matrix and $\widetilde{C}^{\rm diag}_{t_{i'}}$ is independent of $x_{t_{i'}}$. By combining this asymptotic relation, asymptotic relation (3.35), and equality (3.36), we have the asymptotic relation

$$\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\sim\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}-\widehat{P}_{i'}(x_{t_{i'}},0)\widetilde{C}^{\rm diag}_{t_{i'}}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}.\tag{3.37}$$

So we have

$$\begin{aligned}&\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}-\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}))\\&=\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}-\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}))\\&\quad+\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\widehat{P}_{i'}(x_{t_{i'}},0)\widetilde{C}^{\rm diag}_{t_{i'}}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)}))\\&=\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\delta^{\rm IMD}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})\\&\quad-\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\Upsilon^{\rm IMD}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)})).\end{aligned}\tag{3.38}$$

Here we may check the last equality as follows. By the same calculations as in (3.2), we have

$$\begin{aligned}&\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}(\widehat{P}_{i'}(x_{t_{i'}},0)\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))\\&\quad=\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}(\widetilde{\psi}_{i'}\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\widetilde{\psi}_{i'}^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))\\&\quad=\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d((\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widetilde{\psi}_{i'}))\big).\end{aligned}$$

Here, remark that $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ and $\exp(-\Lambda^-_{i'}(x_{t_{i'}}))$ are diagonal, and $\widetilde{\psi}_{i'}=\widehat{P}_{i'}(x_{t_{i'}},0)\times\exp(-\Lambda^-_{i'}(x_{t_{i'}}))$. This is equal to

$$\begin{aligned}&\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}d\big(\exp(-\Lambda^-_{i'}(x_{t_{i'}}))^{-1}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{P}_{i'}(x_{t_{i'}},0))\exp(-\Lambda^-_{i'}(x_{t_{i'}}))\big)\big)\\&\qquad+\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\Big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\big(\exp(-\Lambda^-_{i'}(x_{t_{i'}}))^{-1}\hat{\delta}\big(\exp(-\Lambda^-_{i'}(x_{t_{i'}}))\big)\big)\Big)\\&\quad=\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\Lambda^-_{i'}(x_{t_{i'}})\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{P}_{i'}(x_{t_{i'}},0))\big)\\&\qquad-\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{P}_{i'}(x_{t_{i'}},0))\,d\Lambda^-_{i'}(x_{t_{i'}})\big)\\&\qquad+\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\big(\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{P}_{i'}(x_{t_{i'}},0))\big)\big)\\&\qquad+\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\big(\hat{\delta}(-\Lambda^-_{i'}(x_{t_{i'}}))\big)\big)\\&\quad=\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\big(\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}\hat{\delta}(\widehat{P}_{i'}(x_{t_{i'}},0))\big)\big)\\&\qquad+\mathrm{res}_{x=t_{i'}}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\,d\big(\hat{\delta}(-\Lambda^-_{i'}(x_{t_{i'}}))\big)\big).\end{aligned}\tag{3.39}$$

Since $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ is independent of $x_{t_{i'}}$, the last line of (3.39) is zero. Then we have the last equality of (3.38). This means that we may replace $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}$ with $\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$.

Next we will calculate $\mathrm{Tr}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})$. By taking variations of the both sides of $d\widetilde{\psi}_{i'}=-\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}\widetilde{\psi}_{i'}$, we have

$$\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}=-d(\hat{\delta}(\widetilde{\psi}_{i'}))(\widetilde{\psi}_{i'})^{-1}-\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}).\tag{3.40}$$

By the integrable condition (3.19), we have

$$\begin{aligned}&\mathrm{Tr}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})\\&\quad=\mathrm{Tr}\big((d\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}+[\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)},\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}])\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\big)\\&\quad=\mathrm{Tr}(d(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\\&\qquad\quad-\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Tr}\bigl(d(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} - d\widetilde{\psi}_{i'}(\widetilde{\psi}_{i'})^{-1}\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} \\
&\qquad + \Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\, d(\hat{\delta}(\widetilde{\psi}_{i'}))(\widetilde{\psi}_{i'})^{-1} + \Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\bigr) \\
&= \operatorname{Tr}\bigl(d(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} + \Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\, d(\hat{\delta}(\widetilde{\psi}_{i'}))(\widetilde{\psi}_{i'})^{-1} \\
&\qquad + \Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widetilde{\psi}_{i'})\, d((\widetilde{\psi}_{i'})^{-1})\bigr) \\
&\quad + \operatorname{Tr}(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\
&= \operatorname{Tr}\bigl(d(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})\bigr) + \operatorname{Tr}(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})). \qquad (3.41)
\end{aligned}
$$

Here, the third equality of (3.41) is given by (3.40). By combining (3.38) and (3.41), we have the equalities

$$
\begin{aligned}
&\operatorname{res}_{x=t_{i'}} \operatorname{Tr}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} - \delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\
&\quad = \operatorname{res}_{x=t_{i'}} \operatorname{Tr}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}) \\
&\qquad - \operatorname{res}_{x=t_{i'}} \operatorname{Tr}(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\
&\quad = \operatorname{res}_{x=t_{i'}} \operatorname{Tr}\bigl(d(\Upsilon^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1})\bigr) = 0. \qquad (3.42)
\end{aligned}
$$

By combining (3.34) and (3.42), we have

$$
\begin{aligned}
\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}, \hat{\delta}) &= \frac{1}{2}\sum_{i'\in I} \operatorname{res}_{x=t_{i'}} \operatorname{Tr}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} \\
&\qquad\qquad - \delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\
&= 0.
\end{aligned}
$$

Next we will show that $\widehat{\omega}(\delta^{\mathrm{IMD}}_{t_i}, \hat{\delta}) = 0$ for any $\hat{\delta}$. For this purpose, we show that

$$
\begin{aligned}
\widehat{\omega}(\delta^{\mathrm{IMD}}_{t_i}, \hat{\delta}) = \frac{1}{2}\sum_{i'\in I} \operatorname{res}_{x=t_{i'}} \operatorname{Tr}(\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} \\
- \Upsilon^{\mathrm{IMD}}_{t_i}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \qquad (3.43)
\end{aligned}
$$

for any $\hat{\delta}$. Here, $\Upsilon^{\mathrm{IMD}}_{t_i}$ appeared in (3.22). We define $\widehat{\Delta}^{\mathrm{an}}_{i'}$ and U^{an}_t as above. Let $D(\tilde{\boldsymbol{t}}_0)_\varepsilon$ be the pull-back of $D(\tilde{\boldsymbol{t}}_0)$ under the composition (2.18) with respect to the vector field $\partial/\partial t_i$. We define a divisor $D(\tilde{\boldsymbol{t}}_0)_t$ on $\widehat{\Delta}^{\mathrm{an}}_{i'} \times U^{\mathrm{an}}_t$ for this divisor $D(\tilde{\boldsymbol{t}}_0)_\varepsilon$. We take a vector bundle $\widehat{E}^t_1$ on $\widehat{\Delta}^{\mathrm{an}}_{i'} \times U^{\mathrm{an}}_t$ and a relative connection

$$
\widehat{E}^t_1 \longrightarrow \widehat{E}^t_1 \otimes \Omega^1_{\widehat{\Delta}^{\mathrm{an}}_{i'} \times U^{\mathrm{an}}_t / \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}} \times U^{\mathrm{an}}_t}(D(\tilde{\boldsymbol{t}}_0)_t)
$$

corresponding to (3.21). We denote the connection matrix of the relative connection by $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)$, where the expansion of $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}}-\varepsilon\delta^{\mathrm{IMD}}_{t_i}(t_{i'}),\varepsilon)$ with respect to ε is equal to

$$\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}+\varepsilon\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}.$$

Let $\widehat{\Sigma}\subset\widehat{\Delta}^{\mathrm{an}}_{i'}$ be a family of sufficiently small sectors in $\widehat{\Delta}^{\mathrm{an}}_{i'}\to\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. We take a fundamental matrix solution $\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)$ on $\widehat{\Sigma}\times U^{\mathrm{an}}_t$ of the connection

$$d\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(x_{t_{i'}},t)\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)=0$$

with uniform asymptotic relation

$$\Psi_{\widehat{\Sigma}}(x_{t_{i'}},t)\exp(\Lambda^-_{i'}(x_{t_{i'}},t))\sim\widehat{P}_{i'}(x_{t_{i'}},t)\quad(x_{t_{i'}}\to 0,x_{t_{i'}}\in\widehat{\Sigma}).\tag{3.44}$$

Here we define $\Lambda^-_{i'}(x_{t_{i'}},t)$ and $\widehat{P}_{i'}(x_{t_{i'}},t)$ so that the expansions of $\Lambda^-_{i'}(x_{t_{i'}}-\varepsilon\delta^{\mathrm{IMD}}_{t_i}(t_{i'}),\varepsilon)$ and $\widehat{P}_{i'}(x_{t_{i'}}-\varepsilon\delta^{\mathrm{IMD}}_{t_i}(t_{i'}),\varepsilon)$ with respect to ε are

$$\begin{aligned}&\Lambda^-_{i'}(x_{t_{i'}}-\varepsilon\delta^{\mathrm{IMD}}_{t_i}(t_{i'}),\varepsilon)=\Lambda^-_{i'}(x_{t_{i'}})+\varepsilon\delta^{\mathrm{IMD}}_{t_i}(\Lambda^-_{i'}(x_{t_{i'}})),\\&\widehat{P}_{i'}(x_{t_{i'}}-\varepsilon\delta^{\mathrm{IMD}}_{t_i}(t_{i'}),\varepsilon)=\widetilde{\psi}_{i'}\exp(\Lambda^-_{i'}(x_{t_{i'}}))+\varepsilon\delta^{\mathrm{IMD}}_{t_i}\big(\widetilde{\psi}_{i'}\exp(\Lambda^-_{i'}(x_{t_{i'}}))\big).\end{aligned}$$

By the asymptotic relation (3.44) and the same argument as above, we have

$$\delta^{\mathrm{IMD}}_{t_i}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\sim\Upsilon^{\mathrm{IMD}}_{t_i}-\widehat{P}_{i'}(x_{t_{i'}},0)\widetilde{C}^{\mathrm{diag}}_{t_{i'}}\widehat{P}_{i'}(x_{t_{i'}},0)^{-1}$$

as in (3.37). Here, $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ is a diagonal matrix and $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ is independent of $x_{t_{i'}}$. By this asymptotic relation, we may check that (3.43) is as above. By the integrable condition (3.23), we may check that

$$\begin{aligned}\widehat{\omega}(\delta^{\mathrm{IMD}}_{t_i},\hat{\delta})&=\frac{1}{2}\sum_{i'\in I}\mathrm{res}_{x=t_{i'}}\mathrm{Tr}(\delta^{\mathrm{IMD}}_{t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}-\Upsilon^{\mathrm{IMD}}_{t_i}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}))\\&=0\end{aligned}$$

for any $\hat{\delta}$. □

By Theorems 3.4 and 3.10, we have that the 2-form $\widehat{\omega}$ is the isomonodromy 2-form.

§3.6. Hamiltonian systems

First, we define *Hamiltonians* on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ as follows. By Lemma 3.1 we have the diagonalization

$$\Omega^{\rm diag}_{t_i} = \frac{\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix}}{x_{t_i}^{n_i}} dx_{t_i} + \cdots + \frac{\begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix}}{x_{t_i}} dx_{t_i}$$
$$+ \begin{pmatrix} \theta^+_{n_i,t_i} & 0 \\ 0 & \theta^-_{n_i,t_i} \end{pmatrix} dx_{t_i} + \cdots + \begin{pmatrix} \theta^+_{2n_i-1,t_i} & 0 \\ 0 & \theta^-_{2n_i-1,t_i} \end{pmatrix} x_{t_i}^{n_i-1} dx_{t_i} + \cdots .$$

Here we set $\Omega^{\rm diag}_{t_i} := (\Phi_i \Xi_i)^{-1} d(\Phi_i\Xi_i) + (\Phi_i\Xi_i)^{-1} \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0)} (\Phi_i \Xi_i)$. Remark that we have an equation $(d + \Omega^{\rm diag}_{t_i}) \exp(-\Lambda_i(x_{t_i})) = 0$. We set

$$\Lambda^+_i(x_{t_i}) = \begin{pmatrix} \sum_{k=n_i}^\infty \theta^+_{k,t_i} \int x_{t_i}^{-n_i+k}\, dx_{t_i} & 0 \\ 0 & \sum_{k=n_i}^\infty \theta^-_{k,t_i} \int x_{t_i}^{-n_i+k}\, dx_{t_i} \end{pmatrix}.$$

Definition 3.11. For each $i \in I_{\rm un}$ and each l $(0 \le l \le n_i - 2)$, we define rational functions $H_{\theta^\pm_{l,t_i}}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ as

$$\begin{aligned} H_{\theta^+_{l,t_i}} &= -[\text{the coefficient of the } x_{t_i}^{n_i-l-1}\text{-term of the } (1,1)\text{-entry of } \Lambda^+_i(x_{t_i})] \\ &= -\frac{\theta^+_{2n_i-l-2,t_i}}{n_i - l - 1}, \\ H_{\theta^-_{l,t_i}} &= -[\text{the coefficient of the } x_{t_i}^{n_i-l-1}\text{-term of the } (2,2)\text{-entry of } \Lambda^+_i(x_{t_i})] \\ &= -\frac{\theta^-_{2n_i-l-2,t_i}}{n_i - l - 1}. \end{aligned}$$

We call $H_{\theta^\pm_{l,t_i}}$ the *Hamiltonian associated to* $\theta^\pm_{l,t_i}$.

Definition 3.12. For each $i \in \{3, 4, \dots, \nu\}$, put $\hat{\lambda}^{\le 2n_i-1}_{i,\pm}(x) := \sum_{l=0}^{2n_i-1} \theta^\pm_{l,t_i} \int (x - t_i)^{-n_i+l}\, dx$. We define a rational function H_{t_i} on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$ as

$$\begin{aligned} H_{t_i} &:= -\frac{1}{2} \mathrm{res}_{x=t_i} (\mathrm{Tr}(\Omega^{\rm diag}_{t_i})^2) \\ &= -\sum_{l=0}^{n_i-1} \theta^+_{l,t_i} \theta^+_{2n_i-l-1,t_i} - \sum_{l=0}^{n_i-1} \theta^-_{l,t_i} \theta^-_{2n_i-l-1,t_i}. \end{aligned}$$

We call H_{t_i} the *Hamiltonian associated to* t_i.

We will give a description of integrable deformations by using these Hamiltonians. This description is derived by calculation of the isomonodromy 2-form $\widehat{\omega}$.

Now we prepare a lemma for calculation of the isomonodromy 2-form $\widehat{\omega}$. Let t be a component of the divisor D and (U_t, x_t) be a couple of an affine open subset such that $t \in U_t$ and $x_t = x - t$. Let Ω be an element of $\mathfrak{gl}(2,\mathbb{C}) \otimes \Omega^1_{U_t}(D + D_{\mathrm{App}}) \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$, which has an expansion at t as

$$\Omega = \frac{\Omega_0\, dx_t}{x_t^{n_t}} + \frac{\Omega_1\, dx_t}{x_t^{n_t-1}} + \cdots + \frac{\Omega_{n_t-1}\, dx_t}{x_t} + \cdots ,$$

where $\Omega_k \in \mathfrak{gl}(2,\mathbb{C}) \otimes \mathcal{O}_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$. Let g be an element of $\mathcal{E}\mathrm{nd}(\mathcal{O}^{\oplus 2}_{U_t \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})})$ such that g has an expansion at t as

$$g = g_0 + g_1 x_t + \cdots + g_{n_t} x_t^{n_t} + \cdots , \tag{3.45}$$

where $g_k \in \mathcal{E}\mathrm{nd}(\mathcal{O}^{\oplus 2}_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}})$.

Lemma 3.13. *Let $\hat{\delta}_1$ and $\hat{\delta}_2$ be vector fields on an open subset of $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ such that Ω and g are defined on this open subset. Let ψ be a formal solution of $d+\Omega = 0$ at t. We assume that*

- $\Omega_k \in \mathfrak{gl}(2,\mathbb{C}) \otimes \pi^{-1}_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}(\mathcal{O}_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}})$ *for* $k = 0, 1, \ldots, n_t - 1$,
- $g_k \in \mathcal{E}\mathrm{nd}(\pi^{-1}_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}(\mathcal{O}_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}})^{\oplus 2})$ *for* $k = 0, 1, \ldots, n_t - 1$,
- *we can define the inverse matrix* $g_0^{-1} \in \mathcal{E}\mathrm{nd}(\pi^{-1}_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}(\mathcal{O}_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}})^{\oplus 2})$ *of* g_0,
- *the* $(1,1)$*- and* $(2,2)$*-entries of* $g_0^{-1} g_{n_t}$ *vanish,*
- $g_0^{-1}\Omega_0 g_0$ *is a diagonal matrix, and*
- $\hat{\delta}_1(\psi)\psi^{-1}$ *and* $\hat{\delta}_2(\psi)\psi^{-1}$ *are formally meromorphic at* t.

If we set $\Omega' = g^{-1}\, dg + g^{-1}\Omega g$ and $\psi' = g^{-1}\psi$, then the difference

$$\mathrm{res}_{x=t} \operatorname{Tr}(\hat{\delta}(\Omega') \wedge \hat{\delta}(\psi')(\psi')^{-1}) - \mathrm{res}_{x=t} \operatorname{Tr}(\hat{\delta}(\Omega) \wedge \hat{\delta}(\psi)(\psi)^{-1})$$

is a section of $\pi^{-1}_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}(\Omega^2_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}})$.

