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$\mathfrak{sl}_2$ Triples Whose Nilpositive Elements Are in a Space Which Is Spanned by the Real Root Vectors in Rank 2 Symmetric Hyperbolic Kac–Moody Lie Algebras

by

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Abstract

In analogy with the theory of nilpotent orbits in finite-dimensional semisimple Lie algebras, it is known that the principal $\mathfrak{sl}_2$ subalgebras can be constructed in hyperbolic Kac–Moody Lie algebras. We obtained a series of $\mathfrak{sl}_2$ subalgebras in rank 2 symmetric hyperbolic Kac–Moody Lie algebras by extending the aforementioned construction. We present this result and also discuss $\mathfrak{sl}_2$ modules obtained by the action of the $\mathfrak{sl}_2$ subalgebras on the original Lie algebras.

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§1. Introduction

Kac–Moody Lie algebras are generalizations of finite-dimensional simple Lie algebras, and can be divided into three types: finite, affine, and indefinite [Kac90]. Finite-type Kac–Moody Lie algebras are finite-dimensional simple Lie algebras.

In [Dyn57], nilpotent orbits in finite-type Kac–Moody Lie algebras were classified. Nilpotent orbits in a finite-type Kac–Moody Lie algebra $\mathfrak{g}$ are those formed by inner automorphisms acting on nilpotent elements. These orbits are completely classified by weighted Dynkin diagrams.

Moreover, from the Jacobson–Morosov theorem, for a nilpotent element x, we can construct an $\mathfrak{sl}_2$-triple whose nilpositive element is x [CM93, Thm. 3.3.1]. A nilpositive element is an element that is nilpotent and is in the positive root space.

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Classifying nilpotent orbits is equivalent to classifying $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ up to inner automorphisms.

Among the nilpotent orbits in a finite-type Lie algebra, a nilpotent orbit with the largest dimension (as an algebraic variety) is called a principal nilpotent orbit. Let $\{e_i, f_i, h_i\}$ be the Chevalley generators of a finite-type Lie algebra. For each i, take the appropriate a_i's in $\mathbb{C}$ and put

$$X = \sum_i e_i,$$
$$Y = \sum_i a_i f_i,$$
$$H = \sum_i a_i h_i.$$

This $\{X, Y, H\}$ is called an $\mathfrak{sl}_2$-triple corresponding to the principal nilpotent orbit [CM93, Thm. 4.1.6].

For a principal nilpotent orbit of a finite-type Lie algebra, we can create a principal SO(3) subalgebra compatible with the compact involution [Kos59]. We provide further detail: Denote the Cartan matrix of $\mathfrak{g}$ by $A = (A_{ij})$ and the Chevalley generators by $\{e_i, f_i, h_i\}$. Put

$$p_i = \sum_j A_{ij}^{-1} \ (> 0),$$
$$n_i = \sqrt{p_i}.$$

Finally, put

$$J_3 = \sum_j p_j h_j,$$
$$J^+ = \sum_j n_j e_j,$$
$$J^- = \sum_j n_j f_j.$$

This $\{J_3, J^+, J^-\}$ spans a principal SO(3) subalgebra. This means that

$$[J_3, J^{\pm}] = \pm J^{\pm},$$
$$[J^+, J^-] = J_3.$$

In [NO01], principal SO(1, 2) subalgebras of hyperbolic Kac–Moody Lie algebras are constructed and described in terms of the eigenvalues of their Casimir elements. The principal SO(1, 2) subalgebras are constructed for certain indefinite-type Lie algebras which are not hyperbolic [GOW02].

In this paper, we will construct SO$(1,2)$ subalgebras which are not principal in rank-2 symmetric Kac–Moody Lie algebras. However, we will actually construct $\mathfrak{sl}_2$ triples, which correspond to SO$(1,2)$ subalgebras. The discussion using $\mathfrak{sl}_2$ makes it easy to compare with Dynkin–Kostant's theory of nilpotent orbits. The principal SO$(1,2)$ subalgebra is compatible with the compact involution ω_0. This means that when we decompose $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module into a direct sum of irreducible components, these irreducible components are unitary with respect to the Hermitian form in [Kac90, §2.7] except for the SO$(1,2)$ subalgebra itself. We will search for SO$(1,2)$ subalgebras that are compatible with ω_0.

In fact, we will search for $\mathfrak{sl}_2$-triples corresponding to SO$(1,2)$ subalgebras. Since it is difficult to grasp the behavior of the imaginary root vectors, we construct and classify the $\mathfrak{sl}_2$-triples under the condition that the nilpositive elements lie in the space spanned by the real roots. This condition makes it possible to explicitly calculate them. Although the meaning of this condition is unclear, we remark that this condition is automatically satisfied for principal SO$(1,2)$ subalgebras.

We will construct most of the $\mathfrak{sl}_2$-triples that satisfy these conditions. In particular, we classified them in all cases where the neutral element H is dominant. For these cases, we will calculate the weighted Dynkin diagrams and the range of the eigenvalues on $\mathfrak{h}$ in the adjoint actions of the Casimir elements. We will also calculate some of the components that appear when $\mathfrak{g}$ is decomposed by the action of each $\mathfrak{sl}_2$-triple.

§2. General theory of Kac–Moody Lie algebras

In the following, we consider Kac–Moody Lie algebras on $\mathbb{C}$. Denote by $\mathfrak{g}$ or $\mathfrak{g}(A)$ a Kac–Moody Lie algebra for the Cartan matrix A. Let A be an $n \times n$ matrix. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The root space with respect to the root α is written as $\mathfrak{g}_\alpha$. We denote the Chevalley generators of $\mathfrak{g}$ by e_i, f_i, h_i $(i = 0, \dots, n-1)$, and the simple roots of $\mathfrak{g}$ as α_i $(i = 0, \dots, n-1)$. In this case, $\langle h_i, \alpha_j \rangle = a_{ij}$. We write $\mathfrak{n}^+$ for the subalgebra generated by the e_i and $\mathfrak{n}^-$ for the subalgebra generated by the f_i. We also denote by ω the Chevalley involution on $\mathfrak{g}$. We denote by $\mathcal{W}$ the Weyl group of $\mathfrak{g}$.

