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Crossed Product Interpretation of the Double Shuffle Lie Algebra Attached to a Finite Abelian Group

by

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Abstract

Racinet studied a scheme associated with the double shuffle and regularization relations between multiple polylogarithm values at Nth roots of unity and constructed a group scheme attached to the situation; he also showed it to be the specialization for $G = \mu_N$ of a group scheme DMR_0^G attached to a finite abelian group G. Then Enriquez and Furusho proved that DMR_0^G can be essentially identified with the stabilizer of a coproduct element arising in Racinet's theory with respect to the action of a group of automorphisms of a free Lie algebra attached to G. We reformulate Racinet's construction in terms of crossed products. Racinet's coproduct can then be identified with a coproduct $\widehat{\Delta}_G^{\mathcal{M}}$ defined on a module $\widehat{\mathcal{M}}_G$ over an algebra $\widehat{\mathcal{W}}_G$, which is equipped with its own coproduct $\widehat{\Delta}_G^{\mathcal{W}}$, and the group action on $\widehat{\mathcal{M}}_G$ extends to a compatible action of $\widehat{\mathcal{W}}_G$. We then show that the stabilizer of $\widehat{\Delta}_G^{\mathcal{M}}$, hence DMR_0^G, is contained in the stabilizer of $\widehat{\Delta}_G^{\mathcal{W}}$ thus generalizing a result of Enriquez and Furusho [Selecta Math. (N.S.) 29 (2023), article no. 3]. This yields an explicit group scheme containing DMR_0^G, which we also express in the Racinet formalism.

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§0. Introduction

A *multiple L-value* (MLV in short) is a complex number defined by the series

$$L_{(k_1,\dots,k_r)}(z_1,\dots,z_r) \coloneqq \sum_{0<m_1<\cdots<m_r} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}, \tag{0.1}$$

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where $r, k_1, \dots, k_r$ are positive integers and $z_1, \dots, z_r$ in μ_N the group of Nth roots of unity in $\mathbb{C}$ with N a positive integer. The series (0.1) converges if and only if $(k_r, z_r) \neq (1,1)$. These values have been defined by Goncharov [Gon98, Gon01] and studied by many others like Arakawa and Kaneko [ArKa], and appear as a generalization of the so-called multiple zeta values which in turn generalize the special values of the Riemann zeta function. Among the relations satisfied by the MLVs, our main interests here are the *double shuffle* and *regularization* ones. Understanding these relations has been greatly improved thanks to Racinet's work [Rac].

Essentially, he attached to each pair (G, ι) of a finite cyclic group G and a group injection $\iota\colon G \to \mathbb{C}^\times$, a $\mathbb{Q}$-scheme DMR^ι which associates to each commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a set $\mathsf{DMR}^\iota(\mathbf{k})$ that can be decomposed as a disjoint union of sets $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ $(\lambda \in \mathbf{k})$. For any $\lambda \in \mathbf{k}$, $\mathsf{DMR}^\iota_\lambda(\mathbf{k})$ is a subset of the algebra of noncommutative power series $\mathbf{k}\langle\langle X\rangle\rangle$ over formal noncommutative variables x_0 and $(x_g)_{g\in G}$ satisfying the following conditions:

(i) group-likeness for the coproduct $\widehat{\Delta}\colon \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle^{\widehat{\otimes}2}$ for which the elements x_0 and $(x_g)_{g\in G}$ are primitive;

(ii) group-likeness of the image in $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ of a suitable correction of the element for the coproduct $\widehat{\Delta}_\star\colon \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes}2}$ (see [Rac, Def. 2.3.1]);

(iii) conditions on the degree 1 and 2 terms of the element.

The double shuffle and regularization relations on MLVs are then encoded in the statement that a suitable generating series of these values belongs to the set $\mathsf{DMR}^{\iota_{\mathrm{can}}}_{i2\pi}(\mathbb{C})$ where $\iota_{\mathrm{can}}\colon G = \mu_N \to \mathbb{C}^\times$ is the canonical embedding. Racinet also proved that for any pair (G, ι), the set $\mathsf{DMR}^\iota_0(\mathbf{k})$ equipped with the product $\circledast$ given in (1.10) is a group that is independent of the choice of the embedding ι, so we denote it $\mathsf{DMR}^G_0(\mathbf{k})$. The pair $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ is a group (see Proposition-Definition 1.2) which contains DMR^G_0 as a subgroup. Thanks to [Rac, Thm. I], the sets $\mathrm{DMR}^\iota_\lambda(\mathbf{k})$ have a torsor structure over $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$. This motivates the study of this group.

In order to improve the understanding of the group $(\mathsf{DMR}^G_0(\mathbf{k}), \circledast)$, Enriquez and Furusho related this group with the stabilizer $\mathsf{Stab}(\widehat{\Delta}_\star)(\mathbf{k})$ of the coproduct $\widehat{\Delta}_\star$ in [EF18] for an action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ (see Section 1.2.2).

In addition, Racinet's work also introduced a subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$ of $\mathbf{k}\langle\langle X\rangle\rangle$ spanned by the words ending with x_g for some $g \in G$. It is identified, as a $\mathbf{k}$-module, with $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ and is equipped with a coproduct $\mathbf{k}\langle\langle Y\rangle\rangle \to \mathbf{k}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2}$ compatible with $\widehat{\Delta}_\star$. For this reason, the former coproduct also has the same notation in [Rac]. However, we will adopt distinct notation for these two coproducts, by denoting the coproducts on $\mathbf{k}\langle\langle Y\rangle\rangle$ and $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ by $\widehat{\Delta}_\star^{\mathrm{alg}}$ and $\widehat{\Delta}_\star^{\mathrm{mod}}$ respectively.

The situation, detailed in Section 1, may be summarized by the diagram

$$\mathbf{k}\langle\langle Y\rangle\rangle \hookrightarrow \mathbf{k}\langle\langle X\rangle\rangle \curvearrowright \mathbf{k}\langle\langle X\rangle\rangle \twoheadrightarrow \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \tag{0.2}$$

where the first arrow is an algebra morphism, the second is the module structure of the algebra $\mathbf{k}\langle\langle X\rangle\rangle$ on itself and the last is a module morphism. The three last terms of sequence (0.2) are equipped with compatible actions of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, while the first and last terms are equipped with the compatible coproducts $\widehat{\Delta}_\star^{\mathrm{alg}}$ and $\widehat{\Delta}_\star^{\mathrm{mod}}$. The stabilizer group construction of [EF18] is then based on the fourth term of (0.2).

When $G = \{1\}$, it was proved in [EF21, Part 2, §3] that the subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$ of $\mathbf{k}\langle\langle X\rangle\rangle$ is stable under the action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle$. One can therefore construct the stabilizer group $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ of $\widehat{\Delta}_\star^{\mathrm{alg}}$ with respect to the action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$. By [EF23, §3.1], one then has the inclusion $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$.

However, if $G \neq \{1\}$, the previous group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle$ no longer restricts to an action on $\mathbf{k}\langle\langle Y\rangle\rangle$ (see Proposition 2.15). This forbids a direct generalization of the result of [EF23]. Such a generalization is obtained in Section 2 by introducing an algebra containing $\mathbf{k}\langle\langle X\rangle\rangle$, namely, the crossed product algebra $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ (see Definition 2.2) and developing a formalism on it which is parallel to Racinet's. In this framework, there is a subalgebra $\widehat{\mathcal{W}}_G$ of $\widehat{\mathcal{V}}_G$ isomorphic to the algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ (see Proposition 2.6) and a quotient module $\widehat{\mathcal{M}}_G$ of the left-regular $\widehat{\mathcal{V}}_G$-module isomorphic to the module $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ (see Proposition 2.7). The algebra $\widehat{\mathcal{W}}_G$ is equipped with a bialgebra coproduct $\widehat{\Delta}_G^{\mathcal{W}}$ and the module $\widehat{\mathcal{M}}_G$ is equipped with a compatible coalgebra coproduct $\widehat{\Delta}_G^{\mathcal{M}}$. The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ acts compatibly on the algebra $\widehat{\mathcal{V}}_G$ and on its regular left module. In contrast to the situation with $\mathbf{k}\langle\langle Y\rangle\rangle \subset \mathbf{k}\langle\langle X\rangle\rangle$, the action on the algebra $\widehat{\mathcal{V}}_G$ restricts to the subalgebra $\widehat{\mathcal{W}}_G$, while the action on the left regular $\widehat{\mathcal{V}}_G$-module induces an action of the quotient module $\widehat{\mathcal{M}}_G$. This can be summarized in the following diagram:

$$\widehat{\mathcal{W}}_G \hookrightarrow \widehat{\mathcal{V}}_G \curvearrowright \widehat{\mathcal{V}}_G \twoheadrightarrow \widehat{\mathcal{M}}_G. \tag{0.3}$$

This situation allows us to define two stabilizers: one denoted $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$ and another denoted $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$. One shows that the latter group is a generalization of the group with the same notation defined in [EF23] for $G = \{1\}$. One also shows the inclusion (see Theorem 2.32, generalizing [EF23, Thm. 3.1])

$$\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k}).$$

In Section 3 we identify $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$ with $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k})$ (see Theorem 3.8). We also identify the group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ with an explicit group $\mathsf{Stab}(\widehat{\Delta}_*^{\mathrm{alg}})(\mathbf{k})$ expressed in Racinet's formalism by working out the suitable isomorphisms (see Theorem 3.12).

In Section 4 we show that the group functors $\mathbf{k} \mapsto \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$ and $\mathbf{k} \mapsto \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ are affine $\mathbb{Q}$-group subschemes of $\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ and study their Lie algebras. We show that these are stabilizer Lie algebras corresponding to the Lie algebra actions which are the infinitesimal versions of the $\mathbb{Q}$-group scheme morphisms obtained from the previous actions of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, which are made explicit (see Propositions 4.6 and 4.10).

Notation. Throughout this paper, G is a finite abelian group whose product will be denoted multiplicatively. For a commutative $\mathbb{Q}$-algebra $\mathbf{k}$, a $\mathbf{k}$-algebra A, an element $x \in A$ and a left A-module M we consider

- $\ell_x \colon M \to M$ to be the $\mathbf{k}$-module endomorphism defined by $m \mapsto xm$ and if x is invertible, then ℓ_x is an automorphism;
- $r_x \colon A \to A$ to be the $\mathbf{k}$-module endomorphism defined by $a \mapsto ax$ and if x is invertible, then r_x is an automorphism;
- $\mathrm{ad}_x \colon A \to A$ to be the $\mathbf{k}$-module endomorphism given by $a \mapsto [x, a] = xa - ax$;
- $\mathrm{Ad}_x \colon A \to A$ to be the $\mathbf{k}$-algebra automorphism defined by $a \mapsto xax^{-1}$ with $x \in A^\times$.

§1. Racinet's formalism of the double shuffle theory

In this part we recall from [Rac] the basic formalism of the double shuffle theory, the main ingredients being presented in Section 1.1. In Sections 1.2 and 1.3 we introduce the double shuffle group and the double shuffle Lie algebra respectively, and we recall from [EF18] the stabilizer interpretation of both objects.

§1.1. Basic objects of Racinet's formalism

Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. Let $\mathbf{k}\langle\langle X\rangle\rangle$ be the free noncommutative associative series algebra with unit over the alphabet $X := \{x_0\} \sqcup \{x_g \mid g \in G\}$. It is complete graded with $\deg(x_0) = \deg(x_g) = 1$ for $g \in G$. This algebra is endowed with a Hopf algebra structure for the coproduct $\widehat{\Delta} \colon \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle^{\widehat{\otimes}2}$, which is the unique morphism of topological $\mathbf{k}$-algebras given by $\widehat{\Delta}(x_g) = x_g \otimes 1 + 1 \otimes x_g$, for any $g \in G \sqcup \{0\}$ ([Rac, §2.2.3]). Then let $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ be the set of grouplike elements of $\mathbf{k}\langle\langle X\rangle\rangle$ for the coproduct $\widehat{\Delta}$ (see (1.6)). It is a group for the product of the algebra $\mathbf{k}\langle\langle X\rangle\rangle$.

The group G acts on the set X, the permutation t_g corresponding to $g \in G$ being given by $t_g(x_0) = x_0$, $t_g(x_h) = x_{gh}$ for $h \in G$. This action extends to an action by $\mathbf{k}$-algebra automorphisms on $\mathbf{k}\langle\langle X\rangle\rangle$ ([Rac, §3.1.1]), which will also be denoted $g \mapsto t_g$. By checking on generators one can verify the identity

$$\forall g \in G, \quad \widehat{\Delta} \circ t_g = t_g^{\otimes 2} \circ \widehat{\Delta}, \tag{1.1}$$

since both sides are given as a composition of $\mathbf{k}$-algebra morphisms. As a consequence of (1.1), for any $g \in G$, the $\mathbf{k}$-algebra automorphism $t_g \colon \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle$ restricts to a group automorphism $t_g \colon \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

Throughout the paper, let us denote by $\mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}^{\{\text{words in } x_0,\, (x_g)_{g\in G}\}}$, $v \mapsto ((v|w))_w$ the map such that $v = \sum_w (v|w)w$ (the empty word is equal to 1).

Each word in X can be uniquely written as

$$(x_0^{n_1} x_{g_1} x_0^{n_2} x_{g_2} \cdots x_0^{n_r} x_{g_r} x_0^{n_{r+1}})_{\substack{r, n_1, \ldots, n_{r+1} \in \mathbb{Z}_{\geq 0} \\ g_1, \ldots, g_r \in G}}.$$

This family forms a topological $\mathbf{k}$-module basis of $\mathbf{k}\langle\langle X\rangle\rangle$. Let $\mathbf{q}$ be the $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by ([Rac, §2.2.7])

$$\begin{aligned} &\mathbf{q}(x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2} \cdots x_0^{n_r-1} x_{g_r} x_0^{n_{r+1}-1}) \\ &\quad = x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2 g_1^{-1}} \cdots x_0^{n_r-1} x_{g_r g_{r-1}^{-1}} x_0^{n_{r+1}-1}. \end{aligned} \tag{1.2}$$

For $(n, g) \in \mathbb{Z}_{>0} \times G$, set $y_{n,g} := x_0^{n-1} x_g$. Let $Y := \{y_{n,g} \mid (n, g) \in \mathbb{Z}_{>0} \times G\}$. We define $\mathbf{k}\langle\langle Y\rangle\rangle$ to be the topological free $\mathbf{k}$-algebra over Y, where for every $(n, g) \in \mathbb{Z}_{>0} \times G$, the element $y_{n,g}$ is of degree n. One can show that $\mathbf{k}\langle\langle Y\rangle\rangle$ is equal to the $\mathbf{k}$-subalgebra $\mathbf{k} \oplus \bigoplus_{g\in G} \mathbf{k}\langle\langle X\rangle\rangle x_g$ of $\mathbf{k}\langle\langle X\rangle\rangle$ ([Rac, §2.2.5] and [EF18, §2.2]).

One denotes by $\mathbf{q}_Y$ the $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle Y\rangle\rangle$ given by ([Rac, §2.2.7])

$$\mathbf{q}_Y(y_{n_1,g_1} \cdots y_{n_r,g_r}) := y_{n_1,g_1} y_{n_2,g_2 g_1^{-1}} \cdots y_{n_r, g_r g_{r-1}^{-1}}. \tag{1.3}$$

Let $\widehat{\Delta}_\star^{\text{alg}} \colon \mathbf{k}\langle\langle Y\rangle\rangle \to (\mathbf{k}\langle\langle Y\rangle\rangle)^{\widehat{\otimes} 2}$ be the unique topological $\mathbf{k}$-algebra morphism such that for any $(n, g) \in \mathbb{Z}_{>0} \times G$,

$$\widehat{\Delta}_\star^{\text{alg}}(y_{n,g}) = y_{n,g} \otimes 1 + 1 \otimes y_{n,g} + \sum_{\substack{k=1 \\ h \in G}}^{n-1} y_{k,h} \otimes y_{n-k, gh^{-1}}. \tag{1.4}$$

The map $\widehat{\Delta}_\star^{\text{alg}}$ is called the *harmonic coproduct* ([Rac, §2.3.1]) and endows $\mathbf{k}\langle\langle Y\rangle\rangle$ with a Hopf algebra structure. Moreover, one can easily check that the action t on $\mathbf{k}\langle\langle X\rangle\rangle$ restricts to an action on $\mathbf{k}\langle\langle Y\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms.

The topological $\mathbf{k}$-module quotient $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is a left $\mathbf{k}\langle\langle Y\rangle\rangle$-module free of rank 1. The topological $\mathbf{k}$-module morphism $\pi_Y\colon \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is a surjective map and its restriction to $\mathbf{k}\langle\langle Y\rangle\rangle$ is a bijective map. It follows that there is a topological $\mathbf{k}$-module morphism

$$\widehat{\Delta}_\star^{\mathrm{mod}}\colon \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes}2}$$

uniquely defined by the condition that the diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\widehat{\Delta}_\star^{\mathrm{alg}}} & (\mathbf{k}\langle\langle Y\rangle\rangle)^{\widehat{\otimes}2} \\ {\scriptstyle \pi_Y}\big\downarrow & & \big\downarrow{\scriptstyle (\pi_Y)^{\otimes 2}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\widehat{\Delta}_\star^{\mathrm{mod}}]{} & (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes}2} \end{array} \tag{1.5}$$

commutes. This equips $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ with a cocommutative coassociative coalgebra structure.

The $\mathbf{k}$-module automorphism $\mathbf{q}$ of $\mathbf{k}\langle\langle X\rangle\rangle$ preserves the submodule $\mathbf{k}\langle\langle X\rangle\rangle x_0$ and, therefore, induces a $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ denoted $\bar{\mathbf{q}}$, which is intertwined with the $\mathbf{k}$-module automorphism $\mathbf{q}_Y$ of $\mathbf{k}\langle\langle Y\rangle\rangle$ via the identification $\mathbf{k}\langle\langle Y\rangle\rangle \simeq \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

§1.2. The double shuffle group $\mathsf{DMR}_0^G(\mathbf{k})$

1.2.1. The group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$. Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. Recall that the set of grouplike elements of $\mathbf{k}\langle\langle X\rangle\rangle$ for the coproduct $\widehat{\Delta}$ is

$$\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) = \left\{\Psi \in \mathbf{k}\langle\langle X\rangle\rangle^\times \;\middle|\; \widehat{\Delta}(\Psi) = \Psi \otimes \Psi\right\}. \tag{1.6}$$

For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, let aut_Ψ be the topological $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by ([EF18, §4.1.3] based on [Rac, §3.1.2])

$$x_0 \mapsto x_0 \quad \text{and for } g \in G, \quad x_g \mapsto \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g). \tag{1.7}$$

Define S_Ψ to be the topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle$ given by ([EF18, (5.15)] based on [Rac, (3.1.2.1)])

$$S_\Psi := \ell_\Psi \circ \mathrm{aut}_\Psi\,. \tag{1.8}$$

Lemma 1.1. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the $\mathbf{k}$-algebra automorphism aut_Ψ is a bialgebra automorphism of $(\mathbf{k}\langle\langle X\rangle\rangle, \widehat{\Delta})$.*

Proof. Both aut_Ψ and $\widehat{\Delta}$ are $\mathbf{k}$-algebra automorphisms. So, using identity (1.1), one can check on generators that

$$\widehat{\Delta} \circ \mathrm{aut}_\Psi = (\mathrm{aut}_\Psi)^{\otimes 2} \circ \widehat{\Delta}, \tag{1.9}$$

which is the wanted result. □

Proposition-Definition 1.2 ([Rac, Prop. 3.1.6]). *The pair* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *is a group, where for* $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$,

$$\Psi \circledast \Phi := S_\Psi(\Phi). \tag{1.10}$$

A proof of this claim is already available in Racinet's paper; however, considering the way it has been stated (using categorical considerations), it might be hard to read. Thus, we find it useful to rewrite it here. We will then need this result:

Lemma 1.3. *For* $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, *we have*

(i) $\mathrm{aut}_{\Psi\circledast\Phi} = \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi$;
(ii) $S_{\Psi\circledast\Phi} = S_\Psi \circ S_\Phi$.

This, in turn, uses the following technical lemma, which can be easily obtained by checking this identity on generators:

Lemma 1.4. *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *and* $g \in G$, *we have* $\mathrm{aut}_\Psi \circ t_g = t_g \circ \mathrm{aut}_\Psi$.

