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A Weighted Version of Saitoh's Conjecture

by

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Abstract

In this article, we prove a weighted version of Saitoh's conjecture. As an application, we prove a weighted version of Saitoh's conjecture for higher derivatives.

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§1. Introduction

Let D be a planar regular region with n boundary components which are analytic Jordan curves (see [18, 22]). Let $H_2^{(c)}(D)$ (see [18]) denote the analytic Hardy class on D defined as the set of all analytic functions $f(z)$ on D such that the subharmonic functions $|f(z)|^2$ have harmonic majorants $U(z)$:

$$|f(z)|^2 \leq U(z) \quad \text{on } D.$$

Then each function $f(z) \in H_2^{(c)}(D)$ has Fatou's nontangential boundary value a.e. on ∂D belonging to $L^2(\partial D)$ (see [5]).

Kernel functions associated with various norms have been shown to play a fundamental role in several branches of mathematical analysis (see [2, 16]). Let us recall two reproducing kernels on D.

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Let λ be a positive continuous function on ∂D. We call $K_\lambda(z,\overline{w})$ (see [15]) the weighted Szegö kernel if

$$f(w) = \frac{1}{2\pi}\int_{\partial D} f(z)\overline{K_\lambda(z,\overline{w})}\lambda(z)\,|dz|$$

holds for any $f \in H_2^{(c)}(D)$. Let $G_D(p,t)$ be the Green function on D, and let $\partial/\partial v_p$ denote the derivative along the outer normal unit vector v_p. For fixed $t \in D$, $\frac{\partial G_D(p,t)}{\partial v_p}$ is positive and continuous on ∂D because of the analyticity of the boundary (see [18, 8]). When $\lambda(p) = (\frac{\partial G_D(p,t)}{\partial v_p})^{-1}$ on ∂D, $\widehat{K}_t(z,\overline{w})$ denotes $K_\lambda(z,\overline{w})$, which is the so-called conjugate Hardy H^2 kernel on D (see [18]). When $t = w$ and $z = w$, $\widehat{K}(z)$ denotes $\widehat{K}_t(z,\overline{w})$ for simplicity.

Let ρ be a positive Lebesgue measurable function on D, which satisfies that there exists $a_U > 0$ such that $\rho^{-a_U} \in L^1(U)$ for any open subset $U \Subset D\backslash Z$, where Z is a discrete subset of D. We denote by $B_\rho(z,\overline{w})$ the weighted Bergman kernel on D with the weight ρ (see [17]) if

$$f(w) = \int_D f(z)\overline{B_\rho(z,\overline{w})}\rho(z)$$

holds for any holomorphic function f on D satisfying $\int_D |f(z)|^2 < +\infty$. Denote

$$B_\rho(z) := B_\rho(z,\bar{z}).$$

When $\rho \equiv 1$, $B(z)$ denotes $B_\rho(z)$ for simplicity.

Let $c_\beta(z)$ be the logarithmic capacity which is defined by

$$c_\beta(z) := \exp\lim_{w\to z}(G_D(w,z) - \log|w-z|).$$

In [22], Yamada listed the following conjectures on $c_\beta(z)$, $B(z)$ and $\widehat{K}(z)$.

Conjecture 1.1. *If $n > 1$, then*

$$c_\beta(z)^2 < \pi B(z) < \widehat{K}(z). \tag{1.1}$$

The left part of inequality (1.1) is the so-called Suita conjecture (see [20]) and the right part of inequality (1.1) is the so-called Saitoh conjecture (see [18]).

The original form of the Suita conjecture (see [20]) was posed on open Riemann surfaces admitting nontrivial Green functions. Błocki [3] proved the "$\leq$" part of the Suita conjecture on bounded planar domains. Guan–Zhou [13] proved the "$\leq$" part of the Suita conjecture on open Riemann surfaces. In [14], Guan–Zhou proved a necessary and sufficient condition for $c_\beta(z)^2 = \pi B(z)$ to hold on open Riemann surfaces, which completed the proof of the Suita conjecture.

In [8], Guan proved Saitoh's conjecture:

Theorem 1.2 ([8]). *If $n > 1$, then $\widehat{K}(z) > \pi B(z)$.*

We recall some notation (see [7], see also [14, 11, 10]). Let $p\colon \Delta \to D$ be the universal covering from the unit disc Δ to D, and let $z_0 \in D$. We call the holomorphic function f on Δ a multiplicative function, if there is a character χ, which is the representation of the fundamental group of D, such that $g^* f = \chi(g) f$, where $|\chi| = 1$ and g is an element of the fundamental group of D. Denote the set of such f by $\mathcal{O}^{\chi}(D)$.

It is known that for any function u on D with value $[-\infty, +\infty)$ such that e^u is locally the modulus of a holomorphic function, there exist a character χ_u and a multiplicative function $f_u \in \mathcal{O}^{\chi_u}(D)$, such that $|f_u| = p^*(e^u)$. If $u_1 - u_2 = \log |f|$, where f is a holomorphic function on Ω, then $\chi_{u_1} = \chi_{u_2}$. For the Green function $G_D(\cdot, z_0)$, denote $\chi_{z_0} := \chi_{G_D(\cdot, z_0)}$ and $f_{z_0} := f_{G_D(\cdot, z_0)}$. Note that D is conformally equivalent to the unit disc (i.e. $n = 1$) if and only if $\chi_{z_0} \equiv 1$ (see [20]).

Let u be a harmonic function on D, and let $\rho = e^{-2u}$. Yamada [22] posed the following weighted version of the Suita conjecture, which is the so-called extended Suita conjecture.

Conjecture 1.3. *The inequality $c_\beta^2(z_0) \le \pi \rho(z_0) B_\rho(z_0)$ holds for any $z_0 \in D$, and equality holds if and only if $\chi_{z_0} = \chi_{-u}$.*

In [14], Guan–Zhou proved the extended Suita conjecture. More general weighted versions of Suita conjecture can be found in [9, 11], and a weighted version of the Suita conjecture for higher derivatives can be found in [10].

In the present article, we consider weighted versions of Saitoh's conjecture.

§1.1. Main result

Let D be a planar regular region with n boundary components which are analytic Jordan curves, and let $z_0 \in D$.

Let ψ be a Lebesgue measurable function on $\overline{D}$, which satisfies that ψ is subharmonic on D, $\psi|_{\partial D} \equiv 0$ and the Lelong number $v(dd^c\psi, z_0) > 0$, where $d^c = \frac{\partial - \bar{\partial}}{2\pi\sqrt{-1}}$. Assume that $\psi \in C^1(U \cap \overline{D})$ for an open neighborhood U of ∂D and $\partial\psi/\partial v_p$ is positive on ∂D, where $\partial/\partial v_p$ denotes the derivative along the outer normal unit vector v_p. Let φ be a Lebesgue measurable function on $\overline{D}$ satisfying that $\varphi + 2\psi$ is subharmonic on D, the Lelong number

$$v(dd^c(\varphi + 2\psi), z_0) \ge 2$$

and φ is continuous at z for any $z \in \partial D$. Assume that one of the following two statements holds:

(a) $(\psi - p_0 G_D(\cdot, z_0))(z_0) > -\infty$, where $p_0 = v(dd^c(\psi), z_0) > 0$;

(b) $\varphi + 2a\psi$ is subharmonic near z_0 for some $a \in [0, 1)$.

