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Movable Singularity of Some Hamiltonian Systems Associated with Blowup Phenomena

by

Masafumi Yoshino

Abstract

The motivation for this paper comes from a blowup problem for a semi-linear wave equation and a heat equation. Following an idea of J. Leray, we study a radially symmetric self-similar solution which has singularities on the characteristic cone. This naturally leads to the study of a Hamiltonian system called a profile equation. The novelty of this paper is that we focus on the movable singularity of the Hamiltonian system and we use Borel summability in constructing a singular solution. By "movable singularity" we mean that the singularity does not appear in the coefficients of the equation and depends on the respective solution. In the proof of our theorem we reduce the Hamiltonian system to a simpler form by a method similar to the so-called Birkhoff reduction. We obtain the parametrization of a singular solution by an elementary function. We also give applications to a semi-linear wave equation and a heat equation.

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§1. Introduction

In this paper we are interested in the blowup of a semi-linear wave equation and a semi-linear heat equation. (cf. Section 6). We study it using the movable singularity of the deduced Hamiltonian system, called a profile equation. As for the blowup of solutions of equations in mathematical physics, there are many works from the viewpoint of real analysis. (See [4] and the references therein.) In this paper we consider a radially symmetric self-similar solution following an idea of J. Leray. Such a solution gives a singular solution with singularity on some characteristic cone. By simple calculations, the solution satisfies a certain Hamiltonian system. (cf. Section 6). Our main idea in constructing a singular solution is to use a solution with a movable singular point of the reduced Hamiltonian system. In fact, in the

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M. Yoshino: Department of Mathematics, Hiroshima University, Hiroshima 739-8526, Japan;
e-mail: yoshinom@hiroshima-u.ac.jp

preceding paper [6] we considered a semi-linear wave equation. Then a semi-linear Heun equation appeared as the reduced Hamiltonian system. We constructed a self-similar singular solution by virtue of a solution with a movable singular point of the semi-linear Heun equation.

In this paper we give a new method to treat the Hamiltonian system, which is based on Borel summability. This also gives an alternate proof of [6] in the case of a semi-linear wave equation. We also apply our method to the study of a semi-linear heat equation. We remark that a close relation between the movable singularity and the elliptic function has already been observed in the classical work [1] for the Painlevé equation. Among the many related works, we cite the recent work [3], where an exact asymptotic theory and the elliptic function play an important role. (See also [2].) We consider the Hamiltonian system H which is the perturbation of some Hamiltonian system H_0. We assume that H_0 has a solution with a movable pole or a movable algebraic branch point. Our main result is the construction of the solution of H with a movable pole or a movable algebraic branch point. We apply our result to the construction of a radially symmetric self-similar solution of a semi-linear wave equation with singularity on a certain characteristic cone. The position of the characteristic cone is given by the movable singular point of the solution of the profile equation. The proof of the main theorem is achieved by expressing the solution of H as the pull-back of that of H_0 by some diffeomorphism on the phase space. The existence of the diffeomorphism is shown by the Borel summability method. The method has the advantage that we also get information on the location of the movable singular point and the property of the singular point.

This paper is organized as follows. In Section 2 we consider the movable singularity of some Hamiltonian system. In Section 3 we show the solvability of a homology equation. In Section 4 we prove the Borel summability of a formal solution of a homology equation in a domain of phase space. Using the preparations in Sections 3 and 4 we prove Theorem 2.1 in Section 5. In Section 6 we give examples of the blowup problem. In the appendix we give the proof of Borel summability.

§2. Statement of results

Let $q = (q_1, \dots, q_n)$ and $p = (p_1, \dots, p_n)$ be the variables in $(q, p) \in \mathbb{C}^n \times \mathbb{C}^n$. Let $H \equiv H(q, p)$ be an entire function of q and p. Consider the Hamiltonian system

$$\frac{dq_j}{dz} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dz} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \dots, n. \tag{2.1}$$

For a function F, define the Hamiltonian vector field χ_F as

$$\chi_F := \sum_{j=1}^{n} \Big(\frac{\partial F}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial F}{\partial q_j} \frac{\partial}{\partial p_j} \Big). \tag{2.2}$$

Define the Poisson bracket of F and G as $\{F, G\} := \chi_F G$. If $\chi_H F = 0$, then we say that F is the first integral of χ_H. We say that the Hamiltonian system is integrable if there exist n first integrals ϕ_j in involution, $\{\phi_i, \phi_j\} = 0$ such that the ϕ_j are almost everywhere functionally independent.

In what follows we consider Hamiltonian systems with $n = 2$. Let $\lambda \neq 0$ be a constant. For integers $i > j \geq 1$, set $\nu = i - j$. Define

$$H_0 := \lambda q_2 p_2 + q_2^i p_2^j. \tag{2.3}$$

Let H_1 be a polynomial of q_2 and p_2 with coefficients analytic in q_1 in the domain $\Omega_1 \subset \mathbb{C}$ containing the origin. Assume that the constant term of H_1 with respect to q_2 and p_2 vanishes. Set $H := H_0(q_2, p_2) + H_1(q_1, q_2, p_2)$. Consider

$$\frac{dq_2}{dq_1} = \frac{\partial H}{\partial p_2}, \quad \frac{dp_2}{dq_1} = -\frac{\partial H}{\partial q_2}. \tag{2.4}$$

Let $K_0 > 0$, $\varepsilon_0 > 0$ and $r_0 > 0$ be constants. Define Σ_0 as

$$\begin{aligned} \Sigma_0 := \big\{ (q_2, p_2) \bigm| q_2^{-1} = \alpha r,\ p_2 = \beta r,\ |\alpha| < \varepsilon_0,\ |\beta| < \varepsilon_0, \\ 0 < r < r_0,\ |q_2^{-1}| \leq K_0,\ |p_2| \leq K_0 \big\}. \end{aligned} \tag{2.5}$$

Suppose

$$\frac{\partial^2 H}{\partial p_2^2}(t, q_2, p_2) \neq 0, \quad \text{for all } t \in \Omega_1, \text{ for all } (q_2, p_2) \in \Sigma_0. \tag{2.6}$$

Consider the equation of (u_3, u_4):

$$\frac{\partial H}{\partial q_2}(z, u_3, u_4) = 0, \quad \frac{\partial H}{\partial p_2}(z, u_3, u_4) = 0, \quad z \in \Omega_1. \tag{2.7}$$

Assume that

(AS1) Equations (2.7) have a solution $(u_3, u_4) = (w_0(z), \widetilde{w}_0(z))$, which is holomorphic in $z \in \Omega_1$ and which does not vanish identically, such that $w_0(0) = 0$, $\widetilde{w}_0(0) = 0$.

Then we have the following theorem:

Theorem 2.1. *Suppose that (AS1) and (2.6) are satisfied. Assume $j = 1$. Then (2.4) has a solution (q_2, p_2) having a movable pole or a movable branch point in Ω_1.*

Remark 2.2. (a) We note that H is independent of p_1. The nonautonomous system (2.4) is equivalent to the autonomous one with Hamiltonian $p_1 + H$. Assumption (AS1) is used to transform (2.4) into the Hamiltonian system for H_0. In fact, we construct the formal power series of a certain parameter by (AS1). The Borel sum of the series gives the transformation.

(b) Our method also works for H_0 with resonance. Define

$$H_0(q_2, p_2) = \lambda q_2 p_2 + c q_2^2 p_2^2 + \varepsilon_2 (q_2^4 + p_2^4), \tag{2.8}$$

where $\lambda \neq 0$, c and $\varepsilon_2 > 0$, $\varepsilon_2 \neq 1$ are constants. Then we have the following theorem:

Theorem 2.3. *Let H_0 be given by (2.8). Assume that (AS1) and (2.6) are satisfied. Then (2.4) has a solution (q_2, p_2) having a movable pole or a movable branch point in Ω_1.*

§3. Homology equation

Let $x = (\tilde{q}_1, \tilde{p}_1, \tilde{q}_2, \tilde{p}_2)$ and $y = (q_1, p_1, q_2, p_2)$ be the variables. We write $x = (x_1, \ldots, x_4)$ and $y = (y_1, \ldots, y_4)$, for simplicity. Let H be given in (2.4). Consider the map $x = u(y)$, $u = (u_1, \ldots, u_4)$ which transforms χ_{p_1+H} to $\chi_{p_1+H_0}$. Define

$$X_0 := \chi_{p_1+\lambda q_2 p_2}. \tag{3.1}$$

The component vector of X_0 with respect to the basis $\partial/\partial q_1, \partial/\partial p_1, \partial/\partial q_2, \partial/\partial p_2$ is given by $(1, 0, \lambda q_2, -\lambda p_2) =: \Lambda(y)$. Set

$$f(x) := H(\tilde{q}_1, \tilde{q}_2, \tilde{p}_2) - \lambda \tilde{q}_2 \tilde{p}_2 \tag{3.2}$$

(f is independent of $x_2 = \tilde{p}_1$) and define

$$R := \chi_f, \quad S := \chi_{q_2^i p_2^j}. \tag{3.3}$$

Let $r(x)$ and $s(y)$ be the component vectors of R and S, respectively. The first and the second components of $s(y)$ vanish. The term $\Lambda(u(y))$ is independent of x_2. Then we have the following lemma:

Lemma 3.1. *Let $u = u(y)$ satisfy*

$$\Lambda(y)\nabla u + s(y)\nabla u = r(u) + \Lambda(u) \tag{3.4}$$

such that $(\nabla u)^{-1}$ exists. Then $x = u(y)$ transforms $(\Lambda(y) + s(y))\frac{\partial}{\partial y}$ to $(\Lambda(x) + r(x))\frac{\partial}{\partial x}$. Here, ∇u is the nabla of u with respect to $(y_1, \ldots, y_4)$ and $\Lambda(u) = (1, 0, \lambda u_3, -\lambda u_4)$.