Proof of Lemma 1.3. It is enough to prove the identity (i) on generators. Since for $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ we have $\mathrm{aut}_\Psi(x_0) = x_0$, identity (i) is immediately true for x_0. Then, for $g \in G$, we have

$$\begin{aligned}\mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi(x_g) &= \mathrm{aut}_\Psi \circ \mathrm{Ad}_{t_g(\Phi^{-1})}(x_g) = \mathrm{Ad}_{\mathrm{aut}_\Psi(t_g(\Phi^{-1}))} \circ \mathrm{aut}_\Psi(x_g)\\ &= \mathrm{Ad}_{\mathrm{aut}_\Psi(t_g(\Phi^{-1}))} \circ \mathrm{Ad}_{t_g(\Psi^{-1})}(x_g) = \mathrm{Ad}_{t_g(\mathrm{aut}_\Psi(\Phi^{-1}))t_g(\Psi^{-1})}(x_g)\\ &= \mathrm{Ad}_{t_g(\mathrm{aut}_\Psi(\Phi^{-1})\Psi^{-1})}(x_g) = \mathrm{Ad}_{t_g((\Psi\circledast\Phi)^{-1})}(x_g) = \mathrm{aut}_{\Psi\circledast\Phi}(x_g),\end{aligned}$$

where the fourth equality is obtained by applying Lemma 1.4. This concludes the proof of identity (i). Finally, by using the latter, we get

$$\begin{aligned}S_\Psi \circ S_\Phi &= \ell_\Psi \circ \mathrm{aut}_\Psi \circ \ell_\Phi \circ \mathrm{aut}_\Phi = \ell_\Psi \circ \ell_{\mathrm{aut}_\Psi(\Phi)} \circ \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi\\ &= \ell_{\Psi\,\mathrm{aut}_\Psi(\Phi)} \circ \mathrm{aut}_\Psi \circ \mathrm{aut}_\Phi = \ell_{\Psi\circledast\Phi} \circ \mathrm{aut}_{\Psi\circledast\Phi} = S_{\Psi\circledast\Phi},\end{aligned}$$

thus, establishing identity (ii). □

Proof of Proposition-Definition 1.2. From Lemma 1.1, we deduce that $\circledast$ has its image in $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. Next, thanks to identity (ii) in Lemma 1.3, the product $\circledast$ is associative. Indeed, for Ψ, Φ and $\Lambda \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$(\Psi \circledast \Phi) \circledast \Lambda = S_{\Psi\circledast\Phi}(\Lambda) = S_\Psi(S_\Phi(\Lambda)) = S_\Psi(\Phi \circledast \Lambda) = \Psi \circledast (\Phi \circledast \Lambda).$$

Finally, the other group axioms being easy to check, this proves Proposition 1.2. □

Corollary 1.5. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by*

(i) *topological* $\mathbf{k}$*-algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle), \quad \Psi \mapsto \mathrm{aut}_\Psi, \tag{1.11}$$

(ii) *topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle), \quad \Psi \mapsto S_\Psi. \tag{1.12}$$

Proof. This result is exactly Lemma 1.3. □

Next we aim to give a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on the topological $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ which is compatible with its action S on $\mathbf{k}\langle\langle X\rangle\rangle$. It is important to notice that this action is not given by compatibility using π_Y but by the following:

Proposition-Definition 1.6 ([EF18, §5.4]). *For* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, there is a unique topological* $\mathbf{k}$*-module automorphism* S^Y_Ψ *of* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{S_\Psi} & \mathbf{k}\langle\langle X\rangle\rangle \\ \downarrow{\scriptstyle \bar{\mathbf{q}}\circ\pi_Y} & & \downarrow{\scriptstyle \bar{\mathbf{q}}\circ\pi_Y} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[S^Y_\Psi]{} & \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \end{array} \tag{1.13}$$

commutes.

Corollary 1.7. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ *by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0), \quad \Psi \mapsto S^Y_\Psi. \tag{1.14}$$

Proof. We have

$$S^Y_\Psi \circ S^Y_\Phi \circ \bar{\mathbf{q}} \circ \pi_Y = S^Y_\Psi \circ \bar{\mathbf{q}} \circ \pi_Y \circ S_\Phi = \bar{\mathbf{q}} \circ \pi_Y \circ S_\Psi \circ S_\Phi = \bar{\mathbf{q}} \circ \pi_Y \circ S_{\Psi\circledast\Phi},$$

and, by uniqueness of the $\mathbf{k}$-module automorphism $S^Y_{\Psi\circledast\Phi}$, we obtain

$$S^Y_\Psi \circ S^Y_\Phi = S^Y_{\Psi\circledast\Phi}.$$ □

Let $\Gamma\colon \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}[[x]]^\times$, $\Psi \mapsto \Gamma_\Psi$ be the function given by ([Rac, (3.2.1.2)])

$$\Gamma_\Psi(x) := \exp\left(\sum_{n\geq 2} \frac{(-1)^n}{n} (\Psi | x_0^{n-1} x_1) x^n\right). \tag{1.15}$$

This function is related to the classical gamma function as established in [Fur11, p. 344], thanks to [Dri90]. Moreover, it satisfies the following property:

Lemma 1.8. *For $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have $\Gamma_{\Psi\circledast\Phi} = \Gamma_\Psi \Gamma_\Phi$.*

Proof. Lemma 4.12 in [EF18] says that the map $(-|x_0^{n-1}x_1)\colon (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to (\mathbf{k}, +)$ is a group morphism, for any $n \in \mathbb{Z}_{>0}$. The result is then obtained by straightforward computations. □

We then define the following topological $\mathbf{k}$-module automorphism of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$:

$$ {}^{\Gamma}S_\Psi^Y := \ell_{\Gamma_\Psi^{-1}(x_1)} \circ S_\Psi^Y. \tag{1.16}$$

Corollary 1.9. *There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ by topological $\mathbf{k}$-module automorphisms*

$$ (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0), \quad \Psi \mapsto {}^{\Gamma}S_\Psi^Y. \tag{1.17}$$

Proof. It follows from Corollary 1.7 and Lemma 1.8. □

The above automorphism is related to an automorphism introduced in [EF18].

Proposition 1.10. *For any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the $\mathbf{k}$-module automorphism ${}^{\Gamma}S_\Psi^Y$ is equal to the $\mathbf{k}$-module automorphism $S_{\Theta(\Psi)}^Y$ with $\Theta\colon (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to ((\mathbf{k}\langle\langle X\rangle\rangle)^\times, \circledast)$ being the group morphism given by ([EF18, Prop. 4.13])*

$$ \Theta(\Psi) := \Gamma_\Psi^{-1}(x_1)\Psi \exp(-(\Psi|x_0)x_0). \tag{1.18}$$

Remark 1.11. Note that the product $\circledast$ extends to a product on $\mathbf{k}\langle\langle X\rangle\rangle^\times$. See [EF18, Lem. 4.1] and [Rac, §3.1.2].

Proof of Proposition 1.10. Let $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $v \in \mathbf{k}\langle\langle X\rangle\rangle$. First, we have

$$ S_{\Theta(\Psi)}(v) = \Theta(\Psi)\,\mathrm{aut}_{\Theta(\Psi)}(v) = \big(\Gamma_\Psi^{-1}(x_1)\Psi \exp(-(\Psi|x_0)x_0)\big)\,\mathrm{aut}_{\Theta(\Psi)}(v). $$

Moreover, one can check on generators that

$$ \mathrm{aut}_{\Theta(\Psi)} = \mathrm{Ad}_{\exp((\Psi|x_0)x_0)} \circ \mathrm{aut}_\Psi . $$

Therefore, one obtains

$$ S_{\Theta(\Psi)}(v) = \Gamma_\Psi^{-1}(x_1)\Psi\,\mathrm{aut}_\Psi(v)\exp(-(\Psi|x_0)x_0) = \Gamma_\Psi^{-1}(x_1) S_\Psi(v) \exp(-(\Psi|x_0)x_0). $$

Consequently,

$$ \begin{aligned} {}^{\Gamma}S_\Psi^Y(\bar{\mathbf{q}} \circ \pi_Y(v)) &= \Gamma_\Psi^{-1}(x_1) S_\Psi^Y(\bar{\mathbf{q}} \circ \pi_Y(v)) = \Gamma_\Psi^{-1}(x_1)\big(\bar{\mathbf{q}} \circ \pi_Y(S_\Psi(v))\big) \\ &= \bar{\mathbf{q}} \circ \pi_Y(\Gamma_\Psi^{-1}(x_1) S_\Psi(v)) = \bar{\mathbf{q}} \circ \pi_Y(S_{\Theta(\Psi)}(v)). \end{aligned} $$

This establishes the identity ${}^{\Gamma}S_\Psi^Y = S_{\Theta(\Psi)}^Y$, thanks to Proposition-Definition 1.6. □

1.2.2. The group $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$. Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, set $\Psi_\star := \bar{\mathbf{q}} \circ \pi_Y(\Gamma_\Psi^{-1}(x_1)\Psi) \in \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$.

Proposition-Definition 1.12 ([Rac, Def. 3.2.1 and Thm. I]). *If G is a cyclic group, we define $\mathsf{DMR}_0^G(\mathbf{k})$ to be the set of $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ such that*

(i) $(\Psi|x_0) = (\Psi|x_1) = 0$;
(ii) $\widehat{\Delta}_\star^{\mathrm{mod}}(\Psi_\star) = \Psi_\star \otimes \Psi_\star$;
(iii) *if* $|G| \in \{1, 2\}$, $(\Psi|x_0x_1) = 0$;
(iv) *if* $|G| \geq 3$, *for all* $g \in G$, $(\Psi|x_g - x_{g^{-1}}) = 0$.

The pair $(\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$ is a subgroup of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.

Remark 1.13. (i) The notation DMR is for “double mélange et régularization”, which is French for “double shuffle and regularization”.
(ii) Definition 3.2.1 in [Rac] gives sets $\mathsf{DMR}_\lambda^\iota(\mathbf{k})$ where $\lambda \in \mathbf{k}$ and $\iota\colon G \to \mathbb{C}^*$ is a group embedding (therefore G is cyclic). If $|G| \in \{1, 2\}$, the embedding ι is unique; and if $|G| \geq 3$, for $\lambda = 0$, condition (iv) does not depend on the choice of ι. For this reason, the embedding ι does not appear in this paper's notation.

Thanks to Corollary 1.9, there is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on the $\mathbf{k}$-module $\mathrm{Mor}_{\mathbf{k}\text{-mod}}^{\mathrm{cont}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes} 2})$ via

$$\Psi \cdot D := ((^\Gamma S_\Psi^Y)^{\otimes 2}) \circ D \circ (^\Gamma S_\Psi^Y)^{-1}, \tag{1.19}$$

with $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $D \in \mathrm{Mor}_{\mathbf{k}\text{-mod}}^{\mathrm{cont}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes} 2})$. In particular, the stabilizer of $D = \widehat{\Delta}_\star^{\mathrm{mod}}$ is the subgroup ([EF18, §5.4])

$$\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \;\middle|\; (^\Gamma S_\Psi^Y)^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{mod}} = \widehat{\Delta}_\star^{\mathrm{mod}} \circ {}^\Gamma S_\Psi^Y\right\}. \tag{1.20}$$

Proposition 1.14 ([EF18, Thm. 1.2]). *If G is a cyclic group, we have*

$$\mathsf{DMR}_0^G(\mathbf{k}) = \left\{\Psi \in \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \;\middle|\; (\Psi|x_0) = (\Psi|x_1) = 0\right\}. \tag{1.21}$$

Since the condition $(\Psi|x_0) = (\Psi|x_1) = 0$ defines a subgroup of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$, Proposition 1.14 then identifies $\mathsf{DMR}_0^G(\mathbf{k})$ with the intersection of two subgroups of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$.

1.2.3. An affine $\mathbb{Q}$-group scheme structure. Recall that an affine $\mathbb{Q}$-group scheme is a functor G from the category of commutative $\mathbb{Q}$-algebras to the category of groups which is representable by a Hopf $\mathbb{Q}$-algebra (see, for example, [Wat, §1.2]).

Proposition 1.15. *The following assignments are affine $\mathbb{Q}$-group schemes:*

(i) $\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$;

(ii) $\mathsf{DMR}_0^G \colon \mathbf{k} \mapsto (\mathsf{DMR}_0^G(\mathbf{k}), \circledast)$;

(iii) $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}}) \colon \mathbf{k} \mapsto (\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}), \circledast)$.

Proof. (i) See [EF18, Lem. 4.6]. (ii) See [Rac, Thm. I]. (iii) See [EF18, Lem. 5.1]. □

Therefore, Proposition 1.14 provides an inclusion of affine $\mathbb{Q}$-group schemes

$$\mathsf{DMR}_0^G \subset \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}}) \subset \big(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle), \circledast)\big). \tag{1.22}$$

§1.3. The double shuffle Lie algebra $\mathfrak{dmr}_0^G$

Recall from [Wat, Thm. 12.2] that there exists a functor **Lie** from the category of affine $\mathbb{Q}$-group schemes to the category of $\mathbb{Q}$-Lie algebras such that

$$\mathbf{Lie}(\mathsf{G}) = \ker\big(\mathsf{G}(\mathbb{Q}[\varepsilon]/(\varepsilon^2)) \to \mathsf{G}(\mathbb{Q})\big),$$

for any $\mathbb{Q}$-group scheme G. In this section we provide an explicit formulation of the Lie algebras obtained by applying the functor **Lie** to the inclusions (1.22).

1.3.1. The Lie algebra $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$. Let $\widehat{\mathfrak{Lib}}(X)$ be the free complete graded $\mathbb{Q}$-Lie algebra over the alphabet X. One can identify the $\mathbb{Q}$-algebra $\mathbb{Q}\langle\langle X \rangle\rangle$ with the enveloping algebra of $\widehat{\mathfrak{Lib}}(X)$ ([Rac, §2.2.3]). Therefore, $\widehat{\mathfrak{Lib}}(X)$ is identified with the Lie subalgebra of primitive elements in $\mathbb{Q}\langle\langle X \rangle\rangle$ for the coproduct $\widehat{\Delta}$. Namely,

$$\widehat{\mathfrak{Lib}}(X) \simeq \big\{\psi \in \mathbb{Q}\langle\langle X \rangle\rangle \mid \widehat{\Delta}(\psi) = \psi \otimes 1 + 1 \otimes \psi\big\}. \tag{1.23}$$

For $\psi \in \widehat{\mathfrak{Lib}}(X)$, let d_ψ be the derivation of $\mathbb{Q}\langle\langle X \rangle\rangle$ given by ([Rac, (3.1.12.2)])

$$d_\psi(x_0) = 0, \quad \text{and for } g \in G, \quad d_\psi(x_g) = [x_g, t_g(\psi)], \tag{1.24}$$

and let s_ψ be the $\mathbb{Q}$-linear endomorphism of $\mathbb{Q}\langle\langle X \rangle\rangle$ given by ([Rac, (3.1.12.1)])

$$s_\psi := \ell_\psi + d_\psi. \tag{1.25}$$

We then define a Lie algebra bracket on $\widehat{\mathfrak{Lib}}(X)$ as ([Rac, (3.1.10.2)])

$$\forall\, \psi_1, \psi_2 \in \widehat{\mathfrak{Lib}}(X), \quad \langle \psi_1, \psi_2 \rangle := s_{\psi_1}(\psi_2) - s_{\psi_2}(\psi_1). \tag{1.26}$$

1.3.2. The Lie algebra $(\mathfrak{dmr}_0^G, \langle\cdot,\cdot\rangle)$. Let us define $\gamma \colon \mathbb{Q}\langle\langle X \rangle\rangle \to \mathbb{Q}[[x]]$, $\psi \mapsto \gamma_\psi$, where

$$\gamma_\psi(x) := \sum_{n \geq 2} \frac{(-1)^n}{n} (\psi | x_0^{n-1} x_1) x^n, \tag{1.27}$$

and for $\psi \in \mathbb{Q}\langle\langle X\rangle\rangle$, set

$$\psi_\star := \bar{\mathbf{q}} \circ \pi_Y(-\gamma_\psi(x_1) + \psi) \in \mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0.$$

Proposition-Definition 1.16 ([Rac, Defs. 3.3.1, 3.3.8 and Prop. 4.A.i]). *The set* $\mathfrak{dmr}_0^G$ *of elements* $\psi \in \widehat{\mathfrak{Lib}}(X)$ *such that*

(i) $(\psi|x_0) = (\psi|x_1) = 0$;

(ii) $\widehat{\Delta}_\star^{\mathrm{mod}}(\psi_\star) = \psi_\star \otimes 1 + 1 \otimes \psi_\star$;

(iii) $(\psi_\star|x_0^{n-1}x_g) = (-1)^{n-1}(\psi_\star|x_0^{n-1}x_{g^{-1}})$ *for* $(n,g) \in \mathbb{Z}_{>0} \times G$;

is a complete graded Lie subalgebra of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$.

Remark 1.17. According to [Rac, Props. 3.3.3 and 3.3.7], it is enough to have (iii) in these cases:

$$\begin{cases} \text{for } (n,g) = (2,1) & \text{if } |G| = 2, \\ \text{for } n = 1 \text{ and any } g \in G & \text{if } |G| \geq 3, \end{cases}$$

since this identity is always true for all the other cases.

1.3.3. Relation of $\mathfrak{dmr}_0^G$ with a stabilizer Lie algebra.

Proposition 1.18 ([Rac, (3.1.9.2)]). *There exists a Lie algebra action of* $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ *by* $\mathbb{Q}$*-linear endomorphisms on* $\mathbb{Q}\langle\langle X\rangle\rangle$ *given by*

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle), \quad \psi \mapsto s_\psi. \tag{1.28}$$

Proposition-Definition 1.19 ([Rac, §4.1.1] and [EF18, Lem. 2.2]). *For* $\psi \in \widehat{\mathfrak{Lib}}(X)$, *there exists a unique* $\mathbb{Q}$*-linear endomorphism* s_ψ^Y *of* $\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0$ *such that the diagram*

$$\begin{array}{ccc} \mathbb{Q}\langle\langle X\rangle\rangle & \xrightarrow{\ s_\psi\ } & \mathbb{Q}\langle\langle X\rangle\rangle \\ {\scriptstyle \bar{\mathbf{q}}\circ\pi_Y}\downarrow & & \downarrow{\scriptstyle \bar{\mathbf{q}}\circ\pi_Y} \\ \mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0 & \xrightarrow[s_\psi^Y]{} & \mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0 \end{array}$$

commutes. Moreover, there is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ *by* $\mathbb{Q}$*-linear endomorphisms on* $\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0$ *given by*

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0), \quad \psi \mapsto s_\psi^Y. \tag{1.29}$$

Remark 1.20. For $\psi \in \widehat{\mathfrak{Lib}}(X)$, Racinet defined s_ψ^Y as a $\mathbb{Q}$-linear endomorphism of $\mathbb{Q}\langle\langle Y\rangle\rangle$. Even if this paper proceeds differently, Racinet's notation is kept so the reader may refer to [Rac].

For $\psi \in \widehat{\mathfrak{Lib}}(X)$ we consider the following $\mathbb{Q}$-linear endomorphism on $\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0$:

$$ {}^{\gamma}s_{\psi}^{Y} := \ell_{-\gamma_{\psi}(x_1)} + s_{\psi}^{Y}. \tag{1.30} $$

The following result is an analogue of Proposition 1.10:

Lemma 1.21. *For any $\psi \in \widehat{\mathfrak{Lib}}(X)$, the $\mathbb{Q}$-linear endomorphism ${}^{\gamma}s_{\psi}^{Y}$ is equal to $s_{\theta(\psi)}^{Y}$, where $\theta\colon (\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to (\mathbb{Q}\langle\langle X\rangle\rangle, \langle\cdot,\cdot\rangle)$ is the Lie algebra morphism given by ([EF18, Prop. 2.5])*

$$ \theta(\psi) := -\gamma_{\psi}(x_1) + \psi - (\psi|x_0)x_0. \tag{1.31} $$

Remark 1.22. One can equip $\mathbb{Q}\langle\langle X\rangle\rangle$ with the bracket $\langle\cdot,\cdot\rangle$ as described in (1.26).