Proof. Put $\hat{\delta}(\Omega) \wedge \hat{\delta}(\psi)\psi^{-1} := \hat{\delta}_1(\Omega)\hat{\delta}_2(\psi)(\psi)^{-1} - \hat{\delta}_1(\psi)(\psi)^{-1}\hat{\delta}_2(\Omega)$. By Lemma 3.8, we have the equality

$$\begin{aligned}
&\operatorname{Tr}(\hat{\delta}(\Omega') \wedge \hat{\delta}(\psi')(\psi')^{-1}) - \operatorname{Tr}(\hat{\delta}(\Omega) \wedge \hat{\delta}(\psi)\psi^{-1}) \\
&\qquad = - \operatorname{Tr}(\hat{\delta}_1(\Omega')\tilde{u}^{(2)} - \tilde{u}^{(1)}\hat{\delta}_2(\Omega')) - \operatorname{Tr}(\hat{\delta}_1(\Omega)u^{(2)} - u^{(1)}\hat{\delta}_2(\Omega)) \\
&\qquad\quad + \operatorname{Tr}\big(d(\psi^{-1}u^{(1)}\hat{\delta}_2(\psi) - \psi^{-1}u^{(2)}\hat{\delta}_1(\psi))\big),
\end{aligned} \tag{3.46}$$

where $u^{(i)} := \hat{\delta}_i(g)g^{-1}$ and $\tilde{u}^{(i)} := g^{-1}\hat{\delta}_i(g)$ for $i \in \{1,2\}$. We set $\hat{\delta}_1 = \delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})}$. We will show that the residue of (3.46) at t vanishes. We consider the residues of the first term and the second term of the right-hand side of (3.46). We calculate $\hat{\delta}_2(\Omega')$ as

$$\begin{aligned}\hat{\delta}_2(\Omega') &= \hat{\delta}_2((g_0^{-1}\Omega_0 g_0)(x-t)^{-n_t})\,dx + \cdots \\ &= g_0^{-1}\Omega_0 g_0 \hat{\delta}_2((x-t)^{-n_t})\,dx + \hat{\delta}_2(g_0^{-1}\Omega_0 g_0)(x-t)^{-n_t}\,dx + \cdots \\ &= n_{t_i}\hat{\delta}_2(t) g_0^{-1}\Omega_0 g_0 (x-t)^{-n_t-1}\,dx + \hat{\delta}_2(g_0^{-1}\Omega_0 g_0)(x-t)^{-n_t}\,dx + \cdots .\end{aligned}$$

So we have that the variation $\hat{\delta}_2(\Omega')$ has a pole of order n_t+1 at t and the leading coefficient of $\hat{\delta}_2(\Omega')$ is the diagonal matrix $n_t\hat{\delta}_2(t)g_0^{-1}\Omega_0 g_0$. Since δ_1 is an element of $\Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm tra}}\times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm tra}}\times T_{\boldsymbol{\theta}})}$, we may check that $\delta_1(\Omega')\tilde{u}^{(2)}$ is holomorphic at t. We calculate the residue of the first term as

$$\begin{aligned}-\mathrm{res}_{x=t}\operatorname{Tr}(\delta_1(\Omega')\tilde{u}^{(2)} - \tilde{u}^{(1)}\hat{\delta}_2(\Omega')) &= \mathrm{res}_{x=t}\operatorname{Tr}(\tilde{u}^{(1)}\hat{\delta}_2(\Omega')) \\ &= n_t\hat{\delta}_2(t)\operatorname{Tr}(g_0^{-1}\delta_1(g_{n_t})g_0^{-1}\Omega_0 g_0) \\ &= n_t\hat{\delta}_2(t)\operatorname{Tr}(\delta_1(g_0^{-1}g_{n_t})g_0^{-1}\Omega_0 g_0).\end{aligned} \tag{3.47}$$

Since the diagonal entries of $g_0^{-1}g_{n_t}$ vanish and $g_0^{-1}\Omega_0 g_0$ is a diagonal matrix, the residue (3.47) is zero. Next we calculate the residue of the second term as

$$\begin{aligned}&-\mathrm{res}_{x=t}\operatorname{Tr}(\delta_1(\Omega)u^{(2)} - u^{(1)}\hat{\delta}_2(\Omega)) \\ &\quad= \mathrm{res}_{x=t}\operatorname{Tr}(\delta_1(g)g^{-1}\hat{\delta}_2(\Omega)) \\ &\quad= \mathrm{res}_{x=t}\operatorname{Tr}\big(g^{-1}\delta_1(g)(\hat{\delta}_2(g^{-1}\Omega g) - \hat{\delta}_2(g^{-1})\Omega g - g^{-1}\Omega\hat{\delta}_2(g))\big) \\ &\quad= n_t\hat{\delta}_2(t)\operatorname{Tr}(g_0^{-1}\delta_1(g_{n_t})g_0^{-1}\Omega_0 g_0) = 0.\end{aligned}$$

Here, remark that $\delta_1(g)\in O(x_t^{n_t})$. Finally, the residue of the third term of (3.46) at t is zero, since $\operatorname{Tr}(\psi^{-1}u^{(1)}\hat{\delta}_2(\psi) - \psi^{-1}u^{(2)}\hat{\delta}_1(\psi))$ is formally meromorphic at t. Then we have that the residue of (3.25) at t vanishes. We obtain the assertion of this lemma. □

Theorem 3.14. *Set $P(x;\boldsymbol{t}) := \prod_{i=1}^{\nu}(x-t_i)^{n_i}$ and $D_i(x;\boldsymbol{t},\boldsymbol{\theta}) := D_i(x)$ for $i\in I$. We put*

$$\begin{aligned}\widehat{\omega}' &:= \sum_{j=1}^{n-3} d\left(\frac{p_j}{P(q_j;\boldsymbol{t})} - \sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}} - D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})\right)\wedge dq_j \\ &\quad + \sum_{i\in I_{\rm un}}\sum_{l=0}^{n_i-2}(dH_{\theta^+_{l,t_i}}\wedge d\theta^+_{l,t_i} + dH_{\theta^-_{l,t_i}}\wedge d\theta^-_{l,t_i}) + \sum_{i\in\{3,4,\ldots,\nu\}} dH_{t_i}\wedge dt_i.\end{aligned} \tag{3.48}$$

Then the difference $\widehat{\omega} - \widehat{\omega}'$ is a section of $\pi^*_{\boldsymbol{t}_{\rm ra},\boldsymbol{\theta}_0}(\Omega^2_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\rm ra}}\times T_{\boldsymbol{\theta}}})$.

Proof. Recall that $\widehat{\omega}$ is

$$\frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_i)\psi_i^{-1})$$
$$+\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}).$$

The plan of the proof is as follows. First we will consider the first term of this formula. We calculate the residue at t_i for some (local) gauge transformation of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. We need to consider the difference between the residue after taking the gauge transformation and the residue before taking the gauge transformation. Here, the residue before taking the gauge transformation is just $\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_i)\psi_i^{-1})$. To consider this difference, we will use Lemma 3.13. By calculation of the residue after taking the gauge transformation, we may derive the second part of (3.48):

$$\sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}(dH_{\theta^+_{l,t_i}}\wedge d\theta^+_{l,t_i}+dH_{\theta^-_{l,t_i}}\wedge d\theta^-_{l,t_i})+\sum_{i\in\{3,4,\dots,\nu\}}dH_{t_i}\wedge dt_i.$$

Second we will calculate the second term (the residue of $\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1})$ at $x=q_j$) by the same argument as in the proof of Theorem 3.4. Then we may derive the first part of (3.48):

$$\sum_{j=1}^{n-3}d\left(\frac{p_j}{P(q_j;\boldsymbol{t})}-\sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}}-D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})\right)\wedge dq_j.$$

First, we consider the residue of $\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_i)\psi_i^{-1})$ at t_i. Now we take diagonalizations of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ until some degree term at each point t_i. For $i\in I$, we put

$$\Xi_i^{\leq 2n_i-1}(x_{t_i}):=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{s=1}^{2n_i-1}\begin{pmatrix}(\xi^{(i)}_s)_{11}&(\xi^{(i)}_s)_{12}\\(\xi^{(i)}_s)_{21}&(\xi^{(i)}_s)_{22}\end{pmatrix}x_{t_i}^s.$$

Here, the coefficient matrices of $\Xi_i^{\leq 2n_i-1}$ appear in Lemma 3.1 as the coefficient matrices of Ξ_i for the connection $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. We put

$$\widetilde{\Omega}_i:=(\Phi_i\Xi_i^{\leq 2n_i-1})^{-1}\,d(\Phi_i\Xi_i^{\leq 2n_i-1})+(\Phi_i\Xi_i^{\leq 2n_i-1})^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\Phi_i\Xi_i^{\leq 2n_i-1}),$$
$$\widetilde{\psi}_i:=(\Phi_i\Xi_i^{\leq 2n_i-1})^{-1}\psi_i,$$

where ψ_i is the formal solution as in Lemma 3.1. We may describe $\widetilde{\Omega}_i$ as

$$\begin{aligned}\widetilde{\Omega}_i = &\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix} \frac{dx}{(x-t_i)^{n_i}} + \cdots + \begin{pmatrix} \theta^+_{n_i-1,t_i} & 0 \\ 0 & \theta^-_{n_i-1,t_i} \end{pmatrix} \frac{dx}{x-t_i} \\ &+ \begin{pmatrix} \theta^+_{n_i,t_i} & 0 \\ 0 & \theta^-_{n_i,t_i} \end{pmatrix} dx + \cdots + \begin{pmatrix} \theta^+_{2n_i-1,t_i} & 0 \\ 0 & \theta^-_{2n_i-1,t_i} \end{pmatrix} (x-t_i)^{n_i-1}\, dx \\ &+ O(x-t_i)^{n_i}.\end{aligned}$$

The residue part $\theta^\pm_{n_i-1,t_i}$ of $\widetilde{\Omega}_i$ is constant on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}$. So we have $\hat{\delta}(\theta^\pm_{n_i-1,t_i}) = 0$ for any $\hat{\delta} \in \Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\rm ra}} \times T_{\boldsymbol{\theta}}}$. We may check that the variation $\hat{\delta}_1(\widetilde{\Omega}_i)$ is equal to

$$\begin{aligned}&\begin{pmatrix} \frac{\hat{\delta}_1(\theta^+_{0,t_i})}{(x-t_i)^{n_i}} & 0 \\ 0 & \frac{\hat{\delta}_1(\theta^-_{0,t_i})}{(x-t_i)^{n_i}} \end{pmatrix} dx + \cdots + \begin{pmatrix} \frac{\hat{\delta}_1(\theta^+_{n_i-2,t_i})}{(x-t_i)^2} & 0 \\ 0 & \frac{\hat{\delta}_1(\theta^-_{n_i-2,t_i})}{(x-t_i)^2} \end{pmatrix} dx \\ &\quad + \begin{pmatrix} \hat{\delta}_1(\theta^+_{n_i,t_i}) & 0 \\ 0 & \hat{\delta}_1(\theta^-_{n_i,t_i}) \end{pmatrix} dx + \cdots \\ &\quad + \begin{pmatrix} \hat{\delta}_1(\theta^+_{2n_i-1,t_i}) & 0 \\ 0 & \hat{\delta}_1(\theta^-_{2n_i-1,t_i}) \end{pmatrix} (x-t_i)^{n_i-1}\, dx \\ &\quad + n_i\hat{\delta}_1(t_i) \begin{pmatrix} \frac{\theta^+_{0,t_i}}{(x-t_i)^{n_i+1}} & 0 \\ 0 & \frac{\theta^-_{0,t_i}}{(x-t_i)^{n_i+1}} \end{pmatrix} dx + \cdots + \hat{\delta}_1(t_i) \begin{pmatrix} \frac{\theta^+_{n_i-1,t_i}}{(x-t_i)^2} & 0 \\ 0 & \frac{\theta^-_{n_i-1,t_i}}{(x-t_i)^2} \end{pmatrix} dx \\ &\quad - \hat{\delta}_1(t_i) \begin{pmatrix} \theta^+_{n_i+1,t_i} & 0 \\ 0 & \theta^-_{n_i+1,t_i} \end{pmatrix} dx - \cdots \\ &\quad - (n_i-1)\hat{\delta}_1(t_i) \begin{pmatrix} \theta^+_{2n_i-1,t_i} & 0 \\ 0 & \theta^-_{2n_i-1,t_i} \end{pmatrix} (x-t_i)^{n_i-2}\, dx \\ &\quad - \hat{\delta}_1(t_i) \begin{pmatrix} * & * \\ * & * \end{pmatrix} (x-t_i)^{n_i-1}\, dx + O(x-t_i)^{n_i}.\end{aligned}$$

We define $\hat{\lambda}^{\leq 2n_i-1}_{i,\pm}(x_{t_i})$ as

$$\begin{aligned}\hat{\lambda}^{\leq 2n_i-1}_{i,\pm}(x_{t_i}) = &\frac{\theta^\pm_{0,t_i}}{-n_i+1}(x-t_i)^{-n_i+1} + \cdots + \frac{\theta^\pm_{n_i-2,t_i}}{-1}(x-t_i)^{-1} \\ &+ \theta^\pm_{n_i-1,t_i} \log(x-t_i) + \theta^\pm_{n_i,t_i}(x-t_i) + \cdots \\ &+ \frac{\theta^\pm_{2n_i-1,t_i}}{n_i}(x-t_i)^{n_i}.\end{aligned}$$

On the other hand, the variation $\hat{\delta}_2(\widetilde{\psi}_i)\widetilde{\psi}_i^{-1}$ is equal to

$$\begin{aligned}
&\delta(g_i^{\leq 2n_i-1})(g_i^{\leq 2n_i-1})^{-1} + g_i^{\leq 2n_i-1}\begin{pmatrix} -\hat{\delta}_2(\hat{\lambda}_i^+(x_{t_i})) & 0 \\ 0 & -\hat{\delta}_2(\hat{\lambda}_i^-(x_{t_i})) \end{pmatrix}(g_i^{\leq 2n_i-1})^{-1} \\
&\qquad = \begin{pmatrix} -\hat{\delta}_2(\hat{\lambda}_{i,+}^{\leq 2n_i-1}(x_{t_i})) & 0 \\ 0 & -\hat{\delta}_2(\hat{\lambda}_{i,-}^{\leq 2n_i-1}(x_{t_i})) \end{pmatrix} \\
&\qquad\quad + \hat{\delta}_2(t_i)\begin{pmatrix} * & * \\ * & * \end{pmatrix} x_{t_i}^{n_i} + O(x_{t_i}^{n_i+1}).
\end{aligned}$$

Here we set

$$g_i^{\leq 2n_i-1} := (\Phi_i \Xi_i^{\leq 2n_i-1})^{-1}\Phi_i\Xi_i.$$

Since $\hat{\delta}_2(\hat{\lambda}_{i,\pm}^{\leq 2n_i-1}(x_{t_i}))$ is equal to

$$\begin{aligned}
&\frac{\hat{\delta}_2(\theta^{\pm}_{0,t_i})}{-n_i+1}(x-t_i)^{-n_i+1} + \cdots + \frac{\hat{\delta}_2(\theta^{\pm}_{n_i-2,t_i})}{-1}(x-t_i)^{-1} \\
&\quad + \hat{\delta}_2(\theta^{\pm}_{n_i,t_i})(x-t_i) + \cdots + \frac{\hat{\delta}_2(\theta^{\pm}_{2n_i-1,t_i})}{n_i}(x-t_i)^{n_i} \\
&\quad + \theta^{\pm}_{0,t_i}(-\hat{\delta}_2(t_i))(x-t_i)^{-n_i} + \cdots + \theta^{\pm}_{n_i-2,t_i}(-\hat{\delta}_2(t_i))(x-t_i)^{-2} \\
&\quad + \theta^{\pm}_{n_i-1,t_i}(-\hat{\delta}_2(t_i))(x-t_i)^{-1} \\
&\quad + \theta^{\pm}_{n_i,t_i}(-\hat{\delta}_2(t_i)) + \cdots + \theta^{\pm}_{2n_i-1,t_i}(-\hat{\delta}_2(t_i))(x-t_i)^{n_i-1},
\end{aligned}$$

we may check that the residue $\mathrm{Tr}(\hat{\delta}_1(\widetilde{\Omega}_i)\hat{\delta}_2(\widetilde{\psi}_i)\widetilde{\psi}_i^{-1})$ at t_i coincides with

$$\begin{aligned}
&\sum_{\substack{l\in\{0,1,\ldots,2n_i-2\} \\ \setminus\{n_i-1\}}} \left(\hat{\delta}_1(\theta^+_{l,t_i})\frac{\hat{\delta}_2(\theta^+_{2n_i-l-2,t_i})}{n_i-l-1}\right) \\
&\quad + \sum_{\substack{l\in\{0,1,\ldots,2n_i-2\} \\ \setminus\{n_i-1\}}} \left(\hat{\delta}_1(\theta^-_{l,t_i})\frac{\hat{\delta}_2(\theta^-_{2n_i-l-2,t_i})}{n_i-l-1}\right) \\
&\quad + \sum_{\substack{l\in\{0,1,\ldots,2n_i-1\} \\ \setminus\{n_i\}}} \left((n_i-l)\theta^+_{l,t_i}\hat{\delta}_1(t_i)\frac{\hat{\delta}_2(\theta^+_{2n_i-l-1,t_i})}{n_i-l} - \hat{\delta}_1(\theta^+_{l,t_i})\theta^+_{2n_i-l-1,t_i}\hat{\delta}_2(t_i)\right) \\
&\quad + \sum_{\substack{l\in\{0,1,\ldots,2n_i-1\} \\ \setminus\{n_i\}}} \left((n_i-l)\theta^-_{l,t_i}\hat{\delta}_1(t_i)\frac{\hat{\delta}_2(\theta^-_{2n_i-l-1,t_i})}{n_i-l} - \hat{\delta}_1(\theta^-_{l,t_i})\theta^-_{2n_i-l-1,t_i}\hat{\delta}_2(t_i)\right) \\
&\quad + R_i\hat{\delta}_1(t_i)\hat{\delta}_2(t_i),
\end{aligned}$$

where R_i is a rational function on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. We may check that

$$\sum_{\substack{l\in\{0,1,\dots,2n_i-2\}\\ \setminus\{n_i-1\}}} \left(\hat{\delta}_1(\theta^{\pm}_{l,t_i})\frac{\hat{\delta}_2(\theta^{\pm}_{2n_i-l-2,t_i})}{n_i-l-1}\right) = \sum_{l=0}^{n_i-2}\hat{\delta}_1(\theta^{\pm}_{l,t_i})\frac{\hat{\delta}_2(\theta^{\pm}_{2n_i-l-2,t_i})}{n_i-l-1} - \sum_{l=0}^{n_i-2}\frac{\hat{\delta}_1(\theta^{\pm}_{2n_i-l-2,t_i})}{n_i-l-1}\hat{\delta}_2(\theta^{\pm}_{l,t_i})$$

and

$$\sum_{\substack{l\in\{0,1,\dots,2n_i-1\}\\ \setminus\{n_i\}}} \left((n_i-l)\theta^{\pm}_{l,t_i}\hat{\delta}_1(t_i)\frac{\hat{\delta}_2(\theta^{\pm}_{2n_i-l-1,t_i})}{n_i-l} - \hat{\delta}_1(\theta^{\pm}_{l,t_i})\theta^{\pm}_{2n_i-l-1,t_i}\hat{\delta}_2(t_i)\right) = \hat{\delta}_1(t_i)\hat{\delta}_2\left(\sum_{l=0}^{n_i-1}\theta^{\pm}_{l,t_i}\theta^{\pm}_{2n_i-l-1,t_i}\right) - \hat{\delta}_1\left(\sum_{l=0}^{n_i-1}\theta^{\pm}_{l,t_i}\theta^{\pm}_{2n_i-l-1,t_i}\right)\hat{\delta}_2(t_i),$$

since $\hat{\delta}(\theta^{\pm}_{n_i-1,t_i}) = 0$. Remark that $\hat{\delta}_1(t_i) = \hat{\delta}_2(t_i) = 0$ for $i = 0, 1, \infty$. Then we have

$$\begin{aligned}\frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\hat{\delta}(\widetilde{\Omega}_i)\wedge\hat{\delta}(\widetilde{\psi}_i)\widetilde{\psi}_i^{-1}) &= \Bigg(\sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}(dH_{\theta^{+}_{l,t_i}}\wedge d\theta^{+}_{l,t_i} + dH_{\theta^{-}_{l,t_i}}\wedge d\theta^{-}_{l,t_i}) \\ &\quad + \sum_{i\in\{3,4,\dots,\nu\}} dH_{t_i}\wedge dt_i\Bigg)(\hat{\delta}_1,\hat{\delta}_2).\end{aligned} \tag{3.49}$$

This is just the second part of (3.48). We may take Φ_i and $\Xi_i^{\leq 2n_i-1}(x_{t_i})$ so that $\Phi_i\Xi_i^{\leq 2n_i-1}(x_{t_i})$ satisfies the assumption of Lemma 3.13 (see Remark 3.2). Since $\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$ also satisfies the assumption of Lemma 3.13, the difference between the residue (3.49) and the residue of

$$\frac{1}{2}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_i)\psi_i^{-1})$$

at t_i is zero if $\delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$.