The Cartan matrix A is called symmetrizable when there exist an invertible diagonal matrix $D = \operatorname{diag}(\varepsilon_0, \dots, \varepsilon_{n-1})$ and a symmetric matrix B such that $A = DB$. A Kac–Moody Lie algebra whose Cartan matrix is symmetrizable is called a symmetrizable Lie algebra. From [Kac90, Thm. 2.2], a symmetrizable Lie algebra has a $\mathbb{C}$-valued nondegenerate symmetric bilinear form $(\cdot \mid \cdot)$ called the standard form.

We fix a real form $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{h}$ and define the antilinear automorphism ω_0 in $\mathfrak{g}$ by

$$\begin{aligned} \omega_0(e_i) &= -f_i, \\ \omega_0(f_i) &= -e_i \qquad (i = 0, \dots, n-1), \\ \omega_0(h) &= -h \qquad (h \in \mathfrak{h}_{\mathbb{R}}). \end{aligned}$$

The automorphism ω_0 is called the compact involution of $\mathfrak{g}$. From [Kac90, §2.7], when $\mathfrak{g}$ is symmetrizable, we can define the nondegenerate Hermitian form $(\cdot \mid \cdot)_0$ on $\mathfrak{g}$ by

$$(x \mid y)_0 = -(\omega_0(x) \mid y).$$

§3. SO(1, 2) subalgebras in hyperbolic Kac–Moody Lie algebras

In this section, we will briefly recall the theory of SO(1, 2) subalgebras in the hyperbolic Kac–Moody Lie algebras from [NO01]. An SO(1, 2) subalgebra of $\mathfrak{g}$ is the 3-dimensional subalgebra spanned by the three nonzero elements $J^+ \in \mathfrak{n}^+$, $J^- \in \mathfrak{n}^-$, $J_3 \in \mathfrak{h}$, satisfying

$$\begin{aligned} [J_3, J^{\pm}] &= \pm J^{\pm}, \\ [J^+, J^-] &= -J_3. \end{aligned}$$

A representation of an SO(1, 2) subalgebra is called unitary if the representation space V has a Hermitian scalar product $(\cdot, \cdot)$ and satisfies the following two conditions:

(i) The actions of J^+ and J^- are adjoint, and the action of J_3 is self-adjoint. That is, for any $x, y \in V$,

$$\begin{aligned} ([J^+, x], y) &= (x, [J^-, y]), \\ ([J_3, x], y) &= (x, [J_3, y]). \end{aligned}$$

(ii) The Hermitian scalar product $(\cdot, \cdot)$ is positive definite.

A hyperbolic Kac–Moody Lie algebra is a Kac–Moody Lie algebra such that the Cartan matrix A is of indefinite type and symmetrizable, the Dynkin diagram is connected, and any proper connected subdiagram is finite or affine.

Put

$$p_i = -\sum_j A^{-1}_{ij}.$$

Since $p_i \geq 0$ for any i, we put

$$n_i = \sqrt{p_i}.$$

We may construct a principal $\mathrm{SO}(1,2)$ subalgebra of the hyperbolic Kac–Moody Lie algebra $\mathfrak{g}$ as follows:

$$J_3 = -\sum_j p_j h_j,$$
$$J^+ = \sum_j n_j e_j,$$
$$J^- = \sum_j n_j f_j.$$

When $\mathfrak{g}$ is decomposed into the direct sum of irreducible modules by the adjoint action of the principal $\mathrm{SO}(1,2)$ subalgebra, these irreducible modules except for $\mathrm{SO}(1,2)$ itself are infinite-dimensional and unitary [NO01].

In the case of the indefinite-type Kac–Moody Lie algebras which are not hyperbolic, the principal $\mathrm{SO}(1,2)$ subalgebras can be constructed in the same way if $p_i \geq 0$ for any i [GOW02].

On the other hand, an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ with three nonzero elements $X \in \mathfrak{n}^+$, $Y \in \mathfrak{n}^-$, $H \in \mathfrak{h}$ such that

$$[H, X] = 2X,$$
$$[H, Y] = -2Y,$$
$$[X, Y] = H.$$

The subalgebra spanned by X, Y, H in $\mathfrak{g}$ is called an $\mathfrak{sl}_2$ subalgebra. An $\mathfrak{sl}_2$-triple can be constructed from the generators of an $\mathrm{SO}(1,2)$ subalgebra by setting

$$J^+ = \frac{1}{\sqrt{2}} X,$$
$$J^- = -\frac{1}{\sqrt{2}} Y,$$
$$J_3 = \frac{1}{2} H.$$

In the principal $\mathrm{SO}(1,2)$ subalgebra, we have $J^- = -\omega_0(J^+)$.

Lemma 3.1 (Cf. [Kac90, §2.7 and Thm. 2.2]). *Suppose that $\mathfrak{g}$ is symmetrizable. For any $u \in \mathfrak{g}$, $\operatorname{ad} u$ and $-\operatorname{ad} \omega_0(u)$ are adjoint to each other with respect to $(\cdot \mid \cdot)_0$. That is, for any $x, y \in \mathfrak{g}$,*

$$([u, x] \mid y)_0 = -(x \mid [\omega_0(u), y])_0.$$

Proof. By the definition of the Hermitian form $(\cdot \mid \cdot)_0$, let $(\cdot \mid \cdot)$ be the standard form and we have

$$\begin{aligned}([u,x] \mid y)_0 &= -(\omega_0([u,x]) \mid y)\\ &= -([\omega_0(u), \omega_0(x)] \mid y)\\ &= ([\omega_0(x), \omega_0(u)] \mid y),\\ -(x \mid [\omega_0(u), y])_0 &= (\omega_0(x) \mid [\omega_0(u), y]).\end{aligned}$$

From [Kac90, Thm. 2.2(a)], the standard form is invariant and we have

$$([\omega_0(x), \omega_0(u)] \mid y) = (\omega_0(x) \mid [\omega_0(u), y]).$$

Therefore

$$([u,x] \mid y)_0 = -(x \mid [\omega_0(u), y])_0.$$ □

From Lemma 3.1, even in the case of nonprincipal $\mathrm{SO}(1,2)$ subalgebras, if $J^- = -\omega_0(J^+)$ holds, then the unitarity condition (i) is satisfied with respect to the Hermitian form $(\cdot \mid \cdot)_0$ when $\mathfrak{g}$ is considered as a representation space with adjoint actions. We will show that the converse is also true.