Proof of Lemma 1.21. Let $\psi \in \widehat{\mathfrak{Lib}}(X)$ and $a \in \mathbb{Q}\langle\langle X\rangle\rangle$. First, we have

$$ s_{\theta(\psi)}(a) = \theta(\psi)a + d_{\theta(\psi)}(a) = (-\gamma_{\psi}(x_1) + \psi - (\psi|x_0)x_0)a + d_{\theta(\psi)}(a). $$

Moreover, one can check on generators that

$$ d_{\theta(\psi)} = \mathrm{ad}_{(\psi|x_0)x_0} + d_{\psi}. $$

Therefore, one obtains

$$ \begin{aligned} s_{\theta(\psi)}(a) &= (-\gamma_{\psi}(x_1) + \psi - (\psi|x_0)x_0)a + \mathrm{ad}_{(\psi|x_0)x_0}(a) + d_{\psi}(a) \\ &= -\gamma_{\psi}(x_1)a + s_{\psi}(a) - (\psi|x_0)ax_0. \end{aligned} $$

Consequently,

$$ \begin{aligned} {}^{\gamma}s_{\psi}^{Y}(\bar{\mathbf{q}} \circ \pi_Y(a)) &= -\gamma_{\psi}(x_1)(\bar{\mathbf{q}} \circ \pi_Y(a)) + s_{\psi}^{Y}(\bar{\mathbf{q}} \circ \pi_Y(a)) \\ &= \bar{\mathbf{q}} \circ \pi_Y(-\gamma_{\psi}(x_1)a) + \bar{\mathbf{q}} \circ \pi_Y(s_{\psi}(a)) \\ &= \bar{\mathbf{q}} \circ \pi_Y(-\gamma_{\psi}(x_1)a + s_{\psi}(a)) = \bar{\mathbf{q}} \circ \pi_Y(s_{\theta(\psi)}(a)). \end{aligned} $$

This establishes the identity ${}^{\gamma}s_{\psi}^{Y} = s_{\theta(\psi)}^{Y}$, thanks to Proposition-Definition 1.19. □

Proposition 1.23. *There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ by $\mathbb{Q}$-linear endomorphisms on $\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0$ by*

$$ (\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0), \quad \psi \mapsto {}^{\gamma}s_{\psi}. \tag{1.32} $$

Proof. Thanks to [EF18, §2.5], the map $\psi \mapsto s_{\theta(\psi)}$ is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0$. The result then follows from Lemma 1.21. □

The space of $\mathbb{Q}$-linear morphisms

$$\mathrm{Mor}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0, (\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes}2})$$

is then equipped with an action of the Lie algebra $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ given by ([EF18, §2.5])

$$\psi \cdot D := ({}^{\gamma}s_{\psi}^{Y} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}s_{\psi}^{Y}) \circ D - D \circ {}^{\gamma}s_{\psi}^{Y}, \tag{1.33}$$

where $\psi \in \widehat{\mathfrak{Lib}}(X)$ and $D \in \mathrm{Mor}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0, (\mathbb{Q}\langle\langle X\rangle\rangle/\mathbb{Q}\langle\langle X\rangle\rangle x_0)^{\widehat{\otimes}2})$.

The stabilizer Lie algebra of $D = \widehat{\Delta}_{\star}^{\mathrm{mod}}$ is then the Lie subalgebra of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ given by ([EF18, §2.5])

$$\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}}) := \left\{\psi \in \widehat{\mathfrak{Lib}}(X) \;\middle|\; ({}^{\gamma}s_{\psi}^{Y} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}s_{\psi}^{Y}) \circ \widehat{\Delta}_{\star}^{\mathrm{mod}} = \widehat{\Delta}_{\star}^{\mathrm{mod}} \circ {}^{\gamma}s_{\psi}^{Y}\right\}. \tag{1.34}$$

It is related to the Lie algebra $\mathfrak{dmr}_0^G$ as follows:

Proposition 1.24. *We have*

$$\mathfrak{dmr}_0^G = \left\{\psi \in \mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}}) \;\middle|\; (\psi|x_0) = (\psi|x_1) = 0\right\}. \tag{1.35}$$

Proof. Thanks to Lemma 1.21, the stabilizer Lie algebra $\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}})$ is identified with the stabilizer Lie algebra given in [EF18]. Therefore, the wanted equality is stated in [EF18, Thm. 3.10] ($\mathfrak{dmr}_0^G$ being denoted by $\mathfrak{dmr}_0$ in [EF18]). □

1.3.4. Exponential maps.

Proposition 1.25. *We have (equalities of* $\mathbf{k}$*-Lie algebras)*

(i) $\mathbf{Lie}(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)) = (\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$.
(ii) $\mathbf{Lie}(\mathsf{DMR}_0^G, \circledast) = (\mathfrak{dmr}_0^G, \langle\cdot,\cdot\rangle)$, *where G is a cyclic group.*
(iii) $\mathbf{Lie}(\mathsf{Stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}}), \circledast) = (\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}}), \langle\cdot,\cdot\rangle)$.

Proof. (i) See [EF18, §4.1.4]. (ii) See [Rac, §3.3.8]. (iii) See [EF18, (5.12)]. □

Let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. Let us denote $\widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) := \widehat{\mathfrak{Lib}}(X)\widehat{\otimes}\mathbf{k}$. Let $\mathrm{cbh}_{\langle\cdot,\cdot\rangle} \colon \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \times \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \to \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X)$ be the map defined by $\mathrm{cbh}_{\langle\cdot,\cdot\rangle}(\psi,\phi) := \mathrm{mor}_{\psi,\phi}(\mathrm{cbh})$, where cbh in $\widehat{\mathfrak{Lib}}_{\mathbb{Q}}(a,b)$ is the Campbell–Baker–Hausdorff series ([EF18, §4.1.2]) $\mathrm{cbh} = \log(\exp(a)\exp(b))$ with $\log \colon 1 + \mathbb{Q}\langle\langle a,b\rangle\rangle \to \mathbb{Q}\langle\langle a,b\rangle\rangle_0$ and $\mathrm{mor}_{\psi,\phi}$ is the Lie algebra morphism $\widehat{\mathfrak{Lib}}_{\mathbb{Q}}(a,b) \to (\widehat{\mathfrak{Lib}}_{\mathbf{k}}(X), \langle\cdot,\cdot\rangle)$, $a \mapsto \psi$, $b \mapsto \phi$. We then define $\exp_{\circledast}^{\mathbf{k}} \colon \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ to be the exponential map; it intertwines $\mathrm{cbh}_{\langle\cdot,\cdot\rangle}$ and $\circledast$. The following proposition recalls from [Rac, §3.1.8] and [DeGo, Rem. 5.14], the explicit form of $\exp_{\circledast}^{\mathbf{k}}$ as well as gives a proof of this statement.

Proposition 1.26. *For a commutative* $\mathbb{Q}$*-algebra* $\mathbf{k}$ *and* $\psi \in \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X)$*, we have the following:*

(i) *the exponential map* $\exp_{\circledast}^{\mathbf{k}} : \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ *is a bijection;*

(ii) $S_{\exp_{\circledast}^{\mathbf{k}}(\psi)} = \exp(s_\psi)$*, where* $\psi \mapsto s_\psi$ *is the map* $\widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \to \mathrm{End}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle)$ *obtained from the map* $\widehat{\mathfrak{Lib}}(X) \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle)$ *in* (1.25) *by tensoring with* $\mathbf{k}$ *and* $\exp$ *is the usual exponential of an endomorphism;*

(iii) $\exp_{\circledast}^{\mathbf{k}}(\psi) = \exp(s_\psi)(1)$.

Proof. (i) See [EF18, §4.1.4 and §4.1.5].

(ii) The assignment $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle)$ is an affine $\mathbb{Q}$-group scheme and the map $\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle)$, $\Psi \mapsto S_\Psi$ defines an affine $\mathbb{Q}$-group scheme morphism from $\mathbf{k} \mapsto \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ to $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle)$. Using the usual dual number formalism, one sees that the associated $\mathbb{Q}$-Lie algebra morphism is $\widehat{\mathfrak{Lib}}(X) \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle\langle X\rangle\rangle)$, $\psi \mapsto s_\psi$. As a consequence, for any $\psi \in \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X)$, $S_{\exp_{\circledast}^{\mathbf{k}}(\psi)} = \exp(s_\psi)$.

(iii) It follows by applying the latter equality to 1, using the identity $S_\Psi(1) = \Psi$ for any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. □

To conclude this part, let us note that the bijection of the map $\exp_{\circledast} : \widehat{\mathfrak{Lib}}_{\mathbf{k}}(X) \to \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ implies that we have an identification between the group actions defined in Section 1.2 with the exponential of the Lie algebra actions of the current subsection.

§2. A crossed product formulation of the double shuffle theory

We construct a crossed product version of the double shuffle formalism. The relevant algebras and modules are introduced in Section 2.1:

(i) an algebra $\widehat{\mathcal{V}}_G$ defined by generators and relations, which is then identified with a crossed product algebra involving Racinet's formal series algebra $\mathbf{k}\langle\langle X\rangle\rangle$;

(ii) a bialgebra $(\widehat{\mathcal{W}}_G, \widehat{\Delta}_G^{\mathcal{W}})$ isomorphic to the bialgebra $(\mathbf{k}\langle\langle Y\rangle\rangle, \widehat{\Delta}_\star^{\mathrm{alg}})$, where $\widehat{\mathcal{W}}_G$ is a subalgebra of $\widehat{\mathcal{V}}_G$;

(iii) a coalgebra $(\widehat{\mathcal{M}}_G, \widehat{\Delta}_G^{\mathcal{M}})$ isomorphic to the coalgebra $(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, \widehat{\Delta}_\star^{\mathrm{mod}})$, where $\widehat{\mathcal{M}}_G$ has a $\widehat{\mathcal{V}}_G$-module structure inducing a free rank-one $\widehat{\mathcal{W}}_G$-module structure on it, compatible with the coproducts $\widehat{\Delta}_G^{\mathcal{W}}$ and $\widehat{\Delta}_G^{\mathcal{M}}$.

In Sections 2.2 and 2.3 we construct actions of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on these objects by algebra and module automorphisms. This leads us in Section 2.4 to define the stabilizer groups of the coproducts $\widehat{\Delta}_G^{\mathcal{W}}$ and $\widehat{\Delta}_G^{\mathcal{M}}$ and show in Theorem 2.32 that the stabilizer of the latter is included in the stabilizer of the former.

§2.1. The algebra $\widehat{\mathcal{V}}_G$, the bialgebra $(\widehat{\mathcal{W}}_G, \widehat{\Delta}_G^{\mathcal{W}})$ and the coalgebra $(\widehat{\mathcal{M}}_G, \widehat{\Delta}_G^{\mathcal{M}})$

2.1.1. The algebras $\widehat{\mathcal{V}}_G$ and $\widehat{\mathcal{W}}_G$ and the module $\widehat{\mathcal{M}}_G$. Let $\widehat{\mathcal{V}}_G^{\mathbf{k}}$ (or simply $\widehat{\mathcal{V}}_G$ if there is no risk of ambiguity) be the complete graded topological $\mathbf{k}$-algebra generated by $\{e_0, e_1\} \sqcup G$, where e_0 and e_1 are of degree 1 and elements $g \in G$ are of degree 0 satisfying the relations

(i) $g \cdot h = gh$;
(ii) $1 = 1_G$;
(iii) $g \cdot e_0 = e_0 \cdot g$;

for any $g, h \in G$, where "$\cdot$" is the algebra multiplication which we will no longer denote if there is no risk of ambiguity.

Remark 2.1. The notation e_0 and e_1 is inspired by [EF21] which in turn is inspired by [DeTe].

Set $\widehat{\mathcal{W}}_G^{\mathbf{k}} := \mathbf{k} \oplus \widehat{\mathcal{V}}_G^{\mathbf{k}} e_1$ (or simply $\widehat{\mathcal{W}}_G$ if there is no risk of ambiguity). It is a graded topological $\mathbf{k}$-subalgebra of $\widehat{\mathcal{V}}_G$.

Next, the quotient

$$\widehat{\mathcal{M}}_G^{\mathbf{k}} := \widehat{\mathcal{V}}_G^{\mathbf{k}} \Big/ \Big(\widehat{\mathcal{V}}_G^{\mathbf{k}} e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G^{\mathbf{k}}(g-1)\Big)$$

(or simply $\widehat{\mathcal{M}}_G$ if there is no risk of ambiguity) is a topological $\mathbf{k}$-module. It is also a topological $\widehat{\mathcal{V}}_G$-module and, by restriction, a topological $\widehat{\mathcal{W}}_G$-module. Let $1_{\mathcal{M}}$ be the class of $1 \in \widehat{\mathcal{V}}_G$ in $\widehat{\mathcal{M}}_G$. The map $- \cdot 1_{\mathcal{M}} \colon \widehat{\mathcal{V}}_G \to \widehat{\mathcal{M}}_G$ is a surjective topological $\mathbf{k}$-module morphism whose kernel is $\widehat{\mathcal{V}}_G e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G(g-1)$.

2.1.2. The algebra $\widehat{\mathcal{V}}_G$ as a crossed product. First, let us introduce the basic material about the crossed product of an algebra by a group acting by algebra automorphisms.

Definition 2.2. Let A be a $\mathbf{k}$-algebra such that the group G acts on A by $\mathbf{k}$-algebra automorphisms. Let us denote this action by $G \times A \ni (g, a) \mapsto a^g \in A$.

The *crossed product algebra* of the $\mathbf{k}$-algebra A by the group G denoted $A \rtimes G$ is the $\mathbf{k}$-algebra $(A \otimes \mathbf{k}G, *)$, where $*$ is the product given by

$$\sum_{g \in G} (a_g \otimes g) * \sum_{h \in G} (b_h \otimes h) := \sum_{k \in G} \Bigg(\sum_{g,h \in G | gh = k} a_g b_h^g \Bigg) \otimes k, \tag{2.1}$$

for $a_g, b_g \in A$ with $g \in G$ ([Bou, Chap. 3, p. 180, Exer. 11]).

Proposition 2.3 (Universal property of the crossed product algebra). *For any* $\mathbf{k}$*-algebra* B*, there is a natural bijection between the set* $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(A \rtimes G, B)$ *and the set of pairs* $(f, \tau) \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}(A, B) \times \mathrm{Mor}_{\mathrm{grp}}(G, B^\times)$ *such that* $f(a^g) = \tau(g) f(a) \tau(g)^{-1}$.

Proof. Indeed, given a $\mathbf{k}$-algebra morphism $\beta \colon A \rtimes G \to B$ we consider

- the $\mathbf{k}$-algebra morphism $f \colon A \to B$ given for any $a \in A$ by $f(a) = \beta(a \otimes 1)$;
- the group morphism $\tau \colon G \to B^\times$ given for any $g \in G$ by $\tau(g) = \beta(1 \otimes g)$.

These morphisms verify

$$\begin{aligned}
\tau(g) f(a) \tau(g^{-1}) &= \beta(1 \otimes g) \beta(a \otimes 1) \beta(1 \otimes g^{-1}) \\
&= \beta((1 \otimes g) * (a \otimes 1) * (1 \otimes g^{-1})) \\
&= \beta((a^g \otimes g) * (1 \otimes g^{-1})) \\
&= \beta(a^g \otimes 1) = f(a^g).
\end{aligned}$$

This shows that the map $\beta \mapsto (f, \tau)$ is well defined. Now let us define a converse map in order to get a bijection. Given any pair (f, τ) of morphisms satisfying the conditions of the proposition, we set $\beta \colon a \otimes g \mapsto f(a)\tau(g)$ for any $a \otimes g \in A \rtimes G$. This is a $\mathbf{k}$-algebra morphism. Indeed, for any $a \otimes g$ and $b \otimes h \in A \rtimes G$,

$$\begin{aligned}
\beta((a \otimes g) * (b \otimes h)) &= \beta(ab^g \otimes gh) \\
&= f(ab^g)\tau(gh) = f(a) f(b^g) \tau(g) \tau(h) \\
&= f(a) \tau(g) f(b) \tau(g)^{-1} \tau(g) \tau(h) \\
&= f(a) \tau(g) f(b) \tau(h) \\
&= \beta(a \otimes g) \beta(b \otimes h).
\end{aligned}$$

Thus the map $(f, \tau) \to \beta$ is also well defined. Finally, one can easily check that the composition of the two maps on both sides gives the identity. □

Now recall that $g \mapsto t_g$ defines an action of G on $\mathbf{k}\langle\langle X \rangle\rangle$ by $\mathbf{k}$-algebra automorphisms ([Rac, §3.1.1]). We can then consider the crossed product algebra $\mathbf{k}\langle\langle X \rangle\rangle \rtimes G$ for this action.

Proposition 2.4. *We describe the presentation of the crossed product algebra* $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$*:*

(i) *There is a unique* $\mathbf{k}$*-algebra morphism* $\alpha\colon \widehat{\mathcal{V}}_G \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ *such that* $e_0 \mapsto x_0 \otimes 1$, $e_1 \mapsto -x_1 \otimes 1$ *and* $g \mapsto 1 \otimes g$.

(ii) *There is a unique* $\mathbf{k}$*-algebra morphism* $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$ *such that* $x_0 \otimes 1 \mapsto e_0$ *and for* $g \in G$, $x_g \otimes 1 \mapsto -ge_1g^{-1}$ *and* $1 \otimes g \mapsto g$.

(iii) *The morphisms* α *and* β *given respectively in* (i) *and* (ii) *are isomorphisms which are the inverse of one another.*

Proof. (i) We verify that the images by the morphism α of the generators of $\widehat{\mathcal{V}}_G$ satisfy the relations of $\widehat{\mathcal{V}}_G$:

- For $g, h \in G$, $\alpha(g) * \alpha(h) = (1 \otimes g) * (1 \otimes h) = 1t_g(1) \otimes gh = 1 \otimes gh = \alpha(gh)$.
- $\alpha(1_G) = 1 \otimes 1_G = \alpha(1)$.
- For $g \in G$, $\alpha(g) * \alpha(e_0) = (1 \otimes g) * (x_0 \otimes 1) = 1t_g(x_0) \otimes g = x_0 \otimes g$. On the other hand, we have $\alpha(e_0) * \alpha(g) = (x_0 \otimes 1) * (1 \otimes g) = x_0t_1(1) \otimes g = x_0 \otimes g$. Thus $\alpha(g) * \alpha(e_0) = \alpha(e_0) * \alpha(g)$.

(ii) First, since for any $g \in G$, the element $-ge_1g^{-1}$ is of degree 1, there is a unique $\mathbf{k}$-algebra morphism $f\colon \mathbf{k}\langle\langle X\rangle\rangle \to \widehat{\mathcal{V}}_G$ such that $x_0 \mapsto e_0$, $x_g \mapsto -ge_1g^{-1}$. Second, there is a unique group morphism $\tau\colon G \to \widehat{\mathcal{V}}_G^{\times}$ given by $g \mapsto g$. Next, for any $g \in G$, the maps $\mathbf{k}\langle\langle X\rangle\rangle \to \widehat{\mathcal{V}}_G$ defined by $a \mapsto f(t_g(a))$ and $a \mapsto \tau(g)f(a)\tau(g)^{-1}$ are $\mathbf{k}$-algebra morphisms that are equal by restriction on generators x_h ($h \in \{0\} \sqcup G$) of $\mathbf{k}\langle\langle X\rangle\rangle$. Indeed,

$$\tau(g)f(x_0)\tau(g)^{-1} = ge_0g^{-1} = e_0gg^{-1} = e_0 = f(x_0) = f(t_g(x_0)),$$

and for $h \in G$,

$$\tau(g)f(x_h)\tau(g)^{-1} = g(-he_1h^{-1})g^{-1} = -ghe_1(gh)^{-1} = f(x_{gh}) = f(t_g(x_h)).$$

We then have for any $g \in G$ and any $a \in \mathbf{k}\langle\langle X\rangle\rangle$, $f(t_g(a)) = \tau(g)f(a)\tau(g)^{-1}$. Finally, according to the universal property of crossed products, the pair (f, τ) gives a unique $\mathbf{k}$-algebra morphism $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$, $a \otimes g \mapsto f(a)\tau(g)$ which verifies $\beta(x_0 \otimes 1) = f(x_0)\tau(1) = e_0$, $\beta(x_g \otimes 1) = f(x_g)\tau(1) = -ge_1g^{-1}$ and $\beta(1 \otimes g) = f(1)\tau(g) = g$, for $g \in G$.

(iii) It is enough to show that the compositions of α and β give the identity. First, since $\beta \circ \alpha\colon \widehat{\mathcal{V}}_G \to \widehat{\mathcal{V}}_G$, it is enough to compute it on generators. We have $e_0 \mapsto x_0 \otimes 1 \mapsto e_0$, $e_1 \mapsto -x_1 \otimes 1 \mapsto e_1$ and $g \mapsto 1 \otimes g \mapsto g$. Thus $\beta \circ \alpha = \mathrm{id}_{\widehat{\mathcal{V}}_G}$.

For the converse, we show that $\alpha \circ \beta \in \mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle X\rangle\rangle \rtimes G, \mathbf{k}\langle\langle X\rangle\rangle \rtimes G)$ and the identity of $\mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ have the same image via the bijection of the universal property of crossed products. The image of the identity is the pair

$$f_{\mathrm{id}}\colon a \mapsto a \otimes 1 \quad \text{and} \quad \tau_{\mathrm{id}}(g) = 1 \otimes g.$$

Next, let us compute the image of $\alpha \circ \beta$. The $\mathbf{k}$-algebra morphism f is given for any $a \in \mathbf{k}\langle\langle X\rangle\rangle$ by

$$f(a) = \alpha \circ \beta(a \otimes 1).$$

Since it is a $\mathbf{k}$-algebra morphism, it is enough to determine it on x_g, $g \in \{0\} \sqcup G$. We have

$$f(x_0) = \alpha \circ \beta(x_0 \otimes 1) = \alpha(e_0) = x_0 \otimes 1,$$

and for $g \in G$,

$$\begin{aligned} f(x_g) &= \alpha \circ \beta(x_g \otimes 1) = \alpha(-ge_1g^{-1}) \\ &= -\alpha(g) * \alpha(e_1) * \alpha(g^{-1}) \\ &= -(1 \otimes g) * (-x_1 \otimes 1) * (1 \otimes g^{-1}) \\ &= (t_g(x_1) \otimes g) * (1 \otimes g^{-1}) = x_g \otimes 1. \end{aligned}$$

We then deduce that for any $a \in \mathbf{k}\langle\langle X\rangle\rangle$, $f(a) = a \otimes 1$. Next, the group morphism $\tau\colon G \to (\mathbf{k}\langle\langle X\rangle\rangle \rtimes G)^{\times}$ is given for any $g \in G$ by

$$\tau(g) = \alpha \circ \beta(1 \otimes g) = \alpha(g) = 1 \otimes g.$$

Finally, by uniqueness of the images we conclude that $\alpha \circ \beta = \mathrm{id}_{\mathbf{k}\langle\langle X\rangle\rangle \rtimes G}$. □

2.1.3. The bialgebra $(\widehat{\mathcal{W}}_G, \widehat{\Delta}_G^{\mathcal{W}})$ and the coalgebra $(\widehat{\mathcal{M}}_G, \widehat{\Delta}_G^{\mathcal{M}})$.

