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Lattices of Logmodular Algebras

by

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Abstract

A subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{M}$ is logmodular (resp. has factorization) if the set $\{a^*a;\ a \in \mathcal{M}$ is invertible with $a, a^{-1} \in \mathcal{A}\}$ is dense in (resp. equal to) the set of all positive and invertible elements of $\mathcal{M}$. In this paper, we show that the lattice of projections in a (separable) von Neumann algebra $\mathcal{M}$ whose ranges are invariant under a logmodular algebra in $\mathcal{M}$, is a commutative subspace lattice. Further, if $\mathcal{M}$ is a factor then this lattice is a nest. As a special case, it follows that all reflexive (in particular, completely distributive CSL) logmodular subalgebras of type I factors are nest algebras, thus answering in the affirmative a question by Paulsen and Raghupathi (Trans. Amer. Math. Soc. **363** (2011) 2627–2640). We also give a complete characterization of logmodular subalgebras in finite-dimensional von Neumann algebras.

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§1. Introduction

The well-known Cholesky factorization theorem states that any positive and invertible $n \times n$ matrix can be written as U^*U for some invertible upper triangular $n \times n$ matrix U (so the inverse U^{-1} is also upper triangular). We then say that the algebra of upper triangular matrices has *factorization* in M_n, the algebra of all $n \times n$ complex matrices. Using this or otherwise, one can show that an algebra consisting of block upper triangular matrices (with respect to some orthonormal basis) also admits such factorization in M_n. Is there any other algebra in M_n with this property? Paulsen and Raghupathi [20] showed that any algebra in M_n containing all

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diagonal matrices, has factorization in M_n if and only if it is unitarily equivalent to an algebra of block upper triangular matrices (they actually studied the notion known as logmodular algebra, which is our interest in this paper as well). The general characterization was settled by Juschenko [15], who showed that up to a change of basis, all algebras with factorization in M_n contain the diagonal algebra, thus showing that algebras of block upper triangular matrices are all that have factorization in M_n.

A natural question that arises is to what extent these results generalize to infinite-dimensional settings. We say that a (non-self-adjoint) subalgebra $\mathcal{A}$ has *factorization* in a C^*-algebra $\mathcal{M}$ if every positive and invertible element in $\mathcal{M}$ can be expressed as a^*a for some invertible a with $a, a^{-1} \in \mathcal{A}$. The specific interest of research would be to characterize all subalgebras which have factorization in a type I factor.

One of the most well-studied algebras having factorization is the class of nest algebras. Let $\mathcal{H}$ be a complex and separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded operators on $\mathcal{H}$. Let $\mathcal{E}$ be a collection of closed subspaces of $\mathcal{H}$ totally ordered with respect to inclusion (such a collection is called a *nest*), and let $\operatorname{Alg}\mathcal{E}$ (called a *nest algebra*) denote the algebra of all operators in $\mathcal{B}(\mathcal{H})$ which leave all subspaces in $\mathcal{E}$ invariant. Note that an algebra of block upper triangular matrices is nothing but $\operatorname{Alg}\mathcal{E}$ for some nest $\mathcal{E}$ on $\mathbb{C}^n$, and vice versa. So a generalization of Cholesky factorization would be to ask whether $\operatorname{Alg}\mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ for a nest $\mathcal{E}$ on $\mathcal{H}$. Gohberg and Krein [10] appear to be the first who studied factorization along nest algebras, mainly examining positive and invertible operators "close" to the identity operator. Arveson [3] considered the factorization property of nest algebras arising out of nests of order type $\mathbb{Z}$. It was Larson [17] who investigated the factorization property of arbitrary nest algebras, and he proved in particular that if $\mathcal{E}$ is a countable complete nest on $\mathcal{H}$, then $\operatorname{Alg}\mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$. Again, what can be said about the converse? That is, if a subalgebra $\mathcal{A}$ has factorization in $\mathcal{B}(\mathcal{H})$, is it of the form $\operatorname{Alg}\mathcal{E}$ for some countable complete nest $\mathcal{E}$? In this paper, we show that this is indeed the case if we also assume that $\mathcal{A}$ is reflexive (see Theorem B below).

More generally, the factorization property of subalgebras of arbitrary von Neumann algebras is considered. A classical result says that the Hardy algebra $\mathcal{H}^\infty(\mathbb{T})$ on the unit circle has factorization in $L^\infty(\mathbb{T})$, i.e. for any non-negative element $f \in L^\infty(\mathbb{T})$ with $1/f \in L^\infty(\mathbb{T})$, there is an element $h \in \mathcal{H}^\infty(\mathbb{T})$ with $1/h \in \mathcal{H}^\infty(\mathbb{T})$ such that $f = \bar{h}h$. Some other function algebras like weak*-Dirichlet algebras introduced by Srinivasan and Wang [27] have factorization. Taking a cue from analytic function algebras, Arveson [2] introduced the theory of finite maximal subdiagonal algebras as a non-commutative variant and considered many results analogous to the classical Hardy space theory, showing in particular that they have the factorization property. Later, several authors examined such algebras in different settings. For more about algebras with factorization see [2, 3, 17, 18, 21, 25, 8, 10], and for some closely related properties see [22, 23, 1, 14, 19, 26], to name a few.

An algebra with the factorization property is a particular case of a more general class of algebras called logmodular algebras. We say that a subalgebra $\mathcal{A}$ is *logmodular* in a C^*-algebra $\mathcal{M}$ if any positive and invertible element in $\mathcal{M}$ can be approximated in norm by elements of the form a^*a, where a is invertible with $a, a^{-1} \in \mathcal{A}$. It is immediate that all logmodular algebras have factorization as well. The notion of logmodularity was first introduced by Hoffman [12] for subalgebras of commutative C^*-algebras, whose main idea was to generalize some classical results of analytic function theory in the unit disc. Blecher and Labuschagne [5] extended this notion to subalgebras of non-commutative C^*-algebras. They studied completely contractive representations on such algebras and their extension properties. Paulsen and Raghupathi [20] also studied representations of logmodular algebras and explored conditions under which contractive representations are automatically completely contractive. In [15], Juschenko gave a complete characterization of all logmodular subalgebras of M_n. See [6] for a beautiful survey on logmodular algebras arising out of tracial subalgebras and their relation to finite subdiagonal algebras among others. They show how most results generalized in the 1960s from the Hardy space on the unit disc to more general function algebras generalize further to the non-commutative situation, though more sophisticated proof techniques had to be developed for the purpose. We list some additional references on logmodular algebras in [12, 13, 9, 5, 6, 20, 15].

In this article, our aim is to understand the behavior of lattices of subspaces (or projections) invariant under logmodular subalgebras of a (separable) von Neumann algebra, and to use it to characterize reflexive logmodular algebras. Our main result is as follows (see Theorem 3.1):

Theorem A. *Let $\mathcal{M}$ be a von Neumann algebra with separable predual and let $\mathcal{A}$ be a logmodular subalgebra of $\mathcal{M}$. Then the lattice $\mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$ of projection is a commutative subspace lattice. Moreover, if $\mathcal{M}$ is a factor then $\mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$ is a nest.*

Here, $\mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$ denotes the lattice of those projections in $\mathcal{M}$ whose ranges are invariant under elements of $\mathcal{A}$. Our proof relies on the structure of two subspaces provided by Halmos [11].

An immediate consequence of the above result answers a conjecture from [20, p. 2630] which asks whether every completely distributive CSL logmodular algebra of $\mathcal{B}(\mathcal{H})$ is a nest algebra. We give an affirmative solution to this question while proving it for the larger class of reflexive subalgebras of $\mathcal{B}(\mathcal{H})$ (see Corollary 3.4).

Theorem B. *All reflexive (hence completely distributive CSL) logmodular algebras in $\mathcal{B}(\mathcal{H})$ are nest algebras.*

See Section 2 for precise definitions of reflexive and CSL algebras. Although reflexive subalgebras form a large class, whether the reflexivity assumption in the above result can be dropped (or replaced with some weaker assumptions) remains an open question. Instead, we attempt to explore some sufficient criteria under which an algebra with factorization and logmodularity is automatically reflexive and is a nest algebra. In particular, we prove the following result with a condition put on the atoms of a lattice of invariant subspaces (see Theorem 5.3). For the definition of atoms, see Section 2.

Theorem C. *Let $\mathcal{A}$ be a weakly closed algebra having factorization in $\mathcal{B}(\mathcal{H})$. If all the atoms of the lattice of invariant subspaces of $\mathcal{A}$ are finite-dimensional, then $\mathcal{A}$ is reflexive and hence $\mathcal{A}$ is a nest algebra.*

Next we consider logmodular subalgebras of finite-dimensional von Neumann algebras. Since all subalgebras having factorization in a finite-dimensional von Neumann algebra are logmodular as well, we use Theorem C to extend the result of Juschenko [15] on matrix algebras and give a complete characterization of all logmodular subalgebras of finite-dimensional von Neumann algebras as follows (see Corollary 5.7):

Theorem D. *A subalgebra of a finite-dimensional von Neumann algebra is logmodular if and only if it is a nest subalgebra.*

Finally, we discuss an example of a subalgebra in a von Neumann algebra (certainly infinite-dimensional), which has factorization but is not a nest subalgebra.

It may be remarked here that we have found a surprising application of our results about the factorization property of algebras in $\mathcal{B}(\mathcal{H})$ in the study of C^*-convexity and C^*-extreme points of the spaces of normal unital completely positive maps on von Neumann algebras taking values in $\mathcal{B}(\mathcal{H})$. This result is part of the paper [4] by the authors, and the second author's PhD thesis [16].

§2. Definitions and examples

All Hilbert spaces considered in this paper are complex and separable. Throughout, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. By subspaces, projections and operators on $\mathcal{H}$, we mean closed subspaces, orthogonal projections and bounded operators respectively. We write $E^{\perp}$ or $\mathcal{H} \ominus E$ for the orthogonal complement of E in $\mathcal{H}$. The projection onto a subspace E is denoted by P_E. For any projection p, we write $p^{\perp}$ for the projection $1 - p$, where 1 is the identity operator on $\mathcal{H}$. If $\{p_i\}_{i \in \Lambda}$ is a collection of projections, then $\bigvee_{i \in \Lambda} p_i$ denotes the projection onto the smallest subspace containing ranges of all p_i, and $\bigwedge_{i \in \Lambda} p_i$ denotes the projection onto the intersection of ranges of all p_i. For any operator x in $\mathcal{B}(\mathcal{H})$, $\ker x$ and $\mathcal{R}(x)$ denote the kernel and range of x respectively. All algebras considered will be subalgebras of $\mathcal{B}(\mathcal{H})$, which are always assumed to be norm closed, and contain the identity operator which we shall denote by 1. Unless said otherwise, convergence of any sequence of operators is taken in norm topology.

We briefly recall some basic notions of von Neumann algebra theory. A *von Neumann algebra* is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing 1 and closed in weak operator topology (WOT). A von Neumann algebra $\mathcal{M}$ is called a *factor* if its center $\mathcal{M} \cap \mathcal{M}'$ is trivial. Here $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$. Let p, q be two projections in $\mathcal{M}$. Then p and q are said to be *(Murray–von Neumann) equivalent*, and denoted $p \sim q$, if there exists a partial isometry $v \in \mathcal{M}$ such that $v^*v = p$ and $vv^* = q$. We say $p \preceq q$ if there is a projection $q_1 \in \mathcal{M}$ such that $q_1 \leq q$ and $p \sim q_1$. Here $\leq$ denotes the usual order of self-adjoint operators, while $<$ will denote the strict order. A projection $p \in \mathcal{M}$ is called *finite* if the only projection q in $\mathcal{M}$ such that $q \leq p$ and $q \sim p$ is p. The von Neumann algebra $\mathcal{M}$ is called *finite* if $1 \in \mathcal{M}$ is finite. Note that if p is a projection in $\mathcal{M}$, then $p\mathcal{M}p$ is a von Neumann algebra which is $*$-isomorphic to a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is the range subspace of p. See [7] for more details on these topics.