Let c be a positive Lebesgue measurable function on $[0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $[0, +\infty)$, $\lim_{t\to 0+0} c(t) = c(0) = 1$ and $\int_0^{+\infty} c(t)e^{-t}\,dt < +\infty$.

Denote

$$\rho := e^{-\varphi}c(-2\psi) \quad \text{and} \quad K_{\rho,\psi}(z) := K_{\rho(\frac{\partial\psi}{\partial v_p})^{-1}}(z, \bar{z}),$$

and assume that ρ has a positive lower bound on any compact subset of $D\backslash Z$, where $Z \subset \{\psi = -\infty\}$ is a discrete subset of D.

We present a weighted version of Saitoh's conjecture as follows:

Theorem 1.4. *Assume that $B_\rho(z_0) > 0$. Then*

$$K_{\rho,\psi}(z_0) \geq \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B_\rho(z_0)$$

holds, and the equality holds if and only if the following statements hold:

(1) *$\varphi + 2\psi = 2G_D(\cdot, z_0) + 2u$, where u is a harmonic function on D;*

(2) *$\psi = p_0 G_D(\cdot, z_0)$, where $p_0 = v(dd^c(\psi), z_0) > 0$;*

(3) *$\chi_{z_0} = \chi_{-u}$, where χ_{-u} and χ_{z_0} are the characters associated to the functions $-u$ and $G_D(\cdot, z_0)$ respectively.*

Remark 1.5. Let p be the universal covering from the unit disc Δ to D. When statements (1)–(3) in Theorem 1.4 hold,

$$K_{\rho,\psi}(\cdot, \overline{z_0}) = \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B_\rho(\cdot, \overline{z_0}) = c_1(p_*(f_{z_0}))'p_*(f_u),$$

where $K_{\rho,\psi}(\cdot, \overline{z_0})$ denotes $K_{\rho(\frac{\partial\psi}{\partial v_p})^{-1}}(\cdot, \overline{z_0})$, c_1 is a constant, f_u is a holomorphic function on Δ such that $|f_u| = p^*(e^u)$ and f_{z_0} is a holomorphic function on Δ such that $|f_{z_0}| = p^*(e^{G_D(\cdot,z_0)})$. We prove the remark in Section 3.

Remark 1.6. For any $z_0 \in D$, there exists $u \in C(\overline{D})$ such that u is harmonic on D and $\chi_{z_0} = \chi_{-u}$. In fact, $u(z) := \log|z - z_0| - G_D(z, z_0)$ is harmonic on D and $\chi_{z_0} = \chi_{-u}$.

Let λ be any positive continuous function on ∂D. By solving the Dirichlet problem, there exists $u \in C(\overline{D})$ satisfying that $u|_{\partial D} = -\frac{1}{2}\log\lambda$ and u is harmonic on D. When $\psi = G_D(\cdot, z_0)$, $\widehat{K}_\lambda(z_0)$ denotes $K_{\lambda,\psi}(z_0)$.

Theorem 1.4 implies the following corollary.

Corollary 1.7. *The inequality* $\widehat{K}_\lambda(z_0) \geq \pi B_{e^{-2u}}(z_0)$ *holds for any* $z_0 \in D$*, and the equality holds if and only if* $\chi_{z_0} = \chi_{-u}$*.*

Note that $\chi_{z_0} \equiv 1$ holds if and only if $n = 1$ (see [20]); then the above corollary is Theorem 1.2 when $\lambda \equiv 1$ and $u \equiv 0$.

§1.2. Applications: The weighted version of Saitoh's conjecture for higher derivatives

Let D be a planar regular region with n boundary components which are analytic Jordan curves, and let $z_0 \in D$.

Let ψ be a Lebesgue measurable function on $\overline{D}$, which satisfies that ψ is subharmonic on D, $\psi|_{\partial D} \equiv 0$ and the Lelong number $v(\psi, z_0) > 0$. Assume that $\psi \in C^1(U \cap \overline{D})$ for an open neighborhood U of ∂D and $\partial\psi/\partial v_p$ is positive on ∂D. Let φ be a Lebesgue measurable function on $\overline{D}$ satisfying that $\varphi + 2\psi$ is subharmonic on D, the Lelong number

$$v(dd^c(\varphi + 2\psi), z_0) \geq 2(k+1)$$

and φ is continuous at z for any $z \in \partial D$. Assume that one of the following two statements holds:

(a) $(\psi - p_0 G_D(\cdot, z_0))(z_0) > -\infty$, where $p_0 = v(dd^c(\psi), z_0) > 0$;

(b) $\varphi + 2a\psi$ is subharmonic near z_0 for some $a \in [0, 1)$.

Let k be a nonnegative integer. Let c be a positive Lebesgue measurable function on $[0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $[0, +\infty)$, $\lim_{t\to 0+0} c(t) = c(0) = 1$ and $\int_0^{+\infty} c(t)e^{-t}\, dt < +\infty$.

Denote

$$\rho := e^{-\varphi} c(-2\psi),$$

and assume that ρ has a positive lower bound on any compact subset of $D \backslash Z$, where $Z \subset \{\psi = -\infty\}$ is a discrete subset of D.

Let us consider two kernel functions for higher derivatives. Denote

$$B_\rho^{(k)}(z_0) := \sup\Big\{\Big|\frac{f^{(k)}(z_0)}{k!}\Big|^2 : f \in \mathcal{O}(D),\ \int_D |f|^2 \rho \leq 1 \\ \text{and } f(z_0) = \cdots = f^{(k-1)}(z_0) = 0\Big\}.$$

When $\rho \equiv 1$, $B_\rho^{(k)}(z_0)$ is the Bergman kernel for higher derivatives (see [2, 4]). When $k = 0$, $B_\rho^{(k)}(z_0)$ is the weighted Bergman kernel $B_\rho(z_0)$ (see Section 1.1). Denote

$$K_{\rho,\psi}^{(k)}(z_0) := \sup\Big\{\Big|\frac{f^{(k)}(z_0)}{k!}\Big|^2 : f \in H_2^{(c)}(D),\ \int_{\partial D} |f|^2 \rho \Big(\frac{\partial\psi}{\partial v_z}\Big)^{-1} |dz| \leq 1 \\ \text{and } f(z_0) = \cdots = f^{(k-1)}(z_0) = 0\Big\}.$$

In particular, when $k = 0$, $K^{(k)}_{\rho,\psi}(z_0)$ is the weighted Szegö kernel $K_{\rho,\psi}(z_0)$ (see Section 1.1).

