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Some More Fano Threefolds with a Multiplicative Chow–Künneth Decomposition

by

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Abstract

We exhibit several families of Fano threefolds with a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. As a consequence, a certain tautological subring of the Chow ring of powers of these threefolds injects into cohomology. As a by-product of the argument, we observe that double covers of projective spaces admit a multiplicative Chow–Künneth decomposition.

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§1. Introduction

Given a smooth projective variety Y over $\mathbb{C}$, let $A^i(Y) := CH^i(Y)_{\mathbb{Q}}$ denote the Chow groups of Y (i.e. the groups of codimension i algebraic cycles on Y with $\mathbb{Q}$-coefficients, modulo rational equivalence). The intersection product defines a ring structure on $A^*(Y) = \bigoplus_i A^i(Y)$, the *Chow ring* of Y [14].

In the special case of K3 surfaces, this ring structure has remarkable properties:

Theorem 1.1 (Beauville–Voisin [3]). *Let S be a projective K3 surface. The $\mathbb{Q}$-subalgebra*

$$\langle A^1(S), c_j(S)\rangle \subset A^*(S)$$

injects into cohomology under the cycle class map.

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Theorem 1.2 (Voisin [47], Yin [49]). *Let S be a projective K3 surface, and $m \in \mathbb{N}$. The $\mathbb{Q}$-subalgebra*

$$R^*(S^m) := \langle A^1(S), \Delta_S \rangle \quad \subset A^*(S^m)$$

(generated by pullbacks of divisors and pullbacks of the diagonal $\Delta_S \subset S \times S$) injects into cohomology under the cycle class map for all $m \leq 2 \dim H^2_{\mathrm{tr}}(S, \mathbb{Q}) + 1$ (where $H^2_{\mathrm{tr}}(S, \mathbb{Q})$ denotes the transcendental part of cohomology). Moreover, $R^*(S^m)$ injects into cohomology for all $m \in \mathbb{N}$ if and only if S is Kimura finite-dimensional.

The Chow ring of abelian varieties also has an interesting property: there is a multiplicative splitting, defined by the Fourier transform [1].

Motivated by the particular behavior of K3 surfaces and abelian varieties, Beauville [2] has conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting. In the wake of Beauville's "splitting property conjecture", Shen–Vial [41] have introduced the concept of *multiplicative Chow–Künneth decomposition* (we will abbreviate this to "MCK decomposition"). With the concept of MCK decomposition, it is possible to make concrete sense of this elusive "splitting property conjecture" of Beauville.

It is hard to understand precisely which varieties admit an MCK decomposition. To give an idea of what is known, hyperelliptic curves have an MCK decomposition [41, Exm. 8.16], but the very general curve of genus ≥ 3 does not have an MCK decomposition [12, Exm. 2.3]; K3 surfaces have an MCK decomposition, but certain high-degree surfaces in $\mathbb{P}^3$ do not have an MCK decomposition (cf. the examples given in [37], cf. also Section 2 below).

In this note we will focus on Fano threefolds and ask the following question:

Question 1.3. Let X be a Fano threefold with Picard number 1. Does X admit an MCK decomposition?

The restriction on the Picard number is necessary to rule out a counterexample of Beauville [2, Exms. 9.1.5]. The answer to Question 1.3 is affirmative for cubic threefolds [8, 12], for intersections of 2 quadrics [26], for intersections of a quadric and a cubic [27], and for prime Fano threefolds of genus 8 [25] and of genus 10 [31].

The main result of this paper answers Question 1.3 for several more families of Fano threefolds:

Theorem (= Theorem 4.1). *The following smooth Fano threefolds have a multiplicative Chow–Künneth decomposition:*

- *hypersurfaces of weighted degree* 6 *in weighted projective space* $\mathbb{P}(1^3, 2, 3)$*;*
- *quartic double solids;*
- *sextic double solids;*
- *double covers of a quadric in* $\mathbb{P}^4$ *branched along the intersection with a quartic;*
- *special Gushel–Mukai threefolds.*

In Table 1 (at the end of this paper), we have listed all Fano threefolds of Picard number 1 and what is known about MCK for them.

To prove Theorem 4.1, we provide a general criterion (Proposition 3.3) that may be useful in other situations. For example, using this criterion we also prove the following:

Proposition (= Proposition 3.6). *Let* X *be a smooth projective variety such that* $X \to \mathbb{P}^n$ *is a double cover ramified along a smooth divisor* $D \subset \mathbb{P}^n$ *of degree* $d > n$*. Then* X *admits an MCK decomposition.*

As a consequence of Theorem 4.1, we obtain an injectivity result similar to Theorem 1.2:

Corollary (cf. Theorem 5.1). *Let* Y *be a Fano threefold as in Theorem 4.1, and* $m \in \mathbb{N}$*. Let*

$$R^*(Y^m) := \langle h, \Delta_Y \rangle \quad \subset A^*(Y^m)$$

be the $\mathbb{Q}$*-subalgebra generated by pullbacks of the polarization* $h \in A^1(Y)$ *and pullbacks of the diagonal* $\Delta_Y \in A^3(Y \times Y)$*. The cycle class map induces injections*

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \textit{for all } m \in \mathbb{N}.$$

Conventions. In this paper, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(Y)$ the Chow group of j-dimensional cycles on Y with $\mathbb{Q}$-coefficients; for Y smooth of dimension n the notation $A_j(Y)$ and $A^{n-j}(Y)$ is used interchangeably. The notation $A^j_{\text{hom}}(Y)$ will be used to indicate the subgroup of homologically trivial cycles. For a morphism $f \colon X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f.

The contravariant category of Chow motives (i.e. pure motives with respect to rational equivalence as in [40, 35]) will be denoted $\mathcal{M}_{\text{rat}}$.

§2. MCK decomposition

Definition 2.1 (Murre [33, 34]). Let X be a smooth projective variety of dimension n. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in } A^n(X \times X),$$

such that the π_X^i are mutually orthogonal idempotents and $(\pi_X^i)_* H^*(X, \mathbb{Q}) = H^i(X, \mathbb{Q})$.

(NB. "CK decomposition" is shorthand for "Chow–Künneth decomposition".)

Remark 2.2. Murre has conjectured that any smooth projective variety should have a CK decomposition [33, 34, 17].

