Publ. RIMS Kyoto Univ. **60** (2024), 583–606
DOI 10.4171/PRIMS/60-3-6

The Principle of Limiting Amplitude for Perturbed Wave Equations in an Exterior Domain

by

Kiyoshi Mochizuki and Hideo Nakazawa

Abstract

In this paper we consider the dissipative wave propagation problem in an exterior domain. Uniform estimates and Hölder conditions of the resolvent are studied for the reduced wave operator without dissipation. Based on these results, the validity of the principle of the limiting amplitude is proved for the wave propagation problem with dissipation.

Mathematics Subject Classification 2020: 35L05 (primary); 35B40, 35J05 (secondary).
Keywords: principles of limiting absorption and amplitude, dissipative wave equation, magnetic field, uniform resolvent estimates, local Hölder continuity of resolvent.

§1. Introduction

Let Ω be an exterior domain in $\mathbf{R}^n$ with star-shaped complement with respect to the origin 0 and smooth boundary $\partial\Omega$ (the case $\Omega = \mathbf{R}^n$ is not excluded when $n \geq 3$). We consider in Ω the wave propagation problem

$$\begin{gathered} \partial_t^2 w + b_0(x)\partial_t w - \Delta_b w + c(x)w = g(x)e^{-i\omega t}, \quad (x,t) \in \Omega \times \mathbf{R}_+, \\ w(x,0) = 0, \quad \partial_t w(x,0) = 0, \quad x \in \Omega, \\ w(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbf{R}_+, \end{gathered} \tag{1.1}$$

where $\partial_t = \partial/\partial t$ in $t \in \mathbf{R}_+ = (0,\infty)$, Δ_b is the magnetic Laplacian

$$\Delta_b = \nabla_b \cdot \nabla_b = \sum_{j=1}^{n} (\partial_j + ib_j(x))^2,$$

Communicated by K. Nakanishi. Received May 12, 2022. Revised October 3, 2022.

K. Mochizuki: Department of Mathematics, Chuo University, Kasuga, Bunkyo, Tokyo 112-8551, Japan (Emeritus Tokyo Metropolitan University);
e-mail: mochizuk@math.chuo-u.ac.jp
H. Nakazawa: Department of Mathematics, Nippon Medical School, Kyonan-cho, Musashino, Tokyo 180-0023, Japan;
e-mail: nakazawa-hideo@nms.ac.jp

with $i = \sqrt{-1}$, $\partial_j = \partial/\partial x_j$ and real-valued smooth functions $b_j(x)$ $(j = 1, \dots, n)$ of $x \in \mathbf{R}^n$, the scalar potential $c(x)$ is a real and $b_0(x)$ is a nonnegative bounded continuous function, $\omega \neq 0$ is a real number and $g(x)$ belongs to some weighted L^2-space in Ω. Thus, $b_0(x)\partial/\partial t$ represents a dissipation (friction term).

The principle of limiting amplitude states that every solution of the above problem tends as $t \to \infty$ to the steady state

$$e^{-i\omega t} v(x, \omega)$$

in an appropriate topology, and v satisfies

$$-\Delta_b v + c(x)v - \omega^2 v - i\omega b_0(x)v = g(x) \text{ in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{1.2}$$

This principle has been justified by many authors from various standpoints and by different methods; see e.g., [2, 4, 13, 14, 15, 16] for wave equations and [5, 7] for first-order hyperbolic systems. In these works, results are limited to self-adjoint systems (i.e., the case $b_0(x) = 0$ in the above problem), and most important properties reduce to show low-frequency estimates of solutions to the stationary problem.

The dissipative wave equation (1.1) has been studied by Mizohata–Mochizuki [6] in the whole $\mathbf{R}^3$ with no magnetic potentials. The aim of this paper is to extend the results of [6] to (1.1) with magnetic potentials and in an exterior domain Ω. Note that our results include the case $n = 2$. The existence of the dissipative term makes the problem slightly complicated. Since the spectral theory (Stone's formula) does not apply to this case, our proof is as in [6] restricted to the use of the Laplace inversion formula. So high-frequency estimates for (1.2) also play an important role. In this sense, our theory is based on the uniform resolvent estimates of the self-adjoint operator $L = -\Delta_b + c(x)$. Note that in [12] a similar problem is treated, when $L = -\Delta$ and $|b_0(x)|$ is small and decays suitably. In our general case also, the condition $b_0(x) \geq 0$ is able to be replaced by the smallness of $|b_0(x)|$ (see Remark 3 in Section 4).

The uniform resolvent estimates for L have been developed in Mochizuki [9] (for $n \geq 3$) and Mochizuki–Nakazawa [11] (for $n = 2$), which we shall examine precisely here. The results are applied among ordinary tools of functional identities to show a Hölder continuity of solutions $u(\cdot, z)$ to the stationary problem

$$-\Delta_b u + c(x)u - zu = f(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0 \tag{1.3}$$

(cf. e.g., Roach–Zhang [14]). This Hölder condition is available to the dissipative problem (1.1) under a suitable decay condition on $b_0(x)$.

In Section 2 we consider the stationary equation (1.3) with $z = \kappa^2$, where $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C};\ \operatorname{Im}\kappa > 0\}$. The uniform resolvent estimates developed in [9, 11] are summarized in Theorem 1 and a necessary smoothness property for $i\kappa(L-\kappa^2)$ is proved there (Corollary 1). In Section 3, we shall show that the solution $u(x,z)$ of (1.3) satisfy a local Hölder continuity as a function of z (Theorem 2 and Corollary 2). The validity of the principle of limiting amplitude (Theorem 3) is demonstrated in Section 4. Finally, in Section 5 a concise proof of Theorem 1 is given.

§2. Preliminaries

We list the notation which will be used freely in the sequel:

- For $x \in \mathbf{R}^n$, $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\tilde{x} = x/r = x/|x|$.
- $\partial_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, $\partial_r = \partial/\partial_r$, $\nabla = (\partial_1, \ldots, \partial_n)$, $b(x) = (b_1(x), \ldots, b_n(x))$, $\nabla_b = \nabla + b(x)$, $\nabla \times b(x) = (\partial_j b_k(x) - \partial_k b_j(x))_{j<k}$.
- For $z \in \mathbf{C}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote its real and imaginary parts, respectively.
- $\sqrt{z}$ denotes the branch of the square root of $z \in \mathbf{C}$ with $\operatorname{Im} \sqrt{z} \geq 0$.
- $\Omega_s = \{x \in \Omega;\ |x| < s\}$, $\Omega'_s = \{x \in \Omega;\ |x| > s\}$, $S_s = \{x \in \mathbf{R}^n;\ |x| = s\}$;
- $L^2(G)$, $G \subset \Omega$, is the usual L^2-space with inner product $(f,g)_G = \int_G f(x)\overline{g(x)}\,dx$ and norm $\|f\|_G = \sqrt{(f,f)_G}$; in the case that $G = \Omega$, we simply write $\int_\Omega dx = \int dx$.
- $H^j(\Omega)$ $(j = 1, 2, \ldots)$ are the usual Sobolev spaces on Ω.
- $H^2_{\mathrm{loc}}(\overline{\Omega})$ is the space of H^2-functions on each compact set of $\overline{\Omega} = \Omega \cup \partial\Omega$.
- For a smooth function $\psi(x) \geq 0$, $L^2_\psi(G)$ is a class of functions such that

$$\|f\|^2_{\psi,G} = \int_G \psi(x)|f(x)|^2\,dx < \infty.$$

- The weight function $\mu = \mu(r) > 0$ is used to be a smooth decreasing function of $r > 0$ such that

$$\mu''(r) \geq 0 \quad \text{and} \quad \|\mu\|_1 = \int_0^\infty \mu(s)\,ds < \infty.$$

- Also, the weight function $\xi(r) = (1 + [r])^{-2}$ is used, where

$$[r] = \begin{cases} r, \\ r(1 + \log r/r_0), \end{cases} \qquad [n-2] = \begin{cases} n-2 & \text{when } n \geq 3, \\ 1 & \text{when } n = 2, \end{cases}$$

  and $\log r/r_0 = \log(r/r_0)$ with $r_0 > 0$ satisfying $S_{r_0} \subset \mathbf{R}^2 \setminus \Omega$. Without loss of generality we can assume $\xi(r) \leq \mu(r) \leq \xi(r)^{1/2}$.

Now we define the operator $L = -\Delta_b + c(x)$ acting in $L^2(\Omega)$ as

$$\mathcal{D}(L) = \bigl\{u \in L^2(\Omega) \cap H^2_{\mathrm{loc}}(\overline{\Omega});\ (-\Delta_b + c)u \in L^2(\Omega),\ u|_{\partial\Omega} = 0\bigr\}. \tag{2.1}$$

As is well known (see e.g., Mochizuki [10]), if

$$\max\{|\nabla \times b(x)|, |c(x)|\} = o(r^{-1}) \quad \text{as } r \to \infty, \tag{2.2}$$

then L is self-adjoint and its essential spectrum fills the whole nonnegative half-line $[0, \infty)$. If we require

(UC) the operator $-\Delta_b + c(x)$ verifies the unique continuation property,

there are no positive eigenvalues. Moreover, the continuous spectrum is absolutely continuous if we strengthen the decay condition (2.2) as $O(\mu)$. The absolute continuity is verified by establishing the principle of limiting absorption in $L^2_\mu(\Omega)$.

For $z = \kappa^2$, $\kappa \in \mathbf{C}_+$, the resolvent operator of L is defined by $R(z) = (L - z)^{-1}$. For $f(x) \in L^2(\Omega)$ the function $u = u(x, \kappa) = R(\kappa^2)f$ gives a unique solution in $L^2(\Omega)$ of the boundary-value problem

$$-\Delta_b u + c(x)u - \kappa^2 u = f(x) \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.3}$$

Let $I = [\lambda_1, \lambda_2]$ be an interval in $\mathbf{R}_+$. For a small $\nu_0 > 0$ we set

$$\begin{aligned} &\Gamma_\pm = \Gamma_\pm(I, \nu_0) = \big\{z \in \mathbf{C};\ \operatorname{Re} z \in I,\ 0 < \pm \operatorname{Im} z \le \nu_0\big\}, \\ &\bar\Gamma_\pm = \Gamma_\pm \cup I. \end{aligned} \tag{2.4}$$

The principle of limiting absorption asserts the existence of the limit of $R(z)$ as $z \to \lambda \in I$. Since λ is in the spectrum of L, it cannot converge to a limit in the uniform operator topology, and it is necessary to adopt a weaker topology.