Second we calculate the residue of

$$\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1})$$

at $x = q_j$. First, $\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})$ is described at $x = q_j$ as

$$\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) = \begin{pmatrix} 0 & \hat{\delta}(1/P(x;\boldsymbol{t})) \\ \hat{\delta}(c_0) & \hat{\delta}(d_0)\end{pmatrix},$$

where

$$\hat{\delta}(1/P(x;\boldsymbol{t})) = \sum_{i=1}^{\nu} \frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}(t_i) + O(x-q_j),$$

$$\hat{\delta}(c_0) = \frac{p_j\hat{\delta}(q_j)}{(x-q_j)^2} + \frac{\hat{\delta}(p_j)}{(x-q_j)} + O(x-q_j)^0,$$

$$\hat{\delta}(d_0) = -\frac{\hat{\delta}(q_j)}{(x-q_j)^2} - \sum_{k\neq j}\frac{\hat{\delta}(q_k)}{(q_j-q_k)^2} + \sum_{i=3}^{\nu}\Big(\frac{\partial d_0}{\partial t_i}(q_j)\hat{\delta}(t_i)\Big) + \sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}\Big(\frac{\partial d_0}{\partial\theta^{\pm}_{l,t_i}}(q_j)\hat{\delta}(\theta^{\pm}_{l,t_i})\Big) + O(x-q_j).$$

Second, we consider $\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}$. By Lemma 2.11, we have

$$\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1} = \hat{\delta}(\Phi_{q_j}\Xi_{q_j}(x))(\Phi_{q_j}\Xi_{q_j}(x))^{-1} + (\Phi_{q_j}\Xi_{q_j}(x))\begin{pmatrix}0 & 0\\ 0 & \frac{-\hat{\delta}(q_j)}{x-q_j}\end{pmatrix}(\Phi_{q_j}\Xi_{q_j}(x))^{-1}.$$

By using Lemma 2.11, we have the equality

$$\Phi_{q_j}\Xi_{q_j}(x) = \begin{pmatrix}1 & 0\\ p_j & 1\end{pmatrix} + \begin{pmatrix}-\frac{p_j}{P(q_j;\boldsymbol{t})} & -\frac{1}{2P(q_j;\boldsymbol{t})}\\ -\frac{p_j^2}{P(q_j;\boldsymbol{t})} & \frac{p_j}{2P(q_j;\boldsymbol{t})} - \sum_{i=1}^{\nu}\frac{D_i(q_j)}{(q_j-t_i)^{n_i}} + \sum_{k\neq j}\frac{1}{q_j-q_k} - D_\infty(q_j)\end{pmatrix}(x-q_j) + O(x-q_j)^2.$$

By this description of $\Phi_{q_j}\Xi_{q_j}(x)$, we may check that the constant term of the expansion of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ at q_j has the description

$$\begin{pmatrix}0 & 0\\ \delta(p_j) & 0\end{pmatrix} - \delta(q_j)\begin{pmatrix}-\frac{p_j}{P(q_j;\boldsymbol{t})} & -\frac{1}{2P(q_j;\boldsymbol{t})}\\ -\frac{p_j^2}{P(q_j;\boldsymbol{t})} & *\end{pmatrix}.$$

Since

$$\sum_{i=1}^{\nu}\frac{D_i(x)}{(x-t_i)^{n_i}} + D_\infty(x) = d_0 - \sum_{j=1}^{n-3}\frac{-1}{x-q_j},$$

the coefficient of the $(x-q_j)$-term of the expansion of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ has the description

$$\begin{pmatrix}* & 0\\ * & \frac{\delta(p_j)}{2P(q_j;\boldsymbol{t})} + \sum_{k\neq j}\frac{\delta(q_k)}{(q_j-q_k)^2}\end{pmatrix} - \delta(q_j)\begin{pmatrix}* & *\\ * & *\end{pmatrix} + \begin{pmatrix}* & -\frac{1}{2}\sum_{i=1}^{\nu}\frac{\partial}{\partial t_i}\big(\frac{1}{P(q_j;\boldsymbol{t})}\big)\hat{\delta}(t_i)\\ * & x^3_{22}\end{pmatrix}.$$

Here, the $(2,2)$-entry x_{22}^3 of the third matrix is

$$\frac{p_j}{2}\sum_{i=3}^{\nu}\frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}(t_i)-\sum_{i=3}^{\nu}\Big(\frac{\partial d_0}{\partial t_i}(q_j)\hat{\delta}(t_i)\Big)-\sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}\Big(\frac{\partial d_0}{\partial\theta^{\pm}_{l,t_i}}(q_j)\hat{\delta}(\theta^{\pm}_{l,t_i})\Big),$$

and we put the entries having $\delta(q_j)$ together in the second matrices. Moreover, we may check that the constant term of the expansion of $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ at q_j is $\left(\begin{smallmatrix}1&0\\-p_j&1\end{smallmatrix}\right)$ and the coefficient of the $(x-q_j)$-term of the expansion of $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ has the description

$$-\begin{pmatrix}-\frac{p_j}{2P(q_j;\boldsymbol{t})} & -\frac{1}{2P(q_j;\boldsymbol{t})}\\ * & \frac{p_j}{P(q_j;\boldsymbol{t})}-\sum_{i=1}^{\nu}\frac{D_i(q_j)}{(q_j-t_i)^{n_i}}+\sum_{k\neq j}\frac{1}{q_j-q_k}-D_\infty(q_j)\end{pmatrix}.$$

By the calculation of $\delta(\Phi_{q_j}\Xi_{q_j}(x))$ and $(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$, we may show that $\delta(\Phi_{q_j}\Xi_{q_j}(x))(\Phi_{q_j}\Xi_{q_j}(x))^{-1}$ is

$$\begin{pmatrix}* & \frac{\hat{\delta}(q_j)}{2P(q_j;\boldsymbol{t})}\\ * & *\end{pmatrix}+\begin{pmatrix}* & x_{12}\\ * & \frac{\hat{\delta}(p_j)}{P(q_j;\boldsymbol{t})}+\sum_k\frac{\hat{\delta}(q_k)}{(q_j-q_k)^2}+x_{22}\end{pmatrix}(x-q_j)+O(x-q_j)^2.$$

Here we put

$$\begin{aligned}x_{12}&:=f_{12}\hat{\delta}(q_j)-\frac{1}{2}\sum_{i=3}^{\nu}\frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}(t_i),\\ x_{22}&:=f_{22}\hat{\delta}(q_j)-\sum_{i=3}^{\nu}\Big(\frac{\partial d_0}{\partial t_i}(q_j)\hat{\delta}(t_i)\Big)\\ &\qquad-\sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}\Big(\frac{\partial d_0}{\partial\theta^{\pm}_{l,t_i}}(q_j)\hat{\delta}(\theta^{\pm}_{l,t_i})\Big)+\frac{p_j}{2}\sum_{i=3}^{\nu}\frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}(t_i),\end{aligned}$$

where f_{12} and f_{22} are rational functions on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. Moreover, we may show that

$$\begin{aligned}&(\Phi_{q_j}\Xi_{q_j}(x))\begin{pmatrix}0&0\\0&\frac{-\hat{\delta}(q_j)}{x-q_j}\end{pmatrix}(\Phi_{q_j}\Xi_{q_j}(x))^{-1}\\ &\quad=\frac{\begin{pmatrix}0&0\\p_j\hat{\delta}(q_j)&-\hat{\delta}(q_j)\end{pmatrix}}{x-q_j}+\begin{pmatrix}* & \frac{\hat{\delta}(q_j)}{2P(q_j;\boldsymbol{t})}\\ * & g_{22}^{(0)}\hat{\delta}(q_j)\end{pmatrix}+\begin{pmatrix}* & g_{12}^{(1)}\hat{\delta}(q_j)\\ * & g_{22}^{(1)}\hat{\delta}(q_j)\end{pmatrix}(x-q_j)\\ &\qquad+O(x-q_j)^2,\end{aligned}$$

where $g_{22}^{(0)}$, $g_{12}^{(1)}$, and $g_{22}^{(1)}$ are rational functions on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. Finally, we have

$$\begin{aligned}
&\frac{1}{2}\mathrm{res}_{x=q_j} \mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}) \\
&\quad = \frac{\hat{\delta}_1(p_j)\hat{\delta}_2(q_j)}{P(q_j;\boldsymbol{t})} - \frac{\hat{\delta}_2(p_j)\hat{\delta}_1(q_j)}{P(q_j;\boldsymbol{t})} + \sum_{k\neq j} \frac{\hat{\delta}_1(q_k)\hat{\delta}_2(q_j)}{(q_j-q_k)^2} - \sum_{k\neq j} \frac{\hat{\delta}_1(q_j)\hat{\delta}_2(q_k)}{(q_j-q_k)^2} \\
&\qquad + p_j \sum_{i=3}^{\nu} \frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)(\hat{\delta}_1(t_i)\hat{\delta}_2(q_j) - \hat{\delta}_2(t_i)\hat{\delta}_1(q_j)) \\
&\qquad - \sum_{i=3}^{\nu} \frac{\partial d_0}{\partial t_i}(q_j)(\hat{\delta}_1(t_i)\hat{\delta}_2(q_j) - \hat{\delta}_2(t_i)\hat{\delta}_1(q_j)) \\
&\qquad - \sum_{i\in I_{\mathrm{un}}} \sum_{l=0}^{n_i-2} \frac{\partial d_0}{\partial \theta^{\pm}_{l,t_i}}(q_j)(\hat{\delta}_1(\theta^{\pm}_{l,t_i})\hat{\delta}_2(q_j) - \hat{\delta}_2(\theta^{\pm}_{l,t_i})\hat{\delta}_1(q_j)).
\end{aligned}$$

Moreover, this is equal to

$$\begin{aligned}
&\Bigg(\frac{\hat{\delta}_1(p_j)}{P(q_j;\boldsymbol{t})} + p_j \sum_{i=3}^{\nu} \frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}_1(t_i) \\
&\qquad - \sum_{i=3}^{\nu} \frac{\partial d_0}{\partial t_i}(q_j)\hat{\delta}_1(t_i) - \sum_{i\in I_{\mathrm{un}}} \sum_{l=0}^{n_i-2} \frac{\partial d_0}{\partial \theta^{\pm}_{l,t_i}}(q_j)\hat{\delta}_1(\theta^{\pm}_{l,t_i})\Bigg)\hat{\delta}_2(q_j) \\
&\quad - \Bigg(\frac{\hat{\delta}_2(p_j)}{P(q_j;\boldsymbol{t})} + p_j \sum_{i=3}^{\nu} \frac{\partial}{\partial t_i}\Big(\frac{1}{P(q_j;\boldsymbol{t})}\Big)\hat{\delta}_2(t_i) \\
&\qquad - \sum_{i=3}^{\nu} \frac{\partial d_0}{\partial t_i}(q_j)\hat{\delta}_2(t_i) - \sum_{i\in I_{\mathrm{un}}} \sum_{l=0}^{n_i-2} \frac{\partial d_0}{\partial \theta^{\pm}_{l,t_i}}(q_j)\hat{\delta}_2(\theta^{\pm}_{l,t_i})\Bigg)\hat{\delta}_1(q_j) \\
&\quad - \sum_{k\neq j} \frac{\hat{\delta}_1(q_k)\hat{\delta}_2(q_j) - \hat{\delta}_1(q_j)\hat{\delta}_2(q_k)}{(q_j-q_k)^2}.
\end{aligned}$$

In the first and second terms of this formula, the exterior derivative of

$$\frac{p_j}{P(q_j;\boldsymbol{t})} - \sum_{i=1}^{\nu} \frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}} - D_{\infty}(q_j;\boldsymbol{t},\boldsymbol{\theta})$$

on the extended moduli space $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ appears. Here, remark that

$$\sum_{i=1}^{\nu} \frac{D_i(x;\boldsymbol{t},\boldsymbol{\theta})}{(x-t_i)^{n_i}} + D_{\infty}(x;\boldsymbol{t},\boldsymbol{\theta}) = d_0 - \sum_{j=1}^{n-3} \frac{-1}{x-q_j}$$

and the coefficients of $D_i(x;\boldsymbol{t},\boldsymbol{\theta})$ and $D_\infty(x;\boldsymbol{t},\boldsymbol{\theta})$ are independent of the parameters $\{(q_j,p_j)\}_{j=1,2,\ldots,n-3}$. Since

$$\sum_{j=1}^{n-3}\sum_{k\neq j}\frac{\hat\delta_1(q_k)\hat\delta_2(q_j)-\hat\delta_1(q_j)\hat\delta_2(q_k)}{(q_j-q_k)^2}=0,$$

we have

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\,\mathrm{Tr}(\hat\delta(\widehat\Omega^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat\delta(\psi_{q_j})\psi_{q_j}^{-1})\\&\quad=\Bigg(\sum_{j=1}^{n-3}d\bigg(\frac{p_j}{P(q_j;\boldsymbol{t})}-\sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}}-D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})\bigg)\wedge dq_j\Bigg)(\hat\delta_1,\hat\delta_2).\end{aligned}$$

We obtain the assertion of this theorem. □

Corollary 3.15. *Set* $\eta_j := \frac{p_j}{P(q_j;\boldsymbol{t})}-\sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}}-D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})$. *The vector fields* $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ *(*$i\in I_{\mathrm{un}}$ *and* $l=0,1,\ldots,n_i-2$*) and* $\delta^{\mathrm{IMD}}_{t_i}$ *(*$i=3,4,\ldots,\nu$*) have the following Hamiltonian description:*

$$\begin{aligned}\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}&=\frac{\partial}{\partial\theta^{\pm}_{l,t_i}}-\sum_{j=1}^{n-3}\left(\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\right),\\\delta^{\mathrm{IMD}}_{t_i}&=\frac{\partial}{\partial t_i}-\sum_{j=1}^{n-3}\left(\frac{\partial H_{t_i}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{t_i}}{\partial q_j}\frac{\partial}{\partial\eta_j}\right),\end{aligned}\tag{3.50}$$

respectively.

Proof. We can put

$$\begin{aligned}\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}&=\frac{\partial}{\partial\theta^{\pm}_{l,t_i}}+\sum_{j=1}^{n-3}\left(X^j_{\theta^{\pm}_{l,t_i}}\frac{\partial}{\partial q_j}+Y^j_{\theta^{\pm}_{l,t_i}}\frac{\partial}{\partial\eta_j}\right),\\\delta^{\mathrm{IMD}}_{t_i}&=\frac{\partial}{\partial t_i}+\sum_{j=1}^{n-3}\left(X^j_{t_i}\frac{\partial}{\partial q_j}+Y^j_{t_i}\frac{\partial}{\partial\eta_j}\right).\end{aligned}$$

By Theorem 3.14, the terms of dq_j, $d\eta_j$ ($j=1,2,\ldots,n-3$) of the 1-forms $\widehat\omega(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}},*)$ and $\widehat\omega(\delta^{\mathrm{IMD}}_{t_i},*)$ are

$$\begin{aligned}&\sum_{j=1}^{n-3}(-X^j_{\theta^{\pm}_{l,t_i}}\,d\eta_j+Y^j_{\theta^{\pm}_{l,t_i}}\,dq_j)-\sum_{j=1}^{n-3}\left(\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial\eta_j}\,d\eta_j+\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial q_j}\,dq_j\right),\\&\sum_{j=1}^{n-3}(-X^j_{t_i}\,d\eta_j+Y^j_{t_i}\,dq_j)-\sum_{j=1}^{n-3}\left(\frac{\partial H_{t_i}}{\partial\eta_j}\,d\eta_j+\frac{\partial H_{t_i}}{\partial q_j}\,dq_j\right),\end{aligned}$$

respectively. By Theorem 3.10, we have the equalities

$$X^j_{\theta^{\pm}_{l,t_i}} = -\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial \eta_j}, \qquad Y^j_{\theta^{\pm}_{l,t_i}} = \frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial q_j},$$

$$X^j_{t_i} = -\frac{\partial H_{t_i}}{\partial \eta_j}, \qquad Y^j_{\theta^{\pm}_{l,t_i}} = \frac{\partial H_{t_i}}{\partial q_j}.$$

Then we have the Hamiltonian description (3.50). □

Remark. In [15], the Hamiltonian systems of the two-dimensional (degenerated) Garnier systems have been described by using the coordinates $(q_j, \tilde{\eta}_j)_{0\leq j\leq 2}$, where $\tilde{\eta}_j := -\frac{p_j}{P(q_j;\boldsymbol{t})}$. In these cases, the 2-form

$$d(-\sum_{i=1}^{\nu} \frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}} - D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})) \wedge dq_j,$$

which comes from the residue at an apparent singularity, is canceled by some terms of the 2-form

$$\sum_{i\in I}\sum_{l=0}^{n_i-2}(dH_{\theta^+_{l,t_i}} \wedge d\theta^+_{l,t_i} + dH_{\theta^-_{l,t_i}} \wedge d\theta^-_{l,t_i}),$$

which comes from the residues at unramified irregular singular points.

§4. Ramified irregular singularities

In this section we assume that $I_{\mathrm{ra}} \neq \emptyset$. For $i \in I_{\mathrm{ra}}$, the leading coefficient $\Omega_{t_i}(0)$ is a nontrivial Jordan block. In Section 4.1 we define a 2-form on the fiber $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ by Krichever's formula [16, Sect. 5]. Remark that $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ is isomorphic to the moduli space $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. We show that this 2-form coincides with the symplectic form (1.2). In Section 4.2 we will construct horizontal lifts of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$. Let $\partial/\partial\theta_{l',t_i}$ ($i \in I_{\mathrm{ra}}$, $l' = 0, 1, \dots, 2n_i - 3$) be the vector fields on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. By the construction of the horizontal lifts, we have the vector field $\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ determined by the integrable deformations with respect to $\partial/\partial\theta_{l',t_i}$. Remark that $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ is isomorphic to the extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. In Section 4.3 we define a 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by Krichever's formula. We show that this 2-form is the isomonodromy 2-form. In Section 4.4 we calculate this 2-form on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by using Diarra–Loray's global normal form. Then we obtain an explicit formula for this 2-form.

For $i \in I \setminus I_{\rm ra}$, we fix a compatible framing Φ_i and take $\Xi_i(x_{t_i})$ as in Lemma 3.1. For each $i \in I_{\rm ra}$, we consider the leading coefficient of $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$ at t_i:

$$\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} = \begin{pmatrix} 0 & \prod_{j\neq i}(t_i-t_j)^{-n_j} \\ -\frac{\theta_{0,t_i}^2}{4}\prod_{j\neq i}(t_i-t_j)^{n_j} & \theta_{0,t_i} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + [\text{higher-order terms}].$$

This leading coefficient at t_i is independent of $\{(q_j,p_j)\}_{j=1,2,\dots,n-3}$. We fix $\Phi_i \in \mathrm{GL}(2,\mathbb{C})$ so that

$$\Phi_i^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi_i = \begin{pmatrix} \frac{\theta_{0,t_i}}{2} & \frac{\theta_{1,t_i}}{2} \\ 0 & \frac{\theta_{0,t_i}}{2} \end{pmatrix} \frac{dx_{t_i}}{x_{t_i}^{n_i}} + [\text{higher-order terms}]. \tag{4.1}$$

We call the matrix Φ_i a *compatible framing at* t_i. If we have another Φ_i' such that the leading coefficient matrix of $(\Phi_i')^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}\Phi_i'$ is an upper triangular matrix as in (4.1), then there exists an upper triangular matrix

$$C_{t_i} = \begin{pmatrix} c_{t_i,11} & c_{t_i,12} \\ 0 & c_{t_i,11} \end{pmatrix}$$

such that $\Phi_i' = \Phi_i C_{t_i}$. We define ζ_i as $x_{t_i} = \zeta_i^2$. Let M_{ζ_i} be the matrix (2.9). For the compatible framing Φ_i, there exist a unique

- formal transformation

$$\Xi_i(x_{t_i}) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{s=1}^{\infty} \begin{pmatrix} (\xi_s^{(i)})_{11} & (\xi_s^{(i)})_{12} \\ (\xi_s^{(i)})_{21} & (\xi_s^{(i)})_{22} \end{pmatrix} x_{t_i}^s, \tag{4.2}$$

  and
- $\theta_{l',t_i} \in \Gamma(\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\rm ra}}, \mathcal{O}_{\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\rm ra}}})$ ($l' \geq 2n_i - 2$ and $i \in I_{\rm ra}$)

such that

(1) we have the equality

$$\begin{aligned} &(\Phi_i\Xi_i(x_{t_i}))^{-1}\, d(\Phi_i\Xi_i(x_{t_i})) + (\Phi_i\Xi_i(x_{t_i}))^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\Phi_i\Xi_i(x_{t_i})) \\ &\quad = \begin{pmatrix} \alpha_i & \beta_i \\ x_{t_i}\beta_i & \alpha_i - \frac{dx_{t_i}}{2x_{t_i}} \end{pmatrix}, \end{aligned} \tag{4.3}$$

where we set

$$\begin{cases} \alpha_i := \dfrac{\theta_{0,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \dfrac{\theta_{2l,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \dfrac{\theta_{2n_i-2,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}} + \cdots, \\ \beta_i := \dfrac{\theta_{1,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \dfrac{\theta_{2l+1,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \dfrac{\theta_{2n_i-3,t_i}}{2}\dfrac{dx_{t_i}}{x_{t_i}^2} + \cdots, \end{cases}$$

(2) there exists a formal power series $\xi(\zeta_i) \in \Gamma(\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}, \mathcal{O}_{\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}})[[\zeta_i]]$ such that

$$M_{\zeta_i}^{-1}\Xi_i(\zeta_i^2)M_{\zeta_i} = \begin{pmatrix} 1 & \zeta_i\xi(\zeta_i) \\ -\zeta_i\xi(-\zeta_i) & 1 \end{pmatrix}. \tag{4.4}$$

Indeed, the $\zeta_i^{-2n_i+1}$- and $\zeta_i^{-2n_i+2}$-terms of the expansion of

$$(\Phi_i M_{\zeta_i})^{-1}\, d(\Phi_i M_{\zeta_i}) + (\Phi_i M_{\zeta_i})^{-1}\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2)(\Phi_i M_{\zeta_i}) \tag{4.5}$$

at $\zeta_i = 0$ are diagonal. The eigenvalues of the $\zeta_i^{-2n_i+2}$-term are distinct. After the $\zeta_i^{-2n_i+2}$-term, we can diagonalize (4.5) by a matrix as in Lemma 3.1. By the argument as in the proof of [5, Prop. 10], we may check that, in this situation, this matrix has a form as in the right-hand side of (4.4). Moreover, by the gauge transformation of this diagonal matrix by $M_{\zeta_i}^{-1}$, we have the right-hand side of (4.3). We may check that

$$M_{\zeta_i}\begin{pmatrix} 1 & \zeta_i\xi(\zeta_i) \\ -\zeta_i\xi(-\zeta_i) & 1 \end{pmatrix} M_{\zeta_i}^{-1}$$

is invariant under replacing ζ_i with $-\zeta_i$. So we have $\Xi_i(x_{t_i})$. By Lemma 3.1, such a $\Xi_i(x_{t_i})$ is unique.