Definition 2.3 (Shen–Vial [41]). Let X be a smooth projective variety of dimension n, and let $\Delta_X^{\mathrm{sm}} \in A^{2n}(X \times X \times X)$ denote the class of the small diagonal

$$\Delta_X^{\mathrm{sm}} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.$$

An *MCK decomposition* is defined as a CK decomposition $\{\pi_X^i\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi_X^k \circ \Delta_X^{\mathrm{sm}} \circ (\pi_X^i \times \pi_X^j) = 0 \quad \text{in } A^{2n}(X \times X \times X) \text{ for all } i + j \neq k.$$

(NB. "MCK decomposition" is shorthand for "multiplicative Chow–Künneth decomposition".)

Remark 2.4. The small diagonal (when considered as a correspondence from $X \times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{\mathrm{sm}} : h(X) \otimes h(X) \to h(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

Let us assume X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

By definition, this decomposition is multiplicative if for any i, j the composition

$$h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{\mathrm{sm}}} h(X) \quad \text{in } \mathcal{M}_{\mathrm{rat}}$$

factors through $h^{i+j}(X)$.

If X has an MCK decomposition, then setting

$$A^i_{(j)}(X) := (\pi_X^{2i-j})_* A^i(X),$$

one obtains a bigraded ring structure on the Chow ring: i.e. the intersection product sends $A^i_{(j)}(X) \otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

It is conjectured that for any X with an MCK decomposition, one has

$$A^i_{(j)}(X) \stackrel{??}{=} 0 \ \text{ for } j < 0, \quad A^i_{(0)}(X) \cap A^i_{\mathrm{hom}}(X) \stackrel{??}{=} 0;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition [33, 34].

For more background on the concept of MCK, and for examples of varieties with an MCK decomposition, we refer to [41, Sect. 8], as well as [46, 42, 13, 21, 32, 22, 23, 24, 12, 29, 27, 30, 36].

§3. A general criterion

We develop a general criterion for having an MCK. The criterion hinges on the *Franchetta property* for families of varieties, which is defined as follows:

Definition 3.1. Let $\mathcal{X} \to B$ be a smooth projective morphism, where $\mathcal{X}$, B are smooth quasi-projective varieties, and let us write X_b for the fiber over $b \in B$. We say that $\mathcal{X} \to B$ has the *Franchetta property in codimension* j if the following holds: for every $\Gamma \in A^j(\mathcal{X})$ such that the restriction $\Gamma|_{X_b}$ is homologically trivial for the very general $b \in B$, the restriction $\Gamma|_b$ is zero in $A^j(X_b)$ for all $b \in B$.

We say that $\mathcal{X} \to B$ has the *Franchetta property* if $\mathcal{X} \to B$ has the Franchetta property in codimension j for all j.

This property is studied in [39, 5, 10, 11].

Definition 3.2. Given a family $\mathcal{X} \to B$ as in Definition 3.1, we use the shorthand

$$\mathrm{GDA}^j_B(X_b) := \mathrm{Im}(A^j(\mathcal{X}) \to A^j(X_b)) \quad \subset A^j(X_b)$$

($\mathrm{GDA}^*()$ stands for the "generically defined cycles".)

The Franchetta property for $\mathcal{X} \to B$ means that the generically defined cycles inject into cohomology.

Proposition 3.3. *Let $\mathcal{X} \to B$ be a family of smooth projective varieties of relative dimension n, with fiber X_b. Assume the following:*

(i) *the family $\mathcal{X} \times_B \mathcal{X} \to B$ has the Franchetta property;*
(ii) *there exists a projective quotient variety P (i.e. $P = P'/G$ where P' is smooth projective and $G \subset \mathrm{Aut}(P')$ is a finite cyclic group) with trivial Chow groups (i.e. $A^*_{\mathrm{hom}}(P) = 0$), such that $X_b \to P$ is a double cover with branch locus a smooth ample divisor, for all $b \in B$.*

Then X_b admits an MCK decomposition, for all $b \in B$.

Proof. We have the following Lefschetz-type result in cohomology:

Lemma 3.4. *Let $X_b \to P$ be as in the proposition. Then pullback*

$$H^i(P,\mathbb{Q}) \to H^i(X_b,\mathbb{Q})$$

is an isomorphism for $i < n$, and injective for $i = n$.

Proof. In the case that P is smooth, this is a result of Cornalba [6]. The general case is readily deduced from this: assume $P = P'/G$ where P' is smooth projective and $G \subset \mathrm{Aut}(P')$ is a finite cyclic group, and consider the fiber square

$$\begin{array}{ccc} X'_b & \longrightarrow & X_b \\ \downarrow & & \downarrow \\ P' & \longrightarrow & P. \end{array}$$

Cornalba's result applies to the double cover of the left-hand vertical arrow, and so the pullback

$$H^i(P',\mathbb{Q}) \to H^i(X'_b,\mathbb{Q})$$

is an isomorphism for $i < n$, and injective for $i = n$. The G-action on P' lifts to X'_b, and taking G-invariants we find that

$$H^i(P,\mathbb{Q}) = H^i(P',\mathbb{Q})^G \to H^i(X'_b,\mathbb{Q})^G = H^i(X_b,\mathbb{Q})$$

is an isomorphism for $i < n$, and injective for $i = n$. ☐

Since $H^*(P,\mathbb{Q})$ is algebraic (this is a general fact for any variety with trivial Chow groups, cf. [19]), this implies that also $H^i(X_b,\mathbb{Q})$ is algebraic, for all $i \neq n$. More precisely, for $i \neq n$ odd, one has $H^i(X_b,\mathbb{Q}) = 0$ while for $i < n$ even, one has isomorphisms

$$A^{i/2}(P) \cong H^i(X_b,\mathbb{Q}),$$

induced by pullback. This implies that for $i < n$ the Künneth components $\pi^i_{X_b}$ are algebraic, and generically defined. To define the Künneth components $\pi^i_{X_b}$ explicitly, let $p\colon X_b \to P$ denote the projection morphism, and let π^i_P denote the (unique) CK decomposition of P. One can then define

$$\begin{aligned} \pi^i_{X_b} &:= 1/2\, {}^t\Gamma_p \circ \pi^i_P \circ \Gamma_p && \text{if } i < n, \\ \pi^i_{X_b} &:= \pi^{2n-i}_{X_b} && \text{if } i > n, \\ \pi^{n,\mathrm{fix}}_{X_b} &:= 1/2\, {}^t\Gamma_p \circ \pi^n_P \circ \Gamma_p, && \end{aligned}$$

$$\pi^{n,\mathrm{var}}_{X_b} := \Delta_{X_b} - \sum_{j\neq n} \pi^j_{X_b} - \pi^{n,\mathrm{fix}}_{X_b},$$
$$\pi^n_{X_b} := \pi^{n,\mathrm{fix}}_{X_b} + \pi^{n,\mathrm{var}}_{X_b} \qquad \in A^n(X_b \times X_b).$$

(Note that $\pi^n_{X_b} = 0$ in case n is odd.) The notation is meant to remind the reader that $\pi^{n,\mathrm{fix}}_{X_b}$ and $\pi^{n,\mathrm{var}}_{X_b}$ are projectors on the fixed part, resp. the variable part of cohomology in degree n.

