Publ. RIMS Kyoto Univ. **60** (2024), 629–649
DOI 10.4171/PRIMS/60-3-8

Multivector Fields on Quaternionic Kähler Manifolds

by

Takayuki Moriyama and Takashi Nitta

Abstract

In this paper we define a differential operator as a modified Dirac operator. Using the operator, we introduce a quaternionic k-vector field on a quaternionic Kähler manifold and show that any quaternionic k-vector field corresponds to a holomorphic k-vector field on the twistor space.

Mathematics Subject Classification 2020: 53C28 (primary); 53C26 (secondary).
Keywords: quaternionic Kähler manifolds, twistor methods.

§1. Introduction

Deformation quantization is constructed on any symplectic manifold [7, 8, 17]. Kontsevich generalized the construction to Poisson manifolds [13]. A Poisson structure is given by a 2-vector field whose Schouten bracket vanishes. In complex geometry, Hitchin studied holomorphic Poisson structures [10]. He showed that a holomorphic Poisson structure is deeply related to generalized Kähler manifolds. We constructed a family of real Poisson structures on S^4 from holomorphic Poisson structures on $\mathbb{C}P^3$ [15], where S^4 is a typical example of quaternionic Kähler manifolds and $\mathbb{C}P^3$ is the twistor space.

Let (M, g) be a quaternionic Kähler manifold, that is, a $4n$-dimensional Riemannian manifold whose holonomy group is reduced to a subgroup of $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. Let E and H denote the associated bundles with the canonical representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively. Then $TM \otimes \mathbb{C} = E \otimes_{\mathbb{C}} H$. Levi-Civita connection induces the covariant derivative $\nabla\colon \Gamma(\wedge^k E \otimes S^m H) \to$

Communicated by K. Ono. Received September 1, 2022. Revised February 7, 2023.

T. Moriyama: Department of Mathematics, Mie University, Kurimamachiya 1577, Tsu 514-8507, Japan;
e-mail: takayuki@edu.mie-u.ac.jp
T. Nitta: Department of Mathematics, Mie University, Kurimamachiya 1577, Tsu 514-8507, Japan;
e-mail: nitta@edu.mie-u.ac.jp

© 2024 Research Institute for Mathematical Sciences, Kyoto University.
This work is licensed under a CC BY 4.0 license.

$\Gamma(\wedge^k E \otimes S^m H \otimes E^* \otimes H^*)$. By the Clebsch–Gordan formula, the Dirac operator $\mathfrak{D}_{\wedge^k E}$ is defined as the $\wedge^k E \otimes E^* \otimes S^{m+1}H$-part of ∇. Baston considered a complex associated with the operator $\mathfrak{D}_{\wedge^0 E}$ (he used the notation D instead) and another operator F on a quaternionic manifold [4]. He proved that the cohomology corresponds to Dolbeault cohomology on the twistor space Z. Nagatomo and the second author provided a vanishing theorem of the cohomology on quaternionic Kähler manifolds [16]. A k-vector field contained in the kernel of $\mathfrak{D}_{\wedge^k E}$ is lifted to a holomorphic k-vector field on Z. However, any holomorphic k-vector field on Z does not correspond to such a k-vector field on M. We consider the trace map $\mathrm{tr}\colon \wedge^k E \otimes E^* \to \wedge^{k-1}E$ and define an operator $\mathfrak{D}^0_{\wedge^k E}$ as the traceless part of $\mathfrak{D}_{\wedge^k E}$. We remark that, in the case of $k = 2n$, the operator $\mathfrak{D}^0_{\wedge^{2n} E}$ vanishes.

Definition 1.1. A section X of $\wedge^k E \otimes S^k H$ is a *quaternionic k-vector field* on M if $\mathfrak{D}^0_{\wedge^k E}(X) = 0$ for $1 \le k \le 2n-1$ and $\mathfrak{D}_{\wedge^{2n-1}E} \circ \mathrm{tr} \circ \mathfrak{D}_{\wedge^{2n}E}(X) = 0$ for $k = 2n$.

A quaternionic 1-vector field is a vector field preserving the quaternionic structure. In [2, 6, 14], the authors studied quaternionic 1-vector fields and provided characterizations of $\mathbb{H}P^n$. A quaternionic k-vector field is a sort of generalization of such a vector field. In the case of positive scalar curvature, there are many quaternionic Kähler orbifolds [5, 9]. For this reason, we consider a sheaf of quaternionic k-vector fields. Let $\mathcal{Q}(\wedge^k E \otimes S^k H)$ be the sheaf of quaternionic k-vector fields on M and $\widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$ that of holomorphic $(k,0)$-vector fields defined in the pull-back of open sets by the projection from Z to M. The main theorem is the following:

Theorem 1.2. *The sheaf $\mathcal{Q}(\wedge^k E \otimes S^k H)$ is isomorphic to $\widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$. In particular, any global quaternionic k-vector field on M corresponds to a global holomorphic $(k,0)$-vector field on Z.*

The Schouten–Nijenhuis bracket induces graded Lie algebra structures on $\bigoplus_k \mathcal{Q}(\wedge^k E \otimes S^k H)$ and $\bigoplus_k \widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$.

Theorem 1.3. *The isomorphism $\mathcal{Q}(\wedge^k E \otimes S^k H) \cong \widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$ preserves the structures of graded Lie algebras. In particular, the space of global quaternionic k-vector fields on M is isomorphic to that of global holomorphic $(k,0)$-vector fields on Z as graded Lie algebras.*

The space $\mathcal{Q}(\wedge^k E \otimes S^k H)$ admits a real structure τ. A τ-invariant element of $\mathcal{Q}(\wedge^k E \otimes S^k H)$ is a real k-vector field on M. We also have a real structure $\hat{\tau}$ on $\widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$. Let $\mathcal{Q}(\wedge^k E \otimes S^k H)^{\tau}$ be the sheaf of quaternionic real k-vector fields and $\widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)^{\hat{\tau}}$ that of $\hat{\tau}$-invariant elements of $\widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$. Graded Lie algebra structures are induced in those sheaves.

Theorem 1.4. *The sheaf $\mathcal{Q}(\wedge^k E \otimes S^k H)^\tau$ is isomorphic to $\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)^{\hat\tau}$. The isomorphism preserves the structures of graded Lie algebras. In particular, the space of global quaternionic real k-vector fields on M is isomorphic to that of global holomorphic and $\hat\tau$-invariant $(k,0)$-vector fields on Z as graded Lie algebras.*

§2. Preliminaries

§2.1. Quaternionic Kähler manifolds

Let (M, g) be a Riemannian manifold of dimension $4n$. A subbundle Q of $\mathrm{End}(TM)$ is called an *almost quaternionic structure* if there exists a local basis I, J, K of Q such that $I^2 = J^2 = K^2 = -\mathrm{id}$ and $K = IJ$. A pair (Q, g) is an *almost quaternionic Hermitian structure* if any section φ of Q satisfies $g(\varphi X, Y) + g(X, \varphi Y) = 0$ for $X, Y \in TM$. For $n \geq 2$, if the Levi-Civita connection ∇ preserves Q, then (Q, g) is called a *quaternionic Kähler structure*, and (M, Q, g) a *quaternionic Kähler manifold.* A Riemannian manifold is a quaternionic Kähler manifold if and only if the holonomy group is reduced to a subgroup of $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$. Alekseevskii [1] shows that a quaternionic Kähler manifold is Einstein and the curvature of Q is described by the scalar curvature (we also refer to [11, 18]). For $n = 1$, since $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)$ is $\mathrm{SO}(4)$, a manifold satisfying the above condition is just an oriented Riemannian manifold. A 4-dimensional oriented Riemannian manifold M is said to be a *quaternionic Kähler manifold* if it is Einstein and self-dual.

The symplectic group $\mathrm{Sp}(n)$ acts on the right $\mathbb{H}$-module $\mathbb{H}^n$ by $A\xi$ for $A \in \mathrm{Sp}(n)$ and $\xi \in \mathbb{H}^n$. On the other hand, $\mathrm{Sp}(1)$ has an action on the left $\mathbb{H}$-module $\mathbb{H}$ by $\xi\bar{q}$ for $q \in \mathrm{Sp}(1)$ and $\xi \in \mathbb{H}$. Let E, H denote the associated bundles with the representations $\mathrm{Sp}(n)$, $\mathrm{Sp}(1)$ on $\mathbb{H}^n$, $\mathbb{H}$, respectively. Then E is the right $\mathbb{H}$-module bundle and H is the left $\mathbb{H}$-module bundle. The dual representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ induce the left $\mathbb{H}$-module bundle E^* and the right $\mathbb{H}$-module bundle H^*. Then $TM = E \otimes_{\mathbb{H}} H$ and $T^*M = H^* \otimes_{\mathbb{H}} E^*$. The $\mathbb{H}$-bundles E, H are regarded as the $\mathbb{C}$-vector bundles with anti $\mathbb{C}$-linear maps J_E, J_H satisfying $J_E^2 = -\mathrm{id}_E$, $J_H^2 = -\mathrm{id}_H$. Then there exist symplectic structures ω_E, ω_H on E, H which are compatible with J_E, J_H, respectively. The correspondences $e \mapsto \omega_E(\cdot, e)$, $h \mapsto \omega_H(\cdot, h)$ provide the $\mathbb{C}$-isomorphisms $E \cong E^*$, $H \cong H^*$, which are denoted by $\omega_E^\sharp$, $\omega_H^\sharp$. The tangent space TM is the real form of $E \otimes_{\mathbb{C}} H$ with respect to the real structure $J_E \otimes J_H$:

$$TM \otimes \mathbb{C} = E \otimes_{\mathbb{C}} H.$$

The tensor product $\omega_E \otimes \omega_H$ is the complexification of the Riemannian metric g. The technique is called *EH-formalism* and was introduced by Salamon [18].

§2.2. The twistor space

The quaternionic structure Q is considered as a subbundle of the real vector bundle $\mathrm{End}_{\mathbb{H}}(H)$. We identify $\mathrm{End}_{\mathbb{H}}(H)$ with the real form of $\mathrm{End}_{\mathbb{C}}(H) = H \otimes_{\mathbb{C}} H^*$. Let u be an $\mathbb{H}$-frame of H. We define local sections I, J, K of $\mathrm{End}_{\mathbb{H}}(H)$ as $I(hu) = hiu$, $J(hu) = hju$, $K(hu) = hku$ for any $h \in \mathbb{H}$. Then $\{I, J, K\}$ is a local basis of Q and represented by elements

$$(1)\quad I = i(u\otimes u^* - ju\otimes(ju)^*),\quad J = ju\otimes u^* - u\otimes(ju)^*,\quad K = i(ju\otimes u^* + u\otimes(ju)^*)$$

of $\mathrm{End}_{\mathbb{C}}(H)$ for the $\mathbb{C}$-frame $\{u, ju\}$ of H. Let Z be a sphere bundle

$$Z = \left\{ aI + bJ + cK \in Q \;\middle|\; a^2 + b^2 + c^2 = 1 \right\}$$

over M. Let $f\colon Z \to M$ denote the projection. The bundle Z is called a *twistor space* of the quaternionic Kähler manifold M.