Proposition 2.5. *The family*

$$(e_0^{n_1-1}g_1e_1 \cdots e_0^{n_r-1}g_re_1e_0^{n_{r+1}-1}g_{r+1})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\dots,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$*-module* $\widehat{\mathcal{V}}_G$*.*

Proof. Since the family

$$((-1)^r x_0^{n_1-1}x_{g_1} \cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}-1})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\dots,g_r\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle$, it follows that the family

$$((-1)^r x_0^{n_1-1}x_{g_1} \cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}-1} \otimes g_1\cdots g_rg_{r+1})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\dots,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle \otimes \mathbf{k}G$. Thus, its image by the bijection β defined in Proposition 2.4(ii) is a basis of $\widehat{\mathcal{V}}_G$. Moreover, for $r \in \mathbb{Z}_{\geq 0}, n_1, \dots, n_{r+1} \in \mathbb{Z}_{>0}$

and $g_1, \dots, g_{r+1} \in G$, we have

$$\begin{aligned} & x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1} \otimes g_1 \cdots g_r g_{r+1} \\ & \quad = (x_0^{n_1-1} \otimes 1) * (x_{g_1} \otimes 1) * \cdots * (x_0^{n_r-1} \otimes 1) * (x_{g_1\cdots g_r} \otimes 1) \\ & \qquad * (x_0^{n_{r+1}-1} \otimes 1) * (1 \otimes g_1) * \cdots * (1 \otimes g_r) * (1 \otimes g_{r+1}). \end{aligned} \tag{2.2}$$

Then

$$\begin{aligned} & \beta((-1)^r x_0^{n_1-1} x_{g_1} \cdots x_0^{n_r-1} x_{g_1\cdots g_r} x_0^{n_{r+1}-1} \otimes g_1 \cdots g_r g_{r+1}) \\ & \quad = (-1)^r \beta(x_0^{n_1-1} \otimes 1)\beta(x_{g_1} \otimes 1) \cdots \beta(x_0^{n_r-1} \otimes 1)\beta(x_{g_1\cdots g_r} \otimes 1) \\ & \qquad \cdot \beta(x_0^{n_{r+1}-1} \otimes 1)\beta(1 \otimes g_1) \cdots \beta(1 \otimes g_r)\beta(1 \otimes g_{r+1}) \\ & \quad = e_0^{n_1-1} g_1 e_1 g_1^{-1} \cdots e_0^{n_r-1} g_1 \cdots g_r e_1 g_1^{-1} \cdots g_r^{-1} e_0^{n_{r+1}-1} g_1 \cdots g_r g_{r+1} \\ & \quad = e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_1^{-1} \cdots g_{r-1}^{-1} g_1 \cdots g_{r-1} g_r e_1 e_0^{n_{r+1}-1} g_1^{-1} \\ & \qquad \cdots g_r^{-1} g_1 \cdots g_r g_{r+1} \\ & \quad = e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1}, \end{aligned} \tag{2.3}$$

where the first equality comes from (2.2) and the fact that $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$ is a $\mathbf{k}$-algebra morphism. The second equality is obtained by computing the images of appropriate elements by β. The third equality is a consequence of the equality $ge_0 = e_0 g$ for any $g \in G$ and the last one comes from the fact that the group G is abelian. □

Proposition 2.6. *We have the following:*

(i) *The family*

$$\{1\} \cup (e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1} e_1)_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_r,n_{r+1}\in\mathbb{Z}_{>0}, \\ g_1,\dots,g_r,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$*-module* $\widehat{\mathcal{W}}_G$*.*

(ii) *The* $\mathbf{k}$*-subalgebra* $\widehat{\mathcal{W}}_G$ *is topologically freely generated by the family*

$$Z = \left\{ z_{n,g} := -e_0^{n-1} g e_1 \;\middle|\; (n,g) \in \mathbb{Z}_{>0} \times G \right\},$$

where $\deg(z_{n,g}) = n$*.*

Proof. (i) First, $\widehat{\mathcal{W}}_G$ is the image of the $\mathbf{k}$-module morphism $\mathbf{k} \oplus \widehat{\mathcal{V}}_G \to \widehat{\mathcal{V}}_G$, $(\lambda, v) \mapsto \lambda + ve_1$. Second, according to Proposition 2.5, the family

$$(1,0), (0, e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_r,n_{r+1}\in\mathbb{Z}_{>0}, \\ g_1,\dots,g_r,g_{r+1}\in G}}$$

is a basis of the $\mathbf{k}$-module $\mathbf{k} \oplus \widehat{\mathcal{V}}_G$. Moreover, the image of this basis by this $\mathbf{k}$-module morphism is the family

$$\{1\} \cup (e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_r-1} g_r e_1 e_0^{n_{r+1}-1} g_{r+1} e_1)_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_r,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\dots,g_r,g_{r+1}\in G}}$$

which is free since it is contained in a basis of the target. This implies that this family is a basis of the image of the previous morphism which is $\widehat{\mathcal{W}}_G$.

(ii) Let $\mathbf{k}\langle\langle Z\rangle\rangle$ be the topological free algebra over the letters $z_{n,g}$ ($n \in \mathbb{Z}_{>0}$, $g \in G$), which we view as free variables with $\deg(z_{n,g}) = n$. Then there is a unique $\mathbf{k}$-algebra morphism $\mathbf{k}\langle\langle Z\rangle\rangle \to \widehat{\mathcal{W}}_G$ given by $z_{n,g} \mapsto -e_0^{n-1} g e_1$. Let us show that it is an isomorphism:

The free $\mathbf{k}$-module $\mathbf{k}\langle\langle Z\rangle\rangle$ has basis

$$\{1\} \cup (z_{n_1,g_1} \cdots z_{n_{r+1},g_{r+1}})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0}\\ g_1,\dots,g_{r+1}\in G}}$$

and, as a $\mathbf{k}$-module, $\widehat{\mathcal{W}}_G$ has basis

$$\{1\} \cup (e_0^{n_1-1} g_1 e_1 \cdots e_0^{n_{r+1}-1} g_{r+1} e_1)_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\dots,g_{r+1}\in G}}$$

according to (i). One computes the image by $z_{n,g} \mapsto -e_0^{n-1} g e_1$ of the former basis and finds it to be equal to the latter basis. Therefore, $z_{n,g} \mapsto -e_0^{n-1} g e_1$ induces a bijection between the two bases – up to appropriate signs – and then a bijection between $\mathbf{k}\langle\langle Z\rangle\rangle$ and $\widehat{\mathcal{W}}_G$. Hence, $z_{n,g} \mapsto -e_0^{n-1} g e_1$ is a $\mathbf{k}$-algebra isomorphism between $\mathbf{k}\langle\langle Z\rangle\rangle$ and $\widehat{\mathcal{W}}_G$. □

So, from now on, by abuse of notation, we will identify elements of $\widehat{\mathcal{W}}_G$ with elements of $\mathbf{k}\langle\langle Z\rangle\rangle$ by the $\mathbf{k}$-algebra isomorphism $z_{n,g} \mapsto -e_0^{n-1} g e_1$.

Proposition 2.7. *There exists a $\mathbf{k}$-module isomorphism $\kappa\colon \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G$ uniquely determined by the condition that the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathcal{M}}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\kappa]{} & \widehat{\mathcal{M}}_G \end{array} \tag{2.4}$$

commutes.

We will prove this proposition by using the following general lemma. In this lemma, for any $\mathbf{k}$-module M and any submodule M', let us denote the canonical projection by $\mathrm{can}(M, M')\colon M \to M/M'$.

Lemma 2.8. *Let $f\colon M \to N$ be a $\mathbf{k}$-module morphism. Let M' be a submodule of M and N', N'' two submodules of N such that*

(i) $f(M') \subset N' \subset f(M') + N''$ *and*
(ii) $\mathrm{can}(N, N'') \circ f\colon M \to N/N''$ *is a $\mathbf{k}$-module isomorphism.*

Then there is a unique $\mathbf{k}$-module morphism $\bar{f}\colon M/M' \to N/(N' + N'')$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \mathrm{can}(M,M')}\downarrow & & \downarrow{\scriptstyle \mathrm{can}(N,N'+N'')} \\ M/M' & \xrightarrow[\bar{f}]{} & N/(N'+N'') \end{array} \tag{2.5}$$

commutes. Moreover, $\bar{f}$ is a $\mathbf{k}$-module isomorphism.

Proof. Thanks to (i), $f(M') \subset N'$. This implies that $f(M')+N'' \subset N'+N''$. From (i) again, we have $N' \subset f(M') + N''$. This implies that $N' + N'' \subset f(M') + N''$. Therefore

$$f(M') + N'' = N' + N''. \tag{2.6}$$

Next, from (ii), we have that $\mathrm{can}(N, N'') \circ f\colon M \to N \to N/N''$ is an isomorphism. One checks that it restricts to an isomorphism from M' to $(f(M') + N'')/N''$. Thanks to equality (2.6), this yields an isomorphism from M' to $(N' + N'')/N''$. This allows us to construct a unique $\mathbf{k}$-module morphism $\tilde{f}\colon M/M' \to (N/N'')/((N' + N'')/N'')$ such that the lower square of the diagram

$$\begin{array}{ccc} M' & \xrightarrow{\mathrm{can}(N,N'')\circ f_{|M'}} & (N'+N'')/N'' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\mathrm{can}(N,N'')\circ f} & N/N'' \\ {\scriptstyle \mathrm{can}(M,M')}\downarrow & & \downarrow{\scriptstyle \mathrm{can}(N/N'',(N'+N'')/N'')} \\ M/M' & \xrightarrow[\tilde{f}]{} & (N/N'')/((N'+N'')/N'') \end{array} \tag{2.7}$$

commutes. Moreover, since $\mathrm{can}(N,N'') \circ f\colon M \to N/N''$ is an isomorphism, so is $\tilde{f}\colon M/M' \to (N/N'')/((N' + N'')/N'')$. Finally, we construct an isomorphism $\bar{f}\colon M/M' \to N/(N' + N'')$ by composing $\tilde{f}$ with the inverse map $(N/N'')/((N' + N'')/N'') \simeq N/(N' + N'')$ given by the third isomorphism theorem. Thanks to diagram (2.7), the isomorphism $\bar{f}\colon M/M' \to N/(N' + N'')$ is such that diagram (2.5) commutes. $\square$

Proof of Proposition 2.7. This is done by application of Lemma 2.8 for $M = \mathbf{k}\langle\langle X\rangle\rangle$, $N = \widehat{\mathcal{V}}_G$, $M' = \mathbf{k}\langle\langle X\rangle\rangle x_0$, $N' = \widehat{\mathcal{V}}_G e_0$, $N'' = \sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$ and $f = \beta\circ(-\otimes 1)$. It, therefore, suffices to prove that criteria (i) and (ii) of Lemma 2.8 are satisfied.

Criterion (i). $\beta(\mathbf{k}\langle\langle X\rangle\rangle x_0\otimes 1)\subset\widehat{\mathcal{V}}_G e_0\subset\beta(\mathbf{k}\langle\langle X\rangle\rangle x_0\otimes 1)+\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$.

For the first inclusion, we have for any $a\in\mathbf{k}\langle\langle X\rangle\rangle$,

$$\beta(ax_0\otimes 1)=\beta(a\otimes 1)\beta(x_0\otimes 1)=\beta(a\otimes 1)e_0\in\widehat{\mathcal{V}}_G e_0.$$

Therefore, $\beta(\mathbf{k}\langle\langle X\rangle\rangle x_0\otimes 1)\subset\widehat{\mathcal{V}}_G e_0$.

For the second inclusion, by using the basis of $\widehat{\mathcal{V}}_G$ described in Proposition 2.5, we have for $r\in\mathbb{Z}_{\geq 0}$, $n_1,\dots,n_{r+1}\in\mathbb{Z}_{>0}$ and $g_1,\dots,g_{r+1}\in G$,

$$\begin{aligned}
&(e_0^{n_1-1}g_1e_1\cdots e_0^{n_r-1}g_re_1e_0^{n_{r+1}-1}g_{r+1})e_0\\
&\quad= e_0^{n_1-1}g_1e_1\cdots e_0^{n_r-1}g_re_1e_0^{n_{r+1}}g_{r+1}\\
&\quad=(-1)^r\beta(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}}\otimes g_1\cdots g_{r+1})\\
&\quad=(-1)^r\beta((x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}-1})x_0\otimes 1)g_1\cdots g_{r+1}\\
&\quad=(-1)^r\beta((x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}-1})x_0\otimes 1)\\
&\qquad+(-1)^r\beta((x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}x_0^{n_{r+1}-1})x_0\otimes 1)(g_1\cdots g_{r+1}-1),
\end{aligned}$$

where the first equality comes from the relation $ge_0=e_0g$ for any $g\in G$, the second from computation (2.3) and the third from the fact that $ax_0\otimes g=(ax_0\otimes 1)*(1\otimes g)$ for any $a\in\mathbf{k}\langle\langle X\rangle\rangle$ and any $g\in G$. Finally, the last equality shows that we obtain an element of $\beta(\mathbf{k}\langle\langle X\rangle\rangle x_0\otimes 1)+\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$, thus proving the claimed inclusion.

Criterion (ii). $\mathrm{can}(\widehat{\mathcal{V}}_G,\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1))\circ\beta\circ(-\otimes 1)\colon\mathbf{k}\langle\langle X\rangle\rangle\to\widehat{\mathcal{V}}_G/\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$ is an isomorphism.

Let us consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle X\rangle\rangle\otimes\bigoplus_{g\in G}\mathbf{k}G & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G\\
\downarrow & & \|\\
\bigoplus_{g\in G}(\mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G) & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G\\
{\scriptstyle\bigoplus_{g\in G}\beta}\downarrow & & \downarrow{\scriptstyle\beta}\\
\bigoplus_{g\in G}\widehat{\mathcal{V}}_G & \longrightarrow & \widehat{\mathcal{V}}_G,
\end{array}\tag{2.8}$$

where the top horizontal arrow is the tensor product of the identity and the **k**-module morphism

$$\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}G,\quad (h_g)_{g\in G}\mapsto\sum_{g\in G}h_g(g-1),$$

and the bottom horizontal arrow is the **k**-module morphism

$$\bigoplus_{g\in G}\widehat{\mathcal{V}}_G\to\widehat{\mathcal{V}}_G,\quad (v_g)_{g\in G}\mapsto\sum_{g\in G}v_g(g-1).$$

Since the vertical arrows are isomorphisms, they induce an isomorphism between the cokernels of the top and bottom morphisms. We can then extend the above diagram in the following way:

$$\begin{array}{ccccc}
\mathbf{k}\langle\langle X\rangle\rangle\otimes\bigoplus_{g\in G}\mathbf{k}G & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G & \longrightarrow & \operatorname{coker}\Big(\mathbf{k}\langle\langle X\rangle\rangle\otimes\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G\Big) \\
\downarrow & & \| & & \\
\bigoplus_{g\in G}(\mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G) & \longrightarrow & \mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G & & \downarrow \\
\downarrow{\scriptstyle \bigoplus_{g\in G}\beta} & & \downarrow{\scriptstyle\beta} & & \\
\bigoplus_{g\in G}\widehat{\mathcal{V}}_G & \longrightarrow & \widehat{\mathcal{V}}_G & \longrightarrow & \operatorname{coker}\Big(\bigoplus_{g\in G}\widehat{\mathcal{V}}_G\to\widehat{\mathcal{V}}_G\Big).
\end{array}\tag{2.9}$$

On the other hand, we have

$$\operatorname{coker}\left(\bigoplus_{g\in G}\widehat{\mathcal{V}}_G\to\widehat{\mathcal{V}}_G\right)=\widehat{\mathcal{V}}_G\Big/\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$$

and

$$\operatorname{coker}\left(\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}G\right)=\mathbf{k}G\Big/\sum_{g\in G}\mathbf{k}G(g-1)\simeq\mathbf{k}.$$

Therefore,

$$\begin{aligned}\operatorname{coker}\left(\mathbf{k}\langle\langle X\rangle\rangle\otimes\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}G\right)&\simeq\mathbf{k}\langle\langle X\rangle\rangle\otimes\operatorname{coker}\left(\bigoplus_{g\in G}\mathbf{k}G\to\mathbf{k}G\right)\\&\simeq\mathbf{k}\langle\langle X\rangle\rangle\otimes\mathbf{k}\simeq\mathbf{k}\langle\langle X\rangle\rangle.\end{aligned}$$

Thus, the isomorphism between cokernels establishes that $\mathbf{k}\langle\langle X\rangle\rangle$ is isomorphic to $\widehat{\mathcal{V}}_G/\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1)$. Moreover, thanks to the commutativity of diagram (2.9), this isomorphism is exactly $\operatorname{can}(\widehat{\mathcal{V}}_G,\sum_{g\in G}\widehat{\mathcal{V}}_G(g-1))\circ\beta\circ(-\otimes 1)$. $\square$

Corollary 2.9. *We have the following:*

(i) *The diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\varpi} & \widehat{\mathcal{W}}_G \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathcal{M}}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\kappa\circ\bar{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G \end{array} \tag{2.10}$$

commutes, where $\varpi\colon \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G$ *is the* $\mathbf{k}$*-algebra isomorphism uniquely defined by* $y_{n,g} \mapsto z_{n,g}$*.*

(ii) *The map* $-\cdot 1_{\mathcal{M}}\colon \widehat{\mathcal{W}}_G \to \widehat{\mathcal{M}}_G$ *is a* $\mathbf{k}$*-module isomorphism and* $\widehat{\mathcal{M}}_G$ *is free of rank* 1 *as a* $\widehat{\mathcal{W}}_G$*-module.*

Proof. (i) One needs to show the equality of two maps from $\mathbf{k}\langle\langle Y\rangle\rangle$ to $\widehat{\mathcal{M}}_G$. Since these maps are both $\mathbf{k}$-module morphisms, it is enough to show the equality of the images of the elements of a basis of the source module. Such a basis is ([Rac, §2.2.7.])

$$(y_{n_1,g_1}\cdots y_{n_rg_r})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_r\in\mathbb{Z}_{>0},\\ g_1,\dots,g_r\in G}}.$$

For $r\in\mathbb{Z}_{\geq 0}$, $n_1,\dots,n_r\in\mathbb{Z}_{>0}$ and $g_1,\dots,g_r\in G$ we have

$$(-\cdot 1_{\mathcal{M}})\circ\varpi(y_{n_1,g_1}\cdots y_{n_r,g_r}) = z_{n_1,g_1}\cdots z_{n_r,g_r}\cdot 1_{\mathcal{M}}.$$

On the other hand,

$$\begin{aligned} \kappa\circ\bar{\mathbf{q}}^{-1}\circ\pi_Y(y_{n_1,g_1}\cdots y_{n_r,g_r}) &= \kappa(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_1-1}x_{g_1\cdots g_r}) \\ &= \beta(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_1\cdots g_r}\otimes 1)\cdot 1_{\mathcal{M}} \\ &= (-1)^r e_0^{n_1-1}g_1e_1\cdots e_0^{n_r-1}g_re_1g_1^{-1}\cdots g_r^{-1}\cdot 1_{\mathcal{M}} \\ &= (-e_0^{n_1-1}g_1e_1)\cdots(-e_0^{n_r-1}g_re_1)\cdot 1_{\mathcal{M}} \\ &= z_{n_1,g_1}\cdots z_{n_r,g_r}\cdot 1_{\mathcal{M}}, \end{aligned}$$

where the first equality comes from [Rac, §2.2.7], the second from the commutative diagram (2.4), the third from computation (2.3) with $n_{r+1}=1$ and $g_{r+1}=(g_1\cdots g_r)^{-1}$ and the fourth from the fact that for any $v\in\widehat{\mathcal{V}}_G$ and any $g\in G$, we have $vg\cdot 1_{\mathcal{M}} = v\cdot 1_{\mathcal{M}}$.

