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C^*-Simplicity of Relative Profinite Completions of Generalized Baumslag–Solitar Groups

by

Miho Mukohara

Abstract

Suzuki recently gave constructions of non-discrete examples of locally compact C^*-simple groups and Raum showed C^*-simplicity of the relative profinite completions of the Baumslag–Solitar groups by using Suzuki's results. We extend this result to some fundamental groups of graphs of groups called generalized Baumslag–Solitar groups. In this article, we focus on some sufficient condition to show that these locally compact groups are C^*-simple and that KMS-weights of these reduced group C^*-algebras are unique. This condition is an analogue of the Powers averaging property of discrete groups and holds for several currently known constructions of non-discrete C^*-simple groups.

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§1. Introduction

The reduced group C^*-algebra is a C^*-algebra generated by the left regular representation λ of a locally compact group. A locally compact group G is called C^*-simple if its reduced group C^*-algebra $C^*_r(G)$ is a simple C^*-algebra. C^*-simplicity of discrete groups has been studied since Powers proved that the non-commutative free group $\mathbb{F}_2$ is C^*-simple [Po], and as of today, satisfactory characterizations related to boundary actions and the unique trace property of reduced group C^*-algebras have been found for discrete groups [K-K, Haa, BKKO].

On the other hand, C^*-simplicity of non-discrete locally compact groups is not well understood compared to that of discrete groups. The existence of a non-discrete example of a C^*-simple group was questioned by de la Harpe [Har], and Suzuki constructed the first example of it [Su1]. As of today, some examples have been found in [Su1, Su2, R2]. By using the construction in [Su1], Raum [R2]

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M. Mukohara: Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan;
e-mail: miho-mukohara@g.ecc.u-tokyo.ac.jp

showed the relative profinite completion of the Baumslag–Solitar group $\mathrm{BS}(n,m)$ is C^*-simple if $|n|,|m| \geq 2$. Our main result is a generalization of that. The Baumslag–Solitar group $\mathrm{BS}(n,m)$ is generated by two elements a, t with a relation $ta^mt^{-1} = a^n$, and it has a natural action on a tree whose vertex stabilizers and edge stabilizers are isomorphic to $\mathbb{Z}$. Generally, groups with such actions on trees are called generalized Baumslag–Solitar groups. In this article, we show C^*-simplicity of the closures of generalized Baumslag–Solitar groups in the topology of the automorphism groups on trees. Because every locally compact C^*-simple group is totally disconnected [R1], it is natural to consider C^*-simplicity for closed subgroups of automorphism groups on trees.

For discrete groups, Haagerup [Haa] and Kennedy [Ke] proved that a group Γ is C^*-simple if and only if it has the Powers averaging property, that is, for any element a in the reduced group C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ with the canonical trace τ, $\tau(a)$ is approximated by elements in the convex hull of the set $\{\lambda_s a {\lambda_s}^* \in \mathrm{C}^*_{\mathrm{r}}(\Gamma) \mid s \in \Gamma\}$. In the proof of our main theorem, we focus on the canonical conditional expectation E_K from the reduced group C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ of a totally disconnected group G onto the C^*-subalgebra $\mathrm{C}^*_{\mathrm{r}}(K)$ and the averaging projection p_K corresponding to a compact open subgroup K of G. The following condition can be confirmed in a direct way.

Lemma (Lemma 3.1). *Let $(K_\nu)_{\nu\in N}$ be a decreasing net of compact open subgroups of G and $(A_\nu)_{\nu\in N}$ be an increasing net of C^*-subalgebras of $\mathrm{C}^*_{\mathrm{r}}(G)$ with $\bigcap_\nu K_\nu = \{e\}$ and $\overline{\bigcup_\nu A_\nu} = \mathrm{C}^*_{\mathrm{r}}(G)$. If every conditional expectation $E_\nu := E_{K_\nu}$ and averaging projection $p_\nu := p_{K_\nu}$ satisfy the following conditions, then G is C^*-simple:*

(1) $p_\nu \in A_\nu$.

(2) *For any $\varepsilon > 0$ and self-adjoint element $x \in A_\nu$, there are $g_1, g_2, \ldots, g_n \in G$ such that*

$$\left\| \frac{1}{n}\sum_{i=1}^{n} \lambda_{g_i}(x - E_\nu(x))\lambda_{g_i}^* \right\| < \varepsilon, \quad \left\| p_\nu - \frac{1}{n}\sum_{i=1}^{n} \lambda_{g_i} p_\nu \lambda_{g_i}^* \right\| < \varepsilon.$$

Moreover, if $C_{cc}(G) \subset \bigcup_\nu A_\nu$, and each $g_1, g_2, \ldots, g_n$ can be taken in the kernel of the modular function on G, then the Plancherel weight is a unique σ^φ-KMS-weight on $\mathrm{C}^*_{\mathrm{r}}(G)$ up to a scalar multiple.

The assumption of the above lemma is an analogue of the Powers averaging property with respect to conditional expectations, and not only groups in our main theorem, but also two kinds of non-discrete C^*-simple groups constructed by Suzuki [Su1, Su2] satisfy it.

§2. Preliminaries

§2.1. Weights on C*-algebras

Let A be a C*-algebra and A^+ be the set of positive elements in A. A map $\psi\colon A^+ \to [0,\infty]$ is called a weight on A, if for any $x, y \in A^+$ and $r > 0$, we have

$$\psi(x+y) = \psi(x) + \psi(y), \quad \psi(rx) = r\psi(x).$$

For a weight ψ on A, we set

$$\begin{aligned} \mathfrak{m}_\psi^+ &:= \{x \in A^+ \mid \psi(x) < \infty\}, \\ \mathfrak{n}_\psi &:= \{y \in A \mid y^* y \in \mathfrak{m}_\psi^+\}, \\ \mathfrak{m}_\psi &:= \mathfrak{n}_\psi^* \mathfrak{n}_\psi = \mathrm{span}_{\mathbb{C}} \mathfrak{m}_\psi^+, \end{aligned}$$

as in [R1, Def. 2.17]. This $\mathfrak{n}_\psi$ is a left ideal of A and this $\mathfrak{m}_\psi$ is a subalgebra of A. There is a linear functional from $\mathfrak{m}_\psi$ to $\mathbb{C}$, which is an extension of $\psi|_{\mathfrak{m}_\psi^+}$. We say that ψ is densely defined if $\mathfrak{m}_\psi$ is dense in A. In this article, we suppose that every weight ψ is non-zero, lower semi-continuous, and densely defined.

2.1.1. KMS-weights. For a continuous one-parameter group $(\sigma_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of A, the set of analytic elements is dense in A. (See [Ku, Sect. 1].) A weight ψ is called a σ-KMS-weight if

$$\psi \circ \sigma_t = \psi, \quad \psi(x^* x) = \psi(\sigma_{\frac{i}{2}}(x)\sigma_{\frac{i}{2}}(x)^*)$$

for all $t \in \mathbb{R}$ and analytic elements $x \in A$. (See [R1, Sect. 2.6.3].)

The following proposition holds.

Proposition 2.1 ([R1, Prop. 2.24]). *Let $(\sigma_t)_{t\in\mathbb{R}}$ be a one-parameter group of $*$-automorphisms on a* C**-algebra A, and ψ be a σ-KMS-weight. Any analytic elements $x, y \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ satisfy $\psi(xy) = \psi(y\sigma_{-i}(x))$.*

§2.2. Reduced group C*-algebras

In this article, let G be a locally compact group with a left Haar measure μ and λ be a left regular representation on a Hilbert space $L^2(G) := L^2(G,\mu)$. The representation λ extends to a $*$-representation of $C_c(G)$ on $B(L^2(G))$ as follows. For every $\xi \in L^2(G), f \in C_c(G)$, and $g \in G$, we have

$$\lambda(f)\xi(g) := \int_G f(h)\xi(h^{-1}g)\, d\mu(h).$$

The reduced group C*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$ of G is the norm closure of $\lambda(C_c(G))$.

2.2.1. Averaging projections. If K is a compact open subgroup of G, a projection

$$p_K \coloneqq \lambda\Big(\frac{1}{\mu(K)}\chi_K\Big)$$

induced by an indicator function $\chi_K \in C_c(G)$ of K is called the averaging projection. (See [R1, Sect. 2.6.2] and [Su2, Sect. 2].) These averaging projections satisfy $p_K \geq p_L$ for any compact open subgroups K, L with $K \subset L$, since p_K is the orthogonal projection onto the subspace of $\lambda(K)$-fixed points in $L^2(G)$. When G is totally disconnected, a family Ω of compact open subgroups of G generate a neighborhood basis of $e \in G$, and $\{p_K\}_{K\in\Omega}$ give approximate units of $\mathrm{C}^*_{\mathrm{r}}(G)$.

2.2.2. Conditional expectations. For an open subgroup H of G, we may identify $\mathrm{C}^*_{\mathrm{r}}(H)$ with the C^*-subalgebra of $\mathrm{C}^*_{\mathrm{r}}(G)$ generated by $\lambda(C_c(H))$. It is well known that the restriction map $E_K\colon C_c(G) \to C_c(H)$ extends to a faithful conditional expectation from $\mathrm{C}^*_{\mathrm{r}}(G)$ onto $\mathrm{C}^*_{\mathrm{r}}(H)$. (See [B-O, Sect. 2.5].) This conditional expectation is also denoted by E_K in this article.