These projectors define a generically defined CK decomposition for each X_b, i.e. all projectors are in $\mathrm{GDA}^n_B(X_b \times X_b)$. This CK decomposition has the property that

$$\begin{aligned} h^j(X_b) &:= (X_b, \pi^j_{X_b}, 0) = \oplus \mathbb{1}(*) \qquad \forall j \neq n,\\ h^{n,\mathrm{fix}}(X_b) &:= (X_b, \pi^{n,\mathrm{fix}}_{X_b}, 0) = \oplus \mathbb{1}(*) \qquad \text{in } \mathcal{M}_{\mathrm{rat}}. \end{aligned} \tag{1}$$

Let us now proceed to verify that this CK decomposition is MCK. What we need to check is the vanishing

$$\pi^k_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^i_{X_b} \times \pi^j_{X_b}) = 0 \quad \text{in } A^{2n}(X_b \times X_b \times X_b) \text{ for all } i + j \neq k.$$

First, let us assume that at least one of the three integers (i, j, k) is different from n, and $i + j \neq k$. In this case, we have

$$\begin{aligned} \pi^k_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^i_{X_b} \times \pi^j_{X_b}) &= ({}^t\pi^i_{X_b} \times {}^t\pi^j_{X_b} \times \pi^k_{X_b})_* \Delta^{\mathrm{sm}}_{X_b}\\ &= (\pi^{2n-i}_{X_b} \times \pi^{2n-j}_{X_b} \times \pi^k_{X_b})_* \Delta^{\mathrm{sm}}_{X_b}\\ &\hookrightarrow \bigoplus A^*(X_b \times X_b). \end{aligned}$$

Here, the first equality is an application of Lieberman's lemma [35, Lem. 2.1.3], and the inclusion follows from property (1). The resulting cycle in $\bigoplus A^*(X_b \times X_b)$ is generically defined (since the $\pi^*_{X_b}$ and $\Delta^{\mathrm{sm}}_{X_b}$ are) and homologically trivial (since $i + j \neq k$). By assumption (i), the resulting cycle in $\bigoplus A^*(X_b \times X_b)$ is rationally trivial, and so

$$\pi^k_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^i_{X_b} \times \pi^j_{X_b}) = 0 \quad \text{in } A^{2n}(X_b \times X_b \times X_b),$$

as desired.

It remains to treat the case $i = j = k = n$. The decomposition $\pi^n_{X_b} := \pi^{n,\mathrm{fix}}_{X_b} + \pi^{n,\mathrm{var}}_{X_b}$ induces a decomposition

$$\begin{aligned} &\pi^n_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^n_{X_b} \times \pi^n_{X_b})\\ &\quad = \pi^{n,\mathrm{fix}}_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^{n,\mathrm{fix}}_{X_b} \times \pi^{n,\mathrm{fix}}_{X_b})\\ &\qquad + \pi^{n,\mathrm{fix}}_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^{n,\mathrm{fix}}_{X_b} \times \pi^{n,\mathrm{var}}_{X_b})\\ &\qquad + \cdots\\ &\qquad + \pi^{n,\mathrm{var}}_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^{n,\mathrm{var}}_{X_b} \times \pi^{n,\mathrm{var}}_{X_b}) \quad \text{in } A^{2n}(X_b \times X_b \times X_b). \end{aligned}$$

Using property (1) and the Franchetta property for $X_b \times X_b$, all summands containing $\pi^{n,\mathrm{fix}}_{X_b}$ vanish. One is left with the last term. To deal with the last term, we observe that the covering involution $\iota \in \mathrm{Aut}(X_b)$ of the double cover $p\colon X_b \to P$ induces a splitting of the motive

$$\begin{aligned} h(X_b) &= h(X_b)^+ \oplus h(X_b)^- \\ &\coloneqq (X_b, 1/2(\Delta_{X_b} + \Gamma_\iota), 0) \oplus (X_b, 1/2(\Delta_{X_b} - \Gamma_\iota), 0) \quad \text{in } \mathcal{M}_{\mathrm{rat}}, \end{aligned}$$

where Γ_ι denotes the graph of the involution ι. Moreover, there is the equality

$$h^{n,\mathrm{var}}(X_b) = h(X_b)^- \quad \text{in } \mathcal{M}_{\mathrm{rat}}.$$

But the intersection product map

$$h(X_b)^- \otimes h(X_b)^- \xrightarrow{\Delta^{\mathrm{sm}}_{X_b}} h(X_b)$$

factors over $h(X_b)^+$, as is readily seen (cf. Lemma 3.5 below), which is saying exactly that

$$\pi^{n,\mathrm{var}}_{X_b} \circ \Delta^{\mathrm{sm}}_{X_b} \circ (\pi^{n,\mathrm{var}}_{X_b} \times \pi^{n,\mathrm{var}}_{X_b}) = 0 \quad \text{in } A^{2n}(X_b \times X_b \times X_b).$$

This closes the proof, modulo the following lemma (which is probably well known, but we include a proof for completeness):

Lemma 3.5. *Let $X \to P$ be a double cover, where X and P are quotient varieties, and let $\iota \in \mathrm{Aut}(X)$ be the covering involution. Let*

$$h(X)^+ \coloneqq (X, 1/2(\Delta_X + \Gamma_\iota), 0), \quad h(X)^- \coloneqq (X, 1/2(\Delta_X - \Gamma_\iota), 0) \quad \textit{in } \mathcal{M}_{\mathrm{rat}}.$$

The map of motives

$$h(X)^- \otimes h(X)^- \xrightarrow{\Delta^{\mathrm{sm}}_{X}} h(X)$$

factors over $h(X)^+$.