Lemma 3.2. *Suppose that the adjoint action of the* $\mathrm{SO}(1,2)$ *subalgebra on* $\mathfrak{g}$ *satisfies the unitary condition* (i) *for the Hermitian form* $(\cdot \mid \cdot)_0$*. Then* $J^- = -\omega_0(J^+)$*.*

Proof. Since J^+ and $-\omega_0(J^+)$ are adjoint to each other and so are J^+ and J^-, for any $x \in \mathfrak{g}, h \in \mathfrak{h}$, we have

$$\begin{aligned}([J^+, x], h)_0 &= (x, [-\omega_0(J^+), h])_0\\ &= (x, [J^-, h])_0.\end{aligned}$$

Since $(\cdot \mid \cdot)_0$ is nondegenerate, we have

$$[-\omega_0(J^+), h] = [J^-, h],$$

which implies that

$$[h, J^- + \omega_0(J^+)] = 0.$$

Since this holds for any $h \in \mathfrak{h}$, we have $J^- + \omega_0(J^+) \in \mathfrak{h}$. However, from $[J_3, J^\pm] = \pm J^\pm$, we have $J^+, J^- \in \mathfrak{n}^+ \oplus \mathfrak{n}^-$, and hence $J^- + \omega_0(J^+) \in \mathfrak{n}^+ \oplus \mathfrak{n}^-$. From the above, we have $J^- + \omega_0(J^+) = 0$. This proves the lemma. □

Motivated by this, we consider only $\mathrm{SO}(1,2)$ subalgebras that satisfy the condition $J^- = -\omega_0(J^+)$, which is rephrased as $Y = \omega_0(X)$ in terms of $\mathfrak{sl}_2$-triples.

§4. The real roots of a rank 2 symmetric hyperbolic Kac–Moody Lie algebra

Let a be an integer such that $a \geq 3$. Let $\mathfrak{g}$ be a Kac–Moody Lie algebra whose Cartan matrix is

$$\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}.$$

It is of hyperbolic type. Any real root can be expressed as $w(\alpha_0)$ or $w(\alpha_1)$ for some $w \in \mathcal{W}$ [Kac90]. Let r_0 and r_1 be the fundamental reflections for α_0 and α_1. Since $\mathcal{W}$ is generated as a group by r_0 and r_1 [Kac90], any element of $\mathcal{W}$ can be written in the form

$$\begin{matrix} (r_0 r_1)^i, & r_1(r_0 r_1)^i, \\ (r_1 r_0)^i, & r_0(r_1 r_0)^i \end{matrix} \quad (i \in \mathbb{Z}).$$

Lemma 4.1 ([KM95, Prop. 4.4]). *Let $\{F_n\}$ be a sequence determined by*

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = aF_{k+1} - F_k.$$

The real positive roots of $\mathfrak{g}$ are of the form

$$\alpha = F_{k+1}\alpha_0 + F_k\alpha_1$$

or

$$\beta = F_k\alpha_0 + F_{k+1}\alpha_1.$$

We will distinguish between these roots and call them α-type roots and β-type roots, and define types α and β for the root vectors as well.

§5. $\mathfrak{sl}_2$-triples of rank 2 hyperbolic Lie algebras which are compatible with compact involution

Let $\mathfrak{g}$ be a rank 2 hyperbolic Kac–Moody algebra. We want to find $X \in \mathfrak{g}$ in the space spanned by the real root vectors such that X, $Y = \omega_0(X)$, and $H = [X, Y]$ form an $\mathfrak{sl}_2$-triple. For $X \in \mathfrak{g}$ in the space spanned by the real root vectors, X can be written as

$$X = \sum_i c_i E_i \quad (i \in \{0, \dots, n_X - 1\},\ c_i \in \mathbb{C},\ c_i \neq 0,\ E_i \in \mathfrak{g}_{\beta_i},\ E_i \neq 0),$$

where β_i $(i \in \{0, \dots, n_X - 1\})$ are distinct real roots and n_X is a positive integer. We call n_X the *length* of X. First we show the following lemma.

Lemma 5.1. *If the length of X is greater than or equal to 3, a required $\mathfrak{sl}_2$-triple does not exist.*

Proof. We plot the roots on the xy-plane with the α_0 component as x-coordinate and the α_1 component as y-coordinate. If X, $Y = \omega_0(X)$, and $H = [X, Y]$ form an $\mathfrak{sl}_2$-triple, $[H, E_i] = 2E_i$ holds for each i. Put $\beta_i = p\alpha_0 + q\alpha_1$, where $p, q \in \mathbb{R}$. Since $[H, E_i] = (p\alpha_0(H) + q\alpha_1(H))E_i$, we have

$$p\alpha_0(H) + q\alpha_1(H) = 2.$$

Since we can write $H = rh_0 + sh_1$ where $r, s \in \mathbb{C}$, h_0, h_1 are the part of Chevalley generator, we have

$$\begin{pmatrix} r & s \end{pmatrix} \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 2.$$

This represents a line on the xy-plane. In other words, the β_i are collinear on the xy-plane.

On the other hand, from [KM95, Cor. 4.3], the set of real roots Δ^{re} is represented as the set of grid points on the hyperbola

$$\Delta^{\mathrm{re}} = \left\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; x^2 - axy + y^2 = 1\right\}.$$

Therefore, the β_i are on the intersection of this hyperbola and the line. However, since there are at most two intersections of a hyperbola and a line, it is not possible to create the desired $\mathfrak{sl}_2$-triple when $n_X \geq 3$. □

From Lemma 5.1, we only need to consider the case when the length of X is 1 or 2. The multiplicity of a real root is always 1 (cf. [Kac90, Prop. 5.1(a)]), and the real root can be obtained by acting on the simple roots with an element of the Weyl group. Therefore, the real root vectors can be written in the form $cw(e_0)$ or $cw(e_1)$, using $c \in \mathbb{C}$, $w \in \mathcal{W}$. Note that the actions of the Weyl group on the element of $\mathfrak{g}$, which is now written as $w(e_0)$s, are the elements of $\operatorname{Aut}\mathfrak{g}$, which are determined by defining

$$r_i(x) = (\exp(\operatorname{ad} f_i))(\exp(\operatorname{ad} -e_i))(\exp(\operatorname{ad} f_i))$$

for the fundamental reflections r_i $(i \in \{0, 1\})$ (cf. [Kac90, Lem. 3.8]), and $w(x) \in \mathfrak{g}_{w(\alpha)}$ holds for $x \in \mathfrak{g}_\alpha$. We now show the following lemma.