§2.3. The principal bundle $P(H^*)$

Let $p\colon P(H^*) \to M$ be a frame bundle of H^*, whose fiber consists of right $\mathbb{H}$-bases of H^*. Then $P(H^*)$ is a principal $\mathrm{GL}(1, \mathbb{H})$-bundle by the right action. An element u^* of $P(H^*)$ induces the complex structure I in (1) by

$$\wedge^{1,0} T^*_x M = E^*_x \otimes \langle u^* \rangle_{\mathbb{C}}, \quad \wedge^{0,1} T^*_x M = E^*_x \otimes \langle u^* j \rangle_{\mathbb{C}}.$$

We identify each fiber of p with $\mathbb{C}^2 \backslash \{0\}$ by $\mathbb{H} = \mathbb{C} + j\mathbb{C} \cong \mathbb{C}^2$. Thus we have an almost complex structure $\tilde{I}$ on $P(H^*)$. Then $\tilde{I}$ is integrable (cf. [3, Thm. 4.1], [18, Thm. 4.1]). The twistor space Z is regarded as the quotient space $P(H^*)/\mathrm{GL}(1, \mathbb{C})$. We denote by $\pi\colon P(H^*) \to Z$ the quotient map. By the definition, the twistor space Z is a $\mathbb{C}P^1$-bundle over M. A complex structure $\hat{I}$ on Z is induced by $\tilde{I}$.

§3. Lifts of sections of $\wedge^k E \otimes S^m H$ to $P(H^*)$ and Z

We denote by $\mathcal{A}^q$, $\mathcal{A}^q_{P(H^*)}$ and $\mathcal{A}^q_Z$ the sheaves of smooth q-forms on M, $P(H^*)$ and Z, respectively.

§3.1. Lift of $\mathcal{A}^q(\wedge^k E \otimes S^m H)$ to $P(H^*)$

The bundles H and H^* are regarded as bundles of the left $\mathbb{C}$-module and the right $\mathbb{C}$-module, respectively. We denote the complex representation ρ of $\mathrm{GL}(1, \mathbb{H})$ on $\mathbb{H}$ by $\rho(a)h = ah$ for $a \in \mathrm{GL}(1, \mathbb{H})$ and $h \in \mathbb{H}$. Then $S^m H$ is the associated bundle $P(H^*) \times_{\rho^*} S^m \mathbb{H}$ with the dual representation ρ^*. The point $u^* \in P(H^*)$ corresponds to a point u of $P(H)$ by the $\mathbb{H}$-dual. The $\mathbb{H}$-basis u provides the $\mathbb{C}$-basis $\{u, ju\}$ of the $\mathbb{C}$-vector bundle H. Thus, any element u of $P(H)$ is regarded as a $\mathbb{C}$-isomorphism $u\colon \mathbb{H} \to H_{p(u)}$. An element $\xi \in \mathcal{A}^q(\wedge^k E \otimes S^m H)$ induces $\tilde{\xi} \in \mathcal{A}^q_{P(H^*)}(\wedge^k E \otimes S^m \mathbb{H})$ by $\tilde{\xi}_{u^*} = u^{-1}(p^*\xi)_{u^*}$ at each point $u^* \in P(H^*)$. Then $(R_a)^*\tilde{\xi} = \rho^*(a^{-1})\tilde{\xi}$ for any $a \in \mathrm{GL}(1,\mathbb{H})$. We define a sheaf $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$ by

$$\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H}) = \big\{\tilde{\xi} \in p^{-1}p_*(p^*\mathcal{A}^q(\wedge^k E \otimes S^m \mathbb{H})) \;\big|\; (R_a)^*\tilde{\xi} = \rho^*(a^{-1})\tilde{\xi},\ \forall\, a \in \mathrm{GL}(1,\mathbb{H})\big\},$$

where $p^{-1}p_*$ means the inverse image of the direct image of a sheaf by p. By the definition, $\tilde{\mathcal{A}}^q = \tilde{\mathcal{A}}^q(\wedge^0 E \otimes S^0 \mathbb{H})$ is the sheaf of pull-backs of smooth q-forms on M by p. In particular, $\tilde{\mathcal{A}}^0$ is the sheaf of smooth functions on $P(H^*)$ which are constant along each fiber. Then

$$\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H}) = \tilde{\mathcal{A}}^q(\wedge^k E) \otimes_{\tilde{\mathcal{A}}^0} \tilde{\mathcal{A}}^0(S^m \mathbb{H}).$$

The sheaf $\mathcal{A}^q(\wedge^k E \otimes S^m H)$ is isomorphic to $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$ by the correspondence $\xi \mapsto \tilde{\xi}$ (cf. [12, Chap. II, §5]). The Levi-Civita connection induces connections of E, H and the covariant exterior derivative $d^\nabla\colon \mathcal{A}^q(\wedge^k E \otimes S^m H) \to \mathcal{A}^{q+1}(\wedge^k E \otimes S^m H)$. Let $\widetilde{\mathcal{H}}$ be the horizontal subbundle of $TP(H^*)$. We define $d_{\widetilde{\mathcal{H}}}\colon \tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H}) \to \tilde{\mathcal{A}}^{q+1}(\wedge^k E \otimes S^m \mathbb{H})$ by the exterior derivative restricted to $\widetilde{\mathcal{H}}$. Then $\widetilde{d^\nabla \xi} = d_{\widetilde{\mathcal{H}}}\tilde{\xi}$.

We fix a point u_0^* of $P(H^*)$. The complex coordinate (z, w) of the fiber is given by $u_0^*(z + jw)$. A function f on $P(H^*)$ is *a polynomial of degree* $(m-i, i)$ *along the fiber* if $f(u_0^*(z+jw))$ is a polynomial of z, w, $\bar{z}$, $\overline{w}$ of degree m such that $(R_c)^*f = c^{m-i}\bar{c}^i f$ for $c \in \mathrm{GL}(1,\mathbb{C})$. We denote by $\tilde{\mathcal{A}}^0_{(m-i,i)}$ the sheaf of elements of $p^{-1}p_*\mathcal{A}^0_{P(H^*)}(\mathbb{C})$ which are polynomials of degree $(m-i, i)$ along the fiber on $P(H^*)$. We also define a sheaf $\tilde{\mathcal{A}}^q_{(m-i,i)}(\wedge^k E)$ as

$$\tilde{\mathcal{A}}^q_{(m-i,i)}(\wedge^k E) = \tilde{\mathcal{A}}^q(\wedge^k E) \otimes_{\tilde{\mathcal{A}}^0} \tilde{\mathcal{A}}^0_{(m-i,i)}.$$

Let $a_1 a_2 \cdots a_m$ denote the symmetrization $\frac{1}{m!}\sum_{\sigma \in S_m} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)}$ of $a_1 \otimes \cdots \otimes a_m \in \otimes^m \mathbb{H}$, where S_m is the symmetric group of degree m. The set $\{1^m, 1^{m-1}j, 1^{m-2}j^2, \ldots, j^m\}$ is a $\mathbb{C}$-basis of $S^m\mathbb{H}$. Any element $\tilde{\xi}$ of $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$ is written as

$$\tilde{\xi} = \tilde{\xi}_0 1^m + \tilde{\xi}_1 1^{m-1} j + \tilde{\xi}_2 1^{m-2} j^2 + \cdots + \tilde{\xi}_m j^m \tag{2}$$

for $p^{-1}(\wedge^k E)$-valued 1-forms $\tilde{\xi}_0, \ldots, \tilde{\xi}_m$. Each $\tilde{\xi}_i$ is in $\tilde{\mathcal{A}}^q_{(m-i,i)}(\wedge^k E)$. We obtain the following proposition:

Proposition 3.1. *There exist two isomorphisms:*

(i) $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H}) \cong \tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)$ *by* $\tilde{\xi} \mapsto \tilde{\xi}_0$. *Moreover,* $(d_{\widetilde{\mathcal{H}}}\tilde{\xi})_0 = d_{\widetilde{\mathcal{H}}}\tilde{\xi}_0$ *for any* $\tilde{\xi} \in \tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$.

(ii) $\mathcal{A}^q(\wedge^k E \otimes S^m H) \cong \tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)$ *by* $\xi \mapsto \tilde{\xi}_0$. *Moreover,* $(\widetilde{d^\nabla \xi})_0 = d_{\widetilde{\mathcal{H}}}\tilde{\xi}_0$ *for any* $\xi \in \mathcal{A}^q(\wedge^k E \otimes S^m H)$. □

For $\xi \in \mathcal{A}^q(\wedge^k E \otimes S^m H)$, the element $\tilde{\xi}_0 \in \tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)$ is said to be a *lift to* $P(H^*)$.

§3.2. Lift of $\mathcal{A}^q(\wedge^k E \otimes S^m H)$ to Z

We denote by l a line bundle over Z which is the hyperplane bundle on each fiber $\mathbb{C}P^1$ of f. We define a sheaf $\hat{\mathcal{A}}^0(l^m)$ by

$$\hat{\mathcal{A}}^0(l^m) = \{\zeta \in f^{-1}f_*(\mathcal{A}^0_Z(l^m)) \mid \zeta\colon \text{holomorphic along each fiber of } f\}.$$

We denote by $\hat{\mathcal{A}}^0$ the sheaf $\hat{\mathcal{A}}^0(l^0)$ of functions on Z which are constant along each fiber of f. Let $\hat{\mathcal{A}}^q(\wedge^k E)$ denote the sheaf of pull-backs of $\wedge^k E$-valued q-forms on M by f. We define a sheaf $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$ as

$$\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m) = \hat{\mathcal{A}}^q(\wedge^k E) \otimes_{\hat{\mathcal{A}}^0} \hat{\mathcal{A}}^0(l^m).$$

Any element $\tilde{\xi}_0$ of $\tilde{\mathcal{A}}^q_{(m,0)}$ defines an element of $\hat{\mathcal{A}}^q(l^m)$, which we denote by $\hat{\xi}$. Such an element $\hat{\xi}$ is called a *lift of* ξ *to* Z. The correspondence $\tilde{\xi}_0 \mapsto \hat{\xi}$ provides the isomorphism $\tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E) \cong \hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$. Proposition 3.1 implies the following proposition:

Proposition 3.2. *We have* $\mathcal{A}^q(\wedge^k E \otimes S^m H) \cong \hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$ *by* $\xi \mapsto \hat{\xi}$. □

§3.3. Real structures

We define an anti-$\mathbb{C}$-linear map $\tau\colon \mathcal{A}^q(\wedge^k E \otimes S^m H) \to \mathcal{A}^q(\wedge^k E \otimes S^m H)$ by

$$\tau(\xi) = \sum_i (J^k_E \otimes J^m_H)(v_i) \otimes \overline{\alpha^i}$$

for $\xi = \sum_i v_i \otimes \alpha^i$, where $\{v_i\}$ is a frame of $\wedge^k E \otimes S^m H$ and α^i is a q-form. We denote by $\mathcal{A}^q(\wedge^k E \otimes S^m H)^\tau$ the sheaf of τ-invariant elements of $\mathcal{A}^q(\wedge^k E \otimes S^m H)$. We define an anti-$\mathbb{C}$-linear endomorphism $\tilde{\tau}$ of $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$ by

$$\tilde{\tau}(\beta \otimes 1^{m-i} j^i) = J^k_E \overline{R^*_j \beta} \otimes 1^{m-i} j^i$$

for $\beta \in \tilde{\mathcal{A}}^q(\wedge^k E)$. It induces an endomorphism of $\tilde{\mathcal{A}}^q_{(m-i,i)}(\wedge^k E)$ such that $\tilde{\tau}(\tilde{\xi}) = \widetilde{\tau(\xi)}$ and $\tilde{\tau}(\tilde{\xi}_i) = \widetilde{\tau(\xi)}_i$ for $\xi \in \mathcal{A}^q(\wedge^k E \otimes S^m H)$. Under the representation (2), $\tilde{\xi}$ is $\tilde{\tau}$-invariant if and only if $\tilde{\xi}_i$ is $\tilde{\tau}$-invariant for each i, and $\tilde{\xi}_i = (-1)^{m-i} J_E \overline{\tilde{\xi}_{m-i}}$. Let $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})^{\tilde{\tau}}$ and $\tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)^{\tilde{\tau}}$ denote the sheaves of $\tilde{\tau}$-invariant elements of $\tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})$ and $\tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)$, respectively. Then we have the following proposition:

Proposition 3.3. *We have* $\mathcal{A}^q(\wedge^k E \otimes S^m H)^\tau \cong \tilde{\mathcal{A}}^q(\wedge^k E \otimes S^m \mathbb{H})^{\tilde{\tau}} \cong \tilde{\mathcal{A}}^q_{(m,0)}(\wedge^k E)^{\tilde{\tau}}$ *by* $\xi \mapsto \tilde{\xi} \mapsto \tilde{\xi}_0$. □

The action R_j on $P(H^*)$ induces an anti-holomorphic involution of Z, and we denote it by $R_{[j]}\colon Z \to Z$. An anti-$\mathbb{C}$ linear endomorphism $\hat{\tau}$ of $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$ is defined by

$$\hat{\tau}(\beta_Z) = J_E^k \overline{R^*_{[j]}\beta_Z}$$

for $\beta_Z \in \hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$. Let $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)^{\hat{\tau}}$ denote the sheaf of $\hat{\tau}$-invariant elements of $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$.