To prove the lemma, let $\iota \in \mathrm{Aut}(X)$ denote the covering involution. The motive $h(X)^-$ is defined by the projector

$$\Delta^-_X \coloneqq 1/2(\Delta_X - \Gamma_\iota) \quad \in A^n(X \times X).$$

Plugging this in and developing, it follows that

$$\begin{aligned} &\Delta^-_X \circ \Delta^{\mathrm{sm}}_X \circ (\Delta^-_X \times \Delta^-_X) \\ &\quad = 1/8(\Delta_X - \Gamma_\iota) \circ \Delta^{\mathrm{sm}}_X \circ (\Delta_{X\times X} - \Delta_X \times \Gamma_\iota - \Gamma_\iota \times \Delta_X + \Gamma_\iota \times \Gamma_\iota) \\ &\quad = 1/8(\Delta_X \circ \Delta^{\mathrm{sm}}_X \circ (\Delta_X \times \Delta_X) + \cdots - \Gamma_\iota \circ \Delta^{\mathrm{sm}}_X \circ (\Gamma_\iota \times \Gamma_\iota)) \end{aligned}$$

$$= 1/8(\Delta_X^{\rm sm} - (\mathrm{id}\times\mathrm{id}\times\iota)_*(\Delta_X^{\rm sm}) - (\mathrm{id}\times\iota\times\mathrm{id})_*(\Delta_X^{\rm sm}) - (\iota\times\mathrm{id}\times\mathrm{id})_*(\Delta_X^{\rm sm}) + (\mathrm{id}\times\iota\times\iota)_*(\Delta_X^{\rm sm}) + (\iota\times\mathrm{id}\times\iota)_*(\Delta_X^{\rm sm}) + (\iota\times\iota\times\mathrm{id})_*(\Delta_X^{\rm sm}) - (\iota\times\iota\times\iota)_*(\Delta_X^{\rm sm})) \quad \text{in } A^{2n}(X\times X\times X).$$

Here, the last equality is by virtue of Lieberman's lemma [35, Lem. 2.1.3]. However, we have the equality

$$\Delta_X^{\rm sm} = \{(x,x,x) \mid x \in X\} = (\iota\times\iota\times\iota)_*(\Delta_X^{\rm sm}) \quad \text{in } A^{2n}(X\times X\times X),$$

and so the sum of the first and last summands vanishes. Likewise, we have the equality

$$(\mathrm{id}\times\iota\times\iota)_*(\Delta_X^{\rm sm}) = (\mathrm{id}\times\iota\times\iota)_*(\iota\times\iota\times\iota)_*(\Delta_X^{\rm sm}) = (\iota\times\mathrm{id}\times\mathrm{id})_*(\Delta_X^{\rm sm}) \quad \text{in } A^{2n}(X\times X\times X),$$

and so the other summands cancel each other pairwise. This proves the lemma. □

As a first application of our general criterion, we now proceed to show the following:

Proposition 3.6. *Let X be a smooth projective variety such that $X \to \mathbb{P}^n$ is a double cover ramified along a smooth divisor $D \subset \mathbb{P}^n$, and assume that either* $\dim H^n(X,\mathbb{Q}) > 1$*, or D has degree $d > n$. Then X admits an MCK decomposition.*

Proof. Double covers X as in the proposition are exactly the smooth hypersurfaces of degree $2d$ in the weighted projective space $\mathbb{P} := \mathbb{P}(1^{n+1}, d)$, where $2d := \deg D$. Let

$$B \subset \overline{B} := \mathbb{P}H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2d))$$

denote the Zariski open parametrizing smooth hypersurfaces, and let

$$B \times \mathbb{P} \supset \mathcal{X} \to B$$

denote the universal family. In view of Proposition 3.3, it suffices to check that the family $\mathcal{X} \times_B \mathcal{X} \to B$ has the Franchetta property.

To this end, we remark that the line bundle $\mathcal{O}_{\mathbb{P}}(2d)$ is very ample (cf. Lemma 3.7 below), which means that the setup verifies condition $(*_2)$ of [12, Def. 2.5]. An application of the stratified projective bundle argument [12, Prop. 2.6] then implies that

$$\mathrm{GDA}^*_B(X_b \times X_b) = \langle (p_i)^*(h), \Delta_{X_b} \rangle, \tag{2}$$

where we write $h \in A^1(X_b)$ for the hyperplane class. The excess intersection formula [14, Thm. 6.3] gives an equality

$$\Delta_{X_b} \cdot (p_i)^*(h) = 2d \sum_j (p_1)^*(h^j) \cdot (p_2)^*(h^{n+1-j}) \quad \text{in } A^{n+1}(X_b \times X_b),$$

and so equality (2) reduces to the equality

$$\mathrm{GDA}^*_B(X_b \times X_b) = \langle (p_1)^*(h), (p_2)^*(h) \rangle \oplus \mathbb{Q}[\Delta_{X_b}].$$

The "decomposable part" $\langle (p_1)^*(h), (p_2)^*(h) \rangle$ injects into cohomology, because of the Künneth formula for $H^*(X_b \times X_b, \mathbb{Q})$. The class of the diagonal in cohomology is linearly independent of the decomposable part: indeed, if the diagonal were decomposable it would act as zero on the primitive cohomology

$$H^n_{\mathrm{prim}}(X_b, \mathbb{Q}) := \mathrm{Coker}(H^n(\mathbb{P}^n, \mathbb{Q}) \to H^n(X_b, \mathbb{Q})).$$

But the assumption $\dim H^n(X_b, \mathbb{Q}) > 1$ is equivalent to having $H^n_{\mathrm{prim}}(X_b, \mathbb{Q}) \neq 0$. This proves the Franchetta property for $\mathcal{X} \times_B \mathcal{X} \to B$, and closes the proof.