Remark 3.2. Equation (3.4) is called a homology equation.

Proof of Lemma 3.1.

$$(\Lambda(x)+r(x))\frac{\partial}{\partial x}=(\Lambda(u)+r(u))(\nabla u)^{-1}\frac{\partial}{\partial y}=(\Lambda(y)+s(y))\frac{\partial}{\partial y}. \qquad \square$$

Deduction of the homology equation. Consider the third row of (3.4). Set $w=u_3$, $u=(u_1,\ldots,u_4)$. By $\Lambda(y)=(1,0,\lambda q_2,-\lambda p_2)$ we have

$$\frac{\partial w}{\partial q_1}+\delta w-iq_2^{i-1}p_2^j\frac{\partial w}{\partial p_2}+jq_2^ip_2^{j-1}\frac{\partial w}{\partial q_2}-\lambda w-\mathcal{R}_3(u)=0, \tag{3.5}$$

where $\delta=\lambda(q_2\frac{\partial}{\partial q_2}-p_2\frac{\partial}{\partial p_2})$ and $\mathcal{R}_3(u)=(\partial f/\partial\tilde{p}_2)(u)$.

Consider the first-order terms of the left-hand side of (3.5) except for the first term. They are given by the Hamiltonian vector field of H_0. By taking t as the time variable we write it as the system of ordinary differential equations

$$\dot{q}_2=\lambda q_2+jq_2^ip_2^{j-1},\quad \dot{p}_2=-\lambda p_2-ip_2^jq_2^{i-1}. \tag{3.6}$$

Set $\nu=i-j$. By the change of unknown functions

$$q_2=1/r,\quad p_2=rv, \tag{3.7}$$

we have

$$\dot{v}=-\nu r^{-\nu}v^j,\quad \dot{r}=-\lambda r-jv^{j-1}r^{1-\nu}. \tag{3.8}$$

Because $\lambda q_2p_2+q_2^ip_2^j$ is the first integral of (3.6), there exists a constant c_1 such that $\lambda q_2p_2+q_2^ip_2^j=c_1$. Hence, by (3.7) we have $\lambda v+v^j/r^\nu=c_1$. By (3.8) we have $\dot{v}-\lambda\nu v+c_1\nu=0$, from which we have

$$v=c_1\lambda^{-1}+c_2e^{\lambda\nu t}, \tag{3.9}$$

where c_2 is a constant.

By (3.8) and $\dot{v}-\lambda\nu v+c_1\nu=0$, (3.5) can be written as

$$\frac{\partial w}{\partial q_1}+\nu(\lambda v-c_1)\frac{\partial w}{\partial v}-(\lambda r+jv^{j-1}r^{1-\nu})\frac{\partial w}{\partial r}-\lambda w-\mathcal{R}_3(u)=0. \tag{3.10}$$

In order to solve (3.10) we introduce a new equation with parameters η and h:

$$\begin{aligned} &h\Big(r^{\nu-1}\eta\Big(\frac{\partial u_3}{\partial q_1}+\nu(\lambda v-c_1)\frac{\partial u_3}{\partial v}\Big)+(1-\lambda\eta r^\nu)\frac{\partial u_3}{\partial r}\Big)\\ &\qquad -r^{\nu-1}(\lambda u_3+\mathcal{R}_3(u))=0. \end{aligned} \tag{3.11}$$

We also consider the equation for u_4 similar to (3.11). For the moment we assume that u_1 is given. Assume $j = 1$. If we have the solution of the system of equation (3.11) and u_4, then we have the solution of (3.10) if we set $\eta = h^{-1}$ with

$$h = -1, \tag{3.12}$$

and multiply (3.10) by $r^{\nu-1}$. Therefore, we consider the solvability of (3.11).

We introduce the renormalization variable $\tilde{q}_1$. Consider the curve $(q_1, v(q_1))$ in (q_1, v)-space, where $v(q_1)$ is given by (3.9) with $q_1 = t$. Then the vector field $\frac{\partial}{\partial q_1} + \nu(\lambda v - c_1)\frac{\partial}{\partial v}$ is written as $\partial/\partial\tilde{q}_1$, where $\tilde{q}_1$ is the characteristic variable. Then we can write (3.11) in the form

$$h\Big(r^{\nu-1}\eta\frac{\partial u_3}{\partial\tilde{q}_1} + (1-\lambda\eta r^\nu)\frac{\partial u_3}{\partial r}\Big) - r^{\nu-1}(\lambda u_3 + \mathcal{R}_3(u)) = 0, \tag{3.13}$$

where $u = (u_1, 0, u_3, u_4)$. Similarly, u_4 satisfies

$$h\Big(r^{\nu-1}\eta\frac{\partial u_4}{\partial\tilde{q}_1} + (1-\lambda\eta r^\nu)\frac{\partial u_4}{\partial r}\Big) + r^{\nu-1}(\lambda u_4 + \mathcal{R}_4(u)) = 0, \tag{3.14}$$

where $\mathcal{R}_4(u) = (\partial f/\partial\tilde{q}_2)(u)$.

Consider the solvability of the reduced homology equation (3.13)–(3.14). First we show the following:

Lemma 3.3. *The variable u_1 is given by*

$$u_1 = -\frac{q_1}{2} - \frac{1}{2\lambda\nu}\log(\nu\lambda v - c_1\nu) - \frac{2}{\lambda\nu}\log(1-\lambda\eta r^\nu). \tag{3.15}$$

Proof. Because the first component of $r(u)$ vanishes, we replace $\lambda u_3 + \mathcal{R}_3(u)$ in (3.11) with 1. Then the first column of (3.4) gives the equation for $w := u_1$:

$$r^{\nu-1}\eta\frac{\partial w}{\partial\tilde{q}_1} + (1-\lambda\eta r^\nu)\frac{\partial w}{\partial r} - r^{\nu-1}\eta = 0,$$

from which the assertion follows. □

We replace v in (3.15) with $v + v_0$, where $\nu\lambda v_0 - c_1\nu = 1$. Then we are reduced to the case $-c_1\nu = 1$. In the following we assume $-c_1\nu = 1$ for the sake of simplicity. Insert u_1 into $\mathcal{R}_3(u)$ and $\mathcal{R}_4(u)$ in (3.13) and (3.14), respectively. Then we have the following theorem:

Theorem 3.4. *Suppose (AS1). Then, for every $K_0 > 0$, there exist neighborhoods U_0 of $q_1 = 0$ in Ω_1, U_2 of $v = 0$ and Ω_0 of $r = 0$, respectively, $h_0 > 0$ and a sector S_0 with opening equal to or greater than π such that, if $h \in S_0$, $|h| < h_0$ and $|\eta r| < K_0$, then (3.13)–(3.14) has an analytic solution (u_3, u_4) for $(q_1, v, r) \in U_0 \times U_2 \times \Omega_0$.*

Theorem 3.4 follows from Theorem 4.1 which follows below.

Remark 3.5. We can take h_0 in Theorem 3.4 arbitrarily large. (cf. Remark A.6).

Next we consider the solvability of (3.4). We use the variables q_1, p_1, v and r. Note that, if u_1, u_3 and u_4 are given, then u_2 is given by the integration of the second equation of (3.4). By Theorem 3.4 we have the following theorem:

Theorem 3.6. *Suppose (AS1) and $j = 1$. Then, for every $K_0 > 0$, there exist neighborhoods U_0 of $q_1 = 0$ in Ω_1, U_1 of $p_1 = p_1^{(0)}$, U_2 of $v = 0$ and Ω_0 of $r = 0$, $h_0 > 0$ and a sector S_0 with opening equal to or greater than π such that, if $h \in S_0$, $|h| < h_0$ and $|r| < K_0|h|$ for $h = -1$, then (3.4) has an analytic solution $u = (u_1, \dots, u_4)$ for $(q_1, p_1, v, r) \in U_0 \times U_1 \times U_2 \times \Omega_0$.*

Remark 3.7. Let $u = (u_1, \dots, u_4)$ be a solution of (3.4). Then (u_3, u_4) is the pull-back of a certain symplectic transformation if (2.6) holds. In particular, $(\nabla u)^{-1}$ exists. For the sake of completeness we give the proof in the appendix.

§4. Borel summability

§4.1. Definition

Let U_0, U_2 and Ω_0 be the domain in $\mathbb{C}$. Consider the formal power series of h,

$$v(q_1, s, r, h) = \sum_{n=1}^{\infty} v_n(q_1, s, r)h^n, \tag{4.1}$$

where the $v_n(q_1, s, r)$ are holomorphic in the domain $(q_1, s, r) \in U_0 \times U_2 \times \Omega_0$. Define the formal Borel transform $\mathcal{B}$ as

$$\mathcal{B}(v)(q_1, s, r, y) := \sum_{n=1}^{\infty} \frac{v_n(q_1, s, r)}{(n-1)!} y^{n-1}, \tag{4.2}$$

where y is a dual variable of h. Denote the set of nonnegative real numbers by $\mathbb{R}_+$. Let $\operatorname{dist}(z, \mathbb{R}_+ e^{i\xi})$ be the distance from z to $\mathbb{R}_+ e^{i\xi}$. For $\tau > 0$ and the direction ξ, define

$$E(\xi, \tau) := \{z \in \mathbb{C} \mid \operatorname{dist}(z, \mathbb{R}_+ e^{i\xi}) < \tau/2\}. \tag{4.3}$$

Define the sector with direction $\xi \in \mathbb{R}$ and opening $\tau > 0$ as $\{z \in \mathbb{C}\backslash 0 \mid |\arg z - \xi| < \tau/2\}$.

