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BC_2-Type Multivariable Matrix Functions and Matrix Spherical Functions

by

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Abstract

Matrix spherical functions associated to the compact symmetric pair $(\mathrm{SU}(m+2), \mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(m)))$, $m \geq 2$, having a reduced root system of type BC_2, are studied. We consider an irreducible K-representation (π, V) arising from the $\mathrm{U}(2)$-part of K, and the induced representation $\mathrm{Ind}_K^G \pi$ splits multiplicity-free. The corresponding spherical functions, i.e. $\Phi\colon G \to \mathrm{End}(V)$ satisfying $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$, are studied by examining certain leading terms which involve hypergeometric functions. This is done explicitly using the action of the radial part of the Casimir operator on these functions and their leading terms. To suitably grouped matrix spherical functions we associate two-variable matrix orthogonal polynomials giving a matrix analogue of Koornwinder's 1970s two-variable orthogonal polynomials, which are Heckman–Opdam polynomials for BC_2. In particular, we find explicit orthogonality relations with the matrix polynomials being eigenfunctions to an explicit second-order matrix partial differential operator. The scalar part of the matrix weight is less general than Koornwinder's weight.

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§1. Introduction

Spherical functions on compact symmetric spaces and orthogonal polynomials have been known to be closely related ever since the work of É. Cartan; see e.g. [6, 10, 38]. The notion of a spherical function taking values in a matrix algebra goes back to the initial introduction of the notion of spherical functions; see e.g. [6, Introduction] and references given there. In the case of a matrix spherical

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function for a compact symmetric space of rank one, there is a connection to matrix orthogonal polynomials. One of the first papers in this direction is Koornwinder [26] introducing vector-valued polynomials, which can be written as matrix orthogonality; see also [21, 22]. The vector polynomials are evaluated in an explicit way in terms of the representations of SU(2); see [26, Prop. 3.2]. Another seminal paper making this connection to matrix polynomials explicit is Grünbaum, Pacharoni and Tirao [7], where the rank-one symmetric space $(\mathrm{SU}(3), \mathrm{U}(2))$ is studied. The approach of [7] relies on invariant differential operators on the corresponding homogeneous space. Since then several other approaches have been explored, and many other rank-one cases have been studied in detail. For this paper the approach of [21, 22, 23] is the most relevant; see Grünbaum, Pacharoni, Tirao [13, Chap. 13] for other approaches and references.

Scalar spherical functions on symmetric spaces have been vastly generalised in the work of Heckman and Opdam; see Heckman's lecture notes [9], or Heckman and Opdam [27, Chap. 8]. The root multiplicities, i.e. dimensions of root spaces, arising from the symmetric spaces are considered to be more general continuous parameters, and the second-order partial differential operator extending the radial part of the Casimir operator for the symmetric space plays an important role. A first important step was taken by Koornwinder in the 1970s, who studied several sets of orthogonal polynomials in two variables, generalising the spherical functions arising for types A_2 and BC_2. As a first step for a matrix generalisation, matrix spherical functions and the corresponding matrix orthogonal polynomials need to be considered. For type A_n this is done in [23], and the purpose of this paper is to study matrix spherical functions and the corresponding matrix orthogonal polynomials for type BC_2. A possible next step is more general parameters: one possibility is to use shift operators for the classical case of BC_2 (see Opdam [30, §2]) and to employ the same shift operator in the matrix case as well. This has been done successfully in the rank-one case to go from matrix Chebyshev polynomials to matrix Gegenbauer polynomials; see [19]. We expect that this interpretation can lead to more properties of the corresponding matrix orthogonal polynomials studied in this paper. Moreover, the relation to possible applications in mathematical physics needs to be investigated; see e.g. [33] for more information and references given there.

In this paper we study matrix spherical functions for the compact symmetric pair $(G, K) = (\mathrm{SU}(m+2), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(m)))$, and we study matrix spherical functions and corresponding matrix orthogonal polynomials as described in Section 1.1 for the case of an irreducible representation of K arising from the U(2)-component in K. The results of Section 1.1 follow [23, Part I], but there are slight variations on this approach; see [31, §9]. In fact, we use the classification of [31] in order to find the right K-representations satisfying the multiplicity-free Condition 1.1, but [31] gives more possibilities, i.e. also involving other K-representations. In this paper we restrict to the K-representations arising from the $\mathrm{U}(2)$-block in K with a slight assumption on this representation. In this paper we show that instead of studying the more complicated matrix spherical functions, we can study the simpler leading terms of matrix spherical functions. The leading terms turn out to be homogeneous polynomials, and homogeneity considerations allow us to prove some results, e.g. on the indecomposability of the corresponding matrix weight and the explicit derivation of the second-order matrix partial differential equation. Initially, we study the leading terms of matrix spherical functions for labels in $B(\mu)$ as defined in Condition 1.2; see Theorem 4.5. Note that to study matrix spherical functions explicitly we need explicit control over the K-intertwiner embedding a specific K-representation into a larger irreducible G-representation. This is in general hard to do explicitly, but this approach is used successfully in [23] for the symmetric pair corresponding to the group case for type A. In this paper we take an alternative approach and we construct the embedding of the specific K-representation into a larger tensor product G-representation containing the required irreducible G-representation as a constituent in the decomposition. Then we have to show that the embedding indeed "sees" the appropriate irreducible G-representation. Of course, there are many ways to do this, and in this paper we motivate the choice we make as follows. First, it leads to a leading term whose components are homogeneous polynomials, and second, the radial part of the Casimir operator on the leading term has a simple expression; see Lemma 6.5. The approach taken is motivated by the van Pruijssen preprint [36].

In order to make explicit the connection between leading terms and matrix spherical functions, we need the action of the radial part of the Casimir operator as an operator acting on matrix-valued functions on A. For completeness this action is derived in the Appendix. For matrix spherical functions corresponding to elements from $B(\mu)$ as in Condition 1.2, we find an explicit expression in this way involving the leading terms; see Proposition 6.4. Then in Section 8 we obtain the leading terms for the general case, and we show that the radial part of the Casimir operator acts in a lower triangular way with respect to the partial ordering. This is analogous to the case for the (scalar) Heckman–Opdam polynomials; see [9, §1.3]. The main result is Theorem 7.1 in which we explicitly give the matrix orthogonality for the corresponding family of two-variable orthogonal polynomials with an explicit matrix weight on a region bounded by two straight lines and a parabola; see Figure 1. Theorem 7.1 also states that these matrix polynomials are eigenfunctions of a second-order matrix partial differential operator.

We now describe the content of the paper in brief. In the remaining part of the introduction we briefly recall in Section 1.1 the set-up to go from matrix spherical functions to matrix orthogonal polynomials, where the number of variables is equal to the rank of the compact symmetric space. This follows [23, Part I]. In Section 2 we briefly describe the structure theory and notation for the compact symmetric pair $(G, K) = (\mathrm{SU}(m+2), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(m)))$, $m > 2$, and we show that the conditions in Section 1.1 are satisfied in this case. In Section 3 we develop the building blocks for the leading terms. These are essentially the leading terms in the case of the K-representation in Section 1.1 corresponding to the trivial representation and to the natural representation of the U(2)-block in K. Building on this we study the leading term for matrix spherical functions corresponding to $B(\mu)$ as in Condition 1.2. The leading terms can be fully described in terms of single-variable Krawtchouk polynomials, and hence as single-variable hypergeometric functions. Next, in Section 4 we use the radial part of the Casimir operator to give an explicit expression for the matrix spherical functions corresponding to $B(\mu)$ in terms of the leading terms. In Section 5 we describe the two-variable matrix weight, and we show that the weight is indecomposable and that its determinant is nonvanishing on the interior of the integration region. In Section 7 we describe two-variable matrix orthogonal polynomials, and we describe the corresponding eigenvalue equation involving a second-order matrix partial differential operator. We have chosen the coordinates in Theorem 7.1 to match the notation of Koornwinder [25, 24]; see also [32]. Theorem 7.1 generalises the results of [25, 24, 32] to the matrix case, but the scalar part of the weight measure in [25, 24, 32] is more general than that in Theorem 7.1. Theorem 7.1 also contains the case [9, Chap. 5] for this particular symmetric pair (corresponding to the case $a = 0$ in the notation of Section 7). In Section 8 we then derive the leading term for general matrix spherical functions, and we show that the radial part of the Casimir operator acts in a lower triangular fashion on such a leading term. Finally, in Section 9 we discuss briefly the remaining cases of K-representations of this type.

In the course of several proofs we have to manipulate several expressions involving functions in two variables. We have used computer algebra, in particular Maple and Maxima, to check these computations.

§1.1. General set-up

In this subsection we recall notation and the necessary results. We follow [23, Part 1], but see also [31, §11], [35]. We consider a compact symmetric pair (G, K) and for its structure theory and results we refer to [10]. For the explicit case $(G, K) = (\mathrm{SU}(m+2), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(m)))$ the structure theory is explicitly given in Section 2. We label the representations of G, respectively K, by the highest weights P_G^+, respectively P_K^+, and such a representation is denoted by $(\pi_\lambda^G, V_\lambda^G)$, $\lambda \in P_G^+$, and similarly for K. We now fix $\mu \in P_K^+$.

In order to apply the general approach of [23] we need to establish three conditions.

Condition 1.1. $\mathrm{Ind}_K^G \pi_\mu^K$ splits multiplicity-free.

By Frobenius reciprocity this is equivalent to $[\pi_\lambda^G|_K : \pi_\mu^K] \leq 1$ for all $\lambda \in P_G^+$, and we put

$$P_G^+(\mu) = \{\lambda \in P_G^+ \mid [\pi_\lambda^G|_K : \pi_\mu^K] = 1\}. \tag{1.1}$$

So, if Condition 1.1 holds, we have

$$\mathrm{Ind}_K^G \pi_\mu^K = \bigoplus_{\lambda \in P_G^+(\mu)} V_\lambda^G.$$

For $\lambda \in P_G^+(\mu)$ we define the corresponding matrix spherical functions

$$\Phi_\lambda^\mu \colon G \to \mathrm{End}(V_\mu^K), \quad \Phi_\lambda^\mu(g) = p \circ \pi_\lambda^G(g) \circ j, \tag{1.2}$$

where $j \in \mathrm{Hom}_K(V_\mu^K, V_\lambda^G)$ is the unitary intertwiner and $p = j^*$ is the corresponding K-equivariant orthogonal projection. Then (1.2) is independent of the choice of j and we have

$$\Phi_\lambda^\mu(k_1 g k_2) = \pi_\mu^K(k_1)\Phi_\lambda^\mu(g)\pi_\mu^K(k_2) \quad \forall\, k_1, k_2 \in K,\ \forall\, g \in G. \tag{1.3}$$

The space of regular functions $\Phi \colon G \to \mathrm{End}(V_\mu^K)$ satisfying the left and right K transformation behaviour as in (1.3) is denoted by E^μ. Using the Peter–Weyl decomposition we see that $\{\Phi_\lambda^\mu \mid \lambda \in P_G^+(\mu)\}$ forms a linear basis for E^μ. Then E^0 is the space of scalar continuous bi-K-invariant functions, and E^μ is an E^0-module. Moreover, Schur orthogonality gives

$$\int_G \mathrm{Tr}\big(\Phi_\lambda^\mu(g)(\Phi_{\lambda'}^\mu(g))^*\big)\, dg = \delta_{\lambda,\lambda'} \frac{(\dim V_\mu^K)^2}{\dim V_\lambda^G}, \quad \lambda, \lambda' \in P_G^+(\mu). \tag{1.4}$$

Note that the integrand is a bi-K-invariant function, so contained in E^0.

Let A be the abelian subgroup and $M = Z_K(A)$ as in [23, §2]. By the Cartan decomposition, $G = KAK$, and by (1.3) it suffices to consider

$$\Phi_\lambda^\mu|_A \colon A \to \mathrm{End}_M(V_\mu^K), \tag{1.5}$$

since $\pi_\mu^K(m)\Phi_\lambda^\mu(a) = \Phi_\lambda^\mu(ma) = \Phi_\lambda^\mu(am) = \Phi_\lambda^\mu(a)\pi_\mu^K(m)$. So we need to know the decomposition

$$V_\mu^K|_M \cong \bigoplus_{i=1}^N V_{\sigma_i}^M, \tag{1.6}$$

where $\sigma_i \in P_M^+$ are the corresponding highest weights for M. The decomposition (1.6) is again a multiplicity-free decomposition; see [35] and also [4, 17].

Note that if the representation π_μ^K induces multiplicity-free, then also its dual (or contragredient) representation $(\pi_\mu^K)^* = \pi_{\mu^*}^K$ induces multiplicity-free, where μ^* corresponds to the highest weight of the dual representation. Then $P_G^+(\mu^*)$ consists of those G-representations for which the dual is in $P_G^+(\mu)$, i.e. $P_G^+(\mu^*) = \{\lambda^* \mid \lambda \in P_G^+(\mu)\}$, where λ^* corresponds to the highest weight of the dual of the G-representation with highest weight λ. Then we obtain

$$(1.7)\quad (\Phi_{\lambda^*}^{\mu^*}(a)v^*)(v) = v^*(\Phi_\lambda^\mu(a^{-1})v), \quad a \in A,\ v \in V_\mu^K,\ v^* \in \mathrm{Hom}(V_\mu^K, \mathbb{C}) = V_{\mu^*}^K.$$

Note that if Condition 1.1 holds, then it also holds for the dual $\mu^* \in P_K^+$. Moreover, taking duals gives an involution on the spherical weights $P_G^+(0)$.

Condition 1.2. There exists a set of weights $B(\mu) \subset P_G^+$, so that for $\lambda \in P_G^+(\mu)$ there exist unique elements $\nu \in B(\mu)$ and $\lambda_{\mathrm{sph}} \in P_G^+(0)$ with $\lambda = \nu + \lambda_{\mathrm{sph}}$. The restriction map of the torus of $G^{\mathbb{C}}$ to the torus of $M^{\mathbb{C}}$ gives a bijection $B(\mu) \xrightarrow{\cong} \{\sigma \in P_M^+ \mid [V_\mu^K|_M : V_\sigma^M] = 1\}$.

Assuming Condition 1.2 is satisfied for $\mu \in P_K^+$, then Condition 1.2 is also satisfied for the dual K-representation with highest weight μ^*.

Taking $\mu = 0$, $P_G^+(0)$ corresponds to the spherical weights, and $P_G^+(0) = \bigoplus_{i=1}^n \mathbb{N}\lambda_i$, where $\lambda_1, \ldots, \lambda_n$ are the generators for the spherical weights and n is the rank of the compact symmetric space (G, K). We let $\phi_i = \Phi_{\lambda_i}^0 : G \to \mathbb{C}$, which generate the algebra of bi-K-invariant polynomials on G. For $\lambda = \sum_{i=1}^n d_i\lambda_i \in P_G^+(0)$ we put $|\lambda| = \sum_{i=1}^n d_i$. We use the notation $P_G(\lambda)$ for all the weights occurring in the G-representation π_λ^G of highest weight $\lambda \in P_G^+$, and similarly for other groups.

Condition 1.3. For all weights $\nu \in B(\mu)$, for all generators λ_i of the spherical weights $P_G^+(0)$ and for all weights $\eta \in P_G(\lambda_i)$ such that $\nu + \eta \in P_G^+(\mu)$, we have by Condition 1.2 a unique $\nu' \in B(\mu)$ such that $\nu + \eta = \nu' + \lambda$ with $\lambda \in P_G^+(0)$. Then $|\lambda| \leq 1$.

Note that if Condition 1.3 holds for μ, then it also holds for the dual $\mu^* \in P_K^+$.

Assuming Conditions 1.1, 1.2 and 1.3, one can show that for a spherical weight $\lambda_{\mathrm{sph}} = \sum_{r=1}^n d_r\lambda_r \in P_G^+(0)$, $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exist unique n-variable polynomials $p_{\nu_i,v_r;\mathbf{d}}^\mu$ of total degree $|\mathbf{d}| = |\lambda_{\mathrm{sph}}|$ so that for $\lambda = \nu_i + \lambda_{\mathrm{sph}} \in P_G^+(\mu)$ and $a \in A$,

$$(1.8)\qquad \Phi_\lambda^\mu(a) = \Phi_{\nu_i+\lambda_{\mathrm{sph}}}^\mu(a) = \sum_{r=1}^N p_{\nu_i,\nu_r;\mathbf{d}}^\mu(\phi_1(a), \ldots, \phi_n(a))\Phi_{\nu_r}^\mu(a)$$

using a slightly different labelling from [23]. Using this expansion in the orthogonality relations (1.4) and reducing the integral for bi-K-invariant functions to an integral over A (see [10, Prop. X.1.19]), we find the matrix orthogonality relations

$$\int_A P_{\mathbf{d}}(\phi(a))W(\phi(a))\big(P_{\mathbf{d}'}(\phi(a))\big)^*|\delta(a)|\,da = c\delta_{\mathbf{d},\mathbf{d}'}H_{\mathbf{d}}, \tag{1.9}$$

where

$$\begin{aligned}
(H_{\mathbf{d}})_{i,j} &= \delta_{i,j}\frac{(\dim V_\mu^K)^2}{\dim V^G_{\nu_i+\lambda_{\mathrm{sph}}}},\\
\phi(a) &= (\phi_1(a),\ldots,\phi_n(a)),\\
P_{\mathbf{d}}(\phi(a)) &= \big(p_{\nu_i,\nu_j;\mathbf{d}}(\phi_1(a),\ldots,\phi_n(a))\big)_{i,j=1}^N,\\
W(\phi(a)) &= \big(\mathrm{Tr}\big(\Phi^\mu_{\nu_i}(a)(\Phi^\mu_{\nu_j}(a))^*\big)\big)_{i,j=1}^N
\end{aligned}$$

and $c>0$ is determined by $c=\int_A|\delta(a)|\,da$ and δ is given in [10, Prop. X.1.19], where it is denoted by D_*.

§2. Structure theory and multiplicity-free triples

In this section we specialise to the compact symmetric pair $(G,K)=(\mathrm{SU}(m+2),\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(m)))$, $m>2$, for which we study matrix spherical functions and the related orthogonal polynomials in detail. First we describe the structure theory (see e.g. [10]) needed in order to associate the corresponding orthogonal polynomials in Section 2.1. In the remaining part we show that for explicit K-representations the conditions of [23, Part I] are satisfied in this case.