The case $d > n$ is a special case where $H^n_{\mathrm{prim}}(X_b, \mathbb{Q}) \neq 0$, because it is known that the geometric genus of X_b is [9, Sect. 3.5.4]

$$p_g(X_b) = \binom{d-1}{n}.$$

It remains to prove the following, which we have used above:

Lemma 3.7. *Let $\mathbb{P} := \mathbb{P}(1^{n+1}, d)$. The sheaf $\mathcal{O}_{\mathbb{P}}(d)$ is locally free and very ample.*

The assertion about the sheaf being locally free is just because d is a multiple of the weights of $\mathbb{P}$ (cf. [7, Rem. 1.8]). As for the very ampleness, we apply Delorme's criterion [7, Prop. 2.3(iii)] (cf. also [4, Thm. 4.B.7]). To prove very ampleness of $\mathcal{O}_{\mathbb{P}}(d)$, we need to prove that the integer E as defined in [7] and [4] is equal to 0.

Let us write $x_0, \ldots, x_n, y$ for the weighted homogeneous coefficients of $\mathbb{P}$, where x_j and y have weight 1, resp. d. It is readily seen that every monomial in x_j, y of (weighted) degree $m + dk$ (where m is a positive multiple of d, and k is any positive integer) is divisible by a monomial of (weighted) degree dk. This means that the integer E defined in [7, 4] is 0, and so [7, Prop. 2.3(iii)] implies the very ampleness of $\mathcal{O}_{\mathbb{P}}(d)$.

This proves the lemma, and ends the proof of the proposition. □

Here is another sample application of our general criterion:

Proposition 3.8. *Let $X \subset \mathbb{P}(1^n, 2, 3)$ be a smooth hypersurface of (weighted) degree 6. Assume $\dim H^n(X, \mathbb{Q}) > 1$. Then X has an MCK decomposition.*

Proof. The varieties X as in the proposition are exactly the smooth double covers of $\mathbb{P} := \mathbb{P}(1^n, 2)$ branched along a (weighted) degree 6 divisor (cf. [20, Rem. 2.3] and for $n = 3$ also [15, Thm. 4.2]). Let $\mathcal{X} \to B$ denote the family of such double covers. We are going to check that the family $\mathcal{X} \times_B \mathcal{X} \to B$ has the Franchetta property. Proposition 3.8 is then a special case of our general criterion Proposition 3.3.

Let $\overline{\mathcal{X}} \to \overline{B} \cong \mathbb{P}^r$ denote the universal family of all (possibly singular) hypersurfaces of weighted degree 6 in $\mathbb{P}$. The line bundle $\mathcal{O}_{\mathbb{P}}(6)$ is very ample (cf. Lemma 3.9 below), and so the projection

$$\overline{\mathcal{X}} \times_{\overline{B}} \overline{\mathcal{X}} \to \mathbb{P} \times \mathbb{P}$$

has the structure of a stratified projective bundle (with strata the diagonal $\Delta_{\mathbb{P}}$ and its complement). One can thus use the stratified projective bundle argument [12, Prop. 2.6] to deduce the identity

$$\begin{aligned} \mathrm{GDA}^*_B(X \times X) &= \langle (p_i)^* \, \mathrm{GDA}^*_B(X), \Delta_X \rangle \\ &= \langle (p_i)^*(h), \Delta_X \rangle \end{aligned}$$

(here, $h \in A^1(X)$ denotes the restriction to X of an ample generator of $A^1(\mathbb{P}) \cong \mathbb{Q}$).

Since $X \subset \mathbb{P}$ is a hypersurface, the excess intersection formula gives

$$\Delta_X \cdot (p_i)^*(h) = \Delta_{\mathbb{P}}|_X \quad \in \langle (p_i)^*(h) \rangle.$$

The above identification thus simplifies to

$$\mathrm{GDA}^*_B(X \times X) = \langle (p_i)^*(h) \rangle \oplus \mathbb{Q}[\Delta_X].$$

The assumption that $\dim H^n(X, \mathbb{Q}) > 1$ implies that the diagonal Δ_X is linearly independent in cohomology from the decomposable classes $\langle (p_i)^*(h) \rangle$ (indeed, the decomposable classes act as zero on the primitive cohomology of X, while the diagonal acts as the identity). This shows that $\mathrm{GDA}^*_B(X \times X)$ injects into cohomology, as requested.

Lemma 3.9. *Let* $\mathbb{P} := \mathbb{P}(1^n, 2, 3)$. *The sheaf* $\mathcal{O}_{\mathbb{P}}(6)$ *is (locally free and) very ample.*

The assertion about the sheaf being locally free is just because 6 is a multiple of all the weights (cf. [7, Rem. 1.8]). As for the very ampleness, we apply Delorme's criterion [7, Prop. 2.3(iii)] (cf. also [4, Thm. 4.B.7]). To prove very ampleness of $\mathcal{O}_{\mathbb{P}}(6)$, we need to prove that the integer E defined in [7] and [4] is equal to 0.

Let us write $x_1, \ldots, y, z$ for the weighted homogeneous coefficients of $\mathbb{P}$, where y and z have weight 2, resp. 3. We need to check that every monomial in x_j, y, z of (weighted) degree $6 + 6k$ is divisible by a monomial of (weighted) degree $6k$ (if this is the case, then $E = 0$ and [7, Prop. 2.3(iii)] implies the very ampleness of $\mathcal{O}_{\mathbb{P}}(6)$). In the case that the monomial contains z^2, it is divisible by z^2 and so the condition is satisfied. Assume now the monomial contains only one z. In the case that the monomial contains y^3 it is divisible by y^3. Next, if the monomial contains y (or y^2) it is divisible by zyx_j (for some j) and so the condition is satisfied. A monomial in z and x_j obviously satisfies the condition. Finally, monomials in x_j satisfy the condition.

This proves the lemma, and ends the proof of the proposition. □

§4. Main result

Theorem 4.1. *The following Fano threefolds admit an MCK decomposition:*

(i) *hypersurfaces of weighted degree* 6 *in weighted projective space* $\mathbb{P}(1^3, 2, 3)$*;*
(ii) *quartic double solids;*
(iii) *sextic double solids;*
(iv) *double covers of a quadric in* $\mathbb{P}^4$ *branched along the intersection with a quartic;*
(v) *special Gushel–Mukai threefolds.*

Proof. Cases (ii) and (iii) are immediate applications of Proposition 3.6. Case (i) is a special case of Proposition 3.8.