We now define some notation relevant to our results. Fix a von Neumann algebra $\mathcal{M}$, which is always assumed to be acting on a separable Hilbert space. Let $\mathcal{A}$ be a norm closed subalgebra (not necessarily self-adjoint) of $\mathcal{M}$. We denote by $\mathcal{A}^*$ the set $\{x \in \mathcal{M};\ x^* \in \mathcal{A}\}$, and by $\mathcal{A}^{-1}$ the set $\{x \in \mathcal{A};\ x \text{ is invertible with } x^{-1} \in \mathcal{A}\}$. Let $\mathcal{M}_+^{-1}$ denote the set of all positive and invertible elements of $\mathcal{M}$. Note that all this notation makes sense for any C^*-algebra. But our main focus in this paper lies in von Neumann algebras.

For $\mathcal{M}$ and $\mathcal{A}$ as above, let $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ denote the lattice of all projections in $\mathcal{M}$ whose ranges are invariant under every operator of $\mathcal{A}$, i.e.

$$\operatorname{Lat}_{\mathcal{M}} \mathcal{A} = \{p \in \mathcal{M};\ p = p^2 = p^* \text{ and } ap = pap \ \forall\, a \in \mathcal{A}\}.$$

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, we denote $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ simply by $\operatorname{Lat} \mathcal{A}$. Note that if $\mathcal{A}$ is also considered as a subalgebra of $\mathcal{B}(\mathcal{H})$ (where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$), then we have $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} =$

$\mathcal{M} \cap \operatorname{Lat} \mathcal{A}$. Also, $0, 1 \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ and $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is closed under the operations $\vee$ and $\wedge$ of arbitrary subcollection, as well as closed under strong operator topology (SOT).

Dually, let $\mathcal{E}$ be a collection of projections in $\mathcal{M}$ (which may not be a lattice), and let $\operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ (or $\operatorname{Alg} \mathcal{E}$ when $\mathcal{M} = \mathcal{B}(\mathcal{H})$) denote the algebra of all operators in $\mathcal{M}$ which leave the range of every projection of $\mathcal{E}$ invariant, i.e.

$$\operatorname{Alg}_{\mathcal{M}} \mathcal{E} = \big\{ x \in \mathcal{M};\ xp = pxp\ \forall\, p \in \mathcal{E} \big\}.$$

Again we note that $\operatorname{Alg}_{\mathcal{M}} \mathcal{E} = \mathcal{M} \cap \operatorname{Alg} \mathcal{E}$. It is clear that $\operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ is a unital subalgebra of $\mathcal{M}$, which is closed in weak operator topology.

Following [12, 5], we now consider the following definitions:

Definition 2.1. Let $\mathcal{A}$ be a subalgebra of a C^*-algebra $\mathcal{M}$. Then $\mathcal{A}$ is called *logmodular or has logmodularity* in $\mathcal{M}$ if the set $\{a^*a;\ a \in \mathcal{A}^{-1}\}$ is norm dense in $\mathcal{M}_+^{-1}$. The algebra $\mathcal{A}$ is said to have *factorization or strong logmodularity* in $\mathcal{M}$ if $\{a^*a;\ a \in \mathcal{A}^{-1}\} = \mathcal{M}_+^{-1}$.

It is immediate that any algebra having factorization is logmodular. Below we collect some known and straightforward results about logmodular algebras whose proofs are simple, and so are left to the readers (see [5, Prop. 4.6]).

Proposition 2.2. *Let $\phi\colon \mathcal{M} \to \mathcal{N}$ be a $*$-isomorphism between two C^*-algebras, and let $\mathcal{A}$ be a subalgebra of $\mathcal{M}$. Then $\mathcal{A}$ has logmodularity (resp. factorization) in $\mathcal{M}$ if and only if $\phi(\mathcal{A})$ has logmodularity (resp. factorization) in $\mathcal{N}$. In particular, if U is an appropriate unitary, then $U^*\mathcal{A}U$ has logmodularity (resp. factorization) in $U^*\mathcal{M}U$ if and only if $\mathcal{A}$ has logmodularity (resp. factorization) in $\mathcal{M}$.*

Proposition 2.3 ([5, Prop. 4.1]). *Let $\mathcal{A}$ be a subalgebra of a C^*-algebra $\mathcal{M}$. Then the following are true:*

(1) *$\mathcal{A}$ has factorization in $\mathcal{M}$ if and only if $\mathcal{A}^*$ has factorization in $\mathcal{M}$ if and only if for every invertible element $x \in \mathcal{M}$, there exist unitaries $u, v \in \mathcal{M}$ and invertible elements $a, b \in \mathcal{A}^{-1}$ such that $x = ua = bv$.*
(2) *$\mathcal{A}$ is logmodular in $\mathcal{M}$ if and only if $\mathcal{A}^*$ is logmodular in $\mathcal{M}$ if and only if for each invertible element $x \in \mathcal{M}$, there exist nets $\{u_n\}$, $\{v_n\}$ of unitaries in $\mathcal{M}$ and invertible elements $\{a_n\}$, $\{b_n\}$ in $\mathcal{A}^{-1}$ such that $x = \lim_n u_n a_n = \lim_n b_n v_n$.*

There are plenty such algebras known in the literature. The following are examples of logmodular algebras in commutative C^*-algebras.

Example 2.4 (Function algebras). A classical result of Szegö ([7, Thm. 25.13]) says that the Hardy algebra $\mathcal{H}^\infty(\mathbb{T})$ has factorization in $L^\infty(\mathbb{T},\mu)$. Here, $\mathbb{T}$ is the unit circle, μ is the one-dimensional Lebesgue measure on $\mathbb{T}$ and $\mathcal{H}^\infty(\mathbb{T})$ is the algebra of all essentially bounded functions on $\mathbb{T}$ whose negative Fourier coefficients are zero.

More generally, let m be a probability measure, and let $\mathcal{A}$ be a unital subalgebra of $L^\infty(m)$ satisfying the following:

(i) $\int fg\,dm = \int f\,dm \int g\,dm$ for all $f,g\in\mathcal{A}$;

(ii) if $h\in L^1(m)$ with $h\geq 0$ a.e. and $\int fh\,dm = \int f\,dm$ for all $f\in\mathcal{A}$, then $h=1$ a.e.

Let $\mathcal{H}^2(m)$ be the closure of $\mathcal{A}$ in the Hilbert space $L^2(m)$, and let $\mathcal{H}^\infty(m) = \mathcal{H}^2(m)\cap L^\infty(m)$. Then the proof of [13, Thm. 4] says that $\mathcal{H}^\infty(m)$ has factorization in $L^\infty(m)$. The algebra $\mathcal{H}^\infty(m)$ satisfies many other equivalent conditions analogous to classical Hardy space theory (see [27, Thm. 3.1] for details). Also see [12, 13, 27] for more concrete examples of such measures and algebras.

Example 2.5 (Dirichlet algebras). A closed unital subalgebra $\mathcal{A}$ of a commutative C^*-algebra $C(X)$ is called a *Dirichlet algebra* if $\mathcal{A}+\bar{\mathcal{A}}$ is uniformly dense in $C(X)$ (equivalently, $\Re\mathcal{A}$ is uniformly dense in $\Re C(X)$), where $\Re\mathcal{A}$ (resp. $\Re C(X)$) denotes the set of real parts of the functions in $\mathcal{A}$ (resp. $C(X)$). If $\mathcal{A}$ is a Dirichlet algebra, then since $\log|\mathcal{A}^{-1}|\subseteq\Re\mathcal{A}$, it is immediate that $\log|\mathcal{A}^{-1}|$ is dense in $\Re C(X)$; hence $\mathcal{A}$ is a logmodular subalgebra of $C(X)$. But some Dirichlet algebras may not have factorization. For example, consider the algebra $A(\mathbb{D})$ of all continuous functions on the closed unit disc $\overline{\mathbb{D}}$ which is holomorphic on the open unit disc $\mathbb{D}$. Then $A(\mathbb{D})$ is a Dirichlet algebra when considered as the subalgebra of $C(\mathbb{T})$, which is a consequence of Fejér–Riesz theorem on factorization of positive trigonometric polynomials, but $A(\mathbb{D})$ does not have factorization in $C(\mathbb{T})$. On the other hand, $\mathcal{H}^\infty(\mathbb{T})$ is an example of an algebra which has factorization in $L^\infty(\mathbb{T})$, but which is not a Dirichlet algebra. See [12] for details of these facts and more concrete examples of Dirichlet algebras.

To see examples and other properties of non-commutative algebras having factorization and logmodularity, we recall some notions to this end. Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{E}$ be a lattice of projections in $\mathcal{M}$ (i.e. $p\wedge q$ and $p\vee q\in\mathcal{E}$ whenever $p,q\in\mathcal{E}$). Then $\mathcal{E}$ is called *complete* if $0,1\in\mathcal{E}$, and $\bigvee_{i\in\Lambda} p_i$ and $\bigwedge_{i\in\Lambda} p_i\in\mathcal{E}$ for any arbitrary family $\{p_i\}_{i\in\Lambda}$ in $\mathcal{E}$. The lattice $\mathcal{E}$ is called a *commutative subspace lattice* (*CSL*) if projections of $\mathcal{E}$ commute with one another. Moreover, $\mathcal{E}$ is called a *nest* if $\mathcal{E}$ is totally ordered by the usual operator ordering, i.e. for any $p, q \in \mathcal{E}$, either $p \le q$ or $q \le p$ holds true. We remark here that some authors assume a nest or a CSL to be always complete.

A subalgebra $\mathcal{A}$ of the von Neumann algebra $\mathcal{M}$ is called a *nest subalgebra* of $\mathcal{M}$ (or a *nest algebra* when $\mathcal{M} = \mathcal{B}(\mathcal{H})$) if $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ for a nest $\mathcal{E}$ in $\mathcal{M}$. Further, $\mathcal{A}$ is called $\mathcal{M}$-*reflexive* (or *reflexive* when $\mathcal{M} = \mathcal{B}(\mathcal{H})$) if $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$. Any subalgebra of $\mathcal{M}$ of the form $\operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ for some collection $\mathcal{E}$ of projections in $\mathcal{M}$, is always $\mathcal{M}$-reflexive. In particular, a nest subalgebra of $\mathcal{M}$ is $\mathcal{M}$-reflexive. Note that if $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, then it is possible that a subalgebra of $\mathcal{M}$ can be reflexive in $\mathcal{B}(\mathcal{H})$, but it need not be $\mathcal{M}$-reflexive.

We state the following celebrated result by Larson [17] regarding the factorization property of nest algebras.

Theorem 2.6 ([17, Thm. 4.7]). *Let $\mathcal{E}$ be a complete nest on a separable Hilbert space $\mathcal{H}$. Then $\operatorname{Alg} \mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{E}$ is countable.*

The following are some examples of algebras having factorization in noncommutative von Neumann algebras.