§2.1. Structure theory

From now on we take $G=\mathrm{SU}(m+2)$, $m>2$, $K=\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(m))$ embedded block-diagonally. We view $\mathrm{U}(2)\subset K$ as a subgroup as the upper-left (2×2)-block of K. The abelian subgroup is $A=\{a_{\mathbf{t}}=a_{(t_1,t_2)}\mid t_1,t_2\in\mathbb{R}\}$, with

$$a_{\mathbf{t}}=a_{(t_1,t_2)}=\begin{pmatrix}
\cos t_1 & 0 & 0\cdots 0 & 0 & i\sin t_1\\
0 & \cos t_2 & 0\cdots 0 & i\sin t_2 & 0\\
0 & 0 & 1\cdots 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0\cdots 1 & 0 & 0\\
0 & i\sin t_2 & 0\cdots 0 & \cos t_2 & 0\\
i\sin t_1 & 0 & 0\cdots 0 & 0 & \cos t_1
\end{pmatrix}, \tag{2.1}$$

where the middle block is the $((m-2)\times(m-2))$-identity matrix. Then $M = Z_K(A)$ is given by matrices m which are block diagonal of size 2×2, $(m-2)\times(m-2)$, 2×2 of the form

$$m = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} e^{is_1} & 0 \\ 0 & e^{is_2} \end{pmatrix}, \quad M_1 \in \mathrm{U}(m-2), \quad D_2 = \begin{pmatrix} e^{is_2} & 0 \\ 0 & e^{is_1} \end{pmatrix} \tag{2.2}$$

with $\det(m) = 1$.

As the torus of $G^{\mathbb{C}}$ we take the diagonal elements, and we take this also as the torus of $K^{\mathbb{C}}$. Explicitly,

$$T_{G^{\mathbb{C}}} = T_{K^{\mathbb{C}}} = \bigl\{\mathrm{diag}(t_1,\dots,t_{m+2}) \bigm| t_k \in \mathbb{C},\ \textstyle\prod_{i=1}^{m+2} t_i = 1\bigr\}. \tag{2.3}$$

We take the torus of $M^{\mathbb{C}}$ as contained in the torus of $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$:

$$\begin{aligned} T_{M^{\mathbb{C}}} &= \bigl\{\mathrm{diag}(t_1,\dots,t_{m+2}) \bigm| t_{m+1} = t_2,\ t_{m+2} = t_1,\ \textstyle\prod_{i=1}^{m+2} t_i = 1\bigr\} \\ &\subset T_{G^{\mathbb{C}}} = T_{K^{\mathbb{C}}}. \end{aligned} \tag{2.4}$$

By $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$ and $\mathfrak{a}$ we denote the corresponding complexified Lie algebras of G, K, M and A. Then the root system Δ of $\mathfrak{g}$ is of type A_{m+1}, and we denote the standard simple roots α_i, $1 \le i \le m+1$. We put $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, $H_i = E_{i+1,i+1} - E_{i,i}$, where $E_{i,j}$ is the matrix with all zeros except the (i,j)th entry. The roots and positive roots are denoted by $Q_G = \bigoplus_{i=1}^{m+1} \mathbb{Z}\alpha_i$ and $Q_G^+ = \bigoplus_{i=1}^{m+1} \mathbb{N}\alpha_i$. The partial order $\sigma \preccurlyeq \eta$ is $\eta - \sigma \in Q_G^+$.

With this choice of positive roots, we define the fundamental weights for G, K and M by

$$\begin{aligned} &\omega_i\colon T_{G^{\mathbb{C}}} = T_{K^{\mathbb{C}}} \to \mathbb{C}, && \omega_i(\mathrm{diag}(t_1,\dots,t_{m+2})) = \prod_{j=1}^{i} t_j, \quad 1 \le i < m+2, \\ &\eta_i\colon T_{M^{\mathbb{C}}} \to \mathbb{C}, && \eta_i(\mathrm{diag}(t_1,t_2,\dots,t_m,t_2,t_1)) = \prod_{j=1}^{i} t_j, \quad 1 \le i < m. \end{aligned}$$

Note that η_1, η_2 are characters of M. Then we find

$$\omega_i|_{T_{M^{\mathbb{C}}}} = \eta_i \ (1 \le i < m), \quad \omega_m|_{T_{M^{\mathbb{C}}}} = -\eta_2, \quad \omega_{m+1}|_{T_{M^{\mathbb{C}}}} = -\eta_1. \tag{2.5}$$

Then we have

$$P_K^+ = \bigl\{\textstyle\sum_{i=1}^{m+1} a_i\omega_i \bigm| a_2 \in \mathbb{Z},\ a_i \in \mathbb{N},\ i \ne 2\bigr\}, \quad P_G^+ = \textstyle\bigoplus_{i=1}^{m+1} \mathbb{N}\omega_i. \tag{2.6}$$

Considering $\mathrm{U}(2) \subset K$, we see that the $\mathrm{U}(2)$ representations correspond to the elements of P_K^+ with $a_j = 0$ for $j \ge 3$.

The reduced root system is of type BC$_2$, and the corresponding reduced Weyl group is generated by s_1 and s_2, and we put $n_1, n_2 \in N_K(A)$ by

$$n_1 = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & I_{m-2} & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad n_2 = \mathrm{diag}(1, -i, \underbrace{1, \ldots, 1}_{m-2}, i, 1) \tag{2.7}$$

using the notation of Appendix A.1 for the flip J_2. Then $n_1 a_{\mathbf{t}} n_1^{-1} = a_{s_1\mathbf{t}}$ and $n_2 a_{\mathbf{t}} n_2^{-1} = a_{s_2\mathbf{t}}$ with $s_1\mathbf{t} = (t_2, t_1)$ and $s_2\mathbf{t} = (t_1, -t_2)$.

§2.2. Multiplicity-free triples

The triple (G, K, μ), $\mu \in P_K^+$, is a multiplicity-free triple if Condition 1.1 is satisfied. Since (G, K) is a symmetric pair, the triple $(G, K, 0)$, where $\mu = 0$ corresponds to the trivial K-representation, is a multiplicity-free triple. Then we have the spherical weights

$$P_G^+(0) = \mathbb{N}\lambda_1 \oplus \mathbb{N}\lambda_2, \quad \lambda_1 = \omega_1 + \omega_{m+1}, \ \lambda_2 = \omega_2 + \omega_m; \tag{2.8}$$

see Krämer [28, Table 1]. More generally, the multiplicity-free triples and the set $P_G^+(\mu)$ for a multiplicity-free triple (G, K, μ) are determined by Pezzini and van Pruijssen [31]. We focus on representations of K that correspond to representations of $\mathrm{U}(2) \subset K$, i.e. we assume $\mu = a\omega_1 + b\omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$.

Proposition 2.1. *The triple (G, K, μ), with $\mu = a\omega_1 + b\omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$, is multiplicity-free. Moreover, $P_G^+(\mu) = B(\mu) + P_G^+(0)$. In the case $b \in \mathbb{N}$ we have*

$$B(\mu) = \{\nu_i = \nu_i(\mu) = (a-i)\omega_1 + (i+b)\omega_2 + i\omega_{m+1} \mid 0 \le i \le a\}.$$

In the case $b \le -a$ we have

$$B(\mu) = \{\nu_i = \nu_i(\mu) = (a-i)\omega_1 + (-i-b)\omega_m + i\omega_{m+1} \mid 0 \le i \le a\},$$

and in the case $-a < b < 0$ we have

$$\begin{aligned} B(\mu) = &\{\nu_i = \nu_i(\mu) = (a-i)\omega_1 + (-i-b)\omega_m + i\omega_{m+1} \mid 0 \le i < -b\} \\ &\cup \{\nu_i = \nu_i(\mu) = (a-i)\omega_1 + (b+i)\omega_2 + i\omega_{m+1} \mid -b \le i \le a\}. \end{aligned}$$

Remark 2.2. Recall that the G-representation of highest weight ω_i can be realised in the exterior power $\Lambda^i V$, where $V = \mathbb{C}^{m+2}$ is the natural G-representation. It follows that $\omega_i^* = \omega_{m+2-i}$, and this determines λ^*. For the spherical weights (see (2.8)), $\lambda_1^* = \lambda_1$ and $\lambda_2^* = \lambda_2$. The dual of the K-representation of highest weight $\mu = a\omega_1 + b\omega_2$ is the K-representation of highest weight $\mu^* = a\omega_1 - (a+b)\omega_2$. Indeed, the map $v \mapsto \langle v, v_\mu \rangle$, with v_μ the highest weight vector of V_μ^K, is the lowest weight vector of $(V_\mu^K)^*$ of weight $-a\omega_1 - b\omega_2$. Then $B(\mu^*) = (B(\mu))^*$, and more precisely, in the notation of Proposition 2.1, $\nu_i(\mu^*) = (\nu_{a-i}(\mu))^*$.

Proof of Proposition 2.1. The proof is a verification using the results and notation of [31], in particular [31, Table B.2.1]. As noted after [31, Def. 9.1], we have $[\pi_\lambda^G|_K : \pi_\mu^K] = 1$ if and only if $(\lambda, -\mu)$ is an element of the so-called extended weight monoid $\widetilde{\Gamma}(G/P)$, $P \subset K^{\mathbb{C}}$ corresponding parabolic subgroup, corresponding to G/K. Now we use [31, Table B.2.1] to see that the elements of $\widetilde{\Gamma}(G/P)$ are nonnegative integral linear combinations of

$$(\omega_1 + \omega_{m+1}, 0), \quad (\omega_1, -\omega_1), \quad (\omega_2, -\omega_2), \quad (\omega_m, \omega_2), \quad (\omega_{m+1}, \omega_2 - \omega_1).$$

We then see that $(\lambda, 0) \in \widetilde{\Gamma}(G/K)$ if and only if $\lambda \in P_G^+(0)$, i.e. λ is a spherical weight. It is now a straightforward calculation to determine the $\lambda \in P_G^+$ satisfying $(\lambda, a\omega_1 + b\omega_2) \in \widetilde{\Gamma}(G/K)$. $\square$

Note that the representation of K with highest weight $\mu = a\omega_1 + b\omega_2$ has dimension $a+1$. Denoting the highest weight vector by v_μ we see that V_μ^K has an orthogonal basis $\{v_k = F_1^k \cdot v_\mu \mid 0 \le k \le a\}$ by considering the representation as a U(2)-representation. It follows that, taking $m \in M$ as in (2.2), we have $\pi_\mu^K(m)v_k = e^{i(a+b-k)s_1} e^{i(b+k)s_2} v_k$, so that this corresponds to the M-weight $(a-2k)\eta_1 + (b+k)\eta_2$. So

$$V_\mu^K|_M = \bigoplus_{k=0}^{a} V_{\sigma_k}^M, \quad \sigma_k = \sigma_k(\mu) = (a-2k)\eta_1 + (b+k)\eta_2, \tag{2.9}$$

splits multiplicity-free into 1-dimensional M-representations. Since the M-representations are 1-dimensional, we find $\sigma_k^* = -\sigma_k$ and $\sigma_k(\mu^*) = \sigma_{a-k}(\mu)^*$.

In any of the cases of Proposition 2.1 we have $\nu_i(\mu)|_{T_{M^{\mathbb{C}}}} = \sigma_i(\mu)$ using (2.5). This leads to Corollary 2.3.

Corollary 2.3. *For $\mu = a\omega_1 + b\omega_2 \in P_K^+$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$, Conditions* 1.1 *and* 1.2 *are satisfied.*

Proof. The statement can obtained by analysing more carefully the extended weight monoid of [31] used in the proof of Proposition 2.1, but it can also be done directly having the $B(\mu)$ at hand. Assume $\lambda \in P_G^+(\mu)$ can be written as $\nu_i + \lambda_{\mathrm{sph}} = \nu_j + \lambda'_{\mathrm{sph}}$, with $\lambda_{\mathrm{sph}} = n_1\lambda_1 + n_2\lambda_2$, $\lambda'_{\mathrm{sph}} = m_1\lambda_1 + m_2\lambda_2$. Assume first $\mu = a\omega_1 + b\omega_2$ with $b \in \mathbb{N}$. Then we have, using Proposition 2.1 and (2.8),

$$\begin{aligned} 0 = \nu_i + \lambda_{\mathrm{sph}} - \nu_j - \lambda'_{\mathrm{sph}} &= (j - i + n_1 - m_1)\omega_1 \\ &+ (i - j + n_2 - m_2)\omega_2 + (n_2 - m_2)\omega_m + (i - j + n_1 - m_1)\omega_{m+1}. \end{aligned}$$

This gives $n_2 = m_2$, $i = j$ and $n_1 = m_1$, and uniqueness follows. The case $b \leq -a$ follows by duality, and the case $-a < b < 0$ can be proved similarly, taking into account the different cases in Proposition 2.1.

The fact that the restriction map gives an isomorphism of $B(\mu)$ and the set of irreducible M-modules in $V_\mu^K|_M$ follows from (2.9) and (2.5). □

§2.3. Condition 1.3

In order to able to apply the general theory described in Section 1.1 we need to check Condition 1.3. Recall that $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ with the convention $\omega_0 = \omega_{m+2} = 0$, which gives

$$\lambda_1 = \sum_{i=1}^{m+1} \alpha_i, \quad \lambda_2 = \alpha_1 + \alpha_{m+1} + 2\sum_{i=2}^{m} \alpha_i. \tag{2.10}$$

Note that for weights $\eta \in P_G(\lambda_i)$, we have $\eta \preccurlyeq \lambda_i$, or $\lambda_i - \eta \in Q_G^+$, so that the coefficient of α_1 (or α_{m+1}) in η is less than or equal to 1. Moreover, we see from (2.10) that for $\lambda \in P_G^+(0)$ the degree $|\lambda|$ is equal to the coefficient of α_1 (or α_{m+1}) in λ.

Proposition 2.4. *For $\mu = a\omega_1 + b\omega_2 \in P_K^+$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, Condition 1.3 satisfied.*

Proof. We first assume $b \in \mathbb{N}$. Let $\eta \in P_G(\lambda_i)$, and assume $\nu_i + \eta = \nu_j + \lambda_{\mathrm{sph}} \in P_G^+(\mu)$ with $\lambda_{\mathrm{sph}} \in P_G^+(0)$. Then

$$\lambda_{\mathrm{sph}} - \eta = \nu_i - \nu_j = (i-j)(\omega_2 + \omega_{m+1} - \omega_1).$$

Since $\omega_2 + \omega_{m+1} - \omega_1 = \sum_{k=2}^{m+1} \alpha_k$, we see that in the expansion of simple roots, the coefficient of α_1 in λ_{sph} equals the coefficient of α_1 in η, which is less than or equal to 1. Since this coefficient is nonnegative and since $|\lambda_{\mathrm{sph}}|$ is the coefficient of α_1, we get that $|\lambda_{\mathrm{sph}}| \leq 1$. This proves Condition 1.3 in the case $b \in \mathbb{N}$. By duality it follows for $b \leq -a$.

In the case $-a < b < 0$, Proposition 2.1 gives two possible forms for ν_i and ν_j. In the case that they have the same form, a similar argument to above proves $|\lambda_{\mathrm{sph}}| \leq 1$. Assume $\nu_i = (a-i)\omega_1 + (-i-b)\omega_m + i\omega_{m+1}$ for $0 \leq i \leq -b$ and $\nu_j = (a-j)\omega_1 + (b+j)\omega_2 + j\omega_{m+1}$ for $-b \leq j \leq a$. Then

$$\begin{aligned}\lambda_{\mathrm{sph}} - \eta = \nu_i - \nu_j &= (j-i)\omega_1 - (b+j)\omega_2 - (b+i)\omega_m + (i-j)\omega_{m+1} \\ &= -(b+i)(\omega_1 + \omega_m - \omega_{m+1}) - (b+j)(-\omega_1 + \omega_2 + \omega_{m+1}),\end{aligned}$$

and now we additionally use $\omega_1 + \omega_m - \omega_{m+1} = \sum_{k=1}^{m} \alpha_k$. Since $-(b+i) \geq 0$ and $-(b+j) \leq 0$, the coefficient of α_{m+1} in $\lambda_{\mathrm{sph}} - \eta = \nu_i - \nu_j$ is nonpositive. Since the coefficient of α_{m+1} in η is at most 1, the coefficient of α_{m+1} in λ_{sph} is at most 1, and thus $|\lambda_{\mathrm{sph}}| \leq 1$. The other situation, with the form of the ν_i and ν_j interchanged, can be proved analogously. $\square$

Note that we have the following corollary of the proof of Proposition 2.4.

Corollary 2.5. *For $\mu = a\omega_1 + b\omega_2 \in P_K^+$, $a \in \mathbb{N}$, $b \in \mathbb{N}$, we have $\nu_i(\mu) \succ \nu_j(\mu)$ for $i > j$.*

A similar statement holds for $b \leq -a$, but not for $-a < b < 0$. Then not all elements can be compared in the partial ordering.

The reduced Weyl group $W = N_K(A)/M$ acts on the M-types in V_μ^K. Let $n_w \in N_K(A)$ be a representative of w. Then (1.3) shows

$$\Phi_\lambda^\mu(a_{w\mathbf{t}}) = \Phi_\lambda^\mu(n_w a_\mathbf{t} n_w^{-1}) = \pi_\mu^K(n_w)\Phi_\lambda^\mu(a_\mathbf{t})\pi_\mu^K(n_w^{-1}) \in \mathrm{End}_M(V_\mu^K).$$

For $T \in \mathrm{End}_M(V_\mu^K)$, the action $w \cdot T = \pi_\mu^K(n_w)T\pi_\mu^K(n_w^{-1})$ is well defined, and preserves orthogonal projections, and so it induces an action of W on the M-types in V_μ^K. In this case, the decomposition (2.9) splits into 1-dimensional M-representations. From (2.7), we see that $s_2 \in W$ acts trivially on the M-types, since it commutes with M. For s_1 we see that it acts on the characters as $s_1 \cdot \eta_1 = \eta_2 - \eta_1$, $s_1 \cdot \eta_2 = \eta_2$ leading to

$$s_1 \cdot \sigma_k = \sigma_{a-k}, \quad s_2 \cdot \sigma_k = \sigma_k \tag{2.11}$$

§3. Special cases

In this section we give the simplest cases of embedding of K-representations in tensor products of G-representations in order to obtain the leading term. The first case concerns the zonal spherical functions for the weights λ_1 and λ_2 generating the spherical weights. This is based on suitable embeddings of the K-fixed vector in a twofold tensor product. Next we find the embedding for the fundamental K-representation $V_{\omega_1}^K$ in a twofold tensor product of G-representations. This will be used in Section 4 to obtain the leading terms of special matrix spherical functions.

We first prove Lemma 3.1, which we use on several occasions.

Lemma 3.1. *For $i \leq j$ we have*

$$V_{\omega_i}^G \otimes V_{\omega_j}^G \cong \bigoplus_{r=0}^{\min(i,m+2-j)} V_{\omega_{i-r}+\omega_{j+r}}^G,$$

with the convention $\omega_0 = 0 = \omega_{m+2}$.