Definition 1. For $\mu = \mu(r)$ given above we choose here $\varphi = (\int_r^\infty \mu(\tau)\, d\tau)^{-1}$. A solution $u = u(x, \kappa)$ of (2.3) with $z = \kappa^2 \in \bar\Gamma_\pm$ is said to satisfy the radiation condition if we have

$$\int_{\Omega'_{R_1}} \mu(r)|u|^2\, dx < \infty \quad \text{and} \quad \int_{\Omega'_{R_1}} \varphi(r)'|\tilde{x} \cdot \theta|^2\, dx < \infty, \tag{2.5$_\pm$}$$

where $R_1 > 0$ is chosen to satisfy $\partial\Omega \subset \Omega_{R_1}$ and $\theta = \theta(x, \sqrt{z}) = \theta(x, \kappa)$ is a vector-valued function

$$\theta = \nabla_b u + \tilde{x}\Big(\frac{n-1}{2r}u - i\sqrt{z}\Big)u. \tag{2.6}$$

When $z = \lambda \pm i0$, solutions $u(x, \sqrt{\lambda} + i0)$ and $u(x, -\sqrt{\lambda} + i0)$ satisfy the same equation. They are distinguished as outgoing (+) and incoming (−) solutions.

For $\lambda \in I$, every solution $u \in H^2_{\text{loc}}(\Omega)$ of the generalized eigenvalue problem

$$(-\Delta_b + c(x) - \lambda)u = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0$$

satisfies the following growth property: if the support in Ω of u is not compact, then

$$\liminf_{s\to\infty} \int_{S_s} |\tilde{x}\cdot\theta(x, \pm\sqrt{\lambda}+i0)|^2\, dS \neq 0$$

(see Jäger–Rejto [3]). Since $\varphi(r)' \notin L^1(\mathbf{R}_+)$, this contradicts the radiation condition, and (UC) is applied to show the uniqueness of the solution of (2.3) and $(2.5)_\pm$ when $\kappa = \pm\sqrt{\lambda}+i0$. As is well known, (UC) is guaranteed for $-\Delta_b + c(x)$ if $b_j(x) \in C^2(\Omega)$ and $c(x)$ is Hölder continuous in Ω.

When $z \in \Gamma_\pm$ solutions of (2.3) and $(2.5)_\pm$ are also unique and coincide with the L^2-solution $R(z)f$. Moreover, there exists $C = C(\Gamma_\pm) > 0$ such that

$$\int_{\Omega'_R} \mu|R(z)f|^2\, dx \leq C\varphi(R)^{-1} \int \mu^{-1}|f|^2\, dx \quad \text{for } R \geq R_1, \tag{2.7}$$

$$\int \mu|R(z)f|^2\, dx + \int_{\Omega'_{R_1}} \varphi'|\theta|^2\, dx \leq C \int \mu^{-1}|f|^2\, dx. \tag{2.8}$$

These resolvent estimates and the above uniqueness results imply, with the help of the Rellich compactness criterion, the existence of the limit

$$u(\pm\sqrt{\lambda}+i0) = \lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)f \quad \text{in } L^2_\mu(\Omega), \tag{2.9$_\pm$}$$

which gives the unique solution of (2.3) and $(2.5)_\pm$ for $\kappa = \pm\sqrt{\lambda}+i0$. Thus, $R(z)$ is continuously extended to $\bar{\Gamma}_\pm$ as an operator from $L^2_{\mu^{-1}}(\Omega)$ to $L^2_\mu(\Omega)$ (cf. e.g., Mochizuki [8]).

Note that the assertions of [3] and [8] are summarized in [10, Chaps. 3–5] for more general second-order elliptic equations in an exterior domain.

To proceed with problem (1.1), the constant $C = C(\Gamma_\pm)$ in (2.7) and (2.8) should be improved to be chosen independent of $z \in \mathbf{C}\setminus\mathbf{R}$. To this end we add a smallness of the coefficients.

(BC.1) Assume that $c(x) = c_0(x) + c_1(x)$ and $\nabla\times b(x)$ and $c_0(x)$ are small: for $\varepsilon_0 > 0$ small,

$$\{|\nabla\times b(x)|^2 + |c_0(x)|^2\}^{1/2} \leq \varepsilon_0[r]^{-2};$$

$c_1(x)$ is not necessarily small but satisfies

$$c_1(x) \geq 0 \quad \text{and} \quad \partial_r\{rc_1(x)\} \leq 0.$$

Under these conditions we shall prove (in the last Section 5) the following theorem which represents a uniform resolvent estimate:

Theorem 1. *Assume (BC.1) and (UC). Let $u = R(\kappa^2)f$. Then we have the following:*

(i) *There exists* $C_1 = C_1(\varepsilon_0) > 0$ *such that*

$$P_{\partial\Omega}(u) + \int \frac{\operatorname{Im}\kappa r + 1}{4[r]^2}|u|^2\,dx \le C_1 \int [r]^2|f|^2\,dx, \quad \forall\,\kappa \in \mathbf{C}_+, \tag{2.10}$$

where

$$P_{\partial\Omega}(u) = -\frac{1}{2}\int_{\partial\Omega}(\nu\cdot x)|\nu\cdot\nabla u|^2\,dS,$$

with the outer unit normal $\nu = \nu(x)$ *to the boundary* $\partial\Omega$.

(ii) *There exists* $C_2 = C_2(\varepsilon_0, \|\mu\|_1) > 0$ *such that*

$$\int \mu\{|\nabla_b u|^2 + |\kappa u|^2\}\,dx \le C_2 \int \max\{[r]^2, \mu^{-1}\}|f|^2\,dx \quad \forall\,\kappa \in \mathbf{C}_+,$$

where $\mu = \mu(r)$ *is a weight function given above.*

As a corollary of this theorem we have the following:

Corollary 1. *There exists* $C > 0$ *such that*

$$\int \mu(r)^{3/2}|\kappa u|^2\,dx \le C\int \mu(r)^{-3/2}|f|^2\,dx, \quad \forall\,\kappa \in \mathbf{C}_+.$$

Proof. Since $[r]^2 \le \mu(r)^{-2}$ and L is self-adjoint, we have from Theorem 1(ii),

$$\|\mu^{1/2}R(\kappa^2)\mu\|^2 \le |\kappa|^{-2}C_2, \quad \|\mu R(\kappa^2)\mu^{1/2}\|^2 \le |\kappa|^{-2}C_2$$

for any $\kappa \in \mathbf{C}_+$. Interpolation between these two inequalities gives the desired conclusion. □

Remark 1. (i) If $n \ge 3$ and $c_0(x)$ satisfies a stronger condition

$$|c_0(x)|^2 \le -\varepsilon_0\frac{n-1}{2r}\mu(r)\mu'(r),$$

then (see [8]) there exists $C_3 = C_3(\varepsilon_0, \|\mu\|_{L^1}) > 0$ such that for any $\kappa \in \mathbf{C}_+ \cup \mathbf{R}$,

$$\int \mu\{|\nabla_b u|^2 + |\kappa u|^2\}\,dx \le C_3\int \mu^{-1}|f|^2\,dx.$$

(ii) The case $\Omega = \mathbf{R}^2$ is excluded in the above theorem. But in the special case of Laplacian $L = -\Delta$ in $\mathbf{R}^2$, it is known (see Barcelo–Ruiz–Vega [1]) that, for any $\delta > 0$, there exists $C_4 = C_4(\delta) > 0$ such that

$$\int (1+r)^{-1-\delta}\{|\nabla u|^2 + |\kappa u|^2\}\,dx \le C_4\int (1+r)^{1+\delta}|u|^2\,dx.$$

§3. Local Hölder continuity of $R(z)f$

To enter into the proof of the principle of limiting amplitude we need one more theorem: the local Hölder continuity of the resolvent $R(z)$. For this aim, we apply the results of Theorem 1 and Corollary 1 to a functional identity for solutions of (2.3) under slightly stronger conditions on the coefficients:

(BC.2) Assume that $c(x) = c_0(x) + c_1(x)$ and $\nabla \times b(x)$ and $c_0(x)$ are small: for $\varepsilon_0 > 0$ small,

$$\{|\nabla \times b(x)|^2 + |c_0(x)|^2\}^{1/2} \le \varepsilon_0 (1+r)^{-1} \mu(r);$$

$c_1(x)$ is not necessarily small but satisfies for some $C_5 > 0$,

$$0 \le c_1(x) \le C_5 \mu(r) \text{ and } -C_5 \mu(r) \le \partial_r \{r c_1(x)\} \le 0.$$

For $f,\, g \in L^2(\Omega)$ let $u = R(\kappa^2) f$ and $v = R(\bar{\kappa}^2) g$ (note that $\bar{v} = R(\kappa^2)\bar{g}$). Consider the functional

$$J = \Delta_b u(\varphi \tilde{x} \cdot \overline{\nabla_b v}) + \overline{\Delta_b v}(\varphi \tilde{x} \cdot \nabla_b u). \tag{3.1}$$

We can follow a similar argument to the proof of Proposition 1 in Section 5 to obtain

$$\begin{aligned} J &= \nabla \cdot \{\nabla_b u(\varphi \tilde{x} \cdot \overline{\nabla_b v}) + \overline{\nabla_b v}(\varphi \tilde{x} \cdot \nabla_b u)\} - \nabla \cdot \{\varphi \tilde{x}(\nabla_b u \cdot \overline{\nabla_b v})\} \\ &\quad + \Big(\varphi \frac{n-1}{r} + \varphi'\Big)(\nabla_b u \cdot \overline{\nabla_b v}) - 2\varphi'(\tilde{x} \cdot \overline{\nabla_b v})(\tilde{x} \cdot \nabla_b u) \\ &\quad - \frac{2\varphi}{r}\{\nabla_b u \cdot \overline{\nabla_b v} - (\tilde{x} \cdot \nabla_b u)(\tilde{x} \cdot \overline{\nabla_b v})\} \\ &\quad - i\varphi\{(\tilde{x} \times \nabla_b u) \cdot (\nabla \times b)\bar{v} + (\tilde{x} \times \overline{\nabla_b v})(\nabla \times b) u\}. \end{aligned}$$