(ii) First, the following maps are $\mathbf{k}$-module isomorphisms:

- $\varpi\colon \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G$: it sends the basis

$$(y_{n_1,g_1}\cdots y_{n_r,g_r})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\dots,n_r\in\mathbb{Z}_{>0},\\ g_1,\dots,g_r\in G}}$$

of the $\mathbf{k}$-module $\mathbf{k}\langle\langle Y\rangle\rangle$ to the basis

$$(z_{n_1,g_1}\cdots z_{n_r,g_r})_{\substack{r\in\mathbb{Z}_{\geq 0},n_1,\dots,n_r\in\mathbb{Z}_{>0},\\ g_1,\dots,g_r\in G}}$$

of the $\mathbf{k}$-module $\widehat{\mathcal{W}}_G$ (where the latter family is a basis of $\widehat{\mathcal{W}}_G$ thanks to Proposition 2.6).

- $\pi_Y\colon \mathbf{k}\langle\langle Y\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$: see [Rac, §2.2.5].
- $\kappa\circ\bar{\mathbf{q}}^{-1}\colon \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G$: see Proposition 2.7 and [Rac, §2.2.7].

Next, the diagram (2.10) commutes, thanks to (i). This allows us to conclude that the map $-\cdot 1_{\mathcal{M}}\colon \widehat{\mathcal{W}}_G \to \widehat{\mathcal{M}}_G$ is a $\mathbf{k}$-module isomorphism and that $\widehat{\mathcal{M}}_G$ is a free $\widehat{\mathcal{W}}_G$-module of rank 1. □

Remark 2.10. If $G\neq\{1\}$, the composed algebra morphisms

$$\mathbf{k}\langle\langle Y\rangle\rangle \xrightarrow{\varpi} \widehat{\mathcal{W}}_G \hookrightarrow \widehat{\mathcal{V}}_G \quad\text{and}\quad \mathbf{k}\langle\langle Y\rangle\rangle \hookrightarrow \mathbf{k}\langle\langle X\rangle\rangle \xrightarrow{\beta\circ(-\otimes 1)} \widehat{\mathcal{V}}_G$$

do not coincide. This is motivated by the presence of $\bar{\mathbf{q}}^{-1}$ in diagram (2.10) whereas it is missing in diagram (2.4).

Now we are able to put more structure on $\widehat{\mathcal{W}}_G$ and $\widehat{\mathcal{M}}_G$. More precisely, we are going to define a coproduct on $\widehat{\mathcal{W}}_G$ and a coproduct on $\widehat{\mathcal{M}}_G$.

Proposition-Definition 2.11. *We consider the coproducts on $\widehat{\mathcal{W}}_G$ and on $\widehat{\mathcal{M}}_G$:*

(i) *There exists a unique topological $\mathbf{k}$-algebra morphism $\widehat{\Delta}^{\mathcal{W}}_G\colon \widehat{\mathcal{W}}_G \to \widehat{\mathcal{W}}_G^{\widehat{\otimes}2}$ such that for any $(n,g)\in\mathbb{Z}_{>0}\times G$,*

$$\widehat{\Delta}^{\mathcal{W}}_G(z_{n,g}) = z_{n,g}\otimes 1 + 1\otimes z_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} z_{k,h}\otimes z_{n-k,gh^{-1}}. \tag{2.11}$$

The pair $(\widehat{\mathcal{W}}_G,\widehat{\Delta}^{\mathcal{W}}_G)$ is then a topological bialgebra.

(ii) *There exists a unique topological $\mathbf{k}$-module morphism $\widehat{\Delta}^{\mathcal{M}}_G\colon \widehat{\mathcal{M}}_G \to \widehat{\mathcal{M}}_G^{\widehat{\otimes}2}$ such that the diagram*

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G & \xrightarrow{\widehat{\Delta}^{\mathcal{W}}_G} & (\widehat{\mathcal{W}}_G)^{\widehat{\otimes}2} \\ {\scriptstyle -\cdot 1_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathcal{M}}^{\otimes 2}} \\ \widehat{\mathcal{M}}_G & \xrightarrow[\widehat{\Delta}^{\mathcal{M}}_G]{} & (\widehat{\mathcal{M}}_G)^{\widehat{\otimes}2} \end{array} \tag{2.12}$$

commutes. The pair $(\widehat{\mathcal{M}}_G,\widehat{\Delta}^{\mathcal{M}}_G)$ is then a cocommutative coassociative coalgebra.

(iii) *For any* $w \in \widehat{\mathcal{W}}_G$ *and any* $m \in \widehat{\mathcal{M}}_G$ *we have*

$$\widehat{\Delta}_G^{\mathcal{M}}(w \cdot m) = \widehat{\Delta}_G^{\mathcal{W}}(w) \cdot \widehat{\Delta}_G^{\mathcal{M}}(m). \tag{2.13}$$

Proof. (i) This is a consequence of Proposition 2.6(ii).

(ii) This is a consequence of (i) and Corollary 2.9(ii).

(iii) Since $- \cdot 1_{\mathcal{M}} \colon \widehat{\mathcal{W}}_G \to \widehat{\mathcal{M}}_G$ is a $\mathbf{k}$-module isomorphism, for $m \in \widehat{\mathcal{M}}_G$ there exists a unique $w' \in \widehat{\mathcal{W}}_G$ such that $m = w' \cdot 1_{\mathcal{M}}$. We then have

$$\begin{aligned}\widehat{\Delta}_G^{\mathcal{M}}(w \cdot m) &= \widehat{\Delta}_G^{\mathcal{M}}(ww' \cdot 1_{\mathcal{M}}) = \widehat{\Delta}_G^{\mathcal{W}}(ww') \cdot 1_{\mathcal{M}}^{\otimes 2} \\ &= \widehat{\Delta}_G^{\mathcal{W}}(w)\widehat{\Delta}_G^{\mathcal{W}}(w') \cdot 1_{\mathcal{M}}^{\otimes 2} = \widehat{\Delta}_G^{\mathcal{W}}(w) \cdot (\widehat{\Delta}_G^{\mathcal{W}}(w') \cdot 1_{\mathcal{M}}^{\otimes 2}) \\ &= \widehat{\Delta}_G^{\mathcal{W}}(w) \cdot \widehat{\Delta}_G^{\mathcal{M}}(w' \cdot 1_{\mathcal{M}}) = \widehat{\Delta}_G^{\mathcal{W}}(w) \cdot \widehat{\Delta}_G^{\mathcal{M}}(m),\end{aligned}$$

where the first and the fourth equalities come from $- \cdot 1_{\mathcal{M}} \colon \widehat{\mathcal{W}}_G \to \widehat{\mathcal{M}}_G$ being a $\widehat{\mathcal{W}}_G$-module isomorphism, the second and the fifth from the commutative diagram (2.12) and the third from the fact that $\widehat{\Delta}_G^{\mathcal{W}}$ is a $\mathbf{k}$-algebra morphism. □

§2.2. Actions of the group $(\mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle), \circledast)$ by automorphisms

We recall that the map $\beta \colon \mathbf{k}\langle\langle X \rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$ is the $\mathbf{k}$-algebra isomorphism given in Proposition 2.4(ii).

2.2.1. Actions of $(\mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle), \circledast)$ by algebra automorphisms.

Proposition-Definition 2.12. *Let* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X \rangle\rangle)$. *There exists a unique topological* $\mathbf{k}$*-algebra automorphism* $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ *of* $\widehat{\mathcal{V}}_G$ *extending the automorphism* aut_Ψ *of* $\mathbf{k}\langle\langle X \rangle\rangle$ *of* (1.7) *such that*

$$e_0 \mapsto e_0; \quad e_1 \mapsto \beta(\Psi^{-1} \otimes 1)e_1\beta(\Psi \otimes 1); \quad g \mapsto g, \ \ \text{for } g \in G, \tag{2.14}$$

Proof. First, let us verify that the images by the morphism $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ of the generators of $\widehat{\mathcal{V}}_G$ satisfy the relations of $\widehat{\mathcal{V}}_G$. Indeed, for $g, h \in G$ we have

- $\mathrm{aut}_\Psi^{\mathcal{V},(0)}(g) \cdot \mathrm{aut}_\Psi^{\mathcal{V},(0)}(h) = g \cdot h = gh = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(gh)$;
- $\mathrm{aut}_\Psi^{\mathcal{V},(0)}(1_G) = 1_G = 1 = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(1)$;
- $\mathrm{aut}_\Psi^{\mathcal{V},(0)}(g) \cdot \mathrm{aut}_\Psi^{\mathcal{V},(0)}(e_0) = g \cdot e_0 = e_0 \cdot g = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(e_0) \cdot \mathrm{aut}_\Psi^{\mathcal{V},(0)}(g)$.

This proves the existence and uniqueness of the algebra endomorphism $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$. Next, in order to prove that $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ is an automorphism, we show that the diagram

$$\begin{array}{ccc} \mathbf{k}\langle\langle X \rangle\rangle \rtimes G & \xrightarrow{\beta} & \widehat{\mathcal{V}}_G \\ {\scriptstyle \mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G}} \downarrow & & \downarrow {\scriptstyle \mathrm{aut}_\Psi^{\mathcal{V},(0)}} \\ \mathbf{k}\langle\langle X \rangle\rangle \rtimes G & \xrightarrow[\beta]{} & \widehat{\mathcal{V}}_G \end{array} \tag{2.15}$$

commutes, where aut_Ψ is the $\mathbf{k}$-algebra automorphism in (1.7). Since all arrows of diagram (2.15) are $\mathbf{k}$-algebra morphisms, it is enough to check the commutativity on generators:

- $\mathrm{aut}_\Psi^{\mathcal{V},(0)} \circ \beta(x_0 \otimes 1) = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(e_0) = e_0$ and $\beta \circ (\mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G})(x_0 \otimes 1) = \beta(\mathrm{aut}_\Psi(x_0) \otimes 1) = \beta(x_0 \otimes 1) = e_0$.
- For $g \in G$, $\mathrm{aut}_\Psi^{\mathcal{V},(0)} \circ \beta(1 \otimes g) = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(g) = g$ and $\beta \circ (\mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G})(1 \otimes g) = \beta(\mathrm{aut}_\Psi(1) \otimes g) = \beta(1 \otimes g) = g$.
- For $g \in G$, we have

$$\mathrm{aut}_\Psi^{\mathcal{V},(0)} \circ \beta(x_g \otimes 1) = \mathrm{aut}_\Psi^{\mathcal{V},(0)}(-ge_1g^{-1}) = -g\beta(\Psi^{-1} \otimes 1)e_1\beta(\Psi \otimes 1)g^{-1}$$

and

$$\begin{aligned}
&\beta \circ (\mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G})(x_g \otimes 1) \\
&\quad = \beta(\mathrm{aut}_\Psi(x_g) \otimes 1) \\
&\quad = \beta(t_g(\Psi^{-1})x_g t_g(\Psi) \otimes 1) \\
&\quad = \beta((1 \otimes g) * (\Psi^{-1} \otimes 1) * (1 \otimes g^{-1}) * (x_g \otimes 1) * (1 \otimes g) * (\Psi \otimes 1) * (1 \otimes g^{-1})) \\
&\quad = \beta(1 \otimes g)\beta(\Psi^{-1} \otimes 1)\beta(1 \otimes g^{-1})\beta(x_g \otimes 1)\beta(1 \otimes g)\beta(\Psi \otimes 1)\beta(1 \otimes g^{-1}) \\
&\quad = g\beta(\Psi^{-1} \otimes 1)g^{-1}(-ge_1g^{-1})g\beta(\Psi \otimes 1)g^{-1} \\
&\quad = -g\beta(\Psi^{-1} \otimes 1)e_1\beta(\Psi \otimes 1)g^{-1}.
\end{aligned}$$

Therefore, $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ is an automorphism thanks to the commutativity of diagram (2.15) and the fact that $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$ and $\mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G}\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \mathbf{k}\langle\langle X\rangle\rangle \rtimes G$ are $\mathbf{k}$-algebra isomorphisms.

Finally, the automorphism $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ of $\widehat{\mathcal{V}}_G$ extends the automorphism aut_Ψ of $\mathbf{k}\langle\langle X\rangle\rangle$. Indeed, combining diagram (2.15) with the commutative diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{-\otimes 1} & \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \\
{\scriptstyle \mathrm{aut}_\Psi}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}_\Psi \otimes \mathrm{id}_{\mathbf{k}G}} \\
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[-\otimes 1]{} & \mathbf{k}\langle\langle X\rangle\rangle \rtimes G,
\end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G \\
{\scriptstyle \mathrm{aut}_\Psi}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}_\Psi^{\mathcal{V},(0)}} \\
\mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G.
\end{array} \tag{2.16}$$

□

Definition 2.13. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define $\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ to be the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{V}}_G$ given by

$$\mathrm{aut}_{\Psi}^{\mathcal{V},(1)} := \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} . \tag{2.17}$$

Proposition 2.14. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\widehat{\mathcal{V}}_G$ *by*

(i) $\mathbf{k}$-*algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G), \quad \Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(0)},$$

(ii) $\mathbf{k}$-*algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G), \quad \Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(1)} .$$

(iii) *Both group actions induce actions of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\mathbf{k}\langle\langle X\rangle\rangle$ *by* $\mathbf{k}$-*algebra automorphisms, the former by* $\Psi \mapsto \mathrm{aut}_{\Psi}$ (1.11) *and the latter by* $\Psi \mapsto \mathrm{Ad}_{\Psi} \circ \mathrm{aut}_{\Psi}$.

Proof. (i) Let us show that for any $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(0)} = \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} .$$

It suffices to prove this identity on generators. Since for $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ we have $\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(e_0) = e_0$ and $\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(g) = g$, this is immediately true for e_0 and $g \in G$. Moreover,

$$\begin{aligned}
\mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(0)}(e_1) &= \beta((\Psi \circledast \Phi)^{-1} \otimes 1)e_1\beta((\Psi \circledast \Phi) \otimes 1) \\
&= \beta(\mathrm{aut}_{\Psi}(\Phi^{-1})\Psi^{-1} \otimes 1)e_1\beta(\Psi\,\mathrm{aut}_{\Psi}(\Phi) \otimes 1) \\
&= \beta(\mathrm{aut}_{\Psi}(\Phi^{-1}) \otimes 1)\beta(\Psi^{-1} \otimes 1)e_1\beta(\Psi \otimes 1)\beta(\mathrm{aut}_{\Psi}(\Phi) \otimes 1) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(\beta(\Phi^{-1} \otimes 1))\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(e_1)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(\beta(\Phi \otimes 1)) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(\beta(\Phi^{-1} \otimes 1)e_1\beta(\Phi \otimes 1)) = \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)}(e_1),
\end{aligned}$$

where the fourth equality comes from the commutativity of diagram (2.15).

(ii) Using identity (i), we get

$$\begin{aligned}
\mathrm{aut}_{\Psi}^{\mathcal{V},(1)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(1)} &= \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{Ad}_{\beta(\Phi\otimes 1)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{Ad}_{\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(\beta(\Phi\otimes 1))} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \mathrm{Ad}_{\beta(\Psi\otimes 1)\beta(\mathrm{aut}_{\Psi}(\Phi)\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \mathrm{Ad}_{\beta((\Psi\circledast\Phi)\otimes 1)} \circ \mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(0)} = \mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(1)} .
\end{aligned}$$

(iii) The first part of the claim is a consequence of the fact that for any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the automorphism $\mathrm{aut}_\Psi^{\mathcal{V},(0)}$ extends the automorphism aut_Ψ thanks to Proposition-Definition 2.12. The same proposition-definition allows us to obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G \\ \downarrow{\scriptstyle \mathrm{Ad}_\Psi\circ\mathrm{aut}_\Psi} & & \downarrow{\scriptstyle \mathrm{aut}_\Psi^{\mathcal{V},(1)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G \end{array} \tag{2.18}$$

This implies that the automorphism $\mathrm{aut}_\Psi^{\mathcal{V},(1)} = \mathrm{Ad}_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(0)}$ extends the automorphism $\mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi$; thus it proves the second part of the claim. □

We are now in a position to prove a claim from the introduction which can be formulated in the following way:

Proposition 2.15. *If $G \neq \{1\}$, the action of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle X\rangle\rangle$ by $\mathbf{k}$-algebra automorphisms $\Psi \mapsto \mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi$ does not restrict to an action on the topological subalgebra $\mathbf{k}\langle\langle Y\rangle\rangle$.*

In order to prove this proposition, we will need the following lemma:

Lemma 2.16. *Let $g \in G$, $u \in \mathbf{k}\langle\langle X\rangle\rangle^\times$ and $v \in \mathbf{k}\langle\langle X\rangle\rangle$. Then $ux_gv \in \mathbf{k}\langle\langle Y\rangle\rangle$ if and only if $v \in \mathbf{k}\langle\langle Y\rangle\rangle$.*

Proof. If $v \in \mathbf{k}\langle\langle Y\rangle\rangle$, then, since $ux_g \in \mathbf{k}\langle\langle Y\rangle\rangle$ and $\mathbf{k}\langle\langle Y\rangle\rangle$ is an algebra, we have that $ux_gv \in \mathbf{k}\langle\langle Y\rangle\rangle$. Conversely, by the decomposition $\mathbf{k}\langle\langle X\rangle\rangle = \mathbf{k}\langle\langle X\rangle\rangle x_0 \oplus \mathbf{k}\langle\langle Y\rangle\rangle$, there exist $a \in \mathbf{k}\langle\langle X\rangle\rangle$ and $b \in \mathbf{k}\langle\langle Y\rangle\rangle$ such that $v = ax_0 + b$. Then

$$ux_gv = ux_g(ax_0 + b) = ux_gax_0 + ux_gb.$$

Since ux_g and b belong to the algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ then $ux_gb \in \mathbf{k}\langle\langle Y\rangle\rangle$. Since, by assumption, $ux_gv \in \mathbf{k}\langle\langle Y\rangle\rangle$, then $ux_gax_0 \in \mathbf{k}\langle\langle Y\rangle\rangle$. By the previous direct sum decomposition, this implies that $ux_gax_0 = 0$. Since u is invertible this is equivalent to $x_gax_0 = 0$ which implies that $a = 0$ thanks to $\mathbf{k}\langle\langle X\rangle\rangle$ being an integral domain. Finally,

$$v = ax_0 + b = b \in \mathbf{k}\langle\langle Y\rangle\rangle.$$ □

Proof of Proposition 2.15. Since $G \neq \{1\}$, let $g \neq 1$ be an element of G. Let us set $\Psi = \exp([x_1, x_0]) \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$. We have

$$\mathrm{Ad}_\Psi \circ \mathrm{aut}_\Psi(x_g) = \Psi t_g(\Psi^{-1})x_gt_g(\Psi)\Psi^{-1}.$$

Set $u = \Psi t_g(\Psi^{-1})$ and $v = t_g(\Psi)\Psi^{-1}$. One checks that $u \in \mathbf{k}\langle\langle X\rangle\rangle^\times$ and $v \in \mathbf{k}\langle\langle X\rangle\rangle$. One may therefore apply Lemma 2.16 with these values of u and v and obtain that $ux_gv = \Psi t_g(\Psi^{-1})x_g t_g(\Psi)\Psi^{-1}$ belongs to $\mathbf{k}\langle\langle Y\rangle\rangle$ if and only if $v = t_g(\Psi)\Psi^{-1}$ is in $\mathbf{k}\langle\langle Y\rangle\rangle$. On the other hand, one has

$$t_g(\Psi)\Psi^{-1} = \exp([x_g, x_0])\exp(-[x_1, x_0]) = 1 + [x_g - x_1, x_0] + \text{terms of order} > 2.$$

The order 2 term has $\mathbf{k}\langle\langle Y\rangle\rangle$ component equal to $x_0(x_1 - x_g)$ and $\mathbf{k}\langle\langle X\rangle\rangle x_0$ component equal to $(x_g - x_1)x_0$; the latter being nonzero, $t_g(\Psi)\Psi^{-1}$ is, therefore, not in $\mathbf{k}\langle\langle Y\rangle\rangle$ which implies, by Lemma 2.16, that $\Psi t_g(\Psi^{-1})x_g t_g(\Psi)\Psi^{-1} \notin \mathbf{k}\langle\langle Y\rangle\rangle$. □

Proposition-Definition 2.17. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the automorphism $\mathrm{aut}_\Psi^{\mathcal{V},(1)} : \widehat{\mathcal{V}}_G \to \widehat{\mathcal{V}}_G$ restricts to a topological $\mathbf{k}$-algebra automorphism on the $\mathbf{k}$-subalgebra $\widehat{\mathcal{W}}_G$ which will be denoted $\mathrm{aut}_\Psi^{\mathcal{W},(1)}$. Moreover, there is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{W}}_G$ by $\mathbf{k}$-algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G), \quad \Psi \mapsto \mathrm{aut}_\Psi^{\mathcal{W},(1)}. \tag{2.19}$$