Proof. Observe that the fundamental weights for the root system of type A are minuscule weights (see e.g. [2, p. 230]), and for this case one has the multiplicity-free decomposition

$$V^G_{\omega_i} \otimes V^G_{\omega_j} \cong \bigoplus_{\substack{w \in W/W_{\omega_j} \\ \omega_i + w\omega_j \in P^+_G}} V^G_{\omega_i + w\omega_j},$$

where $W = S_{m+2}$ is the Weyl group for G and $W_{\omega_j} = \{w \in W \mid w\omega_j = \omega_j\} = S_j \times S_{m+2-j}$ is the stabiliser subgroup; see e.g. [14, Prop. 1], [29, Cor. 3.5]. Since W_{ω_j} is a parabolic subgroup, we can take coset representatives of minimal length [11, §1.10]. Such an element is determined by a sequence $k_1 < k_2 < \cdots < k_j$ of numbers from $\{1, 2, \ldots, m+2\}$, defined by $w(j) = k_j$ and extended such that w has minimal length. Using the expression for ω_j as in [2, Planche I], we get $w\omega_j = \sum_{p=1}^{j} \omega_{k_p} - \omega_{k_p - 1}$. It remains to determine the choices leading to $\omega_i + w\omega_j \in P^+_G = \bigoplus_{i=1}^{m+1} \mathbb{N}\omega_i$. It follows that the sequence $\{k_1, k_2, \ldots, k_j\}$ can have at most one hole. Keeping track of these possibilities yields the result. $\square$

§3.1. Spherical functions on A

We first construct explicit generators for the algebra of spherical functions for (G, K). The natural representation $V^G_{\omega_1} = V = \mathbb{C}^{m+2}$ of G is equipped with the standard orthonormal basis $(e_1, \ldots, e_{m+2})$. Recall that $V^G_{\omega_j} \cong \Lambda^j V$.

Lemma 3.2. *We have $V^G_{\omega_1} \otimes V^G_{\omega_{m+1}} \cong V^G_{\lambda_1} \oplus V^G_0$ and define*

$$v_1 = e_1 \otimes e_2 \wedge e_3 \wedge \cdots \wedge e_{m+2} - e_2 \otimes e_1 \wedge e_3 \wedge \cdots \wedge e_{m+2} \in V^G_{\omega_1} \otimes V^G_{\omega_{m+1}}.$$

Then v_1 is a K-invariant vector, i.e. v_1 is contained in the 2-dimensional space $(V^G_{\lambda_1})^K \oplus (V^G_0)^K$ and v_1 has a nonzero component in $(V^G_{\omega_1+\omega_{m+1}})^K = (V^G_{\lambda_1})^K$.

Proof. The tensor product decomposition follows from Lemma 3.1. From (2.8) we know that $0, \lambda_1 = \omega_1 + \omega_{m+1} \in P^+_G(0)$, so that $(V^G_{\omega_1+\omega_{m+1}})^K$ and $(V^G_0)^K$ are 1-dimensional. It is a straightforward calculation to check that v_1 is a K-fixed vector, and the easiest way is to check that $E_i \cdot v_1 = 0$, $i \in \{1, \ldots, m+1\} \setminus \{2\}$ and $H_i \cdot v_1 = 0$, $i \in \{1, 2, \ldots, m+1\}$. Note that $E_2 \cdot v_1 \neq 0$, so that v is not contained in $(V^G_0)^K \cong \mathbb{C}$, and so has a nonzero component in $(V^G_{\omega_1+\omega_{m+1}})^K$. $\square$

Having v_1 given explicitly in Lemma 3.2 we can calculate the corresponding matrix entry restricted to A explicitly using $a_{\mathbf{t}}$ in (2.1), and we obtain

$$\langle (\pi^G_{\omega_1} \otimes \pi^G_{\omega_{m+1}})(a_{\mathbf{t}}) v_1, v_1 \rangle = \cos^2 t_1 + \cos^2 t_2.$$

Lemma 3.3. *Define $\psi_1 \colon A \to \mathbb{C}$, $\psi_1(a_{\mathbf{t}}) = \cos^2 t_1 + \cos^2 t_2$ and let $\phi_1 \colon A \to \mathbb{C}$ be the spherical function associated to $V^G_{\lambda_1}$. Then there exists a positive constant ξ^1_1 and a nonnegative constant ξ^0_1, so that $\psi_1 = \xi^1_1 \phi_1 + \xi^0_1$ as functions on A.*

The constants ξ_1^1, ξ_1^0 can be calculated explicitly; see Lemma 6.2. Moreover, we can also consider the identity as an identity for functions on G by interpreting the matrix entries as functions on G.

Proof of Lemma 3.3. Put $(V_{\lambda_1}^G)^K = \mathbb{C}\hat{v}$, $\|\hat{v}\| = 1$, and let $V_0^G = \mathbb{C}\tilde{v}$. Then $\phi_1(a_{\mathbf{t}}) = \langle \pi_{\lambda_1}^G(a_{\mathbf{t}})\hat{v}, \hat{v}\rangle$. In Lemma 3.2 we see that $v_1 = a\hat{v} + b\tilde{v}$ with $0 \neq a \in \mathbb{C}$. Then

$$\begin{aligned}\psi_1(a_{\mathbf{t}}) &= \langle (\pi_{\omega_1}^G \otimes \pi_{\omega_{m+1}}^G)(a_{\mathbf{t}})v, v\rangle \\ &= |a|^2\langle \pi_{\lambda_1}^G(a_{\mathbf{t}})\hat{v}, \hat{v}\rangle + |b|^2\langle \pi_0^G(a_{\mathbf{t}})\tilde{v}, \tilde{v}\rangle = |a|^2\phi_1(a_{\mathbf{t}}) + |b|^2,\end{aligned}$$

proving the result. □

In order to find the second spherical function, we proceed similarly.

Lemma 3.4. *We have $V_{\omega_2}^G \otimes V_{\omega_m}^G \cong V_{\lambda_2}^G \oplus V_{\lambda_1}^G \oplus V_0^G$ and define*

$$v_2 = e_1 \wedge e_2 \otimes e_3 \wedge \cdots \wedge e_{m+2} \in V_{\omega_2}^G \otimes V_{\omega_m}^G.$$

Then v_2 is a K-invariant vector, and v_2 has a nonzero component in $(V_{\lambda_2}^G)^K$. Moreover,

$$\psi_2\colon A \to \mathbb{C}, \quad \psi_2(a_{\mathbf{t}}) = \langle (\pi_{\omega_2}^G \otimes \pi_{\omega_m}^G)(a_{\mathbf{t}})v_2, v_2\rangle = (\cos t_1)^2(\cos t_2)^2$$

and $\psi_2 = \xi_2^2\phi_2 + \xi_2^1\phi_1 + \xi_2^0$, where ϕ_2 is the spherical function corresponding to $\lambda_2 \in P_G^+(0)$ and the constants $\xi_2^2 > 0$ and ξ_2^1 and ξ_2^0 are nonnegative.

The proof of Lemma 3.4 follows the lines of the proofs of Lemmas 3.2 and 3.3. It is possible to calculate the constants ξ_2^i explicitly; see Lemma 6.2.

Proof of Lemma 3.4. The tensor product decomposition follows from Lemma 3.1. The K-invariance of v_2 follows from $E_i \cdot v_2 = 0$, $i \in \{1, \ldots, m+1\} \setminus \{2\}$ and $H_i \cdot v_2 = 0$, $i \in \{1, \ldots, m+1\}$, which follows straightforwardly. Then the matrix entry can be calculated using (2.1), and this gives the statement of the explicit expression for $\psi_2(a_{\mathbf{t}})$. Since v_2 is a linear combination of the K-fixed vectors of $V_{\lambda_2}^G$, $V_{\lambda_1}^G$ and V_0^G, we find analogously that ψ_2 is a linear combination of ϕ_1, ϕ_2 and the constant with nonnegative coefficients. Since the function $(\cos t_1)^2(\cos t_2)^2$ is not a linear combination of $(\cos t_1)^2 + (\cos t_2)^2$ and the constants, the coefficient of ϕ_2 has to be nonzero. □

Remark 3.5. Note that $A \cap M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and so the spherical functions, satisfying $\phi(ma_{\mathbf{t}}) = \phi(a_{\mathbf{t}})$ for $m \in A \cap M$, show that the spherical functions in Lemmas 3.2 and 3.4, have to be invariant under $(\cos t_1, \cos t_2) \mapsto (\pm\cos t_1, \pm\cos t_2)$ for all choices of signs.

§3.2. The special case $\mu = \omega_1$

From Proposition 2.1 we know that $B(\omega_1)$ consists of ω_1 and $\omega_2 + \omega_{m+1}$. The K-equivariant map $V^K_{\omega_1} \to V^G_{\omega_1}$ is the standard embedding, which sends the highest weight vector for K to the highest weight vector for G in the natural representation. We need to understand the K-equivariant map $V^K_{\omega_1} \to V^G_{\omega_2+\omega_{m+1}}$, and it suffices to understand the K-highest weight vector in $V^G_{\omega_2+\omega_{m+1}}$. We proceed as in Section 3.1.

Lemma 3.6. *We have $V^G_{\omega_2} \otimes V^G_{\omega_{m+1}} \cong V^G_{\omega_2+\omega_{m+1}} \oplus V^G_{\omega_1}$ and we define the vector*

$$v_0 = e_1 \wedge e_2 \otimes e_1 \wedge e_3 \wedge \cdots \wedge e_{m+2} \in V^G_{\omega_2} \otimes V^G_{\omega_{m+1}}.$$

Then v_0 is a K-highest weight vector of weight ω_1. The vector v_0 has a nonzero component in $V^G_{\omega_2+\omega_{m+1}}$.

It follows from the tensor product decomposition and Proposition 2.1 for $\mu = \omega_1$ that there is a 2-dimensional space of K-highest weight vectors of weight ω_1. It is possible to explicitly write down a linearly independent vector and give the K-highest weight vectors of weight ω_1 in $V^G_{\omega_2+\omega_{m+1}}$ and $V^G_{\omega_1}$.

Proof of Lemma 3.6. Lemma 3.1 proves the first statement. Note that both representations in the direct sum correspond to $B(\omega_1) = \{\omega_1, \omega_2 + \omega_{m+1}\}$. It is straightforward to check that $E_i \cdot v_0 = 0$, $i \in \{1, \ldots, m+1\} \setminus \{2\}$ and $H_1 \cdot v_0 = v_0$, $H_i \cdot v_0 = 0$, $i \in \{2, \ldots, m+1\}$, so that v_0 is a K-highest weight vector of weight ω_1. Note that the K-highest weight vector of weight ω_1 in $V^G_{\omega_1}$ is also a G-highest weight vector of weight ω_1, but $E_2 \cdot v_0 \neq 0$. So the vector v_0 has a nonzero component in $V^G_{\omega_2+\omega_{m+1}}$. $\square$

§4. The leading term of matrix spherical functions for $B(\mu)$

We focus on the case $\mu = a\omega_1 + b\omega_2$, with $b \in \mathbb{N}$, and then discuss the case $b < 0$ briefly in Section 9. In this case, $\nu_i = (a-i)\omega_1 + i(\omega_2 + \omega_{m+1}) + b\omega_2$, $0 \le i \le a$; see Proposition 2.1. Instead of trying to determine the K-equivariant embedding $V^K_\mu \to V^G_{\nu_i}$, we embed V^K_μ in a much bigger G-representation containing $V^G_{\nu_i}$, in which we can identify a K-highest weight of weight μ that "sees" $V^G_{\nu_i}$, i.e. has a nonzero component in $V^G_{\nu_i}$.

Recall that the representation $V^G_{N\omega_1}$ can be realised in the space of polynomials in variables $(x_1, x_2, \ldots, x_{m+2})$ which are homogeneous of degree N. Its G-highest weight vector is x_1^N. Now define the tensor product representation with specific element u:

$$U^G_{\nu_i} = V^G_{(a-i)\omega_1} \otimes (V^G_{\omega_2} \otimes V^G_{\omega_{m+1}})^{\otimes i} \otimes (V^G_{\omega_2})^{\otimes b},$$
$$u = x_1^{a-i} \otimes \underbrace{v_0 \otimes \cdots \otimes v_0}_{i \text{ times}} \otimes \underbrace{e_1 \wedge e_2 \otimes \cdots \otimes e_1 \wedge e_2}_{b \text{ times}}, \tag{4.1}$$

where v_0 is as in Lemma 3.6. Then u is a K-highest weight vector of weight $\mu = a\omega_1 + b\omega_2$ by Lemma 3.6, since $e_1 \wedge e_2 \in V^G_{\omega_2}$ is the G- and K-highest weight vector of weight ω_2. Moreover,

$$U^G_{\nu_i} = V^G_{\nu_i} \oplus \bigoplus_{\lambda \prec \nu_i} n_\lambda V^G_\lambda, \tag{4.2}$$

for certain multiplicities n_λ. Since we are only interested in $\lambda \in P^+_G(\mu)$, we need Lemma 4.1.

Lemma 4.1. *Let* $\mu = a\omega_1 + b\omega_2$ *with* $b \in \mathbb{N}$. *Then* $\{\lambda \in P^+_G(\mu) \mid \lambda \preccurlyeq \nu_i\} = \{\nu_0, \ldots, \nu_i\}$.

Proof. Using the ideas and identities of Section 2.3 we assume $\nu_j + n_1\lambda_1 + n_2\lambda_2 \preccurlyeq \nu_i$, $n_1, n_2 \in \mathbb{N}$. Writing

$$\begin{aligned}
&\nu_i - (\nu_j + n_1\lambda_1 + n_2\lambda_2)\\
&\quad = (i-j)(-\omega_1 + \omega_2 + \omega_{m+1}) - n_1(\omega_1 + \omega_{m+1}) - n_2(\omega_2 + \omega_m)\\
&\quad = (-n_1 - n_2)\alpha_1 + (i - j - n_1 - n_2)\alpha_{m+1} + (i - j - n_1 - 2n_2)\sum_{k=2}^{m} \alpha_k,
\end{aligned}$$

we see that this is in Q^+_G if and only if $n_1 = n_2 = 0$ and $i \geq j$. $\square$

Our next objective is to give an explicit expression for the matrix-valued spherical function associated to the K-equivariant embedding $V^K_\mu \to U^G_{\nu_i}$, which maps the highest weight vector of V^K_μ to u. In order to describe the result, we need the Krawtchouk polynomials; see e.g. [12, §6.2], [18, §9.11]. The Krawtchouk polynomials are defined as a terminating hypergeometric series and are generated by a generating function:

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n, -x\\ -N\end{matrix}; \frac{1}{p}\right), \quad N \in \mathbb{N},\ x, n \in \{0, 1, \ldots, N\},$$
$$\sum_{n=0}^{N} \binom{N}{n} K_n(x;p,N) t^n = \left(1 - \frac{1-p}{p} t\right)^x (1+t)^{N-x}. \tag{4.3}$$

Note that the Krawtchouk polynomials are self-dual: $K_n(x;p,N) = K_x(n;p,N)$, and $K_0(x;p,N) = 1 = K_n(0;p,N)$.

Proposition 4.2. *Let $\mu = a\omega_1 + b\omega_2$, $b \in \mathbb{N}$, $\nu_i = (a-i)\omega_1 + i(\omega_2 + \omega_{m+1}) + b\omega_2$ and $U^G_{\nu_i}$ be the representation defined in (4.1). Then, for $k, l \in \{0, 1, \dots, a\}$,*

$$\langle \pi_{U^G_{\nu_i}}(a_{\mathbf{t}}) F_1^k \cdot u, F_1^l \cdot u\rangle = \delta_{k,l} \|F_1^k \cdot u\|^2 (\cos t_1)^{a+b-k} (\cos t_2)^{b+2i+k} K_i\Big(k; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a\Big),$$

with $a_{\mathbf{t}}$ as in (2.1) and the Krawtchouk polynomials as in (4.3).

Remark 4.3. We make the following observations related to Proposition 4.2:

(i) The fact that we get zero for $k \neq l$ follows from the fact that matrix spherical functions restricted to A are M-intertwiners and the vectors $F^k \cdot u$ correspond to different M-types for different k. Indeed, u spans a 1-dimensional M-representation of weight $\sigma_0(\mu) = a\eta_1 + b\eta_2$ by (4.1) and Lemma 3.6, and more generally $F^k \cdot u$ corresponds to the 1-dimensional M-representation of weight $\sigma_k(\mu) = (a-2k)\eta_1 + (b+k)\eta_2$; see (2.9) for the K-representation generated by u.

(ii) For $k = l$, the right-hand side is a polynomial in $(\cos t_1, \cos t_2)$, and it is a homogeneous polynomial of degree $a + 2b + 2i$ in $(\cos t_1, \cos t_2)$. Note that the degree of homogeneity is independent of k. Indeed, for $k = l$, the right-hand side of Proposition 4.2 equals

$$\|F_1^k \cdot u\|^2 (\cos t_1)^{a+b-k} \sum_{p=0}^{\min(i,k)} \frac{(-i)_p (-k)_p}{p!(-a)_p} (\cos^2 t_2 - \cos^2 t_1)^p (\cos t_2)^{b+2i+k-2p},$$

using (4.3) and the notation for Pochhammer symbols $(x)_p = \prod_{i=0}^{p-1}(x+i)$; see e.g. [1, 12, 18].

(iii) Using $\Phi(ma_{\mathbf{t}}) = \pi^K_\mu(m)\Phi(a_{\mathbf{t}})$ for $m \in A \cap M$, the decomposition (2.9) and $\sigma_k(\mathrm{diag}(\zeta_1, \zeta_2, 1, \dots, 1, \zeta_2, \zeta_1)) = \zeta_1^{a+b-k}\zeta_2^{b+k}$ for $\zeta_i \in \mathbb{Z}/2\mathbb{Z}$, we see that the right-hand side has to be invariant up to $(-1)^{a+b-k}$ under $\cos t_1 \mapsto -\cos t_1$ and invariant up to $(-1)^{b+k}$ under $\cos t_2 \mapsto -\cos t_2$; cf. Remark 3.5. This also follows directly from the explicit expression of Proposition 4.2.