Let U_{t_i} be an affine open subset on $\mathbb{P}^1$ for $i \in I_{\mathrm{ra}}$ so that x_{t_i} is a coordinate on U_{t_i}. Let U_{ζ_i} be the inverse image of U_{t_i} under the map $\operatorname{Spec}\mathbb{C}[\zeta_i] \to \operatorname{Spec}\mathbb{C}[x_{t_i}]$ by $x_{t_i} = \zeta_i^2$. Let

$$f_{\zeta_i} \colon U_{\zeta_i} \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \longrightarrow U_{t_i} \times \mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$$

be the map induced by $\operatorname{Spec}\mathbb{C}[\zeta_i] \to \operatorname{Spec}\mathbb{C}[x_{t_i}]$. We consider the pull-back

$$(f_{\zeta_i}^* E_{n-2}|_{U_{t_i}\times\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}}, f_{\zeta_i}^*\widetilde{\nabla}^{(n-1)}_{\mathrm{DL}}|_{U_{t_i}\times\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}}).$$

Let $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2)$ be the pull-back of the connection matrix $\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}|_{U_{t_i}\times\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}}$ under the map $f_{\zeta_i}^*$. Now we take a formal fundamental matrix solution of $d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2) = 0$ as follows. We have the diagonalization

$$\begin{aligned} &M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1}\begin{pmatrix} \alpha_i & \beta_i \\ x_{t_i}\beta_i & \alpha_i - \frac{dx_{t_i}}{2x_{t_i}} \end{pmatrix} M_{\zeta_i} \\ &\quad = \sum_{l=0,1,\ldots}\begin{pmatrix} \frac{\theta_{2l,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-1}} & 0 \\ 0 & \frac{\theta_{2l,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-1}} \end{pmatrix} + \sum_{l=0,1,\ldots}\begin{pmatrix} \frac{\theta_{2l+1,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-2}} & 0 \\ 0 & -\frac{\theta_{2l+1,t_i}\,d\zeta_i}{\zeta_i^{2(n_i-l)-2}} \end{pmatrix}. \end{aligned} \tag{4.6}$$

We set

$$
\begin{aligned}
\hat{\lambda}_{i,\pm}(\zeta_i) &:= \sum_{l'=0}^{\infty} (\pm 1)^{l'} \theta_{l',t_i} \int \zeta_i^{-2n_i+l'+1}\, d\zeta_i, \\
\Lambda_i(\zeta_i) &:= \begin{pmatrix} \hat{\lambda}_{i,+}(\zeta_i) & 0 \\ 0 & \hat{\lambda}_{i,-}(\zeta_i) \end{pmatrix}, \\
\psi_{\zeta_i} &:= \Phi_i \Xi_i(\zeta_i^2) M_{\zeta_i} \exp(-\Lambda_i(\zeta_i)).
\end{aligned} \tag{4.7}
$$

Then ψ_{ζ_i} is a formal matrix solution of $d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2) = 0$, that is, $(d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\psi_{\zeta_i} = 0$.

For $i \in I_{\mathrm{ra}}$, we take a tuple $(\Phi_i, \Xi_i(x_{t_i}))$ of a compatible framing and a formal transformation as above. We may give another formal fundamental matrix solution ψ'_{ζ_i} as follows. If we set

$$
\widetilde{C}_{t_i}(x_{t_i}) := \begin{pmatrix} c_{t_i,\mathrm{odd}}(x_{t_i}) & c_{t_i,\mathrm{even}}(x_{t_i}) \\ x_{t_i} c_{t_i,\mathrm{even}}(x_{t_i}) & c_{t_i,\mathrm{odd}}(x_{t_i}) \end{pmatrix} = \begin{pmatrix} c_{t_i,1} & c_{t_i,2} \\ 0 & c_{t_i,1} \end{pmatrix} + \begin{pmatrix} c_{t_i,3} & c_{t_i,4} \\ c_{t_i,2} & c_{t_i,3} \end{pmatrix} x_{t_i} + \cdots,
$$

then

$$
\begin{aligned}
&(\Phi_i \Xi_i(\zeta_i^2) \widetilde{C}_{t_i}(x_{t_i}) M_{\zeta_i})^{-1} d\big((\Phi_i \Xi_i(\zeta_i^2) \widetilde{C}_{t_i}(x_{t_i}) M_{\zeta_i})\big) \\
&\quad + (\Phi_i \Xi_i(\zeta_i^2) \widetilde{C}_{t_i}(x_{t_i}) M_{\zeta_i})^{-1} \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)} (\Phi_i \Xi_i(\zeta_i^2) \widetilde{C}_{t_i}(x_{t_i}) M_{\zeta_i})
\end{aligned}
$$

is also diagonal, since $M_{\zeta_i}^{-1} \widetilde{C}_{t_i}(\zeta_i^2) M_{\zeta_i}$ is diagonal. Let $\tilde{c}_{t_i,11}(\zeta_i)$ and $\tilde{c}_{t_i,22}(\zeta_i)$ be the formal power series such that

$$
M_{\zeta_i}^{-1} \widetilde{C}_{t_i}(\zeta_i^2) M_{\zeta_i} = \begin{pmatrix} \tilde{c}_{t_i,11}(\zeta_i) & 0 \\ 0 & \tilde{c}_{t_i,22}(\zeta_i) \end{pmatrix}.
$$

We define $\Lambda'_{i'}(\zeta_{i'})$ by

$$
\Lambda'_{i'}(\zeta_{i'}) = \Lambda_{i'}(\zeta_{i'}) + \begin{pmatrix} \int \tilde{c}_{t_i,11}(\zeta_i)^{-1} d(\tilde{c}_{t_i,11}(\zeta_i)) & 0 \\ 0 & \int \tilde{c}_{t_i,22}(\zeta_i)^{-1} d(\tilde{c}_{t_i,22}(\zeta_i)) \end{pmatrix}.
$$

If we set

$$
\psi'_{\zeta_i} := \Phi_i \Xi_i(\zeta_i^2) \widetilde{C}_{t_i}(\zeta_i^2) M_{\zeta_i} \exp(-\Lambda'_i(\zeta_i)), \tag{4.8}
$$

we have another formal solution $(d + \Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\psi'_{\zeta_i} = 0$. There exists a diagonal matrix $\widetilde{C}_{t_i}$ such that $\psi'_{\zeta_i} = \psi_{\zeta_i} \widetilde{C}_{t_i}$ and $\widetilde{C}_{t_i}$ is independent of ζ_i.

§4.1. Symplectic structure

Definition 4.1. Let δ_1 and δ_2 be vector fields on

$$
\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}} \subset \mathrm{Sym}^{(n-3)}(\mathrm{Tot}(\Omega^1_{\mathbb{P}^1}(D))),
$$

which is isomorphic to the moduli space $\mathfrak{Conn}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}$. We fix a (formal) fundamental matrix solution ψ_i of $(d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_i=0$ at $x=t_i$ for $i\in I\setminus I_{\mathrm{ra}}$ as in Lemma 3.1. We fix a fundamental matrix solution ψ_{q_j} of $(d+\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\psi_{q_j}=0$ at $x=q_j$ as in Lemma 2.11. Moreover, we take ψ_{ζ_i} defined in (4.7). We define a 2-form ω on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ as

$$\begin{aligned}\omega(\delta_1,\delta_2)&:=\frac{1}{2}\sum_{i\in I\setminus I_{\mathrm{ra}}}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})\\&\quad+\frac{1}{4}\sum_{i\in I_{\mathrm{ra}}}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big)\\&\quad+\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\,\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_{q_j})\psi_{q_j}^{-1}).\end{aligned}\tag{4.9}$$

As in Section 3.1, we may check that the residue of $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ at $\zeta_i=0$ is well defined. Moreover, we may also check that the right-hand side of (4.9) is independent of the choice of ψ_{q_j} and a formal solution ψ_i ($i\in I\setminus I_{\mathrm{ra}}$). Let ψ'_{ζ_i} be another fundamental matrix solution (4.8). There exists a diagonal matrix $\widetilde{C}_{t_i}$ such that $\psi'_{\zeta_i}=\psi_{\zeta_i}\widetilde{C}_{t_i}$ and $\widetilde{C}_{t_i}$ is independent of ζ_i. By the same argument, to check the independency of the choice of ψ_i, we may check that the residue of $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1})$ at $\zeta_i=0$ is equal to $\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi'_{\zeta_i})(\psi'_{\zeta_i})^{-1})$ at $\zeta_i=0$.

Lemma 4.2. *For any vector field δ on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$, the formal series $\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i}=\zeta_i^2$.*

Proof. We consider $\psi_{\zeta_i}M_{\zeta_i}^{-1}$. We decompose $\Lambda_i(\zeta_i)$ into the odd-degree part, the even-degree part, and the logarithmic term:

$$\begin{aligned}\Lambda_i(\zeta_i)&=\begin{pmatrix}\hat{\lambda}_{\mathrm{odd}}(\zeta_i)&0\\0&-\hat{\lambda}_{\mathrm{odd}}(\zeta_i)\end{pmatrix}+\begin{pmatrix}\hat{\lambda}_{\mathrm{even}}(\zeta_i)&0\\0&\hat{\lambda}_{\mathrm{even}}(\zeta_i)\end{pmatrix}\\&\quad+\begin{pmatrix}\theta_{2n_i-2,t_i}\log(\zeta_i)&0\\0&\theta_{2n_i-2,t_i}\log(\zeta_i)\end{pmatrix}.\end{aligned}$$

Since

$$\begin{aligned}M_{\zeta_i}\Lambda_i(\zeta_i)M_{\zeta_i}^{-1}&=\begin{pmatrix}0&\hat{\lambda}_{\mathrm{odd}}(\zeta_i)/\zeta_i\\\zeta_i\hat{\lambda}_{\mathrm{odd}}(\zeta_i)&0\end{pmatrix}+\begin{pmatrix}\hat{\lambda}_{\mathrm{even}}(\zeta_i)&0\\0&\hat{\lambda}_{\mathrm{even}}(\zeta_i)\end{pmatrix}\\&\quad+\frac{1}{2}\begin{pmatrix}\theta_{2n_i-2,t_i}\log(\zeta_i^2)&0\\0&\theta_{2n_i-2,t_i}\log(\zeta_i^2)\end{pmatrix},\end{aligned}$$

we may check that $M_{\zeta_i}\Lambda_i(\zeta_i)M_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$. Since

$$\begin{aligned}\psi_{\zeta_i}M_{\zeta_i}^{-1} &= \Phi_i\Xi_i(\zeta_i^2)M_{\zeta_i}\exp(-\Lambda_{i'}(\zeta_{i'}))M_{\zeta_i}^{-1}\\ &= \Phi_i\Xi_i(\zeta_i^2)\exp(-M_{\zeta_i}\Lambda_{i'}(\zeta_{i'})M_{\zeta_i}^{-1}),\end{aligned}$$

we have that $\psi_{\zeta_i}M_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$. Since M_{ζ_i} is independent of the parameters of $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$, we have $\delta(M_{\zeta_i}) = 0$. Then we have $\delta(\psi_{\zeta_i}M_{\zeta_i}^{-1})(\psi_{\zeta_i}M_{\zeta_i}^{-1})^{-1} = \delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$. Finally, we have that $\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$. □

Remark. In the proof of this lemma, we check that $\psi_{\zeta_i}M_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$. If we set $\psi_i := \psi_{\zeta_i}M_{\zeta_i}^{-1}$ for $i \in I_{\mathrm{ra}}$, we have

$$\begin{aligned}&\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big)\\ &\qquad= \frac{1}{2}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1}).\end{aligned}$$

So we may define the 2-form ω as

$$\begin{aligned}\omega(\delta_1,\delta_2) :=& \frac{1}{2}\sum_{i\in I}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_i)\psi_i^{-1})\\ &+\frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\,\mathrm{Tr}(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)})\wedge\delta(\psi_{q_j})\psi_{q_j}^{-1}).\end{aligned}$$

In this definition of ω, the variable ζ_i disappears.

Theorem 4.3. *Let ω be the 2-form on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$ defined by (4.9) in Definition 4.1. The 2-form ω coincides with*

$$\sum_{j=1}^{n-3} d\Big(\frac{p_j}{P(q_j)}\Big)\wedge dq_j.$$

Proof. Let δ be a vector field on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$. Since t_i $(i \in I)$, $\theta^{\pm}_{l,t_i}$ $(i \in I_{\mathrm{un}}$, $0 \le l \le n_i - 1)$, and θ_{l',t_i} $(i \in I_{\mathrm{ra}}$, $0 \le l' \le 2n_i - 2)$ are constants on $\mathcal{M}_{\boldsymbol{t}_0,\boldsymbol{t}_{\mathrm{ra}}}$, we have $\delta(t_i) = 0$, $\delta(\theta^{\pm}_{l,t_i}) = 0$, and $\delta(\theta_{l',t_i}) = 0$. By equalities (2.10) and (2.11), we have $\delta(C_i) = \delta(D_i) = 0$ for $i = 1, 2, \dots, \nu, \infty$. Here, C_i and D_i are the polynomials in (2.6). We compute the residue of the trace of $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ at $\zeta_i = 0$. First, we consider the expansion of $\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))$ at $\zeta_i = 0$. Since $\delta(C_i) = \delta(D_i) = 0$ for $i = 1, 2, \dots, \nu, \infty$, we have $\delta(c_2) = O(x^0_{t_i})$ and $\delta(d_2) = O(x^0_{t_i})$. Second, we consider $\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$. By the definition (4.7), we have that $\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$

coincides with

$$\Phi_i\Xi_i(\zeta_i^2)\delta(M_{\zeta_i})M_{\zeta_i}^{-1}(\Phi_i\Xi_i(\zeta_i^2))^{-1}+\delta(\Phi_i\Xi_i(\zeta_i^2))(\Phi_i\Xi_i(\zeta_i^2))^{-1}$$
$$+(\Phi_i\Xi_i(\zeta_i^2)M_{\zeta_i})\begin{pmatrix}-\delta(\hat{\lambda}_{i,+}(\zeta_i)) & 0\\ 0 & -\delta(\hat{\lambda}_{i,-}(\zeta_i))\end{pmatrix}(\Phi_i\Xi_i(\zeta_i^2)M_{\zeta_i})^{-1}.$$

Since $\delta(M_{\zeta_i})=0$ and $\delta(\hat{\lambda}_{i,\pm}(\zeta_i))=O(\zeta_i)$, we have

$$\mathrm{res}_{\zeta_i=0}\operatorname{Tr}\big(\delta(\Omega^{(n-2)}_{(\boldsymbol{t}_0,\boldsymbol{\theta},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\delta(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big)=0.$$

The remaining residues are calculated as in the proof of Theorem 3.4. Then we obtain

$$\omega(\delta_1,\delta_2)=\sum_{j=1}^{n-3}\left(\frac{\delta_1(p_j)\delta_2(q_j)}{P(q_j)}-\frac{\delta_2(p_j)\delta_1(q_j)}{P(q_j)}\right),$$

which means that ω coincides with $\sum_{j=1}^{n-3}d(\frac{p_j}{P(q_j)})\wedge dq_j$. □

§4.2. Integrable deformations associated to θ_{l',t_i} for $i\in I_{\mathrm{ra}}$

First we fix $i\in I_{\mathrm{ra}}$ and $l'\in\{0,1,2,\dots,2n_i-3\}$. Let

$$\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}=\begin{cases}d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}),\\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}),\end{cases}$$

be the family (2.14). Let θ_{l',t_i} be the natural coordinate of $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ and $\partial/\partial\theta_{l',t_i}$ be the vector field on $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$ associated to θ_{l',t_i}. We will construct a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial\theta_{l',t_i}$.

By using the explicit form of $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL,ext}}$, we take a family of compatible framings of $\widetilde{\nabla}^{(n-2)}_{\mathrm{DL,ext}}$ at $\tilde{t}_{i'}$ for $i'\in I_{\mathrm{ra}}$. We denote this family of compatible framings at $\tilde{t}_{i'}$, for $i'\in I_{\mathrm{ra}}$, by $\Phi_{i'}$. Let $\Xi_{i'}(x_{t_{i'}})$ be the formal transformation (4.2) for $i'\in I_{\mathrm{ra}}$. If $i'\in I\setminus I_{\mathrm{ra}}$, let $\Phi_{i'}$ and $\Xi_{i'}(x_{t_{i'}})$ be a compatible framing at $\tilde{t}_{i'}$ and the (formal) transformation with respect to $\Phi_{i'}$ appearing in Lemma 3.1, respectively. Let $\widetilde{G}$ be the matrix defined in (2.7). For each $i'\in I$, we denote formal expansion of $\widetilde{G}^{-1}\Phi_{i'}\Xi_{i'}(x_{t_{i'}})$ at $x_{t_{i'}}=0$ by

$$\widetilde{G}^{-1}\Phi_{i'}\Xi_{i'}(x_{t_{i'}})=P_{i',0}+P_{i',1}x_{t_{i'}}+P_{i',2}x_{t_{i'}}^2+\cdots.$$

Set

$$P_{i'}:=P_{i',0}+P_{i',1}x_{t_{i'}}+P_{i',2}x_{t_{i'}}^2+\cdots+P_{i',2n_{i'}-1}x_{t_{i'}}^{2n_{i'}-1}(\text{for } i'\in I),$$
$$P_{\nu+1}:=\mathrm{id}.$$

We take an affine open covering $\{\widehat{U}_{i'}\}_{i'\in I\cup\{\nu+1\}}$ of $\mathbb{P}^1\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})$ as in Section 3.3. By using the matrices $P_{i'}$, we define a new trivialization $\widehat{\varphi}_{i'}$ of $\widehat{E}_1$ on $\widehat{U}_{i'}$ for each $i' \in I \cup \{\nu+1\}$ as in Section 3.3. Let $\widehat{\Omega}_{i'}$ be the connection matrix of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ under the new trivialization $\widehat{\varphi}_{i'}$:

$$\widehat{\Omega}_{i'} = P_{i'}^{-1}\, dP_{i'} + P_{i'}^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{U}_{i'}} P_{i'} \quad \text{for } i' \in (I \cup \{\nu+1\}) \setminus \{\infty\},$$
$$\widehat{\Omega}_\infty = (G_1 P_\infty)^{-1}\, d(G_1 P_\infty) + (G_1P_\infty)^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{\widehat{U}_\infty}(G_1 P_\infty).$$

Remark that $M_{\zeta_{i'}}^{-1}\, dM_{\zeta_{i'}} + M_{\zeta_{i'}}^{-1}\widehat{\Omega}_{i'} M_{\zeta_{i'}}$ is diagonal until the $\zeta_{i'}^{2n_{i'}-3}$-terms for each $i' \in I_{\mathrm{ra}}$.