Proof. For $w = \lambda + ve_1 \in \widehat{\mathcal{W}}_G$, we have

$$\begin{aligned}\mathrm{aut}_\Psi^{\mathcal{V},(1)}(w) &= \lambda + \mathrm{aut}_\Psi^{\mathcal{V},(1)}(v)\beta(\Psi\otimes 1)\beta(\Psi^{-1}\otimes 1)e_1\beta(\Psi\otimes 1)\beta(\Psi^{-1}\otimes 1)\\ &= \lambda + \mathrm{aut}_\Psi^{\mathcal{V},(1)}(v)e_1 \in \widehat{\mathcal{W}}_G.\end{aligned}$$

This implies that $\mathrm{aut}_\Psi^{\mathcal{V},(1)}$ induces a $\mathbf{k}$-algebra endomorphism of $\widehat{\mathcal{W}}_G$. Moreover, the pullback of this endomorphism under the $\mathbf{k}$-module isomorphism $\mathbf{k} \times \widehat{\mathcal{V}}_G \to \widehat{\mathcal{W}}_G$, $(\lambda, v) \mapsto \lambda + ve_1$ is the pair $(\mathrm{id}, \mathrm{aut}_\Psi^{\mathcal{V},(1)})$, which is a $\mathbf{k}$-module automorphism. This implies that $\mathrm{aut}_\Psi^{\mathcal{W},(1)}$ is a $\mathbf{k}$-module automorphism, and therefore a $\mathbf{k}$-algebra automorphism. Thanks to this, the second part of this result can be deduced from Proposition 2.14(ii), by restriction on $\widehat{\mathcal{W}}_G$. □

2.2.2. Action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ by module automorphisms.

Definition 2.18. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define $\mathrm{aut}_\Psi^{\mathcal{V},(10)}$ to be the topological $\mathbf{k}$-module automorphism of $\widehat{\mathcal{V}}_G$ given by

$$\mathrm{aut}_\Psi^{\mathcal{V},(10)} := \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(0)}. \tag{2.20}$$

Remark 2.19. Let us notice that, for any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we also have

$$\mathrm{aut}_\Psi^{\mathcal{V},(10)} = \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(0)} = \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{Ad}_{\beta(\Psi^{-1}\otimes 1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(1)} = r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(1)}.$$

Proposition 2.20. *There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{V}}_G$ by topological $\mathbf{k}$-module automorphisms given by*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{V}}_G), \quad \Psi \mapsto \mathrm{aut}_\Psi^{\mathcal{V},(10)}.$$

Proof. For $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned}
\mathrm{aut}_{\Psi}^{\mathcal{V},(10)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(10)} &= \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \ell_{\beta(\Phi\otimes 1)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \ell_{\beta(\Psi\otimes 1)} \circ \ell_{\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(\beta(\Phi\otimes 1))} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \ell_{\beta(\Psi\otimes 1)\beta(\mathrm{aut}_{\Psi}(\Phi)\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)} \circ \mathrm{aut}_{\Phi}^{\mathcal{V},(0)} \\
&= \ell_{\beta((\Psi\circledast\Phi)\otimes 1)} \circ \mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(0)} = \mathrm{aut}_{\Psi\circledast\Phi}^{\mathcal{V},(10)},
\end{aligned}$$

where the last equality comes from the commutativity of diagram (2.15) and from Proposition 2.14(i). □

Lemma 2.21. *For any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have the following identities:*

(i) *For all $a, b \in \widehat{\mathcal{V}}_G$, $\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(ab) = \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(b)$.*

(ii) *For all $a, b \in \widehat{\mathcal{V}}_G$, $\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(ab) = \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(b)$.*

Proof. Let $a, b \in \widehat{\mathcal{V}}_G$. We have

$$\begin{aligned}
\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(ab) &= \ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(ab) \\
&= \ell_{\beta(\Psi\otimes 1)}(\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(b)) \\
&= (\ell_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(a))\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(b) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(b),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(ab) &= r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(ab) \\
&= r_{\beta(\Psi\otimes 1)}(\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(b)) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(a)(r_{\beta(\Psi\otimes 1)} \circ \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(b)) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(b).
\end{aligned}$$

□

Proposition 2.22. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the $\mathbf{k}$-module automorphism $\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}$ preserves the submodule $\widehat{\mathcal{V}}_G e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G(g-1)$.*

Proof. Using Lemma 2.21(i), we obtain for any $a \in \widehat{\mathcal{V}}_G$ and $(b_g)_{g\in G} \in (\widehat{\mathcal{V}}_G)^G$,

$$\begin{aligned}
&\mathrm{aut}_{\Psi}^{\mathcal{V},(10)}\left(ae_0 + \sum_{g\in G} b_g(g-1)\right) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(a)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(e_0) + \sum_{g\in G} \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(b_g)\,\mathrm{aut}_{\Psi}^{\mathcal{V},(0)}(g-1) \\
&= \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(a)e_0 + \sum_{g\in G} \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}(b_g)(g-1) \in \widehat{\mathcal{V}}_G e_0 + \sum_{g\in G} \widehat{\mathcal{V}}_G(g-1).
\end{aligned}$$

□

Proposition-Definition 2.23. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, there is a unique* $\mathbf{k}$*-module automorphism* $\mathrm{aut}_\Psi^{\mathcal{M},(10)}$ *of* $\widehat{\mathcal{M}}_G$ *such that the diagram*

$$\begin{array}{ccc} \widehat{\mathcal{V}}_G & \xrightarrow{\mathrm{aut}_\Psi^{\mathcal{V},(10)}} & \widehat{\mathcal{V}}_G \\ {\scriptstyle -\cdot 1_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathcal{M}}} \\ \widehat{\mathcal{M}}_G & \xrightarrow[\mathrm{aut}_\Psi^{\mathcal{M},(10)}]{} & \widehat{\mathcal{M}}_G \end{array} \tag{2.21}$$

commutes.

Proof. It follows from Proposition 2.22. □

Lemma 2.24. *For any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have*

(i) *for all $(a, m) \in \widehat{\mathcal{V}}_G \times \widehat{\mathcal{M}}_G$, $\mathrm{aut}_\Psi^{\mathcal{M},(10)}(a \cdot m) = \mathrm{aut}_\Psi^{\mathcal{V},(1)}(a) \cdot \mathrm{aut}_\Psi^{\mathcal{M},(10)}(m)$;*
(ii) *for all $(w, m) \in \widehat{\mathcal{W}}_G \times \widehat{\mathcal{M}}_G$, $\mathrm{aut}_\Psi^{\mathcal{M},(10)}(w \cdot m) = \mathrm{aut}_\Psi^{\mathcal{W},(1)}(w) \cdot \mathrm{aut}_\Psi^{\mathcal{M},(10)}(m)$.*

Proof. The first identity is proved by using a combination of Proposition-Definition 2.23 and Lemma 2.21(ii). The second identity can be deduced from the first by restriction on the subalgebra $\widehat{\mathcal{W}}_G$. □

Corollary 2.25. *There is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{M}}_G$ by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-mod}}^{\mathrm{cont}}(\widehat{\mathcal{M}}_G), \quad \Psi \mapsto \mathrm{aut}_\Psi^{\mathcal{M},(10)}. \tag{2.22}$$

Proof. It is a combination of Proposition-Definition 2.23 and Proposition 2.20. □

§2.3. The cocycle Γ and twisted actions

To an element $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, one associates $\Gamma_\Psi \in \mathbf{k}[[x]]^\times$ (see (1.15)). Then $\Gamma_\Psi(-e_1)$ is an invertible element of $\widehat{\mathcal{V}}_G$.

Definition 2.26. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the topological $\mathbf{k}$-algebra automorphism of $\widehat{\mathcal{V}}_G$:

$$^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{V},(1)} := \mathrm{Ad}_{\Gamma_\Psi^{-1}(-e_1)} \circ \mathrm{aut}_\Psi^{\mathcal{V},(1)}. \tag{2.23}$$

Proposition-Definition 2.27. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the automorphism $^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{V},(1)}$ restricts to a topological* $\mathbf{k}$*-algebra automorphism of the subalgebra $\widehat{\mathcal{W}}_G$ denoted $^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{W},(1)}$.*

Proof. It follows from Proposition-Definition 2.17 and the fact that $\Gamma_\Psi(-e_1)$ is an invertible element of $\widehat{\mathcal{W}}_G$. □

Proposition 2.28. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on*

(i) $\widehat{\mathcal{V}}_G$ *by topological* $\mathbf{k}$*-algebra automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{V}}_G), \quad \Psi \mapsto {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}, \tag{2.24}$$

(ii) $\widehat{\mathcal{W}}_G$ *by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{W}}_G), \quad \Psi \mapsto {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi}. \tag{2.25}$$

Proof. (i) It follows from Proposition 2.14(ii), Lemma 1.8 and the fact that e_1 is invariant under $\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}$ for any $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$.

(ii) It follows from (i) thanks to Proposition-Definition 2.17. □

Definition 2.29. For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we define the following topological $\mathbf{k}$-module automorphism of $\widehat{\mathcal{M}}_G$:

$${}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi} := \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Psi}. \tag{2.26}$$

Lemma 2.30. *For any* $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$*, we have*

(i) *for all* $(a, m) \in \widehat{\mathcal{V}}_G \times \widehat{\mathcal{M}}_G$*,* ${}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}(a \cdot m) = {}^{\Gamma}\mathrm{aut}^{\mathcal{V},(1)}_{\Psi}(a) \cdot {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}(m)$*;*
(ii) *for all* $(w, m) \in \widehat{\mathcal{W}}_G \times \widehat{\mathcal{M}}_G$*,* ${}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}(w \cdot m) = {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi}(w) \cdot {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}(m)$*.*

Proof. It follows by a computation involving Lemma 2.24. □

Proposition 2.31. *There is a group action of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ *on* $\widehat{\mathcal{M}}_G$ *by topological* $\mathbf{k}$*-module automorphisms*

$$(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G), \quad \Psi \mapsto {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi}. \tag{2.27}$$

Proof. For any $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we have

$$\begin{aligned}
{}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi\circledast\Phi} &= \ell_{\Gamma^{-1}_{\Psi\circledast\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Psi\circledast\Phi} \\
&= \ell_{\Gamma^{-1}_{\Psi}(-e_1)\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Psi} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Phi} \\
&= \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \ell_{\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Psi} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Phi} \\
&= \ell_{\Gamma^{-1}_{\Psi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Psi} \circ \ell_{\Gamma^{-1}_{\Phi}(-e_1)} \circ \mathrm{aut}^{\mathcal{M},(10)}_{\Phi} \\
&= {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Psi} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{M},(10)}_{\Phi},
\end{aligned}$$

where the second equality uses Lemma 1.8 and Corollary 2.25, and the fourth equality comes from the following computation:

$$\begin{aligned}\ell_{\Gamma_\Phi^{-1}(-e_1)} \circ \mathrm{aut}_\Psi^{\mathcal{M},(10)}(m) &= \Gamma_\Phi^{-1}(-e_1)\,\mathrm{aut}_\Psi^{\mathcal{M},(10)}(m)\\ &= \mathrm{aut}_\Psi^{\mathcal{V},(1)}(\Gamma_\Phi^{-1}(-e_1))\,\mathrm{aut}_\Psi^{\mathcal{M},(10)}(m)\\ &= \mathrm{aut}_\Psi^{\mathcal{M},(10)}(\Gamma_\Phi^{-1}(-e_1)m)\\ &= \mathrm{aut}_\Psi^{\mathcal{M},(10)} \circ \ell_{\Gamma_\Phi^{-1}(-e_1)}(m),\end{aligned}$$

for any $m \in \widehat{\mathcal{M}}_G$, where the second equality uses the fact e_1 is invariant under $\mathrm{aut}_\Psi^{\mathcal{V},(1)}$ and the third equality comes from Lemma 2.24. $\square$

§2.4. The stabilizer groups $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ and $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$

Using Proposition 2.28, we define the following group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G, (\widehat{\mathcal{W}}_G)^{\hat{\otimes}2})$:

$$\Psi \cdot D^{\mathcal{W}} := ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{\otimes 2} \circ D^{\mathcal{W}} \circ ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{-1}, \tag{2.28}$$

with $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $D^{\mathcal{W}} \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G, (\widehat{\mathcal{W}}_G)^{\hat{\otimes}2})$. In particular, the stabilizer of $D^{\mathcal{W}} = \widehat{\Delta}_G^{\mathcal{W}}$ is the subgroup

$$\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \;\middle|\; ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{W}} = \widehat{\Delta}_G^{\mathcal{W}} \circ {}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)}\right\}. \tag{2.29}$$

Similarly, Proposition 2.31 provides a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on the $\mathbf{k}$-module $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G, (\widehat{\mathcal{M}}_G)^{\hat{\otimes}2})$:

$$\Psi \cdot D^{\mathcal{M}} := ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2} \circ D^{\mathcal{M}} \circ ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{-1}. \tag{2.30}$$

In particular, the stabilizer of $D^{\mathcal{M}} = \widehat{\Delta}_G^{\mathcal{M}}$ is the subgroup

$$\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \;\middle|\; ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M}} = \widehat{\Delta}_G^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}\right\}. \tag{2.31}$$

We then have the following inclusion of subgroups:

Theorem 2.32. $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. Let $\Psi \in \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$. First, let us notice that

$$\begin{aligned}(\Gamma_\Psi^{-1}(-e_1)\beta(\Psi \otimes 1) \cdot 1_{\mathcal{M}})^{\otimes 2} &= ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}(1_{\mathcal{M}}))^{\otimes 2}\\ &= ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M}}(1_{\mathcal{M}})\\ &= \widehat{\Delta}_G^{\mathcal{M}} \circ {}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}(1_{\mathcal{M}}),\end{aligned} \tag{2.32}$$

where the last equality follows from the assumption on Ψ. Then for any $w \in \widehat{\mathcal{W}}_G$,

$$\begin{aligned}
&\widehat{\Delta}_G^{\mathcal{W}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)}(w)) \cdot (\Gamma_\Psi^{-1}(-e_1)\beta(\Psi \otimes 1) \cdot 1_{\mathcal{M}})^{\otimes 2} \\
&\quad = \widehat{\Delta}_G^{\mathcal{W}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)}(w)) \cdot \widehat{\Delta}_G^{\mathcal{M}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}(1_{\mathcal{M}})) \\
&\quad = \widehat{\Delta}_G^{\mathcal{M}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)}(w) \cdot {}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}(1_{\mathcal{M}})) \\
&\quad = \widehat{\Delta}_G^{\mathcal{M}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)}(w \cdot 1_{\mathcal{M}})) \\
&\quad = ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2} \circ \widehat{\Delta}_G^{\mathcal{M}}(w \cdot 1_{\mathcal{M}}) \\
&\quad = ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2}(\widehat{\Delta}_G^{\mathcal{W}}(w) \cdot \widehat{\Delta}_G^{\mathcal{M}}(1_{\mathcal{M}})) \\
&\quad = ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{\otimes 2}(\widehat{\Delta}_G^{\mathcal{W}}(w)) \cdot ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{M},(10)})^{\otimes 2}(\widehat{\Delta}_G^{\mathcal{M}}(1_{\mathcal{M}})) \\
&\quad = ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{\otimes 2}(\widehat{\Delta}_G^{\mathcal{W}}(w)) \cdot (\Gamma_\Psi^{-1}(-e_1)\beta(\Psi \otimes 1) \cdot 1_{\mathcal{M}})^{\otimes 2}, \qquad (2.33)
\end{aligned}$$

where the first and seventh equalities come from (2.32), the second and the fifth from Proposition-Definition 2.11(iii), the third and the sixth from Lemma 2.30 and the fourth from the fact that $\Psi \in \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$. Next, since $\Gamma_\Psi^{-1}(-e_1)\beta(\Psi \otimes 1)$ is invertible in $\widehat{\mathcal{V}}_G$, the map $\widehat{\mathcal{W}}_G \to \widehat{\mathcal{M}}_G$, $w \mapsto w\Gamma_\Psi^{-1}(-e_1)\beta(\Psi \otimes 1) \cdot 1_{\mathcal{M}}$ is an isomorphism of left $\widehat{\mathcal{W}}_G$-modules. Consequently, identity (2.33) implies that

$$\forall w \in \widehat{\mathcal{W}}_G, \quad ({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)})^{\otimes 2}(\widehat{\Delta}_G^{\mathcal{W}}(w)) = \widehat{\Delta}_G^{\mathcal{W}}({}^\Gamma\mathrm{aut}_\Psi^{\mathcal{W},(1)}(w)), \qquad (2.34)$$

thus establishing that $\Psi \in \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$. □

§3. The stabilizer groups in terms of Racinet's formalism

In this part we translate the inclusion of stabilizers in Theorem 2.32 into Racinet's formalism. In Section 3.1 we relate the various $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-actions, the ones which we recalled from [Rac] in Section 1 and the ones we constructed in Section 2. This allows us to identify the group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})$ from (2.31) with the group $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})$ from [EF18]. In Section 3.2 we transport the action of the group $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\widehat{\mathcal{W}}_G$ given in Proposition 2.28(ii) into an action of the same group on the algebra $\mathbf{k}\langle\langle Y\rangle\rangle$ and express the latter action in terms of Racinet's formalism. This enables us to identify the stabilizer group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})$ given in (2.29) with a group $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})$ defined in the framework of Racinet's formalism. The inclusion of stabilizers from Theorem 2.32 is then expressed as the inclusion $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}}) \subset \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})$ (see Corollary 3.13).

§3.1. Identification of the subgroups $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})$ and $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})$

3.1.1. A $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module isomorphism. Let us recall $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{\mathcal{V}}_G$, the $\mathbf{k}$-algebra isomorphism given in Proposition 2.4(ii).

Lemma 3.1. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G \\ {\scriptstyle S_\Psi}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}_\Psi^{\mathcal{V},(10)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G \end{array} \tag{3.1}$$

commutes.

Proof. Thanks to identities (1.8) and (2.20), this is done by composing the bottom of diagram (2.16) with the following commutative diagram:

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow{\beta\circ(-\otimes 1)} & \widehat{\mathcal{V}}_G \\ {\scriptstyle \ell_\Psi}\downarrow & & \downarrow{\scriptstyle \ell_{\beta(\Psi\otimes 1)}} \\ \mathbf{k}\langle\langle X\rangle\rangle & \xrightarrow[\beta\circ(-\otimes 1)]{} & \widehat{\mathcal{V}}_G. \end{array}$$

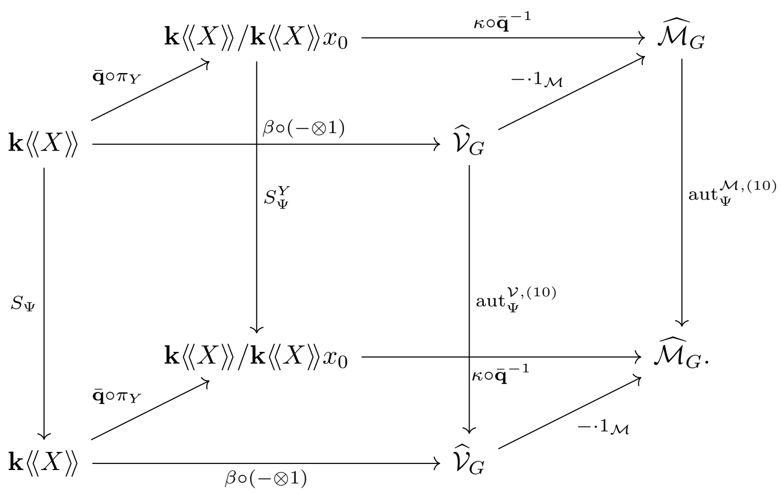
□

Lemma 3.2. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\kappa\circ\bar{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G \\ {\scriptstyle S_\Psi^Y}\downarrow & & \downarrow{\scriptstyle \mathrm{aut}_\Psi^{\mathcal{M},(10)}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\kappa\circ\bar{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G \end{array} \tag{3.2}$$

commutes.

Proof. Let us consider the following cube:



First, the left (resp. right) side commutes by definition of S^Y_Ψ (resp. $\mathrm{aut}^{\mathcal{M},(10)}_\Psi$). Then the upper and lower sides are exactly the same square, which is commutative thanks to Proposition 2.7. Finally, Lemma 3.1 gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of $\bar{\mathbf{q}} \circ \pi_Y$ implies that the back side of the cube commutes, which is exactly diagram (3.2). ☐

Proposition 3.3. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, the diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\kappa\circ\bar{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G \\ {\scriptstyle \ulcorner S^Y_\Psi}\downarrow & & \downarrow{\scriptstyle \ulcorner \mathrm{aut}^{\mathcal{M},(10)}_\Psi} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\kappa\circ\bar{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G \end{array} \tag{3.3}$$

commutes.

Remark 3.4. It follows from diagram (3.3) that $\kappa \circ \bar{\mathbf{q}}^{-1}$ is an isomorphism between the $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-modules $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ and $\widehat{\mathcal{M}}_G$.

For the $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module structure of $\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ (resp. $\widehat{\mathcal{M}}_G$), see Corollary 1.9 (resp. Proposition 2.31).