2.2.3. The Plancherel weight. Let Δ be the modular function of G. The modular flow $(\sigma^{\varphi}_t)_{t\in\mathbb{R}}$ on $\mathrm{C}^*_{\mathrm{r}}(G)$ is defined as

$$\sigma^{\varphi}_t(f)(g) \coloneqq \Delta(g)^{it} f(g),$$

for any $f \in C_c(G)$ and $g \in G$. The map $\varphi\colon C_c(G) \ni f \mapsto f(e) \in \mathbb{C}$ extends to a σ^{φ}-KMS-weight on $\mathrm{C}^*_{\mathrm{r}}(G)$. (See [R1, Sect. 2.6.2].) This is a restriction of the Plancherel weight on the group von Neumann algebra $L(G)$ (see [Ta] for a definition), which is also called the Plancherel weight. When G is totally disconnected, for a set Ω of all compact open subgroups in G, we set a subalgebra

$$C_{cc}(G) \coloneqq \bigcup_{K\in\Omega} p_K C_c(G) p_K$$

of $\mathrm{C}^*_{\mathrm{r}}(G)$ as in [Su2, Sect. 4]. The following propositions about general σ^{φ}-KMS-weights are known. We assume that G is a totally disconnected group and $(\sigma^{\varphi}_t)_{t\in\mathbb{R}}$ is a modular flow on $\mathrm{C}^*_{\mathrm{r}}(G)$ in the following propositions.

Proposition 2.2 ([Su2, Sect. 44]). *If ψ is a σ^{φ}_t-KMS-weight on $\mathrm{C}^*_{\mathrm{r}}(G)$, then $C_{cc}(G) \subset \mathfrak{m}_{\psi}$.*

Proposition 2.3 ([R1, Lem. 42.23], [Su2, proof of Thm. 44.1]). *If a σ^{φ}_t-KMS weight ψ on $\mathrm{C}^*_{\mathrm{r}}(G)$ satisfies $\psi(\lambda_g p_K) = 0$ for all $g \in G \setminus K$ and compact subgroups K, then $\frac{1}{\psi(p_K)\mu(K)}\psi$ is the Plancherel weight on $\mathrm{C}^*_{\mathrm{r}}(G)$ for any compact open subgroup K.*

§2.3. Constructions of non-discrete C*-simple groups

In this subsection we explain two ways to construct non-discrete examples of C*-simple groups, which were found by Suzuki [Su1, Su2].

The first examples are constructed by the following proposition.

Proposition 2.4 ([Su1, Prop.]). *Let G be a locally compact group. Assume we have a decreasing sequence $(K_n)_{n=1}^{\infty}$ of compact open subgroups of G and an increasing sequence $(L_n)_{n=1}^{\infty}$ of clopen subgroups of G with the following properties:*

- *Each L_n contains K_n and normalizes it.*
- *The quotient groups L_n/K_n are* C**-simple.*
- *The intersection $\bigcap_{n=1}^{\infty} K_n$ is the trivial subgroup $\{e\}$.*
- *The union $\bigcup_{n=1}^{\infty} L_n$ is equal to G.*

Then G is C**-simple and has the unique trace property.*

The proposition above shows C*-simplicity of locally compact groups like $(\bigoplus_{n=1}^{\infty} \Gamma_n) \rtimes \prod_{n=1}^{\infty} F_n$. (See [Su1, Thm.].) Here, F_n are isomorphic to $\mathbb{Z}_2$ and discrete groups Γ_n with $\mathbb{Z}_2$ actions are induced by the splitting short exact sequence

$$0 \longrightarrow \Gamma_n \longrightarrow \mathbb{Z} * \mathbb{Z}_2 \rightleftarrows \mathbb{Z}_2 \longrightarrow 0.$$

The second construction is established in [Su2, Sect. 3]. For a totally disconnected group G and the set Ω of all compact open subgroups of G, let Υ_n, $n \in \mathbb{N}$, be pairwise distinct copies of the group

$$\bigoplus_{K \in \Omega} \bigoplus_{G/K} \mathbb{Z}_2,$$

equipped with a G action induced by the left translation action on G/K. Similarly, let Ξ_n, $n \in \mathbb{N}$, be pairwise distinct copies of $\mathbb{Z}$ with the trivial G-action. Set

$$\Gamma_1 \coloneqq \Upsilon_1, \quad \Lambda_1 \coloneqq \Gamma_1 * \Xi_1,$$
$$\Gamma_{n+1} \coloneqq \Lambda_n \times \Upsilon_{n+1}, \quad \Lambda_{n+1} \coloneqq \Gamma_{n+1} * \Xi_{n+1},$$

for all $n \in \mathbb{N}$. Define Λ to be the inductive limit of the sequence,

$$\Gamma_1 < \Lambda_1 < \Gamma_2 < \Lambda_2 < \cdots$$

of discrete groups with canonical G actions, and set

$$\mathcal{G} \coloneqq \Lambda \rtimes G.$$

The following theorems hold.

Theorem 2.5 ([Su2, Thm. 43.1]). *The locally compact group $\mathcal{G}$ is* C^**-simple.*

Theorem 2.6 ([Su2, Thm. 44.1]). *Up to scalar multiple, the Plancherel weight φ is the only σ^φ-KMS weight on $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$. When $\mathcal{G}$ is non-unimodular, there is no tracial weight on $\mathrm{C}^*_\mathrm{r}(\mathcal{G})$.*

§2.4. Graphs of groups

In this subsection we introduce the basic notation for graphs of groups and their fundamental groups. We use the same notation and definitions as in [Se, H-P].

Let X be a connected graph with a vertex set $V(X)$ and an edge set $E(X) \subset V(X) \times V(X)$. For an edge $x \in E(X)$, vertices $o(x)$ and $t(x)$ denote the origin and the terminus of x. We write $y = \bar{x}$ when edges x and y satisfy $o(x) = t(y)$ and $o(y) = t(x)$. In this article, every edge $x \in E(X)$ has $\bar{x} \in E(X)$ for every graph X. An orientation A of X is a subset of $E(X)$ containing exactly one of x, $\bar{x}$. When X is a graph with orientation A, we define the function $e \colon E(X) \to \{0, 1\}$ as

$$e(x) := \begin{cases} 0, & x \in A, \\ 1, & \bar{x} \in A. \end{cases}$$

Definition 2.7 ([H-P, Sect. 44]). A graph of groups (G, Y) consists of the following:

- a non-empty connected graph Y;
- two families of groups $(G_P)_{P \in V(Y)}$ and $(G_y)_{y \in E(Y)}$ with $G_y = G_{\bar{y}}$ for all $y \in E(Y)$;
- a family of monomorphisms $\{\iota_y \colon G_y \to G_{t(y)}\}_{y \in E(Y)}$.

2.4.1. The group $F(G, Y)$. We define the group $F(G, Y)$ for a graph of groups (G, Y) as follows. (See [Se, Sect. 5.1].) Let Γ be the free product of the vertex groups $(G_P)_{P \in V(Y)}$ and the free group with basis $E(Y)$. The group $F(G, Y)$ is the quotient of Γ by the normal subgroup generated by

$$y\bar{y} \quad \text{and} \quad y\iota_y(a)y^{-1}(\iota_{\bar{y}}(a))^{-1}$$

for all $y \in E(Y)$ and $a \in G_y$.

2.4.2. Words of type c. Let $c = \{y_1, y_2, \ldots, y_n\}$ be a path of Y, where $y_1, y_2, \ldots, y_n$ are edges of Y with $t(y_i) = o(y_{i+1})$. For a sequence $\mu = (r_0, r_1, \ldots, r_n)$ of elements $r_0 \in G_{o(y_1)}$ and $r_i \in G_{t(y_i)}$, the element

$$r_0 y_1 r_1 y_2 \cdots y_n r_n$$

of $F(G, Y)$ is said to be associated with the word (c, μ) and denoted by $|c, \mu|$. (See [Se, Def. 9].)

A pair (c, μ) is called reduced if either of the following holds:

- The length n of c is not equal to 0 and $r_{i-1} \notin \iota_{\bar{y}_i}(G_{y_i})$ for every index i with $y_i = \bar{y}_{i-1}$.
- The length $n = 0$ and $r_0 \neq 0$.

When (c, μ) is reduced, we have $|c, \mu| \neq e$. (See [Se, Thm. 11].)

2.4.3. The fundamental groups $\pi_1(G, Y)$. Let P_0 be an element of $V(Y)$. The group $\pi_1(G, Y, P_0)$ is a subgroup of $F(G, Y)$ generated by the set of elements $|c, \mu|$ of closed paths c from P_0 to P_0. This is called the fundamental group of (G, Y) at P_0. (See [Se, Sect. 5.1].)

Let T be a maximal subtree of Y. The fundamental group $\pi_1(G, Y, T)$ of (G, Y) at T is the quotient of $F(G, Y)$ by the normal subgroup generated by the set of edges of T. In $\pi_1(G, Y, T)$, the element induced by $y \in E(Y)$ is denoted by g_y. (See [Se, Sect. 5.1].)

The following proposition holds.

Proposition 2.8 ([Se, Prop. 420]). *For any maximal subtree T of Y and $P_0 \in V(Y)$, the canonical inclusion $i_{P_0} \colon \pi_1(G, Y, P_0) \to F(G, Y)$ and the canonical quotient map $q_T \colon F(G, Y) \to \pi_1(G, Y, T)$ induce an isomorphism of $\pi_1(G, Y, P_0)$ onto $\pi_1(G, Y, T)$.*

This proposition shows that the isomorphism classes of $\pi_1(G, Y, P_0)$ and $\pi_1(G, Y, T)$ are independent of the choice of P_0 and T. Thus we write $\pi_1(G, Y)$ instead of $\pi_1(G, Y, P_0)$ or $\pi_1(G, Y, T)$ if no confusion arises. When c is a closed path of Y, we have $q_T(|c, \mu|) \neq e$ for a reduced form (c, μ). (See [Se, Cor. 3].)

Example 2.9 (Amalgamated free products). If a connected graph Y consists of two vertices $\{P, Q\}$ and two edges $\{y, \bar{y}\}$, then the fundamental group $\pi_1(G, Y)$ is isomorphic to the amalgamated free product $G_P *_{G_y} G_Q$ with respect to the inclusions ι_y and $\iota_{\bar{y}}$. (See [Se, Sect. 1.2] for the definition.) In this case, every element $g \in G_P *_{G_y} G_Q \setminus \{e\}$ has a form

$$g = a_1 a_2 \cdots a_n c \quad \text{with} \quad a_i \in (G_P \setminus \iota_y(G_y)) \cup (G_Q \setminus \iota_{\bar{y}}(G_y)),\ c \in G_y.$$

In addition, the sequence $(a_1, a_2, \ldots, a_n, c)$ can be chosen as follows:

- When $n = 0$, this c is not equal to e.
- We have $a_{i+1} \in G_P \setminus \iota_y(G_y)$ if and only if $a_i \in G_Q \setminus \iota_{\bar{y}}(G_y)$ for any i.