We say that $v(q_1, s, r, h)$ is (fine) Borel summable in the direction ξ if there exists $\theta > 0$ such that $B(v)(q_1, s, r, y)$ converges when $(q_1, s, r) \in U_0 \times U_2 \times \Omega_0$ and y is in some neighborhood of $y = 0$, and $B(v)(q_1, s, r, y)$ is analytically continued to $U_0 \times U_2 \times \Omega_0 \times E(\xi, \theta)$ with exponential type of order 1 in $y \in E(\xi, \theta)$ for $(q_1, s, r) \in U_0 \times U_2 \times \Omega_0$. Namely, there exist $K_0 > 0$ and $K_2 > 0$ such that

$$|B(v)(q_1, s, r, y)| \le K_0 e^{K_2|y|}, \quad y \in E(\xi, \theta), \quad (q_1, s, r) \in U_0 \times U_2 \times \Omega_0.$$

For simplicity, we denote the analytic continuation with the same notation. Define the Borel sum of $v(q_1, s, r, h)$, $V(q_1, s, r, h)$ as the Laplace transform

$$V(q_1, s, r, h) := \int_0^{\infty e^{i\xi}} e^{-yh^{-1}} B(v)(q_1, s, r, y)\, dy. \tag{4.4}$$

§4.2. Construction of a formal solution

Let q_1 be in some neighborhood of the origin. For $K_0 > 0$ let r satisfy $|\eta r| \le K_0$. Substitute u_1 in (3.15) into (3.13) and (3.14). Construct the formal solution of (3.13)–(3.14), (u_3, u_4) given by

$$u_3 = \sum_{n=0}^{\infty} w_n(q_1, v, r) h^n, \quad u_4 = \sum_{n=0}^{\infty} \widetilde{w}_n(q_1, v, r) h^n. \tag{4.5}$$

Here, w_0 and $\widetilde{w}_0$ are determined, respectively, by

$$\lambda w_0 + \mathcal{R}_3(u_1, w_0, \widetilde{w}_0) = 0, \quad \lambda \widetilde{w}_0 + \mathcal{R}_4(u_1, w_0, \widetilde{w}_0) = 0, \tag{4.6}$$

where w_0 (resp. $\widetilde{w}_0$) is a holomorphic function of u_1 in Ω_1 by (AS1). Then we determine w_n and $\widetilde{w}_n$ $(n = 1, 2, \ldots)$ inductively.

We prove that w_n and $\widetilde{w}_n$ $(n \ge 1)$ are holomorphic functions of r^ν at the origin. Since the proof is the same, we prove the assertion for w_n. Consider w_1. Multiply (3.13) by r. We calculate w_1 by dividing the next quantity by r^ν,

$$\Big(r^\nu \eta \frac{\partial w_0}{\partial \widetilde{q}_1} + (1 - \lambda \eta r^\nu) r \frac{\partial w_0}{\partial r} \Big), \tag{4.7}$$

and by inverting the bounded quantity appearing from $\lambda + \nabla \mathcal{R}_3$ (resp. $\lambda + \nabla \mathcal{R}_4$). We note that (4.7) is equal to r^ν times some holomorphic function. Indeed, since w_0 is analytic in u_1, (3.15) implies that $r \frac{\partial w_0}{\partial r}$ is divisible by r^ν. It follows that w_1 is analytic in r^ν.

Next we consider w_2. Similarly, we investigate $r \frac{\partial w_1}{\partial r}$. Since $r \frac{\partial}{\partial r}(1 - \lambda \eta r^\nu)^{-1}$ is divisible by r^ν, $r \frac{\partial w_1}{\partial r}$ has the same property. Hence w_2 is analytic in r^ν. Inductively, we see that the w_n are analytic in r^ν.

By the same argument as that of [5, Prop. 2] we have the Gevrey estimate of w_n. Namely, there exist constants $K_1 > 0$ and $\rho > 0$ independent of n such that $|w_n| \le K_1 \rho^n n!$.

§4.3. Summability

Let Ω_0 be a neighborhood of $r = 0$. Then we have the following theorem:

Theorem 4.1. *Suppose (AS1). Then, for every* $K_0 > 0$*, there exist neighborhoods* U_0 *of* $q_1 = 0$ *in* Ω_1*,* U_2 *of* $v = 0$ *and* $r_0 > 0$ *such that* (3.13)–(3.14) *is Borel summable in the direction* 0 *if* $(q_1, v, r) \in U_0 \times U_2 \times (\Omega_0 \cap \{r \mid |r| < r_0,\ |\eta r| < K_0\})$*.*

The proof of Theorem 4.1 is similar to the argument in [7]. For the sake of completeness we give the proof in the appendix.

§5. Proof of Theorem 2.1

§5.1. Lemma

Let H_0 be given by (2.3). Consider the Hamiltonian system for $p_1 + H_0$. Set $\nu = i - j$. By the change of variables (3.7) we have (3.8). Then we have the following lemma:

Lemma 5.1. *The solution of* (3.6) *is given by* (3.7)*,* (3.9) *and* $r = (-\nu v^j/\dot{v})^{1/\nu}$*. The solution* (q_2, p_2) *of* (3.6) *has infinitely many movable poles (resp. branch points) at the point given by* $v = c_1\lambda^{-1} + c_2 e^{\lambda\nu t} = 0$ *if* $\nu = 1$ *(resp.* $\nu \geq 2$*). As* t *tends to the singular point from some sector,* q_2 *tends to infinity while* p_2 *tends to zero.*

§5.2. Proof of Theorem 2.1

Proof. The proof consists of three steps.

Step 1. Let $j = 1$. We consider $\chi_{p_1+H_0}$. By Lemma 5.1 the Hamiltonian equation for H_0 has a singular solution q_2, p_2,

$$q_2 = 1/r, \quad p_2 = rv, \quad v = v(t) = c_1/\lambda + c_2 e^{\lambda\nu t}, \quad r = (-\nu v/\dot{v})^{1/\nu}, \tag{5.1}$$

where $\dot{v} = dv/dt$ and $t = q_1$. Since the singular point t_0 of q_2 satisfies $v(t_0) = 0$ by Lemma 5.1, we have

$$e^{\lambda\nu t_0} = -c_1/(\lambda c_2), \quad t_0 = (\lambda\nu)^{-1}\log(-c_1/(c_2\lambda)), \quad \dot{v}(t_0) = -\nu c_1. \tag{5.2}$$

We choose c_1 and c_2 such that $-c_1/c_2\lambda\nu$ is in a small neighborhood of 1. Then we see that t_0 is in a neighborhood of the origin. The orbit for H_0 is contained in Σ_0 since v and r vanish at t_0 and t is in some neighborhood of t_0.

Step 2. Suppose that the conditions of Theorem 3.4 are satisfied. By Lemma 3.1, (2.6) and Remark 3.7, χ_{p_1+H} is the pull-back of $\chi_{p_1+H_0}$. Hence we have a solution of the Hamiltonian system for H. We note that the origin corresponds to infinity in the new coordinate. By (AS1), the diffeomorphism preserves the origin at the points $q_1 = 0$. Since the flow of $\chi_{p_1+H_0}$ goes to the origin when approaching a singular point t_0, the Hamiltonian equation for $p_1 + H$ has a singular point z_1 in some neighborhood of $q_1 = 0$. Hence, it is sufficient to verify the conditions $|\eta r| < K_0$, $|r| < r_0$, $|h| < h_0$ and $h \in S_0$ in Theorem 3.4.

Step 3. By (5.1) we have $r = O(q_1^{1/\nu})$ as $q_1 \to 0$. We have $|r| < r_0$ if q_1 is sufficiently small. Choose the bisecting direction of S_0 such that $h \in S_0$. The condition $|h| < h_0$ follows from Remark 3.5 since h_0 can be taken sufficiently large. We show $|\eta r| < K_0$. By definition it is equivalent to $|r| < K_0|h|$. By (5.1), v tends to zero with order $c_1\nu q_1$ as $q_1 \to 0$. Consider h given by (3.12). By definition, h is $O(q_1^{j-1}) = O(1)$ as $q_1 \to 0$ since $j = 1$. Then we have $|r| < K_0|h|$ if q_1 is sufficiently small. □

We summarize the result that we have proved in this section:

Theorem 5.2. *Suppose that* (AS1) *and* (2.6) *are satisfied. Assume* $j = 1$. *Then the Hamiltonian system of* H_0 *has the solution with a movable pole at* $t = t_0$ *if* $\nu = 1$, *while the solution has a movable algebraic branch point if* $\nu > 1$, *where* t_0 *is given by* (5.2). *For every* $t_1 \in \mathbb{C}$, *the Hamiltonian system of* H_0 *has a solution with a movable pole or a movable algebraic branch point at* $t = t_1$. *There exist a neighborhood* Ω_1 *of* $(q_1, p_1, r, v) = (0, p_1^{(0)}, 0, 0)$ *and the diffeomorphism* ϕ_0 *defined on* Ω_1 *and a singular solution* y_0 *of the Hamiltonian system* $H_0 + p_1$ *such that* y_0 *has a movable pole or an algebraic branch point at* t_0 *in a neighborhood of* $q_1 = 0$ *and* $\phi_0(y_0)$ *is the solution of the Hamiltonian system of* $H + p_1$ *having a movable pole or a movable algebraic branch point at* t_2 *for some* t_2 *in a neighborhood of* $q_1 = 0$.