Proof of Proposition 3.3. This is done by composing the bottom of diagram (3.2) with the following diagram:

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\kappa\circ\bar{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G \\ {\scriptstyle \ell_{\Gamma^{-1}_\Psi(x_1)}}\downarrow & & \downarrow{\scriptstyle \ell_{\Gamma^{-1}_\Psi(-e_1)}} \\ \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow[\kappa\circ\bar{\mathbf{q}}^{-1}]{} & \widehat{\mathcal{M}}_G \end{array}$$

The above diagram is commutative because we have

$$\begin{aligned} \ell_{\Gamma^{-1}_\Psi(-e_1)} \circ \kappa \circ \bar{\mathbf{q}}^{-1} \circ \bar{\mathbf{q}} \circ \pi_Y &= \ell_{\Gamma^{-1}_\Psi(-e_1)} \circ (- \cdot 1_\mathcal{M}) \circ \beta \circ (- \otimes 1) \\ &= (- \cdot 1_\mathcal{M}) \circ \ell_{\beta(\Gamma^{-1}_\Psi(x_1)\otimes 1)} \circ \beta \circ (- \otimes 1) \\ &= (- \cdot 1_\mathcal{M}) \circ \beta \circ (- \otimes 1) \circ \ell_{\Gamma^{-1}_\Psi(x_1)} \\ &= \kappa \circ \bar{\mathbf{q}}^{-1} \circ \bar{\mathbf{q}} \circ \pi_Y \circ \ell_{\Gamma^{-1}_\Psi(x_1)} \\ &= \kappa \circ \bar{\mathbf{q}}^{-1} \circ \ell_{\Gamma^{-1}_\Psi(x_1)} \circ \bar{\mathbf{q}} \circ \pi_Y, \end{aligned}$$

where the first and fourth equalities come from the commutativity of diagram (2.4), the second from the fact that $- \cdot 1_\mathcal{M} \colon \widehat{\mathcal{V}}_G \to \widehat{\mathcal{M}}_G$ is a $\widehat{\mathcal{V}}_G$-module morphism, the third from the fact that $\beta \circ (- \otimes 1) \colon \mathbf{k}\langle\langle X\rangle\rangle \to \widehat{\mathcal{V}}_G$ is a $\mathbf{k}$-algebra morphism and the last from the fact that $\pi_Y : \mathbf{k}\langle\langle X\rangle\rangle \to \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0$ is $\mathbf{k}\langle\langle X\rangle\rangle$-module morphism and that for any $a \in \mathbf{k}\langle\langle X\rangle\rangle$, $\mathbf{q}(x_1 a) = x_1\mathbf{q}(a)$.

Finally, since $\bar{\mathbf{q}} \circ \pi_Y$ is a surjective $\mathbf{k}$-module morphism, it follows that

$$\ell_{\Gamma_\Psi^{-1}(-e_1)} \circ \kappa \circ \bar{\mathbf{q}}^{-1} = \kappa \circ \bar{\mathbf{q}}^{-1} \circ \ell_{\Gamma_\Psi^{-1}(x_1)},$$

which is the wanted result. □

3.1.2. An isomorphism of coalgebras. Let us recall $\varpi : \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G$ the $\mathbf{k}$-algebra isomorphism given in Corollary 2.9(i).

Lemma 3.5. *The diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle Y\rangle\rangle & \xrightarrow{\varpi} & \widehat{\mathcal{W}}_G \\ {\scriptstyle \widehat{\Delta}_\star^{\mathrm{alg}}}\downarrow & & \downarrow{\scriptstyle \widehat{\Delta}_G^{\mathcal{W}}} \\ \mathbf{k}\langle\langle Y\rangle\rangle^{\otimes 2} & \xrightarrow[\varpi^{\otimes 2}]{} & \widehat{\mathcal{W}}_G^{\otimes 2} \end{array} \tag{3.4}$$

commutes.

Remark 3.6. It follows from diagram (3.4) that the map ϖ is a bialgebra isomorphism.

Proof of Lemma 3.5. Since all arrows on diagram (3.4) are $\mathbf{k}$-algebra morphisms, it is enough to work on generators. For $(n, g) \in \mathbb{Z}_{>0} \times G$ we have

$$\begin{aligned} \varpi^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{alg}}(y_{n,g}) &= \varpi^{\otimes 2}\Bigg(y_{n,g} \otimes 1 + 1 \otimes y_{n,g} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} y_{k,h} \otimes y_{n-k,gh^{-1}}\Bigg) \\ &= z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}}. \end{aligned}$$

On the other hand,

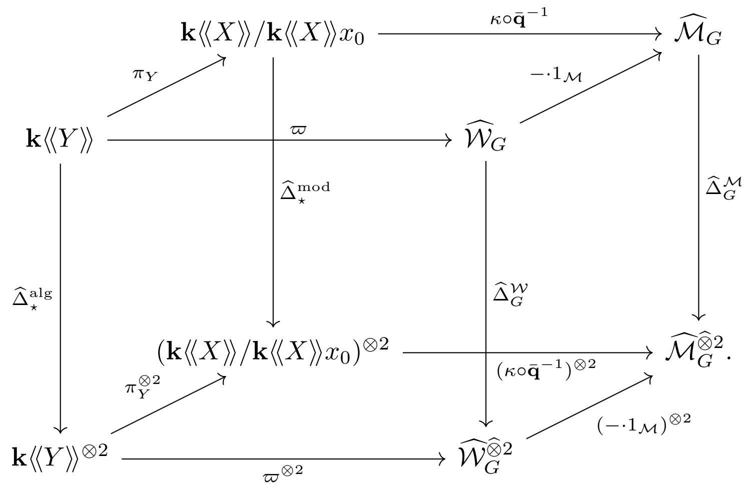
$$\widehat{\Delta}_G^{\mathcal{W}} \circ \varpi(y_{n,g}) = \widehat{\Delta}_G^{\mathcal{W}}(z_{n,g}) = z_{n,g} \otimes 1 + 1 \otimes z_{n,g} + \sum_{\substack{k=1 \\ h\in G}}^{n-1} z_{k,h} \otimes z_{n-k,gh^{-1}}.$$ □

Lemma 3.7. *The diagram*

$$\begin{array}{ccc} \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 & \xrightarrow{\kappa\circ\bar{\mathbf{q}}^{-1}} & \widehat{\mathcal{M}}_G \\ {\scriptstyle \widehat{\Delta}_\star^{\mathrm{mod}}}\downarrow & & \downarrow{\scriptstyle \widehat{\Delta}_G^{\mathcal{M}}} \\ (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\otimes 2} & \xrightarrow[(\kappa\circ\bar{\mathbf{q}}^{-1})^{\otimes 2}]{} & \widehat{\mathcal{M}}_G^{\otimes 2} \end{array} \tag{3.5}$$

commutes.

Proof. Let us consider the following cube:



First, the left (resp. right) side commutes by definition of $\widehat{\Delta}^{\mathrm{mod}}_{\star}$ (resp. $\widehat{\Delta}^{\mathcal{M}}_G$). Then the upper side commutes thanks to Corollary 2.9. Since the lower side is exactly the tensor square of the upper side, it is commutative. Finally, Lemma 3.5 gives us the commutativity of the front side. This collection of commutativities together with the surjectivity of π_Y implies that the back side of the cube commutes, which is exactly diagram (3.5). $\square$

3.1.3. Identification of stabilizer groups.

Theorem 3.8. $\mathsf{Stab}(\widehat{\Delta}^{\mathcal{M}}_G)(\mathbf{k}) = \mathsf{Stab}(\widehat{\Delta}^{\mathrm{mod}}_{\star})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. Thanks to Proposition 3.3, the map $\kappa\circ\bar{\mathbf{q}}^{-1}\colon \mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0 \to \widehat{\mathcal{M}}_G$ is a $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module isomorphism. So it induces a $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module isomorphism

$$\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\otimes 2}) \to \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G, \widehat{\mathcal{M}}_G^{\widehat{\otimes}2})$$

given by

$$\Delta \mapsto (\kappa\circ\bar{\mathbf{q}}^{-1})^{\otimes 2}\circ\Delta\circ(\kappa\circ\bar{\mathbf{q}}^{-1})^{-1},$$

where the $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module structure on the $\mathbf{k}$-module $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G, \widehat{\mathcal{M}}_G^{\widehat{\otimes}2})$ (resp. $\mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-mod}}(\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0, (\mathbf{k}\langle\langle X\rangle\rangle/\mathbf{k}\langle\langle X\rangle\rangle x_0)^{\otimes 2})$) is defined in (2.30) (resp. (1.19)). Moreover, thanks to Lemma 3.7, the coproduct $\widehat{\Delta}^{\mathrm{mod}}_{\star}$ is sent to the coproduct $\widehat{\Delta}^{\mathcal{M}}_G$ via this isomorphism. Thus, they have the same stabilizer. $\square$

§3.2. The stabilizer group $\mathsf{Stab}(\widehat{\boldsymbol{\Delta}}^{\mathcal{W}}_G)$ in Racinet's formalism

Proposition-Definition 3.9. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$, we consider the $\mathbf{k}$-algebra automorphism of $\mathbf{k}\langle\langle Y\rangle\rangle$ given by*

$$ {}^{\Gamma}\mathrm{aut}^{Y}_{\Psi} := \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi} \circ \varpi. \tag{3.6} $$

Then there is a group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathbf{k}\langle\langle Y\rangle\rangle$ by topological $\mathbf{k}$-algebra automorphisms given by

$$ \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathrm{Aut}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle), \quad \Psi \mapsto {}^{\Gamma}\mathrm{aut}^{Y}_{\Psi}. \tag{3.7} $$

Proof. For $\Psi, \Phi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ we have

$$ \begin{aligned} {}^{\Gamma}\mathrm{aut}^{Y}_{\Psi\circledast\Phi} &= \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi\circledast\Phi} \circ \varpi \\ &= \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Phi} \circ \varpi \\ &= \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Psi} \circ \varpi \circ \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}^{\mathcal{W},(1)}_{\Phi} \circ \varpi \\ &= {}^{\Gamma}\mathrm{aut}^{Y}_{\Psi} \circ {}^{\Gamma}\mathrm{aut}^{Y}_{\Phi}. \end{aligned} $$

□

We aim to give an explicit formulation of the action ${}^{\Gamma}\mathrm{aut}^{Y}$ in terms of Racinet's objects. Recall from Section 1.1 that for any $g \in G$ and any $a \in \mathbf{k}\langle\langle X\rangle\rangle$, $ax_g \in \mathbf{k}\langle\langle Y\rangle\rangle$. We then have the following lemma:

Lemma 3.10. *Let $g \in G$. For any $a \in \mathbf{k}\langle\langle X\rangle\rangle$ we have $\beta(ax_g \otimes g) = \varpi \circ \mathbf{q}_Y(ax_g)$.*

Proof. It is enough to show this on a basis of the $\mathbf{k}$-module $\mathbf{k}\langle\langle X\rangle\rangle$. Let us take the family

$$ (x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_2}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1})_{\substack{r\in\mathbb{Z}_{\geq 0}, n_1,\ldots,n_{r+1}\in\mathbb{Z}_{>0},\\ g_1,\ldots,g_r\in G}} $$

as such a basis. For $r \in \mathbb{Z}_{\geq 0}$, $n_1, \ldots, n_{r+1} \in \mathbb{Z}_{>0}$ and $g_1, \ldots, g_r \in G$ we have

$$ \begin{aligned} &x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}x_g \otimes g \\ &\quad = (x_0^{n_1-1}\otimes 1) * (x_{g_1}\otimes 1) * \cdots * (x_0^{n_r-1}\otimes 1) * (x_{g_r}\otimes 1) \\ &\qquad * (x_0^{n_{r+1}-1}\otimes 1) * (x_g \otimes 1) * (1\otimes g). \end{aligned} \tag{3.8} $$

Therefore, we obtain

$$ \begin{aligned} &\beta(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}x_g \otimes g) \\ &\quad = (-1)^{r+1} e_0^{n_1-1} g_1 e_1 g_1^{-1} e_0^{n_2-1} g_2 e_1 g_2^{-1} \cdots e_0^{n_{r-1}-1} g_{r-1} e_1 g_{r-1}^{-1} \\ &\qquad \cdot e_0^{n_r-1} g_r e_1 g_r^{-1} e_0^{n_{r+1}-1} g e_1 g^{-1} g \end{aligned} $$

$$
\begin{aligned}
&= (-1)^{r+1} e_0^{n_1-1} g_1 e_1\, e_0^{n_2-1} g_1^{-1} g_2 e_1 \cdots e_0^{n_{r-1}-1} g_{r-2}^{-1} g_{r-1} e_1 \\
&\quad \cdot e_0^{n_r-1} g_{r-1}^{-1} g_r e_1\, e_0^{n_{r+1}-1} g_r^{-1} g e_1 \\
&= z_{n_1,g_1} z_{n_2,g_1^{-1}g_2} \cdots z_{n_{r-1},g_{r-2}^{-1}g_{r-1}} z_{n_r,g_{r-1}^{-1}g_r} z_{n_{r+1},g_r^{-1}g},
\end{aligned}
$$

where the first equality comes from the computation (3.8) and the fact that $\beta\colon \mathbf{k}\langle\langle X\rangle\rangle \rtimes G \to \widehat{V}_G$ is a $\mathbf{k}$-algebra morphism and the second equality comes from the fact for any $i \in \{2, \dots, r\}$, $g_i^{-1} e_0 = e_0 g_i^{-1}$. On the other hand,

$$
\begin{aligned}
&\varpi \circ \mathbf{q}_Y(x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2} \cdots x_0^{n_{r-1}-1} x_{g_{r-1}} x_0^{n_r-1} x_{g_r} x_0^{n_{r+1}-1} x_g) \\
&\quad = \varpi(y_{n_1,g_1} y_{n_2,g_1^{-1}g_2} \cdots y_{n_{r-1},g_{r-2}^{-1}g_{r-1}} y_{n_r,g_{r-1}^{-1}g_r} y_{n_{r+1},g_r^{-1}g}) \\
&\quad = z_{n_1,g_1} z_{n_2,g_1^{-1}g_2} \cdots z_{n_{r-1},g_{r-2}^{-1}g_{r-1}} z_{n_r,g_{r-1}^{-1}g_r} z_{n_{r+1},g_r^{-1}g}. \qquad \square
\end{aligned}
$$

Proposition 3.11. *For $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $(n,g) \in \mathbb{Z}_{>0} \times G$ we have*

$$
{}^{\Gamma}\mathrm{aut}_\Psi^Y(y_{n,g}) = \mathbf{q}_Y\big(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1} t_g(\Psi^{-1}\Gamma_\Psi(x_1))x_g\big). \tag{3.9}
$$

Proof. Let us start with the following computation:

$$
\begin{aligned}
{}^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{W},(1)}(z_{n,g}) &= -\Gamma_\Psi^{-1}(-e_1)\beta(\Psi\otimes 1)e_0^{n-1} g \beta(\Psi^{-1}\otimes 1)e_1\Gamma_\Psi(-e_1) \\
&= -\Gamma_\Psi^{-1}(-e_1)\beta(\Psi\otimes 1)e_0^{n-1} g \beta(\Psi^{-1}\otimes 1)\Gamma_\Psi(-e_1)e_1 \\
&= \beta\big((\Gamma_\Psi^{-1}(x_1)\otimes 1) * (\Psi\otimes 1) * (x_0^{n-1}\otimes 1) * (1\otimes g) \\
&\qquad * (\Psi^{-1}\otimes 1) * (\Gamma_\Psi(x_1)\otimes 1) * (x_1\otimes 1)\big) \\
&= \beta\big(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1} t_g(\Psi^{-1}\Gamma_\Psi(x_1))x_g \otimes g\big) \\
&= \varpi\circ\mathbf{q}_Y\big(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1} t_g(\Psi^{-1}\Gamma_\Psi(x_1))x_g\big),
\end{aligned}
$$

where t_g is the $\mathbf{k}$-algebra automorphism given in Section 1.1, and the last equality comes from Lemma 3.10. Thanks to this, we have for any $(n,g) \in \mathbb{Z}_{>0} \times G$,

$$
\begin{aligned}
{}^{\Gamma}\mathrm{aut}_\Psi^Y(y_{n,g}) &= \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{W},(1)} \circ \varpi(y_{n,g}) \\
&= \varpi^{-1} \circ {}^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{W},(1)}(z_{n,g}) \\
&= \varpi^{-1}\circ\varpi\circ\mathbf{q}_Y\big(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1} t_g(\Psi^{-1}\Gamma_\Psi(x_1))x_g\big) \\
&= \mathbf{q}_Y\big(\Gamma_\Psi^{-1}(x_1)\Psi x_0^{n-1} t_g(\Psi^{-1}\Gamma_\Psi(x_1))x_g\big). \qquad \square
\end{aligned}
$$

Using Proposition 3.9, we define the following group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$:

$$
\Psi \cdot D := ({}^{\Gamma}\mathrm{aut}_\Psi^Y)^{\otimes 2} \circ D \circ ({}^{\Gamma}\mathrm{aut}_\Psi^Y)^{-1}, \tag{3.10}
$$

with $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ and $D \in \mathrm{Mor}^{\mathrm{cont}}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$. In particular, the stabilizer of $\widehat{\Delta}_\star^{\mathrm{alg}}$ is the subgroup

$$\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) := \left\{\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \mid ({}^{\Gamma}\mathrm{aut}_\Psi^Y)^{\otimes 2} \circ \widehat{\Delta}_\star^{\mathrm{alg}} = \widehat{\Delta}_\star^{\mathrm{alg}} \circ {}^{\Gamma}\mathrm{aut}_\Psi^Y\right\}. \tag{3.11}$$

Theorem 3.12. $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k}) = \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. Thanks to Proposition-Definition 3.9, the map $\varpi\colon \mathbf{k}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G$ is an isomorphism of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-modules. So it induces a $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-module isomorphism $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\otimes 2}) \to \mathrm{Mor}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G, \widehat{\mathcal{W}}_G^{\otimes 2})$ which is given by

$$\Delta \mapsto \varpi^{\otimes 2} \circ \Delta \circ \varpi^{-1}.$$

Moreover, thanks to Lemma 3.5, the coproduct $\widehat{\Delta}_\star^{\mathrm{alg}}$ is sent to the coproduct $\widehat{\Delta}_G^{\mathcal{W}}$ via this isomorphism. Thus, they have the same stabilizer. □

Corollary 3.13. $\mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{mod}})(\mathbf{k}) \subset \mathsf{Stab}(\widehat{\Delta}_\star^{\mathrm{alg}})(\mathbf{k})$ *(as subgroups of* $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$*).*

Proof. It follows immediately from Theorem 2.32 thanks to Theorems 3.8 and 3.12. □

§4. Affine group scheme and Lie algebraic aspects

In this part we show that the results obtained in Sections 2 and 3 fit into the framework of affine $\mathbb{Q}$-group schemes and we make explicit the associated Lie algebraic aspects. More precisely, we use the result of [EF18, Lem. 5.1] to show that the stabilizer group functors $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})$ and $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})$ are affine $\mathbb{Q}$-group schemes, whose Lie algebras are stabilizer Lie algebras which we make explicit. In order to carry out this program, in Section 4.1 we define Lie algebra actions of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ by derivations and by endomorphisms. From this, we derive in Section 4.2 endomorphism actions on $\widehat{\mathcal{M}}_G$ that lead us to an explicit form of the Lie algebra of $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})$ that we show to be equal to the Lie algebra $\mathfrak{stab}(\widehat{\Delta}_\star^{\mathrm{mod}})$ of (1.34). In Section 4.3 we define derivation actions on $\widehat{\mathcal{W}}_G$ that make explicit the Lie algebra $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$ of $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})$ which we show to contain $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}})$. In Section 4.4 we identify $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$ with a Lie algebra stabilizer $\mathfrak{stab}(\widehat{\Delta}_\star^{\mathrm{alg}})$ defined in Racinet's formalism by considering the infinitesimal version of the $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$-action ${}^{\Gamma}\mathrm{aut}^Y$ given in Section 3.2. We conclude by the inclusion $\mathfrak{stab}(\widehat{\Delta}_\star^{\mathrm{mod}}) \subset \mathfrak{stab}(\widehat{\Delta}_\star^{\mathrm{alg}})$.