On the other hand, the definitions of u and v give, with a simple calculation,

$$\begin{aligned} J &= \nabla \cdot \{\tilde{x}\varphi(c_1 - \kappa^2) u\bar{v}\} - \Big(\frac{n-1}{r}\varphi + \varphi'\Big)(c_1 - \kappa^2) u\bar{v} - \varphi \partial_r c_1 u \bar{v} \\ &\quad + \varphi(c_0 u - f)(\tilde{x} \cdot \overline{\nabla_b v}) + \varphi(c_0 \bar{v} - \bar{g})(\tilde{x} \cdot \nabla_b u). \end{aligned}$$

Choose $\varphi = r$ in these expressions of J. Then

$$\begin{aligned} J &= \nabla \cdot \{r \nabla_b u(\tilde{x} \cdot \overline{\nabla_b v}) + r\overline{\nabla_b v}(\tilde{x} \cdot \nabla_b u)\} - \nabla \cdot \{r\tilde{x}(\nabla_b u \cdot \overline{\nabla_b v})\} \\ &\quad + (n-2)(\nabla_b u \cdot \overline{\nabla_b v}) - ir\{(\tilde{x} \times \nabla_b u) \cdot (\nabla \times b)\bar{v} + (\tilde{x} \times \overline{\nabla_b v})(\nabla \times b) u\} \\ &= \nabla \cdot \{\tilde{x} r(c_1 - \kappa^2) u\bar{v}\} - n(c_1 - \kappa^2) u\bar{v} - r\partial_r c_1 u\bar{v} \\ &\quad + r(c_0 u - f)(\tilde{x} \cdot \overline{\nabla_b v}) + r(c_0 \bar{v} - \bar{g})(\tilde{x} \cdot \nabla_b u). \end{aligned}$$

So, integrating over Ω_R (R large), we have

$$\begin{aligned}
&\int_{S_R} r\{2(\tilde{x}\cdot\nabla_b u)(\tilde{x}\cdot\overline{\nabla_b v}) - \nabla_b u\cdot\overline{\nabla_b v}\}\,dS + \int_{\partial\Omega} r(\nu\cdot\tilde{x})(\nu\cdot\nabla u)(\nu\cdot\overline{\nabla v})\,dS \\
&\qquad + \int_{\Omega_R}(n-2)\nabla_b u\cdot\overline{\nabla_b v}\,dx - i\int r(\nabla\times b)\cdot\{(\tilde{x}\times\nabla_b u)\bar{v} + (\tilde{x}\times\overline{\nabla_b v})u\}\,dx \\
&\quad = \int_{S_R} r(c_1-\kappa^2)u\bar{v}\,dS + \int_{\Omega_R}\big[n(\kappa^2-c_1)u\bar{v} - r\partial_r c_1 u\bar{v} - r(c_0u+f)(\tilde{x}\cdot\overline{\nabla_b v}) \\
&\qquad\qquad - r(c_0\bar{v}+\bar{g})(\tilde{x}\cdot\nabla_b u)\big]\,dx.
\end{aligned}$$

Let $R\to\infty$ in this equality. Then noting

$$\begin{aligned}
\int_{\Omega_R}(n-2)\nabla_b u\cdot\overline{\nabla_b v}\,dx &= \frac{n-2}{2}\int_{S_R}\{(\tilde{x}\cdot\nabla_b u)\bar{v} + u(\tilde{x}\cdot\overline{\nabla_b v})\}\,dS \\
&\quad + (n-2)\int_{\Omega_R}\Big\{(\kappa^2-c)u\bar{v} + \frac{1}{2}(f\bar{v}+u\bar{g})\Big\}\,dx,
\end{aligned}$$

we conclude the equation

$$\begin{aligned}
2\kappa^2\int u\bar{v}\,dx = \int\Big[&\{2c_1 + r\partial_r c_1 + (n-2)c_0\}u\bar{v} \\
&+ \frac{n-2}{2}(f\bar{v}+u\bar{g}) - ir(\nabla\times b)\{(\tilde{x}\times\nabla_b u)\bar{v} - u(\tilde{x}\times\overline{\nabla_b v})\} \\
&+ r\{(c_0u+f)(\tilde{x}\cdot\overline{\nabla_b v}) + (c_0\bar{v}+\bar{g})(\tilde{x}\cdot\nabla_b u)\}\Big]\,dx \\
&+ \int_{\partial\Omega} r(\nu\cdot\tilde{x})(\nu\cdot\nabla u)(\nu\cdot\overline{\nabla v})\,dS. \qquad (3.2)
\end{aligned}$$

Lemma 1. *Assume that* $(1+r)f(x)$, $(1+r)g(x)\in L^2_{\mu^{-1}}(\Omega)$. *Then we have*

$$\begin{aligned}
\left|\kappa^2\int_\Omega u\bar{v}\,dx\right| &\le C\{(1+|\kappa|^{-2})\|f\|_{\xi^{-1}} + \|rf\|_{\mu^{-1}}\}\|g\|_{\xi^{-1}} \\
&\quad + C\{(1+|\kappa|^{-2})\|g\|_{\xi^{-1}} + \|rg\|_{\mu^{-1}}\}\|f\|_{\xi^{-1}}. \qquad (3.3)
\end{aligned}$$

Proof. Let us estimate each term on the right-hand side of (3.2). By assumptions (BC.2) on $b(x)$ and $c(x)$ we have, from Theorem 1(ii),

$$\left|\int\{2c_1 + r\partial_r c_1 + (n-2)c_0\}u\bar{v}\,dx\right| \le C\|u\|_\mu\|v\|_\mu \le C|\kappa|^{-2}\|f\|_{\xi^{-1}}\|g\|_{\xi^{-1}}.$$

Similarly, we have

$$\frac{n-2}{2}\left|\int(f\bar{v}+u\bar{g})\,dx\right| \le C\{\|f\|_{\mu^{-1}}\|v\|_\mu + \|u\|_\mu\|g\|_{\mu^{-1}}\} \le C|\kappa|^{-1}\|f\|_{\xi^{-1}}\|g\|_{\xi^{-1}}$$

and

$$\left|\int ir(\nabla \times b)\{(\tilde{x} \times \nabla_b u)\bar{v} - u(\tilde{x} \times \overline{\nabla_b v})\}\, dx\right|$$
$$\leq C|\kappa|^{-1}\{\|f\|_{\xi^{-1}}\|g\|_{\xi^{-1}} + \|f\|_{\xi^{-1}}\|g\|_{\xi^{-1}}\},$$

where we have used $r|\nabla \times b| \leq \varepsilon_0\mu(r)$, and

$$\left|\int r\{(c_0 u + f)(\tilde{x} \cdot \overline{\nabla_b v}) + (c_0\bar{v} + \bar{g})(\tilde{x} \cdot \nabla_b u)\}\, dx\right|$$
$$\leq C\{\|r(c_0 u + f)\|_{\mu^{-1}}\|\tilde{x} \cdot \nabla_b v\|_\mu + \|r(c_0 v + g)\|_{\mu^{-1}}\|\tilde{x} \cdot \nabla_b u\|_\mu\}$$
$$\leq C\{\|u\|_\mu + \|rf\|_{\mu^{-1}}\}\|g\|_{\xi^{-1}} + C\{\|v\|_\mu + \|rg\|_{\mu^{-1}}\}\|f\|_{\xi^{-1}}$$
$$\leq C\{(|\kappa|^{-1}\|f\|_{\xi^{-1}} + \|rf\|_{\mu^{-1}})\|g\|_{\xi^{-1}} + (|\kappa|^{-1}\|g\|_{\xi^{-1}} + \|rg\|_{\mu^{-1}})\|f\|_{\xi^{-1}}\},$$

where we have used $r|c_0| \leq \varepsilon_0\mu(r)$. Finally, if we remember that $\partial\Omega$ is star shaped, then $P_{\partial\Omega}(u)$ in Theorem 1(i) becomes nonnegative, and it follows that

$$\left|\int_{\partial\Omega} (\nu \cdot x)(\nu \cdot \nabla u)(\nu \cdot \nabla v)\, dS\right| \leq C_1\|f\|_{\xi^{-1}}\|g\|_{\xi^{-1}}.$$

Summarizing these inequalities, we conclude the assertion of the lemma. □

Theorem 2. *Assume (UC) and (BC.2). Let $z, z' \in \bar{\Gamma}_\pm$ with $|z - z'| < 1$. Then there exists $C_6 > 0$ independent of z, z' such that*

$$\|\{R(z) - R(z')\}f\|_{\xi_1^{1/p}} \leq C_6(1 + |z|^{-2/p})|z - z'|^\delta\|f\|_{\xi_1^{-1}},$$

where $\xi_1(r) = \xi(r)\mu(r)$, p, q satisfy $1 > \frac{1}{p} = 1 - \frac{1}{q} > \frac{1}{3}$ and $\delta = 1 - \frac{3}{2q} > 0$.