Proof of Proposition 4.2. We put $a_{\mathbf{t}}(r,s) = \exp(sE_1) a_{\mathbf{t}} \exp(rF_1)$. Then using the unitarity of the representation $U^G_{\nu_i}$ we obtain

$$\langle \pi_{U^G_{\nu_i}}(a_{\mathbf{t}}) F_1^k \cdot u, F_1^l \cdot u\rangle = \frac{\partial^k}{\partial r^k}\Big|_{r=0} \frac{\partial^l}{\partial s^l}\Big|_{s=0} \langle \pi_{U^G_{\nu_i}}(a_{\mathbf{t}}(r,s))u, u\rangle. \tag{4.4}$$

Now,

$$a_{\mathbf{t}}(r,s) = \begin{pmatrix} A & 0 & B \\ 0 & I & 0 \\ C & 0 & D \end{pmatrix}, \quad A = \begin{pmatrix} \cos t_1 + rs\cos t_2 & s\cos t_2 \\ r\cos t_2 & \cos t_2 \end{pmatrix},$$

$$B = \begin{pmatrix} si\sin t_2 & i\sin t_1 \\ i\sin t_2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} ri\sin t_2 & i\sin t_2 \\ i\sin t_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \cos t_2 & 0 \\ 0 & \cos t_1 \end{pmatrix},$$

and we can calculate the action of $a_{\mathbf{t}}(r,s)$ on each of the factors in $u \in U^G_{\nu_i}$. We get

$$\begin{aligned} \langle a_{\mathbf{t}}(r,s)\cdot x_1^{a-i}, x_1^{a-i}\rangle &= (\cos t_1 + rs\cos t_2)^{a-i}\langle x_1^{a-i}, x_1^{a-i}\rangle, \\ \langle a_{\mathbf{t}}(r,s)\cdot v_0, v_0\rangle &= \cos t_1\cos t_2(\cos t_2 + rs\cos t_1)\langle v_0, v_0\rangle, \\ \langle a_{\mathbf{t}}(r,s)\cdot e_1\wedge e_2, e_1\wedge e_2\rangle &= \cos t_1 \cos t_2\langle e_1\wedge e_2, e_1\wedge e_2\rangle, \end{aligned}$$

and this gives

$$\begin{aligned} &\langle \pi_{U^G_{\nu_i}}(a_{\mathbf{t}}(r,s))u,u\rangle \\ &\quad = (\cos t_1 + rs\cos t_2)^{a-i}(\cos t_2 + rs\cos t_1)^{i}(\cos t_1\cos t_2)^{b+i}\langle u,u\rangle. \end{aligned} \tag{4.5}$$

The first two factors can be expanded in terms of Krawtchouk polynomials using the generating function of (4.3), and this gives

$$\begin{aligned} &\frac{\langle \pi_{U^G_{\nu_i}}(a_{\mathbf{t}}(r,s))u,u\rangle}{\langle u,u\rangle} \\ &\quad = (\cos t_1)^{a+b}(\cos t_2)^{b+2i}\sum_{n=0}^{a}\binom{a}{n}K_n(i;\frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1},a)\Bigl(rs\frac{\cos t_2}{\cos t_1}\Bigr)^n. \end{aligned}$$

Now the statement of the proposition follows using (4.4). □

Remark 4.4. Note that the right-hand side of (4.5) is a polynomial of the product rs. This follows from (4.4) being zero for $k \neq l$, and this follows from the fact that $a_{\mathbf{t}}$ commutes with M and $F^k\cdot u$ and $F^l\cdot u$ realise different 1-dimensional M-representations for $k\neq l$; cf. Remark 4.3(i).

We can now collect the results of this section into Theorem 4.5.

Theorem 4.5. *Let $\mu = a\omega_1 + b\omega_2$, $a,b\in\mathbb{N}$, and let $\nu_i = (a-i)\omega_1 + i(\omega_2 + \omega_{m+1}) + b\omega_2 \in B(\mu)$, $i\in\{0,\dots,a\}$. Let v_μ be the highest weight vector of V^K_μ, and define $j\colon V^K_\mu \to U^G_{\nu_i}$ to be the K-equivariant map sending $v_\mu \mapsto u$. Then*

$$Q^{\mu}_{\nu_i}: G\to \mathrm{End}(V^K_\mu), \quad g\mapsto j^*\circ \pi_{U^G_{\nu_i}}(g)\circ j$$

is a matrix spherical function and restricted to A we have

$$Q^\mu_{\nu_i}(a_{\mathbf{t}})(F_1^k \cdot v_\mu) = q^\mu_{\nu_i,\sigma_k}(a_{\mathbf{t}}) F_1^k \cdot v_\mu,$$
$$q^\mu_{\nu_i,\sigma_k}(a_{\mathbf{t}}) = (\cos t_1)^{a+b-k}(\cos t_2)^{b+2i+k} K_i\Big(k; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a\Big).$$

Moreover, as matrix spherical functions on G we have

$$Q^\mu_{\nu_i} = \sum_{r=0}^{i} a^i_r \Phi^\mu_{\nu_r}, \quad a^i_r \in \mathbb{C}, \; a^r_r \neq 0.$$

So we see that the transition of the elements $\Phi^\mu_{\nu_i}$, $i \in \{0,\dots,a\}$ to $Q^\mu_{\nu_i}$, $i \in \{0,\dots,a\}$ is given by a triangular matrix with nonzero diagonal entries. Hence, $(Q^\mu_{\nu_i})_{i=0}^a$ and $(\Phi^\mu_{\nu_i})_{i=0}^a$ span the same space of matrix spherical functions, from which the matrix part W of the weight as in (1.9) can be obtained.

Corollary 4.6. *For $i \in \{0,\dots,a\}$ we have $\Phi^\mu_{\nu_i} = \sum_{r=0}^{i} d^i_r Q^\mu_{\nu_r}$ with $d^i_r \in \mathbb{C}$ and $d^r_r \neq 0$.*

Corollary 4.6 and the degree consideration of Remark 4.3 motivate us to consider the explicit matrix spherical function $Q^\mu_{\nu_i}$ as the leading term of the matrix spherical function $\Phi^\mu_{\nu_i}$ of Section 1.1. Note that $d^r_r = (a^r_r)^{-1}$.

Proof of Theorem 4.5. The first statement follows from the general set-up in Section 1.1 and Proposition 4.2. For the last statement we recall that $\{\Phi^\mu_\lambda \mid \lambda \in P^+_G(\mu)\}$ forms a basis for matrix spherical functions; see Section 1.1. By (4.2) and Lemma 4.1 we find that the only matrix spherical functions of type μ occurring in $U^G_{\nu_i}$ are $\Phi^\mu_{\nu_r}$, $r \in \{0,\dots,i\}$. It remains to show that $a^r_r \neq 0$.

In the case $r=0$ we have $Q^\mu_{\nu_0} = \Phi^\mu_{\nu_0}$ since both are the identity in $\mathrm{End}(V^K_\mu)$ for the identity in G, so $a^0_0 = 1$. Assume that $a^i_i \neq 0$ for $i \in \{0,\dots,r-1\}$, $1 \le r \le a$, and $a^r_r = 0$. We show that this leads to a contradiction. Indeed, then $Q^\mu_{\nu_r}$ can be expressed in terms of $\Phi^\mu_{\nu_j}$, $j<r$, which in turn can be expressed in terms of $Q^\mu_{\nu_j}$, $j<r$. Hence, there is a nontrivial linear dependence between the matrix spherical functions $\sum_{j=0}^r c_j Q^\mu_{\nu_j} = 0$. Evaluating at $a_{\mathbf{t}}$, acting on the K-highest weight vector $v_\mu \in V^K_\mu$ and taking inner products with v_μ and using the first part of the theorem, i.e. Proposition 4.2, we get a nontrivial linear dependence of the form

$$\sum_{j=0}^{r} c_j (\cos t_1)^{a+b}(\cos t_2)^{b+2j} = 0 \quad \forall t_1, t_2.$$

This is the required contradiction. ∎

§5. The matrix weight

We keep $\mu = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{N}$ fixed. Then we identify $\mathrm{End}_M(V_\mu^K) \cong \mathbb{C}^{a+1}$ by Schur's lemma and (2.9) and we set

$$\phi^\mu_{\lambda,\sigma_k}(a_\mathbf{t})\colon A \to \mathbb{C}, \quad \Phi^\mu_\lambda(a_\mathbf{t})|_{V^M_{\sigma_k}} = \phi^\mu_{\lambda,\sigma_k}(a_\mathbf{t})\,\mathrm{Id}_{V^M_{\sigma_k}} \tag{5.1}$$

for $\lambda \in P^+_G(\mu)$, $k \in \{0, \dots, a\}$. Note that W-invariance leads to (see (2.11))

$$\phi^\mu_{\lambda,\sigma_k}(s_1 a_\mathbf{t}) = \phi^\mu_{\lambda,\sigma_{a-k}}(a_\mathbf{t}), \quad \phi^\mu_{\lambda,\sigma_k}(s_2 a_\mathbf{t}) = \phi^\mu_{\lambda,\sigma_k}(a_\mathbf{t}), \tag{5.2}$$

and similarly for $q^\mu_{\lambda,\sigma_k}(a_\mathbf{t})$ because of Theorem 4.5. The nontrivial action for $q^\mu_{\lambda,\sigma_k}(a_\mathbf{t})$ corresponds to Pfaff's transformation formula for ${}_2F_1$-series; see e.g. [1, Thm. 2.2.5].

We define the lower triangular matrix L by $L_{i,j} = d^i_j$, $j \leq i$, with d^i_j as in Corollary 4.6. Then L is invertible. Upon defining the matrices Φ_0 and Q_0 on A by $(\Phi_0)_{i,j} = \phi^\mu_{\nu_i,\sigma_j}$ and $(Q_0)_{i,j} = q^\mu_{\nu_i,\sigma_j}$, we see that Corollary 4.6 can be rephrased as $\Phi_0 = LQ_0$, and we calculate L explicitly in Proposition 6.7. Moreover, $\Phi_0(s_1 a_\mathbf{t}) = \Phi_0(a_\mathbf{t})J$, where $J_{i,j} = 1$ if $i + j = a$ and $J_{i,j} = 0$ otherwise, and similarly $Q_0(s_1 a_\mathbf{t}) = Q_0(a_\mathbf{t})J$ by (5.2).

As a function on A we see that the matrix weight W in (1.9) can be written as $\Phi_0\Phi_0^*$, for which each matrix entry is a polynomial in (ϕ_1, ϕ_2). Note that the weight W is a matrix function on A which is invariant for the action of the reduced Weyl group. We switch from the matrix weight W on A to the matrix weight $S = Q_0(Q_0)^*$, so that $W = LSL^*$ as functions on A for the constant lower triangular matrix L. Note that S as a matrix function on A is invariant for the action of the reduced Weyl group. Note that S is a polynomial in (ψ_1, ψ_2) and we have for the matrix entries $S_{i,j}$ of the weight S,

$$\begin{aligned} S_{i,j}(\psi_1(a_\mathbf{t}), \psi_2(a_\mathbf{t})) &= \sum_{k=0}^{a} q^\mu_{\nu_i,\sigma_k}(a_\mathbf{t})\overline{q^\mu_{\nu_j,\sigma_k}(a_\mathbf{t})} \\ &= \sum_{k=0}^{a} (\cos t_1)^{2a+2b-2k}(\cos t_2)^{2b+2k+2i+2j} \\ &\qquad \times K_k\Big(i; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a\Big) K_k\Big(j; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a\Big), \end{aligned} \tag{5.3}$$

and by this expression we see that $S_{i,j}(\psi_1(a_\mathbf{t}), \psi_2(a_\mathbf{t}))$ is a homogeneous polynomial in $(\cos t_1, \cos t_2)$ of degree $2a + 4b + 2i + 2j$. The simplest nonscalar cases for

$a = 1$ and $a = 2$ give the following expressions for $S(\psi_1, \psi_2)$:

$$\psi_2^b \begin{pmatrix} \psi_1 & 2\psi_2 \\ 2\psi_2 & \psi_1\psi_2 \end{pmatrix} \quad \text{and} \quad \psi_2^b \begin{pmatrix} \psi_1^2 - \psi_2 & \frac{3}{2}\psi_1\psi_2 & 3\psi_2^2 \\ \frac{3}{2}\psi_1\psi_2 & 2\psi_2^2 + \frac{1}{4}\psi_1^2\psi_2 & \frac{3}{2}\psi_1\psi_2^2 \\ 3\psi_2^2 & \frac{3}{2}\psi_1\psi_2^2 & \psi_2^2(\psi_1^2 - \psi_2) \end{pmatrix}. \tag{5.4}$$

Note that in (5.4) the matrix part of S is determined by a, and the b-dependence is only in the scalar part ψ_2^b. This follows in general from (5.3).

Proposition 5.1. *The matrix weight S is indecomposable, i.e.*

$$\mathcal{A} = \{T \in M_{a+1}(\mathbb{C}) \mid TS(\psi_1(a_{\mathbf{t}}), \psi_2(a_{\mathbf{t}})) = S(\psi_1(a_{\mathbf{t}}), \psi_2(a_{\mathbf{t}}))T^* \ \forall\, t_1, t_2\} = \mathbb{R}\,\mathrm{Id},$$
$$\mathcal{A}' = \{T \in M_{a+1}(\mathbb{C}) \mid TS(\psi_1(a_{\mathbf{t}}), \psi_2(a_{\mathbf{t}})) = S(\psi_1(a_{\mathbf{t}}), \psi_2(a_{\mathbf{t}}))T \ \forall\, t_1, t_2\} = \mathbb{C}\,\mathrm{Id}\,.$$

Remark 5.2. These notions of indecomposability of the matrix weight for multivariable weights have not yet been introduced, but it follows the definition of the single variable case [20, 34], which can be generalised directly. Note that $\mathcal{A}'$, which is denoted A in [20], is a $*$-algebra, and $\mathcal{A}$ is a real vector space. The corresponding vector spaces for the weight $W = LSL^*$ are then also trivial, which follows directly for $\mathcal{A}$ and the invertibility of L. For $\mathcal{A}'$ this follows from [20, Thm 2.3].

Proof of Proposition 5.1. Recall that the degree of $S_{i,j}$ as a homogeneous polynomial in $(\cos t_1, \cos t_2)$ is $2a + 4b + 2i + 2j$. Assume $T \in \mathcal{A}'$ so that $ST = TS$. We consider the (i, j)th entry:

$$\sum_{k=0}^{a} S_{i,k}(\cos t_1, \cos t_2) T_{k,j} = \sum_{r=0}^{a} T_{i,r} S_{r,j}(\cos t_1, \cos t_2) \quad \forall\, t_1, t_2.$$

Consider this a polynomial identity in $(\cos t_1, \cos t_2)$ and consider the total degree of both sides. Assume that $i < j$; then we see that $T_{i,r} = 0$ for $r > a + i - j$. Taking $j = a$, we see that $T_{i,r} = 0$ for $r > i$. So T is lower triangular. A similar deduction for $i > j$ shows that T is upper triangular, and so T is diagonal. Then we obtain $S_{i,j}(\cos t_1, \cos t_2)T_{j,j} = T_{i,i}S_{i,j}(\cos t_1, \cos t_2)$, and since $S_{i,j}(\cos t_1, \cos t_2)$ is a nonzero function, we find $T_{i,i} = T_{j,j}$. So T is a multiple of the identity.

Assume $T \in \mathcal{A}$ so that $TS = ST^*$. We consider the (i, j)th entry:

$$\sum_{k=0}^{a} S_{i,k}(\cos t_1, \cos t_2) \overline{T_{j,k}} = \sum_{r=0}^{a} T_{i,r} S_{r,j}(\cos t_1, \cos t_2) \quad \forall\, t_1, t_2.$$

Arguing as in the previous case, we see that $i < j$ leads to T being lower triangular. This gives $\sum_{k=0}^{j} S_{i,k}(\cos t_1, \cos t_2)\overline{T_{j,k}} = \sum_{r=0}^{i} T_{i,r}S_{r,j}(\cos t_1, \cos t_2)$. Considering the homogeneous part of highest degree $2a + 4b + 2i + 2j$ gives $\overline{T_{j,j}} = T_{i,i}$, so that each diagonal entry is equal to the same real number. Next, comparing the homogeneous part of the same degree leads to $S_{i,k}(\cos t_1, \cos t_2)\overline{T_{j,k}} = T_{i,k+i-j}S_{k+i-j,j}(\cos t_1, \cos t_2)$, so that in the case $j > i$ we get $\overline{T_{j,k}} = 0$ for $0 \leq k < j-i$. Taking $i = 0$ shows that T is upper triangular. Hence, T is a real multiple of the identity. □

Next we calculate the determinant of S. For this it suffices to calculate the determinant of Q_0 for which we use the orthogonality properties of the Krawtchouk polynomials. Recall e.g. [12, §6.2], [18, §9.11], using the notation of (4.3), the orthogonality relations

$$\sum_{x=0}^{N} w(x;p,N)K_n(x;p,N)K_m(x;p,N) = \delta_{m,n}h(n;p,N), \tag{5.5}$$

where

$$w(x;p,N) = \binom{N}{x}p^x(1-p)^{N-x}, \quad h(n;p,N) = \frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n,$$

which is a positive finite discrete measure for $0 < p < 1$. Rewriting shows that the matrix

$$B = \left(\frac{\sqrt{w(x;p,N)}}{\sqrt{h(n;p,N)}}K_n(x;p,N)\right)_{n,x=0}^{N}$$

is an orthogonal matrix, so of determinant ± 1. Writing B as the product of a diagonal matrix times the matrix whose entries are the Krawtchouk polynomials times a diagonal matrix, and introducing additional parameters gives

$$\begin{aligned} &\det(t^n s^x K_n(x;p,N))_{n,x=0}^{N} \\ &= \pm(st)^{\frac{1}{2}N(N+1)}\left(\prod_{n=0}^{N} h(n;p,N)\right)^{\frac{1}{2}}\left(\prod_{x=0}^{N} w(x;p,N)\right)^{-\frac{1}{2}}. \end{aligned} \tag{5.6}$$

Proposition 5.3. *For $\mu = a\omega_1 + b\omega_2$, $a, b \in \mathbb{N}$, $a_{\mathbf{t}} \in A$ we have*

$$\begin{aligned} \det(S(a_{\mathbf{t}})) = \left(\prod_{n=0}^{a}\binom{a}{n}\right)^{-2} &(\cos t_1 \cos t_2)^{2b(a+1)} \\ &\times (\cos t_1 \cos t_2(\cos^2 t_1 - \cos^2 t_2))^{a(a+1)}. \end{aligned}$$

Proof. With $Q_0(a_{\mathbf{t}})_{i,j} = q^{\mu}_{\nu_i,\sigma_j}(a_{\mathbf{t}})$, $0 \leq i, j \leq a$, expressed in Theorem 4.5 in terms of Krawtchouk polynomials, we take out the terms independent of i, j, and we next apply (5.6) to get

$$\det(Q_0(a_{\mathbf{t}})) = \pm(\cos^{a+b} t_1 \cos^b t_2)^{a+1}\Big(\frac{\cos t_2}{\cos t_1}\Big)^{\frac{1}{2}a(a+1)}(\cos^2 t_2)^{\frac{1}{2}a(a+1)} \\ \times \Big(\prod_{n=0}^{a}\binom{a}{n}\Big)^{-1}\Big(\frac{1-p}{p}\Big)^{\frac{1}{2}a(a+1)}(1-p)^{-\frac{1}{2}a(a+1)},$$

with $p = \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}$ as in Proposition 4.2, using that $\frac{(-a)_n}{(-1)^n n!} = \binom{a}{n}$. Here we assume for the time being that $0 < p < 1$, so that all square roots are well defined. Simplifying gives

$$\det(Q_0(a_{\mathbf{t}})) = \pm\Big(\prod_{n=0}^{a}\binom{a}{n}\Big)^{-1}(\cos t_1 \cos t_2)^{b(a+1)} \\ \times (\cos t_1 \cos t_2(\cos^2 t_1 - \cos^2 t_2))^{\frac{1}{2}a(a+1)},$$

and this proves the statement for $0 < p < 1$. Since we know all entries of S are polynomial in $(\cos t_1, \cos t_2)$, cf. Remark 4.3(ii), the determinant of S is polynomial in $(\cos t_1, \cos t_2)$ and the result holds for all $a_{\mathbf{t}}$. □

Remark 5.4. Now, by the results of Section 1.1 and [10, Prop. X.1.19], we have (1.9) involving the matrix weight W, hence S. In this case, $\delta\colon A \to \mathbb{R}$ is given by

$$\delta(a_{\mathbf{t}}) = (\sin t_1)^{2(m-2)}(\sin t_2)^{2(m-2)}\sin(2t_1)\sin(2t_2)\sin^2(t_1+t_2)\sin^2(t_1-t_2)$$
$$= 4(\sin t_1)^{2m-3}(\sin t_2)^{2m-3}\cos t_1\cos t_2(\cos^2 t_1 - \cos^2 t_2)^2 \tag{5.7}$$

(see [10, §X.5]) using Appendix A.1. In particular, from Proposition 5.3 and (5.7) we see that $\det(S(a_{\mathbf{t}})) = 0$ implies $\delta(a_{\mathbf{t}}) = 0$.