Lemma 5.2. *For $w \in \mathcal{W}$, $w\omega_0 = \omega_0 w$.*

Proof. It is sufficient to show that when $w = r_i$, we get $r_i^{-1}\omega_0 r_i \omega_0^{-1} = \mathrm{id}$. We have

$$\begin{aligned} r_i^{-1} &= (\exp(\operatorname{ad} -f_i))(\exp(\operatorname{ad} e_i))(\exp(\operatorname{ad} -f_i)), \\ \omega_0 r_i \omega_0^{-1} &= \big(\exp(\operatorname{ad} \omega_0(f_i))\big)\big(\exp(\operatorname{ad} \omega_0(-e_i))\big)\big(\exp(\operatorname{ad} \omega_0(f_i))\big) \\ &= (\exp(\operatorname{ad} -e_i))(\exp(\operatorname{ad} f_i))(\exp(\operatorname{ad} -e_i)). \end{aligned}$$

If we consider this in terms of the $\mathrm{SL}(2,\mathbb{C})$ representation of $\operatorname{Aut}\mathfrak{g}$, we get

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \exp \operatorname{ad} e_i,$$
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mapsto \exp \operatorname{ad} -e_i,$$
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto \exp \operatorname{ad} f_i,$$
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mapsto \exp \operatorname{ad} -f_i.$$

Therefore we have

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mapsto r_i^{-1}\omega_0 r_i \omega_0^{-1}.$$

Calculating the left-hand side, we get

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we have $r_i^{-1}\omega_0 r_i \omega_0^{-1} = \mathrm{id}$. □

Lemma 5.3. *When the length of X is 1, a required $\mathfrak{sl}_2$ triple does not exist.*

Proof. In the following, we denote the complex conjugate of a complex number z by $\bar{z}$ and the absolute value by $|z|$. When the length of X is 1, we can write $X = cw(e_0)$ or $X = cw(e_1)$ for some $c \in \mathbb{C}$ and $w \in \mathcal{W}$. When $X = cw(e_0)$ holds, from Lemma 5.2, $Y = -\bar{c}w(f_0)$ and $H = -|c|^2 w(h_0)$. To satisfy the condition that X, Y, H form an $\mathfrak{sl}_2$-triple, $[H, X] = 2X$ should hold. Now we have

$$\begin{aligned} [H, X] &= [-|c|^2 w(h_0), cw(e_0)] \\ &= -|c|^2 2cw(e_0). \end{aligned}$$

Therefore we have $|c|^2 = -1$. Since there is no such complex number, the $\mathfrak{sl}_2$-triple cannot be constructed in this case. Since the same is true for $X = cw(e_1)$, the $\mathfrak{sl}_2$-triple cannot be constructed when $n_X = 1$. □

Next we consider the case where the length of X is 2. First, consider the case when $X = c_0 w(e_0) + c_1 w'(e_1)$ for some $c_0, c_1 \in \mathbb{C}$ and $w, w' \in \mathcal{W}$. Let k, l, m, and

n be integers such that $w(e_0) \in \mathfrak{g}_{k\alpha_0+l\alpha_1}$, $w'(e_1) \in \mathfrak{g}_{m\alpha_0+n\alpha_1}$. Using Lemma 5.2, we have

$$Y = -\overline{c_0}w(f_0) - \overline{c_1}w'(f_1)$$

and

$$\begin{aligned} H = &- |c_0|^2(kh_0 + lh_1) - |c_1|^2(mh_0 + nh_1) \\ &- c_0\overline{c_1}[w(e_0), w'(f_1)] - \overline{c_0}c_1[w'(e_1), w(f_0)]. \end{aligned}$$

For the root space to which $w'(e_1)$ and $w(f_0)$ belong, the sum of their roots is not 0. The same is true for $w(e_0)$ and $w'(f_1)$. Therefore, for $H \in \mathfrak{h}$ to hold, it should hold that

$$[w(e_0), w'(f_1)] = 0, \quad [w'(e_1), w(f_0)] = 0.$$

This condition holds when $w = (r_0r_1)^x$, $w' = (r_1r_0)^y$, or $w = r_1(r_0r_1)^x$, $w' = r_0(r_1r_0)^y$. When this condition holds, we have

$$H = (-k|c_0|^2 - m|c_1|^2)h_0 + (-l|c_0|^2 - n|c_1|^2)h_1.$$

If $[H, X] = 2X$, then X, Y, H form an $\mathfrak{sl}_2$ triple. From the fact that

$$\begin{aligned} [H, X] = &(-k|c_0|^2 - m|c_1|^2)(2k - al)c_0w(e_0) \\ &+ (-l|c_0|^2 - n|c_1|^2)(-ak + 2l)c_0w(e_0) \\ &+ (-k|c_0|^2 - m|c_1|^2)(2m - an)c_1w'(e_1) \\ &+ (-l|c_0|^2 - n|c_1|^2)(-am + 2n)c_1w'(e_1), \end{aligned}$$

it should be satisfied that

$$(akl - 2k^2 + akl - 2l^2)|c_0|^2 + (alm - 2km + akn - 2ln)|c_1|^2 = 2$$

and

$$(akn - 2km + alm - 2ln)|c_0|^2 + (amn - 2m^2 + amn - 2n^2)|c_1|^2 = 2.$$

Let

$$\begin{aligned} A &= 2akl - 2k^2 - 2l^2, \\ B &= alm - 2km + akn - 2ln, \\ C &= 2amn - 2m^2 - 2n^2. \end{aligned}$$

Then

$$\begin{aligned} A|c_0|^2 + B|c_1|^2 &= 2, \\ B|c_0|^2 + C|c_1|^2 &= 2. \end{aligned}$$

If $B^2 - AC \neq 0$,

$$|c_0|^2 = \frac{2(B-C)}{B^2-AC}, \quad |c_1|^2 = \frac{2(B-A)}{B^2-AC}.$$

In addition, a straightforward calculation yields the following lemma.