Proof. We consider only the case $z, z' \in \bar{\Gamma}_+$. The resolvent equation

$$R(z) - R(z') = (z - z')R(z)R(z') \tag{3.4}$$

shows that $\frac{dR(z)f}{dz} = R(z)\{R(z)f\}$ in $L^2(\Omega)$. So we have

$$\left(\frac{dR(z)f}{dz}, \bar{g}\right) = (R(z)f, \overline{R(z)g}) = \int_\Omega uv\, dx, \tag{3.5}$$

and hence, by use of (3.3) in the above lemma with $z = \kappa^2$,

$$\left|\left(\frac{dR(z)f}{dz}, \bar{g}\right)\right| \leq C|z|^{-1}\{(1 + |z|^{-1})\|f\|_{\xi^{-1}} + \|rf\|_{\mu^{-1}}\}\|g\|_{\xi^{-1}}$$
$$+ C|z|^{-1}\{(1 + |z|^{-1})\|g\|_{\xi^{-1}} + \|rg\|_{\mu^{-1}}\}\|f\|_{\xi^{-1}}.$$

Here, choose $g = \xi_1 \frac{dR(z)f}{dz}$. Then since $(1 + r)^2\mu^{-1} \leq \xi_1(r)^{-1}$, the inequality

$$\left\|\frac{dR(z)f}{dz}\right\|_{\xi_1} \leq C(1 + |z|^{-2})\|f\|_{\xi_1^{-1}} \tag{3.6}$$

follows. On the other hand, the Green formula for the solution $u = R(z)f$ of (2.3) gives

$$\operatorname{Im} z\|R(z)f\|^2 = -\operatorname{Im}(f, R(z)f).$$

Using this equality twice, we have

$$\begin{aligned}\Big\|\frac{dR(z)f}{dz}\Big\| \le (\operatorname{Im} z)^{-1}\|R(z)f\| &\le (\operatorname{Im} z)^{-3/2}\{\|f\|_{\xi^{-1}}\|R(z)f\|_{\xi}\}^{1/2}\\ &\le C(\operatorname{Im} z)^{-3/2}\|f\|_{\xi^{-1}}.\end{aligned} \tag{3.7}$$

So, the Hölder inequality with $1 > \frac{1}{p} = 1 - \frac{1}{q} > \frac{1}{3}$ and (3.6), (3.7) show

$$\begin{aligned}\int_\Omega \xi_1^{1/p}\Big|\frac{dR(z)f}{dz}\Big|^2 dx &\le \Big\{\int_\Omega \xi_1\Big|\frac{dR(z)f}{dz}\Big|^2 dx\Big\}^{1/p}\Big\{\int_\Omega\Big|\frac{dR(z)f}{dz}\Big|^2 dx\Big\}^{1/q}\\ &\le C\{(1+|z|^{-2})\|f\|_{\xi_1^{-1}}\}^{2/p}\{(\operatorname{Im} z)^{-3/2}\|f\|_{\xi^{-1}}\}^{2/q}\\ &\le \{C(1+|z|^{-2/p})(\operatorname{Im} z)^{-3/2q}\|f\|_{\xi_1^{-1}}\}^2\end{aligned} \tag{3.8}$$

Now let $z = \lambda + i\varepsilon$, $z' = \lambda' + i\varepsilon'$. Then since

$$R(z) - R(z') = \{R(z) - R(\tilde z)\} + \{R(\tilde z) - R(\tilde z')\} + \{R(\tilde z') - R(z')\},$$

where $\tilde z = \lambda + i\tau$ and $\tilde z' = \lambda' + i\tau$ with $\tau = \max\{\varepsilon, \varepsilon', |\lambda - \lambda'|\}$, (3.8) implies

$$\begin{aligned}\|\{R(z) - R(z')\}f\|_{\xi_1^{1/p}} &\le C|\varepsilon - \tau|\Big\|\frac{dR(\tilde z)f}{d\tau}\Big\|_{\xi_1^{1/p}} + C|\lambda - \lambda'|\Big\|\frac{dR(\tilde z)f}{d\lambda}\Big\|_{\xi_1^{1/p}}\\ &\quad + C|\tau - \varepsilon'|\Big\|\frac{dR(\tilde z')f}{d\tau}\Big\|_{\xi_1^{1/p}}\\ &\le C(1+|z|^{-2/p})\tau(\operatorname{Im}\tilde z)^{-3/2q}\|f\|_{\xi_1^{-1}},\end{aligned}$$

and the desired inequality holds true. □

Corollary 2. *For $\frac{1}{3} < \frac{1}{p} < 1$ and $0 \le s \le 1$ put*

$$\alpha = \frac{1}{p} + \Big(1 - \frac{1}{p}\Big)s, \quad \beta = 1 - \Big(1 - \frac{1}{p}\Big)s. \tag{3.9}$$

Then there exists $C_7 = C_7(\Gamma_\pm) > 0$ independent of α, β such that

$$\|\xi_1^{\alpha/2}\{R(z) - R(z')\}\xi_1^{\beta/2}f\| \le C_7|z - z'|^\delta\|f\|.$$

Proof. It follows from the above theorem that

$$\|\xi_1^{1/2p}\{R(z) - R(z')\}\xi_1^{1/2}f\| \le C(\Gamma_\pm)|z - z'|^\delta\|f\|.$$

So, as in the proof of Corollary 1, the assertion is concluded by the interpolation method. □

§4. The principle of limiting amplitude

Now we return to the wave propagation problem (1.1) with the above two theorems. Here the potentials $c(x) = c_0(x) + c_1(x)$ and $b(x)$ satisfy (BC.2) and (UC), and the coefficient $b_0(x)$ of the friction term is required to be smooth and satisfy

(B_0) there exists $C_8 > 0$ such that

$$0 \le b_0(x) \le C_8\xi_1(r) = C_8(1+[r])^{-2}\mu(r) \quad \text{in } \overline{\Omega}.$$

Under these conditions, it is well known that for each $g(x) \in L^2(\Omega)$, solutions $w(t) = w(x,t)$ of problem (1.1) exist and are unique in the class of differentiable functions of $t > 0$ with values in $L^2(\Omega)$.

For this solution we shall prove the following theorem which establishes the principle of limiting amplitude:

Theorem 3. *Assume (BC.2), (UC) and (B_0). Let $g(x) \in L^2_{\xi_1^{-1}}(\Omega)$. Then, as $t \to \infty$,*

$$w(x,t) = v(x, \omega + i0)e^{-i\omega t} + o(1) \quad \textit{strongly in } L^2_{\xi_1^{1/2}}(\Omega),$$

where $v(x, \omega + i0) \in L^2_{\xi_1^{1/2}}(\Omega)$ is the unique solution of the problem

$$-\Delta_b v + c(x)v - i\kappa b_0(x)v - \kappa^2 v = g(x), \quad v|_{\partial\Omega} = 0, \tag{4.1}$$

with $\kappa = \omega + i0$.

Let

$$\widetilde{w} = \int_0^\infty w(x,t)e^{i\kappa t}\, dt,$$

where $\kappa \in \overline{\mathbf{C}}_+$. Then $\widetilde{w}$ satisfies the reduced equation

$$-\Delta_b\widetilde{w} + c(x)\widetilde{w} - i\kappa b_0(x)\widetilde{w} - \kappa^2\widetilde{w} = \frac{g(x)}{-i(\kappa-\omega)}, \quad \widetilde{w}|_{\partial\Omega} = 0,$$

So, if v solves problem (4.1), then $\widetilde{w} = \frac{v}{-i(\kappa-\omega)}$ and the solution of (1.1) is given by

$$w(x,t) = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\tau+i\sigma_0}^{-\tau+i\sigma_0} \frac{v(x,\kappa)}{\kappa-\omega}e^{-i\kappa t}\, d\kappa, \tag{4.2}$$

where σ_0 is a large positive number.

Put $b_0(x) = a(x)^2$ and let A be a multiplication operator $Ag = a(x)g(x)$. Choose a new unknown $\phi = Av$ in the above (4.1). Then it changes to

$$\phi - i\kappa AR(\kappa^2)A\phi = AR(\kappa^2)g(x). \tag{4.3}$$

Conversely, let $\phi \in L^2(\Omega)$ satisfy this equation. Then the unique solution of (4.1) is given by

$$v = i\kappa R(\kappa^2)A\phi + R(\kappa^2)g. \tag{4.4}$$

In fact, if we denote the right-hand side by h, then we have $(L-\kappa^2)h = i\kappa A\phi + g$. By means of (4.3),

$$\begin{aligned} -i\kappa A\phi &= -i\kappa A\{i\kappa AR(\kappa^2)A\phi + AR(\kappa^2)g\} \\ &= -i\kappa A^2\{i\kappa R(\kappa^2)A\phi + R(\kappa^2)g\} = -i\kappa b(x)h. \end{aligned}$$

Thus, h satisfies the equation $(L-\kappa^2)h = i\kappa b(x)h + g$ showing $h = v$.

Lemma 2. *The operator $AR(\kappa^2)A$ is compact in $L^2(\Omega)$, and for each $\kappa = -\sigma + i\tau$ ($\tau \geq 0$) and $f \in L^2(\Omega)$ we have*

$$\mathrm{Re}[-i\kappa(AR(\kappa^2)Af, f)] \geq 0. \tag{4.5}$$

Proof. The compactness of $AR(\kappa^2)A$ is obvious from the condition (B_0): $A = \xi(r)^{1/2}o(r^{-1/2})$ and the Rellich criterion. The positivity (4.5) is also easily verified. In fact, we have

$$\mathrm{Re}[-i\kappa(R(\kappa^2)Af, Af)] = \int_0^\infty \frac{(\lambda+\sigma^2+\tau^2)\tau}{(\lambda-\sigma^2+\tau^2)^2+(2\tau\sigma)^2}\frac{d}{d\lambda}(E(\lambda)Af, Af)d\lambda \geq 0,$$

where $E(\lambda)$ is the spectral family of the operator L. On the other hand,

$$\begin{aligned} \lim_{\tau\downarrow 0}\mathrm{Re}[-i\kappa(R(\kappa^2)Af, Af)] &= -\frac{|\sigma|}{2i}\big(\{R(\sigma^2-i0)-R(\sigma^2+i0)\}Af, Af\big) \\ &= \pi|\sigma|\frac{d}{d\lambda}(E(\lambda)Af, Af)|_{\lambda=\sigma^2} \geq 0. \end{aligned}$$ □

This lemma shows the uniqueness and the existence of solutions $\phi = \phi(x, \kappa)$ of (4.3) in $L^2(\Omega)$. Moreover, we have the following lemma:

Lemma 3. *Assume (B_0). Then there exists $C > 0$ such that for each $\kappa \in \mathbf{C}_+$,*

$$\|\phi(\kappa)\| \leq C(1+|\kappa|)^{-1}\|g\|_{\xi^{-1}}. \tag{4.6}$$

Proof. By use of Lemma 2 and (4.3) we have

$$\|\phi(\kappa)\| \leq \|AR(\kappa^2)g\|.$$

Then (4.6) is direct since $A(x)^2 \leq \xi(r) \leq \mu(r)$. □

Lemma 4. *Under* (B_0) *there exists* $C > 0$ *such that for all* $\kappa \in \mathbf{C}_+$,

$$\|v(\kappa)\|_\xi + |\kappa|\,\|v\|_\mu \le C\|g\|_{\xi^{-1}}, \tag{4.7}$$

$$\|v(\kappa)\|_{\xi_1^{1/2}} \le C|\kappa|^{-1/2}\|g\|_{\xi^{-1}}. \tag{4.8}$$

Moreover, there exists $C(\Gamma_\pm) > 0$ *such that for all* $\kappa^2, \kappa'^2 \in \Gamma_\pm$,

$$\|v(\kappa) - v(\kappa')\|_{\xi_1^{1/2}} \le C(\Gamma_\pm)|\kappa - \kappa'|^{1/4}\|g\|_{\xi_1^{-1}}. \tag{4.9}$$

Proof. Note (4.6) and $A^2 \le C\xi$. Then we have from (4.4),

$$\begin{aligned}\|v\|_\xi + |\kappa|\,\|v\|_\mu &\le |\kappa|\{\|R(\kappa^2)A\phi\|_\xi + |\kappa|\,\|R(\kappa^2)A\phi\|_\mu\} + \|R(\kappa^2)g\|_\xi + |\kappa|\,\|R(\kappa^2)g\|_\mu \\ &\le C\{|\kappa|\,\|A\phi\|_{\xi^{-1}} + \|g\|_{\xi^{-1}}\} \\ &\le C\{|\kappa|\,\|\phi\| + \|g\|_{\xi^{-1}}\} \le C\|g\|_{\xi^{-1}},\end{aligned}$$

showing (4.7). Inequality (4.8) easily follows from (4.7) since we have

$$\|u\|_{\xi_1^{1/2}} \le C\|u\|_\xi^{1/2}\|u\|_\mu^{1/2}, \quad \forall\, u \in L^2_\mu(\Omega)\ (\subset L^2_{\xi_1^{1/2}}(\Omega)). \tag{4.10}$$

Next, note the identity

$$\begin{aligned}&v(\kappa) - v(\kappa') - i\kappa R(\kappa^2)A\{\phi(\kappa) - \phi(\kappa')\} \\ &\qquad = i(\kappa - \kappa')R(\kappa^2)A\phi(\kappa') + i\kappa'\{R(\kappa^2) - R(\kappa'^2)\}A\phi(\kappa') \\ &\qquad\quad + \{R(\kappa^2) - R(\kappa'^2)\}g.\end{aligned} \tag{4.11}$$

We estimate each term on the right-hand side as follows: Theorem 2 with $p = 2$ shows

$$\|\{R(\kappa^2) - R(\kappa'^2)\}g\|_{\xi_1^{1/2}} \le C(\Gamma_\pm)|\kappa^2 - \kappa'^2|^\delta\|g\|_{\xi_1^{-1}}.$$

Similarly, we have from Theorem 2,

$$\begin{aligned}&|\kappa'|\,\|\{R(\kappa^2) - R(\kappa'^2)\}A\phi(\kappa')\|_{\xi_1^{1/2}} \\ &\qquad \le C(\Gamma_\pm)|\kappa^2 - \kappa'^2|^\delta\|\phi(\kappa')\| \le C(\Gamma_\pm)|\kappa^2 - \kappa'^2|^\delta\|g\|_{\xi^{-1}}.\end{aligned}$$

By use of (4.10), (4.7) and (4.6),

$$|\kappa - \kappa'|\,\|R(\kappa^2)A\phi(\kappa')\|_{\xi_1^{1/2}} \le C|\kappa|^{-1/2}|\kappa - \kappa'|\,\|\phi\| \le C(\Gamma_\pm)|\kappa - \kappa'|\,\|g\|_{\xi^{-1}}.$$

As for the remaining term, note that (4.10) implies

$$|\kappa|\,\|R(\kappa^2)A\{\phi(\kappa) - \phi(\kappa')\}\|_{\xi_1^{1/2}} \le C|\kappa|^{1/2}\|\phi(\kappa) - \phi(\kappa')\|.$$

Here, we multiply by $A(r)$ on both sides of (4.11) and take the L^2-norm. Then, in view of Lemma 2, we obtain

$$\|\phi(\kappa)-\phi(\kappa')\| \le |\kappa-\kappa'|\,\|AR(\kappa^2)A\phi(\kappa')\| + |\kappa'|\,\|A\{R(\kappa^2)-R(\kappa'^2)\}A\phi(\kappa')\| + \|A\{R(\kappa^2)-R(\kappa'^2)\}g\|.$$

Since $A(r)^2 \le \xi(r) \le \xi_1(r)^{1/2}$, the above three estimates are also applicable to this inequality so that

$$C|\kappa|^{1/2}\|\phi(\kappa)-\phi(\kappa')\| \le C(\Gamma_\pm)|\kappa^2-\kappa'^2|^\delta \|g\|_{\xi_1^{-1}}.$$

Estimate (4.9) is thus concluded. □

Proof of Theorem 3. We start from the expression in $L^2_{\xi_1^{1/2}}(\Omega)$,

$$w(x,t) = \frac{1}{2\pi i}\lim_{\rho\to\infty}\int_{\rho+i\tau_0}^{-\rho+i\tau_0} \frac{v(x,\kappa)}{\kappa-\omega} e^{-i\kappa t}\,d\kappa.$$

We recall that $v(\cdot,\kappa)$ is an $L^2_{\xi_1^{1/2}}(\Omega)$-valued analytic function of $\kappa=-\sigma+i\tau$ in $\tau=\operatorname{Im}\kappa>0$. So, by use of the Cauchy integral formula,

$$\begin{aligned}\int_{\rho+i\tau_0}^{-\rho+i\tau_0} & \frac{v(x,\kappa)}{\kappa-\omega} e^{-i\kappa t}\,d\kappa \\ &= -\lim_{\varepsilon\downarrow 0}\int_{-\rho}^{\rho} \frac{v(x,\sigma+i\varepsilon)e^{-i(\sigma+i\varepsilon)t}}{\sigma-\omega+i\varepsilon}\,d\sigma \\ &\quad -\int_0^{\tau_0}\left\{\frac{v(x,\rho+i\tau)e^{-i(\rho+i\tau)t}}{\tau-i(\rho-\omega)} - \frac{v(x,-\rho+i\tau)e^{-i(-\rho+i\tau)t}}{\tau+i(\rho+\omega)}\right\}d\tau.\end{aligned}$$

Here, the second term on the right-hand side tends to 0 in $L^2_{\xi_1^{1/2}}(\Omega)$ as $\rho\to\infty$. Thus, we have

$$\begin{aligned} w(x,t) &= -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{-a+\omega}^{a+\omega} \frac{\{v(x,\sigma+i\varepsilon)-v(x,\omega+i\varepsilon)\}e^{(\varepsilon-i\sigma)t}}{\sigma-\omega+i\varepsilon}\,d\sigma \\ &\quad -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\left\{\lim_{\rho\to\infty}\left(\int_{a+\omega}^{\rho}+\int_{-\rho}^{-a+\omega}\right)\frac{v(x,\sigma+i\varepsilon)e^{(\varepsilon-i\sigma)t}}{\sigma-\omega+i\varepsilon}\,d\sigma\right\} \\ &\quad -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\left\{v(x,\omega+i\varepsilon)e^{(\varepsilon-i\omega)t}\int_{-a}^{a}\frac{e^{-i\sigma t}}{\sigma+i\varepsilon}\,d\sigma\right\} = I_1+I_2+I_3.\end{aligned}$$

Here, a is a small constant satisfying $0<a<|\omega|$. By Hölder continuity and the decay and singularity estimates of $v(\cdot,\kappa)$ in Lemma 4, we can use the Riemann–Lebesgue theorem to see that I_1 and $I_2\to 0$ strongly in $L^2_{\xi_1^{1/2}}(\Omega)$ as $t\to\infty$. Thus, noting

$$\lim_{\varepsilon\downarrow 0}\int_{-a}^{a}\frac{e^{-i\sigma t}}{\sigma+i\varepsilon}\,d\sigma = -2i\lim_{\varepsilon\downarrow 0}\int_0^a \frac{\sigma\sin(\sigma t)+\varepsilon\cos(\sigma t)}{\sigma^2+\varepsilon^2}\,d\sigma$$
$$= -2i\left\{\int_0^a \frac{\sin(\sigma t)}{\sigma}\,d\sigma + \int_0^\infty \frac{1}{\sigma^2+1}\,d\sigma\right\} \to -2\pi i$$

as $t\to\infty$, we obtain from I_3 the desired conclusion. □

Remark 2. Theorem 3 is modified to hold in $L^2_{\mu^{1+\varepsilon}}(\Omega)$ with $\varepsilon=\min\{1,3/p-1\}$. In fact, since $\xi(r)\le\mu(r)\le\xi(r)^{1/2}$ leads us to $\mu^{1+\varepsilon}=\mu^{3/p}\le(\xi\mu)^{1/p}$, it follows that

$$\begin{aligned}(4.12)\quad &\|R(\kappa^2)g\|_{\mu^{1+\varepsilon}} \le \|R(\kappa^2)g\|_{\xi^\varepsilon\mu^{1-\varepsilon}} \le C|\kappa|^{-1+\varepsilon}\|g\|_{\xi^{-1}},\\ &\|\{R(\kappa^2)-R(\kappa'^2)\}g\|_{\mu^{1+\varepsilon}} \le \|\{R(\kappa^2)-R(\kappa'^2)\}g\|_{\xi_1^{1/p}}\\ &\qquad\le C(\Gamma_\pm)|\kappa^2-\kappa'^2|^\delta\|g\|_{\xi_1^{-1}}\ \text{ with } \delta=(3/p-1)/2.\end{aligned}$$

On the other hand, the conditions on $b_0(x)$ and $g(x)$ can be slightly weakened if we make use of Corollary 2. We choose $s=1/2$ there to see $\alpha=\beta=2/3+(3/p-1)/6$. Assume the following:

$(B_0)'$ There exists $C_9>0$ such that

$$0\le b_0(x)\le C_9\xi_1^\alpha \quad \text{in } \overline{\Omega}.$$

We choose $g(x)\in L^2_{\xi_1^{-\alpha}}(\Omega)$ and note that $2\alpha-1=1/p$. Then $\xi_1^\alpha\le\xi^\alpha\mu^{1-\alpha}$ and we have

$$\|R(\kappa^2)g\|_{\xi_1^\alpha}\le\|R(\kappa^2)g\|_{\xi^\alpha\mu^{1-\alpha}}\le C|\kappa|^{-1+\alpha}\|g\|_{\xi^{-1}},$$

which takes the role of (4.12) since $\frac{2}{3}<\alpha<1$. Thus, Theorem 3 holds in $L^2_{\xi_1^\alpha}(\Omega)$ in this case.