Proposition 3.4. *We have* $\mathcal{A}^q(\wedge^k E \otimes S^m H)^\tau \cong \hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)^{\hat{\tau}}$ *by* $\xi \mapsto \hat{\xi}$. □

If $k+m$ is even, then τ and $\hat{\tau}$ are real structures.

§4. Canonical 1-forms on $P(H^*)$ and Z

§4.1. Canonical 1-form on $P(H^*)$

We define a $p^{-1}(E) \otimes \mathbb{H}$-valued 1-form $\tilde{\theta}$ on $P(H^*)$ as

$$\tilde{\theta}_{u^*}(v) = u^{-1}(p_*(v))$$

for $v \in T_{u^*}P(H^*)$ at u^*. The 1-form $\tilde{\theta}$ is called *the canonical* 1*-form* on $P(H^*)$. We define $p^{-1}(E)$-valued 1-forms $\tilde{\theta}_0$ and $\tilde{\theta}_1$ on $P(H^*)$ as $\tilde{\theta} = \tilde{\theta}_0 + \tilde{\theta}_1 j$. Then $\tilde{\theta}_0 \in \tilde{\mathcal{A}}^1_{(1,0)}(E)$ and $\tilde{\theta}_1 \in \tilde{\mathcal{A}}^1_{(0,1)}(E)$ are $(1,0)$- and $(0,1)$-forms, respectively. Moreover, they are $\tilde{\tau}$-invariant, and $\tilde{\theta}_1 = J_E \overline{\tilde{\theta}_0}$. Let A denote the connection form of $P(H^*)$. Then A is written as $A = \eta_0 + j\eta_1$ for complex-valued 1-forms η_0, η_1 on $P(H^*)$. Then η_0 and η_1 are $\tilde{\tau}$-invariant $(1,0)$-forms. We have

$$d^E\tilde{\theta}_0 = -\tilde{\theta}_0 \wedge \eta_0 - \eta_1 \wedge \tilde{\theta}_1, \quad d^E\tilde{\theta}_1 = -\tilde{\theta}_0 \wedge \bar{\eta}_1 - \tilde{\theta}_1 \wedge \bar{\eta}_0. \tag{3}$$

Let s^2_H denote the symmetrization $\otimes^2 H \to S^2H$. We define an S^2H-valued 2-form ω on M as $\omega = \omega_E \otimes s^2_H$. The lift $\widetilde{\omega} \in \tilde{\mathcal{A}}^2_2(S^2\mathbb{H})$ is decomposed as $\widetilde{\omega} = \widetilde{\omega}_0\, 1\cdot 1 + \widetilde{\omega}_1\, 1 \cdot j + \widetilde{\omega}_2\, j \cdot j$ for $\widetilde{\omega}_0 \in \tilde{\mathcal{A}}^2_{(2,0)}$, $\widetilde{\omega}_1 \in \tilde{\mathcal{A}}^2_{(1,1)}$ and $\widetilde{\omega}_2 \in \tilde{\mathcal{A}}^2_{(0,2)}$. Then $\widetilde{\omega}_0$, $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ are $\tilde{\tau}$-invariant, $\widetilde{\omega}_2 = \overline{\widetilde{\omega}_0}$ and $\widetilde{\omega}_1 = -\overline{\widetilde{\omega}_1}$. Moreover,

$$\widetilde{\omega}_0 = \omega_E(\tilde{\theta}_0, \tilde{\theta}_0), \quad \widetilde{\omega}_1 = \omega_E(\tilde{\theta}_0, \tilde{\theta}_1) + \omega_E(\tilde{\theta}_1, \tilde{\theta}_0), \quad \widetilde{\omega}_2 = \omega_E(\tilde{\theta}_1, \tilde{\theta}_1).$$

The endomorphisms I, J, K in (1) induce almost complex structures on M, locally. We define local 2-forms ω_I, ω_J and ω_K on M by $\omega_I(X,Y) = g(IX,Y)$, $\omega_J(X,Y) = g(JX,Y)$ and $\omega_K(X,Y) = g(KX,Y)$ for $X, Y \in TM$. We define a function r on $P(H^*)$ by $r(u^*) = |u^*|$ for $u^* \in P(H^*)$, where $|\cdot|$ means the norm of H^*. Then $i\omega_I = -r^{-2}\widetilde{\omega}_1$ and $\omega_J - i\omega_K = -2r^{-2}\widetilde{\omega}_0$ on $P(H^*)$. We denote by t the scalar curvature of M. The curvature Ω of $P(H^*)$ is given by $\Omega = 2c_n t(i\otimes\omega_I + j\otimes\omega_J + k\otimes\omega_K)$ for a positive number c_n depending on n (cf. [1, 18]). Hence $\Omega = -2c_n tr^{-2}(\widetilde{\omega}_1 + 2j\widetilde{\omega}_0)$. From now on, we set $c = 2c_n t$. Then

$$d\eta_0 = -cr^{-2}\widetilde{\omega}_1 - \eta_1\wedge\bar{\eta}_1, \quad d\eta_1 = -2cr^{-2}\widetilde{\omega}_0 + \eta_0\wedge\eta_1 + \eta_1\wedge\bar{\eta}_0. \tag{4}$$

Equations (3) and (4) induce the integrability of $\tilde{I}$. It follows from $d(r^2\eta_1) = 2(-c\widetilde{\omega}_0 + r^2\eta_0\wedge\eta_1)$ that $r^2\eta_1$ is a holomorphic $(1,0)$-form on $P(H^*)$. If the scalar curvature t is not zero, then $d(r^2\eta_1)$ is a holomorphic symplectic form on $P(H^*)$. Complex structures $\tilde{J}$, $\widetilde{K}$ are provided by definitions similar to that of $\tilde{I}$. Then $(\tilde{I},\tilde{J},-\widetilde{K})$ is a hypercomplex structure on $P(H^*)$. If $t>0$, then $\tilde{g} = r^2(cp^*g + \eta_0\otimes\bar{\eta}_0 + \bar{\eta}_0\otimes\eta_0 + \eta_1\otimes\bar{\eta}_1 + \bar{\eta}_1\otimes\eta_1)$ is a hyperkähler metric. Then $-id(r^2\eta_0)$, $d(r^2\eta_1^{\mathrm{Re}})$, $d(r^2\eta_1^{\mathrm{Im}})$ are Kähler forms with respect to $\tilde{I}$, $\tilde{J}$, $-\widetilde{K}$, respectively. The hyperkähler structure $(\tilde{g},\tilde{I},\tilde{J},-\widetilde{K})$ induces that on $P(H^*)/\mathbb{Z}_2$. This coincides with the hyperkähler structure constructed by Swann [19].

§4.2. Derivatives of canonical forms

We take a torsion-free connection ∇ of $TP(H^*)$ preserving $\tilde{I}$. Let F be a holomorphic vector bundle on $P(H^*)$ and ∇_F a $(1,0)$-connection $\nabla_F\colon F\to F\otimes T^*$ of F. We consider the connection $\nabla_{F\otimes\wedge^q}$ of $F\otimes\wedge^q$ as the map $F\otimes\wedge^q\to F\otimes\wedge^q\otimes T^*$. Then the covariant exterior derivative d^{∇_F} is given by $(-1)^q\wedge\circ\nabla_{F\otimes\wedge^q}$. We remark that the operator $\bar{\partial}_F\colon F\otimes\wedge^{q,0}\to F\otimes\wedge^{q,1}$ satisfies $\bar{\partial}_F = (-1)^q\wedge\circ\nabla^{0,1}_{F\otimes\wedge^q}$. It follows from (3) and (4) that $\nabla^{0,1}_{E\otimes\wedge^1}\tilde{\theta}_0 = \eta_1\otimes\tilde{\theta}_1$, $\nabla^{0,1}\eta_0 = cr^{-2}\omega_E(\tilde{\theta}_0,\tilde{\theta}_1) + \eta_1\otimes\bar{\eta}_1$ and $\nabla^{0,1}\eta_1 = -\eta_1\otimes\bar{\eta}_0$. We define a $p^{-1}(\wedge^k E)$-valued $(k,0)$-form $\tilde{\theta}_0^k$ by the kth wedge $\sum_{i_1,\dots,i_k=1}^{2n} e_{i_1}\wedge\cdots\wedge e_{i_k}\otimes\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}$ of $\tilde{\theta}_0 = \sum_{i=1}^{2n} e_i\otimes\alpha_i$. It implies the following:

Proposition 4.1. *We have*

$$\begin{aligned}
\nabla^{0,1}\tilde{\theta}_0^k &= k\tilde{\theta}_0^{k-1}\wedge\eta_1\wedge_E\tilde{\theta}_1,\\
\nabla^{0,1}(\tilde{\theta}_0^{k-1}\wedge\eta_0) &= -(k-1)\tilde{\theta}_0^{k-2}\wedge\eta_0\wedge\eta_1\wedge_E\tilde{\theta}_1 + \tilde{\theta}_0^{k-1}\\
&\qquad\wedge(cr^{-2}\omega_E(\tilde{\theta}_0,\tilde{\theta}_1) + \eta_1\otimes\bar{\eta}_1),\\
\nabla^{0,1}(\tilde{\theta}_0^{k-1}\wedge\eta_1) &= -\tilde{\theta}_0^{k-1}\wedge\eta_1\otimes\bar{\eta}_0,\\
\nabla^{0,1}(\tilde{\theta}_0^{k-2}\wedge\eta_0\wedge\eta_1) &= -\tilde{\theta}_0^{k-2}\wedge\eta_1\wedge(cr^{-2}\omega_E(\tilde{\theta}_0,\tilde{\theta}_1) - \eta_0\otimes\bar{\eta}_0).
\end{aligned}$$
□

§4.3. Canonical 1-form on Z

The principal $\mathrm{GL}(1,\mathbb{C})$-bundle $\pi\colon P(H^*)\to Z$ is regarded as the frame bundle of l^*. We define $\hat{\theta}_0$ and $\hat{\theta}_1$ as the $f^{-1}(E)\otimes l$-valued $(1,0)$-form and the $f^{-1}(E)\otimes l^{-1}$-valued $(0,1)$-form on Z induced by $\tilde{\theta}_0$ and $r^{-2}\tilde{\theta}_1$, respectively. Let η and $\widehat{\omega}$ be the

l^2-valued $(1,0)$-form and the l^2-valued $(2,0)$-form on Z induced by $r^2\eta_1$ and $\widetilde{\omega}_0$, respectively. The forms $\hat{\theta}_0$, $\hat{\theta}_1$, η and $\widehat{\omega}$ are $\hat{\tau}$-invariant.