Before proving case (iv), let us first state a preparatory lemma:

Lemma 4.2. *Let* $Z \subset \mathbb{P} := \mathbb{P}(1^5, 2)$ *be a smooth weighted hypersurface of degree* 2*. Then*

$$\Delta_Z = \frac{1}{2} \sum_{j=0}^{4} h^j \times h^{4-j} \quad \textit{in } A^4(Z \times Z).$$

Proof. Since Z is a quotient of a non-singular quadric in $\mathbb{P}^5$, Z has trivial Chow groups (i.e. $A^*_{hom}(Z) = 0$). Using [9, Sect. 4.4.2], one can compute the Betti numbers of Z and one finds that they are the same as those of projective space $\mathbb{P}^4$. This means that there is a cohomological decomposition of the diagonal

$$\Delta_Z = \frac{1}{2} \sum_{j=0}^{4} h^j \times h^{4-j} \quad \text{in } H^8(Z \times Z, \mathbb{Q}).$$

Since Z (and hence also $Z \times Z$) has trivial Chow groups, the same decomposition holds modulo rational equivalence, proving the lemma. □

Now, to prove case (iv) of Theorem 4.1, we apply our general criterion Proposition 3.3. Let $\mathbb{P} := \mathbb{P}(1^5, 2)$, and let $\mathcal{Y} \to B$ be the universal family of smooth dimensionally transverse complete intersections of $\mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(4)$, where the base B is a Zariski open

$$B \subset \overline{B} \coloneqq \mathbb{P}H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2) \oplus \mathcal{O}_{\mathbb{P}}(4)).$$

It follows from Lemma 3.7 that $\mathcal{O}_{\mathbb{P}}(2)$ and $\mathcal{O}_{\mathbb{P}}(4)$ are very ample line bundles on $\mathbb{P}$, and so $\overline{\mathcal{Y}} \times_{\overline{B}} \overline{\mathcal{Y}} \to \mathbb{P} \times \mathbb{P}$ is a stratified projective bundle with strata $\Delta_{\mathbb{P}}$ and its complement. The usual stratified projective bundle argument [12, Prop. 2.6] applies, and we find that

$$\begin{aligned} \mathrm{GDA}^*_B(Y \times Y) &= \langle (p_i)^* \, \mathrm{GDA}^*_B(Y), \Delta_Y \rangle \\ &= \langle (p_i)^*(h), \Delta_Y \rangle \end{aligned}$$

(here, $h \in A^1(Y)$ denotes the restriction to Y of an ample generator of $A^1(\mathbb{P}) \cong \mathbb{Q}$). Let $Y = Z \cap Z'$, where Z and $Z' \subset \mathbb{P}$ are hypersurfaces of (weighted) degree 2 and 4. Up to shrinking B, we may assume the hypersurface Z is smooth. Since $Y \subset Z$ is a divisor, the excess intersection formula gives

$$\Delta_Y \cdot (p_i)^*(h) = \Delta_Z|_Y \quad \text{in } A^4(Y \times Y).$$

Using Lemma 4.2, it follows that

$$\Delta_Y \cdot (p_i)^*(h) \in \langle (p_i)^*(h) \rangle.$$

The above identification thus simplifies to

$$\mathrm{GDA}^*_B(Y \times Y) = \langle (p_i)^*(h) \rangle \oplus \mathbb{Q}[\Delta_Y].$$

As before, the fact that the diagonal Δ_Y is linearly independent of the decomposable correspondences in cohomology now shows that

$$\mathrm{GDA}^*_B(Y \times Y) \to H^*(Y \times Y, \mathbb{Q})$$

is injective, and so Y verifies the hypotheses of Proposition 3.3.

The argument for case (v) is similar to that of (iv). First, in view of the spread argument [48, Lem. 3.2], it suffices to establish an MCK decomposition for the *generic* special Gushel–Mukai threefold Y. Thus we may assume that there exists $P \subset \mathrm{Gr}(2,5)$, a smooth complete intersection of Plücker hyperplanes, and a double cover $p\colon Y \to P$ branched along a smooth Gushel–Mukai surface. We now consider the family $\mathcal{Y} \to B$ of all double covers of P branched along smooth Gushel–Mukai surfaces (so $B \subset \overline{B}$ is a Zariski open in the projectivized space of quadratic sections of the cone over P), and we apply our general criterion Proposition 3.3 to this family.

Lemma 4.3. *Let $\mathcal{Y} \to B$ be the family of double covers of P branched along smooth Gushel–Mukai surfaces. The family $\mathcal{Y} \to B$ has the Franchetta property.*

Proof. We consider the family $\overline{\mathcal{Y}} \to \overline{B}$ with the projection to the cone C over P. This is a projective bundle, and so for any fiber $Y = Y_b$ with $b \in B$ we have

$$\mathrm{GDA}^*_B(Y) = \mathrm{Im}(A^*(C) \to A^*(Y)).$$

The condition $b \in B$ means exactly that Y avoids the summit of the cone C, and so (writing $C^\circ \subset C$ for the complement of the summit of the cone) we have

$$\mathrm{GDA}^*_B(Y) = \mathrm{Im}(A^*(C^\circ) \to A^*(Y)). \tag{3}$$

But $C^\circ \to P$ is an affine bundle, and

$$A^*(P) = \mathrm{Im}\big(A^*(\mathrm{Gr}(2,5)) \to A^*(P)\big) = \langle h \rangle,$$

where h denotes the restriction to P of a Plücker hyperplane (this follows from [28, Thm. 3.17], or alternatively from the fact that the derived category of P has a full exceptional collection of length 4 [38]). Thus, (3) reduces to

$$\mathrm{GDA}^*_B(Y) = \langle h \rangle.$$

This proves the Franchetta property for Y. □

Lemma 4.4. *Let $\mathcal{Y} \to B$ be as in Lemma 4.3. The family $\mathcal{Y} \times_B \mathcal{Y} \to B$ has the Franchetta property.*

Proof. Let us consider the family $\overline{\mathcal{Y}} \times_{\overline{B}} \overline{\mathcal{Y}} \to \overline{B}$ with the projection to $C \times C$. This is a stratified projective bundle, with strata Δ_C and its complement. Thus, the stratified projective bundle argument [12, Prop. 2.6] implies that

$$\mathrm{GDA}^*_B(Y \times Y) = \langle \mathrm{Im}(A^*(C^\circ \times C^\circ) \to A^*(Y \times Y)), \Delta_Y \rangle.$$