Generally, a sequence $(a_1, a_2, \ldots, a_n, c)$ of words of an amalgamated free product is said to be reduced if the above conditions hold. Since every sequence of reduced words of $G_P *_{G_y} G_Q$ corresponds to a reduced form (c,μ) of (Y,G), an element $g \in G_P *_{G_y} G_Q$ induced by reduced words is not equal to e.

Example 2.10 (HNN extensions). If a graph Y consists of one vertex P and two edges of loops $\{y,\bar{y}\}$, then the fundamental group $\pi_1(G,Y)$ is isomorphic to the HNN extension $G_P *_\theta$ with respect to the canonical isomorphism θ from $\iota_y(G_y)$ to $\iota_{\bar{y}}(G_y)$. (See [Se, Sect. 1.4] for the definition.) Every element $g \in G_P *_\theta \setminus\{e\}$ has the form

$$g = a_0 t^{\varepsilon_1} a_1 t^{\varepsilon_2} a_2 \cdots t^{\varepsilon_n} a_n$$

with $a_i \in G_P$, $\varepsilon_i \in \{1,-1\}$, and the stable letter t of the HNN extension. We can take these a_i and ε_i as follows:

- If $n = 0$, then $a_0 \neq e$.
- If an index i satisfies $(\varepsilon_{i-1},\varepsilon_i) = (1,-1)$, then $a_{i-1} \notin \iota_y(G_y)$.
- If an index i satisfies $(\varepsilon_{i-1},\varepsilon_i) = (-1,1)$, then $a_{i-1} \notin \iota_{\bar{y}}(G_y)$.

Generally, a sequence $(a_0, t^{\varepsilon_1}, a_1, t^{\varepsilon_2}, a_2, \ldots, t^{\varepsilon_n}, a_n)$ of an HNN-extension is called reduced words if the above conditions hold. For the same reasons as for amalgamated free products, if $g \in G_P *_\theta$ is induced by reduced words, then $g \neq e$.

2.4.4. The universal covering. Fix a maximal subtree T of Y and an orientation $A \subset E(Y)$ of Y. For every $P \in V(Y)$, G_P is a subgroup of $\pi_1(G,Y,T)$ in a natural way. The universal covering $\widetilde{X} := \widetilde{X}(G,Y,T)$ of (G,Y,T) is a graph with a vertex set $V(\widetilde{X}) := \bigsqcup_{p\in V(Y)} \pi_1(G,Y,T)/G_P$ and an edge set $E(\widetilde{X}) := \bigsqcup_{y\in E(Y)} \pi_1(G,Y,T)/\iota_y(G_y)$, where

$$o(g\iota_y(G_y)) = g g_y^{e(y)} G_{o(y)}, \quad t(g\iota_y(G_y)) = g g_y^{1-e(y)} G_{t(y)}$$

for any $g \in \pi_1(G,Y,T)$ and $y \in E(Y)$. This $\widetilde{X}$ has an action of $\pi_1(G,Y,T)$ induced by the left multiplication. It is known that the universal covering $\widetilde{X}$ is a tree. (See [Se, Thm. 12].)

§3. C*-simplicity and the Powers averaging property for conditional expectations

In this section we suppose that G is a totally disconnected locally compact group and σ^φ is the modular flow. First, we give a sufficient condition for C*-simplicity of G.

Lemma 3.1. *Let $(K_\nu)_{\nu\in N}$ be a decreasing net of compact open subgroups of G and $(A_\nu)_{\nu\in N}$ be an increasing net of* C*-*subalgebras of $\mathrm{C}^*_{\mathrm{r}}(G)$ with $\bigcap_\nu K_\nu = \{e\}$*

and $\overline{\bigcup_\nu A_\nu} = \mathrm{C}^*_\mathrm{r}(G)$*. If every conditional expectation* $E_\nu := E_{K_\nu}$ *and averaging projection* $p_\nu := p_{K_\nu}$ *satisfy the following conditions, then* G *is* C*-*simple:*

(1) *The averaging projection* p_ν *is in* A_ν*.*

(2) *For any* $\varepsilon > 0$ *and self-adjoint element* $x \in A_\nu$*, there are* $g_1, g_2, \ldots, g_n \in G$ *such that*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i}(x - E_\nu(x))\lambda_{g_i}^* \right\| < \varepsilon, \quad \left\| p_\nu - \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i} p_\nu \lambda_{g_i}^* \right\| < \varepsilon.$$

Moreover, if $C_{cc}(G) \subset \bigcup_\nu A_\nu$*, and each* $g_1, g_2, \ldots, g_n$ *can be taken in the kernel of the modular function on* G*, then the Plancherel weight is a unique* σ^φ*-KMS-weight on* $\mathrm{C}^*_\mathrm{r}(G)$ *up to a scalar multiple.*

Proof. Let I be a non-zero norm closed two-sided ideal of $\mathrm{C}^*_\mathrm{r}(G)$. To prove C*-simplicity of G, it suffices to show that the averaging projection p_ν is in I for sufficiently large ν, since the net $(p_\nu)_\nu$ gives approximate units of $\mathrm{C}^*_\mathrm{r}(G)$.

Since $\bigcup_\nu A_\nu \cap I \neq 0$, we can take a positive element $x \in \bigcup_\nu A_\nu \cap I$. There is $\nu_0 \in N$ such that $p_\nu x p_\nu \neq 0$ and $x \in A_\nu \cap I$ for any $\nu \geq \nu_0$. Fix $\nu \geq \nu_0$ and we show $p_\nu \in I$. Let φ be the Plancherel weight with respect to the left Haar measure μ. We may assume $\varphi(x) = \frac{1}{\mu(K_\nu)}$ and $x = p_\nu x p_\nu$ by replacing a positive element $x \in A_\nu \cap I$. Since $E_\nu(y) = \mu(K_\nu)\varphi(y)p_\nu$ holds for any $y \in p_\nu \mathrm{C}^*_\mathrm{r}(G) p_\nu$, we have $E_\nu(x) = p_\nu$.

By using assumption (2) of the lemma, for $\varepsilon > 0$ we choose $g_1, g_2, \ldots, g_n \in G$ satisfying

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i}(x - E_\nu(x))\lambda_{g_i}^* \right\| < \varepsilon, \quad \left\| p_K - \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i} p_\nu \lambda_{g_i}^* \right\| < \varepsilon.$$

Thus

$$\left\| p_\nu - \frac{1}{n} \sum_{i=1}^{n} \lambda_{g_i} x \lambda_{g_i}^* \right\| < 2\varepsilon.$$

Since I is a norm closed ideal, we get $p_\nu \in I$.

Next we show the uniqueness of a σ^φ-KMS-weight. Let ψ be a σ^φ-KMS-weight, L be a compact open subgroup of G, and $s \in G \setminus L$. By Proposition 2.3, it suffices to show that $\psi(\lambda_s p_L) = 0$. We prove this equation in the same way as in [Su2, proof of Thm. 4.1]. Set $x_1 := p_L \lambda_s p_L + p_L \lambda_{s^{-1}} p_L$. We can take $\nu \in N$ with $K_\nu \subset L$ and $x_1 \in A_\nu$. For $\varepsilon > 0$, we have $g_1, g_2, \ldots, g_n$ in the kernel of the modular function with

$$\left\| \frac{1}{n} \sum_{j=1}^{n} \lambda_{g_j} x_1 \lambda_{g_j}^* \right\| < \varepsilon, \quad \left\| p_\nu - \frac{1}{n} \sum_{j=1}^{n} \lambda_{g_j} p_\nu \lambda_{g_j}^* \right\| < \varepsilon,$$

since $E_\nu(x_1) = 0$.

Set

$$y := \frac{1}{n}\sum_{j=1}^{n} p_\nu \lambda_{g_j}^* p_\nu \lambda_{g_j} p_\nu \in p_\nu C_c(G) p_\nu;$$

then

$$|\psi(p_\nu - y)| = \left|\psi\left(p_\nu - \frac{1}{n}\sum_{j=1}^{n} p_\nu \lambda_{g_j} p_\nu \lambda_{g_j}^* p_\nu\right)\right| < \psi(p_\nu)\varepsilon.$$

The equation above follows from the KMS-condition of ψ, since $\sigma_{-i}^{\varphi}(p_\nu \lambda_{g_j} p_\nu) = p_\nu \lambda_{g_j} p_\nu$. Similarly,

$$|\psi(yx_1)| = \left|\psi\left(\frac{1}{n}\sum_{j=1}^{n} p_\nu \lambda_{g_j} x_1 \lambda_{g_j}^* p_\nu\right)\right| < \psi(p_\nu)\varepsilon.$$

Therefore, we get

$$\begin{aligned}
|\psi(x_1)| &\le |\psi((p_\nu - y)x_1)| + |\psi(yx_1)| \\
&\le \sqrt{\psi(p_\nu - y)}\sqrt{\psi(x_1^*(p_\nu - y)x_1)} + |\psi(yx_1)| \\
&< 2\psi(p_\nu)\sqrt{\varepsilon} + \psi(p_\nu)\varepsilon,
\end{aligned}$$

since $p_\nu - y \ge 0$ and ψ is a positive functional on $p_\nu\, \mathrm{C}_\mathrm{r}^*(G) p_\nu$. The above inequality holds for every $\varepsilon > 0$, then $\psi(x_1) = \psi(p_L \lambda_s p_L + p_L \lambda_{s^{-1}} p_L) = 0$. Similarly, we have $\psi(ip_L \lambda_s p_L - ip_L \lambda_{s^{-1}} p_L) = 0$, then $\psi(\lambda_s p_L) = \psi(p_L \lambda_s p_L) = 0$. □

Example 3.2. When G is a C*-simple group in Proposition 2.4, the averaging projection p_{K_n} is a central projection of $\mathrm{C}_\mathrm{r}^*(L_n)$ with $\mathrm{C}_\mathrm{r}^*(L_n)p_{K_n} \cong \mathrm{C}_\mathrm{r}^*(L_n/K_n)$. By this isomorphism, the canonical trace of $\mathrm{C}_\mathrm{r}^*(L_n/K_n)$ corresponds to the restriction of E_{K_n} on $\mathrm{C}_\mathrm{r}^*(L_n)p_{K_n}$. Since L_n/K_n is a discrete C*-simple group, for any $x \in \mathrm{C}_\mathrm{r}^*(L_n)p_{K_n}$ and $\varepsilon > 0$ there are $g_1, g_2, \ldots, g_m \in L_n$ such that the inequality

$$\left\|\frac{1}{m}\sum_{j=1}^{m} \lambda_{g_j}(x - E_{K_n}(x))\lambda_{g_j}^*\right\| = \left\|\frac{1}{m}\sum_{j=1}^{m} \lambda_{g_j} x \lambda_{g_j}^* - E_{K_n}(x)\right\| < \varepsilon$$

holds by [Haa, Thm. 4.5].