A connection of l is induced by η_0. Let d^l be the covariant exterior derivative. We obtain

$$d^l\hat{\theta}_0 = -\eta \wedge \hat{\theta}_1, \quad d^l\eta = -2c\widehat{\omega}. \tag{5}$$

If $t \neq 0$, then η is a holomorphic contact form on Z such that l^2 is the contact bundle. Let $g_{\widehat{\mathcal{V}}}$ be a real symmetric 2-form on Z such that $\pi^* g_{\widehat{\mathcal{V}}} = \eta_1 \otimes \bar{\eta}_1 + \bar{\eta}_1 \otimes \eta_1$. If $t > 0$, then $\hat{g} = cf^*g + g_{\widehat{\mathcal{V}}}$ is a Kähler–Einstein metric on Z with positive scalar curvature (cf. [18, Thms. 4.3, 6.1]).

Let ∇ be a torsion-free connection on Z such that $\nabla^{0,1} = \bar{\partial}$. Equation (5) implies that $\nabla^{0,1}\hat{\theta}_0 = \eta \otimes \hat{\theta}_1$ and $\nabla^{0,1}\eta = 0$. We define an $f^{-1}(\wedge^k E) \otimes l^k$-valued $(k,0)$-form $\hat{\theta}_0^k$ as the kth wedge of $\hat{\theta}_0$. Then we have the following proposition:

Proposition 4.2. *We have* $\nabla^{0,1}\hat{\theta}_0^k = k\hat{\theta}_0^{k-1} \wedge \eta \wedge_E \hat{\theta}_1$, *and* $\nabla^{0,1}(\hat{\theta}_0^{k-1} \wedge \eta) = 0$. □

§5. Holomorphic k-vector fields on $P(H^*)$ and Z

§5.1. Holomorphic k-vector fields on $P(H^*)$

Let $\hat{1}$, $\hat{i}$, $\hat{j}$, $\hat{k}$ be fundamental vector fields associated with the elements 1, i, j, k of Lie algebra $\mathrm{gl}(1,\mathbb{H}) = \mathbb{H}$, respectively. We define complex vector fields v_0 and v_1 as $v_0 = \frac{1}{2}(\hat{1} - i\hat{i})$ and $v_1 = \frac{1}{2}(\hat{j} + i\hat{k})$. Then $\{v_0, v_1\}$ is the dual basis of $\{\eta_0, \eta_1\}$. Let X' be a $(1,0)$-vector field on $P(H^*)$. Then X' is decomposed into

$$X' = X'_h + f_0 v_0 + f_1 v_1 \tag{6}$$

for a horizontal vector field X'_h and functions f_0, f_1 on $P(H^*)$.

Lemma 5.1. *The* $(1,0)$*-vector field* X' *is holomorphic if and only if*

(i) $\bar{\partial}(\tilde{\theta}_0(X'_h)) - f_1\tilde{\theta}_1 = 0$,

(ii) $\bar{\partial} f_0 = cr^{-2}\omega_E(\tilde{\theta}_0(X'_h), \tilde{\theta}_1) + f_1\bar{\eta}_1$

under the decomposition (6).

Proof. The vector field X' is holomorphic if and only if $\nabla^{0,1}X' = 0$. The equation is equal to $\tilde{\theta}_0(\nabla^{0,1}X') = 0$, $\eta_0(\nabla^{0,1}X') = 0$ and $\eta_1(\nabla^{0,1}X') = 0$. The first equation induces the third one since $\bar{\partial}^{\nabla}(\tilde{\theta}_0(\nabla^{0,1}X')) = \eta_1(\nabla^{0,1}X') \wedge \tilde{\theta}_1 + \tilde{\theta}_0(\Omega^{(0,2)}_{TP(H^*)}(X')) = \eta_1(\nabla^{0,1}X') \wedge \tilde{\theta}_1$ and the map $\wedge\tilde{\theta}_1 \colon \wedge^{0,1} \to p^{-1}(E) \otimes \wedge^{0,2}$ is injective. Proposition 4.1 implies that $\tilde{\theta}_0(\nabla^{0,1}X') = \bar{\partial}(\tilde{\theta}_0(X'_h)) - f_1\tilde{\theta}_1$ and $\eta_0(\nabla^{0,1}X') = \bar{\partial} f_0 - cr^{-2}\omega_E(\tilde{\theta}_0(X'_h), \tilde{\theta}_1) - f_1\bar{\eta}_1$. It turns out that $\nabla^{0,1}X' = 0$ is equivalent to conditions (i) and (ii). □

Let k be an integer which is greater than 1. Any $(k,0)$-vector X' is decomposed into

$$X' = X'_h + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1 \tag{7}$$

for $X'_h \in \wedge^k \widetilde{\mathcal{H}}^{1,0}$ and $Y_0, Y_1 \in \wedge^{k-1}\widetilde{\mathcal{H}}^{1,0}$ and $Z_0 \in \wedge^{k-2}\widetilde{\mathcal{H}}^{1,0}$. By a proof similar to Lemma 5.1, we obtain the following lemma:

Lemma 5.2. *For $2 \leq k \leq 2n$, the $(k,0)$-vector field X' is holomorphic if and only if*

(i) $\bar\partial(\tilde\theta_0^k(X'_h)) - k^2\tilde\theta_0^{k-1}(Y_1) \wedge_E \tilde\theta_1 = 0$,
(ii) $k^2\bar\partial(\tilde\theta_0^{k-1}(Y_0)) + k^2(k-1)^2\tilde\theta_0^{k-2}(Z_0)\wedge_E\tilde\theta_1 - cr^{-2}\omega_E(\tilde\theta_0^k(X'_h),\tilde\theta_1) - k^2\tilde\theta_0^{k-1}(Y_1)\otimes \bar\eta_1 = 0$,
(iii) $\bar\partial(\tilde\theta_0^{k-1}(Y_1)) + \tilde\theta_0^{k-1}(Y_1) \otimes \bar\eta_0 = 0$,
(iv) $(k-1)^2\bar\partial(\tilde\theta_0^{k-2}(Z_0)) + (k-1)^2\tilde\theta_0^{k-2}(Z_0)\otimes\bar\eta_0 - cr^{-2}\omega_E(\tilde\theta_0^{k-1}(Y_1),\tilde\theta_1) = 0$,

under the decomposition (7). *In particular, in the case $k \neq 2n$, X' is holomorphic if and only if equations* (i), (ii), (iv) *hold.* □

From now on, we extend the decomposition (7) to the case $k = 1$ as $Z_0 = 0$.

Theorem 5.3. *Horizontal k and $(k-1)$-vector fields X'_h, Y_1 satisfy, for $1 \leq k \leq 2n-1$,*

(i) $\bar\partial(\tilde\theta_0^k(X'_h)) - k^2\tilde\theta_0^{k-1}(Y_1) \wedge_E \tilde\theta_1 = 0$,

and for $k = 2n$, (i) *and*

$$\bar\partial(\tilde\theta_0^{2n-1}(r^2 Y_1)) = 0$$

if and only if the $(k,0)$-vector field $X'_h + Y_0\wedge v_0 + Y_1\wedge v_1 + Z_0\wedge v_0\wedge v_1$ is holomorphic for local horizontal $(k-1)$- and $(k-2)$-vector fields Y_0, Z_0 on $P(H^*)$.

Proof. By taking the derivative $\bar\partial$ on (i), we obtain $\bar\partial(r^2\tilde\theta_0^{k-1}(Y_1)) \wedge \tilde\theta_1 = 0$. Since $\wedge\tilde\theta_1 \colon p^{-1}(\wedge^{k-1}E)\otimes\wedge^{0,1} \to p^{-1}(\wedge^k E)\otimes\wedge^{0,2}$ is injective for $1 \leq k \leq 2n-1$, $\bar\partial(r^2\tilde\theta_0^{k-1}(Y_1)) = 0$. The equation is equal to (iii) in Lemma 5.2. It is easy to see that condition (iv) is equivalent to

$$\bar\partial((k-1)^2 r^2\tilde\theta_0^{k-2}(Z_0)) = c\omega_E(r^2\tilde\theta_0^{k-1}(Y_1), r^{-2}\tilde\theta_1). \tag{8}$$

The derivative $\bar\partial$ on the right-hand side of (8) vanishes. By Dolbeault's lemma, there exists an element $Z_0 \in \mathcal{A}^0_{P(H^*)}(\wedge^{k-2}\widetilde{\mathcal{H}}^{1,0})$ satisfying (8), and (iv). In the case $k \neq 1$, we write (ii) as

$$\begin{aligned}\bar\partial(\tilde\theta_0^{k-1}(Y_0)) = {}& k^{-2}c\omega_E(\tilde\theta_0^k(X'_h), r^{-2}\tilde\theta_1) + r^2\tilde\theta_0^{k-1}(Y_1)\otimes r^{-2}\bar\eta_1 \\ & - (k-1)^2 r^2\tilde\theta_0^{k-2}(Z_0)\wedge_E r^{-2}\tilde\theta_1.\end{aligned} \tag{9}$$

The derivative $\bar{\partial}$ on the right-hand side of (9) is provided by

$$cr^{-2}\{\wedge\big(\omega_E(\tilde{\theta}_0^{k-1}(Y_1)\wedge_E\tilde{\theta}_1,\tilde{\theta}_1)\big)-2\tilde{\theta}_0^{k-1}(Y_1)\otimes\omega_E(\tilde{\theta}_1,\tilde{\theta}_1)-\omega_E(\tilde{\theta}_0^{k-1}(Y_1),\tilde{\theta}_1)\wedge\tilde{\theta}_1\}.$$

Then it vanishes. In the case $k=1$, by the same argument, the derivative $\bar{\partial}$ on the right-hand side of (ii) in Lemma 5.1 vanishes. Hence, there exists $Y_0\in\mathcal{A}^0_{P(H^*)}(\wedge^{k-1}\widetilde{\mathcal{H}})$ such that (ii) holds for any $1\le k\le 2n$. It completes the proof. □

§5.2. Holomorphic k-vector fields on Z

The horizontal bundle $\widetilde{\mathcal{H}}$ induces a bundle $\widehat{\mathcal{H}}$ over the twistor space Z. We denote by v the l^{-2}-valued $(1,0)$-vector field on Z induced by the vector field $r^{-2}v_1$ on $P(H^*)$. The vector field v is regarded as the dual of η. A $(k,0)$-vector field X' on Z is given by

$$X'=X'_h+Y\wedge v$$

for $X'_h\in\wedge^k\widehat{\mathcal{H}}^{1,0}$ and $Y\in l^2\otimes\wedge^{k-1}\widehat{\mathcal{H}}^{1,0}$. By the same argument as Theorem 5.3, we have the following theorem:

Theorem 5.4. *For $1\le k\le 2n-1$, the $(k,0)$-vector field X' is holomorphic if and only if*

$$\bar{\partial}^l(\hat{\theta}_0^k(X'_h))-k^2\hat{\theta}_0^{k-1}(Y)\wedge_E\hat{\theta}_1=0.$$

The $(2n,0)$-vector field X' is holomorphic if and only if

$$\bar{\partial}^l(\hat{\theta}_0^{2n}(X'_h))-4n^2\hat{\theta}_0^{2n-1}(Y)\wedge_E\hat{\theta}_1=0,$$
$$\bar{\partial}^l(\hat{\theta}_0^{2n-1}(Y))=0.$$
□

§6. Quaternionic sections

In this section we provide a definition of a quaternionic section of $\wedge^kE\otimes S^mH$. We show that the lifts of the quaternionic section satisfy some $\bar{\partial}$-equations on $P(H^*)$ and Z.