§6. Radial part of the Casimir operator

In order to obtain precise information on matrix spherical functions in their relation to the matrix functions $Q^{\mu}_{\nu_i}$ in Theorem 4.5 and Corollary 4.6, we use the Casimir operator. Since the Casimir operator acts as a multiple of the identity in a representation π^G_λ with scalar $c_\lambda = \langle\lambda,\lambda\rangle + 2\langle\lambda,\rho\rangle$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ (see [16, Prop. 5.28]), we have

$$R^\mu(\Omega)\Phi^\mu_\lambda|_A = c_\lambda \Phi^\mu_\lambda|_A, \tag{6.1}$$

where $R^\mu(\Omega)$ is the radial part of the Casimir operator as in the Appendix. For convenience, the explicit expression for $R^\mu(\Omega)$ is derived in the Appendix. The functions $\Phi^\mu_\lambda|_A$ are eigenfunctions of a much larger class of differential operators arising from a subalgebra of the universal enveloping algebra [5, Chap. 9], but we only use the Casimir operator. The eigenvalues play an important role in order to distinguish the eigenfunctions.

Lemma 6.1. *Let $\lambda_1, \lambda_2 \in P_G^+$ with $\lambda_1 \prec \lambda_2$ and $\lambda_1 \neq \lambda_2$; then $c_{\lambda_1} < c_{\lambda_2}$.*

Proof. Rewriting $c_\lambda = \langle \lambda+\rho, \lambda+\rho\rangle - \langle \rho,\rho\rangle$, then

$$c_{\lambda_2} - c_{\lambda_1} = \langle \lambda_2+\rho, \lambda_2+\rho\rangle - \langle \lambda_1+\rho, \lambda_1+\rho\rangle = \langle \lambda_1+\lambda_2+2\rho, \lambda_2-\lambda_1\rangle,$$

and since $\lambda_1+\lambda_2+2\rho$ is in the interior of the positive Weyl chamber and $\lambda_2-\lambda_1 \in Q_G^+$, the right-hand side is positive. □

As a first application we calculate the constants in Lemmas 3.3 and 3.4.

Lemma 6.2. *With the notation of Section 3 we have as functions on A,*

$$\psi_1 = \frac{2m\phi_1+4}{m+2}, \quad \psi_2 = \frac{m-1}{m+2}\phi_2 + \frac{2(m+1)}{(m+2)^2}\phi_1 + \frac{2(m+1)(2m-1)}{m^2(m+2)^2}.$$

Remark 6.3. The relation is invertible:

$$\phi_1 = \frac{(m+2)\psi_1 - 4}{m}, \quad \phi_2 = \frac{m(m+1)\psi_2 - (m+1)\psi_1 + 2}{m(m-1)}.$$

Proof of Lemma 6.2. In the case $\mu = 0$, $R^0(\Omega)$ is an explicit second-order partial differential operator (see (A.5)), where all terms involving π_μ^K are set to zero. Put $f_1(t_1,t_2) = \psi_1(a_{\mathbf{t}}) = \cos^2 t_1 + \cos^2 t_2$. Then by a trigonometric calculation (or using computer algebra), $R^0(\Omega) f_1 = (2m+4) f_1 - 8$. Since

$$c_{\lambda_1} = \left\langle \omega_1 + \omega_{m+1}, \sum_{n=1}^{m+1} \alpha_n \right\rangle + 2 \sum_{1\leq i<j\leq m+2} \left\langle \omega_1+\omega_{m+1}, \sum_{p=i}^{j-1} \alpha_p \right\rangle = 2m+4$$

using (2.10), we get that $-8a = c_{\lambda_1} b$ when writing $\phi_1 = a\psi_1 + b$ using Lemma 3.3 and $R^0(\Omega)\phi_1 = c_{\lambda_1}\phi_1$. Evaluating at the identity, using $\phi_1(e) = 1$, $\psi_1(e) = 2$, fixes the constant.

For ψ_2, put $f_2(t_1,t_2) = \psi(a_{\mathbf{t}}) = \cos^2 t_1 \cos^2 t_2$. Then we find $R^0(\Omega) f_2 = (4m+4) f_2 - 2 f_1$. In this case $c_{\lambda_2} = 4m+4$, and we identify the expansion by considering the eigenvalue equation and the evaluation at e using the first result as well. □

Next we go back to the situation of $\mu = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{N}$. The basis $(F_1^k \cdot v_\mu)_{k=0}^a$, v_μ being the highest weight vector of V_μ^K, gives the M-decomposition, and

$$\begin{aligned} \pi_\mu^K(E_{1,1}) F_1^k \cdot v_\mu &= (a+b-k) F_1^k \cdot v_\mu, \quad \pi_\mu^K(E_{2,2}) F_1^k \cdot v_\mu = (b+k) F_1^k \cdot v_\mu, \\ \pi_\mu^K(F_1) F_1^k \cdot v_\mu &= F_1^{k+1} \cdot v_\mu, \quad \pi_\mu^K(E_1) F_1^k \cdot v_\mu = k(a-k+1) F_1^{k-1} \cdot v_\mu, \end{aligned} \tag{6.2}$$

and we see that almost all actions of the Lie algebra in the expression for the radial part of the Casimir operator $R^\mu(\Omega)$ of (A.5) commute with the action of M. Only for the third line of expression (A.6) for $R^\mu_m(\Omega)$ corresponding to the middle roots of BC$_2$ do we get a nontrivial interaction of the M-types. Put $G_k = \langle G(\cdot)F_1^k \cdot v_\mu, F_1^k \cdot v_\mu\rangle\colon A \to \mathbb{C}$ for the scalar action of $G\colon A \to \mathrm{End}_M(V^K_\mu)$ on $V^M_{\sigma_k} \subset V^K_\mu$. Then we can rewrite the radial part of the Casimir operator (see Appendix A.3) as

$$
\begin{aligned}
(R^\mu(\Omega)G)_k &= (R^\mu(\Omega_{\mathfrak{m}})G)_k \\
&\quad - \frac{1}{2}\sum_{p=1}^{2} \frac{\partial^2 G_k}{\partial t_p^2} + (R^\mu_s(\Omega)G)_k + (R^\mu_m(\Omega)G)_k + (R^\mu_l(\Omega)G)_k,
\end{aligned}
\tag{6.3}
$$

where the respective parts are given by

$$(R^\mu(\Omega_{\mathfrak{m}})G)_k = \frac{1}{2(m+2)}(m(a+b-k)^2 - 4(a+b-k)(b+k) + m(b+k)^2)G_k$$

for the action corresponding to $\Omega_{\mathfrak{m}}$, the term for the short roots is equal to

$$(R^\mu_s(\Omega)G)_k = -(m-2)\sum_{i=1}^{2} \frac{\cos t_i}{\sin t_i}\frac{\partial G_k}{\partial t_i},$$

the term for the middle roots gives

$$
\begin{aligned}
&(R^\mu_m(\Omega)G)_k \\
&\quad = -\frac{\cos(t_1+t_2)}{\sin(t_1+t_2)}\Big(\frac{\partial G_k}{\partial t_1} + \frac{\partial G_k}{\partial t_2}\Big) - \frac{\cos(t_1-t_2)}{\sin(t_1-t_2)}\Big(\frac{\partial G_k}{\partial t_1} - \frac{\partial G_k}{\partial t_2}\Big) \\
&\qquad - \Big(\frac{\cos(t_1+t_2)}{\sin^2(t_1+t_2)} + \frac{\cos(t_1-t_2)}{\sin^2(t_1-t_2)}\Big)((k+1)(a-k)G_{k+1} + k(a-k+1)G_{k-1}) \\
&\qquad + \Big(\frac{1}{\sin^2(t_1+t_2)} + \frac{1}{\sin^2(t_1-t_2)}\Big)\big(((k+1)(a-k) + k(a-k+1))G_k\big)
\end{aligned}
$$

and the term for the long roots simplifies to

$$(R^\mu_l(\Omega)G)_k = -\sum_{i=1}^{2} \frac{\cos(2t_i)}{\sin(2t_i)}\frac{\partial G_k}{\partial t_i} + \frac{(a+b-k)^2}{2\cos^2 t_1}G_k + \frac{(b+k)^2}{2\cos^2 t_2}G_k.$$

Having described the radial part of the Casimir operator explicitly, we can use the action to make the constants in Theorem 4.5 and Corollary 4.6 explicit.

Proposition 6.4. *As functions $A \to \mathrm{End}_M(V^K_\mu)$ we have*

$$\Phi^\mu_{\nu_i} = \frac{(m+b+i)_i}{(m)_i}\sum_{r=0}^{i} \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)!(1-m-2i-b)_{i-r}} Q^\mu_{\nu_r}.$$

The key ingredient in the proof of Proposition 6.4 is the action of the radial part of the Casimir operator on the functions $Q^{\mu}_{\nu_i}\colon A \to \mathrm{End}_M(V^K_\mu)$.

Lemma 6.5. *For $i \in \{0, \ldots, a\}$ and $Q^{\mu}_{\nu_i}$ as in Theorem 4.5 we have*

$$R^{\mu}(\Omega)Q^{\mu}_{\nu_i} = c_{\nu_i}Q^{\mu}_{\nu_i} - 2i(b+i)Q^{\mu}_{\nu_{i-1}},$$

where c_{ν_i} is the eigenvalue of $\Phi^{\mu}_{\nu_i}$ for $R^{\mu}(\Omega)$:

$$c_{\nu_i} = 2i^2 + 2i(b+m) + (m+1)a + 2mb + \frac{1}{m+2}((m+1)a^2 + 2mb(a+b)).$$

Proof. Note that $R^{\mu}(\Omega)\Phi^{\mu}_{\nu_i} = c_{\nu_i}\Phi^{\mu}_{\nu_i}$ with $c_{\nu_i} = \langle \nu_i, \nu_i\rangle + 2\langle \nu_i, \rho\rangle$, and the explicit value of c_{ν_i} follows by a calculation. This shows that $c_{\nu_i} < c_{\nu_{i+1}}$. This also follows more generally from Corollary 2.5 and Lemma 6.1. Since the transition of the basis of $(\Phi^{\mu}_{\nu_i})_{i=0}^{a}$ to the basis $(Q^{\mu}_{\nu_i})_{i=0}^{a}$ is triangular, we find $R^{\mu}(\Omega)Q^{\mu}_{\nu_i} = c_{\nu_i}Q^{\mu}_{\nu_i} + \sum_{r=0}^{i-1} C_r Q^{\mu}_{\nu_r}$ for certain constants C_r. These constants can be determined considering the action on $V^M_{\sigma_0}$ of this identity using $q^{\mu}_{\nu_i,\sigma_0}(a_{\mathbf{t}}) = (\cos t_1)^{a+b}(\cos t_2)^{b+2i}$ and

$$q^{\mu}_{\nu_i,\sigma_1}(a_{\mathbf{t}}) = \frac{a-i}{a}(\cos t_1)^{a+b-1}(\cos t_2)^{b+2i+1} + \frac{i}{a}(\cos t_1)^{a+b+1}(\cos t_2)^{b+2i-1}, \tag{6.4}$$

where we use $K_1(x;p,N) = 1 - \frac{x}{pN}$ for the Krawtchouk polynomials; see Theorem 4.5. Using this we find by a trigonometric calculation (using computer algebra),

$$(R^{\mu}(\Omega)Q^{\mu}_{\nu_i})_0 = c_{\nu_i}(\cos t_1)^{a+b}(\cos t_2)^{b+2i} - 2i(b+i)(\cos t_1)^{a+b}(\cos t_2)^{b+2i-2}.$$

The right-hand side is $c_{\nu_i}q^{\mu}_{\nu_i,\sigma_0}(a_{\mathbf{t}}) - 2i(b+i)q^{\mu}_{\nu_{i-1},\sigma_0}(a_{\mathbf{t}})$, so that $C_{i-1} = -2i(b+i)$ and $C_r = 0$ for $r < i-1$ since the functions $q^{\mu}_{\nu_i,\sigma_k}$ are independent for $i \in \{0, \ldots, a\}$. $\square$

Remark 6.6. The fact that the right-hand side of Lemma 6.5 consists of just two matrix leading terms makes it possible to derive many explicit results for matrix spherical functions. This is one of the main motivations to consider these specific leading terms.

Proof of Proposition 6.4. Apply $R^{\mu}(\Omega)$ to Corollary 4.6, using that the $\Phi^{\mu}_{\nu_i}$ are eigenfunctions for $R^{\mu}(\Omega)$, Lemma 6.5 and that the $Q^{\mu}_{\nu_i}$ are linearly independent, to find the recursion $d^i_r c_{\nu_i} = d^i_r c_{\nu_r} - d^i_{r+1}2(r+1)(b+r+1)$ for $r < i$. Using the value for c_{ν_i} as in Lemma 6.5 we obtain

$$d^i_r(i-r)(b+m+r+i) = -d^i_{r+1}(r+1)(b+r+1)$$
$$\Longrightarrow \quad d^i_r = \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)!(1-m-2i-b)_{i-r}}d^i_i$$

by iteration and it remains to determine d^i_i. Evaluating at the identity element $e \in A$ and using that $Q^\mu_{\nu_r}$ and $\Phi^\mu_{\nu_i}$ are the identity at e, we find

$$\frac{1}{d^i_i} = \sum_{r=0}^{i} \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)!(1-m-2i-b)_{i-r}} = {}_2F_1\left(\begin{matrix}-i, -i-b\\ 1-m-2i-b\end{matrix}; 1\right) = \frac{(1-m-i)_i}{(1-m-2i-b)_i}$$

by the Chu–Vandermonde summation; see e.g. [1, Cor. 2.2.3], [12, §1.4]. Simplifying d^i_i gives the result. □

As the next step we translate Proposition 6.4 into the transition for the matrix weight W and S. Recall the matrix functions Φ_0 and Q_0 as defined in Section 5.

Proposition 6.7. *We have $\Phi_0 = LQ_0$ with the constant lower triangular matrix L given by $L_{i,j} = 0$ for $j > i$ and*

$$L_{i,j} = (-1)^{i+j}\binom{i}{j}\frac{(m+b+i)_i}{(m)_i}\frac{(b+j+1)_{i-j}}{(m+i+j+b)_{i-j}}, \qquad 0 \le j \le i \le a,$$

and its inverse is the lower triangular matrix given by $(L^{-1})_{i,j} = 0$ for $j > i$ and

$$(L^{-1})_{i,j} = \binom{i}{j}\frac{(m)_j}{(m+b+j)_j}\frac{(b+j+1)_{i-j}}{(m+2j+b-1)_{i-j}}, \qquad 0 \le j \le i \le a.$$

Proof. Recall from Section 5 and Proposition 6.4 that as functions on A we have

$$(\Phi_0)_{i,k} = \phi^\mu_{\nu_i,\sigma_k} = (\Phi^\mu_{\nu_i})_k = \sum_{r=0}^{i} d^i_r (Q^\mu_{\nu_r})_k = \sum_{r=0}^{i} d^i_r q^\mu_{\nu_r,\sigma_k} = \sum_{r=0}^{a} L_{i,r}(Q_0)_{r,k}$$

with $L_{i,r} = d^i_r$ for $i \le r$ and $L_{i,r} = 0$ for $i > r$. Rewriting gives the matrix L.

To show that L^{-1} is as given, we need to show the nontrivial case: for $j \le i$ we have to show $\sum_{r=j}^{i} L_{i,r}(L^{-1})_{r,j} = \delta_{i,j}$. Taking out the r-independent parts, we see that this is equivalent to showing

$$\delta_{i,j} = \sum_{r=j}^{i}(-1)^{i+r}\binom{i}{r}\frac{(b+r+1)_{i-r}}{(m+i+r+b)_{i-r}}\binom{r}{j}\frac{(b+j+1)_{r-j}}{(m+2j+b-1)_{r-j}}.$$

The right-hand side can be rewritten as

$$\frac{(b+i+j)_{i-j}}{(m+i+j+b-1)_{i-j}}\binom{i}{j}(-1)^{i+j}\sum_{k=0}^{i-j}\frac{(j-i)_k(m+i+j+b-1)_k}{k!(m+2j+b)_k}$$

and the sum is a terminating ${}_2F_1$-series at 1, which can be evaluated by the Chu–Vandermonde summation (see e.g. [1, Cor. 2.2.3], [12, §1.4]) as $\frac{(1+j-i)_{i-j}}{(m+2j+b)_{i-j}}$ so that the numerator gives 0 unless $i = j$, in which case we find 1. □

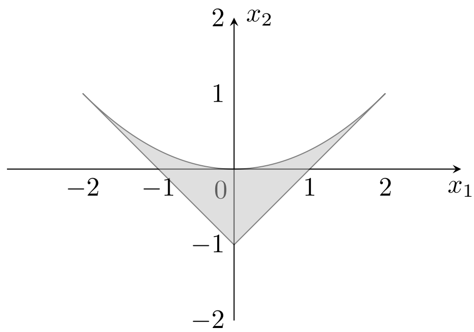

Figure 1. Integration region I of Theorem 7.1.

§7. Matrix orthogonal polynomials in two variables

In Section 5 we established the matrix weight for the polynomials, and in this section we establish some more properties for these matrix orthogonal polynomials in two variables with BC_2-symmetry. In particular, we make the orthogonality relations more explicit. Moreover, we derive the matrix partial differential operator to which these matrix polynomials are eigenfunctions.