The increasing union $\bigcup_n \mathrm{C}_\mathrm{r}^*(L_n)p_{K_n}$ is a dense $*$-subalgebra of $\mathrm{C}_\mathrm{r}^*(G)$ containing $C_{cc}(G)$ and G is unimodular. Thus G satisfies the assumptions of Lemma 3.1.

Example 3.3. Suppose that the discrete group Λ and the C*-simple group $\mathcal{G} := \Lambda \rtimes G$ for a totally disconnected group G are defined as in Theorem 2.5. Let K be a compact open subgroup of G, and $g_1, g_2, \ldots, g_l$ be elements in $\mathcal{G} \setminus K$. Since $\mathcal{G} = \bigcup_n \Lambda_n \rtimes G$, we may assume $g_1, g_2, \ldots, g_l \in \Lambda_{N-1} \rtimes G$ for some N. Each g_i has the form $g_i = s_i h_i$, where s_i is in Λ_{N-1}, h_i is in G. Let $(\delta_{gK})_{gK \in G/K}$ be the canonical generators of $\bigoplus_{G/K}\mathbb{Z}_2 \leq \Upsilon_N = \bigoplus_{K\in\Omega}\bigoplus_{G/K}\mathbb{Z}_2$. For every i, we have $\delta_K g_i \delta_K^{-1} = s_i\delta_K\delta_{h_iK}^{-1}h_i$ and $s_i' := s_i\delta_K\delta_{h_iK}^{-1} \neq e$, because every i satisfies either $s_i \neq e$ or $h_i \notin K$. Since Λ has the subgroup $\Gamma_N * \Xi_N * \Xi_{N+1}$ and $s_i' \in \Gamma_N$, for any $\varepsilon > 0$ there are $t_1, t_2, \dots, t_m \in \Xi_N * \Xi_{N+1}$ such that

$$\left\| \frac{1}{m}\sum_{j=1}^{m} \lambda^{\Lambda}_{t_j s_i' t_j^{-1}} \right\| < \frac{\varepsilon}{l}$$

for all i (see [P-S, proof of Lem. 1.2]), where λ^{Λ} is the left regular representation of Λ. The Hilbert space $L^2(\mathcal{G})$ is isomorphic to $l^2(\Lambda)\otimes L^2(G)$ and the restriction of $\lambda^{\mathcal{G}}$ on Λ is unitary equivalent to $\lambda^{\Lambda}\otimes 1$. Since $(t_j)_j$ and δ_K commute with K, they are in the kernel of the modular function. Moreover, we get

$$\left\| \frac{1}{m}\sum_{j}^{m} \lambda^{\mathcal{G}}_{t_j\delta_K}\left(\sum_{i}^{l} p_K\lambda^{\mathcal{G}}_{g_i}p_K\right)(\lambda^{\mathcal{G}}_{t_j\delta_K})^*\right\| = \left\|\sum_{i}^{l} p_K\left(\frac{1}{m}\sum_{j=1}^{m}\lambda^{\mathcal{G}}_{t_j s_i' t_j^{-1}}\right)\lambda^{\mathcal{G}}_{h_i}p_K\right\| < \varepsilon$$

and

$$p_K = \frac{1}{m}\sum_{j}^{m} \lambda^{\mathcal{G}}_{t_j\delta_K} p_K \lambda^{\mathcal{G}*}_{t_j\delta_K}.$$

Therefore, $\mathcal{G}$ satisfies the assumption of Lemma 3.1 for the net $(K_\nu)_\nu$ of all compact open subgroups of G and the net $(p_{K_\nu}\,\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})p_{K_\nu})_\nu$ of C*-subalgebras of $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G})$.

§4. Generalized Baumslag–Solitar groups and their completions

In this section let (G, Y) be a graph of groups with a connected graph Y, vertex groups $\{G_P\}_{P\in V(Y)}$, edge groups $\{G_y\}_{y\in E(Y)}$, and monomorphisms $\{\iota_y\colon G_y \to G_{t(y)}\}_{y\in E(Y)}$. We suppose that θ_y is a canonical isomorphism from $\iota_y(G_y)$ to $\iota_{\bar{y}}(G_y)$ for any $y \in E(Y)$. Let y be an edge of Y and W be a subgraph of Y, with $E(W) := E(Y)\setminus\{y,\bar{y}\}$ and $V(W) := V(Y)$. Since Y is connected, W is either connected or decomposed into a disjoint union of two non-empty connected graphs W_1 and W_2.

Lemma 4.1. *The following statements hold:*

(a) *If W is connected, then $\pi_1(G, Y)$ is isomorphic to the HNN extension $\pi_1(G|_W, W)*_{\theta_y}$, where $(G|_W, W)$ is a graph of groups with edge groups $\{G_w\}_{w\in E(W)(=E(Y)\setminus\{y,\bar{y}\})}$ and vertex groups $\{G_P\}_{P\in V(W)(=V(Y))}$.*

(b) *If W is disconnected, then $\pi_1(G, Y)$ is isomorphic to the amalgamated free product $\pi_1(G|_{W_1}, W_1)*_{G_y}\pi_1(G|_{W_2}, W_2)$, where $(G|_{W_i}, W_i)$ is a graph of groups with edge groups $\{G_w\}_{w\in E(W_i)(\subset E(Y))}$ and vertex groups $\{G_P\}_{P\in V(W_i)(\subset V(Y))}$.*

Proof. First we suppose W is connected. By definition of HNN extensions, there is the natural isomorphism $f\colon F(G,Y) \to F(G|_W, W)*_{\theta_y}$, where $F(G,Y)$ and $F(G|_W, W)$ are groups defined in Section 2.4.1. Put $P_0 \coloneqq o(y)$ and take a maximal subtree T of Y with $y \notin E(T)$; then the commutative diagram

$$\begin{array}{ccccc}
\pi_1(G,Y,P_0) & \xhookrightarrow{\;i_Y\;} & F(G,Y) & \xrightarrow{\;q_Y\;} & \pi_1(G,Y,T) \\
\Big\downarrow{\scriptstyle f|} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \bar{f}} \\
\pi_1(G|_W,W,P_0)*_{\theta_y} & \xhookrightarrow{(i_W)*_{\theta_y}} & F(G|_W,W)*_{\theta_y} & \xrightarrow{(q_W)*_{\theta_y}} & \pi_1(G|_W,W,T)*_{\theta_y}
\end{array}$$

holds, where i_Y, i_W are the canonical injections and q_Y, q_W are the canonical quotient maps as in Proposition 2.8. By Proposition 2.8, $q_1 \circ i_1$ and $q_2 \circ i_2$ are isomorphisms. Since f is an isomorphism, the induced maps $f|$ and $\bar{f}$ are isomorphisms.

Next, let W be disconnected. We define a graph of groups (H, W') as follows. The graph W' consists of two edges $\{y, \bar{y}\}$ and two vertices $\{P_1 \coloneqq o(y), P_2 \coloneqq t(y)\}$. The edge group and the vertex groups of (H, W') are defined as

$$H_y \coloneqq G_y, \quad H_{P_1} \coloneqq F(G|_{W_1}, W_1), \quad H_{P_2} \coloneqq F(G|_{W_2}, W_2).$$

By using [Se, Lem. 6] repeatedly, we get the natural isomorphism $g\colon F(G,Y) \to F(H,W')$. Fix a maximal subtree T of Y; then $T_1 \coloneqq T \cap Y_1$ and $T_2 \coloneqq T \cap Y_2$ are the maximal subtrees of Y_1 and Y_2. Since $\pi_1(H, W', W')$ is isomorphic to $F(G|_{W_1}, W_1) *_{G_y} F(G|_{W_2}, W_2)$, there is a commutative diagram

$$\begin{array}{ccccc}
F(G,Y) & \xrightarrow{\;q_Y\;} & \pi_1(G,Y,T) & & \\
\Big\downarrow{\scriptstyle g} & & & \searrow^{\bar{g}} & \\
F(H,W') & \xrightarrow{\;q_{W'}\;} & F(G|_{W_1},W_1)*_{G_y}F(G|_{W_2},W_2) & \xrightarrow{\;q\;} & \pi_1(G|_{W_1},W_1,T_1)*_{G_y}\pi_1(G|_{W_2},W_2,T_2),
\end{array}$$

with $q \coloneqq q_{W_1} *_{G_y} q_{W_2}$. Define a subgroup F' of $F(H,W')$ as

$$\begin{aligned}
F' \coloneqq \bigl\{ |c,\mu| \in F(H,W') \bigm| & (c,\mu) \text{ is a word of } (H,W'), \\
& c \text{ is a closed path of } W' \text{ beginning with } P_1, \\
& \mu \text{ is a sequence of } \pi_1(G|_{W_1}, W_1, P_1) \cup \pi_1(G|_{W_2}, W_2, P_2) \bigr\}.
\end{aligned}$$

This F' satisfies $g(\pi_1(G,Y,P_1)) \subset F'$ and the restriction of $q \circ q_{W'}$ on F' is an isomorphism. That is because F' is isomorphic to the fundamental group of the graph of groups (H', W') whose vertex groups and edge groups are

$$H'_y \coloneqq G_y, \quad H'_{P_1} \coloneqq \pi_1(G|_{W_1}, W_1, P_1), \quad H'_{P_2} \coloneqq \pi_1(G|_{W_2}, W_2, 2_2).$$

Since the commutative diagram

$$\begin{array}{ccccc} \pi_1(G,Y,P_0) & \overset{i_Y}{\hookrightarrow} & F(G,Y) & \overset{q_Y}{\twoheadrightarrow} & \pi_1(G,Y,T) \\ {\scriptstyle g|}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle \bar{g}}\downarrow \\ F' & \hookrightarrow & F(H,W') & \overset{q\circ q_{W'}}{\twoheadrightarrow} & \pi_1(G|_{W_1},W_1,T_1) *_{G_y} \pi_1(G|_{W_2},W_2,T_2) \end{array}$$

holds, $\bar{g}$ is an isomorphism. □

Definition 4.2. The fundamental group $\pi_1(G,Y)$ is called a generalized Baumslag–Solitar group if the edge groups $\{G_y\}_{y\in E(Y)}$ and the vertex groups $\{G_P\}_{P\in V(Y)}$ are isomorphic to $\mathbb{Z}$.