§6.1. Definition of quaternionic sections

We identify H with H^* by $\omega^\sharp_{H^*}$. By the Clebsch–Gordan decomposition, the covariant derivative ∇ is regarded as

$$\nabla\colon\Gamma(\wedge^kE\otimes S^mH)\to\Gamma(\wedge^kE\otimes E^*\otimes S^{m+1}H)\oplus\Gamma(\wedge^kE\otimes E^*\otimes S^{m-1}H).$$

The Dirac operator (cf. [4]) is defined as the $\wedge^kE\otimes E^*\otimes S^{m+1}H$-part of ∇:

$$\mathfrak{D}_{\wedge^kE}\colon\Gamma(\wedge^kE\otimes S^mH)\to\Gamma(\wedge^kE\otimes E^*\otimes S^{m+1}H).$$

Let k be a positive integer. Let $(\wedge^k E \otimes E^*)_0$ denote the kernel of the trace map $\mathrm{tr}\colon \wedge^k E \otimes E^* \to \wedge^{k-1} E$. Then $\wedge^k E \otimes E^* = (\wedge^k E \otimes E^*)_0 \oplus (\wedge^{k-1} E) \wedge \mathrm{id}_E$. We define an operator

$$\mathfrak{D}^0_{\wedge^k E}\colon \Gamma(\wedge^k E \otimes S^m H) \to \Gamma((\wedge^k E \otimes E^*)_0 \otimes S^{m+1} H)$$

as the $(\wedge^k E \otimes E^*)_0$-part of $\mathfrak{D}_{\wedge^k E}$. We rescale the trace map as $\frac{1}{2n-k+1}\,\mathrm{tr}$, and also denote it using the same notation tr.

Definition 6.1. Let m be a non-negative integer. A section X of $\wedge^k E \otimes S^m H$ is *quaternionic* if $\mathfrak{D}^0_{\wedge^k E}(X) = 0$ for $1 \le k \le 2n-1$ and $\mathfrak{D}_{\wedge^{2n-1} E} \circ \mathrm{tr} \circ \mathfrak{D}_{\wedge^{2n} E}(X) = 0$ for $k = 2n$.

Any section X of $\wedge^{2n} E \otimes S^m H$ satisfies $\mathfrak{D}^0_{\wedge^{2n} E}(X) = 0$ since $(\wedge^{2n} E \otimes E^*)_0 = \{0\}$. Definition 6.1 is also valid in quaternionic manifolds. The operators $\mathfrak{D}^0_{\wedge^k E}$ and $\mathfrak{D}_{\wedge^{2n-1} E} \circ \mathrm{tr} \circ \mathfrak{D}_{\wedge^{2n} E}$ are commutative with τ. Let $\mathcal{Q}(\wedge^k E \otimes S^m H)$ be the sheaf of quaternionic sections of $\wedge^k E \otimes S^m H$ and $\mathcal{Q}(\wedge^k E \otimes S^m H)^\tau$ that of τ-invariant ones.

§6.2. Lift of quaternionic sections to $P(H^*)$

A map $\widetilde{\omega}^\sharp_{H^*}\colon \tilde{\mathcal{A}}^1_{(m,0)} \to \tilde{\mathcal{A}}^0(E^*) \otimes \tilde{\mathcal{A}}^0_{(1,0)} \otimes \tilde{\mathcal{A}}^0_{(m,0)}$ is induced by $\omega^\sharp_{H^*}\colon \mathcal{A}^1(S^m H) \to \mathcal{A}^0(E^* \otimes H \otimes S^m H)$. By $\tilde{\mathcal{A}}^0_{(1,0)} \otimes \tilde{\mathcal{A}}^0_{(m,0)} = \tilde{\mathcal{A}}^0_{(m+1,0)}$, we have

$$\begin{array}{ccc} \tilde{\mathcal{A}}^1_{(m,0)}(\wedge^k E) & \xrightarrow{\ \widetilde{\omega}^\sharp_{H^*}\ } & \tilde{\mathcal{A}}^0_{(m+1,0)}(\wedge^k E \otimes E^*) \\ \uparrow & & \uparrow \\ \mathcal{A}^1(\wedge^k E \otimes S^m H) & \xrightarrow{\ s^{m+1}_H \circ \omega^\sharp_{H^*}\ } & \mathcal{A}^0(\wedge^k E \otimes E^* \otimes S^{m+1} H). \end{array} \tag{10}$$

Proposition 6.2. *We have* $\widetilde{(\mathfrak{D}_{\wedge^k E}\xi - \zeta \wedge_E \mathrm{id}_E)}_0 = \widetilde{\omega}^\sharp_{H^*}(\bar\partial\tilde\xi_0 - \tilde\zeta_0 \wedge_E r^{-2}\tilde\theta_1)$ *for* $\xi \in \mathcal{A}^0(\wedge^k E \otimes S^m H)$ *and* $\zeta \in \mathcal{A}^0(\wedge^{k-1} E \otimes S^{m+1} H)$.

Proof. It follows from $\mathfrak{D}_{\wedge^k E} = s^{m+1}_H \circ \omega^\sharp_{H^*} \circ \nabla$ and diagram (10) that $(\widetilde{\mathfrak{D}_{\wedge^k E}\xi})_0 = \widetilde{\omega}^\sharp_{H^*}(d_{\widetilde{\mathcal{H}}}\tilde\xi_0)$. Since the kernel of $\widetilde{\omega}^\sharp_{H^*}$ is $\mathcal{A}^0_{P(H^*)}((\widetilde{\mathcal{H}}^*)^{1,0} \otimes \wedge^k E)$, $\widetilde{\omega}^\sharp_{H^*}(d_{\widetilde{\mathcal{H}}}\tilde\xi_0) = \widetilde{\omega}^\sharp_{H^*}(\bar\partial\tilde\xi_0)$. Thus $(\widetilde{\mathfrak{D}_{\wedge^k E}\xi})_0 = \widetilde{\omega}^\sharp_{H^*}(\bar\partial\tilde\xi_0)$. We also have $\widetilde{\omega}^\sharp_{H^*}(\tilde\zeta_0 \wedge_E r^{-2}\tilde\theta_1) = \tilde\zeta_0 \wedge_E \widetilde{\mathrm{id}}_E = (\widetilde{\zeta \wedge_E \mathrm{id}_E})_0$. Hence we finish the proof. □

We denote by $\widetilde{\mathcal{O}}_m(\wedge^k E)$ the kernel of $\bar\partial$ on $\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^k E)$. By Proposition 6.2 and the injectivity of $\widetilde{\omega}^\sharp_{H^*}$ on $\mathcal{A}^0_{P(H^*)}((\widetilde{\mathcal{H}}^*)^{0,1} \otimes \wedge^k E)$, we obtain an isomorphism

$$\operatorname{Ker} \mathfrak{D}_{\wedge^k E} \cong \widetilde{\mathcal{O}}_m(\wedge^k E) \tag{11}$$

by $\xi \mapsto \tilde\xi_0$.

Proposition 6.3. *Let ξ and ζ be elements of $\mathcal{A}^0(\wedge^k E \otimes S^m H)$ and $\mathcal{A}^0(\wedge^{k-1} E \otimes S^{m+1} H)$, respectively. The element ξ is quaternionic and $\zeta = \mathrm{tr} \circ \mathfrak{D}_{\wedge^k E}(\xi)$ if and only if $\bar{\partial}\tilde{\xi}_0 - \tilde{\zeta}_0 \wedge_E r^{-2}\tilde{\theta}_1 = 0$ for $1 \leq k \leq 2n-1$, and $\bar{\partial}\tilde{\xi}_0 - \tilde{\zeta}_0 \wedge_E r^{-2}\tilde{\theta}_1 = 0$, $\bar{\partial}\tilde{\zeta}_0 = 0$ for $k = 2n$.*

Proof. The element ξ is quaternionic and $\zeta = \mathrm{tr} \circ \mathfrak{D}_{\wedge^k E}(\xi)$ if and only if $\mathfrak{D}_{\wedge^k E}\xi - \zeta \wedge_E \mathrm{id}_E = 0$ and, in addition, $\mathfrak{D}_{\wedge^{k-1}E}\zeta = 0$ for $k = 2n$. By Proposition 6.2 and the injectivity of $\widetilde{\omega}^\sharp_{H^*}$ on $(\widetilde{\mathcal{H}}^*)^{0,1}$, $\mathfrak{D}_{\wedge^k E}\xi - \zeta \wedge_E \mathrm{id}_E = 0$ is equal to $\bar{\partial}\tilde{\xi}_0 - \tilde{\zeta}_0 \wedge_E r^{-2}\tilde{\theta}_1 = 0$. Furthermore, the isomorphism (11) implies that $\mathfrak{D}_{\wedge^{k-1}E}\zeta = 0$ is equivalent to $\bar{\partial}\tilde{\zeta}_0 = 0$. □

§6.3. Lift of quaternionic sections to Z

The map $\widetilde{\omega}^\sharp_{H^*}$ induces a map $\widehat{\omega}^\sharp_{H^*} : \hat{\mathcal{A}}^1(\wedge^k E \otimes l^m) \to \hat{\mathcal{A}}^0(\wedge^k E \otimes E^* \otimes l^{m+1})$. There exists a commutative diagram

$$\begin{array}{ccccc}
\hat{\mathcal{A}}^0(\wedge^k E \otimes l^m) & \xrightarrow{d^l_{\widehat{\mathcal{H}}}} & \hat{\mathcal{A}}^1(\wedge^k E \otimes l^m) & \xrightarrow{\widehat{\omega}^\sharp_{H^*}} & \hat{\mathcal{A}}^0(\wedge^k E \otimes E^* \otimes l^{m+1}) \\
\uparrow & & \uparrow & & \uparrow \\
\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^k E) & \xrightarrow{d_{\widetilde{\mathcal{H}}}} & \tilde{\mathcal{A}}^1_{(m,0)}(\wedge^k E) & \xrightarrow{\widetilde{\omega}^\sharp_{H^*}} & \tilde{\mathcal{A}}^0_{(m+1,0)}(\wedge^k E \otimes E^*) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{A}^0(\wedge^k E \otimes S^m H) & \xrightarrow{\nabla} & \mathcal{A}^1(\wedge^k E \otimes S^m H) & \xrightarrow{s_H^{m+1} \circ \omega^\sharp_{H^*}} & \mathcal{A}^0(\wedge^k E \otimes E^* \otimes S^{m+1} H).
\end{array}$$

By the same proof as Proposition 6.2, we obtain the following proposition:

Proposition 6.4. *We have $\mathfrak{D}_{\wedge^k E}\widehat{\xi - \zeta \wedge_E \mathrm{id}_E} = \widehat{\omega}^\sharp_{H^*}(\bar{\partial}^l \hat{\xi} - \hat{\zeta} \wedge_E \hat{\theta}_1)$.* □

We denote by $\widehat{\mathcal{O}}(\wedge^k E \otimes l^m)$ the kernel of $\bar{\partial}^l$ on $\hat{\mathcal{A}}^0(\wedge^k E \otimes l^m)$. Then

$$\mathrm{Ker}\, \mathfrak{D}_{\wedge^k E} \cong \widehat{\mathcal{O}}(\wedge^k E \otimes l^m) \tag{12}$$

by $\xi \mapsto \hat{\xi}$. Proposition 6.4 implies the following:

Proposition 6.5. *Let ξ and ζ be elements of $\mathcal{A}^0(\wedge^k E \otimes S^m H)$ and $\mathcal{A}^0(\wedge^{k-1} E \otimes S^{m+1} H)$, respectively. The element ξ is quaternionic and $\zeta = \mathrm{tr} \circ \mathfrak{D}_{\wedge^k E}(\xi)$ if and only if $\bar{\partial}^l \hat{\xi} - \hat{\zeta} \wedge_E \hat{\theta}_1 = 0$ for $1 \leq k \leq 2n-1$, $\bar{\partial}^l \hat{\xi} - \hat{\zeta} \wedge_E \hat{\theta}_1 = 0$ and $\bar{\partial}^l \hat{\zeta} = 0$ for $k = 2n$.* □

§7. Quaternionic k-vector fields

A quaternionic section of $\wedge^k E \otimes S^k H$ is called a *quaternionic k-vector field* on M. We prove that any quaternionic k-vector field corresponds to a holomorphic $(k, 0)$-vector field on Z.