§6. Example

Consider the semi-linear wave equation

$$U_{tt} - \Delta U - U^\ell = 0, \quad U = U(x,t), \quad x \in \mathbb{R}^n, \tag{6.1}$$

where $U_{tt} = \partial^2 U/\partial t^2$, $\Delta U = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2)U$ and $\ell \geq 2$ is an integer. Consider the self-similar radially symmetric solution

$$U := t^{-\alpha} u(rt^{-1}), \quad \alpha = 2/(\ell - 1), \tag{6.2}$$

where $r^2 = x_1^2 + \cdots + x_n^2$ and $u = u(\rho)$, $\rho = rt^{-1}$. By (6.1), $u = u(\rho)$ satisfies the semi-linear Heun equation

$$(1-\rho^2)u'' + ((n-1)\rho^{-1} + a\rho)u' + bu + u^\ell = 0, \tag{6.3}$$

for some constants a and b, where $u' = du/d\rho$ and so on.

By simple calculations (cf. [6]) we reduce (6.3) to the Hamiltonian system with Hamiltonian

$$H(q_1, q_2, p_2) \coloneqq \frac{1}{2}(p_2^2 + A(q_1)q_2^2) + B(q_1)q_2^{\ell+1}, \tag{6.4}$$

for some rational functions $A(q_1)$ and $B(q_1)$, where $B(q_1) \neq 0$ and A is given by

$$A(y) = \frac{1}{2}\Big(-\tilde{A}' - \frac{\tilde{A}^2}{2} - \frac{\ell+1}{(\ell-1)^2}\frac{2}{1-y^2}\Big), \tag{6.5}$$

$$\tilde{A}(y) = \frac{1}{1-y^2}\Big(\frac{n-1}{y} - \frac{2(\ell+1)y}{\ell-1}\Big). \tag{6.6}$$

We have the following theorem:

Theorem 6.1. *Suppose $\ell = 2$. There exist $z_1 \in \Omega_1$ and a solution U of (6.1) such that U is singular on the cone $z_1 t = r$, $r^2 = x_1^2 + \cdots + x_n^2$.*

Proof. We apply Theorem 2.1 with $j = 1$, $i = 2$, $\ell = 2$ to (6.4). We show $H = H_0 + H_1$ for some H_0 and H_1 given in Theorem 2.1. Let t_0 be such that $A(t_0) \neq 0$. We have

$$\begin{aligned} H &= 2^{-1}(p_2^2 + A(t_0)q_2^2) + B(t_0)q_2^3 + 2^{-1}(A(q_1) - A(t_0))q_2^2 \\ &\quad + (B(q_1) - B(t_0))q_2^3, \end{aligned} \tag{6.7}$$

where $B(t_0) \neq 0$ by the assumption $B(q_1) \neq 0$. By the linear symplectic transformation, we transform $(p_2^2 + A(t_0)q_2^2)/2$ to $\lambda q_2 p_2$ for some constant $\lambda \neq 0$. Then q_2 and p_2 in (6.7) are replaced by $c_1 q_2 + c_2 p_2$ and $\tilde{c}_1 q_2 + \tilde{c}_2 p_2$, respectively, where the nonzero constants c_1, c_2, $\tilde{c}_1$ and $\tilde{c}_2$ satisfy $\tilde{c}_1 c_2 - c_1 \tilde{c}_2 \neq 0$. On the other hand, the term $B(t_0)c_1^2 c_2 q_2^2 p_2$ appears from $B(t_0)(c_1 q_2 + c_2 p_2)^3$. For simplicity we assume $B(t_0)c_1^2 c_2 = 1$. Hence, by (6.7), we can write $H = H_0 + H_1$, where H_0 is given by (2.3). We note

$$H = 2^{-1}(\tilde{c}_1 q_2 + \tilde{c}_2 p_2)^2 + 2^{-1}A(q_1)(c_1 q_2 + c_2 p_2)^2 + B(q_1)(c_1 q_2 + c_2 p_2)^3. \tag{6.8}$$

We construct the solution $u(z)$ of the Hamiltonian system of (6.8) which is singular at $z = z_1$, where z is the time variable. Then, by (6.2), $U(x,t)$ gives the singular solution in the theorem. In fact, by Theorem 5.2, $u(z)$ has a pole at

$z = z_1$. In order to apply Theorem 2.1 with $j = 1$, $i = 2$, $\ell = 2$ to (6.4) we first verify (AS1). Consider (2.7). Set $q_2 = u_3$ and $p_2 = u_4$ in (6.8). Then (2.7) reads

$$\tilde{c}_1(\tilde{c}_1 u_3 + \tilde{c}_2 u_4) + c_1 A(q_1)(c_1 u_3 + c_2 u_4) + 3c_1 B(q_1)(c_1 u_3 + c_2 u_4)^2 = 0, \tag{6.9}$$

$$\tilde{c}_2(\tilde{c}_1 u_3 + \tilde{c}_2 u_4) + c_2 A(q_1)(c_1 u_3 + c_2 u_4) + 3c_2 B(q_1)(c_1 u_3 + c_2 u_4)^2 = 0. \tag{6.10}$$

By the assumption $\tilde{c}_1 c_2 - c_1 \tilde{c}_2 \neq 0$ we have

$$\tilde{c}_1 u_3 + \tilde{c}_2 u_4 = 0. \tag{6.11}$$

Since u_1 and u_2 do not vanish identically, it follows that $c_1 u_3 + c_2 u_4$ does not vanish identically. Hence we obtain

$$A(q_1) + 3B(q_1)(c_1 u_3 + c_2 u_4) = 0. \tag{6.12}$$

Since $B(q_1)$ does not vanish, we can determine u_3 and u_4 from (6.11) and (6.12) as the holomorphic functions of q_1 at $q_1 = z_0$. Choose z_0 and Ω_1 such that $A(z_0) = 0$ and $z_0 \in \overline{\Omega}_1$. This is possible by (6.5). Hence we have (AS1).

Next we show (2.6). Because q_2 and p_2 are sufficiently small, we may consider terms of degree 2 with respect to q_2 and p_2 in (6.5). Hence $\partial^2 H/\partial p_2^2$ is close to the quantity $2(\tilde{c}_2^2 + A(q_1)c_2^2)$. The last quantity does not vanish since $A(q_1)$ is close to $A(z_0) = 0$ and $\tilde{c}_2 \neq 0$. □

Remark 6.2. Consider the semi-linear heat equation

$$u_t = \Delta u + u^2, \tag{6.13}$$

where Δ is an n-dimensional Laplacian. Consider the radially symmetric self-similar solution

$$u = (T-t)^{-1}\phi\Big(\frac{r}{\sqrt{T-t}}\Big), \tag{6.14}$$

where $T > 0$ and $\phi = \phi(s)$ is a function of one variable. By simple computation, ϕ satisfies

$$\phi''(s) + \Big(\frac{s}{2} + \frac{n-1}{s}\Big)\phi'(s) + \phi + \phi^2 = 0. \tag{6.15}$$

Define $C(s) := s/2 + (n-1)/s$ and

$$A(s) = -\frac{C'}{2} - \frac{C^2}{4} + 1, \quad B(s) = \frac{1}{3}\exp\Big(-\frac{1}{2}\int C\,ds\Big). \tag{6.16}$$

Then (6.15) can be written in the Hamiltonian system with Hamiltonian (6.4) with $\ell = 2$. The remaining argument is similar.

Appendix. Borel summability

Appendix A.1. Proof of Theorem 4.1

Convolution. For $0 < \theta < \pi$, set $\Omega := E(\pi, \theta)$. Let $H(\Omega)$ be the set of holomorphic functions in Ω. For $c > 0$, define $\mathcal{H}_c(\Omega)$ as the set of $h \in H(\Omega)$ such that there exists $K \geq 0$ for which

$$|h(z)| \leq K e^{c|\mathrm{Re}\, z|}(1 + |z|)^{-2} \quad \text{for all } z \in \Omega. \tag{A.1}$$

Here, $\mathcal{H}_c(\Omega)$ is the Banach space with norm

$$\|h\|_{\Omega,c} := \sup_{z \in \Omega} |h(z)|(1 + |z|)^2 e^{-c|\mathrm{Re}\, z|}. \tag{A.2}$$

If there is no fear of confusion we write $\|h\|_c$ instead of $\|h\|_{\Omega,c}$. Define the convolution $f * g$ of $f, g \in \mathcal{H}_c(\Omega)$ as

$$(f * g)(z) := \int_0^z f(z - t)g(t)\, dt = \int_0^z f(t)g(z - t)\, dt. \tag{A.3}$$

If the formal Borel transforms of f and g converge, then we have $\mathcal{B}(fg) = \mathcal{B}(f) * \mathcal{B}(g)$. There exists a constant $K_0 > 0$ such that, for every $f, g \in \mathcal{H}_c(\Omega)$, we have $f * g \in \mathcal{H}_c(\Omega)$ with the estimate $\|f * g\|_{\Omega,c} \leq K_0 \|f\|_{\Omega,c} \|g\|_{\Omega,c}$.