For the fixed indices $i \in I_{\mathrm{ra}}$ and $l' \in \{0,1,\dots,2n_i-3\}$, we define a matrix $B_{\theta_{l',t_i}}$ by

$$B_{\theta_{l',t_i}} = \begin{cases} \dfrac{-1}{2(n_i-l-1)} \dfrac{\delta(\theta_{2l,t_i})}{\zeta_i^{2(n_i-1-l)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } l' = 2l \\ \dfrac{-1}{2(n_i-l)-3} \dfrac{\delta(\theta_{2l+1,t_i})}{\zeta_i^{2(n_i-1-l)-1}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } l' = 2l+1. \end{cases}$$

We may check that

$$M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1} = \begin{cases} \dfrac{-1}{2(n_i-l-1)} \delta(\theta_{2l,t_i}) \begin{pmatrix} x_{t_i}^{-n_i+l+1} & 0 \\ 0 & x_{t_i}^{-n_i+l+1} \end{pmatrix} & \text{if } l' = 2l, \\ \dfrac{-1}{2(n_i-l)-3} \delta(\theta_{2l+1,t_i}) \begin{pmatrix} 0 & x_{t_i}^{-n_i+l+1} \\ x_{t_i}^{-n_i+l+2} & 0 \end{pmatrix} & \text{if } l' = 2l+1. \end{cases}$$

In particular, $M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$. We define a vector bundle $(\widehat{E}_1)^{\varepsilon}_{\theta_{l',t_i}}$ on $\mathbb{P}^1 \times (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec} \mathbb{C}[\varepsilon]$ by the same argument as in the construction of $(\widehat{E}_1)^{\varepsilon}_{\theta^{\pm}_{l,t_i}}$. That is, we replace $B_{\theta^{\pm}_{l,t_i}}(x_{t_i})$ in (3.14) with $M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1}$. We define a morphism

$$\nabla^{\varepsilon}_{\partial/\partial\theta_{l',t_i}} : (\widehat{E}_1)^{\varepsilon}_{\theta_{l',t_i}} \longrightarrow (\widehat{E}_1)^{\varepsilon}_{\theta_{l',t_i}} \otimes \widetilde{\Omega}^1_{\partial/\partial t_i}$$

by the same argument as in the construction of $\nabla^{\varepsilon}_{\partial/\partial\theta^{\pm}_{l,t_i}}$ in Section 3.3. That is, we replace $B_{\theta^{\pm}_{l,t_i}}(x_{t_i})$ in (3.15) with $M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1}$. The ε-term of $\nabla^{\varepsilon}_{\partial/\partial\theta_{l',t_i}}|_{\widehat{U}_i^{\varepsilon}}$ for fixed $i \in I_{\mathrm{ra}}$ is

$$\begin{aligned} &d(M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1}) + [\widehat{\Omega}_i, M_{\zeta_i} B_{\theta_{l',t_i}} M_{\zeta_i}^{-1}] \\ &\quad = M_{\zeta_i}\Big(\frac{\partial}{\partial\zeta_i}(B_{\theta_{l',t_i}})\, d\zeta_i + [M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1}\widehat{\Omega}_i M_{\zeta_i}, B_{\theta_{l',t_i}}]\Big) M_{\zeta_i}^{-1} \end{aligned}$$

$$= \begin{cases} \dfrac{\delta(\theta_{2l,t_i})}{2} \begin{pmatrix} x_{t_i}^{-n_i+l} & 0 \\ 0 & x_{t_i}^{-n_i+l} \end{pmatrix} dx_{t_i} \\ \qquad +M_{\zeta_i}[M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1}\widehat{\Omega}_i M_{\zeta_i}, B_{\theta_{l',t_i}}]M_{\zeta_i}^{-1} \quad \text{if } l' = 2l, \\ \dfrac{\delta(\theta_{2l+1,t_i})}{2} \begin{pmatrix} 0 & x_{t_i}^{-n_i+l} \\ x_{t_i}^{-n_i+l+1} & 0 \end{pmatrix} dx_{t_i} \\ \qquad +M_{\zeta_i}[M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1}\widehat{\Omega}_i M_{\zeta_i}, B_{\theta_{l',t_i}}]M_{\zeta_i}^{-1} \quad \text{if } l' = 2l+1. \end{cases}$$

Since $M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1}\widehat{\Omega}_i M_{\zeta_i}$ and $B_{\theta_{l',t_i}}$ are diagonal until the $\zeta_i^{2n_i-3}$-terms, the negative part of the relative connection $\widetilde{\nabla}^{\varepsilon}_{\partial/\partial\theta_{l',t_i}}$ along the divisor $[x_{t_i} = 0]$ is

$$\begin{pmatrix} \alpha_i & \beta_i \\ x_{t_i}\beta_i & \alpha_i - \frac{dx_{t_i}}{2x_{t_i}} \end{pmatrix}.$$

Here, if $l' = 2l$, the entry α_i is

$$\frac{\theta_{0,t_i}}{2}\frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \frac{\theta_{2l,t_i} + \varepsilon\delta(\theta_{2l,t_i})}{2}\frac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \frac{\theta_{2n_i-2,t_i}}{2}\frac{dx_{t_i}}{x_{t_i}},$$

and the coefficients of β_i are independent of ε until the $x_{t_i}^{-2}$-term. If $l' = 2l+1$, the expansion of β_i at t_i until the $x_{t_i}^{-2}$-term is

$$\frac{\theta_{1,t_i}}{2}\frac{dx_{t_i}}{x_{t_i}^{n_i}} + \cdots + \frac{\theta_{2l+1,t_i} + \varepsilon\delta(\theta_{2l+1,t_i})}{2}\frac{dx_{t_i}}{x_{t_i}^{n_i-l}} + \cdots + \frac{\theta_{2n_i-3,t_i}}{2}\frac{dx_{t_i}}{x_{t_i}^2},$$

and the coefficients of α_i are independent of ε until the $x_{t_i}^{-1}$-term.

As in Section 3.3, $\widehat{E}_1^{\varepsilon} \cong (\widehat{E}_1)^{\varepsilon}_{\theta_{l',t_i}}$. If we consider the pull-back of $\nabla^{\varepsilon}_{\partial/\partial\theta_{l',t_i}}$ under this isomorphism, then we have a horizontal lift of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ with respect to $\partial/\partial\theta_{l',t_i}$. If we take a relativization of this horizontal lift, we have a family of connections parametrized by $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$. This family gives a map from the base space $(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon]$ to the moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. By taking composition with $\widehat{\mathrm{App}}$ defined in (2.13), we have a map

$$(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \longrightarrow \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}. \tag{4.10}$$

Definition 4.4. Then we may define the vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ associated to the map (4.10). We denote this vector field on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ by $\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}$.

Let $f^{\mathrm{IMD}}_{\theta_{l',t_i}} \colon (\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}) \times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ be the map induced by the vector field $\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}$. We have

$$\widehat{E}_1^{\varepsilon} = (\mathrm{id} \times f^{\mathrm{IMD}}_{\theta_{l',t_i}})^*\widehat{E}_1.$$

We denote the pull-back of $\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$ under the map $\mathrm{id}\times f^{\mathrm{IMD}}_{\theta_{l'},t_i}$ by

$$\begin{cases} d+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon], \\ d+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \\ \qquad +\varepsilon G_1^{-1}\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1 & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon]. \end{cases} \tag{4.11}$$

As in Section 3.3, we have a lift of $(\mathrm{id}\times f^{\mathrm{IMD}}_{\theta_{l'},t_i})^*\widetilde{\nabla}^{(1)}_{\mathrm{DL,ext}}$:

$$\begin{cases} \hat{d}+\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}+\varepsilon\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) & \\ \qquad +\Upsilon^{\mathrm{IMD}}_{\theta_{l'},t_i}\,d\varepsilon & \text{on } U_0\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon], \\ \hat{d}+G_1^{-1}\,dG_1+G_1^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}G_1 & \\ \qquad +\varepsilon G_1^{-1}\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})G_1 & \\ \qquad +G_1^{-1}\Upsilon^{\mathrm{IMD}}_{\theta_{l'},t_i}G_1\,d\varepsilon & \text{on } U_\infty\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\times\operatorname{Spec}\mathbb{C}[\varepsilon], \end{cases} \tag{4.12}$$

which is a morphism $\widehat{E}_1^{\varepsilon}\to\widehat{E}_1^{\varepsilon}\otimes\widetilde{\Omega}^1_{\partial/\partial t_i}$ with the Leibniz rule and the equality

$$\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})=d\Upsilon^{\mathrm{IMD}}_{\theta_{l'},t_i}+[\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)},\Upsilon^{\mathrm{IMD}}_{\theta_{l'},t_i}], \tag{4.13}$$

which means the integrable condition.

Remark. In this paper, we consider only rank-two connections on $\mathbb{P}^1$. Moreover, we impose some Zariski open conditions, for example the underlying vector bundles are isomorphic to $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)$. That is, we consider only a Zariski open subset of the moduli space of connections constructed in [9]. Horizontal lifts for more general situations are constructed by Inaba [10, Sect. 9].

§4.3. Isomonodromy 2-form

Definition 4.5. Let $\hat{\delta}_1$ and $\hat{\delta}_2$ be vector fields on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$, which is isomorphic to the extended moduli space $\widehat{\mathfrak{Conn}}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}$. For each $i\in I_{\mathrm{ra}}$, we fix a formal fundamental matrix solution ψ_{ζ_i} of $(d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\psi_{\zeta_i}=0$ at $x=t_i$ defined in (4.7). For each $i\in I_{\mathrm{reg}}\cup I_{\mathrm{un}}$, we fix a formal fundamental matrix solution ψ_i of $(d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\psi_i=0$ at $x=t_i$ as in Lemma 3.1. We take a fundamental matrix solution ψ_{q_j} of $(d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\psi_{q_j}=0$ at $x=q_j$ as in Lemma 2.11. We define a

2-form $\widehat{\omega}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ as

$$\begin{aligned}\widehat{\omega}(\hat{\delta}_1, \hat{\delta}_2) := \frac{1}{2} \sum_{i \in I \setminus I_{\mathrm{ra}}} \mathrm{res}_{x=t_i} \mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_i)\psi_i^{-1}) \\ + \frac{1}{4} \sum_{i \in I_{\mathrm{ra}}} \mathrm{res}_{\zeta_i=0} \mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)) \wedge \hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big) \\ + \frac{1}{2} \sum_{j=1}^{n-3} \mathrm{res}_{x=q_j} \mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}).\end{aligned}$$

By the same argument as in Section 4.1, we may check that the residue of $\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\psi_i)\psi_i^{-1}$ at $\tilde{t}_i$ (for $i \in I \setminus I_{\mathrm{ra}}$) and $\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)) \wedge \hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ at $\zeta_i = 0$ (for $i \in I_{\mathrm{ra}}$) are well defined. Moreover, the right-hand side of (3.24) is independent of the choice of ψ_{q_j}, ψ_i $(i \in I \setminus I_{\mathrm{ra}})$, and ψ_{ζ_i} $(i \in I_{\mathrm{ra}})$. By the same argument as in the proof of Lemma 4.2, we may check that the formal series $\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$ for any vector field $\hat{\delta}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$.

Proposition 4.6. *Let $\widetilde{G}$ and $\widetilde{G}_\infty$ be the matrices defined in (2.7). If ∞ is an element of $I_{\mathrm{reg}} \cup I_{\mathrm{un}}$, we set $\widetilde{\psi}_i := \widetilde{G}^{-1}\psi_i$ for any $i \in (I_{\mathrm{reg}} \cup I_{\mathrm{un}}) \setminus \{\infty\}$, $\widetilde{\psi}_{\zeta_i} := \widetilde{G}^{-1}\psi_{\zeta_i}$ for any $i \in I_{\mathrm{ra}}$, and $\widetilde{\psi}_\infty := \widetilde{G}_\infty^{-1}\psi_\infty$. If ∞ is an element of I_{ra}, we set $\widetilde{\psi}_i := \widetilde{G}^{-1}\psi_i$ for any $i \in I_{\mathrm{reg}} \cup I_{\mathrm{un}}$, $\widetilde{\psi}_{\zeta_i} := \widetilde{G}^{-1}\psi_{\zeta_i}$ for any $i \in I_{\mathrm{ra}} \setminus \{\infty\}$, and $\widetilde{\psi}_{\zeta_\infty} := \widetilde{G}_\infty^{-1}\psi_{\zeta_\infty}$. We have the equality*

$$\begin{aligned}\widehat{\omega}(\hat{\delta}_1, \hat{\delta}_2) = \frac{1}{2} \sum_{i \in I \setminus I_{\mathrm{ra}}} \mathrm{res}_{x=t_i} \mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\widetilde{\psi}_i)(\widetilde{\psi}_i)^{-1}) \\ + \frac{1}{4} \sum_{i \in I_{\mathrm{ra}}} \mathrm{res}_{\zeta_i=0} \mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)) \wedge \hat{\delta}(\widetilde{\psi}_{\zeta_i})\widetilde{\psi}_{\zeta_i}^{-1}\big).\end{aligned} \tag{4.14}$$

Proof. Since $\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}$ descends under the ramification $x_{t_i} = \zeta_i^2$, we have

$$\begin{aligned}\frac{1}{4}\mathrm{res}_{\zeta_i=0} \mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)) \wedge \hat{\delta}(\widetilde{\psi}_{\zeta_i})\widetilde{\psi}_{\zeta_i}^{-1}\big) \\ = \frac{1}{2}\mathrm{res}_{x=t_i} \mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) \wedge \hat{\delta}(\widetilde{\psi}_{\zeta_i})\widetilde{\psi}_{\zeta_i}^{-1}).\end{aligned}$$

By this equality and the same argument as in Proposition 3.9, we may check equality (4.14). □

Theorem 4.7. *For the vector field $\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}$, we have $\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}, \hat{\delta}) = 0$ for any vector field $\hat{\delta} \in \Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$.*

Proof. By equality (4.14) we have

$$\begin{aligned}\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i},\hat{\delta}) = \frac{1}{2}\sum_{i'\in I\setminus I_{\mathrm{ra}}} \mathrm{res}_{x=t_{i'}} \mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})\widetilde{\psi}_{i'}^{-1} \\ - \delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widetilde{\psi}_{i'})\widetilde{\psi}_{i'}^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\big) \\ + \frac{1}{4}\sum_{i'\in I_{\mathrm{ra}}} \mathrm{res}_{\zeta_{i'}=0} \mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}_{\zeta_{i'}}^{-1} \\ - \delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}_{\zeta_{i'}}^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big).\end{aligned}$$

We take $i'\in I\setminus I_{\mathrm{ra}}$. We may show that

$$\mathrm{res}_{x=t_{i'}} \mathrm{Tr}(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1} - \delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widetilde{\psi}_{i'})(\widetilde{\psi}_{i'})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) = 0$$

by the same argument as in the proof of Theorem 3.10. So we have

$$\begin{aligned}\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i},\hat{\delta}) = \frac{1}{4}\sum_{i'\in I_{\mathrm{ra}}} \mathrm{res}_{\zeta_{i'}=0} \mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}_{\zeta_{i'}}^{-1} \\ - \delta^{\mathrm{IMD}}_{\theta_{l'},t_i}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}_{\zeta_{i'}}^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big).\end{aligned}$$

We take $i'\in I_{\mathrm{ra}}$. Let $U_{t_{i'}}$ be an affine open subset on $\mathbb{P}^1$ so that $x_{t_{i'}}$ is a coordinate on $U_{t_{i'}}$. We define $\zeta_{i'}$ as $x_{t_{i'}}=\zeta_{i'}^2$. Let $U_{\zeta_{i'}}$ be the inverse image of $U_{t_{i'}}$ under the map $\mathrm{Spec}\,\mathbb{C}[\zeta_{i'}]\to\mathrm{Spec}\,\mathbb{C}[x_{t_{i'}}]$ by $x_{t_{i'}}=\zeta_{i'}^2$. We take an analytic open subset V of $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. We take an inverse image of V under the composition of

$$\hat{f}_{\zeta_{i'}}: U_{\zeta_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}) \longrightarrow U_{t_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})$$

and the projection $p_{U_{t_{i'}}}: U_{t_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})\to\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. Here, $\hat{f}_{\zeta_{i'}}$ is defined by $x_{t_i}=\zeta_i^2$. Let $\widehat{\Delta}^{\mathrm{an}}_{i'}$ ($i'\in I$) be an analytic open subset of the inverse image $(p_{U_{t_{i'}}}\circ\hat{f}_{\zeta_{i'}})^{-1}(V)$ such that $[\zeta_{i'}=0]\cap(p_{U_{t_{i'}}}\circ\hat{f}_{\zeta_{i'}})^{-1}(V)\subset\widehat{\Delta}^{\mathrm{an}}_{i'}$ and the fibers of $(p_{U_{t_{i'}}}\circ\hat{f}_{\zeta_{i'}})|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}:\widehat{\Delta}^{\mathrm{an}}_{i'}\to V$ for each point of V are unit disks such that $\zeta_{i'}$ gives a coordinate of the unit disks. We denote the pull-back of the connection matrix $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{U_{t_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$ under the map $\hat{f}_{\zeta_{i'}}$ by $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)$. We define a matrix $S(\zeta_{i'})$ on $U_{\zeta_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}T_{\boldsymbol{\theta}})$ as

$$S(\zeta_{i'}) := \widetilde{G}^{-1}\Phi_{i'}\begin{pmatrix}1 & 0\\ 0 & \zeta_{i'}\end{pmatrix}.$$

Remark that $\widetilde{G}^{-1}\Phi_{i'}$ is a compatible framing of $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}|_{U_{t_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$ at $t_{i'}$. So we can define the (local) elementary transformation of $\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)$ by $S(\zeta_{i'})$. We denote the elementary transformation by ${}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)$. That is,

$${}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2) = S(\zeta_{i'})^{-1}\,dS(\zeta_{i'}) + S(\zeta_{i'})^{-1}\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)S(\zeta_{i'}).$$

Let ${}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2,t)$ be a connection matrix such that

$$\begin{aligned}{}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2,0) &= {}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)|_{\widehat{\Delta}^{\mathrm{an}}_{i'}},\\ \frac{\partial}{\partial t}{}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2,t)|_{t=0} &= \delta^{\mathrm{IMD}}_{\theta_{l',t_i}}({}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))|_{\widehat{\Delta}^{\mathrm{an}}_{i'}}.\end{aligned}$$

Let $\widehat{\Sigma}\subset\widehat{\Delta}^{\mathrm{an}}_{i'}$ be a family of sufficiently small sectors in $\widehat{\Delta}^{\mathrm{an}}_{i'}\to\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}$. By [18, Thm. 12.1], we may take a fundamental matrix solution ${}'\Psi_{\widehat{\Sigma}}(\zeta_{i'},t)$ on $\widehat{\Sigma}\times U_t^{\mathrm{an}}$ of the differential equation

$$d({}'\Psi_{\widehat{\Sigma}}(\zeta_{i'},t))+{}'\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2,t)({}'\Psi_{\widehat{\Sigma}}(\zeta_{i'},t))=0$$

with uniform asymptotic relation

$${}'\Psi_{\widehat{\Sigma}}(\zeta_{i'},t)\exp(\Lambda^-_{i'}(\zeta_{i'},t))\sim\widehat{P}_{i'}(\zeta_{i'},t)\quad(\zeta_{i'}\to 0,\zeta_{i'}\in\widehat{\Sigma}).$$