§7.1. Horizontal lift of k-vector fields to $P(H^*)$

Let $\widetilde{X}_h$ denote the horizontal lift to $P(H^*)$ of a k-vector field X on M. We denote by $\tilde{\mathcal{A}}_0(\wedge^k \widetilde{\mathcal{H}}^{1,0})$ the sheaf of horizontal $(k, 0)$-vector fields which are $\mathrm{GL}(1, \mathbb{C})$-invariant and holomorphic along each fiber.

Proposition 7.1. *The isomorphism $\mathcal{A}^0(\wedge^k E \otimes S^k H) \cong \tilde{\mathcal{A}}_0(\wedge^k \widetilde{\mathcal{H}}^{1,0})$ is given by $X \mapsto \widetilde{X}_h^{k,0}$. Moreover, $\widetilde{X}_0 = (k!)^{-2}\tilde{\theta}_0^k(\widetilde{X}_h^{k,0})$ for $X \in \mathcal{A}^0(\wedge^k E \otimes S^k H)$.*

Proof. The lift $\widetilde{X}$ of $X \in \mathcal{A}^0(\wedge^k E \otimes S^k H)$ to $P(H^*)$ is related to the horizontal lift $\widetilde{X}_h$ by $\widetilde{X} = (\otimes^k \tilde{\theta})(\widetilde{X}_h)$. The $\wedge^k E \otimes \wedge^k T^* P(H^*) \otimes S^k \mathbb{H}$-part of $\otimes^k \tilde{\theta}$ is $\sum_{i=0}^k (k!)^{-2}\binom{k}{i}\tilde{\theta}_0^{k-i} \wedge \tilde{\theta}_1^i 1^{k-i} j^i$. Hence $\widetilde{X}_0 = (k!)^{-2}\tilde{\theta}_0^k(\widetilde{X}_h) = (k!)^{-2}\tilde{\theta}_0^k(\widetilde{X}_h^{k,0})$. Then $\tilde{\theta}_0^k(\nabla_v^{0,1}\widetilde{X}_h^{k,0}) = \nabla_v^{0,1}(\tilde{\theta}_0^k(\widetilde{X}_h^{k,0})) = 0$ for any tangent vector v along the fiber. By Proposition 4.1, $\tilde{\theta}_0^{k-1} \wedge \eta_0(\nabla_v^{0,1}\widetilde{X}_h^{k,0})$, $\tilde{\theta}_0^{k-1} \wedge \eta_1(\nabla_v^{0,1}\widetilde{X}_h^{k,0})$ and $\tilde{\theta}_0^{k-2} \wedge \eta_0 \wedge \eta_1(\nabla_v^{0,1}\widetilde{X}_h^{k,0})$ also vanish. Hence $\widetilde{X}_h^{k,0}$ is holomorphic along each fiber. Let X' be a horizontal $(k, 0)$-vector field on $P(H^*)$. Then

$$R_c^*(\tilde{\theta}_0^k(X')) = (R_c^*\tilde{\theta}_0^k)((R_{c^{-1}})_* X') = c^k \tilde{\theta}_0^k((R_{c^{-1}})_* X') \tag{13}$$

for any $c \in \mathrm{GL}(1, \mathbb{C})$. Thus the bundle isomorphism $\tilde{\theta}_0^k\colon\ \wedge^k \widetilde{\mathcal{H}}^{1,0} \cong p^{-1}(\wedge^k E)$ induces $\tilde{\mathcal{A}}_0(\wedge^k \widetilde{\mathcal{H}}^{1,0}) \cong \tilde{\mathcal{A}}^0_{(k,0)}(\wedge^k E)$. It follows from Proposition 3.1 that $\mathcal{A}^0(\wedge^k E \otimes S^k H) \cong \tilde{\mathcal{A}}_0(\wedge^k \widetilde{\mathcal{H}}^{1,0})$. We finish the proof. □

Under the irreducible decomposition of $\wedge^k TM$, the horizontal lift of the components except for $\wedge^k E \otimes S^k H$ vanish by $\tilde{\theta}_0^k$. Hence, Proposition 7.1 induces the following:

Corollary 7.2. *Let X be an element of $\mathcal{A}^0(\wedge^k TM)$. The $(k, 0)$-part $\widetilde{X}_h^{k,0}$ of $\widetilde{X}_h$ is $\mathrm{GL}(1, \mathbb{C})$-invariant and holomorphic along each fiber.* □

§7.2. Holomorphic lift of quaternionic k-vector fields to $P(H^*)$

A horizontal $(k, 0)$-vector field X' on $P(H^*)$ is called of $\mathrm{GL}(1, \mathbb{C})$-*order* m if $(R_{c^{-1}})_* X' = c^m X'$ for any $c \in \mathrm{GL}(1, \mathbb{C})$. We define $\tilde{\mathcal{A}}_m(\wedge^k \widetilde{\mathcal{H}}^{1,0})$ as the sheaf of horizontal $(k, 0)$-vector fields which are of $\mathrm{GL}(1, \mathbb{C})$-order m and holomorphic along each fiber. By equation (13), we obtain an isomorphism $\tilde{\mathcal{A}}_m(\wedge^k \widetilde{\mathcal{H}}^{1,0}) \cong \tilde{\mathcal{A}}^0_{(k+m,0)}(\wedge^k E)$ as $X' \mapsto (k!)^{-2}\tilde{\theta}_0^k(X')$. For an element ξ of $\mathcal{A}^0(\wedge^k E \otimes S^{k+m} H)$, there exists a unique element $\widetilde{Y}_\xi$ of $\tilde{\mathcal{A}}_m(\wedge^k\widetilde{\mathcal{H}}^{1,0})$ such that

$$(k!)^{-2}\tilde{\theta}_0^k(\widetilde{Y}_\xi) = \tilde{\xi}_0.$$

Hence, we have

$$\mathcal{A}^0(\wedge^k E \otimes S^{k+m}H) \cong \tilde{\mathcal{A}}_m(\wedge^k\widetilde{\mathcal{H}}^{1,0}) \tag{14}$$

by $\xi \mapsto \widetilde{Y}_\xi$. In the case $m = 0$, the isomorphism is given by Proposition 7.1.

Proposition 7.3. *Let X and ζ be elements of $\mathcal{A}^0(\wedge^k E \otimes S^k H)$ and $\mathcal{A}^0(\wedge^{k-1}E \otimes S^{k+1}H)$, respectively. The k-vector field X is quaternionic and $\zeta = \operatorname{tr}\circ\mathfrak{D}_{\wedge^k E}(X)$ if and only if there exist $Y_0 \in \mathcal{A}^0_{P(H^*)}(\wedge^{k-1}\widetilde{\mathcal{H}}^{1,0})$ and $Z_0 \in \mathcal{A}^0_{P(H^*)}(\wedge^{k-2}\widetilde{\mathcal{H}}^{1,0})$ such that the $(k,0)$-vector field $\widetilde{X}_h^{k,0} + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1$ is holomorphic for $Y_1 = \frac{1}{r^2}\widetilde{Y}_\zeta$.*

Proof. Setting $Y_1 = \frac{1}{r^2}\widetilde{Y}_\zeta$, then we obtain $\tilde{\zeta}_0 = ((k-1)!)^{-2}\tilde{\theta}_0^{k-1}(r^2 Y_1)$. It follows from Proposition 6.3 that X is quaternionic and $\zeta = \operatorname{tr}\circ\mathfrak{D}_{\wedge^k E}(X)$ if and only if $\bar{\partial}\widetilde{X}_0 - \tilde{\zeta}_0 \wedge_E r^{-2}\tilde{\theta}_1 = 0$ for $1 \le k \le 2n-1$, $\bar{\partial}\widetilde{X}_0 - \tilde{\zeta}_0 \wedge_E r^{-2}\tilde{\theta}_1 = 0$ and $\bar{\partial}\tilde{\zeta}_0 = 0$ for $k = 2n$. The condition is equivalent to $\bar{\partial}(\tilde{\theta}_0^k(\widetilde{X}_h^{k,0})) - k^2\tilde{\theta}_0^{k-1}(Y_1) \wedge_E \tilde{\theta}_1 = 0$ for $1 \le k \le 2n-1$, $\bar{\partial}(\tilde{\theta}_0^k(\widetilde{X}_h^{k,0})) - k^2\tilde{\theta}_0^{k-1}(Y_1) \wedge_E \tilde{\theta}_1 = 0$ and $\bar{\partial}(\tilde{\theta}_0^{k-1}(r^2 Y_1)) = 0$ for $k = 2n$. It is equivalent that there exist $Y_0 \in \mathcal{A}^0_{P(H^*)}(\wedge^{k-1}\widetilde{\mathcal{H}}^{1,0})$, $Z_0 \in \mathcal{A}^0_{P(H^*)}(\wedge^{k-2}\widetilde{\mathcal{H}}^{1,0})$ such that $\widetilde{X}_h^{k,0} + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1$ is holomorphic by Theorem 5.3. □

§7.3. Horizontal lift of k-vector fields to Z

Let $\widehat{X}_h$ be the horizontal lift to Z of a k-vector field X on M. The horizontal vector field $\widehat{X}_h$ and the $(k,0)$-part $\widehat{X}_h^{k,0}$ correspond to $\widetilde{X}_h$ and $\widetilde{X}_h^{k,0}$, respectively. We denote by $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0})$ the sheaf of horizontal $(k,0)$-vector fields which are holomorphic along each fiber of $f\colon Z \to M$. Proposition 7.1 induces the following:

Proposition 7.4. *The isomorphism $\mathcal{A}^0(\wedge^k E \otimes S^k H) \cong \hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0})$ is given by $X \mapsto \widehat{X}_h^{k,0}$. Moreover, $\widehat{X} = (k!)^{-2}\hat{\theta}_0^k(\widehat{X}_h^{k,0})$ for $X \in \mathcal{A}^0(\wedge^k E \otimes S^k H)$.* □

Corollary 7.2 implies the following corollary:

Corollary 7.5. *Let X be an element of $\mathcal{A}^0(\wedge^k TM)$. The $(k,0)$-part $\widehat{X}_h^{k,0}$ of $\widehat{X}_h$ is holomorphic along each fiber of f.* □

We consider the holomorphic bundle $\wedge^k\widehat{\mathcal{H}}^{1,0} \otimes l^m$ for a non-negative integer m. Let $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0} \otimes l^m)$ be a sheaf of l^m-valued horizontal smooth $(k,0)$-vector fields which are holomorphic along each fiber. Let $\widehat{\mathcal{O}}(\wedge^k\widehat{\mathcal{H}}^{1,0} \otimes l^m)$ denote the subsheaf of $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0} \otimes l^m)$ of holomorphic l^m-valued horizontal $(k,0)$-vector fields. By the definition of l, we obtain the isomorphism

$$\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)\cong\tilde{\mathcal{A}}_m(\wedge^k\widetilde{\mathcal{H}}^{1,0}). \tag{15}$$

The kth wedge $\hat{\theta}_0^k$ defines a map from $\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m$ to $f^{-1}(\wedge^k E)\otimes l^{k+m}$. The map induces isomorphisms $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)\cong\hat{\mathcal{A}}^0(\wedge^k E\otimes l^{k+m})$ and $\widehat{\mathcal{O}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)\cong\widehat{\mathcal{O}}(\wedge^k E\otimes l^{k+m})$. For an element ξ of $\mathcal{A}^0(\wedge^k E\otimes S^{k+m}H)$, there exists a unique element $\widehat{Y}_\xi$ of $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)$ such that

$$(k!)^{-2}\hat{\theta}_0^k(\widehat{Y}_\xi)=\hat{\xi}.$$

The isomorphisms in (14) and (15) yield

$$\mathcal{A}^0(\wedge^k E\otimes S^{k+m}H)\cong\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m) \tag{16}$$

by $\xi\mapsto\widehat{Y}_\xi$. The isomorphism (12) implies $\operatorname{Ker}\mathfrak{D}_{\wedge^k E}\cong\widehat{\mathcal{O}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)$ by the correspondence.