We present a weighted version of Saitoh's conjecture for higher derivatives as follows:

Corollary 1.8. *Assume that $B^{(k)}_{\rho}(z_0) > 0$. Then*

$$K^{(k)}_{\rho}(z_0) \geq \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B^{(k)}_{\rho}(z_0)$$

holds, and the equality holds if and only if the following statements hold:

(1) $\varphi + 2\psi = 2(k+1)G_D(\cdot, z_0) + 2u$, *where u is a harmonic function on D;*

(2) $\psi = p_0 G_D(\cdot, z_0)$, *where $p_0 = v(dd^c(\psi), z_0) > 0$;*

(3) $\chi^{k+1}_{z_0} = \chi_{-u}$, *where χ_{-u} and χ_{z_0} are the characters associated to the functions $-u$ and $G_D(\cdot, z_0)$ respectively.*

Let λ be an arbitrary positive continuous function on ∂D. By solving the Dirichlet problem, there exists $u \in C(\overline{D})$ satisfying that $u|_{\partial D} = -\frac{1}{2}\log\lambda$ and u is harmonic on D. When $\psi = (k+1)G_D(\cdot, z_0)$, $\widehat{K}^{(k)}_{\lambda}(z_0)$ denotes $K^{(k)}_{\lambda,\psi}(z_0)$.

Corollary 1.8 implies the following corollary:

Corollary 1.9. *The inequality $\widehat{K}^{(k)}_{\lambda}(z_0) \geq \pi B^{(k)}_{e^{-2u}}(z_0)$ holds for any $z_0 \in D$, and the equality holds if and only if $\chi^{k+1}_{z_0} = \chi_{-u}$.*

§2. Preparations

In this section, we make some preparations.

§2.1. A sufficient condition for $f \in H^{(c)}_2(D)$

Let D be a planar regular region with n boundary components which are analytic Jordan curves, and let $z_0 \in D$. Let ψ be as in Theorem 1.4. Let f be a holomorphic function on D. In this section we give a sufficient condition for $f \in H^{(c)}_2(D)$ (i.e. Lemma 2.4).

We recall the following basic formula, and we give a proof for the convenience of readers.

Lemma 2.1. *The equality $\frac{\partial\psi}{\partial v_z} = ((\frac{\partial\psi}{\partial x})^2 + (\frac{\partial\psi}{\partial y})^2)^{\frac{1}{2}}$ holds on ∂D, where $\partial/\partial v_z$ denotes the derivative along the outer normal unit vector v_z.*

Proof. For fixed $z_1 \in \partial D$, as $\frac{\partial\psi}{\partial v_z}$ is positive on D, we can assume that $\frac{\partial\psi}{\partial y} \neq 0$ without loss of generality. Then there exists a neighborhood U_1 of z_1 with coordinates

$(u,v)=(x,\psi(x+\sqrt{-1}y))$. It is clear that

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}=\frac{\partial \psi}{\partial x}\quad\text{and}\quad \frac{\partial v}{\partial y}=\frac{\partial \psi}{\partial y},$$

which implies that

$$\frac{\partial x}{\partial u}=1,\quad \frac{\partial y}{\partial u}=-\frac{\partial\psi/\partial x}{\partial\psi/\partial y},\quad \frac{\partial x}{\partial v}=0\quad\text{and}\quad \frac{\partial y}{\partial v}=\Big(\frac{\partial \psi}{\partial y}\Big)^{-1}.$$

It is clear that

$$v_z=\frac{(\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y})}{((\frac{\partial\psi}{\partial x})^2+(\frac{\partial\psi}{\partial y})^2)^{\frac{1}{2}}},$$

and thus we have $\frac{\partial\psi}{\partial v_z}=((\frac{\partial\psi}{\partial x})^2+(\frac{\partial\psi}{\partial y})^2)^{\frac{1}{2}}$. □

We give a relationship between the superlevel sets of ψ and $G_D(\cdot,z_0)$.

Lemma 2.2. *There exist $t_0>0$ and $C>1$ such that*

$$\big\{z\in D: G_D(z,z_0)\geq -t\big\}\subset\big\{z\in D:\psi(z)\geq -Ct\big\}$$

for any $t\in(0,t_0)$.

Proof. As ∂D is compact, it suffices to prove that for any $z_1\in\partial D$, there exist a neighborhood U of z_1, $t_0>0$ and $C>1$ such that $\{z\in D\cap U: G_D(z,z_0)\geq -t\}\subset\{z\in D\cap U:\psi(z)\geq -Ct\}$ for any $t\in(0,t_0)$.

For fixed $z_1\in\partial D$, as $\frac{\partial G_D(z,z_0)}{\partial v_z}$ is positive on D, we can assume that $\frac{\partial G_D(z,z_0)}{\partial y}\neq 0$ and z_1 is the origin o in $\mathbb{C}$ without loss of generality. Then there exists a neighborhood U_1 of z_1 with coordinates $(u,v)=(x,G_D(x+\sqrt{-1}y,z_0))$. It is clear that

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}=\frac{\partial}{\partial x}G(z,z_0)\quad\text{and}\quad \frac{\partial v}{\partial y}=\frac{\partial}{\partial y}G_D(z,z_0),$$

which implies that

$$\frac{\partial x}{\partial u}=1,\quad \frac{\partial y}{\partial u}=-\frac{\frac{\partial}{\partial x}G(z,z_0)}{\frac{\partial}{\partial y}G_D(z,z_0)},\quad \frac{\partial x}{\partial v}=0\quad\text{and}\quad \frac{\partial y}{\partial v}=\Big(\frac{\partial}{\partial y}G_D(z,z_0)\Big)^{-1}.$$

It is clear that

$$v_z=\frac{(\frac{\partial G_D(z,z_0)}{\partial x},\frac{\partial G_D(z,z_0)}{\partial y})}{((\frac{\partial G_D(z,z_0)}{\partial x})^2+(\frac{\partial G_D(z,z_0)}{\partial y})^2)^{\frac{1}{2}}}$$

on ∂D. Thus, we have

$$\begin{aligned}&\frac{\partial\psi}{\partial u}\cdot\frac{\partial G_D(z,z_0)}{\partial x}+\frac{\partial\psi}{\partial v}\cdot|\nabla G_D(z,z_0)|^2\\&\quad=\Big(\frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial u}\Big)\frac{\partial G_D(z,z_0)}{\partial x}+\Big(\frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial v}\Big)|\nabla G_D(z,z_0)|^2\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\cdot\frac{\frac{\partial}{\partial x}G(z,z_0)}{\frac{\partial}{\partial y}G_D(z,z_0)}\right)\frac{\partial G_D(z,z_0)}{\partial x}\\
&\quad + \frac{\partial\psi}{\partial y}\cdot\left(\frac{\partial}{\partial y}G_D(z,z_0)\right)^{-1}\cdot\left(\left(\frac{\partial G_D(z,z_0)}{\partial x}\right)^2 + \left(\frac{\partial G_D(z,z_0)}{\partial y}\right)^2\right)\\
&= \frac{\partial\psi}{\partial y}\cdot\frac{\partial G_D(z,z_0)}{\partial y} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial G_D(z,z_0)}{\partial x}\\
&= \frac{\frac{\partial\psi}{\partial v_z}}{((\frac{\partial G_D(z,z_0)}{\partial x})^2 + (\frac{\partial G_D(z,z_0)}{\partial y})^2)^{\frac{1}{2}}} > 0
\end{aligned}
$$

on ∂D. Note that $|\nabla G_D(z,z_0)|^2 > 0$ on ∂D. There exist $a \in \mathbb{R}$, $m > 0$, $r_0 > 0$ and $b > 0$ such that