Remark 3. The dissipation condition (B_0) or $(B_0)'$ can be replaced by the following:

$(B_0)''$ There exists a small $\varepsilon_1>0$,

$$|b_0(x)|\le\varepsilon_1\xi_1(r)\ \text{ or }\ \le\varepsilon_1\xi_1^{3/4}(r)\quad\text{in } \overline{\Omega}.$$

Note that $3/4$ is given as α when $s=1/2$ and $p=2$ in Corollary 2. Let A, B be multiplication operators $Af=|b_0(x)|^{1/2}f$, $Bf=(\operatorname{sign}b_0(x))Af$. Put $\phi=Av$ for the solution v of (4.1). Then it satisfies

$$(4.3)'\qquad \phi-i\kappa AR(\kappa^2)B\phi=AR(\kappa^2)g(x).$$

Since $\xi_1^{3/4}(r) \le \mu^{3/2}(r)$, in the account of Corollary 1, we can choose ε_1 small enough to satisfy

$$\sup_{\kappa\in\mathbf{C}_+} \|i\kappa AR(\kappa^2)A\| < 1.$$

Moreover, since $\|AR(\kappa^2)B\| = \|AR(\kappa^2)A\|$, the Neumann series

$$\{1 - i\kappa AR(\kappa^2)B\}^{-1} = \sum_{j=0}^{\infty}[i\kappa AR(\kappa^2)B]^j$$

converges in the operator topology uniformly in $\kappa \in \mathbf{C}_+$. Hence, we are able to reach Theorem 3 in the case of $(B_0)''$ also.

§5. The uniform resolvent estimates

First, remember the vector function $\theta(x,\kappa) = \nabla_b u + \tilde{x}(\frac{n-1}{2r} - i\kappa)u$. By use of this function we rewrite equation (2.3) as

$$-\nabla_b \cdot \theta + \Big(\frac{n-1}{2r} - i\kappa\Big)\tilde{x}\cdot\theta + \Big\{c(x) + \frac{(n-1)(n-3)}{4r^2}\Big\}u = f(x). \tag{5.1}$$

Let $\varphi = \varphi(r) > 0$ be a weight function such that $\varphi(r) = O(r)$ $(r \to \infty)$. We multiply by $\varphi(\tilde{x}\cdot\bar{\theta})$ on both sides of (5.1). Then the real parts give

$$\begin{aligned}&\operatorname{Re}\Big[\varphi\Big\{-\nabla_b\cdot\theta + \Big(\frac{n-1}{2r} - i\kappa\Big)\tilde{x}\cdot\theta + \Big(c(x) + \frac{(n-1)(n-3)}{4r^2}\Big)u\Big\}(\tilde{x}\cdot\bar{\theta})\Big]\\ &\qquad = \operatorname{Re}\{\varphi f(\tilde{x}\cdot\bar{\theta})\}.\end{aligned} \tag{5.2}$$

Note that

$$-\varphi\nabla_b\cdot\theta(\tilde{x}\cdot\bar{\theta}) = -\nabla\cdot[\varphi\theta(\tilde{x}\cdot\bar{\theta})] + \varphi'|\tilde{x}\cdot\theta|^2 + \varphi\theta\cdot\overline{\nabla_b(\tilde{x}\cdot\theta)},$$

and substitute the identities

$$\begin{aligned}-\operatorname{Re}\{\varphi\nabla\cdot\theta(\tilde{x}\cdot\bar{\theta})\} = \operatorname{Re}\Big[&-\nabla\cdot\{\varphi\theta(\tilde{x}\cdot\bar{\theta})\} + \varphi'|\tilde{x}\cdot\theta|^2 + \frac{1}{2}\varphi(|\theta|^2 - |\tilde{x}\cdot\theta|^2)\\ &+ \frac{1}{2}\nabla\cdot\{\tilde{x}\varphi|\theta|^2\} - \frac{n-1}{2r}\varphi|\theta|^2 - \frac{1}{2}\varphi'|\theta|^2\\ &- i\varphi(\tilde{x}\times\theta)(\nabla\times b)\bar{u} + \varphi\Big(\frac{n-1}{2r} - i\kappa\Big)\{|\theta|^2 - |\tilde{x}\cdot\theta|^2\}\Big]\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Re}\varphi\Big(c_1(x) + \frac{(n-1)(n-3)}{4r^2}\Big)u(\tilde{x}\cdot\bar{\theta})\\ &\quad = \frac{1}{2}\nabla\cdot\Big\{\tilde{x}\varphi\Big(c_1 + \frac{(n-1)(n-3)}{4r^2}\Big)|u|^2\Big\} + \Big(\operatorname{Im}\kappa c_1 - \frac{\partial_r(\varphi c_1)}{2\varphi}\Big)|u|^2\\ &\qquad + \Big(\operatorname{Im}\kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi}\Big)\frac{(n-1)(n-3)}{4r^2}|u|^2\end{aligned}$$

in equation (5.2). Integrate both sides over Ω. Then since $\varphi = O(r)$, we have

$$-\liminf_{s\to\infty}\int_{S_s}\varphi\Big\{|\tilde{x}\cdot\theta|^2 - \frac{1}{2}|\theta|^2 - \frac{1}{2}\Big(c_1 + \frac{(n-1)(n-3)}{4r^2}\Big)|u|^2\Big\}\,dS = 0,$$

and the following proposition holds true.

Proposition 1. *The solution $u = R(\kappa^2)f$ satisfies*

$$\begin{aligned}
&-\int_{\partial\Omega}\varphi(\nu\cdot\tilde{x})\frac{1}{2}|\nu\cdot\nabla u|^2\,dS\\
&\quad+\int\varphi\Big[-\Big(\frac{1}{r}-\frac{\varphi'}{\varphi}\Big)|\tilde{x}\cdot\theta|^2 + \Big(\operatorname{Im}\kappa + \frac{1}{r} - \frac{\varphi'}{2\varphi}\Big)\Big\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\\
&\qquad+\Big(\operatorname{Im}\kappa c_1 - \frac{\partial_r(\varphi c_1)}{2\varphi}\Big)|u|^2 + \operatorname{Re}\{-(\nabla\times ib)u\overline{\tilde{x}\times\theta} + c_0 u\overline{\tilde{x}\cdot\theta}\}\Big]\,dx\\
&=\operatorname{Re}\int\varphi f\overline{\tilde{x}\cdot\theta}\,dx.
\end{aligned}$$

Lemma 5. *Under the additional conditions*

$$\frac{\varphi'(r)}{\varphi(r)} \le \frac{1}{r} \quad \text{and} \quad \partial_r(\varphi c_1)(x) \le 0 \tag{5.3}$$

on φ, the solution u satisfies the inequality

$$\begin{aligned}
&\int\Big\{\Big(\operatorname{Im}\kappa\varphi + \frac{\varphi'}{2}\Big)|\theta|^2 + \Big(\operatorname{Im}\kappa\varphi + \frac{\varphi}{r} - \frac{\varphi'}{2}\Big)\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx\\
&\le \int\varphi\{|f| + (|\nabla\times b|^2 + |c_0|^2)^{1/2}|u|\}|\theta|\,dx + \int_{\partial\Omega}\frac{\varphi}{2}(\nu\cdot\tilde{x})|\nu\cdot\nabla u|^2\,dS.
\end{aligned}\tag{5.4}$$

Proof. The lemma is obvious from Proposition 1 if we take note of the inequalities

$$\begin{gathered}
\Big(\frac{1}{r}-\frac{\varphi'}{\varphi}\Big)\{|\theta|^2 - |\tilde{x}\cdot\theta|^2\} \ge 0, \quad \operatorname{Im}\kappa\varphi c_1 - \frac{\partial_r(\varphi c_1)}{2} \ge 0,\\
|-(\nabla\times ib)u\overline{\tilde{x}\times\theta} + c_0 u\overline{\tilde{x}\cdot\theta}| \le (|\nabla\times b|^2 + |c_0|^2)^{1/2}|u|\,|\theta|.
\end{gathered}$$

□

Lemma 6. *For any $\varepsilon > 0$,*

$$\begin{aligned}
&\int\Big(\operatorname{Im}\kappa r + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2}\Big)\Big\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx\\
&\quad\le \frac{1}{4\varepsilon}\int[r]^2|f|^2\,dx + \Big(\frac{\varepsilon_0^2}{\varepsilon} - 2\varepsilon(n-1)(n-3)\Big)\int\frac{1}{4[r]^2}|u|^2\,dx\\
&\qquad+\int_{\partial\Omega}\frac{r}{2}(\nu\cdot\tilde{x})|\nu\cdot\nabla u|^2\,dS.
\end{aligned}$$

Proof. We choose $\varphi = r$ in the above lemma and use the Schwarz inequality. Then noting (BC.1), we have

$$\int \Big(\operatorname{Im}\kappa r + \frac{1}{2}\Big)\Big\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx \\ \leq \int \Big\{\frac{[r]^2}{4\varepsilon}|f|^2 + \frac{\varepsilon_0^2}{4\varepsilon[r]^2}|u|^2\Big\}\,dx + \int 2\varepsilon\frac{r^2}{[r]^2}|\theta|^2\,dx.$$

Hence the desired inequality follows. □

Next, let $H^1_{b,0} = H^1_{b,0}(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^1_b} = \int \{|u(x)|^2 + |\nabla_b u(x)|^2\}\,dx. \tag{5.5}$$