First, the Haar measure on A is $dt_1\,dt_2$ on $[-\pi,\pi]\times[-\pi,\pi]$ and using the invariance under the sign changes, we can reduce to the integral over $[0,\frac{1}{2}\pi]\times[0,\frac{1}{2}\pi]$. Using (5.7) we find for the normalising constant in (1.9),

$$\frac{1}{c}=\int_A |\delta(a)|\,da = 4^2\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}|\delta(a_{\mathbf{t}})|\,dt_1\,dt_2=\frac{32}{m^2(m^2-1)}. \tag{7.1}$$

In order to make the connection to the BC_2-case as originally introduced by Koornwinder [25, 24] (see also [32, 30]), we make an affine change of variable $\psi_1=\frac{1}{2}x_1+1$, $\psi_2=\frac{1}{4}x_2+\frac{1}{4}x_1+\frac{1}{4}$, or, in terms of t_1 and t_2, $x_1=\cos(2t_1)+\cos(2t_2)$, $x_2=\cos(2t_1)\cos(2t_2)$. Then the map sending $(t_1,t_2)\in[0,\frac{1}{2}\pi]\times[0,\frac{1}{2}\pi]$ to (x_1,x_2) is a $2:1$ mapping onto the region bounded by the parabola $x_1^2=4x_2$ and the lines $x_2=x_1-1$, $x_2=1-x_1$; see Figure 1. This is exactly the region of integration for the polynomials studied in [25, 24, 32]. For $\mathbf{d}=(d_1,d_2)\in\mathbb{N}^2$ we define matrix polynomials $R_{\mathbf{d}}$ of size $(a+1)\times(a+1)$ of degree $\mathbf{d}$ by

$$R_{\mathbf{d}}(x_1,x_2)=P_{\mathbf{d}}(\phi_1,\phi_2)L, \tag{7.2}$$

where we use the notation for $P_{\mathbf{d}}$ as in Section 1.1, the affine transformation from (x_1,x_2) to (ψ_1,ψ_2) as given above and the affine transformation from Lemma 6.2, and with L as in Proposition 6.7. Finally, we define the matrix weight

$$S^a(x_1, x_2) = S(\psi_1, \psi_2) \tag{7.3}$$

with $S(\psi_1(a_{\mathbf{t}}), \psi_2(a_{\mathbf{t}}))$ defined in (5.3) for the case $a \in \mathbb{N}$, $b = 0$, and using the coordinate change of (ψ_1, ψ_2) to (x_1, x_2). In the case $a = 1$ we obtain

$$S^1(x_1, x_2) = \begin{pmatrix} \frac{1}{2}x_1 + 1 & \frac{1}{2}(x_1 + x_2 + 1) \\ \frac{1}{2}(x_1 + x_2 + 1) & \frac{1}{4}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) \end{pmatrix},$$

and for $a = 2$ we obtain that $S^2(x_1, x_2)$ equals

$$\begin{pmatrix} (\frac{1}{2}x_1 + 1)^2 - \frac{x_1+x_2+1}{4} & \frac{3}{8}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) & \frac{3}{16}(x_1 + x_2 + 1)^2 \\ \frac{3}{8}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) & \frac{(x_1+x_2+1)(2(x_1+x_2+1)+(\frac{1}{2}x_1+1)^2)}{16} & \frac{3}{32}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1)^2 \\ \frac{3}{16}(x_1 + x_2 + 1)^2 & \frac{3}{32}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1)^2 & \frac{(x_1+x_2+1)^2((\frac{1}{2}x_1+1)^2 - \frac{x_1+x_2+1}{4})}{16} \end{pmatrix}.$$

These examples follow from (5.4) taking $b = 0$ and $\psi_1 = \frac{1}{2}x_1 + 1$, $\psi_2 = \frac{1}{4}x_2 + \frac{1}{4}x_1 + \frac{1}{4}$.

Theorem 7.1. *The matrix polynomials $R_{\mathbf{d}}$ defined by (7.2) are orthogonal on the region of integration I as in Figure 1, and*

$$\begin{aligned} \iint_I & R_{\mathbf{d}}(x_1, x_2) S^a(x_1, x_2) (R_{\mathbf{d}'}(x_1, x_2))^* (1 - x_1 + x_2)^{m-2} (1 + x_1 + x_2)^b \\ & \times (x_1^2 - 4x_2)^{\frac{1}{2}}\, dx_1\, dx_2 \\ & = \delta_{\mathbf{d},\mathbf{d}'} 2^{2m+2b-10} m^2 (m^2 - 1) H_{\mathbf{d}} \end{aligned}$$

where S^a is positive definite on I with positive determinant on the interior of I. Moreover, the weight function is indecomposable. Here the matrix $H_{\mathbf{d}}$ is a diagonal matrix with $(H_{\mathbf{d}})_{k,k} = (a+1)^2 / \dim V^G_{\nu_k + d_1\lambda_1 + d_2\lambda_2}$.

Moreover, the polynomials $R_{\mathbf{d}}$ are eigenfunctions to a second-order matrix partial differential operator:

$$R_{\mathbf{d}} R^0(\Omega) - R_{\mathbf{d}} C^\mu + R_{\mathbf{d}}(\Lambda_0 + S) = \Lambda_{\mathbf{d}} R_{\mathbf{d}},$$

where $\Lambda_{\mathbf{d}} = \mathrm{diag}(c_{\nu_i + d_1\lambda_1 + d_2\lambda_2})_{i=0}^a$, $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$, $\Lambda_0 = \Lambda_{(0,0)}$ and S is the lower triangular matrix with one nonzero subdiagonal with $S_{r,r-1} = -2r(b+r)$. The operator $R^0(\Omega)$ is the second-order partial differential operator acting from the right as the identity times the classical partial differential operator

$$\begin{aligned} & (2x_1^2 - 4x_2 - 4)\frac{\partial^2}{\partial x_1^2} + (-2x_1^2 + 4x_2^2 + 4x_2)\frac{\partial^2}{\partial x_2^2} + 4x_1(x_2 - 1)\frac{\partial^2}{\partial x_1 \partial x_2} \\ & \quad + 2((m+2)x_1 + 2m - 4)\frac{\partial}{\partial x_1} + 2((m-2)x_1 + 2 + (2m+2)x_2)\frac{\partial}{\partial x_2}, \end{aligned}$$

and C^μ is the first-order matrix differential operator $\frac{\partial}{\partial x_1}C_1^\mu + \frac{\partial}{\partial x_2}C_2^\mu$, where C_1^μ and C_2^μ are tridiagonal polynomial matrices of degree 1 *given by*

$$\begin{aligned}
(C_1^\mu)_{r,r} &= 2((a+b+r)x_1 - 2b - 2r),\\
(C_2^\mu)_{r,r} &= 2((b+r)(2x_2 - x_1) + a(x_2+1)),\\
(C_1^\mu)_{r,r-1} = (C_2^\mu)_{r,r-1} &= -r(x_1+x_2+1),\\
(C_1^\mu)_{r,r+1} = (C_2^\mu)_{r,r+1} &= -4(a-r).
\end{aligned}$$

Moreover, the matrix partial differential operator is symmetric with respect to the matrix weight.

Remark 7.2. Note that the scalar part of the weight in Theorem 7.1 is the weight considered by Koornwinder [25, 24] and Sprinkhuizen-Kuyper [32] for the special case $\alpha = m-2$, $\beta = b$, $\gamma = \frac{1}{2}$. Similarly, in the case $\mu = 0$, i.e. $a = b = 0$, the partial differential operator reduces to the partial differential operator studied in [25, 24, 32] up to a scalar multiple for these choices of parameters. The case $a = 0$, $b \in \mathbb{N}$, gives the case of a nontrivial character of K, and this corresponds to Heckman and Schlichtkrull [9, Chap. 5].

Note that in the scalar case, the 2-variable orthogonal polynomials can be expressed in terms of Jacobi polynomials [24, eq. (3.13)], [32, Lem. 3.1]. It is not clear whether in this case we also have an explicit expression for $R_{\mathbf{d}}(x_1, x_2)$ in terms of matrix Jacobi polynomials of a single variable.

Proof of the orthogonality in Theorem 7.1. Observe that the Jacobian for the change of (t_1, t_2) to (x_1, x_2) is given by

$$32|\sin(t_1)\sin(t_2)\cos(t_1)\cos(t_2)(\cos^2(t_1) - \cos^2(t_2))|$$

and $\sin^2(t_1)\sin^2(t_2) = \frac{1}{4}(1-x_1+x_2)$, $(\cos^2(t_1)-\cos^2(t_2))^2 = \frac{1}{4}(x_1^2-4x_2)$. Keeping track of the constants involved, the statements on the orthogonality follow from (1.9) and from Section 5, in particular Proposition 5.3 and Remark 5.4, and $S = Q_0Q_0^*$ being positive. $\square$

In order to prove the statement of Theorem 7.1 concerning the partial differential operator, we need to be able to rewrite the eigenvalue equation of the radial part of the Casimir operator $R^\mu(\Omega)$ acting on the eigenvector $\Phi_\lambda^\mu|_A$ in terms of an operator acting on the polynomials $R_{\mathbf{d}}$. For this we need to conjugate $R^\mu(\Omega)$ with the matrix function Q_0; see [23, §3.2]. We collect the technical results in Lemma 7.3.

Lemma 7.3. *We have for $i = 1, 2$, as matrix-valued functions on A,*

$$\frac{\partial \psi_i}{\partial t_1}\frac{\partial Q_0}{\partial t_1} + \frac{\partial \psi_i}{\partial t_2}\frac{\partial Q_0}{\partial t_2} = C_i(\psi_1, \psi_2)Q_0,$$

where we consider the functions as functions of (t_1, t_2) by evaluating at $a_{\mathbf{t}} \in A$. Here $C_i(\psi_1, \psi_2)$ is a matrix polynomial in (ψ_1, ψ_2) of total degree at most 1, where the nonzero entries are explicitly given by

$$\begin{aligned} C_1(\psi_1, \psi_2)_{r,r} &= 2(a + 2b + 2r - (a + b + r)\psi_1), \\ C_2(\psi_1, \psi_2)_{r,r} &= 2((b + r)\psi_1 - (a + 2b + 2r)\psi_2), \\ C_1(\psi_1, \psi_2)_{r,r-1} = C_2(\psi_1, \psi_2)_{r,r-1} &= 2r\psi_2, \\ C_1(\psi_1, \psi_2)_{r,r+1} = C_2(\psi_1, \psi_2)_{r,r+1} &= 2(a - r). \end{aligned}$$

Note that the tridiagonal matrices coincide on the off-diagonal entries.

There are analogues of Lemma 7.3 with Q_0 replaced by Φ_0 and ψ_i replaced by ϕ_i or x_i; see also the first paragraph of the proof. However, in general it is hard to calculate the right-hand side explicitly. In this case we can do the explicit calculation because of the homogeneity properties of the entries of Q_0 and ψ_1, ψ_2.

Proof of Lemma 7.3. Lemma 3.9 of [23] implies that

$$\frac{\partial \phi_i}{\partial t_1}\frac{\partial \Phi_0}{\partial t_1} + \frac{\partial \phi_i}{\partial t_2}\frac{\partial \Phi_0}{\partial t_2} = C_i'(\phi_1, \phi_2)\Phi_0$$

for a matrix polynomial C_i' in (ϕ_1, ϕ_2) of maximal total degree 1, where we use the adjoint of [23, Lem. 3.9]. Using $\Phi_0 = LQ_0$ and the affine transformation of (ϕ_1, ϕ_2) to (ψ_1, ψ_2) given in Lemma 6.2 proves the general statement of the lemma, and it remains to determine the polynomials C_i.

Take $i = 1$ and consider the (r, s)-entry of the left-hand side of the identity. Since $(Q_0)_{r,s}(a_{\mathbf{t}}) = q^{\mu}_{\nu_r,\sigma_s}(a_{\mathbf{t}})$ is a homogeneous polynomial of degree $a + 2b + 2r$ in $(\cos t_1, \cos t_2)$, we see by an explicit calculation that

$$\frac{\partial \psi_1}{\partial t_1}\frac{\partial (Q_0)_{r,s}}{\partial t_1} + \frac{\partial \psi_1}{\partial t_2}\frac{\partial (Q_0)_{r,s}}{\partial t_2} = 2(a + 2b + 2r)(Q_0)_{r,s} + \mathcal{E}_{r,s}, \tag{7.4}$$

where $\mathcal{E}_{r,s}$ is a homogeneous polynomial of degree $a+2b+2r+2$ in $(\cos t_1, \cos t_2)$. Since ψ_1, respectively ψ_2, is homogeneous of degree 2, respectively 4, and the fact that $C_1(\psi_1, \psi_2)$ is of degree at most 1, we have $\mathcal{E}_{r,s} = a_r\psi_1(Q_0)_{r,s} + b_r\psi_2(Q_0)_{r-1,s} + c_r(Q_0)_{r+1,s}$ for coefficients a_r, b_r and c_r. So we see that $C_1(\psi_1, \psi_2)$ is a tridiagonal matrix, and we have to determine the coefficients. For this we take $s = 0$ and recall from Theorem 4.5 that $(Q_0)_{r,0}(a_{\mathbf{t}}) = q^{\mu}_{\nu_r,\sigma_0}(a_{\mathbf{t}}) = (\cos t_1)^{a+b}(\cos t_2)^{b+2r}$.

Therefore,

$$\begin{aligned}\frac{\partial\psi_1}{\partial t_1}\frac{\partial(Q_0)_{r,0}}{\partial t_1}+\frac{\partial\psi_1}{\partial t_2}\frac{\partial(Q_0)_{r,0}}{\partial t_2}&\\ =2(a+b+2r)(Q_0)_{r,0}-2(a+b)(\cos t_1)^{a+b+2}(\cos t_2)^{b+2r}&\\ -2(b+2r)(\cos t_1)^{a+b}(\cos t_2)^{b+2r+2},&\end{aligned}$$

and comparing with the explicit form of $\mathcal{E}_{r,s}$ we get $a_r+b_r=-2(a+b)$ and $a_r+c_r=-2(b+2r)$. Writing b_r and c_r in terms of a_r, we now take $s=1$ in (7.4) and we use the explicit expression (6.4) in order to obtain by a calculation (using computer algebra) that $a_r=-2a-2b-2r$. This gives the expression for $C_1(\psi_1,\psi_2)$.

In the case $i=2$ we proceed similarly and we get

$$\frac{\partial\psi_2}{\partial t_1}\frac{\partial(Q_0)_{r,s}}{\partial t_1}+\frac{\partial\psi_2}{\partial t_2}\frac{\partial(Q_0)_{r,s}}{\partial t_2}=-2(a+2b+2r)\psi_2(Q_0)_{r,s}+\mathcal{E}_{r,s}, \tag{7.5}$$

where $\mathcal{E}_{r,s}$ is a homogeneous polynomial of degree $a+2b+2r+2$ in $(\cos t_1,\cos t_2)$ as before and hence of the form $\mathcal{E}_{r,s}=a_r\psi_1(Q_0)_{r,s}+b_r\psi_2(Q_0)_{r-1,s}+c_r(Q_0)_{r+1,s}$. So also $C_2(\psi_1,\psi_2)$ is tridiagonal. Taking $s=0$ in (7.5) we find by a calculation that $a_r+b_r=2(b+2r)$ and $a_r+c_r=2(a+b)$ in this case. Eliminating b_r and c_r in terms of a_r and now taking $s=1$ in (7.5) and using the explicit form (6.4), we find by a calculation (using computer algebra) that $a_r=2b+2r$. This gives the expression for $C_2(\psi_1,\psi_2)$. □

In order to derive the partial differential operator of Theorem 7.1, we observe that in this case we can rewrite (1.8) as

$$\Phi^{\mu}_{\nu_i+\lambda_{\mathrm{sph}}}(a_{\mathbf{t}})=\sum_{r=0}^{a}q^{\mu}_{\nu_i,v_r;\mathbf{d}}(\psi_1(a_{\mathbf{t}}),\psi_2(a_{\mathbf{t}}))Q^{\mu}_{\nu_r}(a_{\mathbf{t}}), \tag{7.6}$$

using Theorem 4.5 and Remark 6.3. Note that $q^{\mu}_{\nu_i,v_r;\mathbf{d}}$ is a polynomial of total degree $|\mathbf{d}|$. Note that $q^{\mu}_{\nu_i,v_r;\mathbf{d}}$ are entries of the matrix polynomials $R_{\mathbf{d}}$ up to a change of coordinates. Since $\Phi^{\mu}_{\nu_i+\lambda_{\mathrm{sph}}}(a_{\mathbf{t}})$ is an eigenvector of the radial part $R^{\mu}(\Omega)$ of the Casimir operator, we need to derive the action $R^{\mu}(\Omega)$ on $\mathbf{t}\mapsto f(\psi_1(a_{\mathbf{t}}),\psi(a_{\mathbf{t}}))Q^{\mu}_{\nu_r}(a_{\mathbf{t}})$ for f a 2-variable scalar function. It can be checked from (6.3) that (cf. the proof of [23, Lem. 3.9])

$$\begin{aligned}R^{\mu}(\Omega)(f(\psi_1,\psi_2)Q^{\mu}_{\nu_r})&=f(\psi_1,\psi_2)(R^{\mu}(\Omega)Q^{\mu}_{\nu_r})\\ &\quad+(R^{0}(\Omega)f(\psi_1,\psi_2))Q^{\mu}_{\nu_r}-\sum_{p=1}^{2}\frac{\partial f}{\partial t_p}\frac{\partial Q^{\mu}_{\nu_r}}{\partial t_p},\end{aligned} \tag{7.7}$$

where the first term follows from Lemma 6.5 and the last term can be dealt with using Lemma 7.3 and the chain rule. So we can rewrite $R^\mu(\Omega)(f(\psi_1,\psi_2)Q^\mu_{\nu_r})$ completely in terms of the $Q^\mu_{\nu_s}$: we get an eigenvalue equation for the $q^\mu_{\nu_i,\nu_r;\mathbf{d}}$, which is

$$
\begin{aligned}
c_{\nu_i+\lambda_{\mathrm{sph}}}\sum_{r=0}^{a} q^\mu_{\nu_i,\nu_r;\mathbf{d}}(\psi_1,\psi_2)Q^\mu_{\nu_r} &= \sum_{r=0}^{a}(R^0(\Omega)q^\mu_{\nu_i,\nu_r;\mathbf{d}}(\psi_1,\psi_2))Q^\mu_{\nu_r}\\
&\quad+\sum_{r=0}^{a} q^\mu_{\nu_i,\nu_r;\mathbf{d}}(\psi_1,\psi_2)(c_{\nu_r}Q^\mu_{\nu_r}-2r(b+r)Q^\mu_{\nu_{r-1}})\\
&\quad-\sum_{p=1}^{2}\sum_{r,u=0}^{a}\frac{\partial q^\mu_{\nu_i,\nu_r;\mathbf{d}}}{\partial\psi_p}(\psi_1,\psi_2)C_p(\psi_1,\psi_2)_{r,u}Q^\mu_{\nu_u},
\end{aligned}
\tag{7.8}
$$

where $\lambda_{\mathrm{sph}} = d_1\lambda_1 + d_2\lambda_2$, $\mathbf{d} = (d_1,d_2)\in\mathbb{N}^2$.