$$
m < a\frac{\partial\psi}{\partial u} + b\frac{\partial\psi}{\partial v} < \frac{1}{m} \tag{2.1}
$$

on an open parallelogram $U_2 := \{(u,v) : -r_0 < v < r_0,\ \frac{a}{b}v - r_0 < u < \frac{a}{b}v + r_0\} \Subset U_1$. Note that $\psi|_{\{v=0\}} = \psi|_{\partial D} \equiv 0$. For any $(u,v) \in U_2$, we have $(u - \frac{a}{b}v + ta, tb) \in U_2$ for any $t \in [0, \frac{v}{b}]$ and

$$
\begin{aligned}
\psi(u,v) &= \psi(u,v) - \psi\left(u - \frac{a}{b}v, 0\right)\\
&= \psi\left(u - \frac{a}{b}v + ta, tb\right)\Big|_{t=0}^{t=v/b}\\
&= \int_0^{v/b}\left(a\frac{\partial\psi}{\partial u} + b\frac{\partial\psi}{\partial v}\right)\left(u - \frac{a}{b}v + ta, tb\right)dt.
\end{aligned} \tag{2.2}
$$

Thus, for any $t \in (0, r_0)$, if $G(z,z_0) = v \geq -t$, it follows from inequality (2.1) and equality (2.2) that

$$
\begin{aligned}
\psi(u,v) &= -\int_{v/b}^{0}\left(a\frac{\partial\psi}{\partial u} + b\frac{\partial\psi}{\partial v}\right)\left(u - \frac{a}{b}v + ta, tb\right)dt\\
&\geq \frac{v}{mb}\\
&\geq -\frac{t}{mb},
\end{aligned}
$$

which implies that $\{z \in D\cap U_2 : G_D(z,z_0) \geq -t\} \subset \{z \in D \cap U_2 : \psi(z) \geq -\frac{1}{mb}t\}$ for any $t \in (0, r_0)$.

Thus, Lemma 2.2 holds. □

We recall the following coarea formula.

Lemma 2.3 (See [6]). *Suppose that* Ω *is an open set in* $\mathbb{R}^n$ *and* $u \in C^1(\Omega)$*. Then for any* $g \in L^1(\Omega)$*,*

$$\int_\Omega g(x)|\nabla u(x)|\,dx = \int_{\mathbb{R}} \left(\int_{u^{-1}(t)} g(x)\,dH_{n-1}(x) \right) dt,$$

where H_{n-1} *is the* $(n-1)$*-dimensional Hausdorff measure.*

The following lemma gives a sufficient condition for $f \in H_2^{(c)}(D)$.

Lemma 2.4. *Let* f *be a holomorphic function on* D*. Assume that*

$$\liminf_{r\to 1-0} \frac{\int_{\{z\in D:\psi(z)\geq \log r\}} |f(z)|^2}{1-r} < +\infty; \tag{2.3}$$

then we have $f \in H_2^{(c)}(D)$.

Proof. It follows from Lemma 2.2 and inequality (2.3) that

$$\begin{aligned} &\liminf_{r\to 1-0} \frac{\int_{\{z\in D:e^{G_D(z,z_0)}\geq r\}} |f(z)|^2}{1-r} \\ &\leq \liminf_{r\to 1-0} \frac{\int_{\{z\in D:\psi(z)\geq C\log r\}} |f(z)|^2}{1-r} \\ &= \liminf_{r\to 1-0} \frac{\int_{\{z\in D:\psi(z)\geq C\log r\}} |f(z)|^2}{1-r^C} \times \frac{1-r^C}{1-r} \\ &< +\infty. \end{aligned} \tag{2.4}$$

Denote

$$D_r := \{z \in D : e^{G_D(z,z_0)} < r\},$$

where $r \in (0,1)$. It is well known that $G_D(\cdot, z_0) - \log r$ is the Green function on D_r. By the analyticity of the boundary of D, we have that $G_D(z,w)$ has an analytic extension on $U \times V \backslash \{z = w\}$ and $\frac{\partial G_D(z,z_0)}{\partial v_z}$ is positive and smooth on ∂D, where U is a neighborhood of $\overline{D}$ and $V \Subset D$. Then there exist $r_0 \in (0,1)$ and $C_1 > 0$ such that $\frac{1}{C_1} \leq |\nabla G_D(\cdot, z_0)| \leq C_1$ on $\{z \in D : G_D(z,z_0) > \log r_0\}$, which implies

$$\frac{1}{C_1} \leq \frac{\partial G_D(z,z_0)}{\partial v_z} \leq C_1 \tag{2.5}$$

holds on $\{z \in D : G_D(z,z_0) > \log r_0\}$ (by using Lemma 2.1).

Denote

$$v_r(w) := \frac{1}{2\pi} \int_{\partial D_r} |f|^2 \frac{\partial G_{D_r}(z,w)}{\partial v_z} |dz|,$$

a harmonic function on D_r, where $r \in (r_0, 1)$. As $G_{D_r}(z, z_0) = G_D(z, z_0) - \log r$, we have

$$v_r(z_0) = \frac{1}{2\pi} \int_{\partial D_r} |f|^2 \frac{\partial G_D(z, z_0)}{\partial v_z} |dz|. \tag{2.6}$$

For fixed $r_1 \in (r_0, 1)$, inequality (2.5) implies that

$$\begin{aligned} v_{r_1}(z_0) &\le v_r(z_0) \\ &= \frac{1}{2\pi} \int_{\partial D_r} |f|^2 \frac{\partial G_D(z, z_0)}{\partial v_z} |dz| \\ &\le C_2 \int_{\partial D_r} |f|^2 \Big(\frac{\partial G_D(z, z_0)}{\partial v_z} \Big)^{-1} |dz| \end{aligned} \tag{2.7}$$

holds for any $r \in (r_1, 1)$, where C_2 is a positive constant independent of r_1 and r. Using Lemmas 2.1, 2.3 and inequality (2.4), we have

$$\begin{aligned} v_{r_1}(z_0) &\le \liminf_{r \to 1-0} \frac{C_2 \int_r^1 \big(\int_{\partial D_s} |f|^2 \big(\frac{\partial G_D(z, z_0)}{\partial v_z} \big)^{-1} |dz| \big)\, ds}{1 - r} \\ &= \liminf_{r \to 1-0} \frac{C_2 \int_r^1 \big(\int_{\{e^{G_D(\cdot, z_0)} = s\}} |f|^2 e^{G_D(z, z_0)} |\nabla e^{G_D(z, z_0)}|^{-1} |dz| \big)\, ds}{1 - r} \\ &= \liminf_{r \to 1-0} \frac{C_2 \int_{\{z \in D : e^{G_D(z, z_0)} > r\}} |f|^2 e^{G_D(z, z_0)}}{1 - r} \\ &\le C_3, \end{aligned} \tag{2.8}$$

where C_3 is a positive constant independent of r_1. As $|f|^2$ is subharmonic, we have $|f|^2 \le v_r$ on D_r and $\{v_r\}$ is increasing with respect to r. By Harnack's principle (see [1]), the sequence $\{v_r\}$ converges to a harmonic function v on D, which satisfies $|f(z)|^2 \le v(z)$ for any $z \in D$. Thus, $f \in H_2^{(c)}(D)$. □

§2.2. Concavity property of minimal L^2 integrals

In this section we recall the concavity property of minimal L^2 integrals on open Riemann surfaces and a characterization for the concavity degenerating to linearity ([11], see also [10, 12]).