Suppose that (G,Y) is a graph of groups whose edge groups and vertex groups are isomorphic to $\mathbb{Z}$ and Y consists of one vertex P and two edges y, $\bar{y}$. If $[G_P : \iota_{\bar{y}}(G_y)] = n$ and $[G_{o(y)} : \iota_y(G_y)] = m$, then the generalized Baumslag–Solitar group $\pi_1(G,Y)$ is isomorphic to the Baumslag–Solitar group $\mathrm{BS}(n,m)$.

Let (G,Y) be a general graph of groups whose edge groups and vertex groups are isomorphic to $\mathbb{Z}$. Suppose T is a maximal subtree of Y. For a geodesic path $c \subset T$ from P to Q, we define the integer k_c as follows. Suppose that $c \coloneqq (y_1, y_2, \ldots, y_l)$, $[G_{o(y_i)} : \iota_{\bar{y}_i}(G_{y_i})] = n_i$, and $[G_{t(y_i)} : \iota_{y_i}(G_{y_i})] = m_i$. Set $k_1 \coloneqq m_1$ and $k_i \coloneqq \frac{k_{i-1}m_i}{(k_{i-1},n_i)}$ for any $1 \le i \le l$, where (k_{i-1},n_i) is the greatest common divisor of k_{i-1} and n_i. We put $k_c \coloneqq k_l$ for a geodesic path c with length $l \neq 0$ and $k_c \coloneqq 1$ for a path c with length 0. By a direct computation, we get $[G_{t(y_l)} : G_{t(y_l)} \cap G_{o(y_1)}] = k_c$ and $[G_{o(y_1)} : G_{t(y_l)} \cap G_{o(y_1)}] = k_{\bar{c}}$ in $\pi_1(G,Y,T)$, where $\bar{c} \coloneqq (\bar{y}_l, \bar{y}_{l-1}, \ldots, \bar{y}_1)$.

In addition, if y is an edge of Y with $[G_{o(y)} : \iota_{\bar{y}}(G_y)] = n$ and $[G_{t(y)} : \iota_y(G_y)] = m$, then there is the geodesic path $c \subset T$ from $o(y)$ to $t(y)$. The relative order $[G_{o(y)} : \iota_{\bar{y}}(G_y) \cap \iota_y(G_y)]$ in $\pi_1(G,Y,T)$ can be computed as the least common multiple $k'_{\bar{y}}$ of n and $\frac{k_{\bar{c}}m}{(m,k_c)}$. We define $\kappa_{\bar{y}} \coloneqq \frac{k'_{\bar{y}}}{n}$; then we have $[\iota_{\bar{y}}(G_y) : \iota_{\bar{y}}(G_y) \cap \iota_y(G_y)] = \kappa_{\bar{y}}$ in $\pi_1(G,Y,T)$.

By using above notation, we state our main theorem as follows.

Theorem 4.3. *Let T be a maximal subtree of Y and $\overline{\pi_1(G,Y,T)}$ be the closure of $\pi_1(G,Y,T)$ in the automorphism group of the universal covering $\widetilde{X}(G,Y,T)$. If (G,Y,T) satisfies the following conditions, then $\overline{\pi_1(G,Y,T)}$ is a non-discrete locally compact* C*-*simple group:*

- *The graph Y is not a tree.*
- *Every edge group and vertex group of (G,Y) is isomorphic to $\mathbb{Z}$.*
- *There is an edge $y \in E(Y) \setminus E(T)$ such that $\kappa_{\bar{y}} \neq \kappa_y$.*
- *For any edge z of Y, $\iota_z(G_z)$ is a proper subgroup of $G_{t(z)}$.*

Moreover, the reduced group C^**-algebra* $\mathrm{C}^*_{\mathrm{r}}(\overline{\pi_1(G,Y,T)})$ *has a unique KMS-weight with respect to the modular flow.*

In the case of Baumslag–Solitar groups, the following are known (see [R2]):

- $\overline{\mathrm{BS}(n,m)}$ is discrete if and only if $|n|=|m|$.
- $\overline{\mathrm{BS}(n,m)}$ is C^*-simple if and only if $|n|,|m|\neq 1$.

In Theorem 4.3, the third condition corresponds to $|n|\neq|m|$ and the fourth condition corresponds to $|n|,|m|\neq 1$.

From now on, we fix the graph of groups (G,Y), the maximal subtree T of Y, and $y\in E(Y)\setminus E(T)$ which satisfy the assumptions in Theorem 4.3. In order to prove Theorem 4.3, we prepare some lemmas as follows.

Lemma 4.4. *The action of* $\pi_1 := \pi_1(G,Y,T)$ *on the universal covering* $\widetilde{X} := \widetilde{X}(G,Y,T)$ *is faithful.*

Proof. Suppose that $g\in\pi_1$ acts trivially on $\widetilde{X}$. It suffices to show $g=e$. Since left multiplication of g preserves all elements of $V(\widetilde{X})=\sqcup\pi_1/G_P$, we have $g_y^n g g_y^{-n}\in G_{o(y)}\cap G_{t(y)}$ for all $n\in\mathbb{Z}_{\geq 0}$. Let W be a connected subgraph of Y with $V(W):=V(Y)$, $E(W):=E(Y)\setminus\{y,\bar{y}\}$. By Lemma 4.1, π_1 has the structure of an HNN extension and g_y corresponds to the stable letter. Therefore, we get

$$g\in G_{t(y)}\quad\text{and}\quad g_y^n g g_y^{-n}\in\iota_y(G_y)\cap\iota_{\bar{y}}(G_y)$$

for every $n\in\mathbb{Z}_{\geq 0}$. Since all edge groups of Y are isomorphic to $\mathbb{Z}$, we can take the free generators a of $\iota_y(G_y)$ and b of $\iota_{\bar{y}}(G_y)$ with $g_y a g_y^{-1}=b$. Take the integers $N,M\in\mathbb{Z}_{>0}$ with $\iota_y(G_y)\cap\iota_{\bar{y}}(G_y)=\langle a^N\rangle=\langle b^M\rangle$; then $N\neq M$ holds because of the assumption $\kappa_y\neq\kappa_{\bar{y}}$. By replacing y by $\bar{y}$, we may assume $|\frac{M}{(N,M)}|\neq 1$. We show

$$\langle a\rangle\cap g_y^{-n}(\iota_y(G_y)\cap\iota_{\bar{y}}(G_y))g_y^n=\left\langle a^{(N,M)|\frac{M}{(N,M)}|^n}\right\rangle$$

for any n. The inclusion $\langle a\rangle\cap g_y^{-n}(\iota_y(G_y)\cap\iota_{\bar{y}}(G_y))g_y^n\supset\langle a^{(N,M)|\frac{M}{(N,M)}|^n}\rangle$ follows by a straightforward computation. We prove the converse inclusion by using induction. When $n=1$, it is trivial. We assume that the above equation holds for $n_0\geq 1$ and take $h'\in\langle a\rangle\cap g_y^{-n_0-1}(\iota_y(G_y)\cap\iota_{\bar{y}}(G_y))g_y^{n_0+1}$. Set $h:=g_y^{n_0+1}h'g_y^{-n_0-1}$; then we have $h\in\iota_y(G_y)\cap\iota_{\bar{y}}(G_y)$ and $g_y^{-n_0-1}hg_y^{n_0+1}\in\langle a\rangle$. Since g_y is the stable letter, $g_y^{-n_0}hg_y^{n_0}$ is in $\iota_y(G_y)\cap\iota_{\bar{y}}(G_y)=\langle a^N\rangle$. By the induction hypothesis, we get

$$\begin{aligned}g_y^{-n_0}hg_y^{n_0}\in\left\langle a^{(N,M)|\frac{M}{(N,M)}|^{n_0}}\right\rangle\cap\langle a^N\rangle&=\left\langle a^{(N,M)|\frac{M}{(N,M)}|^{n_0}|\frac{N}{(N,M)}|}\right\rangle\\&=\left\langle b^{(N,M)|\frac{M}{(N,M)}|^{n_0+1}}\right\rangle.\end{aligned}$$

Then $h' = g_y^{-n_0-1} h g_y^{n_0+1} \in \langle a^{(N,M)|\frac{M}{(N,M)}|^{n_0+1}} \rangle$. Thus $\langle a \rangle \cap g_y^{-n}(\iota_y(G_y) \cap \iota_{\bar{y}}(G_y)) g_y^n = \langle a^{(N,M)|\frac{M}{(N,M)}|^n} \rangle$ holds for any n. Since $g_y^n g g_y^{-n} \in \iota_y(G_y) \cap \iota_{\bar{y}}(G_y)$ for every $n \in \mathbb{Z}_{\geq 0}$, we have $g = e$. $\square$

From now on, we put $\pi_1 := \pi_1(G, Y, T)$ and $\widetilde{X} := \widetilde{X}(G, Y, T)$ as used in Lemma 4.4. Let $\mathbf{Aut}(\widetilde{X})$ be the automorphism groups of $\widetilde{X}$. We suppose that $\overline{\Gamma}$ is the closure of Γ in $\mathbf{Aut}(\widetilde{X})$ for any subset $\Gamma \subset \pi_1$. For $g \in \pi_1$ and $P \in V(Y)$, the subgroup of $\mathbf{Aut}(\widetilde{X})$ which stabilizes $gG_P \in V(\widetilde{X})$ is denoted by $\mathbf{Stab}(gG_P)$. If $w \in E(Y)$, then there is a subgraph W of Y with $V(W) = V(Y)$ and $E(W) = E(Y) \setminus y, \bar{y}$. We use the letter $Y \setminus \{y\}$ for this W.