Example 2.7 (Nest subalgebras). As already mentioned above, $\operatorname{Alg} \mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ for any countable complete nest $\mathcal{E}$ in $\mathcal{B}(\mathcal{H})$. More generally, Pitts proved that if $\mathcal{E}$ is a complete nest in a factor $\mathcal{M}$, then $\operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ has factorization in $\mathcal{M}$ if and only if a "certain" subnest $\mathcal{E}_r$ of $\mathcal{E}$ is countable ([21, Thm. 6.4]).

Moreover, if $\mathcal{E}$ is a nest (not necessarily countable) in a finite von Neumann algebra $\mathcal{M}$ (not necessarily a factor), then $\operatorname{Alg}_{\mathcal{M}} \mathcal{E}$ has factorization in $\mathcal{M}$ ([21, Cor. 5.11]).

Example 2.8 (Subdiagonal algebras). Let $\mathcal{A}$ be a subalgebra of a von Neumann algebra $\mathcal{M}$, and let $\phi\colon \mathcal{M} \to \mathcal{M}$ be a faithful normal expectation (i.e. ϕ is positive, $\phi(1) = 1$ and $\phi \circ \phi = \phi$). Then $\mathcal{A}$ is called a *subdiagonal algebra* with respect to ϕ if it satisfies

(i) $\mathcal{A} + \mathcal{A}^*$ is σ-weakly dense in $\mathcal{M}$;
(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$;
(iii) $\phi(\mathcal{A}) \subseteq \mathcal{A} \cap \mathcal{A}^*$.

Moreover, if the von Neumann algebra $\mathcal{M}$ is finite with a distinguished trace τ, then the subdiagonal algebra $\mathcal{A}$ is called *finite* if $\tau \circ \phi = \tau$.

Arveson proved that if $\mathcal{A}$ is a maximal (with respect to ϕ) finite subdiagonal algebra of $\mathcal{M}$, then $\mathcal{A}$ has factorization in $\mathcal{M}$ ([2, Thm. 4.2.1]). A nest subalgebra of a finite von Neumann algebra is an example of a maximal finite subdiagonal algebra ([2, Cor. 3.1.2]). See Example 5.9 for another concrete example of a finite subdiagonal algebra. There are other subdiagonal algebras (not necessarily finite) as well, which are known to have factorization. For example, all subdiagonal algebras arising out of periodic flow have factorization. See [25] for more details of these notions and Corollary 3.11 therein.

We believe that some known facts about subdiagonal algebras can also be deduced from our result. One such is [19, Thm. 5.1], which follows directly from Corollary 3.2. However, we have not explored other possible consequences in depth.

The following are some concrete examples of nest algebras which do not have factorization.

Example 2.9. Let $\mathcal{E}$ be the nest $\{p_t;\ t \in [0,1]\}$ of projections on $L^2([0,1])$, where p_t denotes the projection onto $L^2([0,t])$, considered as a subspace of $L^2([0,1])$. Then $\mathcal{E}$ is complete and uncountable; hence $\operatorname{Alg}\mathcal{E}$ does not have factorization in $\mathcal{B}(L^2([0,1]))$ by Theorem 2.6. Additionally, let $\mathcal{F} = \{p_i;\ i \in \mathbb{Q}\}$ be the nest of projections on $\ell^2(\mathbb{Q})$, where p_i denotes the projection onto the subspace $\overline{\operatorname{span}}\{e_j;\ j \leq i\}$, for the canonical basis $\{e_i;\ i \in \mathbb{Q}\}$ of $\ell^2(\mathbb{Q})$. Although $\mathcal{F}$ is a countable nest, it is easy to verify that its completion is not countable (actually indexed by $\mathbb{R} \sqcup \mathbb{Q}$) and hence $\operatorname{Alg}\mathcal{F}$ does not have factorization in $\mathcal{B}(\ell^2(\mathbb{Q}))$. At this point, we do not know whether these algebras are logmodular.

Below is an example of a non-commutative logmodular algebra which does not have factorization. To the best of our knowledge, this is the first such example in a non-commutative setting. Here we use the fact that positive and invertible operators of the form $I + K$ factor along any nest whenever K is a finite-rank operator, while this is not true in general if K is a compact operator.

Example 2.10. Let $\mathcal{K}(\mathcal{H})$ denote the algebra of compact operators on a separable Hilbert space $\mathcal{H}$, and let $\mathcal{M} = \mathcal{K}(\mathcal{H}) + \mathbb{C}$ be the unitization of $\mathcal{K}(\mathcal{H})$. Let $\mathcal{E}$ be an *uncountable* complete nest of projections on $\mathcal{H}$, and consider the subalgebra of $\mathcal{M}$ by

$$\mathcal{A} \coloneqq \mathcal{M} \cap \operatorname{Alg}\mathcal{E}.$$

Since $\operatorname{Alg}\mathcal{E}$ contains plenty of finite-rank operators ([8, Thm. 3.11]), $\mathcal{A}$ is a non-trivial algebra. It is well established that $\mathcal{A}$ does not have factorization in $\mathcal{M}$; indeed, one can get a compact operator K with $\|K\| < 1$ such that $I + K \neq A^*A$ for any invertible operator A with $A, A^{-1} \in \operatorname{Alg}\mathcal{E}$ ([17, Thm. 4.7]).

We claim that $\mathcal{A}$ is a logmodular subalgebra in $\mathcal{M}$. Let K be a compact operator such that $I + K$ is positive and invertible in $\mathcal{M}$. Get a sequence $\{K_n\}$ of finite-rank operators such that $\lim_n K_n = K$, and $I + K_n$ is positive and invertible for all n. Since each K_n is of finite rank, we can find a compact operator C_n

such that $S_n := I + C_n$ is invertible, $S_n, S_n^{-1} \in \operatorname{Alg} \mathcal{E}$ and $I + K_n = S_n^* S_n$ ([8, Thm. 14.9]). Thus we have $S_n \in \mathcal{A}^{-1}$ and $I + K = \lim_n S_n^* S_n$.

§3. Main results on logmodular algebras

We are now ready to state the main result of this paper. This tells us the behavior of lattices of logmodular algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a logmodular algebra in a von Neumann algebra $\mathcal{M}$. Then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a commutative subspace lattice. Moreover, if $\mathcal{M}$ is a factor, then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a nest.*

We postpone the proof of Theorem 3.1 to the next section, and instead look at some of its consequences first. Since any algebra having factorization is also logmodular, the following corollary is immediate.

Corollary 3.2. *Let $\mathcal{A}$ be an algebra having factorization in a von Neumann algebra $\mathcal{M}$. Then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a commutative subspace lattice. Moreover, if $\mathcal{M}$ is a factor, then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a nest.*

Remark 3.3. If $\mathcal{M}$ is an arbitrary von Neumann algebra which is not a factor, and $\mathcal{A}$ is a subalgebra of $\mathcal{M}$, then we can never expect $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ to be a nest, irrespective of whether $\mathcal{A}$ is logmodular or has factorization. In fact, if $\mathcal{P}_{\mathcal{Z}}$ denotes the lattice of all projections in the center $\mathcal{Z}$ of $\mathcal{M}$, then it is always true that $\mathcal{P}_{\mathcal{Z}} \subseteq \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$. So $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ can never be a nest if the center $\mathcal{Z}$ is non-trivial.

Now let $\mathcal{M}$ be a factor, and let $\mathcal{A}$ be an $\mathcal{M}$-reflexive subalgebra of $\mathcal{M}$. If $\mathcal{A}$ is logmodular in $\mathcal{M}$, then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a nest by Theorem 3.1. Since $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$, it follows that $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$.

We now answer an open question posed by Paulsen and Raghupathi (see [20, p. 2630]) using the above observation. They conjectured that every completely distributive CSL logmodular algebra in $\mathcal{B}(\mathcal{H})$ is a nest algebra. Here a (completely distributive) CSL algebra means an algebra of the form $\operatorname{Alg} \mathcal{E}$, where $\mathcal{E}$ is a (completely distributive) commutative subspace lattice (see [8] for more on completely distributive CSL algebras). Notice that all nests are completely distributive. Since any CSL algebra is a special case of a reflexive algebra, we have thus answered their question in the affirmative. We record it below.

Corollary 3.4. *An $\mathcal{M}$-reflexive logmodular algebra in a factor $\mathcal{M}$ is a nest subalgebra of $\mathcal{M}$. In particular, all reflexive (hence completely distributive CSL) logmodular algebras in $\mathcal{B}(\mathcal{H})$ are nest algebras.*

We next give a complete characterization of all reflexive algebras having factorization in $\mathcal{B}(\mathcal{H})$ by combining Larson's results and ours. If an algebra $\mathcal{A}$ has factorization in $\mathcal{B}(\mathcal{H})$, then $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ also has factorization in $\mathcal{B}(\mathcal{H})$ as $\mathcal{A}$ is contained in $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$. Since $\operatorname{Lat}\mathcal{A}$ is a complete nest, we infer from Theorem 2.6 that $\operatorname{Lat}\mathcal{A}$ is a countable nest. In particular, if $\mathcal{A}$ is reflexive, i.e. $\mathcal{A} = \operatorname{Alg}\mathcal{E}$ for a complete lattice $\mathcal{E}$ of projections in $\mathcal{H}$, then $\mathcal{E}$ is a countable nest because $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}$. Thus we get the following corollary, which is a strengthening of Larson's result, Theorem 2.6.

Corollary 3.5. *Let $\mathcal{E}$ be a complete lattice of projections on a separable Hilbert space $\mathcal{H}$. Then $\operatorname{Alg}\mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{E}$ is a countable nest.*

We recall some crucial terminology to be used later. Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{E}$ be a complete nest in $\mathcal{M}$. For any projection $p \in \mathcal{E}$, let

$$p_- = \vee\{q \in \mathcal{E};\ q < p\} \quad \text{and} \quad p_+ = \wedge\{q \in \mathcal{E};\ q > p\}.$$

An *atom* of $\mathcal{E}$ is a non-zero projection of the form $p - p_-$ for some $p \in \mathcal{E}$ with $p \neq p_-$. Clearly, two distinct atoms are mutually orthogonal. The nest $\mathcal{E}$ is called *atomic* if there is a finite or countably infinite sequence $\{r_n\}$ of atoms of $\mathcal{E}$ such that $\sum_n r_n = 1$, where the sum converges in SOT.

Let $\mathcal{E}$ be a complete nest in $\mathcal{B}(\mathcal{H})$. Let $\{r_n\}$ be the collection of all atoms of $\mathcal{E}$, and let $r = \sum_n r_n$ in SOT convergence. If $r \neq 1$, then it is straightforward to check that the nest $\{p \wedge r^\perp;\ p \in \mathcal{E}\}$ in $\mathcal{B}(\mathcal{R}(r^\perp))$ is complete and has no atom (such nests without any atom are called *continuous*). But then any continuous complete nest has to be uncountable (in fact indexed by $[0, 1]$; see [8, Lem. 13.3]). In particular, if the nest $\mathcal{E}$ is countable, then $r = 1$ and hence $\mathcal{E}$ is atomic. We record this observation in the following remark.

Remark 3.6. If an algebra $\mathcal{A}$ has factorization in $\mathcal{B}(\mathcal{H})$, then $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$ also has factorization in $\mathcal{B}(\mathcal{H})$ as it contains $\mathcal{A}$. Since $\operatorname{Lat}\mathcal{A}$ is complete, it follows from Corollary 3.5 that $\operatorname{Lat}\mathcal{A}$ is a countable nest, and hence by the above discussion, $\operatorname{Lat}\mathcal{A}$ is an atomic nest.