Here we set

$$\begin{aligned}\Lambda^-_{i'}(\zeta_{i'}) &:= \sum_{l'=0}^{2n_{i'}-2}\begin{pmatrix}\theta_{l',t_{i'}}\int\zeta_{i'}^{-2n_{i'}+l'+1}\,d\zeta_{i'} & 0\\ 0 & (-1)^{l'}\theta_{l',t_{i'}}\int\zeta_{i'}^{-2n_{i'}+l'+1}\,d\zeta_{i'}\end{pmatrix},\\ \widehat{P}_{i'}(\zeta_{i'}) &:= \begin{pmatrix}1 & 0\\ 0 & \frac{1}{\zeta_{i'}}\end{pmatrix}\Xi_{i'}(\zeta_{i'}^2)M_{\zeta_{i'}},\end{aligned}$$

and we take

$$\begin{aligned}\widehat{P}_{i'}(\zeta_{i'},t) &= \widehat{P}_{i',0}(t)+\widehat{P}_{i',1}(t)\zeta_{i'}+\cdots,\\ \Lambda^-_{i'}(\zeta_{i'},t) &= \sum_{l'=0}^{2n_{i'}-2}\begin{pmatrix}\theta_{l',t_{i'}}(t)\int\zeta_{i'}^{-2n_{i'}+l'+1}\,d\zeta_{i'} & 0\\ 0 & (-1)^{l'}\theta_{l',t_{i'}}(t)\int\zeta_{i'}^{-2n_{i'}+l'+1}\,d\zeta_{i'}\end{pmatrix},\end{aligned}$$

so that the expansions of $\Lambda^-_{i'}(\zeta_{i'},\varepsilon)$ and $\widehat{P}_{i'}(\zeta_{i'},\varepsilon)$ with respect to ε are

$$\begin{aligned}\Lambda^-_{i'}(\zeta_{i'},\varepsilon) &= \Lambda^-_{i'}(\zeta_{i'})+\varepsilon\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\Lambda^-_{i'}(\zeta_{i'})),\\ \widehat{P}_{i'}(\zeta_{i'},\varepsilon) &= \widehat{P}_{i'}(\zeta_{i'})+\varepsilon\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{P}_{i'}(\zeta_{i'})).\end{aligned}$$

Remark that $\widehat{P}_{i'}(\zeta_{i'})^{-1}$ has no pole at $\zeta_{i'}=0$ and $S(\zeta_{i'})\widehat{P}_{i'}(\zeta_{i'})\exp(-\Lambda^-_{i'}(\zeta_{i'}))=\widetilde{\psi}_{\zeta_{i'}}$. We set

$${}'\widetilde{\psi}_{\zeta_{i'}}:=\widehat{P}_{i'}(\zeta_{i'})\exp(-\Lambda^-_{i'}(\zeta_{i'})).$$

Let $\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)$ be the pull-back of $\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}|_{U_{t_{i'}}\times(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$ under the map $\hat{f}_{\zeta_{i'}}$. We set

$${}'\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2):=S(\zeta_{t_{i'}})^{-1}\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)S(\zeta_{t_{i'}})-S(\zeta_{i'})^{-1}\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(S(\zeta_{i'})).$$

By the same argument as the verification of (3.37), we may check the asymptotic relation

$$\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}({}'\widetilde{\psi}_{\zeta_{i'}})({}'\widetilde{\psi}_{\zeta_{i'}})^{-1} \sim {}'\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2) - \widehat{P}_{i'}(\zeta_{t_{i'}})\widetilde{C}^{\mathrm{diag}}_{i'}\widehat{P}_{i'}(\zeta_{t_{i'}})^{-1}. \tag{4.15}$$

Here, $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ is a diagonal matrix such that $\widetilde{C}^{\mathrm{diag}}_{t_{i'}}$ is independent of $\zeta_{i'}$. By the asymptotic relation (4.15) and the definition of ${}'\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)$, we have

$$\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1} \sim \Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2) - S(\zeta_{t_{i'}})\widehat{P}_{i'}(\zeta_{t_{i'}})\widetilde{C}^{\mathrm{diag}}_{i'}\widehat{P}_{i'}(\zeta_{t_{i'}})^{-1}S(\zeta_{t_{i'}})^{-1}.$$

By this asymptotic relation and the same calculation as in the proof of Theorem 3.10, we may check the equalities

$$\begin{aligned}
&\mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1} \\
&\qquad\qquad - \delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big) \\
&= \mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1} \\
&\qquad\qquad - \Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big) \\
&\quad + \mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}d\big(\widehat{P}_{i'}(\zeta_{i'},0)^{-1}S(\zeta_{t_{i'}})^{-1}\hat{\delta}(S(\zeta_{t_{i'}})\widehat{P}_{i'}(\zeta_{i'},0))\big)\big) \\
&\quad + \mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\widetilde{C}^{\mathrm{diag}}_{t_{i'}}d\big(\hat{\delta}(-\Lambda^{-}_{i'}(\zeta_{i'}))\big)\big) \\
&= \mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1} \\
&\qquad\qquad - \Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big).
\end{aligned} \tag{4.16}$$

By the integrable condition (4.13), we have the equality

$$\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)) = d(\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)) + [\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2), \Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)].$$

We may check that

$$\begin{aligned}
&\mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1} - \Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big) \\
&= \mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(d(\Upsilon^{\mathrm{IMD}}_{\theta_{l',t_i}}(\zeta_{i'}^2)\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})(\widetilde{\psi}_{\zeta_{i'}})^{-1})\big) = 0
\end{aligned} \tag{4.17}$$

by the same calculation as in the proof of Theorem 3.10. By combining (4.16) and (4.17), we obtain

$$\begin{aligned}
\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}, \hat{\delta}) = \frac{1}{4}\sum_{i'\in I_{\mathrm{ra}}} &\mathrm{res}_{\zeta_{i'}=0}\,\mathrm{Tr}\big(\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\hat{\delta}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}^{-1}_{\zeta_{i'}} \\
&\qquad - \delta^{\mathrm{IMD}}_{\theta_{l',t_i}}(\widetilde{\psi}_{\zeta_{i'}})\widetilde{\psi}^{-1}_{\zeta_{i'}}\hat{\delta}(\widehat{\Omega}^{(1)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big) = 0.
\end{aligned}$$

That is, we have the assertion of this theorem. □

As in Sections 3.3 and 3.4, respectively, we may define vector fields $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ ($i \in I_{\mathrm{un}}$, $0 \le l \le n_i - 2$) and $\delta^{\mathrm{IMD}}_{t_i}$ ($i \in \{3,4,\dots,\nu\} \cap (I_{\mathrm{reg}} \cup I_{\mathrm{un}})$). By the same argument as in the proofs of Theorems 3.10 and 4.7, we obtain the following theorem:

Theorem 4.8. *Let $i \in I_{\mathrm{un}}$. For the vector field $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$, we have $\widehat{\omega}(\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}, \hat{\delta}) = 0$ for any vector field $\hat{\delta} \in \Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$. Moreover, for the vector field $\delta^{\mathrm{IMD}}_{t_i}$, we have $\widehat{\omega}(\delta^{\mathrm{IMD}}_{t_i}, \hat{\delta}) = 0$ for any vector field $\hat{\delta} \in \Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$.*

Then we have that the 2-form $\widehat{\omega}$ is the isomonodromy 2-form.

§4.4. Hamiltonian systems

We have the diagonalization

$$\Omega^{\mathrm{diag}}_{t_i}(\zeta_i^2) := (\Phi_i \Xi_i M_{\zeta_i})^{-1}\, d(\Phi_i \Xi_i M_{\zeta_i}) + (\Phi_i \Xi_i M_{\zeta_i})^{-1} \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}}, \boldsymbol{\theta}_0)}(\zeta_i^2)(\Phi_i \Xi_i M_{\zeta_i})$$

and

$$\begin{aligned}\Omega^{\mathrm{diag}}_{t_i}(\zeta_i^2) &= \begin{pmatrix} \theta_{0,t_i} & 0 \\ 0 & \theta_{0,t_i} \end{pmatrix} \frac{d\zeta_i}{\zeta_i^{2n_i-1}} + \cdots + \begin{pmatrix} \theta_{2n_i-2,t_i} & 0 \\ 0 & \theta_{2n_i-2,t_i} \end{pmatrix} \frac{d\zeta_i}{\zeta_i} \\ &\quad + \begin{pmatrix} \theta_{2n_i-1,t_i} & 0 \\ 0 & -\theta_{2n_i-1,t_i} \end{pmatrix} d\zeta_i + \cdots + \begin{pmatrix} \theta_{4n_i-4,t_i} & 0 \\ 0 & \theta_{4n_i-4,t_i} \end{pmatrix} \zeta_i^{2n_i-3}\, dx + \cdots .\end{aligned}$$

Remark that we have an equation $(d + \Omega^{\mathrm{diag}}_{t_i}(\zeta_i^2)) \exp(-\Lambda_i(\zeta_i)) = 0$. We set

$$\Lambda_i^{+}(\zeta_i) = \begin{pmatrix} \sum_{l'=2n_i-1}^{\infty} \theta_{l',t_i} \int \zeta_i^{-2n_i+l'+1}\, d\zeta_i & 0 \\ 0 & \sum_{l'=2n_i-1}^{\infty} (-1)^{l'} \theta_{l',t_i} \int \zeta_i^{-2n_i+l'+1}\, d\zeta_i \end{pmatrix}.$$

Definition 4.9. For each t_i ($i \in I_{\mathrm{ra}}$) and each l' ($0 \le l' \le 2n_i - 3$), we define a rational function $H_{\theta_{l',t_i}}$ on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ as

$$\begin{aligned} H_{\theta_{l',t_i}} &= -[\text{the coefficient of the } \zeta_i^{2(n_i-1)-l'}\text{-term of the } (1,1)\text{-entry of } \Lambda_i^{+}(\zeta_i)] \\ &= -\frac{\theta_{4(n_i-1)-l',t_i}}{2(n_i-1)-l'}. \end{aligned}$$

We call $H_{\theta_{l',t_i}}$ the *Hamiltonian associated to* θ_{l',t_i}.

We define $H_{\theta^{\pm}_{l,t_i}}$ ($i \in I_{\mathrm{un}}$ and $l = 0,1,\dots,n_i-2$) and H_{t_i} ($i \in \{3,4,\dots,\nu\} \cap (I_{\mathrm{reg}} \cup I_{\mathrm{un}})$) as in Definitions 3.11 and 3.12.

Theorem 4.10. *Set* $P(x;\boldsymbol{t}) := \prod_{i=1}^{\nu}(x-t_i)^{n_i}$ *and* $D_i(x;\boldsymbol{t},\boldsymbol{\theta}) := D_i(x)$ *for* $i \in I$. *We put*

$$\begin{aligned}\widehat{\omega}' := \sum_{j=1}^{n-3} d\biggl(\frac{p_j}{P(q_j;\boldsymbol{t})} - \sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}} - D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})\biggr)\wedge dq_j \\ + \sum_{i\in I_{\mathrm{un}}}\sum_{l=0}^{n_i-2}(dH_{\theta^+_{l,t_i}}\wedge d\theta^+_{l,t_i} + dH_{\theta^-_{l,t_i}}\wedge d\theta^-_{l,t_i}) \\ + \sum_{i\in I_{\mathrm{ra}}}\sum_{l'=0}^{2n_i-3} dH_{\theta_{l',t_i}}\wedge d\theta_{l',t_i} + \sum_{\substack{i\in\{3,4,\dots,\nu\}\\ \cap(I_{\mathrm{reg}}\cup I_{\mathrm{un}})}} dH_{t_i}\wedge dt_i.\end{aligned}$$

Then the difference $\widehat{\omega}-\widehat{\omega}'$ *is a section of* $\pi^*_{\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0}(\Omega^2_{(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}}})$.

Proof. Recall that $\widehat{\omega}(\hat{\delta}_1,\hat{\delta}_2)$ is

$$\begin{aligned}\frac{1}{2}\sum_{i\in I\setminus I_{\mathrm{ra}}} \mathrm{res}_{x=t_i}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_i)\psi_i^{-1}) \\ + \frac{1}{4}\sum_{i\in I_{\mathrm{ra}}}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\bigl(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\bigr) \\ + \frac{1}{2}\sum_{j=1}^{n-3}\mathrm{res}_{x=q_j}\,\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\wedge\hat{\delta}(\psi_{q_j})\psi_{q_j}^{-1}).\end{aligned}$$

The plan of this proof is as follows. The calculation of the first and third terms in this formula is the same as in the proof of Theorem 3.14. So we omit the calculation of these terms. Now we consider only the second term

$$\frac{1}{4}\sum_{i\in I_{\mathrm{ra}}}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\bigl(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\bigr).$$

We calculate the residue at $\zeta_i = 0$ for some (local) gauge transformation of $d + \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)$. We need to consider the difference between the residue after taking the gauge transformation and the residue before taking the gauge transformation. Here, the residue before taking the gauge transformation is just the residue of $\mathrm{Tr}(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1})$ at $\zeta_i = 0$. This difference is more complicated than the unramified irregular singular cases, since the $(1,2)$-entry of the residue part of (4.3) is not a piece of the local formal data. We consider this difference by using Lemma 3.8.

In fact, for each $i\in I_{\mathrm{ra}}$, we consider the residue

$$\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\bigl(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\bigr). \tag{4.18}$$

Now we take a diagonalization of $d+\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2)$ until some degree term at $\zeta_i = 0$. We put

$$\Xi_i^{\leq 2n_i-1}(x_{t_i}) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{s=1}^{2n_i-1} \begin{pmatrix} (\xi_s^{(i)})_{11} & (\xi_s^{(i)})_{12} \\ (\xi_s^{(i)})_{21} & (\xi_s^{(i)})_{22} \end{pmatrix} x_{t_i}^s$$

for $i \in I$. Here, the coefficient matrices of $\Xi_i^{\leq 2n_i-1}(x_{t_i})$ appear as in the coefficient matrices of $\Xi_i(x_{t_i})$ defined by (4.2). Moreover, we put

$$\begin{aligned}
\widetilde{\Omega}_i &:= (\Phi_i \Xi_i^{\leq 2n_i-1})^{-1}\, d(\Phi_i \Xi_i^{\leq 2n_i-1}) + (\Phi_i \Xi_i^{\leq 2n_i-1})^{-1} \widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} (\Phi_i \Xi_i^{\leq 2n_i-1}), \\
\widetilde{\Omega}'_{\zeta_i} &:= M_{\zeta_i}^{-1}\, dM_{\zeta_i} + M_{\zeta_i}^{-1} \widetilde{\Omega}_i M_{\zeta_i}, \\
\widetilde{\psi}_{\zeta_i} &:= (\Phi_i \Xi_i^{\leq 2n_i-1})^{-1} \psi_{\zeta_i}, \quad \widetilde{\psi}'_{\zeta_i} := M_{\zeta_i}^{-1} \widetilde{\psi}_{\zeta_i},
\end{aligned}$$

where ψ_{ζ_i} is the formal solution (4.7). We may describe $\widetilde{\Omega}_i$ as

$$\begin{aligned}
\widetilde{\Omega}'_{\zeta_i} = &\begin{pmatrix} \theta_{0,t_i} & 0 \\ 0 & \theta_{0,t_i} \end{pmatrix} \frac{d\zeta_i}{\zeta_i^{2n_i-1}} + \cdots + \begin{pmatrix} \theta_{2n_i-2,t_i} & 0 \\ 0 & \theta_{2n_i-2,t_i} \end{pmatrix} \frac{d\zeta_i}{\zeta_i} \\
&+ \begin{pmatrix} \theta_{2n_i-1,t_i} & 0 \\ 0 & -\theta_{2n_i-1,t_i} \end{pmatrix} d\zeta_i + \cdots + \begin{pmatrix} \theta_{4n_i-4,t_i} & 0 \\ 0 & \theta_{4n_i-4,t_i} \end{pmatrix} \zeta_i^{2n_i-3}\, dx \\
&+ O(\zeta_i^{2n_i-2}).
\end{aligned}$$

The residue part θ_{2n_i-2,t_i} of $\widetilde{\Omega}'_{\zeta_i}$ is constant on $\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$. So we have $\hat{\delta}(\theta_{2n_i-2,t_i}) = 0$ for any $\hat{\delta} \in \Theta_{\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}}$. Then the variation $\hat{\delta}_1(\widetilde{\Omega}'_{\zeta_i})$ is equal to

$$\begin{aligned}
&\begin{pmatrix} \hat{\delta}_1(\theta_{0,t_i}) & 0 \\ 0 & \hat{\delta}_1(\theta_{0,t_i}) \end{pmatrix} \frac{d\zeta_i}{\zeta_i^{2n_i-1}} + \cdots + \begin{pmatrix} \hat{\delta}_1(\theta_{2n_i-3,t_i}) & 0 \\ 0 & -\hat{\delta}_1(\theta_{2n_i-3,t_i}) \end{pmatrix} \frac{d\zeta_i}{\zeta_i^2} \\
&\quad + \begin{pmatrix} \hat{\delta}_1(\theta_{2n_i-1,t_i}) & 0 \\ 0 & -\hat{\delta}_1(\theta_{2n_i-1,t_i}) \end{pmatrix} d\zeta_i + \cdots \\
&\quad + \begin{pmatrix} \hat{\delta}_1(\theta_{4n_i-4,t_i}) & 0 \\ 0 & \hat{\delta}_1(\theta_{4n_i-4,t_i}) \end{pmatrix} \zeta_i^{2n_i-3}\, dx + O(\zeta_i^{2n_i-2}).
\end{aligned}$$

We define $\hat{\lambda}_{i,\pm}^{\leq 4n_i-4}(\zeta_i)$ as

$$\begin{aligned}
&\hat{\lambda}_{i,\pm}^{\leq 4n_i-4}(\zeta_i) \\
&\quad = (\pm 1)^0 \frac{\theta_{0,t_i}}{-2n_i+2} \zeta_i^{-2n_i+2} + \cdots + (\pm 1)^{2n_i-3} \frac{\theta_{2n_i-3,t_i}}{-1} \zeta_i^{-1} + \theta_{2n_i-2,t_i} \log \zeta_i \\
&\qquad + (\pm 1)^{2n_i-1} \theta_{2n_i-1,t_i} \zeta_i + \cdots + (\pm 1)^{4n_i-4} \frac{\theta_{4n_i-4,t_i}}{2n_i-2} \zeta_i^{2n_i-2}.
\end{aligned}$$

On the other hand, the variation $\hat{\delta}_2(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1}$ is equal to

$$\begin{aligned}
&\delta(g^{\leq 2n_i-1}_{\zeta_i})(g^{\leq 2n_i-1}_{\zeta_i})^{-1} \\
&\qquad + (g^{\leq 2n_i-1}_{\zeta_i}) \begin{pmatrix} -\hat{\delta}_2(\hat{\lambda}_{i,+}(\zeta_i)) & 0 \\ 0 & -\hat{\delta}_2(\hat{\lambda}_{i,-}(\zeta_i)) \end{pmatrix} (g^{\leq 2n_i-1}_{\zeta_i})^{-1} \\
&\qquad = \begin{pmatrix} -\hat{\delta}_2(\hat{\lambda}^{\leq 4n_i-4}_{i,+}(\zeta_i)) & 0 \\ 0 & -\hat{\delta}_2(\hat{\lambda}^{\leq 4n_i-4}_{i,-}(\zeta_i)) \end{pmatrix} + O(\zeta_i^{2n_i-1}).
\end{aligned}$$

Here we set

$$g^{\leq 2n_i-1}_{\zeta_i} := (\Phi_i \Xi_i^{\leq 2n_i-1} M_{\zeta_i})^{-1} \Phi_i \Xi_i M_{\zeta_i}.$$