Since $A^*(C^\circ) = \mathrm{Im}(A^*(\mathrm{Gr}(2,5)) \to A^*(C^\circ))$, we find that

$$\mathrm{GDA}^*_B(Y \times Y) = \big\langle \mathrm{Im}\big(A^*(\mathrm{Gr}(2,5) \times \mathrm{Gr}(2,5)) \to A^*(Y \times Y)\big), \Delta_Y \big\rangle.$$

But $A^*(\mathrm{Gr}(2,5) \times \mathrm{Gr}(2,5)) = A^*(\mathrm{Gr}(2,5)) \otimes A^*(\mathrm{Gr}(2,5))$ since the Grassmannian has trivial Chow groups, and so

$$\begin{aligned} \mathrm{GDA}^*_B(Y \times Y) &= \langle GD_B(Y), \Delta_Y \rangle \\ &= \langle h, \Delta_Y \rangle \end{aligned}$$

(where the last equality follows from Lemma 4.3).

To finish the proof of the lemma, we now claim that for any (ordinary or special) Gushel–Mukai threefold Y we have

$$\Delta_Y \cdot h \in \big\langle \mathrm{Im}\big(A^*(\mathrm{Gr}(2,5)) \to A^*(Y)\big)\big\rangle. \tag{4}$$

Combined with Lemma 4.3, this means that for a special Gushel–Mukai threefold Y (and $\mathcal{Y} \to B$ as above) there is the equality

$$\mathrm{GDA}^*_B(Y \times Y) = \langle h \rangle \oplus \mathbb{Q}[\Delta_Y].$$

Then, since the diagonal is linearly independent in cohomology of $\langle h \rangle$ (since $h^{1,2}(Y) \neq 0$), this proves the lemma.

It remains to prove claim (4). Using the spread argument [48, Lem. 3.2], it suffices to prove equality (4) for the very general Gushel–Mukai threefold. Thus, we may assume that Y is ordinary, and moreover that

$$Y = Y' \cap Q,$$

where Q is a quadric and $Y' = \mathrm{Gr}(2,5) \cap H_1 \cap H_2$ is a smooth fourfold (where H_1, H_2 are Plücker hyperplanes) and Y' is such that

$$A^*(Y') = \mathrm{Im}\big(A^*(\mathrm{Gr}(2,5)) \to A^*(Y')\big).$$

(Indeed, the smooth fourfold Y' has trivial Chow groups [28, Cor. 4.6], and the very general Y' has no primitive cohomology, as follows from [28, Lem. 3.15]). The excess intersection formula then implies that

$$\Delta_Y \cdot h = \frac{1}{2} \Delta_{Y'}|_{Y \times Y},$$

and the claim (4) follows. □

Lemma 4.4 being proven, all conditions of Proposition 3.3 are met, and so fibers Y of the family $\mathcal{Y} \to B$ have an MCK decomposition; this settles (v). □

§5. The tautological ring

Theorem 5.1. *Let Y be a Fano threefold of Picard number 1. Assume that Y has an MCK decomposition, and Y is a member of a family $\mathcal{Y} \to B$ such that $\mathcal{Y} \times_B \mathcal{Y} \to B$ has the Franchetta property. For $m \in \mathbb{N}$, let*

$$R^*(Y^m) := \langle (p_i)^*(h), (p_{ij})^*(\Delta_Y) \rangle \quad \subset A^*(Y^m)$$

be the $\mathbb{Q}$-subalgebra generated by pullbacks of the polarization $h \in A^1(Y)$ and pullbacks of the diagonal $\Delta_Y \in A^3(Y \times Y)$. (Here p_i and p_{ij} denote the various *projections from Y^m to Y, resp. to $Y \times Y$). The cycle class map induces injections*

$$R^*(Y^m) \hookrightarrow H^*(Y^m, \mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

Proof. This is inspired by an analogous result for cubic hypersurfaces [11, Sect. 2.3]. In its turn, the result of [11] was inspired by analogous results for hyperelliptic curves [43, 44] (cf. Remark 5.2 below) and for K3 surfaces [47, 49].

Let d denote the degree of Y, and let $2b := \dim H^3(Y, \mathbb{Q})$. As in [11, Sect. 2.3], let us write $o := \frac{1}{d} h^3 \in A^3(Y)$ (the "distinguished zero-cycle") and

$$\tau := \Delta_Y - \frac{1}{d} \sum_{j=0}^{3} h^j \times h^{3-j} \quad \in A^3(Y \times Y)$$

(this cycle τ is nothing but the projector on the motive $h^3(Y)$ considered above). Moreover, let us write

$$\begin{aligned} h_i &:= (p_i)^*(h) && \in A^1(Y^m), \\ o_i &:= (p_i)^*(o) && \in A^3(Y^m), \\ \tau_{i,j} &:= (p_{ij})^*(\tau) && \in A^3(Y^m). \end{aligned}$$

We define the $\mathbb{Q}$-subalgebra

$$\overline{R}^*(Y^m) := \langle o_i, h_i, \tau_{i,j} \rangle \quad \subset H^*(Y^m, \mathbb{Q})$$

(where i ranges over $1 \le i \le m$, and $1 \le i < j \le m$). One can prove (just as [11, Lem. 2.11] and [49, Lem. 2.3]) that the $\mathbb{Q}$-algebra $\overline{R}^*(Y^m)$ is isomorphic to the free graded $\mathbb{Q}$-algebra generated by o_i, h_i, τ_{ij}, modulo the following relations:

$$o_i \cdot o_i = 0, \quad h_i \cdot o_i = 0, \quad h_i^3 = d o_i; \tag{5}$$

$$\tau_{i,j} \cdot o_i = 0, \quad \tau_{i,j} \cdot h_i = 0, \quad \tau_{i,j} \cdot \tau_{i,j} = 2b o_i \cdot o_j; \tag{6}$$

$$\tau_{i,j} \cdot \tau_{i,k} = \tau_{j,k} \cdot o_i; \tag{7}$$

$$\sum_{\sigma \in \mathfrak{S}_{2b+2}} \prod_{i=1}^{b+1} \tau_{\sigma(2i-1), \sigma(2i)} = 0. \tag{8}$$

To prove Theorem 5.1, we need to check that these relations are also verified modulo rational equivalence. The relations (5) take place in $R^*(Y)$ and so they follow from the Franchetta property for Y. The relations (6) take place in $R^*(Y^2)$. The first and the last relations are trivially verified, because Y being Fano one has $A^6(Y^2) = \mathbb{Q}$. As for the second relation of (6), this follows from the Franchetta property for $Y \times Y$. (Alternatively, it is possible to deduce the second relation from the MCK decomposition: indeed, the product $\tau \cdot h_i$ lies in $A^4_{(0)}(Y^2)$, and it is readily checked that $A^4_{(0)}(Y^2)$ injects into $H^8(Y^2,\mathbb{Q})$.)