§4. Proof of the main result

This section is devoted to the proof of our main result (Theorem 3.1). We first discuss some general ingredients required for this. A simple observation which will be used throughout the article is the following remark. Recall that $p^\perp$ denotes the projection $1 - p$ for any projection p.

Remark 4.1. For any subalgebra $\mathcal{A}$ of a von Neumann algebra $\mathcal{M}$,

$$\begin{aligned} p \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A} &\Leftrightarrow ap = pap \;\forall\, a \in \mathcal{A} \\ &\Leftrightarrow pa^* = pa^*p \;\forall\, a \in \mathcal{A} \\ &\Leftrightarrow a^*p^\perp = p^\perp a^* p^\perp \;\forall\, a \in \mathcal{A} \\ &\Leftrightarrow p^\perp \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}^*. \end{aligned}$$

The following proposition says that logmodularity and factorization are preserved under compression of algebras by appropriate projections. This result must be well established, but we sketch an outline of the proof for the purpose of completeness.

Here, $p\mathcal{A}p$ denotes the subspace $\{pap;\ a \in \mathcal{A}\}$ for any projection p and an algebra $\mathcal{A}$. Note that $p\mathcal{A}p$ need not always be an algebra.

Proposition 4.2. *Let $\mathcal{A}$ be an algebra having logmodularity (resp. factorization) in a von Neumann algebra $\mathcal{M}$, and let $p, q \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ be such that $p \geq q$. Then $(p-q)\mathcal{A}(p-q)$ has logmodularity (resp. factorization) in $(p-q)\mathcal{M}(p-q)$.*

Proof. We shall prove only the case of logmodularity. That of factorization follows similarly. So assume that $\mathcal{A}$ is logmodular in $\mathcal{M}$.

Set $r = p - q$. That $r\mathcal{A}r$ is an algebra is a direct consequence of well-known facts about the semi-invariant property of r that the map $x \mapsto rxr$ is an algebra homomorphism from $\mathcal{A}$ onto $r\mathcal{A}r$ (see [7, Lem. 35.6]).

To show that $r\mathcal{A}r$ is logmodular in $r\mathcal{M}r$, fix a positive and invertible element x in $r\mathcal{M}r$. Set $\tilde{x} = x + q + p^\perp$. Note that $x = r\tilde{x}r$. It is clear that $\tilde{x}$ is positive and invertible in $\mathcal{M}$ (as x is positive and invertible). We use logmodularity of $\mathcal{A}$ in $\mathcal{M}$ to get a sequence $\{\tilde{a}_n\}$ in $\mathcal{A}^{-1}$ such that $\tilde{x} = \lim_n \tilde{a}_n^* \tilde{a}_n$. For each n, we have

$$(q\tilde{a}_n q)(q\tilde{a}_n^{-1} q) = q\tilde{a}_n\tilde{a}_n^{-1} q = q \quad \text{and} \quad (q\tilde{a}_n^{-1} q)(q\tilde{a}_n q) = q\tilde{a}_n^{-1}\tilde{a}_n q = q,$$

which is to say that $q\tilde{a}_n q$ is invertible in $q\mathcal{M}q$ with $(q\tilde{a}_n q)^{-1} = q\tilde{a}_n^{-1} q \in q\mathcal{A}q$. In particular, since the sequence $\{\tilde{a}_n^{-1}\}$ is bounded (as $\{(\tilde{a}_n^*\tilde{a}_n)^{-1}\}$ is a convergent sequence), the sequence $\{(q\tilde{a}_n q)^{-1}\}$ is bounded. Note that $q\tilde{x}r = 0$, and since $q\tilde{a}_n^* = q\tilde{a}_n^* q$ for all n, we have

$$0 = q\tilde{x}r = \lim_n q\tilde{a}_n^*\tilde{a}_n r = \lim_n (q\tilde{a}_n^* q)(q\tilde{a}_n r).$$

Multiplying on the left of the above sequence by the bounded sequence $(q\tilde{a}_n^* q)^{-1}$ yields $\lim_n q\tilde{a}_n r = 0$, using which, along with the expression $\tilde{a}_n r = p\tilde{a}_n r$, we get

$$\begin{aligned} x = r\tilde{x}r &= \lim_n r\tilde{a}_n^*\tilde{a}_n r = \lim_n r\tilde{a}_n^*(p\tilde{a}_n r) \\ &= \lim_n r\tilde{a}_n^*(q\tilde{a}_n r) + \lim_n r\tilde{a}_n^*(r\tilde{a}_n r) = \lim_n a_n^* a_n, \end{aligned}$$

where $a_n = r\tilde{a}_n r \in r\mathcal{A}r$. Using the algebra property of $r\mathcal{A}r$, for each n, we have $r = r\tilde{a}_n^{-1}r\tilde{a}_n r = r\tilde{a}_n r\tilde{a}_n^{-1}r$, which shows that $a_n = r\tilde{a}_n r$ is invertible with inverse $r\tilde{a}_n^{-1}r$ in $r\mathcal{A}r$. Thus we get a sequence $\{a_n\}$ of invertible elements with $a_n, a_n^{-1} \in r\mathcal{A}r$ for all n such that $x = \lim_n a_n^* a_n$. Since x is an arbitrary positive and invertible element, we conclude that $r\mathcal{A}r$ is logmodular in $r\mathcal{M}r$. □

We now recall some basic facts about subspaces in a separable Hilbert space. The structure theorems of two subspaces provided by Halmos constitute the main step towards the proof of our result. Following Halmos [11], we say two non-zero subspaces E and F of a Hilbert space are in *generic position* if all the subspaces

$$E \cap F, \quad E \cap F^\perp, \quad E^\perp \cap F, \quad E^\perp \cap F^\perp$$

are zero. We are going to use Halmos's two subspace theorem, which characterizes subspaces in generic position. Recall that P_E denotes the projection onto a subspace E, and $\ker x$ denotes the kernel of any operator x.

Lemma 4.3 ([11, Thm. 2]). *Let E and F be two subspaces in generic position in a separable Hilbert space $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{K}$, a unitary $U\colon \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}$ and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ such that $x^2 + y^2 = 1$, $\ker x = \ker y = 0$ and*

$$UP_EU^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad UP_FU^* = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$

The following proposition, which describes the structure of any two general subspaces, is a well-known direct consequence of Halmos's two subspace theorem.

Proposition 4.4. *Let E and F be two subspaces in a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ (could be zero) and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ with $x^2 + y^2 = 1$ and $\ker x = \ker y = 0$ such that, up to unitary equivalence,*

$$\mathcal{H} = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp \oplus \mathcal{K} \oplus \mathcal{K}$$

and

$$P_E = 1 \oplus 1 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \quad \textit{and} \quad P_F = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$

Here, any of the components in the decomposition could be 0. *Moreover, $P_E P_F = P_F P_E = P_{E \cap F}$ if and only if $\mathcal{K} = \{0\}$.*

When $\mathcal{K} \neq 0$ in the proposition above, we call the components of E and F in the summand $\mathcal{K} \oplus \mathcal{K}$ the *generic part* of E and F respectively.

We are now ready to give the proof of our main result through a series of lemmas. The next two lemmas deal with factor von Neumann algebras only, where we use the following comparison theorem of projections (see [7, Cor. 47.9]):

If $\mathcal{M}$ is a factor and p, q are two non-zero projections in $\mathcal{M}$, then either $p \preceq q$ or $q \preceq p$, i.e. there is a non-zero partial isometry $v \in \mathcal{M}$ such that $v^*v \leq p$ and $vv^* \leq q$. The same is clearly not true for arbitrary von Neumann algebras.

Before going forward, we reiterate that, throughout, convergence of any sequence of operators is taken in norm topology unless stated otherwise.

Lemma 4.5. *Let $\mathcal{M}$ be a factor, and let p, q be non-zero mutually orthogonal projections in $\mathcal{M}$. Then $\mathrm{Alg}_{\mathcal{M}}\{p, q\}$ is not logmodular in $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is a factor and $p, q \in \mathcal{M}$ are non-zero, it follows that there is a non-zero partial isometry $v \in \mathcal{M}$ such that $v^*v \leq p$ and $vv^* \leq q$. In particular, we have

$$v = qv = vp. \tag{4.1}$$

Assume to the contrary that $\mathrm{Alg}_{\mathcal{M}}\{p, q\}$ is logmodular in $\mathcal{M}$. Let $x = 1+\varepsilon(v+v^*)$ for some fixed scalar $0 < \varepsilon < 1$. Then x is positive and invertible in $\mathcal{M}$ (as $\|v + v^*\| = 1$). Hence, there exists a sequence $\{a_n\}$ of invertible elements in $\mathrm{Alg}_{\mathcal{M}}\{p, q\}$ such that $x = \lim_n a_n^* a_n$. Since $pq = 0$, we note from (4.1) that $v^*p = (v^*q)p = 0$; hence we get $qxp = \varepsilon qvp = \varepsilon v$. We also have $a_n p = pa_n p$ and $qa_n^* = qa_n^* q$ for all n; thus it follows that

$$\varepsilon v = qxp = \lim_n qa_n^* a_n p = \lim_n qa_n^* qpa_n p = 0,$$

which is a contradiction, as $v \neq 0$. □

The following lemma proves the second assertion of Theorem 3.1, once we assume the first.

Lemma 4.6. *Let $\mathcal{M}$ be a factor, and let $\mathcal{A}$ be a logmodular algebra in $\mathcal{M}$. If $p, q \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$ are such that $pq = qp$, then either $p \leq q$ or $q \leq p$ holds true.*

Proof. Since p and q are commuting projections, the operators pq, $pq^\perp$ and $p^\perp q$ are projections. We assume without loss of generality that $r := (pq)^\perp \neq 0$ (otherwise $p = pq = q$, so there is nothing to prove).

Consider the compression $r\mathcal{A}r$ of the algebra $\mathcal{A}$. Note that $r^\perp = pq \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$. We know thanks to Proposition 4.2 that the algebra $r\mathcal{A}r$ is logmodular in $r\mathcal{M}r$. It is clear that $pq^\perp, p^\perp q \in r\mathcal{M}r$, and it is straightforward to verify that $pq^\perp, p^\perp q \in \mathrm{Lat}_{r\mathcal{M}r}(r\mathcal{A}r)$ (see Lemma 5.1 below). But $pq^\perp$ and $p^\perp q$ are mutually orthogonal projections, and since $\mathrm{Alg}_{r\mathcal{M}r}\{p^{\perp}q, pq^{\perp}\}$ (which contains $r\mathcal{A}r$) is logmodular in $r\mathcal{M}r$, it follows from Lemma 4.5 that either $pq^{\perp} = 0$ or $p^{\perp}q = 0$. If $pq^{\perp} = 0$, then $p = p(q + q^{\perp}) = pq$ which implies that $p \leq q$. Likewise, $p^{\perp}q = 0$ will imply $q \leq p$. □

The next two lemmas are simple but useful observations.