Let D be a domain in $(q_1, v, r) \in \mathbb{C}^3$ and let $\theta > 0$. Set $\Omega := E(0, \theta)$. Let $H(D, \Omega)$ be the set of all holomorphic functions in $(q_1, v, r) \in D$, $y \in \Omega$. For $c > 0$ let $\mathcal{H}_c(D, \Omega)$ denote the set of $f \equiv f(q_1, v, r, y) \in H(D, \Omega)$ such that there exists $K \geq 0$ satisfying

$$\sup_{(q_1, v, r) \in D} |f(q_1, v, r, y)| \leq K e^{c|\mathrm{Re}\, y|}(1 + |y|)^{-2} \quad \text{for all } y \in \Omega. \tag{A.4}$$

Define $u_3 =: w_0 + v_3$, $u_4 =: \widetilde{w}_0 + v_4$, where $v_3 = O(h)$ and $v_4 = O(h)$. Set $v_3 = v$ (resp. $v_4 = \tilde{v}$) and $w = w_0 + v$ (resp. $\widetilde{w} = \widetilde{w}_0 + \tilde{v}$). Let $\hat{v} := \mathcal{B}(v)(q_1, s, r, y)$ be the formal Borel transform of v, where y is the dual variable of h. Substitute $u_3 = w_0 + v_3$ and $u_4 = \widetilde{w}_0 + v_4$ into (3.13) and cancel the constant term, $h^0 = 1$. Divide the equation by h and apply the formal Borel transform to both sides of the equation. Define

$$\widetilde{\mathcal{R}}_3 = \mathcal{R}_3(u_1, u_3, u_4) - \mathcal{R}_3(u_1, w_0, \widetilde{w}_0) - h\Big(\eta \frac{\partial w_0}{\partial \tilde{q}_1} + r^{1-\nu}(1 - \lambda\eta r^\nu)\frac{\partial w_0}{\partial r}\Big).$$

Because the Borel transform maps the multiplication h^{-1} to $\partial/\partial y$, we have

$$\Big(r^{\nu-1}\eta \frac{\partial \hat{v}}{\partial \tilde{q}_1} + (1 - \lambda\eta r^\nu)\frac{\partial \hat{v}}{\partial r}\Big) - r^{\nu-1}\frac{\partial}{\partial y}(\lambda \hat{v} + \widetilde{\mathcal{R}}_3^*(u_1, \hat{u}_3, \hat{u}_4)) = 0. \tag{A.5}$$

Here, $\widetilde{\mathcal{R}}_3^*$ is given by replacing the product in $\widetilde{\mathcal{R}}_3$ with the convolution product. Similarly, if $\widetilde{w} = u_4$, then we have the equation of $\hat{\tilde{v}}$, for some $\widetilde{\mathcal{R}}_4^*$,

$$\text{(A.6)} \qquad \Big(r^{\nu-1}\eta\frac{\partial\hat{\tilde{v}}}{\partial\tilde{q}_1} + (1-\lambda\eta r^\nu)\frac{\partial\hat{\tilde{v}}}{\partial r}\Big) + r^{\nu-1}\frac{\partial}{\partial y}(\lambda\hat{\tilde{v}} + \widetilde{\mathcal{R}}_4^*(u_1, \hat{u}_3, \hat{u}_4)) = 0.$$

There exists a neighborhood $y = 0, W$ such that $\mathcal{B}(v)$ and $\mathcal{B}(\tilde{v})$ are analytic in $(q_1, v, r, y) \in U_0 \times U_2 \times \Omega_0 \times W$ and the unique analytic solution of (A.5) and (A.6). Theorem 4.1 follows from the next theorem:

Theorem A.1. *Let $c > 0$. For every $K_0 > 0$ there exist an $E(0,\tau) =: \Omega$, neighborhoods U_0 of $q_1 = 0$ and U_2 of $v = 0$ and $\rho_0 > 0$ such that (A.5)–(A.6) has a solution $(\hat{u}_3, \hat{u}_4)$ in $\mathcal{H}_c(D,\Omega) \times \mathcal{H}_c(D,\Omega)$, where $D = U_0 \times U_2 \times (\Omega_0 \cap \{r \mid |r| < \rho_0, |\eta r| < K_0\})$.*

Appendix A.2. Preparation of lemmas

Define $\mathcal{R} := (\mathcal{R}_3, \mathcal{R}_4)$. Let $\nabla\mathcal{R}$ be the Jacobi matrix with respect to (u_3, u_4). Denote by $\operatorname{diag}(\lambda, -\lambda)$ the diagonal matrix with diagonal components given, respectively, by λ, $-\lambda$. Let $\lambda(u_1)$ and $\tilde{\lambda}(u_1)$ be the first and second elements of the diagonal part of $\operatorname{diag}(\lambda, -\lambda) + \nabla\mathcal{R}(u_1, w_0(0), \widetilde{w}_0(0))$, respectively. Consider

$$\text{(A.7)} \qquad \mathcal{J}\widehat{w} \equiv r^{\nu-1}\eta\frac{\partial\widehat{w}}{\partial\tilde{q}_1} + (1-\lambda\eta r^\nu)\frac{\partial\widehat{w}}{\partial r} - r^{\nu-1}\lambda(u_1)\frac{\partial\widehat{w}}{\partial y} = r^{\nu-1}\hat{g},$$

where $\hat{g} \in \mathcal{H}_c(D,\Omega)$. Similarly, define $\widetilde{\mathcal{J}}$ by replacing $\lambda(u_1)$ with $\tilde{\lambda}(u_1)$ in (A.7).

In order to solve (A.7) we use the method of characteristics. In the following we consider the case $\tilde{q}_1 = q_1$. The general case where v exists can be treated by modifying the argument slightly. We remark that the existence time span of a characteristic curve can be taken locally uniformly with respect to the initial value. Indeed, the fact follows from the concrete form of the characteristic curve and the definition of the renormalized variable $\tilde{q}_1$. Consider

$$\text{(A.8)} \qquad \frac{dq_1}{r^{\nu-1}\eta} = \frac{dr}{1-\lambda\eta r^\nu} = \frac{dy}{-r^{\nu-1}\lambda(u_1)}.$$

By integrating (A.8) we have

$$\text{(A.9)} \qquad q_1 = q_1^{(0)} - \frac{1}{\lambda\nu}\log(1-\lambda\eta r^\nu),$$

where $q_1^{(0)}$ is a constant. Substitute (A.9) into (A.8) and solve (A.8) with respect to y. We have

$$\text{(A.10)} \quad y = y_0 - \Phi(r), \quad \Phi(r) := \frac{1}{\eta}\int_0^\sigma \lambda(s - q_1^{(0)})\,ds, \quad \sigma = -\frac{1}{\lambda\nu}\log(1-\lambda\eta r^\nu),$$

where $y_0 = y(0) \in \Omega$ is an initial value of $y = y(r)$ at $r = 0$, and y satisfies (A.8). Indeed, we have

$$\frac{dy}{dr} = -\frac{d\sigma}{dr}\frac{d}{d\sigma}\Phi = -\frac{\lambda\eta\nu r^{\nu-1}}{\lambda\nu(1-\lambda\eta r^{\nu})}\frac{\lambda(\sigma - q_1^{(0)})}{\eta}.$$

Equations (A.9) and (A.10) imply $\sigma - q_1^{(0)} = -q_1 - (2/\lambda\nu)\log(1-\lambda\eta r^{\nu}) = u_1(r)$, which proves the assertion. We have the following lemma:

Lemma A.2. *Suppose that r_0 satisfies $1-\lambda\eta r_0^{\nu} \neq 0$. Assume that $\lambda(u_1)$ is analytic at $u_1 = -q_1^{(0)}$. Then there exists a curve γ_{r_0} on $\mathbb{C}$ passing through r_0 such that* $\operatorname{Im}\Phi$ *is constant on γ_{r_0}, where Φ is given by* (A.10).

Proof. Clearly, the integral $\Phi(r)$ in (A.10) is well defined if r is sufficiently small. By continuity $\lambda(\sigma - q_1^{(0)})$ is close to $\lambda(-q_1^{(0)})$ if σ is sufficiently small. Assume that $\lambda(\sigma - q_1^{(0)})$ is a constant function. Set

$$\mu = -\frac{\lambda(-q_1^{(0)})}{\lambda\nu\eta}, \quad \zeta = 1-\lambda\eta r^{\nu}, \quad \zeta_0 = 1-\lambda\eta r_0^{\nu}.$$

Then $\operatorname{Im}\Phi$ is constant on γ_{r_0} if $\operatorname{Im}(\mu\log\zeta) = \operatorname{Im}(\mu\log\zeta_0)$ on γ_{r_0}. Consider the change of variable $\zeta \mapsto \tilde\zeta$, $\tilde\zeta = \mu\log\zeta$. Set $\tilde\zeta = \tilde x + i\tilde y$ and $\tilde\zeta_0 = \tilde x_0 + i\tilde y_0 := \mu\log\zeta_0$. Define the curve $\tilde\gamma_{r_0}$ as $\tilde y = \tilde y(\tilde x) = \tilde y_0$. Map the curve $\tilde\gamma_{r_0}$ in $\tilde\zeta$-space to γ_{r_0} in ζ-space by the transform $\tilde\zeta = \mu\log\zeta$. Clearly, γ_{r_0} passes r_0 and is the desired one.