Let D be a planar regular region with n boundary components which are analytic Jordan curves. Let ψ be a negative subharmonic function on D, and let φ be a Lebesgue measurable function on D, such that $\varphi + \psi$ is a plurisubharmonic function on D.

Let $z_0 \in D$ be such that $\mathcal{I}(\varphi + \psi)_{z_0} \neq \mathcal{O}_{z_0}$, where $\mathcal{I}(\varphi + \psi)$ is the multiplier ideal sheaf, which is the sheaf of germs of holomorphic functions h such that

$|h|^2e^{-\varphi-\psi}$ is locally integrable. Let f be a holomorphic function on a neighborhood of z_0. Let $\mathcal{F}_{z_0} \supseteq \mathcal{I}(\varphi+\psi)_{z_0}$ be an ideal of $\mathcal{O}_{z_0}$.

Denote

$$\inf\left\{\int_{\{\psi<-t\}} |\tilde{f}|^2e^{-\varphi}c(-\psi) : (\tilde{f}-f, z_0) \in \mathcal{F}_{z_0} \text{ and } \tilde{f} \in \mathcal{O}(\{\psi<-t\})\right\}$$

by $G(t;c)$ (without misunderstanding, we denote $G(t;c)$ by $G(t)$), where $t \in [0,+\infty)$ and c is a nonnegative measurable function on $(0,+\infty)$.

Let c be a positive measurable function c on $(0,+\infty)$, which satisfies that $c(t)e^{-t}$ is decreasing with respect to t, $\int_0^{+\infty} c(s)e^{-s}\,ds < +\infty$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $D\backslash Z$, where $Z \subset \{\psi = -\infty\}$ is a discrete subset of M.

We recall some results about the concavity of $G(t)$, which will be used in the proof of Theorem 1.4.

Theorem 2.5 ([11]). *Let $h(t) = \int_t^{+\infty} c(s)e^{-s}\,ds$. Then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}\,ds)$, $\lim_{t\to T+0} G(t) = G(0)$ and $\lim_{t\to+\infty} G(t) = 0$.*

Lemma 2.6 ([11]). *There exists a unique holomorphic function F on $\{\psi < -t\}$ satisfying $(F-f, z_0) \in \mathcal{F}_{z_0}$ and $G(t;c) = \int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi)$. Furthermore, for any holomorphic function $\widehat{F}$ on $\{\psi < -t\}$ satisfying $(\widehat{F}-f, z_0) \in \mathcal{F}_{z_0}$ and $\int_{\{\psi<-t\}} |\widehat{F}|^2e^{-\varphi}c(-\psi) < +\infty$, we have the equality*

$$\int_{\{\psi<-t\}} |F_t|^2e^{-\varphi}c(-\psi) + \int_{\{\psi<-t\}} |\widehat{F}-F_t|^2e^{-\varphi}c(-\psi) = \int_{\{\psi<-t\}} |\widehat{F}|^2e^{-\varphi}c(-\psi).$$

We recall a necessary condition and a characterization of the concavity degenerating to linearity.

Corollary 2.7 ([11]). *If $G(h^{-1}(r))$ is linear with respect to $r\in[0, \int_0^{+\infty} c(s)e^{-s}\,ds)$, where $h(t) = \int_t^{+\infty} c(s)e^{-s}\,ds$, then there is a unique holomorphic function F on D satisfying $(F-f, z_0) \in \mathcal{F}_{z_0}$ and $G(t;c) = \int_{\{\psi<-t\}} |F|^2e^{-\varphi}c(-\psi)$ for any $t \geq 0$. Furthermore,*

$$\int_{\{-t_1\leq\psi<-t_2\}} |F|^2e^{-\varphi}a(-\psi) = \frac{G(0;c)}{\int_0^{+\infty} c(s)e^{-s}\,ds} \int_{t_2}^{t_1} a(t)e^{-t}\,dt \tag{2.9}$$

for any nonnegative measurable function a on $(0,+\infty)$, where $+\infty \geq t_1 > t_2 \geq 0$.

Theorem 2.8 ([11], see also [12]). *Assume that one of the following two statements holds:*

(a) $(\psi - 2p_0 G_D(\cdot, z_0))(z_0) > -\infty$, *where* $p_0 = \frac{1}{2}v(dd^c(\psi), z_0) > 0$;

(b) $\varphi + a\psi$ *is subharmonic near* z_0 *for some* $a \in [0, 1)$.

Then $G(h^{-1}(r))$ *is linear with respect to* r *if and only if the following statements hold:*

(1) $\psi = 2p_0 G_D(\cdot, z_0)$, *where* $p_0 = \frac{1}{2}v(dd^c(\psi), z_0) > 0$;

(2) $\varphi + \psi = 2\log|g| + 2G_D(\cdot, z_0) + 2u$ *and* $\mathcal{F}_{z_0} = \mathcal{I}(\varphi + \psi)_{z_0}$, *where* g *is a holomorphic function on* D *such that* $\mathrm{ord}_{z_0}(g) = \mathrm{ord}_{z_0}(f)$ *and* u *is a harmonic function on* D;

(3) $\chi_{z_0} = \chi_{-u}$, *where* χ_{-u} *and* χ_{z_0} *are the characters associated to the functions* $-u$ *and* $G_D(\cdot, z_0)$ *respectively.*

Remark 2.9 ([12]). Assume statements (1)–(3) in Theorem 2.8 hold. Let p be the universal covering from unit disc Δ to D. Let f_u be a holomorphic function on Δ such that $|f_u| = p^*(e^u)$, and let f_{z_0} be a holomorphic function on Δ such that $|f_{z_0}| = p^*(e^{G_D(\cdot, z_0)})$. Denote $c_0 := \lim_{z \to z_0} \frac{f}{p_0 g p_*(f_u)(p_*(f_{z_0}))'}$. Then

$$c_0 p_0 g p_*(f_u)(p_*(f_{z_0}))'$$

is the unique holomorphic function F on D such that $(F - f, z_0) \in \mathcal{F}_{z_0}$ and $G(t) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$.

§2.3. Some other required results

Let D be a planar regular region with n boundary components which are analytic Jordan curves, and let $z_0 \in D$.

Lemma 2.10 (See [19]; see also [21]). *The Green function*

$$G_D(z, z_0) = \sup_{v \in \Delta_D^*(z_0)} v(z),$$

where $\Delta_D^*(z_0)$ *is the set of negative subharmonic functions on* D *such that* $v(z) - \log|z - z_0|$ *has a locally finite upper bound near* z_0. *Moreover,* $G_D(z, z_0) - \log|z - z_0|$ *is harmonic on* D.

The following two properties of the weighted Szegö kernel can be found in [15].

Lemma 2.11 ([15]). *Let* λ *be a positive continuous function on* ∂D. *There exists an analytic function* $K_\lambda(z, \overline{w})$ *with the following properties:* $K_\lambda(z, \overline{w})$ *is holomorphic on* $D \times D$; $|K_\lambda(z, \overline{w})|$ *is continuous on* $\overline{D}$ *for fixed* $w \in D$;

$$\int_{\partial D} f(z)\overline{K_\lambda(z, \overline{w})}\lambda(z)\,|dz| = f(w)$$

holds for any $f \in H_2^{(c)}(D)$.