Lemma 7.4. *Define $Q_{\mathbf{d}} = Q_{\mathbf{d}}(\psi_1,\psi_2)$ the matrix polynomial by $(Q_{\mathbf{d}})_{i,j}(\psi_1,\psi_2) = q^\mu_{\nu_i,\nu_j;\mathbf{d}}(\psi_1,\psi_2)$ using (7.6). Then*

$$Q_{\mathbf{d}}R^0(\Omega) - \frac{\partial Q_{\mathbf{d}}}{\partial\psi_1}C_1(\psi_1,\psi_2) - \frac{\partial Q_{\mathbf{d}}}{\partial\psi_2}C_2(\psi_1,\psi_2) + Q_{\mathbf{d}}(\Lambda_0+S) = \Lambda_{\mathbf{d}}Q_{\mathbf{d}},$$

where $\Lambda_{\mathbf{d}}$, Λ_0 and S are as in Theorem 7.1, and $C_i(\psi_1,\psi_2)$, $i=1,2$, are the matrix polynomials of at most degree 1 (see Lemma 7.3). Moreover, $R^0(\Omega)$ is a matrix second-order partial differential operator in (ψ_1,ψ_2) acting entrywise, considered as acting from the right.

Remark 7.5. Note that the radial part $R^0(\Omega)$ acts as a matrix differential operator when considered as multiplied by the identity. This has to be rewritten as a differential operator with respect to the variables (ψ_1,ψ_2), which can be done since the spherical functions are polynomials in (ϕ_1,ϕ_2), hence in (ψ_1,ψ_2); see Vretare [37]. For convenience, we write down the terms of $R^0(\Omega)f$, where f is a scalar polynomial in (ψ_1,ψ_2). Then $R^0(\Omega_{\mathfrak{m}})$ is zero, $-\frac{1}{2}\sum_{p=1}^{2}\frac{\partial^2}{\partial t_p^2}$ in (6.3) corresponds to

$$
\begin{aligned}
(2\psi_1-2)\frac{\partial f}{\partial\psi_1} + (4\psi_2-\psi_1)\frac{\partial f}{\partial\psi_2} + (2\psi_1^2-2\psi_1-4\psi_2)\frac{\partial^2 f}{\partial\psi_1^2}\\
+(4\psi_2^2-2\psi_1\psi_2)\frac{\partial^2 f}{\partial\psi_2^2} + (4\psi_1\psi_2-8\psi_2)\frac{\partial^2 f}{\partial\psi_1\partial\psi_2},
\end{aligned}
$$

$R^0_s(\Omega)f$ corresponds to

$$2(m-2)\psi_1\frac{\partial f}{\partial\psi_1} + 4(m-2)\psi_2\frac{\partial f}{\partial\psi_2},$$

$R_l^0(\Omega)f$ corresponds to

$$(2\psi_1 - 2)\frac{\partial f}{\partial \psi_1} + (4\psi_2 - \psi_1)\frac{\partial f}{\partial \psi_2}$$

and $R_m^0(\Omega)f$ corresponds to

$$(4\psi_1 - 4)\frac{\partial f}{\partial \psi_1} + 4\psi_2\frac{\partial f}{\partial \psi_2}.$$

Proof of Lemma 7.4. Writing (7.8) in matrix notation, we obtain the result of the lemma multiplied by the matrix function Q_0 from the right. Since Q_0 is generically invertible (see the proof of Proposition 5.3), the lemma follows. □

Proof of the partial differential equation in Theorem 7.1. Comparing (7.6) with (7.2) and (1.8), we see that $R_{\mathbf{d}}$ and $Q_{\mathbf{d}}$ are the same up to the change of coordinates from (ψ_1, ψ_2) for $Q_{\mathbf{d}}$ to (x_1, x_2) for $R_{\mathbf{d}}$. Note that $x_1 = 2\psi_1 - 2$, $x_2 = 4\psi_1 - 2\psi_1 + 1$, making this affine change of coordinates give the expression for $R^0(\Omega)$ in (x_1, x_2)-coordinates as given in Theorem 7.1. It remains to make the change of coordinates in the other terms involving first-order differentials, which is straightforward.

Note that the radial part $R^\mu(\Omega)$ of the Casimir operator is symmetric with respect to the inner product $\langle \Phi, \Psi \rangle = \frac{1}{c}\int_A \mathrm{Tr}(\Phi(a)(\Psi(a))^*)|\delta(a)|\, da$ by (6.1), (1.4) for matrix spherical functions Φ, Ψ, and the results given in Section 1.1. Since the second-order matrix partial differential operator is obtained by conjugation by Q_0, we obtain the symmetry. □

§8. The leading term of Φ_λ^μ

In Section 4 we introduced the leading term Q_ν^μ of the matrix spherical functions for Φ_ν^μ for $\nu \in B(\mu)$. Using these results we can determine the leading term Q_λ^μ of the matrix spherical functions for Φ_λ^μ for $\lambda \in P_G^+(\mu)$. We do this by introducing the leading term from an embedding of V_μ^K in a large tensor product representation, similarly to the construction in Section 4. We then show, by using the radial part of the Casimir operator, that this is indeed a leading term by establishing the lower triangularity of the radial part of the Casimir operator on these functions.

Assume as before $\mu = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{N}$ and we take $\lambda \in P_G^+(\mu)$. By Condition 1.2 we can write $\lambda = \nu_i + d_1\lambda_1 + d_2\lambda_2$ with $\nu_i \in B(\mu)$, $d_1, d_2 \in \mathbb{N}$. Generalising the construction of ψ_1, ψ_2 and $Q_{\nu_i}^\mu$ as in Sections 3 and 4, we define the tensor product representation and an explicit element by

$$W_\lambda^G = (V_{\omega_1}^G \otimes V_{\omega_{m+1}}^G)^{\otimes d_1} \otimes (V_{\omega_2}^G \otimes V_{\omega_m}^G)^{\otimes d_2} \otimes U_{\nu_i}^G, \quad w = v_1^{\otimes d_1} \otimes v_2^{\otimes d_2} \otimes u \in W_\lambda^G,$$

using the notation of Lemmas 3.2 and 3.4 and (4.1). Using the results of Sections 3 and 4 we see that w is a K-highest weight vector of highest weight μ in W_λ^G. So we get a K-intertwiner $j\colon V_\mu^K \to W_\lambda^G$ mapping the highest weight vector $v_\mu \in V_\mu^K$ to w.

Proposition 8.1. *Define the matrix spherical function* $Q_\lambda^\mu\colon G \to \mathrm{End}(V_\mu^K)$ *by* $Q_\lambda^\mu(g) = j^* \circ \pi_{W_\lambda^G}^G(g) \circ j$. *Then*

$$Q_\lambda^\mu(a_{\mathbf{t}}) = (\psi_1(a_{\mathbf{t}}))^{d_1}(\psi_2(a_{\mathbf{t}}))^{d_2} Q_{\nu_i}^\mu(a_{\mathbf{t}})$$

and $Q_\lambda^\mu|_A = \sum_{\lambda' \preccurlyeq \lambda; \lambda' \in P_G^+(\mu)} a_{\lambda'} \Phi_{\lambda'}^\mu|_A$ *for constants* $a_{\lambda'}$.

Note in particular, that the action of $Q_\lambda^\mu(a_{\mathbf{t}})$ on the 1-dimensional constituent $V_{\sigma_k}^M$ in V_μ^K is given by

$$(\psi_1(a_{\mathbf{t}}))^{d_1}(\psi_2(a_{\mathbf{t}}))^{d_2} q_{\nu_i,\sigma_k}^\mu(a_{\mathbf{t}}), \tag{8.1}$$

which is a homogeneous polynomial in $(\cos t_1, \cos t_2)$ of degree $2d_1 + 4d_2 + 2a + 4b + 2i$; see Remark 4.3 and Theorem 4.5.

Proof of Proposition 8.1. As noted, w is a highest weight vector for the action of K of highest weight μ, so by construction Q_λ^μ is a matrix spherical function, and by Section 1.1 it is a linear combination of Φ_λ^μ for $\lambda \in P_G^+(\mu)$ by the Peter–Weyl theorem. Since we have the decomposition $W_\lambda^G = \bigoplus_{\lambda' \preccurlyeq \lambda} n_{\lambda'} V_\lambda^G$, with $n_\lambda = 1$, by repeated application of, e.g. [29, Lem. 3.1], the expression for $Q_\lambda^\mu|_A$ follows.

For the proof of the explicit expression, we use the notation in the proof of Proposition 4.2. Since the matrix entry of $a_{\mathbf{t}}(r,s)$ acting on v_i and taking the inner product with v_i is $\psi_i(a_{\mathbf{t}})$ for $i = 1, 2$ by Lemmas 3.2 and 3.4, we find the result from Theorem 4.5. $\square$

In order to understand the decomposition of Q_λ^μ of Proposition 8.1 we calculate the action of the radial part of the Casimir operator on Q_λ^μ as a function on A. Recall (7.7), and take f a polynomial in (ψ_1, ψ_2); this leads to Proposition 8.2. Note that Proposition 8.2 generalises Lemma 6.5, but Lemma 6.5 is used in the proof of Proposition 8.2.

Proposition 8.2. *We have as functions on* A,

$$R^\mu(\Omega) Q_\lambda^\mu = c_\lambda Q_\lambda^\mu + \sum_{\substack{\lambda' \prec \lambda \\ \lambda' \in P_G^+(\mu)}} b_{\lambda'} Q_{\lambda'}^\mu.$$

Corollary 8.3. *In Proposition* 8.1 *we have* $a_\lambda \neq 0$, *so that there exist constants* $b_{\lambda'}$ *with*

$$\Phi_\lambda^\mu = \sum_{\lambda' \preccurlyeq \lambda} b_{\lambda'} Q_\lambda^\mu, \quad b_\lambda \neq 0.$$

The statement of Corollary 8.3 motivates us to call the matrix spherical function Q^μ_λ the leading term of Φ^μ_λ.

Proof of Corollary 8.3. In the case $a_\lambda = 0$ in Proposition 8.1, we have Q^μ_λ in the span of $\Phi^\mu_{\lambda'}$ for $\lambda' \prec \lambda$ which is an invariant space for the radial part of the Casimir $R^\mu(\Omega)$ with eigenvalues $c_{\lambda'}$. By Lemma 6.1 the eigenvalue c_λ is not contained in this set, but applying Propositions 8.2 and 8.1 to $\lambda' \prec \lambda$ shows that the eigenvalue c_λ has to occur, since $R^\mu(\Omega)$ acts in a lower triangular way on the $Q^\mu_{\lambda'}$. This is the required contradiction.

So this means that we can invert the relation of Proposition 8.1, giving the stated expansion. □

Proof of Proposition 8.2. Put $f(\psi_1,\psi_2) = \psi_1^{d_1}\psi_2^{d_2}$; then the first term on the right-hand side of (7.7) follows from Lemma 6.5. For the second term we have by a calculation,

$$\begin{aligned} R^0(\mu) f = {} & 2(d_1^2 + d_1(1+2d_2+m) + 2d_2^2 + 2md_2)\psi_1^{d_1}\psi_2^{d_2} - 2d_2^2\psi_1^{d_1+1}\psi_2^{d_2-1} \\ & - 2d_1(d_1+4d_2+3)\psi_1^{d_1-1}\psi_2^{d_2} - 4d_1(d_1-1)\psi_1^{d_1-2}\psi_2^{d_2+1}, \end{aligned} \tag{8.2}$$

which follows from the explicit expression of the radial part of the Casimir operator, $R^0(\Omega)$, in (ψ_1,ψ_2)-coordinates; see Remark 7.5. For the final term of (7.7), $-\sum_{p=1}^2 \frac{\partial f}{\partial t_p}\frac{\partial Q^\mu_{\nu_r}}{\partial t_p}$, we consider the action on the constituent $V^M_{\sigma_k}$ in V^K_μ, and we find

$$-\sum_{s=1}^{2} \frac{\partial f}{\partial \psi_s}(C_s(\psi,\psi_2)Q_0)_{i,k}$$

using the chain rule and Lemma 7.3. By the explicit expression of Lemma 7.3 this term gives

$$\begin{aligned} & -2(d_1\psi_2 + d_2\psi_1)(i\psi_1^{d_1-1}\psi_2^{d_2}Q^\mu_{\nu_{i-1}} + (a-i)\psi_1^{d_1-1}\psi_2^{d_2-1}Q^\mu_{\nu_{i+1}}) \\ & \quad - 2\big((a+b+2i)d_1\psi_2 + (b+2i)d_2\psi_1^2 \\ & \qquad - (d_1(a+b+i) + d_2(a+2b+2i))\psi_1\psi_2\big)\psi_1^{d_1-1}\psi_2^{d_2-1}Q^\mu_{\nu_i}. \end{aligned} \tag{8.3}$$

So from (7.7), Lemma 6.5 and (8.2), (8.3), we collect the coefficient of Q^μ_λ in $R^\mu(\Omega)Q^\mu_\lambda$ as

$$c_{\nu_i} + 2(d_1^2 + d_1(1+2d_2+m) + 2d_2^2 + 2md_2) + 2(a+b+i)d_1 + 2(a+2b+2i)d_2.$$

Write $\lambda = \nu_i + \lambda_{\mathrm{sph}}$, with $\lambda_{\mathrm{sph}} = d_1\lambda_1 + d_2\lambda_2$. Then the eigenvalue c_λ can be written as

$$c_\lambda = c_{\nu_i} + \langle \lambda_{\mathrm{sph}}, \lambda_{\mathrm{sph}}\rangle + 2\langle \lambda_{\mathrm{sph}}, \nu_i + \rho\rangle.$$

Since $\langle \lambda_{\mathrm{sph}}, \lambda_{\mathrm{sph}} \rangle = 2d_1^2 + 4d_1d_2 + 4d_2^2$, and $\langle \lambda_1, \nu_i + \rho \rangle = a + b + i + m + 1$, $\langle \lambda_2, \nu_i + \rho \rangle = a + 2b + 2i + 2m$, we see that the coefficient of Q^μ_λ in $R^\mu(\Omega)Q^\mu_\lambda$ is c_λ. □

Note that the proof of Proposition 8.2 actually gives a complete expression for the action of $R^\mu(\Omega)$ on Q^μ_λ. For completeness we list in Table 1 the $\lambda' \prec \lambda$ for which $Q^\mu_{\lambda'}$ occurs with a nonzero $b_{\lambda'}$ whose explicit value is listed as well.

Indeed, all λ' satisfy $\lambda' \in P^+_G(\mu)$ and $\lambda' \prec \lambda$, which can be checked using the results of Section 2.

λ'	$b_{\lambda'}$
$(d_1-1)\lambda_1 + d_2\lambda_2 + \nu_i$	$-2d_1(d_1+4d_2+3) - 2d_1(a+2b+2i)$
$(d_1-2)\lambda_1 + (d_2+1)\lambda_2 + \nu_i$	$-2d_1(d_1-1)$
$(d_1+1)\lambda_1 + (d_2-1)\lambda_2 + \nu_i$	$-2d_2^2 - 2d_2(b+i)$
$d_1\lambda_1 + d_2\lambda_2 + \nu_{i-1}$	$-2i(b+i) - 2id_2$
$(d_1-1)\lambda_1 + (d_2+1)\lambda_2 + \nu_{i-1}$	$-2id_1$
$(d_1-1)\lambda_1 + d_2\lambda_2 + \nu_{i+1}$	$-2(a-i)d_1$
$d_1\lambda_1 + (d_2-1)\lambda_2 + \nu_{i+1}$	$-2(a-i)d_2$

Table 1. Table for the remaining coefficients in Proposition 8.2.

§9. The case $\mu = a\omega_1 + b\omega_2$ with b negative

In general, we obtain from (1.7) and $\sigma_k(\mu^*) = \sigma_{a-k}(\mu)$ for $\mu = a\omega_1 + b\omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$ (see Section 2.2) and $a_{\mathbf{t}}^{-1} = a_{-\mathbf{t}}$,

$$q^{\mu^*}_{\nu_r(\mu^*),\sigma_k(\mu^*)}(a_{\mathbf{t}}) = q^{\mu}_{\nu_{a-r}(\mu),\sigma_{a-k}(\mu)}(a_{-\mathbf{t}}),$$

extending the notation (5.1) to more general μ and stressing the dependence on μ and μ^* in the corresponding weights. So, for the corresponding Φ^μ_0, we obtain

$$\Phi^{\mu^*}_0(a_{\mathbf{t}}) = J\Phi^\mu_0(a_{-\mathbf{t}})J, \quad J_{i,j} = \delta_{i+j,a}, \ 0 \le i,j \le a. \tag{9.1}$$

Applying (1.7) to (1.8), using that $\lambda^*_{\mathrm{sph}} = \lambda_{\mathrm{sph}}$ and that spherical functions satisfy $\phi(a_{\mathbf{t}}) = \phi(a_{-\mathbf{t}})$, we obtain for the matrix polynomials $P^\mu_{\mathbf{d}}(\phi_1,\phi_2) = P_{\mathbf{d}}(\phi_1,\phi_2)$ introduced in (1.9),

$$P^{\mu^*}_{\mathbf{d}}(\phi_1,\phi_2) = JP^\mu_{\mathbf{d}}(\phi_1,\phi_2)J. \tag{9.2}$$

The weight function satisfies $W^{\mu^*}(\phi_1,\phi_2) = JW^{\mu}(\phi_1,\phi_2)J$ as follows from (9.1), so we see that the matrix polynomials for $\mu = a\omega_1 + b\omega_2$ and $\mu^* = a\omega_1 - (a+b)\omega_2$ are essentially the same. So this covers the case $b \leq -a$.

It remains to consider the case $-a < b < 0$ with $a \in \mathbb{N}$, $b \in \mathbb{Z}$, and using duality we can restrict to the case $-\frac{1}{2}a \leq b < 0$. However, in this case we cannot extend the method established for the case $b \geq 0$ easily, due to the fact that the bottom splits into two parts. The results for each of these parts cannot be easily related to each other.

Remark 9.1. The case that $\mu^* = \mu$, i.e. $a \in 2\mathbb{N}$ and $b = -\frac{1}{2}a$ or $\mu = 2c\omega_1 - c\omega_2$ for $c \in \mathbb{N}$, exhibits different behaviour. Assume $c \geq 1$; we see that the corresponding spaces A and $\mathcal{A}$ as in Proposition 5.1 for the matrix weight W (see Remark 5.2) are no longer trivial, since $\mathcal{A}'$ and $\mathcal{A}$ both contain J. Calculations for small values of c in $\mu = 2c\omega_1 - c\omega_2$ indicate that we may expect $\mathcal{A}' = \mathbb{C}J \oplus \mathbb{C}\,\mathrm{Id}$ and $\mathcal{A} = \mathbb{R}J \oplus \mathbb{R}\,\mathrm{Id}$ with $\mathcal{A}'$ and $\mathcal{A}$ defined as in Proposition 5.1.