Lemma 4.7. *Let $\{a_n\}$ be a sequence of invertible elements in a C^*-algebra such that $\lim_n a_n^* a_n = 1$. Then $\{a_n^{-1}\}$ is bounded and $\lim_n a_n a_n^* = 1$.*

Proof. Since $\lim_n a_n^* a_n = 1$, it follows that $\lim_n (a_n^* a_n)^{-1} = 1$ and so $\{(a_n^* a_n)^{-1}\}$ is bounded. This implies the first assertion that $\{a_n^{-1}\}$ is bounded. Further, we have $\lim_n a_n^* a_n a_n^* a_n = 1$, and hence

$$0 = \lim_n (a_n^* a_n a_n^* a_n - a_n^* a_n) = \lim_n a_n^* (a_n a_n^* - 1) a_n.$$

Since the sequence $\{a_n^{-1}\}$ is bounded, it follows by multiplying by ${a_n^*}^{-1}$ on the left and by a_n^{-1} on the right of the sequence that $\lim_n (a_n a_n^* - 1) = 0$, as was to be proved. □

Lemma 4.8. *Let an algebra $\mathcal{A}$ have logmodularity (resp. factorization) in a von Neumann algebra $\mathcal{M}$, and let $p, q \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$. If $r = (p \wedge q) \vee (p^{\perp} \wedge q^{\perp})$, then $r^{\perp}\mathcal{A}r^{\perp}$ has logmodularity (resp. factorization) in $r^{\perp}\mathcal{M}r^{\perp}$.*

Proof. Write $r_1 = p \wedge q$ and $r_2 = p^{\perp} \wedge q^{\perp}$. It is clear that $r_1 r_2 = 0$ and $r = r_1 + r_2$. Since $p, q \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$, it follows that $r_1 \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$. Also, we note that $p^{\perp}, q^{\perp} \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}^*$ and hence $r_2 \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}^*$, which is to say that $r_2^{\perp} \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$. Note that $r^{\perp} = 1 - r_2 - r_1 = r_2^{\perp} - r_1$, and so $r_1 \leq r_2^{\perp}$. Both the assertions about logmodularity and factorization now follow from Proposition 4.2. □

Finally, we prove our main theorem in full generality. Recall that $\mathcal{R}(x)$ denotes the range of an operator x.

Proof of Theorem 3.1. Let $\mathcal{A}$ be a logmodular subalgebra of a von Neumann algebra $\mathcal{M}$, and let $p, q \in \mathrm{Lat}_{\mathcal{M}}\,\mathcal{A}$. We have to show that $pq = qp$. The second assertion that $p \leq q$ or $q \leq p$ whenever $\mathcal{M}$ is a factor, will then follow thanks to Lemma 4.6.

Set $r = (p \wedge q) \vee (p^{\perp} \wedge q^{\perp})$. Then $r^{\perp}\mathcal{A}r^{\perp}$ is a logmodular algebra in $r^{\perp}\mathcal{M}r^{\perp}$ by Lemma 4.8. The projections p and q commute with r, and hence with $r^{\perp}$. So if we write

$$p' = r^{\perp}pr^{\perp} \quad \text{and} \quad q' = r^{\perp}qr^{\perp},$$

then it is immediate that p', q' are projections in $r^\perp \mathcal{M} r^\perp$, and we have $p' = p \wedge r^\perp$ and $q' = q \wedge r^\perp$. Observe that

$$pq(p \wedge q) = p \wedge q = qp(p \wedge q) \quad \text{and} \quad pq(p^\perp \wedge q^\perp) = 0 = qp(p^\perp \wedge q^\perp).$$

So we have $pqr = p \wedge q = qpr$, which further yields

$$\begin{aligned} pq &= pq(r + r^\perp) = pqr + pqr^\perp = p \wedge q + (r^\perp pr^\perp)(r^\perp qr^\perp) = p \wedge q + p'q', \\ qp &= qpr + qpr^\perp = p \wedge q + (r^\perp qr^\perp)(r^\perp pr^\perp) = p \wedge q + q'p'. \end{aligned}$$

Therefore, in order to show the required assertion, it is enough to prove that $p'q' = q'p'$. We note that

$$p' \wedge q' = p \wedge q \wedge r^\perp \leq r \wedge r^\perp = 0$$

and

$$\begin{aligned} (r^\perp - p') \wedge (r^\perp - q') &= (r^\perp - pr^\perp) \wedge (r^\perp - qr^\perp) = p^\perp r^\perp \wedge q^\perp r^\perp \\ &= (p^\perp \wedge q^\perp) \wedge r^\perp \leq r \wedge r^\perp = 0. \end{aligned}$$

Here, $r^\perp - p'$ and $r^\perp - q'$ are the orthogonal complements of the projections p' and q' in $r^\perp \mathcal{M} r^\perp$ respectively. Thus if necessary, by replacing the algebras $\mathcal{M}$ and $\mathcal{A}$ by $r^\perp \mathcal{M} r^\perp$ and $r^\perp \mathcal{A} r^\perp$ respectively, and the projections p, q by p', q' respectively, we assume without loss of generality that

$$p \wedge q = 0 = p^\perp \wedge q^\perp, \tag{4.2}$$

so that $r = 0$ and $\mathcal{M} = r^\perp \mathcal{M} r^\perp$. The purpose of reducing $\mathcal{M}$ to $r^\perp \mathcal{M} r^\perp$ is just to avoid multiple cases, and work with 4×4 matrices rather than 6×6 matrices, as we shall see below.

Now assume that $pq \neq qp$, contrary to what we need to show. Then the generic parts of $\mathcal{R}(p)$ and $\mathcal{R}(q)$ in $\mathcal{H}$ are non-zero by Proposition 4.4, where $\mathcal{H}$ is the separable Hilbert space on which the von Neumann algebra $\mathcal{M}$ acts.

We need to deal with the following three cases:

(1) $p \wedge q^\perp \neq 0$ and $p^\perp \wedge q \neq 0$,
(2) $p \wedge q^\perp \neq 0$ and $p^\perp \wedge q = 0$ (or by symmetry, $p \wedge q^\perp = 0$ and $p^\perp \wedge q \neq 0$),
(3) $p \wedge q^\perp = 0$ and $p^\perp \wedge q = 0$.

We proceed towards getting a contradiction assuming the first case. The detailed procedure in the other two cases follows similar lines.

So for the remainder of the proof, we assume that both the projections $p \wedge q^\perp$ and $p^\perp \wedge q$ are non-zero. It then follows from Proposition 4.4 that there exist a non-zero Hilbert space $\mathcal{K}$ and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ satisfying

$x^2+y^2=1$ and $\ker x = 0 = \ker y$ such that, up to unitary equivalence, we have

$$\mathcal{H} = \mathcal{R}(p\wedge q^{\perp}) \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{R}(p^{\perp}\wedge q) \tag{4.3}$$

and

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2 & xy & 0 \\ 0 & xy & y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.4}$$

Since logmodularity is preserved under unitary equivalence by Proposition 2.2, we assume without loss of generality that $\mathcal{M}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{R}(p\wedge q^{\perp}) \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{R}(p^{\perp}\wedge q))$, and p, q have the form given in (4.4). Set

$$\widetilde{\mathcal{K}}_1 = \mathcal{R}(p\wedge q^{\perp}) \oplus \mathcal{K} \quad \text{and} \quad \widetilde{\mathcal{K}}_2 = \mathcal{K} \oplus \mathcal{R}(p^{\perp}\wedge q),$$

so that

$$\mathcal{H} = \widetilde{\mathcal{K}}_1 \oplus \widetilde{\mathcal{K}}_2. \tag{4.5}$$

Throughout the proof, we make use of the decomposition of $\mathcal{H}$ in both (4.3) and (4.5), which should be understood according to the context. Fix a scalar $\alpha \geq 1$, and define the operator $Z \in \mathcal{B}(\mathcal{H})$ by

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & \alpha^2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =: \begin{bmatrix} 1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix}, \tag{4.6}$$

where

$$Z_2 = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \quad \text{and} \quad Z_3 = \begin{bmatrix} \alpha^2+1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here, digressing momentarily, it is imperative to mention that in the case when $p\wedge q^{\perp} \neq 0$ and $p^{\perp}\wedge q = 0$, we would have considered the operator Z as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & \alpha^2+1 \end{bmatrix} \in \mathcal{B}(\mathcal{R}(p\wedge q^{\perp}) \oplus \mathcal{K} \oplus \mathcal{K}),$$

while for the case $p\wedge q^{\perp} = 0$ and $p^{\perp}\wedge q = 0$, we would have considered the following operator for Z:

$$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2+1 \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K}).$$

In both of these two other cases, the computations follow almost the same (and possibly easier) steps as below.

Now we come back to the operator Z as considered in (4.6). It is clear that Z is a positive and invertible operator in $\mathcal{B}(\mathcal{H})$. We claim that $Z \in \mathcal{M}$. Since p and q are in $\mathcal{M}$, it follows that

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = pqp \in \mathcal{M} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = p^{\perp} q p^{\perp} \in \mathcal{M}.$$

Thus, their sum $pqp + p^{\perp} q p^{\perp} \in \mathcal{M}$, which means that

$$T := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (q - p^{\perp} \wedge q) - (pqp + p^{\perp} q p^{\perp}) \in \mathcal{M}.$$

Let $T = U|T|$ be the polar decomposition of T, where $|T|$ denotes the square root of the operator T^*T. As x and y are commuting positive operators, we note that $xy \geq 0$ and $T = T^*$. It is straightforward to check (using uniqueness of polar decomposition) that

$$|T| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & xy & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathcal{M}$ is a von Neumann algebra and $T \in \mathcal{M}$, it follows that $U \in \mathcal{M}$ and so

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \alpha U \in \mathcal{M}.$$

Also, since

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^2 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = p + (\alpha^2 + 1) p^{\perp} - \alpha^2 (p^{\perp} \wedge q) \in \mathcal{M},$$

we conclude that their sum $Z \in \mathcal{M}$, as claimed.

Since $\mathcal{A}$ is logmodular in $\mathcal{M}$ and Z is positive and invertible in $\mathcal{M}$, we get a sequence $\{S_n\}$ of invertible operators in $\mathcal{A}^{-1}$ such that $Z = \lim_n S_n^* S_n$. For each n, we have $S_n p = p S_n p$ and $S_n^{-1} p = p S_n^{-1} p$; hence the operators S_n and S_n^{-1} have the forms

$$S_n = \begin{bmatrix} a_n & b_n & r_n & s_n \\ c_n & d_n & t_n & u_n \\ 0 & 0 & e_n & f_n \\ 0 & 0 & g_n & h_n \end{bmatrix} =: \begin{bmatrix} A_n & B_n \\ 0 & C_n \end{bmatrix}$$

and

$$S_n^{-1} = \begin{bmatrix} a_n' & b_n' & r_n' & s_n' \\ c_n' & d_n' & t_n' & u_n' \\ 0 & 0 & e_n' & f_n' \\ 0 & 0 & g_n' & h_n' \end{bmatrix} =: \begin{bmatrix} A_n' & B_n' \\ 0 & C_n' \end{bmatrix},$$

for appropriate operators $a_n, b_n, \ldots, a_n', b_n', \ldots$. In particular, we get $A_n A_n' = 1 = A_n' A_n$, i.e. A_n is invertible in $\mathcal{B}(\widetilde{\mathcal{K}}_1)$. Similarly, C_n is invertible in $\mathcal{B}(\widetilde{\mathcal{K}}_2)$. Now

$$\begin{bmatrix} 1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix} = Z = \lim_n S_n^* S_n = \lim_n \begin{bmatrix} A_n^* A_n & A_n^* B_n \\ B_n^* A_n & B_n^* B_n + C_n^* C_n \end{bmatrix}. \tag{4.7}$$