Relation (7) takes place in $R^*(Y^3)$ and follows from the MCK relation. Indeed, we have

$$\Delta_Y^{\mathrm{sm}} \circ (\pi_Y^3 \times \pi_Y^3) = \pi_Y^6 \circ \Delta_Y^{\mathrm{sm}} \circ (\pi_Y^3 \times \pi_Y^3) \quad \text{in } A^6(Y^3),$$

which (using Lieberman's lemma) translates into

$$(\pi_Y^3 \times \pi_Y^3 \times \Delta_Y)_* \Delta_Y^{\mathrm{sm}} = (\pi_Y^3 \times \pi_Y^3 \times \pi_Y^6)_* \Delta_Y^{\mathrm{sm}} \quad \text{in } A^6(Y^3),$$

which means that

$$\tau_{1,3} \cdot \tau_{2,3} = \tau_{1,2} \cdot o_3 \quad \text{in } A^6(Y^3).$$

It is left to consider relation (8), which takes place in $R^*(Y^{2b+2})$. To check that this relation is also verified modulo rational equivalence, we observe that relation (8) involves a cycle contained in

$$A^*\big(\mathrm{Sym}^{2b+2}(h^3(Y))\big).$$

But we have vanishing of the Chow motive

$$\mathrm{Sym}^{2b+2} h^3(Y) = 0 \quad \text{in } \mathcal{M}_{\mathrm{rat}},$$

because $\dim H^3(Y,\mathbb{Q}) = 2b$ and $h^3(Y)$ is oddly finite-dimensional in the sense of Kimura [18] (all Fano threefolds are known to have Kimura finite-dimensional motive [45, Thm. 4]). This establishes relation (8), modulo rational equivalence, and ends the proof. □

Remark 5.2. Given a curve C and an integer $m \in \mathbb{N}$, one can define the *tautological ring*

$$R^*(C^m) := \langle (p_i)^*(K_C), (p_{ij})^*(\Delta_C) \rangle \quad \subset A^*(C^m)$$

(where p_i, p_{ij} denote the various projections from C^m to C, resp. $C \times C$). Tavakol has proven [44, Cor. 6.4] that if C is a hyperelliptic curve, the cycle class map induces injections

$$R^*(C^m) \hookrightarrow H^*(C^m,\mathbb{Q}) \quad \text{for all } m \in \mathbb{N}.$$

On the other hand, there are many (non-hyperelliptic) curves for which the tautological ring $R^*(C^3)$ does *not* inject into cohomology (this is related to the non-vanishing of the Ceresa cycle, cf. [44, Rem. 4.2] and also [12, Exm. 2.3 and Rem. 2.4]).

§6. A table

Table 1 lists all Fano threefolds with Picard number 1 (the classification of Fano threefolds is contained in [16]). The last column indicates the existence of an MCK decomposition. Note that a Fano threefold X with $h^{1,2}(X) = 0$ has trivial Chow groups (i.e. $A^*_{hom}(X) = 0$), and so these Fano threefolds have an MCK decomposition for trivial reasons. The asterisks indicate new cases settled in this paper. Question marks indicate cases I am not able to settle.

Label	Index	Degree	$h^{1,2}$	Description	MCK
4	4	1	0	$\mathbb{P}^3$	Trivial
3	3	2	0	$X_2 \subset \mathbb{P}^4$	Trivial
2.1	2	1	21	$X_6 \subset \mathbb{P}(1^3, 2, 3)$	$*$
2.2	2	2	10	$X_4 \subset \mathbb{P}(1^4, 2)$	$*$
2.3	2	3	5	$X_3 \subset \mathbb{P}^4$	[8, 12]
2.4	2	4	2	$X_{(2,2)} \subset \mathbb{P}^5$	[26]
2.5	2	5	0	$\mathrm{Gr}(2, 5) \cap L \subset \mathbb{P}^9$	Trivial
1.2	1	2	52	$X_6 \subset \mathbb{P}(1^4, 3)$	$*$
1.4.a	1	4	30	$X_4 \subset \mathbb{P}^4$	?
1.4.b	1	4	30	$X \xrightarrow{2:1} Q$ with quartic branch locus	$*$
1.6	1	6	20	$X_{(2,3)} \subset \mathbb{P}^5$	[27]
1.8	1	8	14	$X_{(2,2,2)} \subset \mathbb{P}^6$	?
1.10.a	1	10	10	Ordinary Gushel–Mukai threefold	?
1.10.b	1	10	10	Special Gushel–Mukai threefold	$*$
1.12	1	12	7	$\mathrm{OGr}_+(5, 10) \cap L \subset \mathbb{P}^{15}$	?
1.14	1	14	5	$\mathrm{Gr}(2, 6) \cap L \subset \mathbb{P}^{14}$	[25]
1.16	1	16	3	$\mathrm{LGr}(3, 6) \cap L \subset \mathbb{P}^{13}$	?
1.18	1	18	2	$G_2/P \cap L \subset \mathbb{P}^{13}$	[31]
1.22	1	22	0	$V(s) \subset \mathrm{Gr}(3, 7)$	Trivial

Table 1. All Fano threefolds with Picard number 1. Here, $X_{(d_1,\ldots,d_r)}$ denotes a complete intersection of multidegree $(d_1, \ldots, d_r)$, Q is a quadric, and $L \subset \mathbb{P}^r$ is a linear subspace of the appropriate dimension. The notation $\mathrm{LGr}(3, 6)$ and $\mathrm{OGr}_+(5, 10)$ indicates the Lagrangian Grassmannian, resp. a connected component of the orthogonal Grassmannian. In 1.22, $V(s)$ denotes the zero locus of a section of some vector bundle.

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