Note that in the study of matrix orthogonal polynomials of a single variable related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ the weight is also reducible; see [21, Prop. 6.4, Thm. 6.5]. In that case the algebra $\mathcal{A}'$ is also 2-dimensional with a similarly defined nontrivial element. So we see that self-duality of the K-representations in these cases leads to reducibility of the weight for the corresponding matrix orthogonal polynomials. The precise relation requires more attention in general.

Appendix. Radial part of the Casimir operator

In general, the determination of the radial part of an operator arising from a suitable element in the universal enveloping algebra is due to Harish-Chandra in unpublished papers from 1960; see [8]. The result is mainly used for representations of noncompact Lie groups; see [15, Chap. VIII], [38, Chap. 9]. In this case we need to do this for the compact setting, and we derive the explicit expression from the Casimir element in the centre of $U(\mathfrak{g})$. For this we follow Casselman and Miličić [3].

Appendix A.1. Structure theory

In order to calculate the radial part of the Casimir operator following [3], we note that $K = G^\theta$ with $\theta(g) = JgJ$, $J = \mathrm{diag}(-1,-1,1,\ldots,1)$. In order to do the calculation we conjugate to the maximally split case. So we take

$$J' = \begin{pmatrix} 0 & 0 & J_2 \\ 0 & I_{m-2} & 0 \\ J_2 & 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} \frac{1}{\sqrt{2}}I_2 & 0 & \frac{1}{\sqrt{2}}J_2 \\ 0 & I_{m-2} & 0 \\ -\frac{1}{\sqrt{2}}J_2 & 0 & \frac{1}{\sqrt{2}}I_2 \end{pmatrix} \in \mathrm{SU}(m+2), \ u^*J'u = J, \tag{A.1}$$

$\beta \in R$	$\dim \mathfrak{g}_\beta$	$\alpha \in \Delta$ with $\alpha\|_{\mathfrak{a}'} = \beta$
$f_1 - f_2$	2	$\varepsilon_1 - \varepsilon_2$, $\varepsilon_{m+1} - \varepsilon_{m+2}$
$f_1 + f_2$	2	$\varepsilon_i - \varepsilon_{m+i}$, $i = 1, 2$
$2f_i$, $1 \le i \le 2$	1	$\varepsilon_i - \varepsilon_{m+3-i}$
f_i, $1 \le i \le 2$	$2(m-2)$	$\varepsilon_i - \varepsilon_{2+j}$, $\varepsilon_{2+j} - \varepsilon_{m+3-i}$, $1 \le j \le m-2$
$f_2 - f_1$	2	$\varepsilon_2 - \varepsilon_1$, $\varepsilon_{m+2} - \varepsilon_{m+1}$
$-f_1 - f_2$	2	$\varepsilon_{m+i} - \varepsilon_i$, $i = 1, 2$
$-2f_i$, $1 \le i \le 2$	1	$\varepsilon_{m+3-i} - \varepsilon_i$
$-f_i$, $1 \le i \le 2$	$2(m-2)$	$\varepsilon_{2+j} - \varepsilon_i$, $\varepsilon_{m+3-i} - \varepsilon_{2+j}$, $1 \le j \le m-2$

Table 2. The restricted root system of type BC$_2$.

where $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\theta'(g) = J'gJ'$, so that $u\theta(g)u^* = \theta'(u^*gu)$ and $K' = G^{\theta'} = uKu^*$. We use the same notation for the involutions θ and θ' for the complexified Lie algebras. Now $\mathfrak{g} = \mathfrak{sl}(m+2, \mathbb{C})$ has the root system $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \le i \ne j \le m+2}$, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\mathfrak{h}$ is the Cartan subalgebra consisting of the diagonal elements in $\mathfrak{g}$. The matrix $E_{i,j}$ spans $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$; see Section 2.1. Then $\mathfrak{a}' = u\mathfrak{a}u^*$ consists of diagonal matrices $X = \mathrm{diag}(d_1, d_2, 0, \ldots, 0, -d_2, -d_1)$, and we let $f_i(X) = d_i$, $i = 1, 2$. Then the reduced root system R is of type BC$_2$ and the identification is given in Table 2. Then the positive roots of Δ and R correspond to each other. Moreover, $\mathfrak{m}' = u\mathfrak{m}u^* = \mathfrak{m}$. With $A' = uAu^*$, and $a'_{\mathfrak{t}} = ua_{\mathfrak{t}}u^* = \mathrm{diag}(e^{it_1}, e^{it_2}, 1, \ldots, 1, e^{-it_2}, e^{-it_1})$, we have $M' = Z_{K'}(A') = uZ_K(A)u^* = M$. Let $n_1 f_1 + n_2 f_2$ be the character of A sending $a'_{\mathfrak{t}} \mapsto e^{i(n_1 t_1 + n_2 t_2)}$.

Then the root space decomposition for the action of A' is given by

$$\mathfrak{g} = \mathfrak{a}' \oplus \mathfrak{m}' \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta, \quad \mathfrak{g}_\beta = \bigoplus_{\alpha \in \Delta, \alpha|_{\mathfrak{a}'} = \beta} \mathfrak{g}_\alpha,$$

where, for $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$, $\mathfrak{g}_\alpha = \mathbb{C}Y_\alpha$ with $Y_\alpha = E_{i,j}$, where we use the same notation β for the corresponding derivative $\beta\colon \mathfrak{a}' \to \mathbb{C}$. Note that θ' gives an action on Δ by $\theta'(\alpha)(H) = \alpha(\theta'(H))$ for $H \in \mathfrak{h}$. Then $-\theta'$ is an involution of $\{\alpha \in \Delta \mid \alpha|_{\mathfrak{a}'} = \beta\}$ for $\beta \in R$.

Appendix A.2. Casimir element

The Killing form on $\mathfrak{g}$ is given by $B(X, Y) = \mathrm{Tr}(XY)$ up to a positive multiple, and the Casimir element $\Omega = \sum_i X_i X_i^* \in Z(U(\mathfrak{g}))$, where $\{X_i\}_i$ is a basis for $\mathfrak{g}$ and $\{X_i^*\}_i$ its dual basis with respect to B. Put $H_i = E_{i,i} - E_{m+3-i,m+3-i}$, $i = 1, 2$, as the basis for $\mathfrak{a}'$, then $H_i^* = \frac{1}{2} H_i$ and note that $E_{i,j}^* = E_{j,i}$ for $i \ne j$, or $Y_\alpha^* = Y_{-\alpha}$. Observe that $B|_{\mathfrak{m} \times \mathfrak{m}}$ is nondegenerate, and let $\Omega_{\mathfrak{m}}$ be the corresponding Casimir element. So we get

$$\Omega = \Omega_{\mathfrak{m}} + \frac{1}{2}\sum_{i=1}^{2} H_i^2 + \sum_{\beta\in R^+} \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} (Y_\alpha Y_{-\alpha} + Y_{-\alpha}Y_\alpha). \tag{A.2}$$

Now we want to rewrite (A.2) following [3, §2]. So let $a \in A'_{\mathrm{reg}}$, i.e. $\beta(a) \neq \pm 1$ for all $\beta \in R^+$. Define $X^a = \mathrm{Ad}(a^{-1})X$, $X \in U(\mathfrak{g})$, and let $\alpha \in \Delta$ with $\alpha|_{\mathfrak{a}'} = \beta$. Then (see [3, Lem. 2.2])

$$\begin{aligned} X_\alpha &= Y_\alpha + \theta' Y_\alpha = Y_\alpha + Y_{\theta'\alpha} \in \mathfrak{k}', \\ Y_\alpha &= \frac{\beta(a)}{1-\beta(a)^2}(X_\alpha^a - \beta(a)X_\alpha). \end{aligned} \tag{A.3}$$

In order to obtain the infinitesimal Cartan decomposition of the Casimir element $\Gamma_a^{-1}(\Omega)$ (see [3, Thm. 2.1]), we need to write Ω as the sum of elements of the form $X^a H Y$ with $X, Y \in U(\mathfrak{k}')$, $H \in U(\mathfrak{a}')$. Note that the first two terms in (A.2) are of the right form. Using (A.3) we see that

$$\begin{aligned} &\sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} (Y_\alpha Y_{-\alpha} + Y_{-\alpha}Y_\alpha) \\ &\qquad = \frac{-1}{(\beta(a)-\beta(a)^{-1})^2} \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} \big(X_\alpha^a X_{-\alpha}^a + X_{-\alpha}^a X_\alpha^a + X_\alpha X_{-\alpha} + X_{-\alpha}X_\alpha \\ &\qquad\qquad - \beta(a)^{-1} X_\alpha^a X_{-\alpha} - \beta(a) X_{-\alpha}^a X_\alpha \\ &\qquad\qquad - \beta(a) X_\alpha X_{-\alpha}^a - \beta(a)^{-1} X_{-\alpha} X_\alpha^a \big). \end{aligned}$$

Next observe that

$$\sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} X_{-\alpha}^a X_\alpha^a = \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} X_{\theta'\alpha}^a X_{-\theta'\alpha}^a = \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} X_\alpha^a X_{-\alpha}^a,$$

using the involution $-\theta'$ and $X_\alpha = X_{\theta'\alpha}$. Similarly, we can take other terms together. Then only the last two terms are not yet of the right form.

Lemma A.1. *For $\alpha \in \Delta^+$ with $\alpha|_{\mathfrak{a}'} = \beta$ we have*

$$[X_\alpha^a, X_{-\alpha}] + [X_{-\theta'\alpha}^a, X_{\theta'\alpha}] = (\beta(a)^{-1} - \beta(a))(H_\alpha + H_{-\theta'\alpha}) \in \mathfrak{a}',$$

where $H_{e_i - e_j} = E_{i,i} - E_{j,j}$.

Proof. Using (A.3) we rewrite the commutators in terms of the Y_α. The mixed terms cancel and we are left with

$$[X_\alpha^a, X_{-\alpha}] + [X_{-\theta'\alpha}^a, X_{\theta'\alpha}] = (\beta(a)^{-1} - \beta(a))[Y_\alpha, Y_{-\alpha}] + (\beta(a) - \beta(a)^{-1})[Y_{\theta'\alpha}, Y_{-\theta'\alpha}]$$

in terms of commutators of the Y_α. Since the right-hand side is in $\mathfrak{h}$ and in the -1-eigenspace of θ' we see that it is contained in $\mathfrak{a}'$. □

Using this in the expression for the Casimir element leads to the infinitesimal Cartan decomposition for Ω:

$$\begin{aligned}\Omega = \Omega_{\mathfrak{m}} &+ \frac{1}{2}\sum_{i=1}^{2} H_i^2 + \frac{1}{2}\sum_{\beta\in R^+} \frac{\beta(a)+\beta(a)^{-1}}{\beta(a)-\beta(a)^{-1}} \dim \mathfrak{g}_\beta H_\beta \\ &+ 2\sum_{\beta\in R^+} \frac{\beta(a)+\beta(a)^{-1}}{(\beta(a)-\beta(a)^{-1})^2} \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} X_\alpha^a X_{-\alpha} \\ &- 2\sum_{\beta\in R^+} \frac{1}{(\beta(a)-\beta(a)^{-1})^2} \sum_{\substack{\alpha\in\Delta^+\\ \alpha|_{\mathfrak{a}'}=\beta}} X_\alpha^a X_{-\alpha}^a + X_\alpha X_{-\alpha},\end{aligned} \tag{A.4}$$

where $H_\beta = n_1 H_1 + n_2 H_2$ for $\beta = n_1 f_1 + n_2 f_2$.

Appendix A.3. The left-invariant differential operator corresponding to the Casimir element

Let $F\colon G \to \mathrm{End}(V_\mu^{K'})$, where $V_\mu^{K'}$ is the same representation space as V_μ^K, and the action is given by $\pi_\mu^{K'}(k') = \pi_\mu^K(u^*k'u)$, $k' \in K'$. We assume F satisfies $F(k_1'gk_2') = \pi_\mu^{K'}(k_1')F(g)\pi_\mu^{K'}(k_2')$, so that F is determined by its restriction to A' and, since $M' = M$, we have $F\colon A' \to \mathrm{End}_M(V_\mu^{K'})$. Now the action of Ω as a left-invariant operator satisfies $(\Omega\cdot F)|_{A'} = R(\Omega)\cdot(F|_{A'})$, where $R(\Omega)$ is the radial part of the Casimir element. In the decomposition (A.4), $\Omega_{\mathfrak{m}}$ acts as a scalar on each M-type by Schur's lemma. So the action of $\Omega_{\mathfrak{m}}$ on $F|_{A'}$ is by multiplying by a diagonal constant matrix. The second term acts as a second-order differential operator, and the third term as a first-order differential operator by observing that, after putting $f(t_1,t_2) = F(a_{\mathbf{t}}')$ we have $iH_p\cdot f = \frac{\partial f}{\partial t_p}$. The actions of the differential operators do not involve the M-type. Then

$$X_\alpha^a X_{-\alpha}\cdot(F|_{A'}) = \pi_\mu^{K'}(X_\alpha)(F|_{A'})\pi_\mu^{K'}(X_{-\alpha}),$$

and similarly

$$X_\alpha^a X_{-\alpha}^a\cdot(F|_{A'}) = \pi_\mu^{K'}(X_\alpha X_{-\alpha})(F|_{A'})$$

and

$$X_\alpha X_{-\alpha}\cdot(F|_{A'}) = (F|_{A'})\pi_\mu^{K'}(X_\alpha X_{-\alpha})$$

(see [3]), where we use the same notation for the representation of the Lie algebra. In order to calculate these terms, we restrict to the K-representation of highest weight $\mu = a\omega_1 + b\omega_2$, $a\in\mathbb{N}$, $b\in\mathbb{Z}$. We can then read off X_α using Table 2, and next see which entry of $u^*X_\alpha u$ is in the upper left (2×2)-block. Finally, we conjugate back and we find the following expression for the radial part of the Casimir operator for a function $F\colon A\to \mathrm{End}_M(V_\mu^K)$ for $\mu=a\omega_1+b\omega_2$, $a\in\mathbb{N}$, $b\in\mathbb{Z}$, where $G(t_1,t_2)=F(a_{\mathbf{t}})$:

$$\begin{aligned}(R^\mu(\Omega)G)(t_1,t_2) &= R^\mu(\Omega_{\mathfrak{m}})G(t_1,t_2)-\frac{1}{2}\sum_{p=1}^{2}\frac{\partial^2 G}{\partial t_p^2}(t_1,t_2)\\ &\quad + (R_s^\mu(\Omega)G)(t_1,t_2)+(R_m^\mu(\Omega)G)(t_1,t_2)+(R_l^\mu(\Omega)G)(t_1,t_2),\end{aligned}\tag{A.5}$$

where the action is split according to the short, middle and long roots of BC_2. We obtain

$$(R_s^\mu(\Omega)G)(t_1,t_2)=-(m-2)\sum_{i=1}^{2}\frac{\cos t_i}{\sin t_i}\frac{\partial G}{\partial t_i}(t_1,t_2),$$

since for the short roots f_i the element $u^*X_\alpha u$ is not contained in the upper-left (2×2)-block, and so the last three terms in (A.4) do not contribute for the short roots. So the operator $R_s^\mu(\Omega)$ is independent of the K-representation π_μ^K. For the middle roots $f_1\pm f_2$ we get that the operator $R_m^\mu(\Omega)$ is defined by

$$\begin{aligned}(R_m^\mu(\Omega)G)(t_1,t_2) &= -\frac{\cos(t_1+t_2)}{\sin(t_1+t_2)}\Big(\frac{\partial G}{\partial t_1}(t_1,t_2)+\frac{\partial G}{\partial t_2}(t_1,t_2)\Big)\\ &\quad -\frac{\cos(t_1-t_2)}{\sin(t_1-t_2)}\Big(\frac{\partial G}{\partial t_1}(t_1,t_2)-\frac{\partial G}{\partial t_2}(t_1,t_2)\Big)\\ &\quad -\Big(\frac{\cos(t_1+t_2)}{\sin^2(t_1+t_2)}+\frac{\cos(t_1-t_2)}{\sin^2(t_1-t_2)}\Big)\\ &\qquad \times(\pi_\mu^K(E_1)G(t_1,t_2)\pi_\mu^K(F_1)+\pi_\mu^K(F_1)G(t_1,t_2)\pi_\mu^K(E_1))\\ &\quad +\frac{1}{2}\Big(\frac{1}{\sin^2(t_1+t_2)}+\frac{1}{\sin^2(t_1-t_2)}\Big)\\ &\qquad \times(\pi_\mu^K(E_1F_1+F_1E_1)G(t_1,t_2)+G(t_1,t_2)\pi_\mu^K(E_1F_1+F_1E_1)),\end{aligned}\tag{A.6}$$

and for the long roots $2f_i$ we get

$$\begin{aligned}(R_l^\mu(\Omega)G)(t_1,t_2) &= -\sum_{i=1}^{2}\frac{\cos(2t_i)}{\sin(2t_i)}\frac{\partial G}{\partial t_i}(t_1,t_2)\\ &\quad -\sum_{i=1}^{2}\frac{\cos(2t_i)}{\sin^2(2t_i)}\pi_\mu^K(E_{i,i})G(t_1,t_2)\pi_\mu^K(E_{i,i})\\ &\quad +\frac{1}{2}\sum_{i=1}^{2}\frac{1}{\sin^2(2t_i)}(\pi_\mu^K(E_{i,i})^2G(t_1,t_2)+G(t_1,t_2)\pi_\mu^K(E_{i,i})^2).\end{aligned}$$

In order to describe the action of $\Omega_\mathfrak{m}$ we only need the action on the 1-dimensional M-representation $V^M_{\sigma_k}$ occurring in V^K_μ; see (1.6). Let $M_1 = E_{1,1} + E_{m+2,m+2} - \frac{2}{m-2}\sum_{r=3}^m E_{r,r}$ and $M_2 = E_{2,2} + E_{m+1,m+1} - \frac{2}{m-2}\sum_{r=3}^m E_{r,r}$. Then the M_i are orthogonal to the $((m-2)\times(m-2))$-block of M, so that we only need to take the action of M_1 and M_2 into account. Note that M_1, respectively M_2, acts as $a+b-k$, respectively $b+k$ on $V^M_{\sigma_k}$. Since

$$M_1^* = \frac{m}{2(m+2)}M_1 - \frac{1}{m+2}M_2 \quad \text{and} \quad M_1^* = \frac{m}{2(m+2)}M_2 - \frac{1}{m+2}M_1,$$

this gives

$$R^\mu(\Omega_\mathfrak{m})|_{V^M_{\sigma_k}} = \frac{1}{2(m+2)}(m(a+b-k)^2 - 4(a+b-k)(b+k) + m(b+k)^2),$$

for $k \in \{0,\dots,a\}$.

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