Proposition 2. *Let $\eta = \eta(r)$ and $\zeta = \zeta(r)$ be smooth, positive functions of $r > 0$, and let s be chosen large. Then the following identity holds for each $u \in H^1_{b,0}$:*

$$\begin{aligned}\int_{\Omega_s} \zeta\Big\{|\tilde{x}\cdot\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx \\ = \int_{\Omega_s} \zeta|\tilde{x}\cdot\nabla_b u - i\kappa u - \eta u|^2\,dx \\ + \int_{S_s}\zeta\Big(\frac{n-1}{2r}+\eta\Big)|u|^2\,dS - \int_{\Omega_s}\zeta'\Big(\frac{n-1}{2r}+\eta\Big)|u|^2\,dx \\ + \int_{\Omega_s}\zeta\Big\{2\operatorname{Im}\kappa\Big(\frac{n-1}{2r}+\eta\Big)|u|^2 - \Big(\frac{n-1}{r}\eta + \eta' + \eta^2\Big)|u|^2\Big\}\,dx.\end{aligned}$$

Proof. Note the identity

$$\begin{aligned}|\tilde{x}\cdot\theta|^2 &= \Big|\tilde{x}\cdot\nabla_b u + \frac{n-1}{2r}u - i\kappa u - \eta u + \eta u\Big|^2 \\ &= |\tilde{x}\cdot\nabla_b u - i\kappa u - \eta u|^2 + \nabla\cdot\Big\{\tilde{x}\Big(\frac{n-1}{2r}+\eta\Big)|u|^2\Big\} \\ &\quad + 2\operatorname{Im}\kappa\Big(\frac{n-1}{2r}+\eta\Big)|u|^2 - \frac{(n-1)(n-3)}{4r^2}|u|^2 - \Big(\frac{n-1}{r}\eta+\eta'+\eta^2\Big)|u|^2.\end{aligned}$$

Multiply by $\zeta(r)$ on both sides and integrate over Ω_s. Then since $u|_{\partial\Omega} = 0$, we conclude the desired identity. □

Lemma 7. *The following statements hold:*
(i) *If $n \geq 3$, then for any $u \in H^1_{b,0}(\Omega)$,*

$$\int \frac{1}{4r^2}|u|^2\,dx \leq \int |\tilde{x}\cdot\theta|^2\,dx.$$

(ii) *If* $n = 2$*, then for any* $u \in H^1_{b,0}(\Omega)$ *and* $\varepsilon > 0$*,*

$$\int \Big(\operatorname{Im}\kappa r + \frac{1}{2} - 18\varepsilon - 8\varepsilon^2\Big)\frac{1}{4[r]^2}|u|^2\,dx \\ \le \int \Big(\operatorname{Im}\kappa r + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2}\Big)\Big\{|\tilde{x}\cdot\theta|^2 - \frac{1}{4r^2}|u|^2\Big\}\,dx.$$

Proof. (i) We choose $\zeta \equiv 1$ and $\eta = -\frac{n-2}{2r}$ in Proposition 2. Then since

$$\frac{n-1}{2r} + \eta = \frac{1}{2r}, \qquad \frac{n-1}{r}\eta + \eta' + \eta^2 = -\frac{(n-2)^2}{4r^2},$$

letting $s \to \infty$, we have the assertion.

(ii) We choose

$$\zeta = \operatorname{Im}\kappa r + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2} \quad \text{and} \quad \eta = \frac{1}{2[r]}$$

in Proposition 2. Then, by assumption, $\zeta(r) > 0$ and also

$$\liminf_{s\to\infty}\int_{S_s}\zeta\Big(\frac{1}{2r} + \eta\Big)|u|^2\,dS = 0.$$

Moreover, since

$$\frac{1}{r}\eta + \eta' + \eta^2 = \frac{-1}{4[r]^2} = \frac{-1}{4r^2(1+\log r/r_0)^2},$$

it follows that

$$\int \Big(\operatorname{Im}\kappa r + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2}\Big)\Big\{|\tilde{x}\cdot\theta|^2 - \frac{1}{4r^2}|u|^2\Big\}\,dx \\ \ge -\int\Big(\operatorname{Im}\kappa - 2\varepsilon\Big(\frac{r^2}{[r]^2}\Big)'\Big)\Big(\frac{1}{2r} + \eta\Big)|u|^2\,dx \\ + \int\Big(\operatorname{Im}\kappa r + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2}\Big)\Big\{2\operatorname{Im}\kappa\Big(\frac{1}{2r} + \eta\Big)|u|^2 + \frac{1}{4[r]^2}|u|^2\Big\}\,dx.$$

Thus the inequalities

$$2(\operatorname{Im}\kappa)^2 r - 4\varepsilon\operatorname{Im}\kappa\frac{r^2}{[r]^2} \ge -\frac{2\varepsilon^2}{r}\frac{r^4}{[r]^4}, \\ -\Big\{\frac{2\varepsilon^2}{r}\frac{r^4}{[r]^4} + 2\varepsilon\Big(\frac{r^2}{[r]^2}\Big)'\Big\}\Big(\frac{1}{2r} + \eta\Big) \ge -\frac{8(\varepsilon^2 + 2\varepsilon)}{4[r]^2}$$

lead us to the desired conclusion. □

Proof of Theorem 1(i). The case $n \geq 3$: In the inequality of Lemma 6 we choose $\varepsilon < \frac{1}{4}$ and apply Lemma 7(i). Then

$$\int \Big(\frac{1}{2} - 2\varepsilon\Big) \frac{1}{4r^2}|u|^2\,dx + \int \frac{(n-1)(n-3)}{8r^2}|u|^2\,dx \\ \leq \frac{1}{4\varepsilon}\int r^2|f|^2\,dx + \frac{\varepsilon_0^2}{\varepsilon}\int \frac{1}{4r^2}|u|^2\,dx - P_{\partial\Omega}(u),$$

where $P_{\partial\Omega}(u) = -\frac{1}{2}\int_{\partial\Omega}(\nu\cdot x)|\nu\cdot\nabla u|^2\,dS$. Hence

$$\frac{(n-2)^2\varepsilon - 4\varepsilon^2 - 2\varepsilon_0^2}{2\varepsilon}\int \frac{1}{4r^2}|u|^2\,dx \leq \frac{1}{4\varepsilon}\int r^2|f|^2\,dx - P_{\partial\Omega}(u),$$

and the desired inequality holds if ε_0 in (BC.1) is sufficiently small.

The case $n = 2$: We combine Lemmas 6 and 7(ii) to obtain

$$\int \Big(\operatorname{Im}\kappa r + \frac{1}{2} - 18\varepsilon - 8\varepsilon^2\Big)\frac{1}{4[r]^2}|u|^2\,dx \\ \leq \int \Big(\operatorname{Im}\kappa + \frac{1}{2} - 2\varepsilon\frac{r^2}{[r]^2}\Big)\Big\{|\tilde{x}\cdot\theta|^2 - \frac{1}{4r^2}|u|^2\Big\}\,dx \\ \leq \frac{1}{4\varepsilon}\int [r]^2|f|^2\,dx + \Big(\frac{\varepsilon_0^2}{\varepsilon} + 2\varepsilon\Big)\int \frac{1}{4[r]^2}|u|^2\,dx - P_{\partial\Omega}(u)$$

for any $\varepsilon < \frac{1}{4}$, which implies

$$\frac{\varepsilon - 40\varepsilon^2 - 16\varepsilon^3 - 2\varepsilon_0^2}{2\varepsilon}\int \frac{1}{4[r]^2}|u|^2\,dx \leq \frac{1}{4\varepsilon}\int [r]^2|f|^2\,dx - P_{\partial\Omega}(u).$$

The desired inequality then holds if ε_0 is sufficiently small. □

To proceed with the proof of Theorem 1(ii) we need one more proposition.

We multiply by $-\overline{i\kappa u}$ on both sides of (2.3) to obtain

$$\nabla\cdot\{(\nabla_b u)\overline{i\kappa u}\} - \overline{i\kappa}\{|\nabla_b u|^2 + c(x)|u|^2 - \kappa^2|u|^2\} = -f\overline{i\kappa u},$$

Integrate the real part of this equation over Ω_t $(t > r_0)$. By means of the boundary condition $u|_{\partial\Omega} = 0$, it then follows that

$$\frac{1}{2}\int_{S_t}\{-|\nabla_b u - i\kappa u|^2 + |\nabla_b u|^2 + |\kappa u|^2\}\,dS \\ + \operatorname{Im}\kappa\int_{\Omega_t}(|\nabla_b u|^2 + c|u|^2 + |\kappa u|^2)\,dx = -\operatorname{Re}\int_{\Omega_t} f\overline{i\kappa u}\,dx. \tag{5.6}$$

Here,

$$
\begin{aligned}
|\nabla_b u - i\kappa\tilde{x}u|^2 &= \Big|\nabla_b u + \tilde{x}\Big(\frac{n-1}{2r} - i\kappa\Big)u\Big|^2 \\
&\quad + \frac{(n-1)(n-3)}{4r^2}|u|^2 - \operatorname{Im}\kappa\frac{n-1}{r}|u|^2 - \nabla\cdot\Big\{\frac{n-1}{2r}\tilde{x}|u|^2\Big\}.
\end{aligned} \tag{5.7}
$$

Then the following proposition is a direct result of (5.6) multiplied by $\mu(t)$ and integrated over (r_0, ∞).

Proposition 3. *Let $u = R(\kappa^2)f$. Then we have*

$$
\begin{aligned}
&\frac{1}{2}\int\Big\{\Big(\mu\operatorname{Im}\kappa\frac{n-1}{r} - \mu'\frac{n-1}{2r}\Big)|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2)\Big\}\,dx \\
&\qquad + \operatorname{Im}\kappa\int_{r_0}^{\infty}\mu(t)\,dt\int_{\Omega_t}\{|\nabla_b u|^2 + c(x)|u|^2 + |\kappa u|^2\}\,dx \\
&\quad = \frac{1}{2}\int\mu\Big\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx - \operatorname{Re}\int_{r_0}^{\infty}\mu(t)\,dt\int_{\Omega_t} f\overline{i\kappa u}\,dx.
\end{aligned}
$$

Now we return to the inequality of Lemma 5. The Schwarz inequality then implies

$$
\begin{aligned}
&\int\Big\{\Big(\operatorname{Im}\kappa\varphi + \frac{\varphi'}{2} - 2\varepsilon\varphi'\Big)|\theta|^2 + \Big(\operatorname{Im}\kappa\varphi + \frac{\varphi}{r} - \frac{\varphi'}{2}\Big)\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx \\
&\qquad \le \int\frac{\varphi^2}{4\varepsilon\varphi'}|f|^2\,dx + \int\frac{(|\nabla\times b|^2 + |c_0|^2)\varphi^2}{4\varepsilon\varphi'}|u|^2\,dx
\end{aligned} \tag{5.8}
$$

for any $\varepsilon > 0$.