Then we obtain $\lim_n A_n^* A_n = 1$ and since A_n is invertible, it follows from Lemma 4.7 that

$$\lim_n A_n A_n^* = 1. \tag{4.8}$$

We also have $\lim_n A_n^* B_n = Z_2$, which after multiplication by A_n on the left of the sequence and using (4.8) yields $\lim_n (B_n - A_n Z_2) = 0$. But

$$B_n - A_n Z_2 = \begin{bmatrix} r_n & s_n \\ t_n & u_n \end{bmatrix} - \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} r_n - \alpha b_n & s_n \\ t_n - \alpha d_n & u_n \end{bmatrix},$$

and thus we get the following equations:

$$\lim_n (r_n - \alpha b_n) = 0, \tag{4.9}$$

$$\lim_n (t_n - \alpha d_n) = 0. \tag{4.10}$$

If we write $D_n = B_n - A_n Z_2$ for all n, then $\lim_n D_n = 0$ and since $\lim_n A_n^* A_n = 1$, we have

$$\lim_n B_n^* B_n = \lim_n (D_n + A_n Z_2)^* (D_n + A_n Z_2) = \lim_n Z_2^* A_n^* A_n Z_2 = Z_2^* Z_2.$$

This, along with the expression $\lim_n(B_n^*B_n + C_n^*C_n) = Z_3$ from (4.7), further yields

$$\lim_n C_n^*C_n = Z_3 - Z_2^*Z_2 = \begin{bmatrix} \alpha^2+1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.11}$$

Consequently, by computing entries of the matrices $C_n^*C_n$, we get $\lim_n(e_n^*e_n + g_n^*g_n) = 1$; hence there exists $m \in \mathbb{N}$ such that $\|e_n^*e_n\| \leq 2$ for $n \geq m$, which in turn yields

$$e_ne_n^* \leq 2 \quad \forall\, n \geq m. \tag{4.12}$$

Now

$$S_nq = \begin{bmatrix} 0 & b_nx^2 + r_nxy & b_nxy + r_ny^2 & s_n \\ 0 & d_nx^2 + t_nxy & d_nxy + t_ny^2 & u_n \\ 0 & e_nxy & e_ny^2 & f_n \\ 0 & g_nxy & g_ny^2 & h_n \end{bmatrix}$$

and

$$qS_nq = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2d_nx^2 + x^2t_nxy + xye_nxy & x^2d_nxy + x^2t_ny^2 + xye_ny^2 & x^2u_n + xyf_n \\ 0 & xyd_nx^2 + xyt_nxy + y^2e_nxy & xyd_nxy + xyt_ny^2 + y^2e_ny^2 & xyu_n + y^2f_n \\ 0 & g_nxy & g_ny^2 & h_n \end{bmatrix}.$$

Since $S_nq = qS_nq$ for each n, by equating the $(3,2)$ entries of the respective matrices and then using $1 - y^2 = x^2$, we get the expression $x^2e_nxy = xyd_nx^2 + xyt_nxy$. But x is one-to-one and hence x has dense range, so x cancels from both sides of the equation to yield

$$xe_ny = yd_nx + yt_ny.$$

If we set $v_n = t_n - \alpha d_n$ for all n, then the above equation further implies

$$xe_ny = yd_nx + y(\alpha d_n + v_n)y = yd_n(x + \alpha y) + yv_ny = yd_nz + yv_ny,$$

where $z = x + \alpha y$. Since $\alpha \geq 1$, we note that z is positive and invertible (in fact $z^2 = 1 + (\alpha^2 - 1)y^2 + 2\alpha xy \geq 1$), and thus we get

$$yd_n = xe_nyz^{-1} - yv_nyz^{-1}. \tag{4.13}$$

Observe that

$$z^2 = (x + \alpha y)^2 = x^2 + \alpha^2y^2 + 2\alpha xy \geq \alpha^2y^2,$$

and since y and z commute, it follows that

$$y^2z^{-2} \leq 1/\alpha^2. \tag{4.14}$$

Next, by equating the $(1,2)$ entries of S_nq and qS_nq, we get $b_nx^2 + r_nxy = 0$; again since x has dense range, it follows that $b_nx + r_ny = 0$ for all n. So by using (4.9) we have

$$0 = \lim_n (b_nx + r_ny) = \lim_n b_n(x + \alpha y) + \lim_n (r_n - \alpha b_n)y = \lim_n b_n(x + \alpha y).$$

But $x + \alpha y$ is invertible as seen before, so the above equation yields

$$\lim_n b_n = 0. \tag{4.15}$$

Similarly, since each S_n^{-1} also has all these properties, we have

$$\lim_n b'_n = 0. \tag{4.16}$$

Now we compute the $(2,2)$ entry of the matrix $S_nS_n^{-1}$ (with respect to the decomposition of $\mathcal{H}$ in (4.3)), to get $c_nb'_n + d_nd'_n = 1$ for all n. Since $\lim_n b'_n = 0$ from (4.16), it follows that

$$\lim_n d_nd'_n = 1.$$

So there exists $n_0 \in \mathbb{N}$ such that $\|d_nd'_n - 1\| < 1$ for all $n \geq n_0$, which in particular says that $d_nd'_n$ is invertible for all $n \geq n_0$; thus

$$d_nd'_n(d_nd'_n)^{-1} = 1,$$

which implies that d_n is right invertible for all $n \geq n_0$. Likewise, from the $(2,2)$ entry of $S_n^{-1}S_n$ and using $\lim_n b_n = 0$ from (4.15), we get

$$\lim_n d'_nd_n = 1.$$

A similar argument implies that, for large n, d'_nd_n is invertible, which forces that d_n is left invertible. Thus we have shown that d_n is both left and right invertible, which is to say that d_n is invertible, for large n.

Now for each n, the $(2,2)$ entry of the matrix $S_n^*S_n$ (with respect to the decomposition of $\mathcal{H}$ in (4.3)) is $b_n^*b_n + d_n^*d_n$. Since $\lim_n S_n^*S_n = Z$, it then follows that $\lim_n(b_n^*b_n + d_n^*d_n) = 1$, and since $\lim_n b_n = 0$ from (4.15), we get $\lim_n d_n^*d_n = 1$. But d_n is invertible for large n, so it follows from Lemma 4.7 that

$$\lim_n d_nd_n^* = 1. \tag{4.17}$$

Finally, we make use of the expression $\lim_n v_n = 0$ from (4.10), and equations (4.12), (4.13), (4.14) and (4.17) to get

$$\begin{aligned} y^2 &= \lim_n yd_nd_n^*y = \lim_n (yd_n)(yd_n)^* \\ &= \lim_n (xe_nyz^{-1} - yv_nyz^{-1})(xe_nyz^{-1} - yv_nyz^{-1})^* \end{aligned}$$

$$= \lim_n (xe_n yz^{-1})(xe_n yz^{-1})^* = \lim_n xe_n y^2 z^{-2} e_n^* x$$
$$\leq \frac{1}{\alpha^2} \lim_n xe_n e_n^* x \leq \frac{2}{\alpha^2} x^2.$$

Since $\alpha \geq 1$ is arbitrary, it follows by taking $\alpha \to \infty$ that $y = 0$, which is a contradiction. Thus our assumption that $pq \neq qp$ is false. The proof is now complete. □

§5. Reflexivity of algebras with the factorization property

One of the main results of this article says that the lattice of any algebra with the factorization property in a factor is a nest. A natural question that arises is whether algebras having factorization are also nest subalgebras, i.e. are they reflexive? Certainly, we cannot always expect automatic reflexivity of such algebras (see Example 5.9). But then what extra condition can be imposed in order to show that they are reflexive? A result due to Radjavi and Rosenthal [24] says that a weakly closed algebra in $\mathcal{B}(\mathcal{H})$ whose lattice is a nest, is a nest algebra if and only if it contains a maximal abelian self-adjoint algebra (masa). In this section we show that if the lattice of an algebra with factorization in $\mathcal{B}(\mathcal{H})$ has finite-dimensional atoms, then it contains a masa and hence it is reflexive. This fact further helps us in characterizing all logmodular algebras in finite-dimensional von Neumann algebras.

We begin with the following simple lemma. Recall that an algebra $\mathcal{A}$ in $\mathcal{M}$ is called *$\mathcal{M}$-transitive* (simply *transitive* when $\mathcal{M} = \mathcal{B}(\mathcal{H})$) if $\mathrm{Lat}_{\mathcal{M}} \mathcal{A} = \{0, 1\}$. Transitive algebras are very well-studied objects and have attracted deep investigations over the decades. Interested readers can consult Radjavi and Rosenthal [24] for history and some major unsolved open problems on this topic.

Lemma 5.1. *Let $\mathcal{A}$ be an algebra in a von Neumann algebra $\mathcal{M}$ such that $\mathrm{Lat}_{\mathcal{M}} \mathcal{A}$ is a nest, and let $p, q \in \mathrm{Lat}_{\mathcal{M}} \mathcal{A}$ with $p < q$. If $r = q - p$, then $\mathrm{Lat}_{r\mathcal{M}r}(r\mathcal{A}r) = \{s \in r\mathcal{M}r;\ p + s \in \mathrm{Lat}_{\mathcal{M}} \mathcal{A}\}$. In particular, if $p = q_-$ then $r\mathcal{A}r$ is $r\mathcal{M}r$-transitive.*

Proof. As seen in Proposition 4.2, $r\mathcal{A}r$ is a subalgebra of $r\mathcal{M}r$. Let $s \in \mathrm{Lat}_{r\mathcal{M}r}(r\mathcal{A}r)$, and $a \in \mathcal{A}$. Note that $(rar)s = s(rar)s$, and since $rs = s$, it follows that $ras = sas$, using which, along with the conditions $aq = qaq$ and $qs = s$, we have

$$as = aqs = qaqs = qas = pas + ras = pas + sas = (p + s)as. \tag{5.1}$$

Since $sp = 0$ and $ap = pap$, we have $sap = spap = 0$, which along with (5.1) yields

$$(p + s)a(p + s) = pap + sap + (p + s)as = ap + as = a(p + s).$$

Since a is arbitrary in $\mathcal{A}$, it follows that $p+s \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$. Conversely, let $s \in r\mathcal{M}r$ be a projection such that $p+s \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$, and fix $a \in \mathcal{A}$. Then $a(p+s) = (p+s)a(p+s)$, and since $ps = 0 = pr$ and $rs = s$, we have

$$(rar)s = ras = ra(p+s)s = r(p+s)a(p+s)s = s(rar)s.$$

Again as $a \in \mathcal{A}$ is arbitrary, we conclude that $s \in \operatorname{Lat}_{r\mathcal{M}r}(r\mathcal{A}r)$. Thus we have proved the first assertion. Note that if $p = q_-$ then for any $s \in r\mathcal{M}r$, $p+s \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ if and only if $s = 0$ or $s = r$. The second assertion then follows from the first. $\square$

The following proposition is the crux of this section. Recall our convention that all algebras are unital and norm closed.