Lemma 4.5. *For any $P \in Y$, the family of finite intersections of $\{\bar{\pi}_1 \cap \mathbf{Stab}(gG_P)\}_{g \in \pi_1}$ gives a neighborhood basis of e in $\bar{\pi}_1$.*

Proof. By definition of $\bar{\pi}_1$, the family of finite intersections of the sets $\{\bar{\pi}_1 \cap \mathbf{Stab}(gG_Q) \mid g \in \pi_1,\ Q \in V(Y)\}$ is a neighborhood basis of e. Thus it suffices to show that for any $g \in \pi_1$ and $Q \in V(Y)$, there are $g_1, g_2, \ldots, g_n \in \pi_1$ with $\bigcap_{i=1}^n \bar{\pi}_1 \cap \mathbf{Stab}(g_i G_P) \subset \bar{\pi}_1 \cap \mathbf{Stab}(gG_Q)$.

First we assume there is an edge w such that $o(w) = P$ and $t(w) = Q$. Fix $g \in \pi_1$. When the graph $Y \setminus \{w\}$ is disconnected, there are subgraphs W_1 and W_2 with $\pi_1(G|_{W_1}, W_1) *_{G_w} \pi_1(G|_{W_2}, W_2) \cong \pi_1$ as in Lemma 4.1. Suppose $P = o(w) \in V(W_1)$ and $Q = t(w) \in V(W_2)$. We can take $b \in G_Q \setminus \iota_w(G_w) \subset \pi_1(G|_{W_2}, W_2)$. For any $h \in \pi_1$, if $g^{-1}hg \in G_P$ and $b^{-1}g^{-1}hgb \in G_P$ hold, then $g^{-1}hg \in G_w \subset G_Q$. Thus we have

$$\pi_1 \cap \mathbf{Stab}(gG_P) \cap \mathbf{Stab}(gbG_P) \subset \pi_1 \cap \mathbf{Stab}(gG_Q).$$

By taking closures, we get $\bar{\pi}_1 \cap \mathbf{Stab}(gG_P) \cap \mathbf{Stab}(gbG_P) \subset \bar{\pi}_1 \cap \mathbf{Stab}(gG_Q)$ since stabilizers are clopen subgroups of $\mathbf{Aut}(\widetilde{X})$.

When the graph $W = Y \setminus \{w\}$ is connected, we have $\pi_1(G|_W, W)*_{\theta_w} \cong \pi_1$. Thus for any $h \in \pi_1$, if $g^{-1}hg \in G_P$ and $g_w g^{-1} h g g_w^{-1} \in G_P$ hold, then $g^{-1}hg \in \iota_w(G_w)$. We get

$$\pi_1 \cap \mathbf{Stab}(gG_P) \cap \mathbf{Stab}(gg_w^{-1}G_P) \subset \pi_1 \cap \mathbf{Stab}(gG_Q).$$

This leads to $\bar{\pi}_1 \cap \mathbf{Stab}(gG_P) \cap \mathbf{Stab}(gg_w^{-1}G_P) \subset \bar{\pi}_1 \cap \mathbf{Stab}(gG_Q)$.

For a general Q, there is a geodesic path from P to Q in Y. By using the same argument repeatedly, we get $g_1, g_2, \ldots, g_n \in \pi_1$ with $\bigcap_{i=1}^n \bar{\pi}_1 \cap \mathbf{Stab}(g_i G_P) \subset \bar{\pi}_1 \cap \mathbf{Stab}(gG_Q)$. $\square$

Lemma 4.6. *For any $P \in V(Y)$, the closure $\overline{G}_P$ is a non-discrete compact open subgroup of $\bar{\pi}_1$.*

Proof. Since $\overline{G}_P = \bar{\pi}_1 \cap \mathbf{Stab}(G_P)$, it is trivial that $\overline{G}_P$ is open.

By Lemmas 4.4 and 4.5, we have $\bigcap_{g\in\pi_1} \mathbf{Stab}(gG_P) \cap \pi_1 = \{e\}$ for any $P \in V(Y)$. Let S_{π_1/G_P} be the topological group of all bijections from π_1/G_P to π_1/G_P. There is the monomorphism $f\colon \pi_1 \to S_{\pi_1/G_P}$ given by left multiplication. This f is continuous with respect to the relative topology of π_1 induced by $\mathbf{Aut}(\widetilde{X})$ and S_{π_1/G_P} is complete. Thus f extends to a continuous homomorphism from $\bar{\pi}_1$. By Lemma 4.5, the extension of f gives a homeomorphism from $\bar{\pi}_1$ onto $\overline{f(\pi_1)}$. Therefore, it suffices to show that $\overline{f(G_P)} \subset S_{\pi_1/G_P}$ is compact.

Let a be a generator of G_P and $g \in \pi_1$. There is a word (c,μ) of (G,Y) such that c is a closed path from P to P and $g = |c,\mu|$. Suppose that $c = (y_1, y_2, \ldots, y_n)$ and $m_i := [G_{t(y_i)} : \iota_{y_i}(G_{y_i})]$. Since all vertex groups of (G,Y) are isomorphic to $\mathbb{Z}$, we get $ga^{(\prod_{i=1}^n m_i)}g^{-1} \in G_P \setminus \{e\}$. Thus, for any $g \in \pi_1$, the relative order $[G_P : G_P \cap gG_Pg^{-1}]$ is finite. By using the argument in [Tz, Sect. 4], the closure $\overline{f(G_P)} \subset S_{\pi_1/G_P}$ is compact.

In the proof of the compactness of $\overline{G}_P$, we have $\mathbf{Stab}(gG_P) \cap G_P = gG_Pg^{-1} \cap G_P \neq \{e\}$ for any $g \in \pi_1$. Since G_P is isomorphic to $\mathbb{Z}$, the intersection $\bigcap_{i=1}^n \mathbf{Stab}(g_iG_P) \cap \overline{G}_P$ is not a trivial subgroup of $\overline{G}_P$ for any $g_1, g_2, \ldots, g_n \in \pi_1$. Thus $\{e\}$ is not an open subgroup of $\overline{G}_P$ and $\overline{G}_P$ is non-discrete. □

This lemma shows that $\bar{\pi}_1$ is a non-discrete locally compact group. To prove C*-simplicity of $\bar{\pi}_1$, we use the following lemma, which is shown in a similar way to [P-S, proof of Lem. 1.2].

Lemma 4.7. *Let G be a totally disconnected group and K be a compact open subgroup. If for every finite subset $\{g_1, g_2, \ldots, g_l\}$ of $G \setminus K$ there are pairwise disjoint subsets $S_1, S_2, \ldots, S_9$ of G and $z_1, z_2, \ldots, z_9 \in G$ with the following conditions, then property* (2) *in Lemma 3.1 holds for $A_\nu = \mathrm{C}^*_{\mathrm{r}}(G)$ and $K_\nu = K$:*

- *Each S_i satisfies $KS_i = S_i$.*
- *Each z_i commutes with every element of K and satisfies $z_ig_jz_i^{-1}(G \setminus S_i) \subset S_i$ for every g_j.*

Proof. Take a self-adjoint element $f \in C_c(G \setminus K) \subset \mathrm{C}^*_{\mathrm{r}}(G)$. Then there are $g_1, g_2, \ldots, g_l \in G \setminus K$ which have pairwise distinct K-double cosets and f is supported on $\bigcup_{j=1}^l Kg_jK$. We have subsets $S_1, S_2, \ldots, S_9$ of G and $z_1, z_2, \ldots, z_9 \in G$ which satisfy the assumptions of the lemma for $g_1, g_2, \ldots, g_l$. Since $S_1, S_2, \ldots, S_9$ are unions of K-cosets, they are clopen. We define q_i as the orthogonal projection onto $L^2(S_i)$. Then we get $(1-q_i)\lambda_{z_ig_jz_i^{-1}}(1-q_i) = 0$ for any i and j. Since q_i and λ_{z_i} commute with $\{\lambda_k\}_{k\in K}$, $(1-q_i)\lambda_{z_igz_i^{-1}}(1-q_i) = 0$ holds for any i and $g \in \bigcup_{j=1}^l Kg_jK$. Thus the self-adjoint element f satisfies $(1-q_i)\lambda_{z_i} f \lambda^*_{z_i}(1-q_i) = 0$.