Lemma 2.12 ([15]). *Let λ be a positive continuous function on ∂D, and let $f \in H_2^{(c)}(D)$ satisfy $f(z_0) = 1$. Then we have*

$$\int_{\partial D} |M(z)|^2 \lambda(z)\,|dz| \le \int_{\partial D} |f(z)|^2 \lambda(z)\,|dz|, \tag{2.10}$$

where $M(z) := \frac{K_\lambda(z,\overline{z_0})}{K_\lambda(z_0,\overline{z_0})}$. Equality in (2.10) holds if and only if $f(z) \equiv M(z)$.

§3. Proofs of Theorem 1.4 and Remark 1.5

In this section we prove Theorem 1.4 and Remark 1.5.

Proof of Theorem 1.4. We prove Theorem 1.4 in three steps: Firstly, we prove that "$\ge$" holds, secondly we prove the necessity of the characterization and finally we prove the sufficiency of the characterization.

Step 1. Denote

$$\inf\Big\{\int_{\{2\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-2\psi) : \tilde{f}(z_0) = 1 \text{ and } \tilde{f} \in \mathcal{O}(\{2\psi < -t\})\Big\}$$

by $G(t)$ for $t \ge 0$; then we have

$$G(0) = \frac{1}{B_\rho(z_0)},$$

where $\rho = e^{-\varphi} c(-2\psi)$. Lemma 2.6 tells us that there exists a holomorphic function F_0 on D such that $F_0(z_0) = 1$ and $G(0) = \int_D |F_0|^2 e^{-\varphi} c(-2\psi)$. Theorem 2.5 shows that $G(h^{-1}(r))$ is concave, where $h(t) = \int_t^{+\infty} c(s) e^{-s}\,ds$. Note that

$$G(-\log r) \le \int_{\{2\psi<\log r\}} |F_0|^2 e^{-\varphi} c(-2\psi)$$

for $r \in (0,1]$; then we have

$$\begin{aligned} \frac{\int_{\{z\in D: 2\psi(z)\ge \log r\}} |F_0(z)|^2 e^{-\varphi} c(-2\psi)}{\int_0^{-\log r} c(t) e^{-t}\,dt} &\le \frac{G(0) - G(-\log r)}{\int_0^{-\log r} c(t) e^{-t}\,dt} \\ &\le \frac{G(0)}{\int_0^{+\infty} c(t) e^{-t}\,dt}. \end{aligned} \tag{3.1}$$

There exists $r_0 \in (0,1)$ such that $\inf\{e^{-\varphi(z)} c(-\psi(z)) : z \in D \text{ and } 2G_D(z, z_0) \ge \log r_0\} > 0$. As $v(dd^c\psi, z_0) > 0$, it follows from Lemma 2.10 that there exists $r_1 \in (0,1)$ such that $\{z \in D : 2\psi(z) \ge \log r_1\} \subset \{z \in D : 2G_D(z, z_0) \ge \log r_0\}$.

Note that $\lim_{t\to 0+0} c(t) = 1$. Then inequality (3.1) implies that

$$\begin{aligned}
&\liminf_{r\to 1-0} \frac{\int_{\{z\in D:2\psi(z)\geq \log r\}} |F_0(z)|^2}{1-r} \\
&\quad \leq C_1 \liminf_{r\to 1-0} \frac{\int_{\{z\in D:2\psi(z)\geq \log r\}} |F_0(z)|^2 e^{-\varphi} c(-2\psi)}{\int_0^{-\log r} c(t)e^{-t}\,dt} \times \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{1-r} \\
&\quad \leq C_1 \frac{G(0)}{\int_0^{+\infty} c(t)e^{-t}\,dt} \liminf_{r\to 1-0} \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{1-r} \\
&\quad < +\infty.
\end{aligned}$$

Using Lemma 2.4, we have $F_0 \in H_2^{(c)}(D)$.

Note that F_0 has Fatou's nontangential boundary value and $|F_0| \in L^2(\partial D)$. It follows from Fatou's lemma and Lemmas 2.1 and 2.3 that

$$\begin{aligned}
&\int_{\partial D} |F_0|^2 e^{-\varphi} c(-2\psi) \Big(\frac{\partial \psi}{\partial v_z}\Big)^{-1} |dz| \\
&\qquad = \int_{\partial D} |F_0|^2 e^{-\varphi} c(-2\psi) |\nabla \psi|^{-1} |dz| \\
&\qquad \leq \liminf_{r\to 1-0} \frac{\int_{\frac{1}{2}\log r}^{0} \big(\int_{\{z\in D:\psi(z)=s\}} |F_0|^2 e^{-\varphi} c(-2\psi) |\nabla\psi|^{-1} |dz|\big)\,ds}{-\frac{1}{2}\log r} \\
&\qquad = \liminf_{r\to 1-0} \frac{\int_{\{z\in D:2\psi(z)\geq \log r\}} |F_0|^2 e^{-\varphi} c(-2\psi)}{\int_0^{-\log r} c(t)e^{-t}\,dt} \times \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{-\frac{1}{2}\log r} \\
&\qquad = 2 \liminf_{r\to 1-0} \frac{\int_{\{z\in D:2\psi(z)\geq \log r\}} |F_0|^2 e^{-\varphi} c(-2\psi)}{\int_0^{-\log r} c(t)e^{-t}\,dt}. \qquad (3.2)
\end{aligned}$$

As $F_0 \in H_2^{(c)}(D)$, we have

$$1 = F_0(z_0) = \frac{1}{2\pi} \int_{\partial D} F_0(z) \overline{K_{\rho(\frac{\partial\psi}{\partial v_z})^{-1}}(z, \overline{z_0})} \rho \Big(\frac{\partial \psi}{\partial v_z}\Big)^{-1} |dz|.$$

By the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
1 &\leq \frac{1}{(2\pi)^2} \bigg(\int_{\partial D} |F_0|^2 \rho \Big(\frac{\partial \psi}{\partial v_z}\Big)^{-1} |dz|\bigg) \\
&\quad \times \bigg(\int_{\partial D} |K_{\rho(\frac{\partial\psi}{\partial v_z})^{-1}}(z, \overline{z_0})|^2 \rho \Big(\frac{\partial \psi}{\partial v_z}\Big)^{-1} |dz|\bigg) \\
&= \frac{1}{2\pi} \bigg(\int_{\partial D} |F_0|^2 \rho \Big(\frac{\partial \psi}{\partial v_z}\Big)^{-1} |dz|\bigg) \times K_{\rho,\psi}(z_0). \qquad (3.3)
\end{aligned}$$

Combining inequalities (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned}\left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)B_\rho(z_0) &= \frac{\int_0^{+\infty} c(t)e^{-t}\,dt}{G(0)}\\ &\le \limsup_{r\to 1-0} \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{\int_{\{z\in D:2\psi(z)\ge \log r\}} |F_0|^2 e^{-\varphi}c(-2\psi)}\\ &\le 2\left(\int_{\partial D} |F_0|^2 e^{-\varphi}c(-2\psi)\left(\frac{\partial\psi}{\partial v_z}\right)^{-1}|dz|\right)^{-1}\\ &\le \frac{1}{\pi}K_{\rho,\psi}(z_0). \end{aligned}\tag{3.4}$$

Thus, we have proved the inequality part of Theorem 1.4.