Lemma 5.4. $B^2 - AC = (a^2-4)(kn-lm)^2$. □

From Lemma 5.4 and $a > 3$, if $kn - lm \neq 0$, then $B^2 - AC > 0$. Since we have $|c_0|^2 \geq 0$, $|c_1|^2 \geq 0$, we want to find the condition. If $w = (r_0r_1)^x$, $w' = (r_1r_0)^y$, then $k = F_{2x+1}$, $l = F_{2x}$, $m = F_{2y}$, $n = F_{2y+1}$. If $w = r_1(r_0r_1)^x$, $w' = r_0(r_1r_0)^y$, then $k = F_{2x+1}$, $l = F_{2x+2}$, $m = F_{2y+2}$, $n = F_{2y+1}$. We will show the following lemma first.

Lemma 5.5. *For any $i, j \in \mathbb{Z}_{\geq 0}$, if $k = F_{i+1}$, $l = F_i$, $m = F_j$, $n = F_{j+1}$, then $A = C = -2$.*

Proof. If we set $f(i) = 2aF_{i+1}F_i - 2F_{i+1}^2 - 2F_i^2$, we get $f(i+1) = f(i)$ by a simple calculation. Therefore, $f(i) = f(0) = -2$. From this we know that $A = C = -2$. □

From Lemma 5.5, we have

$$\begin{aligned}|c_0|^2 = |c_1|^2 &= \frac{2(B+2)}{B^2-4} \\ &= \frac{2}{B-2}.\end{aligned}$$

Lemma 5.6. *For any $i, j \in \mathbb{Z}_{\geq 0}$, if $k = F_{i+1}$, $l = F_i$, $m = F_j$, $n = F_{j+1}$, then $B > 2$.*

Proof. Since B depends on i and j, we will write it subscripted as B_{ij}. Similarly, we will write k, l, m, n as k_i, l_i, m_j, n_j. We can calculate $B_{00} = a > 2$. It is sufficient to show that B_{ij} is monotonically increasing with respect to i and j. By symmetry, it is sufficient to show it only for i. Since

$$k_{i+1} = ak_i - l_i, \quad l_{i+1} = k_i,$$

we can calculate

$$\begin{aligned}B_{(i+1)j} - B_{ij} &= (a-2)k_im_j + 2l_im_j + (a^2-2)k_in_j - al_in_j \\ &> (a-2)k_im_j + 2l_im_j + (2a-2)k_in_j - al_in_j \qquad (a \geq 3) \\ &= (a-2)k_im_j + 2l_im_j + (a-2)k_in_j + a(k_i - l_i)n_j \\ &> 0 \qquad (a \geq 3,\ k_i > l_i).\end{aligned}$$

This shows monotonicity. □

If we set $i = 2x$ and $j = 2y$ in Lemma 5.6, we have $B > 2$ when $w = (r_0 r_1)^x$, $w' = (r_1 r_0)^y$. Similarly, if we swap k and m, l and n in Lemma 5.6 and set $i = 2x + 1$ and $j = 2y + 1$, we can see that $B > 2$ is also true when $w = r_1(r_0 r_1)^x$, $w' = r_0(r_1 r_0)^y$.

Proposition 5.7. *Let E_0, E_1 be real positive roots of different types (in the sense of α and β types). If we take the appropriate $c_0, c_1 \in \mathbb{C}$, then $X = c_0 E_0 + c_1 E_1$, $Y = \omega_0(X)$, $H = [X, Y]$ form an $\mathfrak{sl}_2$-triple.*

Proof. We can assume that E_0 is of type α and E_1 is of type β. The E_0's can be written in the form $cw(e_0)$ or $cw(e_1)$ using the constants c and $w \in \mathcal{W}$. Considering c_0 and c_1, we only need to consider the root vector of the form $w(e_0)$ or $w(e_1)$. The types of $(r_0 r_1)^i(e_0)$ and $r_0(r_1 r_0)^i(e_1)$ are α, while the types of $(r_1 r_0)^i(e_1)$ and $r_1(r_0 r_1)^i(e_0)$ are β.

When $E_0 = (r_0 r_1)^i(e_0)$, $E_1 = (r_1 r_0)^i(e_1)$ or when $E_0 = r_0(r_1 r_0)^i(e_1)$, $E_1 = r_1(r_0 r_1)^i(e_0)$, we have $B > 2$. Therefore we can take $c_0 = c_1 = \sqrt{2/(B-2)}$ and we obtain the conclusion. When $E_0 = (r_0 r_1)^i(e_0)$, $E_1 = r_1(r_0 r_1)^i(e_0)$ or when $E_0 = r_0(r_1 r_0)^i(e_1)$, $E_1 = (r_1 r_0)^i(e_1)$, we can repeat the discussion of this section in the same way. □

The contents of this section can be summarized as follows.

Theorem 5.8. *Let X be an element of a space spanned by a real positive root vector.*

(1) *If the length of X is not 2, then X, $Y = \omega_0(X)$, $H = [X, Y]$ do not form an $\mathfrak{sl}_2$-triple.*

(2) *Suppose the length of X is 2 and E_0, E_1 are real positive roots of different types (in the sense of types α and β). If we take appropriate $c_0, c_1 \in \mathbb{C}$, then $X = c_0 E_0 + c_1 E_1$, $Y = \omega_0(X)$, $H = [X, Y]$ form an $\mathfrak{sl}_2$-triple.*

Proof. We can see this from Lemmas 5.1 and 5.3 and Proposition 5.7. □

§6. Weighted Dynkin diagrams

The Dynkin diagram of the Kac–Moody Lie algebra we are dealing with is

$$\circ \underset{a}{\longleftrightarrow} \circ.$$

In fact, the two vertices are connected by a line segments, which are abbreviated as shown in the figure above. In this section we compute the weighted Dynkin diagram corresponding to the $\mathfrak{sl}_2$-triple constructed in the previous section.