Generally, if a self-adjoint operator T and a projection q on a Hilbert space $\mathcal{H}$ satisfy $(1-q)T(1-q)=0$, then $|\langle T\xi,\xi\rangle| \le 2\|T\|\,\|q\xi\|$ holds for any unit vectors $\xi \in \mathcal{H}$. Thus we have

$$\left|\left\langle \left(\frac{1}{9}\sum_{i=1}^{9}\lambda_{z_i} f \lambda_{z_i}^*\right)\xi,\xi\right\rangle\right| \le \frac{2}{3}\|f\|,$$

for any unit vector ξ (see [P-S, proof of Lem. 1.2]), then

$$\left\|\frac{1}{9}\sum_{i=1}^{9}\lambda_{z_i} f\lambda_{z_i}^*\right\| \le \frac{2}{3}\|f\|.$$

Since the continuous function $\frac{1}{9}\sum_{i=1}^{9}\lambda_{z_i} f\lambda_{z_i}^*$ is supported on $\bigcup_{i,j} Kz_i g_j z_i^{-1}K$ and the assumptions $z_i g_j z_i^{-1}(G\setminus S_i)\subset S_i$ hold for every i, j, the support of $\frac{1}{9}\sum_{i=1}^{9}\lambda_{z_i} f\lambda_{z_i}^*$ is on $G\setminus K$. By repeating the same process, for any $\varepsilon>0$, we can take $w_1, w_2, \ldots, w_n \in G$ with $\|\frac{1}{n}\sum_{i=1}^{n}\lambda_{w_i} f\lambda_{w_i}^*\| < \varepsilon$ and $\lambda_{w_i} p_K \lambda_{w_i}^* = p_K$. Therefore condition (2) in Lemma 3.1 holds for $A_\nu = \mathrm{C}^*_{\mathrm{r}}(G)$ and $K_\nu = K$. ☐

In the following lemma, we focus on reduced forms of elements in groups of HNN extension. Let $G*_\theta$ be an HNN extension with respect to a discrete group G and an isomorphism θ between subgroups L_1 and L_2 of G. Suppose that the stable letter of $G*_\theta$ is denoted by t. If $g \in G*_\theta$ has two reduced forms

$$c_0 t^{\varepsilon_1} c_1 t^{\varepsilon_2} c_2 \cdots t^{\varepsilon_l} c_l \quad \text{and} \quad d_0 t^{\varepsilon'_1} d_1 t^{\varepsilon'_2} d_2 \cdots t^{\varepsilon'_m} d_m,$$

then we have $l=m$ and $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_l)=(\varepsilon'_1,\varepsilon'_2,\ldots,\varepsilon'_m)$. (See [L-S, Lem. 2.3].) We call this $l=m$ the length of g.

Lemma 4.8. *There is a compact open subgroup K of $\bar{\pi}_1$ which is contained in $\overline{G_{o(y)}}$ and satisfies the conditions of Lemma 4.7.*

Proof. Let a, b be free generators of $G_{t(y)}$ and $G_{o(y)}$. We suppose that $\iota_y(G_y)=\langle a^n\rangle$, $\iota_{\bar{y}}(G_y)=\langle b^m\rangle$, and $g_y a^n g_y^{-1}=b^m$ for some $n,m\in\mathbb{Z}$. We have $|n|,|m|\ge 2$ by assumption. If γ is a geodesic path in T from $o(y)$ to $t(y)$ and H is a subgroup $G_{t(y)}\cap G_{o(y)}=\langle a^{k_\gamma}\rangle=\langle b^{k_{\bar{\gamma}}}\rangle$ of π_1, then the subgroup $\langle a,b\rangle\subset\pi_1$ is isomorphic to $\langle a\rangle *_H \langle b\rangle$. Define

$$N := \frac{n^2 k_{\bar{\gamma}}}{(n, \frac{k_\gamma m}{(m,k_{\bar{\gamma}})})(m,k_{\bar{\gamma}})}$$

and a subgroup $K := \overline{\langle a^N\rangle} \subset \bar{\pi}_1$. By Lemma 4.6, $\overline{\langle a\rangle}$ is a compact open subgroup of $\bar{\pi}_1$; then the finite index subgroup $K=\overline{\langle a^N\rangle}$ of $\overline{\langle a\rangle}$ is also a compact open subgroup of $\bar{\pi}_1$. When we take a finite set $\{g'_1, g'_2, \ldots, g'_s\}$ in $\bar{\pi}_1\setminus K$, there is $g_i\in\pi_1\setminus\langle a^N\rangle$ with $g'_i\in g_iK$ for any i, since $\bar{\pi}_1\setminus K\subset(\pi_1\setminus\langle a^N\rangle)K$. Let W

be a subgraph $Y \setminus \{y\}$ of Y. By the isomorphism $\pi_1(G|_W, W)*_{\theta_y} \cong \pi_1$, π_1 has the structure of an HNN extension and $t \coloneqq g_y$ corresponds to the stable letter. We have $\langle a^N \rangle = \langle a \rangle \cap t^{-2}\langle b \rangle t^2$ by definition of N. Let $l_y\colon \pi_1 \to \mathbb{Z}_{\geq 0}$ be the function such that $l_y(g)$ is the length of g as an element of the HNN extension $\pi_1(G|_W, W)*_{\theta_y}$. Suppose that $L \coloneqq \max\{l_y(g_i) \mid 1 \leq i \leq s\}$.

Set $r_1 \coloneqq at^{-1}bt$. For every $g \in \pi_1$ with $l_y(g) \leq L$, we claim that if $g \notin \bigcup_{i\in\mathbb{Z}} r_1^i \langle a^n \rangle$, then $r_1^L g r_1^{-L}$ has a reduced form which begins with at^{-1} and ends with ta^{-1}. Suppose $g \notin \bigcup_{i\in\mathbb{Z}} r_1^i \langle a^n \rangle$ has the reduced form $c_0 t^{\varepsilon_1} c_1 t^{\varepsilon_2} c_2 \cdots t^{\varepsilon_{l_y(g)}} c_{l_y(g)}$. When $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{l_y(g)}) \neq (-1, 1, -1, 1, \dots, -1, 1)$, the claim is trivial. Thus we may assume $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{l_y(g)}) = (-1, 1, -1, 1, \dots, -1, 1)$. First we assume $l_y(g r_1^{-L}) \geq l_y(r_1^L g)$, and let M be a non-negative integer satisfying $l_y(r_1^L g) = 2L + l_y(g) - 2M$. When $M = 0$, we have a reduced form of $r_1^L g r_1^{-L}$ as

$$(at^{-1}bt) \cdots (at^{-1}bt) c_0 t^{-1} c_1 t c_2 \cdots t c_{l_y(g)} (t^{-1}b^{-1}ta^{-1}) \cdots (t^{-1}b^{-1}ta^{-1}).$$

When $M \neq 0$, we get a reduced form of $r_1^L g$ as

$$\begin{cases} r_1^{(L-\frac{M}{2})} c'_M t^{-1} c_{M+1} t c_{M+2} t^{-1} \cdots t c_{l(g)} & \text{if } M \text{ is even,} \\ r_1^{(L-\frac{M+1}{2})} a t^{-1} c'_M t c_{M+1} t^{-1} c_{M+2} t \cdots t c_{l(g)} & \text{if } M \text{ is odd.} \end{cases}$$

Here we have

$$\begin{cases} c'_M \in \langle a^n \rangle a c_M \text{ and } c_M \notin \langle a^n \rangle & \text{if } M \text{ is even,} \\ c'_M \in \langle b^m \rangle b c_M \text{ and } c_M \notin \langle b^m \rangle & \text{if } M \text{ is odd.} \end{cases}$$

This condition holds even if $M = l(g)$ because of the assumption $g \notin \bigcup_{i\in\mathbb{Z}} r_1^i \langle a^n \rangle$. Thus we get

$$\begin{cases} c'_M a^{-1} \notin \langle a^n \rangle & \text{if } M \text{ is even,} \\ c'_M b^{-1} \notin \langle b^m \rangle & \text{if } M \text{ is odd.} \end{cases}$$

Therefore, we have a reduced form of $r_1^L g r_1^{-L}$ whose length is longer than $|l_y(r_1^L g) - l_y(r_1^{-L})|$. Thus the claim is true when $l_y(g r_1^{-L}) \geq l_y(r_1^L g)$. Similarly, we can prove the claim when $l_y(g r_1^{-L}) \leq l_y(r_1^L g)$.

For $j \in \mathbb{Z}_{>0}$, we define S'_j as the set of all $g \in \pi_1$ with a reduced form $g = c_0 t^{\varepsilon_1} c_1 t^{\varepsilon_2} c_2 \cdots t^{\varepsilon_{l(g)}} c_{l(g)}$ such that

$$(\varepsilon_{2i-1}, \varepsilon_{2i}) = (-1, 1) \quad \text{for every } i \leq j, \quad \text{and} \quad (\varepsilon_{2j+1}, \varepsilon_{2j+2}) = (-1, -1).$$

Suppose $r_2 \coloneqq at^{-2}bt^2$ and $z_j \coloneqq r_1^j r_2 r_1^L$. These $\{z_j\}_j$ commute with a^N and satisfy the following: the inclusions

$$z_j g z_j^{-1} (\pi_1 \setminus S'_j) \subset S'_j$$

hold for every $g \in \pi_1 \setminus \langle a^N \rangle$ with $l_y(g) \leq L$.

This is because $r_2 r_1^L g r_1^{-L} r_2^{-1}$ has a reduced form which begins with at^{-2} and ends with $t^2 a^{-1}$, when g satisfies either $l(g) \leq L$ and $g \notin \bigcup_{i\in\mathbb{Z}} r_1^i \langle a^n \rangle$ or $g \in \bigcup_{i\in\mathbb{Z}} r_1^i \langle a^n \rangle \setminus \langle a^N \rangle$. Thus, if $f \in \pi_1 \setminus S'_j$, then $z_j g z_j^{-1} f$ has a reduced form beginning with the words $r_1^j a t^{-2}$.

Since $\overline{S'_j} = \overline{\langle a^N \rangle} S'_j$, the sets $\{S_j := \overline{S'_j}\}_j$ are pairwise disjoint subsets of $\bar{\pi}_1$ satisfying $KS_j = S_j$. The subsets $S_1, S_2, \dots, S_9$ of $\bar{\pi}_1$ and $z_1, z_2, \dots, z_9 \in \bar{\pi}_1$ satisfy the conditions of Lemma 4.7 for $\{g'_1, g'_2, \dots, g'_s\}$. □

In the proof of the above lemma, both r_1 and r_2 are in the kernel of the modular function of $\bar{\pi}_1$, since a, b are elements in compact open subgroups of $\bar{\pi}_1$. Thus $A_\nu := \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$ and $E_\nu := E_K$ satisfy condition (2) of Lemma 3.1 and every g_i can be taken in the kernel of the modular function.