Step 2. Assume that the equality

$$K_{\rho,\psi}(z_0) = \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B_\rho(z_0)\tag{3.5}$$

holds. Then inequality (3.4) becomes an equality, which shows that

$$\frac{\int_0^{+\infty} c(t)e^{-t}\,dt}{G(0)} = \limsup_{r\to 1-0} \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{\int_{\{z\in D:2\psi(z)\ge \log r\}} |F_0|^2 e^{-\varphi}c(-2\psi)}.$$

Following from the concavity of $G(h^{-1}(r))$, we obtain that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}\,dt)$. Theorem 2.8 shows that the following statements hold:

(1) $\psi = p_0 G_D(\cdot, z_0)$, where $p_0 = v(dd^c(\psi), z_0) > 0$;
(2) $\varphi + 2\psi = 2\log|g| + 2G_D(\cdot, z_0) + 2u_1$, where g is a holomorphic function on D such that $\mathrm{ord}_{z_0}(g) = 0$ and u_1 is a harmonic function on D;
(3) $\chi_{z_0} = \chi_{-u_1}$.

In the following, we will prove that $2\log|g|$ is harmonic on D, a.e., $g(z) \neq 0$ holds for any $z \in D$.

Denote $h := \varphi + 2\psi - 2G_D(\cdot, z_0)$, a function on $\overline{D}$; thus h is subharmonic on D and h is continuous at z for any $z \in \partial D$. By the analyticity of ∂D, there exists $\tilde{h} \in C(\overline{D})$ such that $\tilde{h}|_{\partial D} = h|_{\partial D}$ and $\tilde{h}$ is harmonic on D. As h is subharmonic on D, we have

$$h \le \tilde{h}$$

on D. Denote

$$\tilde{\varphi} := \varphi + \tilde{h} - h.$$

Then we have $\tilde{\varphi}|_{\partial D} = \varphi|_{\partial D}$ and $\tilde{\varphi} + 2\psi = 2G_D(\cdot, z_0) + \tilde{h}$. Denote $\tilde{\rho} := e^{-\tilde{\varphi}}c(-2\psi)$. It is clear that

$$K_{\tilde{\rho},\psi}(z_0) = K_{\rho,\psi}(z_0) \quad \text{and} \quad B_{\tilde{\rho}}(z_0) \geq B_\rho(z_0).$$

Following equality (3.5) and the result in Step 1, we have

$$\frac{K_{\rho,\psi}(z_0)}{\int_0^{+\infty} c(t)e^{-t}\,dt} = \pi B_\rho(z_0) \leq \pi B_{\tilde{\rho}}(z_0) \leq \frac{K_{\tilde{\rho},\psi}(z_0)}{\int_0^{+\infty} c(t)e^{-t}\,dt} = \frac{K_{\rho,\psi}(z_0)}{\int_0^{+\infty} c(t)e^{-t}\,dt},$$

which implies that

$$B_\rho(z_0) = B_{\tilde{\rho}}(z_0).$$

Then we have $\tilde{\rho} = \rho$, i.e. $\tilde{h} = h$, which implies that $2\log|g|$ is harmonic on D. Denote

$$u = \log|g| + u_1,$$

a harmonic function on D. Then we have $\varphi + 2\psi = 2G_D(\cdot, z_0) + 2u$ and $\chi_{z_0} = \chi_{-u_1} = \chi_{-u}$.

Step 3. Assume that statements (1)–(3) hold.

It follows from Theorem 2.8 that $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}\,dt)$. By Corollary 2.7 and Remark 2.9, we get that

$$G(t) = \int_{\{2\psi<-t\}} |F_0|^2 e^{-\varphi} c(-2\psi) \tag{3.6}$$

holds for any $t \geq 0$ and

$$F_0 = c_0(p_*(f_{z_0}))' p_*(f_u),$$

where c_0 is a constant, p is the universal covering from unit disc Δ to D, f_u is a holomorphic function on Δ such that $|f_u| = p^*(e^u)$, and f_{z_0} is a holomorphic function on Δ such that $|f_{z_0}| = p^*(e^{G_D(\cdot,z_0)})$. It follows from equality (3.6) that

$$\frac{\int_0^{+\infty} c(t)e^{-t}\,dt}{G(0)} = \limsup_{r\to 1-0} \frac{\int_0^{-\log r} c(t)e^{-t}\,dt}{\int_{\{z\in D: 2\psi(z)\geq \log r\}} |F_0|^2 e^{-\varphi} c(-2\psi)}. \tag{3.7}$$

As $u = \frac{\varphi}{2} + \psi - G_D(\cdot, z_0)$, we have $u \in C(\overline{D})$, which implies that $p_*(|f_u|) \in C(\overline{D})$. As $G_D(\cdot, z_0)$ can be extended to a harmonic function on a $U\backslash\{z_0\}$, where U is a neighborhood of $\overline{D}$, we have $|(p_*(f_{z_0}))'| \in C(\overline{D})$. Thus, we have

$$|F_0| \in C(\overline{D}).$$

Following from the dominated convergence theorem and Lemma 2.3, we obtain

$$\limsup_{r\to 1-0} \frac{\int_{\{z\in D:2\psi(z)\geq \log r\}}|F_0|^2e^{-\varphi}c(-2\psi)}{\int_0^{-\log r}c(t)e^{-t}\,dt} = \frac{1}{2}\int_{\partial D}|F_0|^2e^{-\varphi}c(-2\psi)\Big(\frac{\partial\psi}{\partial v_z}\Big)^{-1}|dz|. \tag{3.8}$$

Denote $M(z) := \frac{K_\lambda(z,\overline{z_0})}{K_\lambda(z_0,\overline{z_0})}$, where $\lambda = \rho(\frac{\partial\psi}{\partial v_z})^{-1}$. Note that $\int_D |F_0|^2e^{-\varphi}c(-2\psi) < +\infty$ implies that $\int_D e^{-\varphi}c(-2\psi) < +\infty$. Lemma 2.11 shows that $|M(z)| \in C(\overline{D})$; then we have

$$\int_D |M|^2e^{-\varphi}c(-2\psi) < +\infty.$$

Note that $M(z_0) = 1$. By using Lemma 2.6 and inequality (3.6), we have

$$\int_{\{2\psi<-t\}}|M|^2e^{-\varphi}c(-2\psi) = \int_{\{2\psi<-t\}}|F_0|^2e^{-\varphi}c(-2\psi) + \int_{\{2\psi<-t\}}|M-F_0|^2e^{-\varphi}c(-2\psi),$$

which implies that

$$\int_{\{2\psi<-t\}}F_0\overline{F_0-M}e^{-\varphi}c(-2\psi) = 0 \tag{3.9}$$

holds for any $t \geq 0$. Note that $\psi = p_0G_D(\cdot, z_0)$. It follows from Lemma 2.3 and equality (3.9) that there exists $r_1 > 0$ such that

$$\int_{\{z\in D:G_D(z,z_0)=r\}}F_0\overline{F_0-M}e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz| = 0 \tag{3.10}$$

holds for any $r \in (0, r_1)$. Note that $|F_0| \in C(\overline{D})$ and $|M| \in C(\overline{D})$; then it follows from the dominated convergence theorem and equality (3.10) that

$$\int_{\partial D}F_0\overline{F_0-M}e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz| = 0,$$

which implies that

$$\int_{\partial D}|M|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz| = \int_{\partial D}|M-F_0|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz| + \int_{\partial D}|F_0|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz|.$$