A weighted Dynkin diagram is a Dynkin diagram where each vertex is labeled, and the label of vertex i is defined as $\alpha_i(H)$. Since the rank of the Kac–Moody Lie algebra we are considering is 2, $i = 0, 1$.

In the following, let $k = F_{i+1}$, $l = F_i$, $m = F_j$, $n = F_{j+1}$. Rewrite $X = c_0E_0 + c_1E_1$ and let $E_0 \in \mathfrak{g}_{k\alpha_0+l\alpha_1}$, $E_1 \in \mathfrak{g}_{m\alpha_0+n\alpha_1}$. Then E_0 is a root vector of type α and E_1 is a root vector of type β. Now, if $E \in \mathfrak{g}_{x\alpha_0+y\alpha_1}$, then $[H, E] = (x\alpha_0(H) + y\alpha_1(H))E$. Recall that $[H, X] = 2X$. Since $X = c_0E_0 + c_1E_1$, we have

$$[H, E_0] = 2E_0,$$
$$[H, E_1] = 2E_1.$$

From these equations, we have

$$\alpha_0(H) = \frac{2(n-l)}{kn-lm},$$
$$\alpha_1(H) = \frac{2(k-m)}{kn-lm}.$$

Therefore, the weighted Dynkin diagram is

$$\overset{\dfrac{2(F_{j+1}-F_i)}{F_{i+1}F_{j+1}-F_iF_j}}{\circ} \xleftrightarrow[a]{\qquad\qquad} \overset{\dfrac{2(F_{i+1}-F_j)}{F_{i+1}F_{j+1}-F_iF_j}}{\circ}.$$

In the general Kac–Moody Lie algebra, we say $h \in \mathfrak{h}$ is dominant if h satisfies $\alpha_i(h) \geq 0$ for any i. In the finite-type case, for any $\mathfrak{sl}_2$ triple, we can transform it by the action of an appropriate element of the Weyl group so that H is dominant. In the case of a rank 2 hyperbolic Kac–Moody Lie algebra, if H is dominant, then when we decompose $\mathfrak{g}$ into the direct sum of eigenspaces with respect to the adjoint action of H, the dimension of the eigenspaces corresponding to each eigenvalue will be finite.

With this motivation, we aim to classify cases where H is dominant. When H is dominant, each weight in the weighted Dynkin diagram is greater than or equal to 0. We do not consider whether an $\mathfrak{sl}_2$-triple can be constructed when X can be written as the sum of two root vectors of the same type (in the sense of types α and β), but even if it can, we can show that H is not dominant.

Lemma 6.1. *Suppose $X = c_0E_0 + c_1E_1$ and that E_0 and E_1 are real root vectors of the same type (in the sense of types α and β) and the roots to which E_0 and E_1 belong are different. If X, $Y = \omega_0(X)$, $H = [X, Y]$ form an $\mathfrak{sl}_2$-triple, then H is not dominant.*

Proof. Suppose that E_0 and E_1 are both of type α. Let $k = F_{i+1}$, $l = F_i$, $m = F_{j+1}$, $n = F_j$, and write $E_0 \in \mathfrak{g}_{k\alpha_0+l\alpha_1}$, $E_1 \in \mathfrak{g}_{m\alpha_0+n\alpha_1}$. The roots to which E_0 and E_1 belong are different. Therefore $i \neq j$. Repeating the above calculation when E_0 is of type α and E_1 is of type β, we get

$$\alpha_0(H) = \frac{2(n-l)}{kn-lm},$$
$$\alpha_1(H) = \frac{2(k-m)}{kn-lm}.$$

However, $n - l = F_j - F_i$ and $k - m = F_{i+1} - F_{j+1}$ are both not equal to zero and have different signs. Therefore, either $\alpha_0(H)$ or $\alpha_1(H)$ will be negative, and H will not be dominant. □

Lemma 6.2. *Of the $\mathfrak{sl}_2$-triples created by Proposition 5.7, H is dominant if and only if $i = j-1,\ j,\ j+1$.*

Proof. In order for $\alpha_0(H) \geq 0$ to be true, $n - l = F_{j+1} - F_i \geq 0$ should be true. Thus we have $i \leq j+1$.

Also, in order for $\alpha_1(H) \geq 0$ to be true, $k - m = F_{i+1} - F_j \geq 0$ should be true. Thus we have $i + 1 \geq j$.

Putting these together, we get $j - 1 \leq i \leq j + 1$. The converse is obvious. □

Proposition 6.3. *The $\mathfrak{sl}_2$-triple constructed in Proposition 5.7 can be transformed under the appropriate action of an element of a Weyl group so that H is dominant.*

Proof. If H is originally dominant, there is no need to transform it. Otherwise, it becomes $|i-j| \geq 2$.

Now we know

$$\begin{aligned} e_0 &\in \mathfrak{g}_{F_1\alpha_0+F_0\alpha_1} \quad (\text{type } \alpha),\\ r_1(e_0) &\in \mathfrak{g}_{F_1\alpha_0+F_2\alpha_1} \quad (\text{type } \beta),\\ r_0r_1(e_0) &\in \mathfrak{g}_{F_3\alpha_0+F_2\alpha_1} \quad (\text{type } \alpha)\\ &\vdots \end{aligned}$$

and

$$\begin{aligned} e_1 &\in \mathfrak{g}_{F_0\alpha_0+F_1\alpha_1} \quad (\text{type } \beta),\\ r_0(e_1) &\in \mathfrak{g}_{F_2\alpha_0+F_1\alpha_1} \quad (\text{type } \alpha),\\ r_1r_0(e_1) &\in \mathfrak{g}_{F_2\alpha_0+F_3\alpha_1} \quad (\text{type } \beta).\\ &\vdots \end{aligned}$$

Since E_0 is of type α and E_1 is of type β, we can write

$$\begin{aligned} E_0 &= r_0 r_1 r_0 \cdots r_p(r_{1-p}) \in \mathfrak{g}_{F_{i+1}\alpha_0 + F_i\alpha_1}, \\ E_1 &= r_1 r_0 r_1 \cdots r_q(r_{1-q}) \in \mathfrak{g}_{F_j\alpha_0 + F_{j+1}\alpha_1} \end{aligned} \qquad (p, q = 0 \text{ or } 1).$$