§7.4. Holomorphic lift of quaternionic *k*-vector fields to *Z*

By the same argument as Proposition 7.3, we obtain the following proposition:

Proposition 7.6. *Let X and ζ be elements of $\mathcal{A}^0(\wedge^k E\otimes S^kH)$ and $\mathcal{A}^0(\wedge^{k-1}E\otimes S^{k+1}H)$, respectively. The k-vector field X is quaternionic and $\zeta=\operatorname{tr}\circ\mathfrak{D}_{\wedge^k E}(X)$ if and only if the $(k,0)$-vector field $\widehat{X}_h^{k,0}+\widehat{Y}_\zeta\wedge v$ is holomorphic.* □

Let $\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)$ be a sheaf of holomorphic $(k,0)$-vector fields defined in the pull-back of open sets on M by f. Proposition 7.6 induces the following:

Theorem 7.7. *An isomorphism $\mathcal{Q}(\wedge^k E\otimes S^kH)\cong\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)$ is given by $X\mapsto\widehat{X}_h^{k,0}+\widehat{Y}_{\operatorname{tr}\circ\mathfrak{D}_{\wedge^k E}(X)}\wedge v$. In particular, any global quaternionic k-vector field on M corresponds to a global holomorphic $(k,0)$-vector field on Z.* □

§7.5. Holomorphic lift of quaternionic real *k*-vector fields to *Z*

An endomorphism $\hat{\tau}$ of $\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)$ is defined by

$$\hat{\tau}(X')=\overline{(R_{[j]})_*X'}$$

for $X'\in\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)$. Then we obtain an $\mathbb{R}$-isomorphism

$$\mathcal{A}^0(\wedge^k E\otimes S^{k+m}H)^\tau\cong\hat{\mathcal{A}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)^{\hat{\tau}}$$

by $\xi\mapsto\widehat{Y}_\xi$. Then $(\operatorname{Ker}\mathfrak{D}_{\wedge^k E})^\tau\cong\widehat{\mathcal{O}}(\wedge^k\widehat{\mathcal{H}}^{1,0}\otimes l^m)^{\hat{\tau}}$ under the correspondence.

Theorem 7.8. *An $\mathbb{R}$-isomorphism $\mathcal{Q}(\wedge^k E\otimes S^kH)^\tau\cong\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)^{\hat{\tau}}$ is given by $X\mapsto\widehat{X}_h^{k,0}+\widehat{Y}_{\operatorname{tr}\circ\mathfrak{D}_{\wedge^k E}(X)}\wedge v$. In particular, any global quaternionic real k-vector* *field on M corresponds to a global holomorphic and $\hat{\tau}$-invariant $(k,0)$-vector field on Z.* □

§7.6. Example

Let M be the n-dimensional quaternionic projective space $\mathbb{H}P^n$. Then $P(H^*) = \mathbb{C}^{2n+2}\backslash\{0\}$ as a complex manifold. The twistor space Z is $\mathbb{C}P^{2n+1}$. Let $\widetilde{V}_k$ denote the space of $\mathrm{GL}(1,\mathbb{C})$-invariant holomorphic k-vector fields on $\mathbb{C}^{2n+2}\backslash\{0\}$. Then

$$\widetilde{V}_k = \{\sum a_{i_1\cdots i_k j_1\cdots j_k} z_{i_1}\cdots z_{i_k}\tfrac{\partial}{\partial z_{j_1}}\wedge\cdots\wedge\tfrac{\partial}{\partial z_{j_k}} \mid a_{ijkl}\in\mathbb{C}\}.$$

We regard the coefficient $(a_{i_1\cdots i_k j_1\cdots j_k})$ as an element of $\otimes^k\mathbb{C}^{2n+2}\otimes\otimes^k(\mathbb{C}^{2n+2})^*$. We define $S^k\otimes\wedge^k$ as the projection from $\otimes^k\mathrm{gl}(2n+2,\mathbb{C})\cong\otimes^k\mathbb{C}^{2n+2}\otimes\otimes^k(\mathbb{C}^{2n+2})^*$ to $S^k\mathbb{C}^{2n+2}\otimes\wedge^k(\mathbb{C}^{2n+2})^*$. Then $\widetilde{V}_k\cong S^k\otimes\wedge^k(\otimes^k\mathrm{gl}(2n+2,\mathbb{C}))$. The space of holomorphic k-vector fields on $\mathbb{C}P^{2n+1}$ is identified with the quotient space $\widetilde{V}_k/\widetilde{V}_{k-1}\wedge v_0$ (cf. [15, §5.1]). Theorems 7.7 and 7.8 imply that the spaces of quaternionic k-vector fields and real ones are identified with

$$S^k\otimes\wedge^k(\otimes^k\mathrm{gl}(2n+2,\mathbb{C}))/S^k\otimes\wedge^k(\otimes^{k-1}\mathrm{gl}(2n+2,\mathbb{C})\otimes\mathrm{id})$$

and

$$S^k\otimes\wedge^k(\otimes^k\mathrm{gl}(n+1,\mathbb{H}))/S^k\otimes\wedge^k(\otimes^{k-1}\mathrm{gl}(n+1,\mathbb{H})\otimes\mathrm{id}),$$

respectively.

§8. Graded Lie algebra structure on the space of quaternionic *k*-vector fields

§8.1. Coefficients of lifts of $\mathcal{A}^0(\wedge^k E\otimes S^m H)$

Let ξ be an element of $\mathcal{A}^0(\wedge^k E\otimes S^m H)$. For each coefficient $\tilde{\xi}_i$ of $\tilde{\xi}$, $r^{-2i}\tilde{\xi}_i$ is of $\mathrm{GL}(1,\mathbb{C})$-order $m-2i$. It induces a section of l^{m-2i} on Z, which we denote by $\hat{\xi}_i$. Then $\hat{\xi}_0=\hat{\xi}$ by the definition. Since $i_{v_1}d\tilde{\xi}_i = -(i+1)\tilde{\xi}_{i+1}$ for $i=0,1,\ldots,m-1$ and $i_{v_1}d\tilde{\xi}_m=0$, we obtain the following lemma:

Lemma 8.1. *We have $i_v d^l\hat{\xi}_i = -(i+1)\hat{\xi}_{i+1}$ for $i=0,1,\ldots,m-1$ and $i_v d^l\hat{\xi}_m=0$.* □

Let X be an element of $\mathcal{A}^0(\wedge^k E\otimes S^k H)$. As in the proof of Proposition 7.1, each coefficient $\widetilde{X}_i$ of $\widetilde{X}$ is given by $\widetilde{X}_i=(k!)^{-2}\binom{k}{i}(\tilde{\theta}_0^{k-i}\wedge\tilde{\theta}_1^i)(\widetilde{X}_h)$. It yields that

$$\widehat{X}_i=(k!)^{-2}\binom{k}{i}(\hat{\theta}_0^{k-i}\wedge\hat{\theta}_1^i)(\widehat{X}_h) \tag{17}$$

for $i=0,1,\ldots,k$. Let $\widehat{X}_h^{k-i,i}$ denote the $(k-i,i)$-part of $\widehat{X}_h$. Lemma 8.1 and equation (17) imply the following:

Proposition 8.2. *If* $X \in \mathcal{A}^0(\wedge^k E \otimes S^k H)$, *then*

$$i_v d^l((\hat{\theta}_0^{k-i} \wedge \hat{\theta}_1^i)(\widehat{X}_h^{k-i,i})) = -(k-i)(\hat{\theta}_0^{k-i-1} \wedge \hat{\theta}_1^{i+1})(\widehat{X}_h^{k-i-1,i+1})$$

for $i = 0, 1, \ldots, k-1$. *In particular,* $i_v d^l(\hat{\theta}_0^k(\widehat{X}_h^{k,0})) = -k(\hat{\theta}_0^{k-1} \wedge \hat{\theta}_1)(\widehat{X}_h^{k-1,1})$. □

§8.2. The Schouten–Nijenhuis bracket

The Schouten–Nijenhuis bracket $[\ ,\ ]$ is a bilinear map $\mathcal{A}^0(\wedge^k TM) \times \mathcal{A}^0(\wedge^{k'} TM) \to \mathcal{A}^0(\wedge^{k+k'-1} TM)$ such that $[X, X'] = (-1)^{kk'}[X', X]$ and

$$(-1)^{k(k''-1)}[X, [X', X'']] + (-1)^{k'(k-1)}[X', [X'', X]] + (-1)^{k''(k'-1)}[X'', [X, X']] = 0$$

for $X \in \mathcal{A}^0(\wedge^k TM)$, $X' \in \mathcal{A}^0(\wedge^{k'} TM)$ and $X'' \in \mathcal{A}^0(\wedge^{k'} TM)$. If we take a torsion-free affine connection ∇ on M, then $[X, X']$ is given by $\wedge^{k+k'-1}(X \cdot \nabla X' + (-1)^k X' \cdot \nabla X)$, where $\wedge^{k+k'-1}$ is the anti-symmetrization of $\otimes^{k+k'-1} TM$ and the dot $\cdot$ means the contraction of $\wedge^k TM$ with $\wedge^{k'} TM \otimes T^*M$. The pair $(\bigoplus_k \mathcal{A}^0(\wedge^k TM), [\ ,\ ])$ is a graded Lie algebra. For $X \in \mathcal{A}^0(\wedge^k E \otimes S^k H)$, $X' \in \mathcal{A}^0(\wedge^{k'} E \otimes S^{k'} H)$, $[X, X']$ is not always in $\mathcal{A}^0(\wedge^{k+k'-1} E \otimes S^{k+k'-1} H)$ except for the case $k = k' = 1$. We define $[X, X']_Q$ by the $\wedge^{k+k'-1} E \otimes S^{k+k'-1} H$-part of $[X, X']$. Then $(\bigoplus_k \mathcal{A}^0(\wedge^k E \otimes S^k H), [\ ,\ ]_Q)$ is a graded Lie algebra.

Let F be a vector bundle on M and ∇^F a connection of F. If α is an F-valued differential $(k + k' - 1)$-form on M, then

$$i_{[X,X']}\alpha = (-1)^{k'(k+1)} i_X d^F i_{X'} \alpha + (-1)^k i_{X'} d^F i_X \alpha - i_{X \wedge X'} d^F \alpha \tag{18}$$

for $X \in \mathcal{A}^0(\wedge^k TM)$, $X' \in \mathcal{A}^0(\wedge^{k'} TM)$.

§8.3. Bracket for quaternionic k-vector fields

The Schouten–Nijenhuis bracket $[\ ,\ ]$ is defined for holomorphic multi-vector fields. The pair $(\bigoplus_k \mathcal{O}(\wedge^k T^{1,0} Z), [\ ,\ ])$ is a graded Lie algebra. Since $[W, W']$ is in $\widehat{\mathcal{O}}(\wedge^{k+k'-1} T^{1,0} Z)$ for $W \in \widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$, $W' \in \widehat{\mathcal{O}}(\wedge^{k'} T^{1,0} Z)$, we have that $(\bigoplus_k \widehat{\mathcal{O}}(\wedge^k T^{1,0} Z), [\ ,\ ])$ is also a graded Lie algebra.

For $X \in \mathcal{Q}(\wedge^k E \otimes S^k H)$, there exists a holomorphic $(k, 0)$-vector field $\widehat{X}_h^{k,0} + Y \wedge v \in \widehat{\mathcal{O}}(\wedge^k T^{1,0} Z)$ by Theorem 7.7.