Lemma 8. *The inequality*

$$
\int\mu\Big\{|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx \le C\int\max\{[r]^2, \mu^{-1}\}|f|^2\,dx
$$

holds for some $C = C(\varepsilon_0, \|\mu\|_1) > 0$.

Proof. In (5.8) we fix $\varepsilon < \frac{1}{8}$. In the case $n \ge 3$ we choose $\varphi(r) = \int_0^r \mu(\tau)\,d\tau$. Since $r\mu \le \varphi \le \|\mu\|_{L^1}$, by use of (BC.1) we have

$$
\begin{aligned}
&\int\Big\{\frac{1-4\varepsilon}{2}\mu|\theta|^2 + \frac{1}{2}\mu\frac{(n-1)(n-3)}{4r^2}|u|^2\Big\}\,dx \\
&\qquad \le \|\mu\|_{L^1}^2\Big\{\int\frac{\mu^{-1}}{4\varepsilon}|f|^2\,dx + \int\frac{\varepsilon_0^2[r]^{-2}}{4\varepsilon}|u|^2\,dx\Big\}.
\end{aligned}
$$

Hence, the use of Theorem 1(i) leads to the assertion.

In the case $n = 2$ we choose $\varphi = r/(4 + \log r/r_0)^2$ in (5.8). Then since

$$\varphi' = \frac{1}{(4+\log r/r_0)^2} - \frac{2}{(4+\log r/r_0)^3} \geq \frac{1}{2(4+\log r/r_0)^2},$$
$$\frac{\varphi}{r} - \frac{\varphi'}{2} \leq \frac{3}{4(4+\log r/r_0)^2} \quad \text{and} \quad \frac{\varphi^2}{\varphi'} \leq \frac{2r^2}{(4+\log r/r_0)^2},$$

it follows that

$$\int \frac{1-4\varepsilon}{4(4+\log r/r_0)^2}|\theta|^2\,dx - \int\Big\{\operatorname{Im}\kappa r + \frac{3}{4}\Big\}\frac{1}{4r^2(4+\log r/r_0)^2}|u|^2\,dx$$
$$\leq \int \frac{r^2}{2\varepsilon(4+\log r/r_0)^2}|f|^2\,dx + \int \frac{\varepsilon_0^2 r^2}{2\varepsilon[r]^4(4+\log r/r_0)^2}|u|^2\,dx$$

for any $\varepsilon > 0$. Hence we have

$$\int \frac{1-4\varepsilon}{4(4+\log r/r_0)^2}|\theta|^2\,dx \leq \frac{1}{32\varepsilon}\int [r]^2|f|^2\,dx + \int\Big\{\operatorname{Im}\kappa r + \frac{3}{4} + \frac{\varepsilon_0^2}{8\varepsilon}\Big\}\frac{|u|^2}{4[r]^2}\,dx.$$

The use of Theorem 1(i) leads to the assertion if we note

$$\mu \leq \frac{1}{(4+\log r/r_0)^2} \quad \text{and} \quad \frac{(n-1)(n-3)}{4r^2} = \frac{-1}{4r^2} \leq 0$$

in this case. □

Lemma 9. *For each $u \in H^1_{b,0}$ and $s > r_0$ we have*

$$\int_{\Omega_s} \frac{[n-2]^2}{4[r]^2}|u|^2\,dx \leq \int_{\Omega_s} |\tilde{x}\cdot\nabla_b u|^2\,dx.$$

Proof. In the identity

$$|\tilde{x}\cdot\nabla_b u|^2 = \Big|\tilde{x}\cdot\nabla_b u + \frac{[n-2]u}{2[r]}\Big|^2 - \frac{[n-2]^2|u|^2}{4[r]^2} - 2\operatorname{Re}\Big\{\tilde{x}\cdot\nabla_b u\frac{[n-2]\bar{u}}{2[r]}\Big\},$$

the last term on the right-hand side is rewritten as

$$-\nabla\cdot\Big\{\tilde{x}\frac{[n-2]|u|^2}{2[r]}\Big\} + \frac{[n-2]^2|u|^2}{2[r]^2}.$$

Integrate this equation over Ω_t. Then the assertion follows from the identity

$$\int_{\Omega_s} |\tilde{x}\cdot\nabla_b u|^2\,dx = \int_{\Omega_s}\Big|\tilde{x}\cdot\nabla_b u - \frac{[n-2]u}{2[r]}\Big|^2\,dx + \int_{S_s}\frac{[n-2]|u|^2}{2[r]}\,dS$$
$$+ \int_{\Omega_s}\frac{[n-2]^2|u|^2}{4[r]^2}\,dx.$$

□

Proof of Theorem 1(ii). We start from the identity of Proposition 3. By (BC.1),

$$c(x) \geq -\frac{[n-2]^2}{4[r]^2}.$$

Then we have from Lemma 9,

$$\int_{\Omega_r} \{|\nabla_b u|^2 + c(x)|u|^2 + |\kappa u|^2\}\, dx \geq 0,$$

and the following inequality holds:

$$\begin{aligned} &\frac{1}{2}\int \Big\{ \Big(\mu \operatorname{Im} \kappa \frac{n-1}{r} - \mu' \frac{n-1}{2r} \Big) |u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2) \Big\}\, dx \\ &\leq \frac{1}{2}\int \mu \Big\{ |\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2 \Big\}\, dx + \int_0^\infty \mu(t)\, dt \int_{\Omega_t} |f(x)|\, |i\kappa u|\, dx. \end{aligned} \tag{5.9}$$

Here $\mu' \leq 0$ by assumption and we have from the Schwarz inequality,

$$\|\mu\|_{L^1} \int |f|\, |i\kappa u|\, dx \leq \|\mu\|_{L^1}^2 \int \mu^{-1}|f|^2\, dx + \frac{1}{4}\int \mu|\kappa u|^2\, dx.$$

Thus, the assertion is concluded from (5.9) and Lemma 8. □

Acknowledgments

The authors would like to express their gratitude to the referees for carefully reading the manuscript and making valuable suggestions.

The second author was partially supported by a Grant-in-Aid for Scientific Research (C) (no. 16k05248), Japan Society for Promotion of Science.

References

[1] J. A. Barcelo, A. Ruiz and L. Vega, Weighted estimates for the Helmholtz equation and some applications, J. Funct. Anal. **150** (1997), 356–382. Zbl 0890.35028 MR 1479544

[2] D. M. Èĭdus, The principle of limiting amplitude, Uspehi Mat. Nauk **24** (1969), 91–156. Zbl 0197.08102 MR 0601072

[3] W. Jäger and P. Rejto, On a theorem of Mochizuki and Uchiyama about long range oscillating potentials. I, in *Operator theory and its applications (Winnipeg, MB, 1998)*, Fields Institute Commununications 25, American Mathematical Society, Providence, RI, 2000, 305–329. Zbl 1008.35011 MR 1759550

[4] O. A. Ladyženskaya, On the principle of limit amplitude., Uspehi Mat. Nauk (N.S.) **12** (1957), 161–164. Zbl 0078.27902 MR 0090401

[5] P. D. Lax and R. S. Phillips, *Scattering theory*, Pure and Applied Mathematics 26, Academic Press, New York-London, 1967. Zbl 0186.16301 MR 0217440

[6] S. Mizohata and K. Mochizuki, On the principle of limiting amplitude for dissipative wave equations, J. Math. Kyoto Univ. **6** (1966), 109–127. Zbl 0173.37102 MR 0212346

[7] K. Mochizuki, The principle of limiting amplitude for symmetric hyperbolic systems in an exterior domain, Publ. Res. Inst. Math. Sci. **5** (1969), 259–265. Zbl 0206.11001 MR 0437954

[8] K. Mochizuki, On the spectrum of Schrödinger operators with oscillating long-range potentials, in *More progresses in analysis. Proc. 5th ISAAC Congress*, World Scientific, Hackensack, NJ, 2009, 533–542. Zbl 1183.35090

[9] K. Mochizuki, Uniform resolvent estimates for magnetic Schrödinger operators and smoothing effects for related evolution equations, Publ. Res. Inst. Math. Sci. **46** (2010), 741–754. Zbl 1203.35194 MR 2791005

[10] K. Mochizuki, *Spectral and scattering theory for second-order partial differential operators*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2017. Zbl 1377.35003 MR 3676925

[11] K. Mochizuki and H. Nakazawa, Uniform resolvent estimates for magnetic Schrödinger operators in a 2D exterior domain and their applications to related evolution equations, Publ. Res. Inst. Math. Sci. **51** (2015), 319–336. Zbl 1405.35176 MR 3348112

[12] K. Mochizuki and H. Nakazawa, Uniform resolvent estimates for stationary dissipative wave equations in an exterior domain and their application to the principle of limiting amplitude, in *New trends in analysis and interdisciplinary applications*, New Trends in Analysis and Interdisciplinary Applications, Birkhäuser/Springer, Cham, 2017, 521–527. Zbl 1383.35061 MR 3695683

[13] C. S. Morawetz, The limiting amplitude principle, Comm. Pure Appl. Math. **15** (1962), 349–361. Zbl 0196.41202 MR 0151712

[14] G. F. Roach and B. Zhang, The limiting-amplitude principle for the wave propagation problem with two unbounded media, Math. Proc. Cambridge Philos. Soc. **112** (1992), 207–223. Zbl 0841.35060 MR 1162945

[15] H. Tamura, Resolvent estimates at low frequencies and limiting amplitude principle for acoustic propagators, J. Math. Soc. Japan **41** (1989), 549–575. Zbl 0722.35060 MR 1013067

[16] B. R. Vaĭnberg, Principles of radiation, limiting absorption and limiting amplitude in the general theory of partial differential equations, Uspehi Mat. Nauk **21** (1966), 115–194. Zbl 0172.13703 MR 0213701