Next suppose that $\lambda(\sigma - q_1^{(0)})$ is not a constant function. Because $\operatorname{Im}\Phi(r) = \operatorname{Im}\Phi(r_0)$, we have

$$\text{(A.11)} \qquad \operatorname{Im}\left(\eta^{-1}\int_{\sigma_0}^{\sigma}\lambda(s - q_1^{(0)})\,ds\right) = 0, \quad \sigma_0 = -(\lambda\nu)^{-1}\log(1-\lambda\eta r_0^{\nu}).$$

Since $u_1 = \sigma - q_1^{(0)}$, we have

$$\text{(A.12)} \qquad \lambda(s - q_1^{(0)}) = \lambda(\sigma_0 - q_1^{(0)}) + (s-\sigma_0)\tilde\lambda(s-\sigma_0)$$

for some holomorphic function $\tilde\lambda(z)$. By (A.12) and (A.11) we have

$$\text{(A.13)} \qquad \operatorname{Im}(\eta^{-1}\lambda(\sigma_0 - q_1^{(0)})(\sigma-\sigma_0)) + \operatorname{Im}(\eta^{-1}(\sigma-\sigma_0)^2\widetilde{R}(\sigma-\sigma_0)) = 0,$$

where $\widetilde{R}(z)$ is holomorphic at $z = 0$. One sees that $\operatorname{Im}(\eta^{-1}\lambda(-q_1^{(0)})(\sigma-\sigma_0)) = \tilde y - \tilde y_0$. Since $\sigma - \sigma_0$ is a holomorphic function of $\zeta - \zeta_0$ when $\zeta - \zeta_0$ is sufficiently small, the second term on the left-hand side of (A.13) is $O((|\tilde x - \tilde x_0| + |\tilde y - \tilde y_0|)^2)$ when $\tilde x \to \tilde x_0$ and $\tilde y \to \tilde y_0$. By the implicit function theorem we determine $\tilde y \equiv \tilde y(\tilde x)$ locally at $\tilde x_0$, from which we obtain γ_{r_0}. ∎

Lemma A.3. *Assume that $\lambda(u_1)$ is analytic at $u_1 = -q_1^{(0)}$. Suppose that r_0 satisfies $1 - \lambda\eta r_0^\nu \neq 0$. Let γ_{r_0} be given by Lemma A.2, and let $r \in \gamma_{r_0}$. For a given $y \in \Omega$ we set $y_0 := y - \varepsilon_1$, where ε_1 depends on r and is a bounded quantity. Define*

$$P\hat{g} := \int_{r_0}^{r} \hat{g}(u_1(s), s, y_0 - \Phi(s)) \frac{s^{\nu-1}}{1 - \lambda\eta s^\nu} ds, \quad \hat{g} \in \mathcal{H}_c(D, \Omega), \tag{A.14}$$

where the integral is taken along γ_{r_0}. Then $P\hat{g}$ satisfies (A.7). In particular, $P\hat{g}$ is analytic in r in some neighborhood of the origin $r = 0$.

Proof. We prove that (A.14) is well defined. By (A.10) we have $y_0 - \Phi(s) = y + \Phi(r) - \Phi(s)$. By Lemma A.2 we have $\mathrm{Im}(\Phi(r) - \Phi(s)) = 0$ on γ_{r_0}. By the definition of D, $\hat{g}(u_1(s), s, y_0 - \Phi(s))$ in (A.14) is well defined. Since the integrand is continuous for $r - r_0$ sufficiently small, the integral (A.14) converges.

We show that $\widehat{w} := P\hat{g}$ satisfies (A.7). In terms of (A.9) and (A.8) we have

$$\begin{aligned} \hat{g}(u_1(r), r, y_0 - \Phi(r)) \frac{r^{\nu-1}}{1 - \lambda\eta r^\nu} &= \frac{d\widehat{w}}{dr} \\ &= \frac{\partial q_1}{\partial r} \frac{\partial \widehat{w}}{\partial q_1} + \frac{\partial \widehat{w}}{\partial r} + \frac{\partial y}{\partial r} \frac{\partial \widehat{w}}{\partial y} \\ &= \frac{r^{\nu-1}\eta}{1 - \lambda\eta r^\nu} \frac{\partial \widehat{w}}{\partial q_1} + \frac{\partial \widehat{w}}{\partial r} - \frac{r^{\nu-1}\lambda(u_1)}{1 - \lambda\eta r^\nu} \frac{\partial \widehat{w}}{\partial y}. \end{aligned} \tag{A.15}$$

Multiplying both sides by $1 - \lambda\eta r^\nu$, we obtain (A.7). □

Lemma A.4. *Assume that $\lambda(u_1)$ is analytic at $u_1 = -q_1^{(0)}$. Suppose that r_0 satisfies $1 - \lambda\eta r_0^\nu \neq 0$. Let γ_{r_0} be given by Lemma A.2. Let r_1 satisfy $1 - \lambda\eta r_1^\nu = 0$. Assume that there exist a neighborhood W_{r_1} of r_1 and $M_1 > 0$ such that, if $\gamma_{r_0} \subset W_{r_1}$, then $M_1^{-1} \leq |r_1 - s|\,|r_1 - t|^{-1} \leq M_1$ for every $s, t \in \gamma_{r_0}$. Then there exists a constant $c_1 \geq 0$ such that, for every $\hat{g} \in \mathcal{H}(D, \Omega)$ we have*

$$\|P\hat{g}\|_c \leq c_1 \|\hat{g}\|_c, \quad \left\| \frac{\partial}{\partial y}(P\hat{g}) \right\|_c \leq c_1 \|\hat{g}\|_c. \tag{A.16}$$

Proof. We prove the lemma in two steps.

Step 1. We estimate $y_0 - \Phi(s) = y + \Phi(r) - \Phi(s)$, where $r, s \in \gamma_{r_0}$. By (A.10) and the continuity of $\lambda(s - q_1^{(0)})$, there exists $C_0 > 0$ such that

$$|\Phi(r) - \Phi(s)| \leq C_0(|\lambda\nu\eta|)^{-1} \left| \log\left(\frac{1 - \lambda\eta s^\nu}{1 - \lambda\eta r^\nu} \right) \right| =: C_0(|\lambda\nu\eta|)^{-1} A(s, t). \tag{A.17}$$

If γ_{r_0} is outside some neighborhood of r_1 with $1 - \lambda\eta r_1^\nu = 0$, then $A(s, t)$ is bounded. If γ_{r_0} is in a neighborhood of r_1, $1 - \lambda\eta r_1^\nu = 0$, then $1 - \lambda\eta s^\nu = \lambda\eta(r_1^\nu - s^\nu)$ and $1 - \lambda\eta t^\nu = \lambda\eta(r_1^\nu - t^\nu)$. Hence $A(s, t)$ is estimated by $\log(|r_1 - s|\,|r_1 - t|^{-1})$. By assumption, the last term is bounded.

Therefore there exists a constant $K_1 > 0$ independent of η, r and r_0 such that

$$\begin{aligned}\exp(c|\mathrm{Re}(y_0 - \Phi(s))|) &= \exp(c|\mathrm{Re}(y + \Phi(r) - \Phi(s))|) \\ &\le \exp(c|\mathrm{Re}\, y| + c|\mathrm{Re}(\Phi(r) - \Phi(s))|) \\ &\le K_1 \exp(c|\mathrm{Re}\, y|). \end{aligned} \tag{A.18}$$

Similarly, there exists $K_2 > 0$ such that

$$\begin{aligned}(1 + |y_0 - \Phi(s)|)^{-2} &= (1 + |y + \Phi(r) - \Phi(s)|)^{-2} \\ &\le K_2 (1 + |y|)^{-2} \quad \text{for all } y \in \Omega. \end{aligned} \tag{A.19}$$

It follows from (A.18) and (A.19) that

$$\begin{aligned}\|P\hat{g}\|_c &\le \sup\left((1 + |y|)^2 \exp(-c|\mathrm{Re}\, y|) \int \|\hat{g}\|_c \frac{\exp(c|\mathrm{Re}(y_0 - \Phi(s))|)}{(1 + |y_0 - \Phi(s)|)^2} |ds|\right) \\ &\le C_2 \|\hat{g}\|_c \int |ds| \le C_3 \|\hat{g}\|_c \end{aligned} \tag{A.20}$$

for some $C_2 > 0$ and $C_3 > 0$ since the length of the path of the integration $\int |ds|$ is finite.