Lemma 8.3. *If* $X \in \mathcal{Q}(\wedge^k E \otimes S^k H)$ *and* $X' \in \mathcal{Q}(\wedge^{k'} E \otimes S^{k'} H)$, *then*

$$(\widehat{[X, X']_Q})_h^{k+k'-1,0} = [\widehat{X}_h^{k,0} + Y \wedge v, \widehat{X'}_h^{k',0} + Y' \wedge v]_h.$$

Proof. Let k'' denote the integer $k+k'-1$. The horizontal $(k'',0)$-part $(\widehat{[X,X']_Q})_h^{k'',0}$ of $(\widehat{[X,X']_Q})$ is

$$\widehat{[X,X']}_h^{k'',0} = [\widehat{X}_h, \widehat{X'}_h]_h^{k'',0}.$$

Hence, it suffices to show

$$\hat{\theta}_0^{k''}([\widehat{X}_h, \widehat{X'}_h]) = \hat{\theta}_0^{k''}\big([\widehat{X}_h^{k,0} + Y \wedge v, \widehat{X'}_h^{k',0} + Y' \wedge v]\big).$$

We remark that $\hat{\theta}_0^{k''}(\widehat{X}_h) = \binom{k''}{k}\hat{\theta}_0^k(\widehat{X}_h) \wedge_E \hat{\theta}_0^{k'-1}$. It follows from equation (18) and $d^l\hat{\theta}_0^k = k\hat{\theta}_0^{k-1} \wedge \hat{\theta}_1 \wedge \eta$ that

$$\begin{aligned}\hat{\theta}_0^{k''}([\widehat{X}_h, \widehat{X'}_h]) = (-1)^{k'(k+1)}\binom{k''}{k'} i_{\widehat{X}_h} d^l(\hat{\theta}_0^{k'}(\widehat{X'}_h)) \wedge \hat{\theta}_0^{k-1}\\ + (-1)^k\binom{k''}{k} i_{\widehat{X'}_h} d^l(\hat{\theta}_0^{k}(\widehat{X}_h)) \wedge \hat{\theta}_0^{k'-1}.\end{aligned}$$

It turns out that

$$\begin{aligned}\hat{\theta}_0^{k''}([\widehat{X}_h^{k,0}, \widehat{X'}_h^{k',0}]) = (-1)^{k'(k+1)}\binom{k''}{k'} i_{\widehat{X}_h^{k,0}} d^l(\hat{\theta}_0^{k'}(\widehat{X'}_h^{k',0})) \wedge \hat{\theta}_0^{k-1}\\ + (-1)^k\binom{k''}{k} i_{\widehat{X'}_h^{k',0}} d^l(\hat{\theta}_0^{k}(\widehat{X}_h^{k,0})) \wedge \hat{\theta}_0^{k'-1}.\end{aligned}$$

Then

$$\begin{aligned}\hat{\theta}_0^{k''}([\widehat{X}_h, \widehat{X'}_h]) &= \hat{\theta}_0^{k''}([\widehat{X}_h^{k,0}, \widehat{X'}_h^{k',0}])\\ &+ (-1)^{k'(k+1)}\binom{k''}{k'}(k')^2\hat{\theta}_0^{k'-1}(Y') \wedge_E (\hat{\theta}_0^{k-1} \wedge \hat{\theta}_1)(\widehat{X}_h^{k-1,1})\\ &+ (-1)^k\binom{k''}{k}k^2\hat{\theta}_0^{k-1}(Y) \wedge_E (\hat{\theta}_0^{k'-1} \wedge \hat{\theta}_1)(\widehat{X'}_h^{k'-1,1}).\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}&\hat{\theta}_0^{k''}([\widehat{X}_h^{k,0} + Y \wedge v, \widehat{X'}_h^{k',0} + Y' \wedge v])\\ &= (-1)^{k'(k+1)}\binom{k''}{k'} i_{\widehat{X}_h^{k,0}+Y\wedge v} d^l(\hat{\theta}_0^{k'}(\widehat{X'}_h^{k',0})) \wedge \hat{\theta}_0^{k-1}\\ &\quad + (-1)^k\binom{k''}{k} i_{\widehat{X'}_h^{k',0}+Y'\wedge v} d^l(\hat{\theta}_0^{k}(\widehat{X}_h^{k,0})) \wedge \hat{\theta}_0^{k'-1}.\end{aligned}$$

Using Proposition 8.2, then we obtain that $\hat{\theta}_0^{k''}([\widehat{X}_h^{k,0} + Y \wedge v, \widehat{X'}_h^{k',0} + Y' \wedge v])$ is equal to (19). Hence we finish the proof. □

Proposition 8.4. *If* $X \in \mathcal{Q}(\wedge^k E \otimes S^k H)$ *and* $X' \in \mathcal{Q}(\wedge^{k'} E \otimes S^{k'} H)$, *then* $[X, X']_Q$ *is quaternionic.*

Proof. Lemma 8.3 implies that $(\widehat{[X,X']_Q})_h^{k'',0}$ is the horizontal $(k'',0)$-part of the holomorphic k''-vector field

$$\big[\widehat{X}_h^{k,0} + Y\wedge v, \widehat{X'}_h^{k',0} + Y'\wedge v\big].$$

Then $[X,X']_Q$ is quaternionic by Proposition 7.6. □

It yields that $(\bigoplus_{k=1}^{2n}\mathcal{Q}(\wedge^k E\otimes S^k H), [\ ,\ ]_Q)$ is a graded Lie algebra. Proposition 7.6 and Lemma 8.3 imply the following theorem:

Theorem 8.5. *The isomorphism $\mathcal{Q}(\wedge^k E\otimes S^k H)\cong\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)$ as in Theorem 7.7 preserves the structures of graded Lie algebras. In particular, the space of global quaternionic k-vector fields on M is isomorphic to that of global holomorphic $(k,0)$-vector fields on Z as graded Lie algebras.* □

§8.4. Bracket for quaternionic real k-vector fields

The real structure τ on $\mathcal{Q}(\wedge^k E\otimes S^k H)$ is the complex conjugate for k-vector fields on M. It implies that $\tau([X,X']_Q) = [\tau(X),\tau(X')]_Q$ for $X\in\mathcal{Q}(\wedge^k E\otimes S^k H)$ and $X'\in\mathcal{Q}(\wedge^{k'}E\otimes S^{k'}H)$. If X and X' are real, then $[X,X']_Q$ is real. Hence, $(\bigoplus_{k=1}^{2n}\mathcal{Q}(\wedge^k E\otimes S^k H)^\tau, [\ ,\ ]_Q)$ admits a structure of a graded Lie algebra.

Proposition 8.6. *If $W\in\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)$ and $W'\in\widehat{\mathcal{O}}(\wedge^{k'}T^{1,0}Z)$, then $\hat\tau([W,W']) = [\hat\tau(W),\hat\tau(W')]$. Moreover, if W and W' are $\hat\tau$-invariant, then $[W,W']$ is also $\hat\tau$-invariant.*

Proof. Equation (18) implies that $\hat\tau(\alpha([W,W'])) = \hat\tau(\alpha)([\hat\tau(W),\hat\tau(W')])$ for any k''-form α on Z. It yields that $\hat\tau(\alpha)(\hat\tau([W,W'])) = \hat\tau(\alpha)([\hat\tau(W),\hat\tau(W')])$. Since $\hat\tau$ is a real structure, $\alpha(\hat\tau([W,W'])) = \alpha([\hat\tau(W),\hat\tau(W')])$ for any α. Hence, $\hat\tau([W,W']) = [\hat\tau(W),\hat\tau(W')]$. □

It induces that $(\bigoplus_{k=1}^{2n+1}\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)^{\hat\tau}, [\ ,\ ])$ is a graded Lie algebra. By the same argument as Theorem 8.5, we obtain the following theorem:

Theorem 8.7. *The isomorphism $\mathcal{Q}(\wedge^k E\otimes S^k H)^\tau\cong\widehat{\mathcal{O}}(\wedge^k T^{1,0}Z)^{\hat\tau}$ as in Theorem 7.8 preserves the structures of graded Lie algebras. In particular, the space of global quaternionic real k-vector fields on M is isomorphic to that of global holomorphic and $\hat\tau$-invariant $(k,0)$-vector fields on Z as graded Lie algebras.* □

Acknowledgments

The authors would like to thank the referee for their valuable comments. The first-named author is supported by a Grant-in-Aid for Young Scientists (B) #17K14187 from JSPS.

References

[1] D. V. Alekseevskii, Riemannian spaces with exceptional holonomy groups, Funct. Anal. Appl. **2** (1968), 97–105. MR 0231313

[2] D. V. Alekseevsky and S. Marchiafava, Gradient quaternionic vector fields and a characterization of the quaternionic projective space, Vienna Preprint ESI, Vol. 138, 1994.

[3] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461. Zbl 0389.53011 MR 0506229

[4] R. J. Baston, Quaternionic complexes, J. Geom. Phys. **8** (1992), 29–52. Zbl 0764.53022 MR 1165872

[5] C. P. Boyer, K. Galicki and B. M. Mann, Quaternionic reduction and Einstein manifolds, Comm. Anal. Geom. **1** (1993), 229–279. Zbl 0856.53038 MR 1243524

[6] L. David and M. Pontecorvo, A characterization of quaternionic projective space by the conformal-Killing equation, J. Lond. Math. Soc. (2) **80** (2009), 326–340. Zbl 1175.53061 MR 2545255

[7] M. De Wilde and P. B. A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, Lett. Math. Phys. **7** (1983), 487–496. Zbl 0526.58023 MR 0728644

[8] B. Fedosov, *Deformation quantization and index theory*, Mathematical Topics 9, Akademie, Berlin, 1996. Zbl 0867.58061 MR 1376365

[9] K. Galicki and T. Nitta, Nonzero scalar curvature generalizations of the ALE hyper-Kähler metrics, J. Math. Phys. **33** (1992), 1765–1771. Zbl 0757.53007 MR 1158996

[10] N. Hitchin, Instantons, Poisson structures and generalized Kähler geometry, Comm. Math. Phys. **265** (2006), 131–164. Zbl 1110.53056 MR 2217300

[11] S. Ishihara, Quaternion Kählerian manifolds, J. Differential Geometry **9** (1974), 483–500. Zbl 0297.53014 MR 0348687

[12] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. I*, Interscience Tracts in Pure and Applied Mathematics 15, Interscience Publishers (a division of John Wiley & Sons), New York-London, 1963. Zbl 0119.37502 MR 0152974

[13] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. **66** (2003), 157–216. Zbl 1058.53065 MR 2062626

[14] C. LeBrun, Fano manifolds, contact structures, and quaternionic geometry, Internat. J. Math. **6** (1995), 419–437. Zbl 0835.53055 MR 1327157

[15] T. Moriyama and T. Nitta, Some examples of global Poisson structures on S^4, Kodai Math. J. **42** (2019), 223–246. Zbl 1432.53119 MR 3981305

[16] Y. Nagatomo and T. Nitta, Vanishing theorems for quaternionic complexes, Bull. London Math. Soc. **29** (1997), 359–366. Zbl 0901.58059 MR 1435574

[17] H. Omori, Y. Maeda and A. Yoshioka, Weyl manifolds and deformation quantization, Adv. Math. **85** (1991), 224–255. Zbl 0734.58011 MR 1093007

[18] S. Salamon, Quaternionic Kähler manifolds, Invent. Math. **67** (1982), 143–171. Zbl 0486.53048 MR 0664330

[19] A. Swann, Hyper-Kähler and quaternionic Kähler geometry, Math. Ann. **289** (1991), 421–450. Zbl 0711.53051 MR 1096180