If r_0 acts on these elements,

$$\begin{aligned} r_0(E_0) &= \qquad r_1 r_0 \cdots r_p(r_{1-p}) \in \mathfrak{g}_{F_{i-1}\alpha_0 + F_i\alpha_1}, \\ r_0(E_1) &= r_0 r_1 r_0 r_1 \cdots r_q(r_{1-q}) \in \mathfrak{g}_{F_{j+2}\alpha_0 + F_{j+1}\alpha_1}. \end{aligned}$$

Keeping in mind that $r_0(E_0)$ is of type β and $r_0(E_1)$ is of type α, comparing these with the above, we can see that the number corresponding to i is $j+1$. Also, if r_1 acts on E_0 and E_1,

$$\begin{aligned} r_1(E_0) &= r_1 r_0 r_1 r_0 \cdots r_p(r_{1-p}) \in \mathfrak{g}_{F_{i+1}\alpha_0 + F_{i+2}\alpha_1}, \\ r_1(E_1) &= \qquad r_0 r_1 \cdots r_q(r_{1-q}) \in \mathfrak{g}_{F_j\alpha_0 + F_{j-1}\alpha_1}. \end{aligned}$$

In this case, the number corresponding to i is $j-1$ and the number corresponding to j is $i+1$. From this, $i-j$ becomes $j-i+2$ under the action of r_0, and $j-i-2$ under the action of r_1. If $i-j \geq 2$, $|i-j|$ decreases by 2 when r_0 is applied. If $j-i \geq 2$, $|i-j|$ decreases by 2 when r_1 is applied. By repeating this process, we can change $|i-j|$ to 0 or 1. □

Put Lemmas 6.1 and 6.2 together to get the following.

Theorem 6.4. *The $\mathfrak{sl}_2$-triple $\{X, Y, H\}$ of $\mathfrak{g}$, where X is in the space spanned by real root vectors, $Y = \omega_0(X)$, and H is dominant, are all those listed in Lemma 6.2.* □

When $i = j$, the weighted Dynkin diagram is

$$\begin{array}{ccc} \dfrac{2}{F_{i+1}+F_i} & & \dfrac{2}{F_{i+1}+F_i} \\ \circ & \overset{}{\underset{a}{\longleftarrow\!\!\!-\!\!\!-\!\!\!\longrightarrow}} & \circ. \end{array}$$

When $i = j+1$, it is

$$\begin{array}{ccc} 0 & & \dfrac{2}{F_i} \\ \circ & \underset{a}{\longleftrightarrow} & \circ. \end{array}$$

When $i = j-1$, it is

$$\begin{array}{ccc} \dfrac{2}{F_{i+1}} & & 0 \\ \circ & \underset{a}{\longleftrightarrow} & \circ. \end{array}$$

§7. Eigenvalues of the action of the Casimir element on $\mathfrak{h}$

The Casimir element c of a finite-dimensional semisimple Lie algebra $\mathfrak{g}_0$ is the element

$$c = \sum_i x_i y_i \in U(\mathfrak{g}_0),$$

where $(\cdot,\cdot)$ is the Killing form, $\{x_i\}$ is the basis of $\mathfrak{g}_0$, and $\{y_i\}$ is the dual basis with respect to this basis and the Killing form. The term $U(\mathfrak{g}_0)$ represents the universal enveloping algebra of $\mathfrak{g}_0$. When considering the action of $\mathfrak{g}_0$ on a $\mathfrak{g}_0$-module L, the action of the Casimir element is commutative with any action of an element of $\mathfrak{g}_0$. When L is an irreducible module, from Schur's lemma, the action of the Casimir element is a scalar multiplication.

When decomposing $\mathfrak{g}$ by the action of the $\mathfrak{sl}_2$ subalgebra created in the previous section, from $\dim \mathfrak{h} = 2$, there are two irreducible modules that pass through $\mathfrak{h}$. In particular, one is the $\mathfrak{sl}_2$ subalgebra itself. In this section we will find the eigenvalues of the Casimir element when it acts on $\mathfrak{h}$ with adjoint action, and we will determine how many times the action of the Casimir element makes the two irreducible modules. Let c_L be the Casimir element in the $\mathfrak{sl}_2$ subalgebra constructed above,

$$\begin{aligned} c_L &= \frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX \\ &= \frac{1}{8}H^2 + \frac{1}{4}H + \frac{1}{2}YX. \end{aligned}$$

For the $\mathfrak{sl}_2$-triple constructed in Proposition 5.7, we will calculate the eigenvalue of the Casimir element. Let $X = c_0E_0 + c_1E_1$, where E_0 is of type α and E_1 is of type β. Using $p, q \in \{0, 1\}$, we can write $E_0 = w(e_p)$, $E_1 = w'(e_q)$. Let $E_0 \in \mathfrak{g}_{k\alpha_0 + l\alpha_1}$, $E_1 \in \mathfrak{g}_{m\alpha_0 + n\alpha_1}$. We can write $k = F_{i+1}$, $l = F_i$, $m = F_j$, $n = F_{j+1}$. We have

$$\begin{aligned} c_L(h_0) &= \frac{1}{2}[Y, [X, h_0]] \\ &= \frac{1}{2}[(al - 2k)c_0 w(e_p) + (an - 2m)c_1 w'(e_q), \overline{c_0} w(f_p) + \overline{c_1} w'(f_q)]. \end{aligned}$$

Now since both $[w(e_p), w'(f_q)]$ and $[w'(e_q), w(f_p)]$ are 0, we have

$$\begin{aligned} c_L(h_0) &= \frac{1}{2}((al - 2k)|c_0|^2 w(h_p) + (an - 2m)|c_1|^2 w'(h_q)) \\ &= \frac{1}{2}((al - 2k)k + (an - 2m)m)|c_0|^2 h_0 \\ &\quad + \frac{1}{2}((al - 2k)l + (an - 2m)n)|c_0|^2 h_1. \end{aligned}$$