Step 2. We prove the second inequality in (A.16). Because $\widehat{w} = P\hat{g}$ satisfies (A.7) and $\lambda(u_1) \neq 0$ we have

$$\frac{\partial \widehat{w}}{\partial y} = -\frac{\hat{g}}{\lambda(u_1)} + \frac{\eta}{\lambda(u_1)} \frac{\partial \widehat{w}}{\partial q_1} + \frac{1 - \lambda \eta r^{\nu}}{r^{\nu - 1} \lambda(u_1)} \frac{\partial \widehat{w}}{\partial r}. \tag{A.21}$$

The first term on the right-hand side of (A.21) is bounded by a constant times $\|\hat{g}\|_c$. Next consider the third term. By differentiating (A.14) with respect to r we have

$$\begin{aligned}\frac{1 - \lambda \eta r^{\nu}}{r^{\nu - 1} \lambda(u_1)} \frac{\partial \widehat{w}}{\partial r} &= \frac{1 - \lambda \eta r^{\nu}}{r^{\nu - 1} \lambda(u_1)} \hat{g}(u_1(r), r, y_0 - \Phi(r)) \frac{r^{\nu - 1}}{1 - \lambda \eta r^{\nu}} \\ &= \frac{\hat{g}(u_1(r), r, y_0 - \Phi(r))}{\lambda(u_1)}. \end{aligned} \tag{A.22}$$

The last term is estimated by $\|\hat{g}\|_c$ by (A.18) and (A.19).

We estimate the second term in (A.21). Consider $\partial \widehat{w} / \partial q_1$. Since $\widehat{w}$ is analytic in q_1 we consider the directional derivative with respect to q_1 with direction given by (A.8) where r moves in the direction of γ_{r_0}. One may assume $q_1^{(0)}$ is constant since $q_1^{(0)}$ moves in the transversal direction to γ_{r_0}. By (A.8) we have

$$\frac{\partial \widehat{w}}{\partial q_1} = \frac{dr}{dq_1} \frac{\partial \widehat{w}}{\partial r} = \frac{1 - \lambda \eta r^{\nu}}{r^{\nu - 1} \eta} \frac{\partial \widehat{w}}{\partial r}, \tag{A.23}$$

which is estimated by $\|\hat{g}\|_c$ by (A.22). □

Appendix A.3. Construction of an approximate sequence

Let $\mathcal{J}$ be given by (A.7). We solve (A.5)–(A.6) or (3.13)–(3.14). Set $V = (v, \tilde{v}) = (w - w_0, \widetilde{w} - \widetilde{w}_0)$. Expand $\mathcal{R}_3(u_1, w, \widetilde{w})$ and $\mathcal{R}_4(u_1, w, \widetilde{w})$ in the power series at $(w, \widetilde{w}) = (w_0, \widetilde{w}_0)$:

$$\begin{aligned}\mathcal{R}_3(u_1, w, \widetilde{w}) &= \mathcal{R}_3(u_1, w_0, \widetilde{w}_0) + \nabla\mathcal{R}_3(u_1, w_0, \widetilde{w}_0)^{\mathsf{T}}(v, \tilde{v}) \\ &\quad + \sum_{|\beta|\geq 2} r_\beta(u_1, w_0, \widetilde{w}_0)V^\beta,\end{aligned} \tag{A.24}$$

$$\begin{aligned}\mathcal{R}_4(u_1, w, \widetilde{w}) &= \mathcal{R}_4(u_1, w_0, \widetilde{w}_0) + \nabla\mathcal{R}_4(u_1, w_0, \widetilde{w}_0)^{\mathsf{T}}(v, \tilde{v}) \\ &\quad + \sum_{|\beta|\geq 2} \tilde{r}_\beta(u_1, w_0, \widetilde{w}_0)V^\beta.\end{aligned} \tag{A.25}$$

Substitute (A.24), (A.25) and (4.6) into (3.13)–(3.14). Divide the equation by h and apply the formal Borel transform to both sides. Because the product becomes a convolution product in the Borel plane we have

$$\begin{aligned}\frac{\partial}{\partial y}(\mathcal{R}_3^*(u_1, \widehat{w}, \widehat{\widetilde{w}}) - \mathcal{R}_3(u_1, w_0, \widetilde{w}_0)) &= \nabla\mathcal{R}_3(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V} \\ &\quad + \sum_{|\beta|\geq 2} r_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}^{*\beta},\end{aligned} \tag{A.26}$$

where $\widehat{V} = (\hat{v}, \hat{\tilde{v}})$. Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial y}(\mathcal{R}_4^*(u_1, \widehat{w}, \widehat{\widetilde{w}}) - \mathcal{R}_4(u_1, w_0, \widetilde{w}_0)) &= \nabla\mathcal{R}_4(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V} \\ &\quad + \sum_{|\beta|\geq 2} \tilde{r}_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}^{*\beta}.\end{aligned} \tag{A.27}$$

Therefore, by (4.6), (A.5) (resp. (A.6)) is written in the form

$$\begin{aligned}\mathcal{J}\hat{v} + r^{\nu-1}(\lambda(u_1) - \lambda)\frac{\partial \hat{v}}{\partial y} &= r^{\nu-1}\nabla\mathcal{R}_3(u_1, w_0, \widetilde{w}_0)\frac{\partial \widehat{V}}{\partial y} \\ &\quad + \sum_{|\beta|\geq 2} r^{\nu-1} r_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}^{*\beta},\end{aligned} \tag{A.28}$$

$$\begin{aligned}\widetilde{\mathcal{J}}\hat{\tilde{v}} + r^{\nu-1}(\tilde{\lambda}(u_1) + \lambda)\frac{\partial \hat{\tilde{v}}}{\partial y} &= -r^{\nu-1}\nabla\mathcal{R}_4(u_1, w_0, \widetilde{w}_0)\frac{\partial \widehat{V}}{\partial y} \\ &\quad - \sum_{|\beta|\geq 2} r^{\nu-1}\tilde{r}_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}^{*\beta}.\end{aligned} \tag{A.29}$$

We define the approximate sequences $\hat{v}_k$ and $\hat{\tilde{v}}_k$ $(k = 1, 2, \ldots)$. Divide (3.13) (resp. (3.14)) by h. Compare the terms of constant part $h^0 = 1$, $\hat{v}_1$ of v (resp. $\hat{\tilde{v}}_1$):

$\hat{v}_1$ satisfies $\mathcal{J}(\hat{v}_1 + w_0) = 0$ (resp. $\widetilde{\mathcal{J}}(\hat{\tilde{v}}_1 + \widetilde{w}_0) = 0$). Define $\hat{v}_k$ as

$$\hat{v}_1 = -P\mathcal{J}w_0, \tag{A.30}$$

$$\hat{v}_{k+1} = P\Big(r^{\nu-1}\big(\nabla\mathcal{R}_3(u_1, w_0, \widetilde{w}_0) - (\lambda(u_1) - \lambda, 0)\big)\frac{\partial\widehat{V}_k}{\partial y}\Big) + P\Big(\sum_{|\beta|\geq 2} r^{\nu-1} r_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}_k^{*\beta}\Big), \quad k = 1, 2, \ldots. \tag{A.31}$$

Next we define $\widetilde{P}$ which corresponds to $\widetilde{\mathcal{J}}$ by a similar formula to P. Define $\hat{\tilde{v}}_k$ as

$$\hat{\tilde{v}}_1 = -\widetilde{P}\widetilde{\mathcal{J}}\widetilde{w}_0, \tag{A.32}$$

$$\hat{\tilde{v}}_{k+1} = -\widetilde{P}\Big(r^{\nu-1}\big(\nabla\mathcal{R}_4(u_1, w_0, \widetilde{w}_0) + (0, \tilde{\lambda}(u_1) + \lambda)\big)\frac{\partial\widehat{V}_k}{\partial y}\Big) - \widetilde{P}\Big(\sum_{|\beta|\geq 2} r^{\nu-1}\tilde{r}_\beta(u_1, w_0, \widetilde{w}_0)\frac{\partial}{\partial y}\widehat{V}_k^{*\beta}\Big), \quad k = 1, 2, \ldots. \tag{A.33}$$

Note that Lemma A.4 holds for $\widetilde{P}$.

If the limits $\lim_k \hat{v}_k =: \hat{v}$ and $\lim_k \hat{\tilde{v}}_k =: \hat{\tilde{v}}$ exist, then $(v + w_0, \tilde{v} + \widetilde{w}_0)$ is a solution of (3.13)–(3.14). Indeed, recall that $\mathcal{J}P = \mathrm{Id}$ and $\widetilde{\mathcal{J}}\widetilde{P} = \mathrm{Id}$. Letting $k \to \infty$ in (A.31) and (A.33) we verify that $(\hat{v}, \hat{\tilde{v}})$ satisfies (A.28)–(A.29). The definition of the Borel transform implies $(v + w_0, \tilde{v} + \widetilde{w}_0)$ is the solution of (3.13)–(3.14). In order to show the existence of the limits $\lim_k \hat{v}_k$ and $\lim_k \hat{\tilde{v}}_k$ we first show the a priori estimate:

Lemma A.5. *There exist $\varepsilon_1 > 0$ and K_1 independent of k such that, for every $0 < \varepsilon < \varepsilon_1$, there exist a neighborhood U_0 of $q_1 = q_1^{(0)}$ and $r_0 > 0$ such that*

$$\|\hat{v}_k\|_c \leq \varepsilon K_1, \quad \|\hat{\tilde{v}}_k\|_c \leq \varepsilon K_1, \quad \|(\hat{v}_k)_y\|_c \leq \varepsilon K_1, \quad \|(\hat{\tilde{v}}_k)_y\|_c \leq \varepsilon K_1. \\ q_1 \in U_0, \quad r \in \Omega_0 \cap \big\{r \mid |r| < r_0,\ |\eta r| < K_0\big\}, \quad k = 1, 2, \ldots. \tag{A.34}$$

Proof. We prove by induction. Because P and $\partial_y P$ are bounded by Lemma A.4, $\|\hat{v}_1\|_c$ and $\|(\hat{v}_1)_y\|_c$ are estimated by $K\|\mathcal{J}w_0\|_c$ for some $K > 0$. Estimate $\|\mathcal{J}w_0\|_c$: Because w_0 is a function of $u_1 = -q_1 - 2(\lambda\nu)^{-1}\log(1 - \lambda\eta r^\nu)$, we have

$$\mathcal{J}w_0 = r^\nu\eta\frac{\partial w_0}{\partial q_1} + (1 - \lambda\eta r^\nu)r\frac{\partial w_0}{\partial r} = r^\nu\eta(-w_0' + 2w_0') = r^\nu\eta w_0'.$$

Choose the neighborhood of $r = 0$ sufficiently small. Then $r^\nu\eta w_0'$ is bounded by $K_3\varepsilon$ for some $K_3 > 0$. The terms $\|\hat{\tilde{v}}_1\|_c$ and $\|(\hat{\tilde{v}}_1)_y\|_c$ are estimated similarly.