Since

$$\begin{aligned}
&\hat{\delta}_2(\hat{\lambda}^{\leq 4n_i-4}_{i,\pm}(\zeta_i)) \\
&\quad = (\pm 1)^0 \frac{\hat{\delta}_2(\theta_{0,t_i})}{-2n_i+2}\zeta_i^{-2n_i+2} + \cdots + (\pm 1)^{2n_i-3}\frac{\hat{\delta}_2(\theta_{2n_i-3,t_i})}{-1}\zeta_i^{-1} \\
&\qquad + (\pm 1)^{2n_i-1}\hat{\delta}_2(\theta_{2n_i-1,t_i})\zeta_i + \cdots + (\pm 1)^{4n_i-4}\frac{\hat{\delta}_2(\theta_{4n_i-4,t_i})}{2n_i-2}\zeta_i^{2n_i-2},
\end{aligned}$$

we may check that the residue of $\mathrm{Tr}(\hat{\delta}_1(\widetilde{\Omega}'_{\zeta_i})\hat{\delta}_2(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1})$ at $\zeta_i = 0$ is equal to

$$\begin{aligned}
&\sum_{\substack{l' \in \{0,1,\dots,4n_i-4\} \\ \setminus \{2n_i-2\}}} 2\left(\hat{\delta}_1(\theta_{l',t_i})\frac{\hat{\delta}_2(\theta_{4(n_i-1)-l',t_i})}{2(n_i-1)-l'}\right) \\
&\quad = 2\left(\sum_{l'=0}^{2n_i-3}\hat{\delta}_1(\theta_{l',t_i})\frac{\hat{\delta}_2(\theta_{4(n_i-1)-l',t_i})}{2(n_i-1)-l'} - \sum_{l'=0}^{2n_i-3}\frac{\hat{\delta}_1(\theta_{4(n_i-1)-l',t_i})}{2(n_i-1)-l'}\hat{\delta}_2(\theta_{l',t_i})\right).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{4}\sum_{i\in I_{\mathrm{ra}}} \mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\hat{\delta}(\widetilde{\Omega}'_{\zeta_i}) \wedge \hat{\delta}(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1}) \\
&\qquad = \left(\sum_{i\in I_{\mathrm{ra}}}\sum_{l'=0}^{2n_i-3} dH_{\theta_{l',t_i}} \wedge d\theta_{l',t_i}\right)(\hat{\delta}_1, \hat{\delta}_2). \qquad (4.19)
\end{aligned}$$

Now we consider the difference between the residue after taking the gauge transformation and the residue before taking the gauge transformation. Put $g_{\zeta_i} := \Phi_i \Xi_i^{\leq 2n_i-1} M_{\zeta_i}$ and $g := \Phi_i \Xi_i^{\leq 2n_i-1}$. We consider the difference between the residues (4.18) and (4.19) when $\hat{\delta}_1 = \delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})}$:

$$\begin{aligned} &\mathrm{Tr}(\hat{\delta}(\widetilde{\Omega}'_{\zeta_i}) \wedge \hat{\delta}(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1}) - \mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)) \wedge \hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big) \\ &\quad = - \mathrm{Tr}(\delta_1(\widetilde{\Omega}'_{\zeta_i})\tilde{u}^{(2)} - \tilde{u}^{(1)}\hat{\delta}_2(\widetilde{\Omega}'_{\zeta_i})) \\ &\qquad - \mathrm{Tr}\big(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))u^{(2)} - u^{(1)}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_{i'}^2))\big) \\ &\qquad + \mathrm{Tr}\big(d(\psi_{\zeta_i}^{-1}u^{(1)}\hat{\delta}_2(\psi_{\zeta_i}) - \psi_{\zeta_i}^{-1}u^{(2)}\delta_1(\psi_{\zeta_i}))\big). \end{aligned} \tag{4.20}$$

Here we set $u^{(k)} := \hat{\delta}_k(g_{\zeta_i})g_{\zeta_i}^{-1}$ and $\tilde{u}^{(k)} := g_{\zeta_i}^{-1}\hat{\delta}_k(g_{\zeta_i})$ for $k = 1, 2$. We calculate the residue of the second term on the right-hand side of (4.20) at $\zeta_i = 0$. Since t_i $(i \in I_{\mathrm{ra}})$ is not a deformation parameter, $\hat{\delta}_k(t_i) = \hat{\delta}_k(\zeta_i) = 0$ for $k = 1, 2$. Then $u^{(k)}$ coincides with $\hat{\delta}_k(g)g^{-1}$. We expand g as (3.45). Since $g_0, \dots, g_{n_i-2}$ are parametrized by only $(T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}}$ and $\delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})}$, the variations $\delta_1(g_0), \dots, \delta_1(g_{n_i-2})$ vanish.

We will calculate the variation $\delta_1(g_{n_i-1})$. We consider the gauge transformation

$$(g_0 M_{\zeta_i})^{-1}\, d(g_0 M_{\zeta_i}) + (g_0 M_{\zeta_i})^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2)(g_0 M_{\zeta_i}). \tag{4.21}$$

The $\zeta_i^{-2n_i+1}$- and $\zeta_i^{-2n_i+2}$-terms of the expansion of (4.21) at $\zeta_i = 0$ are diagonal. The eigenvalues of the $\zeta_i^{-2n_i+2}$-term are distinct. The terms of this expansion after the $\zeta_i^{-2n_i+2}$-term are diagonalized by the gauge transformation by the right-hand side of (4.4). Since this negative part of (4.21) is independent of $(q_j, p_j)_{1 \le j \le n-3}$, we have that the coefficients of $\zeta_i \xi(\zeta_i)$ in the right-hand side of (4.4) are independent of $(q_j, p_j)_{1\le j\le n-3}$ until the $\zeta_i^{2n_i-3}$-term. So the $(2,1)$-entry of $g_0^{-1}g_{n_i-1}$ is independent of $(q_j, p_j)_{1\le j\le n-3}$. Since $\delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})}$ and the $(2,1)$-entry of $g_0^{-1}g_{n_i-1}$ is independent of $(q_j, p_j)_{1\le j\le n-3}$, we have that the $(2,1)$-entry of $g_0^{-1}\delta_1(g_{n_i-1})$ vanishes. Moreover, we have that $\mathrm{Tr}(g_0^{-1}g_{n_i-1})$ is constant. By comparing the $x_{t_i}^{-1}$-terms of the expansions of both sides of

$$g_0^{-1}g\widetilde{\Omega}_i = g_0^{-1}\, dg + g_0^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)} g_0 (g_0^{-1}g),$$

we have the equality

$$\begin{aligned} &g_0^{-1}g_{n_i-1}\begin{pmatrix} \theta_{0,t_i}/2 & \theta_{1,t_i}/2 \\ 0 & \theta_{0,t_i}/2 \end{pmatrix} + \cdots + g_0^{-1}g_0\begin{pmatrix} \theta_{2n_i-2,t_i}/2 & \theta_{2n_i-1,t_i}/2 \\ \theta_{2n_i-3,t_i}/2 & \theta_{2n_i-2,t_i}/2 \end{pmatrix} \\ &\quad = \begin{pmatrix} \theta_{0,t_i}/2 & \theta_{1,t_i}/2 \\ 0 & \theta_{0,t_i}/2 \end{pmatrix} g_0^{-1}g_{n_i-1} + \cdots + g_0^{-1}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})_{n_i-1}g_0 g_0^{-1}g_0. \end{aligned} \tag{4.22}$$

We consider the variations of the $x_{t_i}^{-1}$-terms of both sides of (4.22). In particular, we focus on the $(1,2)$-entries of both sides. Since $\delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}} \times T_{\boldsymbol{\theta}})}$ and

$\mathrm{Tr}(g_0^{-1}g_{n_i-1})$ is constant, we have explicit descriptions of the $(1,1)$-entry and the $(2,2)$-entry of $g_0^{-1}\delta_1(g_{n_i-1})$. So we have

$$g_0^{-1}\delta_1(g_{n_i-1}) = \begin{pmatrix} -\frac{\delta_1(\theta_{2n_i-1,t_i})}{2\theta_{1,t_i}} & * \\ 0 & \frac{\delta_1(\theta_{2n_i-1,t_i})}{2\theta_{1,t_i}} \end{pmatrix}. \tag{4.23}$$

Since $\delta_1(g_0),\dots,\delta_1(g_{n_i-2})$ vanish, we have

$$g_0^{-1}\delta_1(g) = \begin{pmatrix} -\frac{\delta_1(\theta_{2n_i-1,t_i})}{2\theta_{1,t_i}} & * \\ 0 & \frac{\delta_1(\theta_{2n_i-1,t_i})}{2\theta_{1,t_i}} \end{pmatrix} (x-t_i)^{n_i-1} + O((x-t_i)^{n_i}). \tag{4.24}$$

On the other hand, $\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})$ has the expansion

$$\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}) = \begin{pmatrix} 0 & -\frac{\hat{\delta}_2(\prod_{j\neq i}(t_i-t_j)^{n_j})}{\prod_{j\neq i}(t_i-t_j)^{2n_j}} \\ -\frac{1}{4}\hat{\delta}_2(\theta_{0,t_i}^2\prod_{j\neq i}(t_i-t_j)^{n_j}) & \hat{\delta}_2(\theta_{0,t_i}) \end{pmatrix} \frac{1}{(x-t_i)^{n_i}} + [\text{higher-order terms}].$$

Now we take a compatible framing g_0 as

$$g_0 = \begin{pmatrix} 1 & \frac{\theta_{0,t_i}}{2}\prod_{j\neq i}(t_i-t_j)^{n_j} \\ \frac{\theta_{0,t_i}}{2}\prod_{j\neq i}(t_i-t_j)^{n_j} & (\frac{\theta_{0,t_i}^2}{4}\prod_{j\neq i}(t_i-t_j)^{n_j}+\frac{\theta_{1,t_i}}{2})\prod_{j\neq i}(t_i-t_j)^{n_j} \end{pmatrix}. \tag{4.25}$$

Then the leading term of $g_0^{-1}\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}g_0$ coincides with the leading term of the right-hand side of (4.1). We have the following expansion of $g_0^{-1}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})g_0$ at t_i:

$$\begin{pmatrix} -\frac{\theta_{0,t_i}}{2}\frac{\hat{\delta}_2(\prod_{j\neq i}(t_i-t_j)^{n_j})}{\prod_{j\neq i}(t_i-t_j)^{n_j}} & * \\ 0 & \frac{\theta_{0,t_i}}{2}\frac{\hat{\delta}_2(\prod_{j\neq i}(t_i-t_j)^{n_j})}{\prod_{j\neq i}(t_i-t_j)^{n_j}}+\hat{\delta}_2(\theta_{0,t_i}) \end{pmatrix} \frac{1}{(x-t_i)^{n_i}} + [\text{higher-order terms}]. \tag{4.26}$$

Since $\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}_2(g)g^{-1}$ is holomorphic at t_i, we have the equalities

$$\begin{aligned}
&-\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))u^{(2)} - u^{(1)}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\big) \\
&\quad = -\frac{1}{2}\mathrm{res}_{x=t_i}\,\mathrm{Tr}\big(\delta_1(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\hat{\delta}_2(g)g^{-1} - \delta_1(g)g^{-1}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})\big) \\
&\quad = \frac{1}{2}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta_1(g)g^{-1}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})) \\
&\quad = \frac{1}{2}\mathrm{res}_{x=t_i}\,\mathrm{Tr}\big((g_0^{-1}\delta_1(g))g^{-1}g_0(g_0^{-1}\hat{\delta}_2(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)})g_0)\big)
\end{aligned}$$

$$= \frac{\delta_1(\theta_{2n_i-1,t_i})}{4\theta_{1,t_i}}\hat{\delta}_2(\theta_{0,t_i}) + 2\frac{\delta_1(\theta_{2n_i-1,t_i})}{8}\frac{\theta_{0,t_i}}{\theta_{1,t_i}}\frac{\hat{\delta}_2(\prod_{j\neq i}(t_i-t_j)^{n_j})}{\prod_{j\neq i}(t_i-t_j)^{n_j}}$$

$$= \frac{\delta_1(\theta_{2n_i-1,t_i})}{4}\frac{\hat{\delta}_2(\theta_{0,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j})}{\theta_{1,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j}}. \tag{4.27}$$

Remark that $\delta_1 \in \Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$. The fourth equality follows from equalities (4.24) and (4.26).

Next we calculate the residue of the first term on the right-hand side of (4.20) at $\zeta_i = 0$. We calculate this residue as

$$\begin{aligned}
&-\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\delta_1(\widetilde{\Omega}'_{\zeta_i})\tilde{u}^{(2)} - \tilde{u}^{(1)}\hat{\delta}_2(\widetilde{\Omega}'_{\zeta_i}))\\
&\quad= -\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\delta_1(\widetilde{\Omega}'_{\zeta_i})M_{\zeta_i}^{-1}g^{-1}\hat{\delta}_2(gM_{\zeta_i}) - M_{\zeta_i}^{-1}g^{-1}\hat{\delta}_2(gM_{\zeta_i})\hat{\delta}_2(\widetilde{\Omega}'_{\zeta_i}))\\
&\quad= -\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\delta_1(M_{\zeta_i}\widetilde{\Omega}'_{\zeta_i}M_{\zeta_i}^{-1})g^{-1}\hat{\delta}_2(g) - g^{-1}\hat{\delta}_2(g)\hat{\delta}_2(M_{\zeta_i}\widetilde{\Omega}'_{\zeta_i}M_{\zeta_i}^{-1}))\\
&\quad= -2\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta_1(\widetilde{\Omega}_i)g^{-1}\hat{\delta}_2(g) - g^{-1}\delta_1(g)\hat{\delta}_2(\widetilde{\Omega}_i)).
\end{aligned}$$

Here, the last equality follows from $\delta_1(M_{\zeta_i}^{-1}\,dM_{\zeta_i}) = \hat{\delta}_2(M_{\zeta_i}^{-1}\,dM_{\zeta_i}) = 0$. The coefficients of the expansion of $\delta_1(\widetilde{\Omega}_i)$ at $x = t_i$ vanish until the $x_{t_i}^{-2}$-term. The $(1,2)$-entry of the $x_{t_i}^{-1}$-term of $\widetilde{\Omega}_i$ depends on $(q_j,p_j)_{1\leq j\leq n-3}$ and the other entries of the $x_{t_i}^{-1}$-term of $\widetilde{\Omega}_i$ are independent of $(q_j,p_j)_{1\leq j\leq n-3}$. The $(1,2)$-entry of the $x_{t_i}^{-1}$-term of $\delta_1(\widetilde{\Omega}_i)$ is $\delta_1(\theta_{2n_i-1,t_i})/2$ and the other entries are zero. On the other hand, the $(2,1)$-entry of the leading coefficient of $g^{-1}\hat{\delta}_2(g)$ is

$$\frac{\hat{\delta}_2(\theta_{0,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j})}{(\theta_{1,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j})},$$

since we set g_0 as (4.25). Then the residue of $\mathrm{Tr}(\delta_1(\widetilde{\Omega}_i)g^{-1}\hat{\delta}_2(g))\,dx$ at t_i is

$$\frac{\delta_1(\theta_{2n_i-1,t_i})}{2}\frac{\hat{\delta}_2(\theta_{0,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j})}{\theta_{1,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j}}.$$

We consider the residue of $\mathrm{Tr}(g^{-1}\delta_1(g)\hat{\delta}_2(\widetilde{\Omega}_i))$ at t_i. By (4.23), the residue of $\mathrm{Tr}(g^{-1}\delta_1(g)\hat{\delta}_2(\widetilde{\Omega}_i))$ at t_i vanishes. Then we have

$$\begin{aligned}
&-\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\delta_1(\widetilde{\Omega}'_{\zeta_i})\tilde{u}^{(2)} - \tilde{u}^{(1)}\hat{\delta}_2(\widetilde{\Omega}'_{\zeta_i}))\\
&\quad= -\frac{1}{2}\mathrm{res}_{x=t_i}\,\mathrm{Tr}(\delta_1(\widetilde{\Omega}_i)g^{-1}\hat{\delta}_2(g) - g^{-1}\delta_1(g)\hat{\delta}_2(\widetilde{\Omega}_i))\\
&\quad= -\frac{\delta_1(\theta_{2n_i-1,t_i})}{4}\frac{\hat{\delta}_2(\theta_{0,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j})}{\theta_{1,t_i}\prod_{j\neq i}(t_i-t_j)^{n_j}}.
\end{aligned} \tag{4.28}$$

By combining (4.20), (4.27), and (4.28), we have

$$\begin{aligned}
&\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big)\\
&\quad=\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\hat{\delta}(\widetilde{\Omega}'_{\zeta_i})\wedge\hat{\delta}(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1})\\
&\qquad-\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(d(\psi_{\zeta_i}^{-1}u^{(1)}\hat{\delta}_2(\psi_{\zeta_i})-\psi_{\zeta_i}^{-1}u^{(2)}\delta_1(\psi_{\zeta_i}))\big)\\
&\quad=\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}(\hat{\delta}(\widetilde{\Omega}'_{\zeta_i})\wedge\hat{\delta}(\widetilde{\psi}'_{\zeta_i})(\widetilde{\psi}'_{\zeta_i})^{-1}).
\end{aligned}$$

By equality (4.19) we have

$$\frac{1}{4}\mathrm{res}_{\zeta_i=0}\,\mathrm{Tr}\big(\hat{\delta}(\widehat{\Omega}^{(n-2)}_{(\boldsymbol{t}_{\mathrm{ra}},\boldsymbol{\theta}_0)}(\zeta_i^2))\wedge\hat{\delta}(\psi_{\zeta_i})\psi_{\zeta_i}^{-1}\big)=\Big(\sum_{i\in I_{\mathrm{ra}}}\sum_{l'=0}^{2n_i-3}dH_{\theta_{l',t_i}}\wedge d\theta_{l',t_i}\Big)(\hat{\delta}_1,\hat{\delta}_2)$$

when $\delta_1\in\Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$. So we have $\widehat{\omega}(\delta_1,\hat{\delta}_2)-\widehat{\omega}'(\delta_1,\hat{\delta}_2)=0$ when $\delta_1\in\Theta_{(\widehat{\mathcal{M}}_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})/((T_{\boldsymbol{t}})_{\boldsymbol{t}_{\mathrm{ra}}}\times T_{\boldsymbol{\theta}})}$. Then we obtain the assertion of this theorem. □

By Theorems 4.7, 4.8, and 4.10, we obtain the following corollary:

Corollary 4.11. *Set* $\eta_j:=\frac{p_j}{P(q_j;\boldsymbol{t})}-\sum_{i=1}^{\nu}\frac{D_i(q_j;\boldsymbol{t},\boldsymbol{\theta})}{(q_j-t_i)^{n_i}}-D_\infty(q_j;\boldsymbol{t},\boldsymbol{\theta})$. *The vector fields* $\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}$ *(*$i\in I_{\mathrm{un}}$ *and* $l=0,1,\dots,n_i-2$*),* $\delta^{\mathrm{IMD}}_{t_i}$ *(*$i\in\{3,4,\dots,\nu\}\cap(I_{\mathrm{reg}}\cup I_{\mathrm{un}})$*), and* $\delta_{\theta_{l',t_i}}$ *(*$i\in I_{\mathrm{ra}}$ *and* $l'=0,1,\dots,2n_i-3$*) have the Hamiltonian descriptions*

$$\begin{aligned}
\delta^{\mathrm{IMD}}_{\theta^{\pm}_{l,t_i}}&=\frac{\partial}{\partial\theta^{\pm}_{l,t_i}}-\sum_{j=1}^{n-3}\Big(\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{\theta^{\pm}_{l,t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\Big),\\
\delta^{\mathrm{IMD}}_{t_i}&=\frac{\partial}{\partial t_i}-\sum_{j=1}^{n-3}\Big(\frac{\partial H_{t_i}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{t_i}}{\partial q_j}\frac{\partial}{\partial\eta_j}\Big),\\
\delta^{\mathrm{IMD}}_{\theta_{l',t_i}}&=\frac{\partial}{\partial\theta_{l',t_i}}-\sum_{j=1}^{n-3}\Big(\frac{\partial H_{\theta_{l',t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{\theta_{l',t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\Big).
\end{aligned}$$

§5. Examples

§5.1. Example ($\nu=2$, $n_1=n_2=n_\infty=2$)

We consider the connection $d+\Omega$ on $\mathcal{O}\oplus\mathcal{O}(4)$ with the connection matrix

$$\Omega=\begin{pmatrix}0&\frac{1}{P(x)}\\c_0(x)&d_0(x)\end{pmatrix}dx.$$

Here we put $P(x) \coloneqq x^2(x-1)^2$,

$$c_0(x) \coloneqq \frac{C_0^{(0)} + C_0^{(1)}x}{x^2} + \frac{C_1^{(0)} + C_1^{(1)}(x-1)}{(x-1)^2} + \sum_{j=1}^{3} \frac{p_j}{x-q_j} + \widetilde{C}^{(0)} + \widetilde{C}^{(1)}x + \widetilde{C}^{(2)}x^2 + C_\infty^{(0)}x^3 + C_\infty^{(1)}x^4,$$

$$td_0(z) \coloneqq \frac{D_0^{(0)} + D_0^{(1)}x}{x^2} + \frac{D_1^{(0)} + D_1^{(1)}(x-1)}{(x-1)^2} + \sum_{j=1}^{n-3} \frac{-1}{x-q_j} + D_\infty^{(0)}.$$