Similarly,

$$\begin{aligned} c_L(h_1) &= \frac{1}{2}[Y,[X,h_1]] \\ &= \frac{1}{2}((ak-2l)k+(am-2n)m)|c_0|^2 h_0 \\ &\quad + ((ak-2l)l+(am-2n)n)|c_0|^2 h_1. \end{aligned}$$

The eigenvalues of the action of c_L are eigenvalues of

$$\frac{1}{2}|c_0|^2 \begin{pmatrix} (al-2k)k+(an-2m)m & (ak-2l)k+(am-2n)m \\ (al-2k)l+(an-2m)n & (ak^2l)l+(am-2n)n \end{pmatrix}.$$

For simplicity, let

$$\begin{aligned} P &= (al-2k)k+(an-2m)m, \\ Q &= (ak-2l)k+(am-2n)m, \\ R &= (al-2k)l+(an-2m)n, \\ S &= (ak-2l)l+(am-2n)n. \end{aligned}$$

Solving for

$$\begin{vmatrix} X-P & -Q \\ -R & X-S \end{vmatrix} = 0,$$

we get

$$X = \frac{(P+S) \pm \sqrt{(P+S)^2-4PS+4QR}}{2}$$

and the eigenvalues are

$$\frac{(P+S) \pm \sqrt{(P+S)^2-4PS+4QR}}{4}|c_0|^2.$$

We will write this as $E^{\pm}$.

Lemma 7.1. *We have* $P+S=-4$ *and* $QR-PS=B^2-4$.

Proof. This lemma is shown from

$$\begin{aligned} P+S &= akl-2k^2+amn-2m^2+akl-2l^2+amn-2n^2 \\ &= A+C \\ &= -4 \end{aligned}$$

and

$$\begin{aligned} QR - PS &= (ak - 2l)(an - 2m)(kn - lm) + (al - 2k)(am - 2n)(lm - kn) \\ &= B^2 - AC \quad \text{(from Lemma 5.4)} \\ &= B^2 - 4. \end{aligned}$$ □

From Lemma 7.1, we have

$$\begin{aligned} E^{\pm} &= \frac{(P+S) \pm \sqrt{(P+S)^2 + 4(QR - PS)}}{4} |c_0|^2 \\ &= \frac{-4 \pm 2|B|}{4} |c_0|^2 \\ &= \frac{-2 \pm B}{2} \cdot \frac{2}{B-2} \quad \text{(from Lemma 5.6, } B > 2) \\ &= -\frac{B+2}{B-2}, \ 1. \end{aligned}$$

The eigenvalue 1 corresponds to the $\mathfrak{sl}_2$ subalgebra itself, and the other eigenvalue corresponds to the other irreducible component. Let

$$E^+ = -\frac{B+2}{B-2}$$

and we will consider the range of values of E^+. Since $B > 2$ from Lemma 5.6, E^+ is strictly increasing with respect to B. Also, from Lemma 5.6, B is strictly increasing with respect to i and j respectively. When $i = j = 0$, we have $B = a$ and

$$E^+ = -\frac{a+2}{a-2}.$$

When $B > 2$, we have $E^+ < -1$ and

$$-\frac{a+2}{a-2} \leq E^+ < -1.$$

We will do some more calculations on the value of B.

Lemma 7.2. *For any i,*

$$\lim_{j \to \infty} B = \infty,$$

and for any j,

$$\lim_{i \to \infty} B = \infty.$$

Proof. From symmetry, only the first half needs to be shown. We can calculate that

$$\begin{aligned} B &= F_{i+1}(an-2m) + aF_im - 2F_in \\ &\geq F_{i+1}(3n-2m) + aF_im - 2F_in \\ &= F_{i+1}(n-2m) + aF_im + 2n(F_{i+1}-F_i). \end{aligned}$$

If $i \geq 1$, then $F_{i+1} = aF_i - F_{i-1}$ and $F_i > F_{i-1}$, and if $i = 0$, then $F_{i+1} = aF_i$. From this, we have

$$F_{i+1} > (a-1)F_i$$

and therefore

$$2n(F_{i+1} - F_i) > 2(a-2)nF_i.$$

From $n > 2m$ and $a \geq 3$, we have

$$\begin{aligned} \lim_{j\to\infty} F_{i+1}(n-2m) &= \infty, \\ \lim_{j\to\infty} aF_im &= \infty, \\ \lim_{j\to\infty} 2(a-2)nF_i &= \infty. \end{aligned}$$

Consequently, we have

$$\lim_{j\to\infty} B = \infty.$$ □

From Lemma 7.2, we have

$$\lim_{i\to\infty} E^+ = -1, \quad \lim_{j\to\infty} E^+ = -1.$$

The above can be summarized as follows.

Proposition 7.3. *We have that E^+ is strictly increasing for i, j, and*

$$-\frac{a+2}{a-2} \leq E^+ < -1$$

and

$$\lim_{i\to\infty} E^+ = -1, \quad \lim_{j\to\infty} E^+ = -1.$$ □

From [HT92, §II, Thm. 1.1.3], $E^+ < -1$ means that the module with this eigenvalue is an infinite-dimensional module that is neither a highest module nor a lowest module. We can also see that this module is a principal series representations of $\mathfrak{sl}_2$.

Example 7.4. Let $a = 3$. Then $F_0 = 0$, $F_1 = 1$, $F_2 = 3$, $F_3 = 8, \ldots$. The E^+'s for the $\mathfrak{sl}_2$ triples created in Theorem 5.8 are given in Table 1.

$(k,l)\backslash(m,n)$	(0, 1)	(1, 3)	(3, 8)	(8, 21)	(21, 55)	$\cdots$
(1, 0)	−5	−1.8	−1.25	−1.088889	−1.033058	
(3, 1)	−1.8	−1.25	−1.088889	−1.033058	−1.012500	
(8, 3)	−1.25	−1.088889	−1.033058	−1.012500	−1.004756	$\cdots$
(21, 8)	−1.088889	−1.033058	−1.012500	−1.004756	−1.001814	
(55, 21)	−1.033058	−1.012500	−1.004756	−1.001814	−1.000693	
$\vdots$			$\vdots$			$\ddots$

Table 1. The E^+ for the $\mathfrak{sl}_2$ triples created in Theorem 5.8, $a = 3$

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