Proposition 5.2. *Let $\mathcal{A}$ be an algebra having factorization in a von Neumann algebra $\mathcal{M}$, and let $p, q \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ be such that $p < q$. If $q-p$ has finite-dimensional range, then $q-p \in \mathcal{A}$. In particular, if either p or $p^\perp$ has finite-dimensional range, then $p \in \mathcal{A}$.*

Proof. The second assertion clearly follows from the first. To prove the first assertion, set $r = q-p$. Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Note that

$$\mathcal{H} = \mathcal{R}(p) \oplus \mathcal{R}(r) \oplus \mathcal{R}(q^\perp),$$

and we consider operators of $\mathcal{B}(\mathcal{H})$ with respect to this decomposition. For each $n \in \mathbb{N}$, consider the operator

$$X_n = r + \frac{1}{n} r^\perp = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{bmatrix}.$$

It is clear that each X_n is a positive and invertible operator, and since $r \in \mathcal{M}$ it follows that $X_n \in \mathcal{M}$. So by the factorization property of $\mathcal{A}$ in $\mathcal{M}$, there exists an invertible operator $S_n \in \mathcal{A}^{-1}$ such that $X_n = S_n^* S_n$. Each S_n leaves $\mathcal{R}(p)$ and $\mathcal{R}(q)$ invariant, which equivalently says that S_n has the form

$$S_n = \begin{bmatrix} a_n & b_n & c_n \\ 0 & d_n & e_n \\ 0 & 0 & f_n \end{bmatrix}, \tag{5.2}$$

for appropriate operators $a_n, b_n, \ldots$. We claim that the off-diagonal entries b_n, c_n, e_n are 0 for all n. Since $S_n^{-1} \in \mathcal{A}$, S_n^{-1} leaves $\mathcal{R}(p)$ and $\mathcal{R}(q)$ invariant, meaning that S_n^{-1} is also upper triangular. Consequently, the diagonal entries a_n, d_n, f_n of

S_n are invertible. Now, for all n, we have

$$\begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{bmatrix} = X_n = S_n^* S_n = \begin{bmatrix} a_n^* a_n & a_n^* b_n & a_n^* c_n \\ b_n^* a_n & b_n^* b_n + d_n^* d_n & b_n^* c_n + d_n^* e_n \\ c_n^* a_n & c_n^* b_n + e_n^* d_n & c_n^* c_n + e_n^* e_n + f_n^* f_n \end{bmatrix}.$$

By equating entries of the matrices, we get the expressions $a_n^* b_n = 0$ and $a_n^* c_n = 0$. Since a_n is invertible, it follows that $b_n = 0$ and $c_n = 0$. We also have $b_n^* c_n + d_n^* e_n = 0$, and since $b_n = 0$ and d_n is invertible, it follows that $e_n = 0$. This proves the claim that for all n, the operators b_n, c_n and e_n are 0.

We further have $a_n^* a_n = 1/n$ and $c_n^* c_n + e_n^* e_n + f_n^* f_n = 1/n$ for all n, which imply that $\lim_n a_n = 0$ and $\lim_n f_n = 0$. Also, $b_n^* b_n + d_n^* d_n = 1$, but $b_n = 0$, so we get $d_n^* d_n = 1$. Since $\mathcal{R}(r)$ is finite-dimensional by hypothesis, it follows that d_n is a unitary for every n. By compactness of the unitary group in finite dimensions, we extract a subsequence $\{d_{n_k}\}$ converging to a unitary d in $\mathcal{B}(\mathcal{R}(r))$. Thus we have $\lim_k S_{n_k} = S$, where

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since each $S_{n_k} \in \mathcal{A}$ and $\mathcal{A}$ is norm closed, it follows that $S \in \mathcal{A}$. Note that $\lim_k d_{n_k}^{-1} = \lim_k d_{n_k}^* = d^* = d^{-1}$, using which we obtain

$$\lim_k S_{n_k}^{-1} S = \lim_k \begin{bmatrix} a_{n_k}^{-1} & 0 & 0 \\ 0 & d_{n_k}^{-1} & 0 \\ 0 & 0 & f_{n_k}^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} = \lim_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_{n_k}^* d & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e. $\lim_k S_{n_k}^{-1} S = r$. Since $S_{n_k}^{-1} S \in \mathcal{A}$ (as $S_{n_k}^{-1}$ and $S \in \mathcal{A}$) for all k, we conclude that $r \in \mathcal{A}$, as was required to prove. □

We now discuss a sufficient criterion imposed on the dimension of atoms of the lattice to prove the reflexivity of an algebra having factorization in $\mathcal{B}(\mathcal{H})$. It is clearly not necessary, as any nest algebra arising out of a countable nest has factorization and is reflexive.

Theorem 5.3. *Let $\mathcal{A}$ be a weakly closed algebra having factorization in $\mathcal{B}(\mathcal{H})$. If all the atoms of the lattice* $\operatorname{Lat}\mathcal{A}$ *have finite-dimensional range, then $\mathcal{A}$ is reflexive and hence $\mathcal{A}$ is a nest algebra.*

Proof. We shall show that $\mathcal{A}$ contains a masa. As observed above, this claim, along with the fact that $\operatorname{Lat}\mathcal{A}$ is a nest (from Corollary 3.2), will imply the required assertion that $\mathcal{A}$ is reflexive and a nest algebra ([24, Thm. 9.24]).

Let $\{r_i\}_{i\in\Lambda}$ be the collection of all the atoms of $\operatorname{Lat}\mathcal{A}$ for some finite or countable indexing set Λ (countability of index follows from separability of the Hilbert space $\mathcal{H}$; see Corollary 3.5). Since $\operatorname{Lat}\mathcal{A}$ is atomic (Remark 3.6), it follows that $\sum_{i\in\Lambda} r_i = 1$ in SOT convergence; hence $\mathcal{H} = \bigoplus_{i\in\Lambda}\mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{R}(r_i)$, which satisfies $\mathcal{H}_i \perp \mathcal{H}_j$ for all $i \neq j$. For each $i \in \Lambda$, given r_i is an atom, we have $r_i = p_i - q_i$ for some $p_i, q_i \in \operatorname{Lat}\mathcal{A}$ (where $q_i = p_{i-}$), and as r_i has finite-dimensional range by hypothesis, that $r_i \in \mathcal{A}$ follows from Proposition 5.2.

Recognize the von Neumann algebra $r_i\mathcal{B}(\mathcal{H})r_i$ with $\mathcal{B}(\mathcal{H}_i)$, for each $i \in \Lambda$. Since r_i is an atom, we know from Lemma 5.1 that $r_i\mathcal{A}r_i$ is a transitive subalgebra of $\mathcal{B}(\mathcal{H}_i)$. Then, given that $\mathcal{H}_i$ is finite-dimensional, we invoke Burnside's theorem ([24, Cor. 8.6]) to conclude that $r_i\mathcal{A}r_i = \mathcal{B}(\mathcal{H}_i)$. In other words, this says that $r_i\mathcal{B}(\mathcal{H})r_i = r_i\mathcal{A}r_i$, and since $r_i \in \mathcal{A}$, it follows that

$$r_i\mathcal{B}(\mathcal{H})r_i \subseteq \mathcal{A}. \tag{5.3}$$

Now for each i, let $\mathcal{L}_i$ be a masa in $\mathcal{B}(\mathcal{H}_i)$, and let $\mathcal{L} = \bigoplus_{i\in\Lambda}\mathcal{L}_i$, which is considered a subalgebra of $\mathcal{B}(\mathcal{H})$. It is clear that $\mathcal{L}$ is a masa in $\mathcal{B}(\mathcal{H})$. Note that $\mathcal{L}r_i = r_i\mathcal{L}$ for all $i \in \Lambda$. Also, it follows from (5.3) that $r_i\mathcal{L}r_i \subseteq r_i\mathcal{B}(\mathcal{H})r_i \subseteq \mathcal{A}$, and since $\mathcal{A}$ is WOT closed we have

$$\mathcal{L} = \mathcal{L}\sum_{i\in\Lambda} r_i \subseteq \sum_{i\in\Lambda}\mathcal{L}r_i = \sum_{i\in\Lambda} r_i\mathcal{L}r_i \subseteq \mathcal{A},$$

where the sum above is in WOT. Thus we have shown our requirement that $\mathcal{A}$ contains a masa, completing the proof. □

A nest of projections on a Hilbert space is called *maximal* or *simple* if it is not contained in any larger nest. It is easy to verify that a nest $\mathcal{E}$ is maximal if and only if all atoms in $\mathcal{E}$ are one-dimensional. Thus the following corollary is immediate from Theorem 5.3.

Corollary 5.4. *Let $\mathcal{A}$ be a weakly closed algebra having factorization in $\mathcal{B}(\mathcal{H})$, and let* $\operatorname{Lat}\mathcal{A}$ *be a maximal nest. Then $\mathcal{A}$ is reflexive, and so $\mathcal{A}$ is a nest algebra.*

We emphasize the importance of the above corollary in the following example.

Example 5.5. Consider the Hilbert space $\mathcal{H} = \ell^2(\Gamma)$, for $\Gamma = \mathbb{N}$ or $\mathbb{Z}$, and let $\mathcal{A}$ be the reflexive algebra of upper triangular matrices in $\mathcal{B}(\mathcal{H})$ with respect to the canonical basis $\{e_n\}_{n\in\Gamma}$. Note that $\operatorname{Lat}\mathcal{A} = \{p_n;\ n \in \Gamma\}$, where p_n is the projection onto the subspace $\overline{\operatorname{span}}\{e_m;\ m \leq n\}$. Clearly, $\operatorname{Lat}\mathcal{A}$ is a maximal nest. So if $\mathcal{B}$ is any subalgebra of $\mathcal{A}$ with $\operatorname{Lat}\mathcal{B}$ a nest, then $\operatorname{Lat}\mathcal{A} \subseteq \operatorname{Lat}\mathcal{B}$, which implies by maximality that $\operatorname{Lat}\mathcal{A} = \operatorname{Lat}\mathcal{B}$. Thus an appeal to Corollary 5.4 says that the only subalgebra of $\mathcal{A}$ that has factorization in $\mathcal{B}(\mathcal{H})$ is $\mathcal{A}$.

Next we consider some consequences of the above results for subalgebras of finite-dimensional von Neumann algebras. Let M_n denote the algebra of all $n \times n$ complex matrices for some natural number n. Let $\mathcal{A}$ be a logmodular algebra in M_n. It can readily be verified using compactness of the closed unit ball of M_n that the algebra $\mathcal{A}$ has factorization in M_n as well. Since all atoms of $\operatorname{Lat} \mathcal{A}$ are clearly finite-dimensional, $\mathcal{A}$ is a nest algebra in M_n thanks to Theorem 5.3. Thus we have shown that, up to unitary equivalence, $\mathcal{A}$ is an algebra of block upper triangular matrices in M_n. This assertion was put as a conjecture in [20], and an affirmative answer was given in [15]. We have provided a different solution, and we state it below.

Corollary 5.6. *Let $\mathcal{A}$ be a logmodular algebra in M_n. Then $\mathcal{A}$ is an algebra of block upper triangular matrices up to unitary equivalence.*

Moreover, the corollary above generalizes to any logmodular subalgebras of finite-dimensional von Neumann algebras. This characterizes all logmodular algebras in finite-dimensional von Neumann algebras.

Corollary 5.7. *Let $\mathcal{A}$ be a subalgebra of a finite-dimensional von Neumann algebra $\mathcal{M}$. Then $\mathcal{A}$ is logmodular in $\mathcal{M}$ if and only if $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$. In this case, $\mathcal{A}$ is $\mathcal{M}$-reflexive.*

Proof. That a nest subalgebra has factorization (and hence is logmodular) in any finite von Neumann algebra is a well-known fact of Arveson (see [2, Thm. 4.2.1]; also see Example 2.7).