We set $t_1 \coloneqq 0$, $t_2 \coloneqq 1$, and $t_\infty \coloneqq \infty$. The polar divisor of the connection $d+\Omega$ is $2\cdot t_1 + 2\cdot t_2 + 2\cdot t_\infty + q_1 + q_2 + q_3$. We assume that the leading coefficients $\Omega_{t_i}(0)$ are semi-simple for $i = 1, 2, \infty$. We put $x_{t_i} \coloneqq x - t_i$ for $i = 1, 2$ and $x_{t_\infty} = w$. We fix the formal type of the negative part of $d+\Omega$ for each t_i. That is, we fix $\theta^{\pm}_{l,t_i}$ for $l = 0, 1$ and $i = 1, 2, \infty$, and the negative part of $d+\Omega$ for each t_i is diagonalizable as

$$\frac{\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix}}{x_{t_i}^2} + \frac{\begin{pmatrix} \theta^+_{1,t_i} & 0 \\ 0 & \theta^-_{1,t_i} \end{pmatrix}}{x_{t_i}}$$

by a formal transformation (see Section 2.3). Then the coefficients of c_2 and d_2 are determined as

$$\begin{cases} C_0^{(0)} + C_0^{(1)}x = -\theta^+_{0,0}\theta^-_{0,0} + (2\theta^+_{0,0}\theta^-_{0,0} - \theta^+_{0,0}\theta^-_{1,0} - \theta^-_{0,0}\theta^+_{1,0})x, \\ D_0^{(0)} + D_0^{(1)}x = \theta^+_{0,0} + \theta^-_{0,0} + (\theta^+_{1,0} + \theta^-_{1,0})x, \end{cases} \tag{5.1}$$

$$\begin{cases} C_1^{(0)} + C_1^{(1)}(x-1) = -\theta^+_{0,1}\theta^-_{0,1} - (2\theta^+_{0,1}\theta^-_{0,1} + \theta^+_{0,1}\theta^-_{1,1} + \theta^-_{0,1}\theta^+_{1,1})(x-1), \\ D_1^{(0)} + D_1^{(1)}(x-1) = \theta^+_{0,1} + \theta^-_{0,1} + (\theta^+_{1,1} + \theta^-_{1,1})(x-1), \end{cases} \tag{5.2}$$

$$\begin{cases} C_\infty^{(0)} + C_\infty^{(1)}x = 2\theta^+_{0,\infty}\theta^-_{0,\infty} - \theta^-_{0,\infty}\theta^+_{1,\infty} - \theta^+_{0,\infty}\theta^-_{1,\infty} - (\theta^+_{0,\infty}\theta^-_{0,\infty})x, \\ D_\infty^{(0)} = -\theta^+_{0,\infty} - \theta^-_{0,\infty}. \end{cases} \tag{5.3}$$

Moreover, we assume that q_1, q_2, and q_3 are apparent singularities. We define $\widetilde{C}_{q_j}$ for $j = 1, 2, 3$ so that $\widetilde{C}^{(0)} + \widetilde{C}^{(1)}x + \widetilde{C}^{(2)}x^2$ is equal to

$$\widetilde{C}_{q_1}(x-q_2)(x-q_3) + \widetilde{C}_{q_2}(x-q_1)(x-q_3) + \widetilde{C}_{q_3}(x-q_1)(x-q_2). \tag{5.4}$$

Since q_1, q_2, and q_3 are apparent singularities, we have

$$\widetilde{C}_{q_j} = \frac{1}{Q'(q_j)}\Bigg(\frac{p_j^2}{q_j^2(q_j-1)^2} - \sum_{i=1,2} \frac{D_i(q_j)p_j + C_i(q_j)}{(q_j-t_i)^2} + \sum_{k\in\{1,2,3\}\setminus\{j\}} \frac{p_j - p_k}{q_j - q_k} - D_\infty^{(0)}p_j - C_\infty^{(0)}q_j^3 - C_\infty^{(1)}q_j^4\Bigg) \tag{5.5}$$

for $j=1,2,3$, where we put $Q(x) := (x-q_1)(x-q_2)(x-q_3)$. We determine the matrices Φ_i and Ξ_i as in Lemma 3.1:

$$\Phi_0 = \begin{pmatrix} 1 & \frac{1}{\theta^-_{0,0}} \\ \theta^+_{0,0} & 1 \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 1 & \frac{1}{\theta^-_{0,1}} \\ \theta^+_{0,1} & 1 \end{pmatrix}, \quad \Phi_\infty = \begin{pmatrix} 1 & -\frac{1}{\theta^-_{0,\infty}} \\ -\theta^+_{0,\infty} & 1 \end{pmatrix},$$

$$\Xi^{(0)}_1 = \begin{pmatrix} 0 & -\frac{2\theta^-_{0,0}-\theta^-_{1,0}}{(\theta^+_{0,0}-\theta^-_{0,0})\theta^-_{0,0}} \\ \frac{(2\theta^+_{0,0}-\theta^+_{1,0})\theta^-_{0,0}}{\theta^+_{0,0}-\theta^-_{0,0}} & 0 \end{pmatrix}, \qquad \Xi^{(0)}_2 = \begin{pmatrix} 0 & (\xi^{(2)}_0)_{12} \\ (\xi^{(2)}_0)_{21} & 0 \end{pmatrix},$$

$$\Xi^{(1)}_1 = \begin{pmatrix} 0 & \frac{2\theta^-_{0,1}+\theta^-_{1,1}}{(\theta^+_{0,1}-\theta^-_{0,1})\theta^-_{0,1}} \\ -\frac{(2\theta^+_{0,1}+\theta^+_{1,1})\theta^-_{0,1}}{\theta^+_{0,1}-\theta^-_{0,1}} & 0 \end{pmatrix}, \qquad \Xi^{(1)}_2 = \begin{pmatrix} 0 & (\xi^{(2)}_1)_{12} \\ (\xi^{(2)}_1)_{21} & 0 \end{pmatrix},$$

$$\Xi^{(\infty)}_1 = \begin{pmatrix} 0 & \frac{2\theta^-_{0,\infty}-\theta^-_{1,\infty}}{(\theta^+_{0,\infty}-\theta^-_{0,\infty})\theta^-_{0,\infty}} \\ -\frac{(2\theta^+_{0,\infty}-\theta^+_{1,\infty})\theta^-_{0,\infty}}{\theta^+_{0,\infty}-\theta^-_{0,\infty}} & 0 \end{pmatrix}, \qquad \Xi^{(\infty)}_2 = \begin{pmatrix} 0 & (\xi^{(2)}_1)_{12} \\ (\xi^{(2)}_1)_{21} & 0 \end{pmatrix}.$$

Here, the descriptions of $(\xi^{(2)}_i)_{12}$ and $(\xi^{(2)}_i)_{21}$ are omitted. Set

$$\Xi^{\leq 2}_i := \mathrm{id} + \Xi^{(i)}_1 x_{t_i} + \Xi^{(i)}_2 x^2_{t_i}.$$

Let $\theta^\pm_{2,t_i}$ $(i=1,2,\infty)$ be the coefficient as in Lemma 3.1. That is,

$$\begin{aligned}(\Phi_i\Xi^{\leq 2}_i)^{-1}\, d(\Phi_i\Xi^{\leq 2}_i) &+ (\Phi_i\Xi^{\leq 2}_i)^{-1}\Omega(\Phi_i\Xi^{\leq 2}_i) \\ &= \frac{\begin{pmatrix} \theta^+_{0,t_i} & 0 \\ 0 & \theta^-_{0,t_i} \end{pmatrix}}{x^2_{t_i}} + \frac{\begin{pmatrix} \theta^+_{1,t_i} & 0 \\ 0 & \theta^-_{1,t_i} \end{pmatrix}}{x_{t_i}} + \begin{pmatrix} \theta^+_{2,t_i} & 0 \\ 0 & \theta^-_{2,t_i} \end{pmatrix} + O(x_{t_i})\end{aligned}$$

for $i=0,1,\infty$. Remark that $\Xi^{\leq 2}_i$ is degree 2 in x_{t_i}. This degree is sufficient to define Hamiltonians since there is no parameter corresponding to the positions of irregular singularities. By equations (5.1), (5.2), (5.3), (5.4), and (5.5), we can determine the Hamiltonians $H_{\theta^\pm_{0,0}}$, $H_{\theta^\pm_{0,1}}$, and $H_{\theta^\pm_{0,\infty}}$ as

$$\begin{aligned} H_{\theta^\pm_{0,0}} := -\theta^\pm_{2,0} = \frac{-1}{\theta^\pm_{0,0}-\theta^\mp_{0,0}} \Big(& \theta^\pm_{0,0}\theta^\mp_{0,0} + \theta^\pm_{1,0}\theta^\mp_{1,0} - 2(\theta^\pm_{0,0}\theta^\mp_{1,0} + \theta^\pm_{1,0}\theta^\mp_{0,0}) \\ & + \widetilde{C}^{(0)} + (C^{(0)}_1 - C^{(1)}_1) + (D^{(0)}_1 - D^{(1)}_1 + D^{(0)}_\infty)\theta^\pm_{0,0} \\ & - \frac{p_1-\theta^\pm_{0,0}}{q_1} - \frac{p_2-\theta^\pm_{0,0}}{q_2} - \frac{p_3-\theta^\pm_{0,0}}{q_3} \Big), \end{aligned}$$

$$H_{\theta^{\pm}_{0,1}} := -\theta^{\pm}_{2,1} = \frac{-1}{\theta^{\pm}_{0,1}-\theta^{\mp}_{0,1}}\Big(\theta^{\pm}_{0,1}\theta^{\mp}_{0,1}+\theta^{\pm}_{1,1}\theta^{\mp}_{1,1}+2(\theta^{\pm}_{0,1}\theta^{\mp}_{1,1}+\theta^{\pm}_{1,1}\theta^{\mp}_{0,1}) + (\widetilde{C}^{(0)}+\widetilde{C}^{(1)}+\widetilde{C}^{(2)})+(C_0^{(0)}+C_0^{(1)}) + (C_\infty^{(0)}+C_\infty^{(1)})+(D_0^{(0)}+D_0^{(1)}+D_\infty^{(0)})\theta^{\pm}_{0,1} - \frac{p_1-\theta^{\pm}_{0,1}}{q_1-1}-\frac{p_2-\theta^{\pm}_{0,1}}{q_2-1}-\frac{p_3-\theta^{\pm}_{0,1}}{q_3-1}\Big),$$

$$H_{\theta^{\pm}_{0,\infty}} := -\theta^{\pm}_{2,\infty} = \frac{-1}{\theta^{\pm}_{0,\infty}-\theta^{\mp}_{0,\infty}}\Big(\theta^{\pm}_{0,\infty}\theta^{\mp}_{0,\infty}+\theta^{\pm}_{1,\infty}\theta^{\mp}_{1,\infty}-2(\theta^{\pm}_{0,\infty}\theta^{\mp}_{1,\infty}+\theta^{\pm}_{1,\infty}\theta^{\mp}_{0,\infty}) + \widetilde{C}^{(2)}-(D_0^{(0)}+D_1^{(0)}+D_1^{(1)})\theta^{\pm}_{0,\infty} + (q_1+q_2+q_3)\theta^{\pm}_{0,\infty}\Big).$$

Set

$$\eta_j := \frac{p_j}{q_j^2(q_j-1)^2}-\frac{D_0^{(0)}+D_0^{(1)}q_j}{q_j^2}-\frac{D_1^{(0)}+D_1^{(1)}(q_j-1)}{(q_j-1)^2}-D_\infty^{(0)}$$

for $j=1,2,3$. By Corollary 3.15, the vector field determined by the generalized isomonodromic deformations is described as

$$\frac{\partial}{\partial\theta^{\pm}_{0,t_i}}-\sum_{j=1}^{3}\Big(\frac{\partial H_{\theta^{\pm}_{0,t_i}}}{\partial\eta_j}\frac{\partial}{\partial q_j}-\frac{\partial H_{\theta^{\pm}_{0,t_i}}}{\partial q_j}\frac{\partial}{\partial\eta_j}\Big).$$

§5.2. Example corresponding to Kimura's $L(9/2;2)$

In this section we consider Kimura's family $L(9/2;2)$ of rank-two linear differential equations in [15, p. 37]. We describe the corresponding global normal form (see [5, Sect. 6]) and consider the integrable deformations of the family given by the global normal form. Then we can reproduce Kimura's Hamiltonian $H(9/2)$ from [15, p. 40].

Let D be the effective divisor defined as $D=5\cdot\infty$. We consider the connection $d+\Omega^{(\infty)}$ on $\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(3)$ with

$$\Omega^{(\infty)} = \begin{pmatrix} 0 & -\frac{1}{w^5} \\ c_0^{(\infty)}(w) & d_0^{(\infty)}(w)\end{pmatrix} dw,$$

where

$$c_0^{(\infty)}(w) := -\frac{9}{w^4}-\frac{9t_1}{w^2}-\frac{3t_2}{w}-3K_2-3K_1w-\sum_{i=1}^{2}\frac{p_iw^2}{1-q_iw}, \qquad (5.6)$$
$$d_0^{(\infty)}(w) := \sum_{i=1}^{2}\frac{1}{w(1-q_iw)}-\frac{3}{w}$$

(see [5, Sect. 6]). The polar divisor is $D + q_1 + q_2$. Assume that $w = 1/q_1$ and $w = 1/q_2$ are apparent singularities. Then we can determine K_1 and K_2 as rational functions whose variables are t_1, t_2, q_1, q_2, p_1, p_2.

If we set $\Phi_\infty := \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ and

$$\begin{aligned}\Xi_\infty^{\leq 6} := \mathrm{id} &+ \begin{pmatrix} -\frac{t_1}{4} & 0 \\ 0 & \frac{t_1}{4} \end{pmatrix} w^2 + \begin{pmatrix} -\frac{t_2}{12} & \frac{1}{24} \\ 0 & \frac{t_2}{12} \end{pmatrix} w^3 + \begin{pmatrix} \frac{t_1^2}{8} - \frac{K_2}{12} & -\frac{q_1+q_2}{12} \\ -\frac{1}{24} & -\frac{t_1^2}{8} + \frac{K_2}{12} \end{pmatrix} w^4 \\ &+ \begin{pmatrix} \frac{t_1 t_2}{12} - \frac{K_1}{12} & 0 \\ \frac{q_1+q_2}{12} & -\frac{t_1 t_2}{12} + \frac{K_1}{12} \end{pmatrix} w^5 + \begin{pmatrix} * & * \\ * & * \end{pmatrix} w^6, \end{aligned} \tag{5.7}$$

then we have

$$\begin{aligned}\widetilde{\Omega}_\infty = \frac{\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}}{w^5} &+ \frac{\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}}{w^4} + \frac{\begin{pmatrix} 0 & \frac{3t_1}{2} \\ 0 & 0 \end{pmatrix}}{w^3} + \frac{\begin{pmatrix} 0 & \frac{t_2}{2} \\ \frac{3t_1}{2} & 0 \end{pmatrix}}{w^2} + \frac{\begin{pmatrix} -\frac{1}{4} & b_3 \\ \frac{t_2}{2} & -\frac{1}{4} - \frac{1}{2} \end{pmatrix}}{w} \\ &+ \begin{pmatrix} a_1 & b_4 \\ b_3 & a_1 \end{pmatrix} + O(w)^2, \end{aligned} \tag{5.8}$$

where $\widetilde{\Omega}_\infty\, dw := (\Phi_\infty \Xi_\infty^{\leq 6})^{-1} \Omega^{(\infty)} (\Phi_\infty \Xi_\infty^{\leq 6}) + (\Phi_\infty \Xi_\infty^{\leq 6})^{-1} d(\Phi_\infty \Xi_\infty^{\leq 6})$. We have

$$a_1 = \frac{q_1}{2} + \frac{q_2}{2}, \quad b_3 = -\frac{3t_1^2}{8} + \frac{K_2}{2}, \quad b_4 = -\frac{t_1 t_2}{4} + \frac{K_1}{2}.$$

After ramification $w = \zeta^2$ and the following transformation of $\widetilde{\Omega}_\infty$:

$$\widetilde{\Omega}'_\zeta\, d\zeta := M_\zeta^{-1} (\widetilde{\Omega}_\infty\, dw) M_\zeta + M_\zeta^{-1}\, dM_\zeta, \quad \text{where } M_\zeta = \begin{pmatrix} 1 & 1 \\ \zeta & -\zeta \end{pmatrix},$$

we have an unramified irregular singular point with matrix connection

$$\begin{aligned}\widetilde{\Omega}'_\zeta = \frac{\begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}}{\zeta^8} &+ \frac{\begin{pmatrix} 3t_1 & 0 \\ 0 & -3t_1 \end{pmatrix}}{\zeta^4} + \frac{\begin{pmatrix} t_2 & 0 \\ 0 & -t_2 \end{pmatrix}}{\zeta^2} + \frac{\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}}{\zeta} \\ &+ \begin{pmatrix} 2b_3 & 0 \\ 0 & -2b_3 \end{pmatrix} + \begin{pmatrix} 2a_1 & 0 \\ 0 & 2a_1 \end{pmatrix} \zeta + \begin{pmatrix} 2b_4 & 0 \\ 0 & -2b_4 \end{pmatrix} \zeta^2 + O(\zeta)^3. \end{aligned}$$

We define Hamiltonians

$$\begin{aligned} H_1 &:= -\Big[\text{the coefficient of the } \zeta^3\text{-term of } \textstyle\sum_{k=9}^\infty \theta_{k,\infty} \int \zeta^{k-9}\, d\zeta\Big] \\ &= -\frac{2b_4}{3} = -\frac{K_1}{3} + \frac{t_1 t_2}{6}, \\ H_2 &:= -\Big[\text{the coefficient of the } \zeta\text{-term of } \textstyle\sum_{k=9}^\infty \theta_{k,\infty} \int \zeta^{k-9}\, d\zeta\Big] \\ &= -2b_3 = -K_2 + \frac{3t_1^2}{4}. \end{aligned}$$

Then the 2-form $\widehat{\omega}'$ defined in Theorem 4.10 is described as

$$\begin{aligned}\widehat{\omega}' &= \sum_{i=1,2} dp_i \wedge dq_i + dH_1 \wedge d(3t_1) + dH_2 \wedge d(t_2)\\ &= -\Big(\sum_{i=1,2} d\eta_i \wedge dq_i - d(3H_1) \wedge dt_1 - dH_2 \wedge dt_2\Big)\\ &= -\Big(\sum_{i=1,2} d\eta_i \wedge dq_i + dK_1 \wedge dt_1 + dK_2 \wedge dt_2 - t_1\, dt_1 \wedge dt_2\Big), \qquad (5.9)\end{aligned}$$

where $\eta_i \coloneqq -p_i$ for $i = 1, 2$. By Theorems 4.7 and 4.10, the vector field determined by the integrable deformations is described as

$$\frac{\partial}{\partial t_i} - \sum_{j=1}^{2}\Big(\frac{\partial K_i}{\partial \eta_j}\frac{\partial}{\partial q_j} - \frac{\partial K_i}{\partial q_j}\frac{\partial}{\partial \eta_j}\Big)$$

for $i = 1, 2$. This description is given in [15].

Remark. We may check that $\widehat{\omega} = \widehat{\omega}'$ by the calculation of the right-hand side of (3.25) for (5.6), (5.7), and (5.8). Then the 2-form (5.9) is the isomonodromy 2-form. In fact, we may check the equality

$$\frac{\partial(3H_1)}{\partial t_2} - \frac{\partial H_2}{\partial t_1} - \sum_{i=1,2}\Big(\frac{\partial(3H_1)}{\partial p_i}\frac{\partial H_2}{\partial q_i} - \frac{\partial(3H_1)}{\partial q_i}\frac{\partial H_2}{\partial p_i}\Big) = 0$$

directly.

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