§4.1. Actions of the Lie algebra $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$

Proposition-Definition 4.1. *Let $\psi \in \widehat{\mathfrak{Lib}}(X)$. There exists a unique $\mathbb{Q}$-algebra derivation $\mathrm{der}_{\psi}^{\mathcal{V},(0)}$ of $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ given by*

$$e_0 \mapsto 0, \quad e_1 \mapsto [e_1, \beta(\psi \otimes 1)], \quad g \mapsto 0, \ \text{for } g \in G.$$

There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ by $\mathbb{Q}$-algebra derivations

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}}), \quad \psi \mapsto \mathrm{der}_{\psi}^{\mathcal{V},(0)}.$$

Proof. One can prove that the assignment $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G)$ is a $\mathbb{Q}$-group scheme with Lie algebra $\mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}})$ and that the map $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G)$, $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}$ is a morphism of $\mathbb{Q}$-group schemes from $\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ to the latter $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G)$ using Proposition 2.14(i). One checks that the corresponding morphism of Lie algebras is as announced. □

Proposition-Definition 4.2. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, we define $\mathrm{der}_{\psi}^{\mathcal{V},(1)}$, the $\mathbb{Q}$-algebra derivation of $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ given by*

$$\mathrm{der}_{\psi}^{\mathcal{V},(1)} = \mathrm{ad}_{\beta(\psi\otimes 1)} + \mathrm{der}_{\psi}^{\mathcal{V},(0)}. \tag{4.1}$$

There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ by $\mathbb{Q}$-algebra derivations

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}}), \quad \psi \mapsto \mathrm{der}_{\psi}^{\mathcal{V},(1)}.$$

Proof. Same as the proof of Proposition-Definition 4.1, replacing the morphism $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}$ by $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ and using Proposition 2.14(ii). □

Proposition-Definition 4.3. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, we define $\mathrm{end}_{\psi}^{\mathcal{V},(10)}$ to be the $\mathbb{Q}$-linear endomorphism of $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ given by*

$$\mathrm{end}_{\psi}^{\mathcal{V},(10)} := \ell_{\beta(\psi\otimes 1)} + \mathrm{der}_{\psi}^{\mathcal{V},(0)}. \tag{4.2}$$

There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ by $\mathbb{Q}$-linear endomorphisms

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}}), \quad \psi \mapsto \mathrm{end}_{\psi}^{\mathcal{V},(10)}.$$

Proof. Same as the proof of Proposition-Definition 4.1, replacing $\mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G)$ by $\mathrm{Aut}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{V}}_G)$, $\mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}})$ by $\mathrm{End}_{\mathbb{Q}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}})$ and the morphism $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(0)}$ by the morphism $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(10)}$, and using Proposition 2.20. □

§4.2. The stabilizer Lie algebra $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}})$

Proposition-Definition 4.4. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, there is a unique $\mathbb{Q}$-linear endomorphism $\mathrm{end}_\psi^{\mathcal{M},(10)}$ of $\widehat{\mathcal{M}}_G^{\mathbb{Q}}$ such that the diagram*

$$(4.3)\qquad \begin{array}{ccc} \widehat{\mathcal{V}}_G^{\mathbb{Q}} & \xrightarrow{\mathrm{end}_\psi^{\mathcal{V},(10)}} & \widehat{\mathcal{V}}_G^{\mathbb{Q}} \\ {\scriptstyle -\cdot 1_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle -\cdot 1_{\mathcal{M}}} \\ \widehat{\mathcal{M}}_G^{\mathbb{Q}} & \xrightarrow[\mathrm{end}_\psi^{\mathcal{M},(10)}]{} & \widehat{\mathcal{M}}_G^{\mathbb{Q}} \end{array}$$

commutes. There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{M}}_G$ by $\mathbb{Q}$-linear endomorphisms

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\widehat{\mathcal{M}}_G^{\mathbb{Q}}), \quad \psi \mapsto \mathrm{end}_\psi^{\mathcal{M},(10)}.$$

Proof. The commutative diagram is given by an application of Proposition-Definition 2.23 for $\mathbf{k} = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$ and $\psi \in \ker(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbb{Q}\langle\langle X\rangle\rangle))$.

For the second statement, one first checks that the assignment $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G)$ is an affine $\mathbb{Q}$-group scheme whose Lie algebra is $\mathrm{End}_{\mathbb{Q}}(\widehat{\mathcal{M}}_G^{\mathbb{Q}})$. Then, using Corollary 2.25, one deduces that the map $\Psi \mapsto \mathrm{aut}_\Psi^{\mathcal{M},(10)}$ is a $\mathbb{Q}$-group scheme morphism from $(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast))$ to $(\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G))$. One finally proves that $\mathrm{end}^{\mathcal{M},(10)}$ is its corresponding Lie algebra morphism. □

To $\psi \in \widehat{\mathfrak{Lib}}(X)$, one associates $\gamma_\psi \in \mathbb{Q}[[x]]$ (see (1.27)). Then $\gamma_\psi(-e_1)$ is an element of $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$.

Proposition-Definition 4.5. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, we define the following $\mathbb{Q}$-linear endomorphism of $\widehat{\mathcal{M}}_G^{\mathbb{Q}}$:*

$$(4.4)\qquad {}^{\gamma}\mathrm{end}_\psi^{\mathcal{M},(10)} := \ell_{-\gamma_\psi(-e_1)} + \mathrm{end}_\psi^{\mathcal{M},(10)}.$$

There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{M}}_G^{\mathbb{Q}}$ by $\mathbb{Q}$-linear endomorphisms

$$(4.5)\qquad (\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{End}_{\mathbb{Q}}(\widehat{\mathcal{M}}_G^{\mathbb{Q}}), \quad \psi \mapsto {}^{\gamma}\,\mathrm{end}_\psi^{\mathcal{M},(10)}.$$

Proof. The maps $\Psi \mapsto \mathrm{aut}_\Psi^{\mathcal{M},(10)}$ and $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_\Psi^{\mathcal{M},(10)}$ are $\mathbb{Q}$-group scheme morphisms from $(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast))$ to $(\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G))$. The $\mathbb{Q}$-Lie algebra morphism associated to the former $\mathbb{Q}$-group scheme morphism is $\psi \mapsto \mathrm{end}_\psi^{\mathcal{M},(10)}$ by the proof of Proposition-Definition 4.4. The Lie algebra morphism associated to the latter $\mathbb{Q}$-group scheme morphism takes $\psi \in \widehat{\mathfrak{Lib}}(X)$ to the right-hand side of (4.4) in view of (2.26), and therefore is given by $\psi \mapsto {}^{\gamma}\mathrm{end}_\psi^{\mathcal{M},(10)}$. It follows that the latter map is a Lie algebra morphism. □

Thanks to this result, we are able to provide a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on the space $\mathrm{Mor}_{\mathbb{Q}}(\widehat{\mathcal{M}}_G^{\mathbb{Q}}, (\widehat{\mathcal{M}}_G^{\mathbb{Q}})^{\hat{\otimes}2})$ via

$$\psi \cdot D^{\mathcal{M}} := ({}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)}) \circ D^{\mathcal{M}} - D^{\mathcal{M}} \circ {}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)}. \tag{4.6}$$

In particular, the stabilizer of $D^{\mathcal{M}} = \widehat{\Delta}_G^{\mathcal{M}}$ is the Lie subalgebra

$$\begin{aligned}\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}}) := \{\psi \in \widehat{\mathfrak{Lib}}(X) \mid ({}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)}) \circ \widehat{\Delta}_G^{\mathcal{M}} \\ = \widehat{\Delta}_G^{\mathcal{M}} \circ {}^{\gamma}\mathrm{end}_{\psi}^{\mathcal{M},(10)}\}.\end{aligned} \tag{4.7}$$

For a commutative $\mathbb{Q}$-algebra $\mathbf{k}$, recall the group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$ in (2.31). One then has the following proposition:

Proposition 4.6. *The assignment* $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}}) \colon \mathbf{k} \mapsto \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})(\mathbf{k})$ *is an affine* $\mathbb{Q}$*-group scheme and* $\mathbf{Lie}(\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{M}})) = \mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}})$*.*

Proof. The first statement is obtained by an application of [EF18, Lem. 5.1], where $v = \widehat{\Delta}_G^{\mathcal{M}}$, and the second comes from the fact that the $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$-action on $\mathrm{Mor}_{\mathbb{Q}}(\widehat{\mathcal{M}}_G^{\mathbb{Q}}, (\widehat{\mathcal{M}}_G^{\mathbb{Q}})^{\hat{\otimes}2})$ given in (4.6) is the infinitesimal version of the group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{M}}_G, (\widehat{\mathcal{M}}_G)^{\hat{\otimes}2})$ given in (2.30), for any $\mathbb{Q}$-algebra $\mathbf{k}$. ☐

Corollary 4.7. $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}}) = \mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}})$ *(as Lie subalgebras of* $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$*).*

Proof. It follows from Theorem 3.8 thanks to Propositions 4.6 and 1.25(iii). ☐

§4.3. The stabilizer Lie algebra $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$

Proposition-Definition 4.8. *For* $\psi \in \widehat{\mathfrak{Lib}}(X)$*, we define the* $\mathbb{Q}$*-algebra derivation of* $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$*:*

$${}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{V},(1)} := \mathrm{ad}_{-\gamma_{\psi}(-e_1)} + \mathrm{der}_{\psi}^{\mathcal{V},(1)}. \tag{4.8}$$

There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ *on* $\widehat{\mathcal{V}}_G^{\mathbb{Q}}$ *by* $\mathbb{Q}$*-algebra derivations*

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{V}}_G^{\mathbb{Q}}), \quad \psi \mapsto {}^{\gamma}\,\mathrm{der}_{\psi}^{\mathcal{V},(1)}. \tag{4.9}$$

Proof. The maps $\Psi \mapsto \mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ and $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ are $\mathbb{Q}$-group scheme morphisms from $(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast))$ to $(\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G))$. The $\mathbb{Q}$-Lie algebra morphism associated to the former $\mathbb{Q}$-group scheme morphism is $\psi \mapsto \mathrm{der}_{\psi}^{\mathcal{V},(1)}$ by the proof of Proposition-Definition 4.2. The Lie algebra morphism associated to the latter $\mathbb{Q}$-group scheme morphism takes $\psi \in \widehat{\mathfrak{Lib}}(X)$ to the right-hand side of (4.9) in view of (2.23), and therefore is given by $\psi \mapsto {}^{\gamma}\,\mathrm{der}_{\psi}^{\mathcal{V},(1)}$. It follows that the latter map is a Lie algebra morphism. ☐

Proposition-Definition 4.9. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, the derivation ${}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{V},(1)}$ restricts to a derivation of the subalgebra $\widehat{\mathcal{W}}_G^{\mathbb{Q}}$ denoted ${}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}$. Moreover, there is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\widehat{\mathcal{W}}_G^{\mathbb{Q}}$ by $\mathbb{Q}$-algebra derivations*

$$(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle) \to \mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{W}}_G^{\mathbb{Q}}), \quad \psi \mapsto {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}. \tag{4.10}$$

Proof. One can prove that the assignment $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G)$ is a $\mathbb{Q}$-group scheme with Lie algebra $\mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{W}}_G^{\mathbb{Q}})$. The map $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ is a $\mathbb{Q}$-group scheme morphism from $(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast))$ to $(\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{V}}_G))$ and, by the proof of Proposition-Definition 4.8, its associated $\mathbb{Q}$-Lie algebra is $\psi \mapsto {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{V},(1)}$. Thanks to Proposition 2.28(ii), we obtain the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{W}}_G & \xrightarrow{{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}} & \widehat{\mathcal{W}}_G \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{V}}_G & \xrightarrow[{}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}]{} & \widehat{\mathcal{V}}_G, \end{array}$$

where $\Psi \in \mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle)$ with $\mathbf{k}$ a commutative $\mathbb{Q}$-algebra. Using this diagram for $\mathbf{k} = \mathbb{Q}[\varepsilon]/(\varepsilon^2)$ and $\psi \in \ker(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle) \to \mathcal{G}(\mathbb{Q}\langle\langle X\rangle\rangle))$, one obtains that the derivation ${}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{V},(1)}$ restricts to a derivation on $\widehat{\mathcal{W}}_G^{\mathbb{Q}}$ associated to the automorphism ${}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}$, which is denoted by ${}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}$. Moreover, the diagram states that the $\mathbb{Q}$-group scheme morphism provided by $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{V},(1)}$ defines a $\mathbb{Q}$-group scheme morphism $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}$ from $(\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast))$ to $(\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\widehat{\mathcal{W}}_G))$. Therefore, the map $\psi \mapsto {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}$ from $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ to $\mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\widehat{\mathcal{W}}_G^{\mathbb{Q}})$, which is the infinitesimal version of the latter $\mathbb{Q}$-group scheme morphism, is a $\mathbb{Q}$-Lie algebra morphism. □

Using Proposition-Definition 4.9, one can define the following Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on the space $\mathrm{Mor}_{\mathbb{Q}}(\widehat{\mathcal{W}}_G^{\mathbb{Q}}, (\widehat{\mathcal{W}}_G^{\mathbb{Q}})^{\hat{\otimes}2})$:

$$\psi \cdot D^{\mathcal{W}} := ({}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}) \circ D^{\mathcal{W}} - D^{\mathcal{W}} \circ {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}. \tag{4.11}$$

In particular, the stabilizer of $D^{\mathcal{W}} = \widehat{\Delta}_G^{\mathcal{W}}$ is the Lie subalgebra

$$\begin{aligned} \mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}}) := \big\{\psi \in \widehat{\mathfrak{Lib}}(X) \;\big|\; &({}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}) \circ \widehat{\Delta}_G^{\mathcal{W}} \\ &= \widehat{\Delta}_G^{\mathcal{W}} \circ {}^{\gamma}\mathrm{der}_{\psi}^{\mathcal{W},(1)}\big\}. \end{aligned} \tag{4.12}$$

For a commutative $\mathbb{Q}$-algebra $\mathbf{k}$, recall the group $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ in (2.29).

Proposition 4.10. *The assignment $\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}}) \colon \mathbf{k} \mapsto \mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})(\mathbf{k})$ is an affine $\mathbb{Q}$-group scheme and $\mathbf{Lie}(\mathsf{Stab}(\widehat{\Delta}_G^{\mathcal{W}})) = \mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$.*

Proof. The first statement is obtained by an application of [EF18, Lem. 5.1] where $v = \widehat{\Delta}_G^{\mathcal{W}}$ and the second comes from the fact that the Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\mathrm{Mor}_{\mathbb{Q}}(\widehat{\mathcal{W}}_G^{\mathbb{Q}}, (\widehat{\mathcal{W}}_G^{\mathbb{Q}})^{\widehat{\otimes}2})$ given in (4.11) is the infinitesimal version of the group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-mod}}(\widehat{\mathcal{W}}_G, (\widehat{\mathcal{W}}_G)^{\widehat{\otimes}2})$ given in (2.28), for any $\mathbb{Q}$-algebra $\mathbf{k}$. □

Corollary 4.11. $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{M}}) \subset \mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$ *(as Lie subalgebras of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$).*

Proof. It follows from Theorem 2.32 thanks to Propositions 4.10 and 4.6. □

§4.4. The stabilizer Lie algebra $\mathfrak{stab}(\widehat{\Delta}_G^{\mathcal{W}})$ in Racinet's formalism

Proposition-Definition 4.12. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$, we consider the derivation of $\mathbb{Q}\langle\langle Y\rangle\rangle$ given by*

$$ {}^{\gamma}d_{\psi}^{Y} := \varpi^{-1} \circ {}^{\gamma}\operatorname{der}_{\psi}^{\mathcal{W},(1)} \circ \varpi, \tag{4.13}$$

where ${}^{\gamma}\operatorname{der}_{\psi}^{\mathcal{W},(1)}$ is as in Proposition-Definition 4.9 and $\varpi\colon \mathbb{Q}\langle\langle Y\rangle\rangle \to \widehat{\mathcal{W}}_G^{\mathbb{Q}}$ is the $\mathbb{Q}$-algebra isomorphism of Corollary 2.9(i). There is a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on $\mathbb{Q}\langle\langle Y\rangle\rangle$ by derivations given by

$$\widehat{\mathfrak{Lib}}(X) \to \mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\mathbb{Q}\langle\langle Y\rangle\rangle), \quad \psi \mapsto {}^{\gamma}d_{\psi}^{Y}. \tag{4.14}$$

Proof. One can prove that the assignment $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$ is a $\mathbb{Q}$-group scheme with Lie algebra $\mathrm{Der}_{\mathbb{Q}-\mathrm{alg}}(\mathbb{Q}\langle\langle Y\rangle\rangle)$. Thanks to Proposition-Definition 3.9, the map $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast) \to \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$, $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{Y}$ is a morphism of $\mathbb{Q}$-group schemes from $\mathbf{k} \mapsto (\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ to the latter $\mathbf{k} \mapsto \mathrm{Aut}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle)$. It is related to the morphism of $\mathbb{Q}$-group schemes $\Psi \mapsto {}^{\Gamma}\mathrm{aut}_{\Psi}^{\mathcal{W},(1)}$ of Proposition-Definition 2.27 by (3.6). It follows that the corresponding $\mathbb{Q}$-Lie algebra morphism takes $\psi \in \widehat{\mathfrak{Lib}}(X)$ to the right-hand side of (4.13). The statement then follows from (4.13). □

For any $\psi \in \widehat{\mathfrak{Lib}}(X)$, the derivation ${}^{\gamma}d_{\psi}^{Y}$ can be expressed in the formalism of [Rac] as follows:

Proposition 4.13. *For $\psi \in \widehat{\mathfrak{Lib}}(X)$ and $(n,g) \in \mathbb{Z}_{>0} \times G$ we have*

$$\begin{aligned} {}^{\gamma}d_{\psi}^{Y}(y_{n,g}) &= \mathbf{q}_Y\big((\psi x_0^{n-1} - x_0^{n-1} t_g(\psi))x_g\big) \\ &\quad + \mathbf{q}_Y\big((x_0^{n-1}\gamma_{\psi}(x_g) - \gamma_{\psi}(x_1)x_0^{n-1})x_g\big). \end{aligned} \tag{4.15}$$

Proof. The infinitesimal version of the identity in Proposition 3.11 is given by

$${}^{\gamma}d_{\psi}^{Y}(y_{n,g}) = \mathbf{q}_Y\big(\big((-\gamma_{\psi}(x_1) + \psi)x_0^{n-1} + x_0^{n-1}t_g(\gamma_{\psi}(x_1) - \psi)\big)x_g\big).$$

Identity (4.15) then follows. □

From Proposition 4.12, we define a Lie algebra action of $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$ on the space $\mathrm{Mor}_{\mathbb{Q}}(\mathbb{Q}\langle\langle Y\rangle\rangle, \mathbb{Q}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$ by

$$\psi \cdot D := ({}^{\gamma}d_{\psi}^{Y} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}d_{\psi}^{Y}) \circ D - D \circ {}^{\gamma}d_{\psi}^{Y}. \tag{4.16}$$

In particular, the stabilizer of $D = \widehat{\Delta}_{\star}^{\mathrm{alg}}$ is the Lie subalgebra

$$\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}}) := \left\{\psi \in \widehat{\mathfrak{Lib}}(X) \,\middle|\, ({}^{\gamma}d_{\psi}^{Y} \otimes \mathrm{id} + \mathrm{id} \otimes {}^{\gamma}d_{\psi}^{Y}) \circ \widehat{\Delta}_{\star}^{\mathrm{alg}} = \widehat{\Delta}_{\star}^{\mathrm{alg}} \circ {}^{\gamma}d_{\psi}^{Y}\right\}. \tag{4.17}$$

For a commutative $\mathbb{Q}$-algebra $\mathbf{k}$, recall the group $\mathsf{Stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}})(\mathbf{k})$ in (3.11).

Proposition 4.14. *The assignment* $\mathsf{Stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}}) : \mathbf{k} \mapsto \mathsf{Stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}})(\mathbf{k})$ *is an affine* $\mathbb{Q}$*-group scheme and* $\mathbf{Lie}(\mathsf{Stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}})) = \mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}})$.

Proof. The first statement is a consequence of [EF18, Lem. 5.1] where $v = \widehat{\Delta}_{\star}^{\mathrm{alg}}$ and the second comes from the fact that the $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$-action on $\mathrm{Mor}_{\mathbb{Q}}(\mathbb{Q}\langle\langle Y\rangle\rangle, \mathbb{Q}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$ given in (4.16) is the infinitesimal version of the group action of $(\mathcal{G}(\mathbf{k}\langle\langle X\rangle\rangle), \circledast)$ on $\mathrm{Mor}_{\mathbf{k}\text{-alg}}(\mathbf{k}\langle\langle Y\rangle\rangle, \mathbf{k}\langle\langle Y\rangle\rangle^{\widehat{\otimes}2})$ given in (3.10), for any $\mathbb{Q}$-algebra $\mathbf{k}$. □

Corollary 4.15. $\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}}) = \mathfrak{stab}(\widehat{\Delta}_{G}^{\mathcal{W}})$ *(as Lie subalgebras of* $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$*).*

Proof. It follows from Theorem 3.12 thanks to Propositions 4.14 and 4.10. □

Finally, in Racinet's formalism, this translates to the following corollary:

Corollary 4.16. $\mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{mod}}) \subset \mathfrak{stab}(\widehat{\Delta}_{\star}^{\mathrm{alg}})$ *(as Lie subalgebras of* $(\widehat{\mathfrak{Lib}}(X), \langle\cdot,\cdot\rangle)$*).*

Proof. It follows immediately from Corollary 4.11 thanks to Corollaries 4.7 and 4.15. □

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