Lemma 2.12 tells us that

$$\int_{\partial D}|M|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz|\le\int_{\partial D}|F_0|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz|.$$

Then we have

$$\int_{\partial D}|M|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz|=\int_{\partial D}|F_0|^2e^{-\varphi}\Big(\frac{\partial G_D(\cdot,z_0)}{\partial v_z}\Big)^{-1}|dz|.$$

It follows from Lemma 2.12 that

$$F_0\equiv M. \tag{3.11}$$

Thus, inequality (3.3) becomes an equality, i.e.

$$1=\frac{1}{2\pi}\bigg(\int_{\partial D}|F_0|^2\rho\Big(\frac{\partial\psi}{\partial v_z}\Big)^{-1}|dz|\bigg)\times K_{\rho,\psi}(z_0). \tag{3.12}$$

Combining equalities (3.7), (3.8) and (3.12), we know that inequality (3.4) becomes an equality, i.e.

$$\bigg(\int_0^{+\infty}c(t)e^{-t}\,dt\bigg)B_\rho(z_0)=\frac{1}{\pi}K_{\rho,\psi}(z_0).$$

Then Theorem 1.4 has been proved. □

Proof of Remark 1.5. Assume that statements (1)–(3) in Theorem 1.4 hold. Following the discussions in Step 3 in the proof of Theorem 1.4, we obtain

$$F_0=c_0(p_*(f_{z_0}))'p_*(f_u),\quad F_0\equiv M\quad\text{and}\quad M(z)=\frac{K_\lambda(z,\overline{z_0})}{K_\lambda(z_0,\overline{z_0})},$$

where $\lambda=\rho(\frac{\partial\psi}{\partial v_z})^{-1}$. Thus, we have

$$K_{\rho,\psi}(\cdot,\overline{z_0})=K_{\rho,\psi}(z_0,\overline{z_0})F_0=c_1(p_*(f_{z_0}))'p_*(f_u),$$

where c_1 is a constant. As

$$\int_D\bigg|\frac{B_\rho(\cdot,\overline{z_0})}{B_\rho(z_0,\overline{z_0})}\bigg|^2\rho=\frac{1}{B_\rho(z_0,\overline{z_0})}=G(0),$$

it follows from Lemma 2.6 that

$$\frac{B_\rho(\cdot,\overline{z_0})}{B_\rho(z_0,\overline{z_0})}=F_0.$$

Theorem 1.4 shows that $K_{\rho,\psi}(z_0,\overline{z_0})=(\int_0^{+\infty}c(t)e^{-t}\,dt)\pi B_\rho(z_0,\overline{z_0})$, and thus we obtain

$$K_{\rho,\psi}(\cdot,\overline{z_0})=\bigg(\int_0^{+\infty}c(t)e^{-t}\,dt\bigg)\pi B_\rho(\cdot,\overline{z_0}).$$ □

§4. Proof of Corollary 1.8

In this section we prove Corollary 1.8 by using Theorem 1.4.

Let $\tilde{\varphi} = \varphi - 2k\log|z - z_0|$; then it is clear that $\tilde{\varphi} + 2\psi$ is subharmonic on D and $v(dd^c(\tilde{\varphi} + 2\psi), z_0) \geq 2$. Denote $\tilde{\rho} := e^{-\tilde{\varphi}}c(-2\psi) = |z - z_0|^{2k}\rho$. Note that

$$\begin{aligned} B_\rho^{(k)}(z_0) &= \sup\Big\{\Big|\tfrac{f^{(k)}(z_0)}{k!}\Big|^2 : f \in \mathcal{O}(D),\ \textstyle\int_D |f|^2\rho \leq 1 \\ &\qquad\qquad \text{and } f(z_0) = \cdots = f^{(k-1)}(z_0) = 0\Big\} \\ &= \sup\Big\{|g(z_0)|^2 : g \in \mathcal{O}(D) \text{ and } \textstyle\int_D |g|^2\tilde{\rho} \leq 1\Big\} \\ &= B_{\tilde{\rho}}(z_0), \end{aligned}$$

and

$$\begin{aligned} K_{\rho,\psi}^{(k)}(z_0) &= \sup\Big\{\Big|\tfrac{f^{(k)}(z_0)}{k!}\Big|^2 : f \in H_2^{(c)}(D),\ \textstyle\int_{\partial D} |f|^2\rho\big(\tfrac{\partial\psi}{\partial v_z}\big)^{-1}|dz| \leq 1 \\ &\qquad\qquad \text{and } f(z_0) = \cdots = f^{(k-1)}(z_0) = 0\Big\} \\ &= \sup\Big\{|g(z_0)|^2 : g \in H_2^{(c)}(D) \text{ and } \textstyle\int_{\partial D} |g|^2\tilde{\rho}\big(\tfrac{\partial\psi}{\partial v_z}\big)^{-1}|dz| \leq 1\Big\} \\ &= K_{\tilde{\rho},\psi}(z_0). \end{aligned}$$

Theorem 1.4 tell us that

$$K_{\tilde{\rho},\psi}(z_0) \geq \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B_{\tilde{\rho}}(z_0) \tag{4.1}$$

holds and the equality holds if and only if the following statements hold:

(1) $\tilde{\varphi} + 2\psi = 2G_D(\cdot, z_0) + 2u_1$, where u_1 is a harmonic function on D;

(2) $\psi = p_0G_D(\cdot, z_0)$, where $p_0 = v(dd^c(\psi), z_0) > 0$;

(3) $\chi_{z_0} = \chi_{-u_1}$.

Then inequality (4.1) implies that

$$K_{\rho,\psi}^{(k)}(z_0) \geq \left(\int_0^{+\infty} c(t)e^{-t}\,dt\right)\pi B_\rho^{(k)}(z_0) \tag{4.2}$$

holds. Let $u(z) = u_1(z) + k(\log|z - z_0| - G_D(z, z_0))$ on D; then it follows from Lemma 2.10 that u is harmonic on D if and only if u_1 is harmonic on D. It is clear that $\chi_{-u}\chi_{z_0}^k = \chi_{-u_1}$ when u is harmonic on D. Thus, the equality in (4.2) holds if and only if the following statements hold:

(1) $\varphi + 2\psi = 2(k+1)G_D(\cdot, z_0) + 2u$, where u is a harmonic function on D;

(2) $\psi = p_0G_D(\cdot, z_0)$, where $p_0 = v(dd^c(\psi), z_0) > 0$;

(3) $\chi_{z_0}^{k+1} = \chi_{-u}$.

Thus, Corollary 1.8 holds.

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