Suppose that (A.34) is valid up to some k and consider $\|\hat{v}_{k+1}\|_c$ and $\|(\hat{v}_{k+1})_y\|_c$. Consider $\|\hat{v}_{k+1}\|_c$. We estimate the right-hand side of (A.31). We show that

$$A_0 := r^{\nu-1}\big(\nabla\mathcal{R}_3(u_1, w_0, \widetilde{w}_0) - (\lambda(u_1) - \lambda, 0)\big)$$

is sufficiently small. Indeed, if we replace r with $\varepsilon_0 r$ for sufficiently small ε_0 in (A.5) or (A.6), then we see that ε_0^ν appears in front of the nonlinear term. There is no change in the argument.

By the inductive assumption and the boundedness of P we estimate the first term. As for the second term, by the smallness of $r^{\nu-1}r_\beta(u_1, w_0, \widetilde{w}_0)$ and the estimate of the convolution, together with the assumption, it is bounded by a constant times ε, since $|\beta| \geq 2$.

Next we study $\|(\hat{v}_{k+1})_y\|_c$. Consider the second term on the right-hand side of (A.31). Because $\partial_y P$ is a bounded continuous operator by Lemma A.4, it is estimated by a constant times $\frac{\partial}{\partial y}\widehat{V}_k^{*\beta}$. By the property of convolution and $|\beta| \geq 2$ we estimate $\frac{\partial}{\partial y}\widehat{V}_k^{*\beta}$. Next, consider the first term in (A.31). By Lemma A.4, the y-derivative of the first term is estimated by $\frac{\partial}{\partial y}\widehat{V}_k$, which is estimated by the inductive assumption. The estimates of $\|\hat{\tilde{v}}_{k+1}\|_c$ and $\|(\hat{\tilde{v}}_{k+1})_y\|_c$ are similarly done, using (A.33) instead of (A.31). □

Once we have the a priori estimate, the proof of the convergence of the sequence is almost identical to the proof of [5, Lem. 5.7]. Indeed, the smallness of ε implies that the approximate sequence is a Cauchy sequence.

Remark A.6. The proof of Theorem A.1 and the definition of the Laplace transform yield $h_0 = 1/c$ with $c > 0$ given by Theorem A.1. We show we can take $c > 0$ small. Indeed, consider V_1 defined by (A.30) and (A.32). Since the components of V_1 are given by $r^{\nu-1}\eta w_0'$ and $r^{\nu-1}\eta\widetilde{w}_0'$, we can make V_1 arbitrarily small if we take a sufficiently small neighborhood of $r = 0$ and $\nu \geq 2$. Next, define V_2 by (A.31) and (A.33). One easily sees that V_2 has an estimate like $e^{c|y|}$ for a given small $c > 0$. By induction we see that V_k has the same estimate and the limit has growth order like $e^{c|y|}$.

Proof of Remark 3.7. We prove the case $\tilde{q}_1 = q_1$. The general case is proved by a slight modification, taking into account the variable v. Divide (3.13) by $r^{\nu-1}$. Consider the characteristic equation corresponding to the resultant equation

$$\frac{dq_1}{1} = \frac{\eta r^{\nu-1}\,dr}{1 - \lambda\eta r^\nu}. \tag{A.35}$$

Let t be the independent variable of the solution of (A.35). Then we have

$$\frac{du_3}{dt} = \frac{dq_1}{dt}\frac{\partial u_3}{\partial q_1} + \frac{dr}{dt}\frac{\partial u_3}{\partial r} = \frac{\partial u_3}{\partial q_1} + \frac{1 - \lambda\eta r^\nu}{\eta r^{\nu-1}}\frac{\partial u_3}{\partial r}. \tag{A.36}$$

Multiply (3.13) by $r^{1-\nu}$ and substitute (A.36). Make the same calculation as for u_4 with (3.13) replaced by (3.14). The reduced system of equations is the Hamiltonian system with Hamiltonian $\lambda\tilde{q}_2\tilde{p}_2 + f$. Let (u_3, u_4) be the solution with the initial value $(u_3, u_4) = (\xi, \zeta)$ at $t = 0$. By the Hamilton–Jacobi theory, the map $(\xi, \zeta) \mapsto (u_3, u_4)$ is symplectic if the nondegeneracy condition $(\partial u_3/\partial\zeta)(t, \xi, \zeta) \neq 0$ holds. Changing the coordinate from (t, ξ, ζ) to (q_1, q_2, p_2), u_3 and u_4 give the transformation in Lemma 3.1.

By (2.6) we take $u_1 \in \widetilde{\Omega}_1$, $\xi \in \widetilde{\Omega}_2$ and $\zeta \in \widetilde{\Omega}_3$ such that $(\partial R_3/\partial u_4)(u_1, \xi, \zeta) \neq 0$. Indeed, we have $\partial R_3/\partial u_4 = \partial^2 H/\partial p_2^2$. Then we verify $(\partial u_3/\partial\zeta)(t, \xi, \zeta) \neq 0$. Since (u_3, u_4) is the solution of a Hamiltonian system, it holds that $u_3 = \xi + \tilde{u}_3$, $\tilde{u}_3 = O(t)$ and $u_4 = \zeta + \tilde{u}_4$, $\tilde{u}_4 = O(t)$ as $t \to 0$. Here ξ (resp. ζ) is the initial value of u_3 (resp. u_4). On the other hand, since they are equal to the Borel sum (cf. Theorems 3.4 and 4.1) we have

$$u_3 = u_3^{(0)} + v_3, \quad v_3 = O(h), \quad u_4 = u_4^{(0)} + v_4, \quad v_4 = O(h), \tag{A.37}$$

for h, (3.12). Then $u_3^{(0)}$ and $u_4^{(0)}$ are analytic solutions of the equation (cf. (4.6))

$$\lambda u_3^{(0)} + \mathcal{R}_3(u_1, u_3^{(0)}, u_4^{(0)}) = 0. \tag{A.38}$$

Differentiating (A.38) with respect to ζ we have

$$\begin{aligned}\lambda\frac{\partial u_3^{(0)}}{\partial\zeta} + \frac{\partial\mathcal{R}_3}{\partial u_3}(u_1, u_3^{(0)}, u_4^{(0)})\frac{\partial u_3^{(0)}}{\partial\zeta} + \frac{\partial\mathcal{R}_3}{\partial u_4}(u_1, u_3^{(0)}, u_4^{(0)})\frac{\partial u_4^{(0)}}{\partial\zeta}\\ + \frac{\partial\mathcal{R}_3}{\partial u_1}(u_1, u_3^{(0)}, u_4^{(0)})\frac{\partial u_1}{\partial\zeta} = 0.\end{aligned} \tag{A.39}$$

We have that $|u_3^{(0)} - \xi|$ and $|u_4^{(0)} - \zeta|$ are arbitrarily small if h and t are sufficiently small. Then we have $(\partial\mathcal{R}_3/\partial u_4)(u_1, u_3^{(0)}, u_4^{(0)}) \neq 0$ by assumption. In the last term on the left-hand side of (A.39) there appears

$$-\frac{2}{\lambda\nu}\frac{-\lambda r^{\nu}(\eta)_\zeta}{1 - \lambda\eta r^{\nu}}. \tag{A.40}$$

Consider $(\eta)_\zeta = (1/h)_\zeta = -h^{-2}h_\zeta$. We have

$$\frac{1}{h}\frac{\partial h}{\partial\zeta} = \frac{1}{v^{j-1}}\frac{\partial}{\partial\zeta}v^{j-1} = \frac{j-1}{v}\frac{\partial v}{\partial\zeta}.$$

The right-hand side is a bounded term since the small term q_1 in the denominator and the numerator cancels. Since $r^{\nu}\eta$ is small by the assumption, the term (A.40) is small. Since $(\partial\mathcal{R}_3/\partial u_3)(u_1, u_3^{(0)}, u_4^{(0)})$ is small, it follows from (A.39) that $\partial u_3^{(0)}/\partial\zeta \neq 0$.

To show the assertion by (A.37) we prove that $(\partial v_3/\partial\zeta)$ is sufficiently small. Indeed, the smallness follows if $|h|$ and $|(\partial h/\partial\zeta)|$ are sufficiently small. If $j=1$, then we have the assertion by definition. If $j>1$, then by (3.12) we consider the derivative of v. The derivative of v with respect to η is small by the smallness of c_2 in (3.9). □

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