Proof of Theorem 4.3. Take $g \in \pi_1$. We show that there is a compact open subgroup $K_0 \subset \overline{G}_{t(y)} \cap \mathbf{Stab}(gG_{t(y)})$ such that $A_\nu := \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$ and $E_\nu := E_{K_0}$ satisfy condition (2) of Lemma 3.1. We assume that the following statements hold as in the proof of Lemma 4.8:

- The elements $a, b \in \pi_1$ are free generators of $G_{t(y)}$ and $G_{o(y)}$.
- The integers n, m satisfy $\iota_y(G_y) = \langle a^n \rangle$, $\iota_{\bar{y}}(G_y) = \langle b^m \rangle$, and $ta^n t^{-1} = b^m$, where $t := g_y$.
- The function $l_y \colon \pi_1 \to \mathbb{Z}_{\geq 0}$ gives lengths of elements of π_1 with respect to the identification $\pi_1(G|_W, W) *_{\theta_y} \cong \pi_1$ of π_1.

Since $a \notin \langle b^m \rangle$, there are $c, d \in \{a, e\}$ such that the equation $l_y(t^{\pm 1} c g d t^{\pm 1}) = l_y(g) + 2$ holds. Set $r'_j := g d t^{-j} b t^j d g^{-1} c t^{-j} b t^j c$ for $j \in \mathbb{Z}$ and $K_1 := \overline{\langle a \rangle} \cap \mathbf{Stab}(r'_2 \langle a \rangle)$. By assumptions for c, d, if $h \in \langle a \rangle \cap \mathbf{Stab}(r'_2 \langle a \rangle)$, then we have

$$g^{-1} h g \in \langle a \rangle, \quad (t^{-2} b t^2)^{-1} g^{-1} h g (t^{-2} b t^2) \in \langle a \rangle,$$

and

$$(t^{-2} b t^2)^{-1} h (t^{-2} b t^2) \in \langle a \rangle.$$

This is because $(t^{-2} b t^2)^{-1} h (t^{-2} b t^2) = h$ holds for any $h \in \langle a \rangle \cap \mathbf{Stab}(t^{-2} b t^2 \langle a \rangle)$. We get

$$\langle a \rangle \cap \mathbf{Stab}(r'_2 \langle a \rangle) = \langle a \rangle \cap \mathbf{Stab}(g \langle a \rangle) \cap \mathbf{Stab}(t^{-2} b t^2 \langle a \rangle) \cap g\, \mathbf{Stab}(t^{-2} b t^2 \langle a \rangle) g^{-1}.$$

Thus we have $K_1 \subset \overline{\langle a \rangle} \cap \mathbf{Stab}(gG_{t(y)})$ and ${r'_2}^{-1} h r'_2 = h = {r'_1}^{-1} h r'_1$ for any $h \in K_1$.

For any $f \in \pi_1$ with $l_y(f) \geq i$, we define $\varepsilon(f, i)$ as a sequence $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i)$ of $\{0, 1\}$, when f has a reduced form $c_0 t^{\varepsilon_1} c_1 t^{\varepsilon_2} c_2 \dots t^{\varepsilon_{l(f)}} c_{l(f)}$. By using this definition, we set

$$\begin{aligned}
&w_i := {r'_1}^{-i}{r'_2}^{-2},\\
&U'_i := \{f \in \pi_1 \mid l_y(f) \geq il_y(r'_1)+2 \text{ and } \varepsilon(f, il_y(r'_1)+2) = \varepsilon(w_i, il_y(r'_1)+2)\},\\
&U_i := \overline{U'_i} = K_1 U'_i.
\end{aligned}$$

This $\{U_i\}_{i=1}^{\infty}$ is an infinite family of mutually disjoint clopen subsets of $\bar{\pi}_1$ and we have $w_i h w_i^{-1}(\bar{\pi}_1 \setminus U_i) \subset U_i$ for any $h \in \overline{\langle a \rangle} \setminus K_1$. In the same way as in the proof of Lemma 4.7, we get

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \lambda_{w_i}(x - E_{K_1}(x))\lambda_{w_i}^* \right\| \leq \frac{2}{\sqrt{N}} \|x - E_{K_1}(x)\|$$

for every self-adjoint element $x \in \mathrm{C}^*_{\mathrm{r}}(\overline{\langle a \rangle}) \subset \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$. Let $K \subset \overline{\langle a \rangle}$ be a compact open subgroup of Lemma 4.8 and $K_0 := K \cap K_1$; then $E_{K_0} = E_{K_1} \circ E_K$. For any self-adjoint element $x \in \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$ and $\delta > 0$, there are $N \in \mathbb{Z}_{\geq 0}$ and $g_1, g_2, \ldots, g_M \in \pi_1$ such that they satisfy

$$\left\| \frac{1}{M} \sum_{i=1}^{M} \lambda_{g_i}(x - E_K(x))\lambda_{g_i}^* \right\| \leq \frac{\delta}{2}$$

and

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \lambda_{w_i}(E_K(x) - E_{K_1} \circ E_K(x))\lambda_{w_i}^* \right\| \leq \frac{\delta}{2}.$$

By the proof of Lemma 4.8, we may assume $\{g_i\}_i$ are in the kernel of the modular function and $\{\lambda_{g_i}\}_i$ commute with all elements of $\mathrm{C}^*_{\mathrm{r}}(K)$. Thus we have

$$\begin{aligned}
&\left\| \frac{1}{NM} \sum_{i,j} \lambda_{w_j g_i}(x - E_{K_0}(x))\lambda_{w_j g_i}^* \right\| \\
&\quad \leq \left\| \frac{1}{N} \sum_{j=1}^{N} \lambda_{w_j}(E_K(x) - E_{K_1} \circ E_K(x))\lambda_{w_j}^* \right\| \\
&\qquad + \left\| \frac{1}{N} \sum_{j=1}^{N} \lambda_{w_j} \left(\frac{1}{M} \sum_{i=1}^{N} \lambda_{g_i}(x - E_K(x))\lambda_{g_i}^* \right) \lambda_{w_j}^* \right\| \\
&\quad \leq \delta.
\end{aligned}$$

Moreover, every $w_j g_i$ is in the kernel of the modular function and they satisfy

$$\frac{1}{NM} \sum_{i,j} \lambda_{w_j g_i} p_{K_0} \lambda_{w_j g_i}^* = p_{K_0}.$$

By using Lemmas 3.1 and 4.5, we get Theorem 4.3. □

Remark 4.9. For the group $\bar{\pi}_1$ in Theorem 4.3 and the modular function Δ of $\bar{\pi}_1$, $\ker\Delta$ is an open subgroup of $\bar{\pi}_1$. When the graph Y is countable, $\ker\Delta$ satisfies the conditions in Proposition 2.4, and $\bar{\pi}_1$ is an elementary group in Wesolek's sense [W]. This is shown as follows. Since π_1 is countable, we can take the following decreasing sequence $(K_n)_{n=1}^{\infty}$ of compact open subgroups in $\bar{\pi}_1$ by using the proof of Theorem 4.3:

- $\bigcap_{n=1}^{\infty} K_n = \{e\}$.
- For every n, there is a finite subset F_n of π_1 with $K_n = \bigcap_{h\in F_n} \mathbf{Stab}(hG_{t(y)}) \cap \overline{G}_{t(y)}$.
- For every n, self-adjoint element $x \in \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$, and $\varepsilon > 0$, there are $g_1, g_2, \ldots, g_m \in \pi_1 \cap \ker\Delta$ such that each g_i commutes with all elements of K_n and

$$\left\| \frac{1}{m} \sum_{i=1}^{m} \lambda_{g_i}(x - E_{K_n}(x))\lambda_{g_i}^* \right\| < \varepsilon$$

  holds.

Define the open subset $L_n := \ker\Delta \cap \{g \in \bar{\pi}_1 \mid g^{-1}K_n g \leq \overline{G}_{t(y)}\}$ for every n. Then we can show that L_n is the normalizer $N_{\bar{\pi}_1}(K_n)$ of K_n as follows. Since L_n and $N_{\bar{\pi}_1}(K_n)$ are clopen subsets of $\bar{\pi}_1$, it suffices to show that $L_n \cap \pi_1 \subset N_{\bar{\pi}_1}(K_n)$. Put $K'_n := \pi_1 \cap K_n$, and take $g \in L_n \cap \pi_1$. Then we have $g^{-1}K'_n g = G_{t(y)} \cap g^{-1}K_n g$. Since $g \in \ker\Delta$, we get

$$[G_{t(y)} : g^{-1}K'_n g] = [\overline{G}_{t(y)} : g^{-1}K_n g] = [\overline{G}_{t(y)} : K_n] = [G_{t(y)} : K'_n].$$

Thus we get $K'_n = g^{-1}K'_n g \leq G_{t(y)} \cong \mathbb{Z}$ and $g \in N_{\bar{\pi}_1}(K_n)$. Since $\bigcap_{n=1}^{\infty} K_n = \{e\}$ holds, every $g \in \ker\Delta \cap \pi_1$ has $n \in \mathbb{N}$ with $K_n \leq \mathbf{Stab}(gG_{t(y)})$. This gives $\ker\Delta = \bigcup_{n=1}^{\infty} L_n$. By the third condition of $(K_n)_{n=1}^{\infty}$, the C*-algebra $\mathrm{C}^*_{\mathrm{r}}(L_n/K_n) \cong \mathrm{C}^*_{\mathrm{r}}(L_n)p_{K_n} \subset \mathrm{C}^*_{\mathrm{r}}(\bar{\pi}_1)$ is simple for any n. Therefore, the increasing sequence $(L_n)_{n=1}^{\infty}$ and the decreasing sequence $(K_n)_{n=1}^{\infty}$ satisfy the assumptions of Proposition 2.4. Since $\bar{\pi}_1/\ker\Delta \cong \Delta(\bar{\pi}_1)$ is a countable discrete group, the locally compact group $\bar{\pi}_1$ is elementary in Wesolek's sense [W].

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