Conversely, let $\mathcal{A}$ be logmodular in $\mathcal{M}$. Since $\mathcal{M}$ is a finite-dimensional von Neumann algebra, there exist natural numbers $n_1, \ldots, n_k$ such that $\mathcal{M}$ is $*$-isomorphic to $M_{n_1} \oplus \cdots \oplus M_{n_k}$. In view of Proposition 2.2, we assume without loss of generality that

$$\mathcal{M} = M_{n_1} \oplus \cdots \oplus M_{n_k},$$

which acts on the Hilbert space $\mathcal{H} = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k}$. Using compactness of the unit ball in finite-dimensional algebras, we note that $\mathcal{A}$ has factorization in $\mathcal{M}$.

For $i = 1, \ldots, k$, let p_i denote the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathbb{C}^{n_i}$ (considered as a subspace of $\mathcal{H}$), and let $\mathcal{A}_i = p_i \mathcal{A} p_i$. We claim that

$$\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k.$$

Firstly, note that $p_i \in \mathcal{M} \cap \mathcal{M}'$; hence $p_i \in \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$. This in particular says that $\mathcal{A}_i$ is an algebra. Since p_i has finite-dimensional range, it follows from Proposition 5.2 that $p_i \in \mathcal{A}$. This implies that $\mathcal{A}_i \subseteq \mathcal{A}$ for each i; hence $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_k \subseteq \mathcal{A}$.

On the other hand, by using $\sum_{i=1}^{k} p_i = 1$, we get

$$\mathcal{A} = \mathcal{A}\sum_{i=1}^{k} p_i \subseteq \sum_{i=1}^{k} \mathcal{A}p_i = \sum_{i=1}^{k} p_i\mathcal{A}p_i = \bigoplus_{i=1}^{k} \mathcal{A}_i,$$

proving our claim that $\mathcal{A} = \bigoplus_{i=1}^{k} \mathcal{A}_i$. Note that $M_{n_i} = p_i\mathcal{M}p_i$ for each i. So the algebra $\mathcal{A}_i$ has factorization in M_{n_i} by Proposition 4.2. An appeal to Corollary 5.6 yields

$$\mathcal{A}_i = \operatorname{Alg}_{M_{n_i}} \mathcal{E}_i,$$

for some nest $\mathcal{E}_i$ $(= \operatorname{Lat}_{M_{n_i}} \mathcal{A}_i)$ in M_{n_i}. Consider the lattice

$$\mathcal{E} = \bigoplus_{i=1}^{k} \mathcal{E}_i = \left\{\bigoplus_{i=1}^{k} q_i;\ q_i \in \mathcal{E}_i,\ 1 \le i \le k\right\}$$

in $\mathcal{M}$. It is clear that $\mathcal{E} = \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$, which implies $\mathcal{A} \subseteq \operatorname{Alg}_{\mathcal{M}} \mathcal{E}$. The lattice $\mathcal{E}$ is not a nest if $k \ge 2$. Choose a sublattice, namely $\mathcal{F}$, of $\mathcal{E}$ such that $\mathcal{F}$ is a nest and each element q_i in $\mathcal{E}_i$ appears at least once as the ith coordinate of an element of $\mathcal{F}$. Such an $\mathcal{F}$ can always be chosen: for example, consider the nest $\mathcal{F}_i$ for each i given by

$$\mathcal{F}_i = \left\{e_1 \oplus \cdots \oplus e_{i-1} \oplus q_i \oplus 0 \oplus \cdots \oplus 0;\ q_i \in \mathcal{E}_i\right\} \subseteq \mathcal{E},$$

where e_i denotes the identity of M_{n_i}, and let $\mathcal{F} = \bigcup_i^k \mathcal{F}_i$. Since each $\mathcal{E}_i$ is a nest and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_k$, the sublattice $\mathcal{F}$ is a nest in $\mathcal{M}$, and $\mathcal{F}$ fulfils the requirement. We claim that

$$\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \mathcal{F},$$

which will prove that $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$. Clearly, as $\mathcal{F} \subseteq \mathcal{E}$, we have $\mathcal{A} \subseteq \operatorname{Alg}_{\mathcal{M}} \mathcal{E} \subseteq \operatorname{Alg}_{\mathcal{M}} \mathcal{F}$. Conversely, let $x \in \operatorname{Alg}_{\mathcal{M}} \mathcal{F}$, and let $x = \bigoplus_{i=1}^{k} x_i$ for some $x_i \in M_{n_i}$, $1 \le i \le k$. The way $\mathcal{F}$ has been chosen, each element of $\mathcal{E}_i$ appears as the ith coordinate of some element of $\mathcal{F}$, so it follows that $x_i q = q x_i q$ for all $q \in \mathcal{E}_i$ and $1 \le i \le k$. This shows that $x_i \in \operatorname{Alg}_{M_{n_i}} \mathcal{E}_i = \mathcal{A}_i$; hence $x \in \mathcal{A}$. Therefore, we conclude that $\operatorname{Alg}_{\mathcal{M}} \mathcal{F} \subseteq \mathcal{A}$, proving the claim.

Finally, since $\mathcal{F} \subseteq \mathcal{E} = \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$, it follows that $\operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \operatorname{Alg}_{\mathcal{M}} \mathcal{F} = \mathcal{A}$. Since the other inclusion is obvious, we have $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$, which is to say that $\mathcal{A}$ is $\mathcal{M}$-reflexive. $\square$

More generally, Corollary 5.7 can easily be extended to logmodular subalgebras of those von Neumann algebras which are direct sums of finite-dimensional von Neumann algebras, whose proof goes along the same lines. We record the statement here.

Corollary 5.8. *Let $\mathcal{M}$ be a (possibly countably infinite) direct sum of finite-dimensional von Neumann algebras, and let $\mathcal{A}$ be a weakly closed subalgebra of $\mathcal{M}$. Then $\mathcal{A}$ is logmodular in $\mathcal{M}$ if and only if $\mathcal{A}$ is a nest subalgebra of $\mathcal{M}$. In this case, $\mathcal{A}$ is $\mathcal{M}$-reflexive.*

In general, Corollary 5.7 fails to be true for algebras having factorization (or logmodularity) in infinite-dimensional von Neumann algebra, as the following example suggests.

Example 5.9. Let $\mathcal{A}$ be an algebra having factorization in a von Neumann algebra $\mathcal{M}$ such that $\mathcal{A} \neq \mathcal{M}$, and $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ is a factor. We claim that $\mathcal{A}$ is not $\mathcal{M}$-reflexive. Assume otherwise that $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A}$. Then note that since $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is commutative (by Corollary 3.2), we have $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \operatorname{Alg}_{\mathcal{M}} \operatorname{Lat}_{\mathcal{M}} \mathcal{A} = \mathcal{A}$, which implies that $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D}$. Also, it is easy to verify that $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D}'$ and thus we have $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D} \cap \mathcal{D}' = \mathbb{C}$. It then follows that $\operatorname{Lat}_{\mathcal{M}} \mathcal{A} = \{0, 1\}$, so $\mathcal{A} = \operatorname{Alg}_{\mathcal{M}}\{0, 1\} = \mathcal{M}$, which is not true.

There are plenty such algebras. To see one, let G be a countable discrete ordered group. Let

$$\ell^2(G) = \big\{f \colon G \to \mathbb{C};\ \textstyle\sum_{g\in G} |f(g)|^2 < \infty\big\},$$

and for each $g \in G$, let $U_g \colon \ell^2(G) \to \ell^2(G)$ be the unitary operator defined by

$$U_g f(g') = f(g^{-1}g') \quad \text{for } f \in \ell^2(G) \text{ and } g' \in G.$$

Let $\mathcal{M}$ be the finite von Neumann algebra in $\mathcal{B}(\ell^2(G))$ generated by the family $\{U_g\}_{g\in G}$, called the group von Neumann algebra of G. Note that each element X of $\mathcal{B}(\ell^2(G))$ has a matrix representation (x_{gh}) with respect to the canonical basis of $\ell^2(G)$. Let

$$\mathcal{A} = \big\{X = (x_{gh}) \in \mathcal{M};\ x_{gh} = 0 \text{ for } g > h\big\}.$$

Then $\mathcal{A}$ is an example of a finite maximal subdiagonal algebra in $\mathcal{M}$ with respect to the expectation $\phi \colon \mathcal{M} \to \mathcal{M}$ given by

$$\phi((x_{gh})) = x_{ee}1 \quad \text{for } (x_{gh}) \in \mathcal{M},$$

where e denotes the identity of G (see [2, Exm. 3]). In particular, $\mathcal{A}$ has factorization in $\mathcal{M}$ ([2, Thm. 4.2.1,]). But note that $\mathcal{A} \cap \mathcal{A}^* = \mathbb{C}$ (in fact if $(x_{gh}) \in \mathcal{A} \cap \mathcal{A}^*$, then $x_{gh} = 0$ for all $g \neq h$ and $x_{gg} = x_{g'g'}$ for all $g, g' \in G$), so $\mathcal{A}$ cannot be $\mathcal{M}$-reflexive as discussed above. Moreover, we can choose the ordered group G to be countable with the infinite conjugacy class (ICC) property (e.g. $G = \mathbb{F}_2$, the free group on two generators), so that $\mathcal{M}$ is a factor. In this case, although $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a nest, $\mathcal{A}$ cannot be a nest subalgebra of $\mathcal{M}$ (otherwise $\mathcal{A} \cap \mathcal{A}^*$ would contain the nest and so could not be equal to $\mathbb{C}$).

§6. Concluding remarks

In this paper we have discussed the "universal" or "strong" factorization property for subalgebras of von Neumann algebras. But there are weaker notions of factorization which can also be explored. We say a subalgebra $\mathcal{A}$ has the *weak factorization property* (*WFP*) in a von Neumann algebra $\mathcal{M}$ if for any positive element $x \in \mathcal{M}$, there is an element $a \in \mathcal{A}$ such that $x = a^*a$. Here the invertibility assumption is relaxed.

Power [22] has studied the WFP of nest algebras, where he proved that if a nest $\mathcal{E}$ of projections on a Hilbert space $\mathcal{H}$ is well ordered (i.e. $p \neq p_+ = \bigcap_{q>p} q$ for all $p \in \mathcal{E}$ with $p \neq 1$), then $\operatorname{Alg} \mathcal{E}$ has the WFP in $\mathcal{B}(\mathcal{H})$. Inspired by our result on lattices of algebras with factorization, it appears that lattices of algebras with the WFP in a factor should also be a nest. But it is not clear to us at this point. However, for a subalgebra in a finite von Neumann algebra we can certainly say so. We can follow similar lines of proof along with the fact that any left (or right) invertible element in a finite von Neumann algebra is invertible. We record it here.

Theorem 6.1. *Let $\mathcal{A}$ be a subalgebra of a finite von Neumann algebra (resp. factor) $\mathcal{M}$ having the WFP. Then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a commutative subspace lattice (resp. nest).*

So a natural question is the following:

Question 6.2. *Is the lattice of a subalgebra having the WFP in a von Neumann algebra (resp. factor) a commutative subspace lattice (resp. nest)?*

We conclude with the question of reflexivity of algebras with factorization. We showed that a weakly closed algebra with factorization in $\mathcal{B}(\mathcal{H})$ has a masa and hence is reflexive, if we impose some dimensionality condition on the atoms of its lattice. But we still do not know whether every algebra with factorization in $\mathcal{B}(\mathcal{H})$ has a masa. Thus the following question remains open:

Question 6.3. *Is a weakly closed algebra having factorization in $\mathcal{B}(\mathcal{H})$ automatically reflexive? In particular, is a weakly closed transitive algebra with factorization equal to $\mathcal{B}(\mathcal{H})$?*

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