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Hadamard-type variation formulas for the eigenvalues of the η-Laplacian and applications

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Abstract. We consider an analytic family of Riemannian metrics on a compact smooth manifold M. We assume the Dirichlet boundary condition for the η-Laplacian and obtain Hadamard-type variation formulas for analytic curves of eigenfunctions and eigenvalues. As an application, we show that for a subset of all C^r Riemannian metrics $\mathcal{M}^r$ on M, all eigenvalues of the η-Laplacian are generically simple, for $2 \leq r < \infty$. This implies the existence of a residual set of metrics in $\mathcal{M}^r$ that makes the spectrum of the η-Laplacian simple. Likewise, we show that there exists a residual set of drifting functions η in the space $\mathcal{F}^r$ of all C^r functions on M, that again makes the spectrum of the η-Laplacian simple, for $2 \leq r < \infty$. Besides, we provide a precise information about the complement of these residual sets as well as about the structure of the set of deformations of a Riemannian metric (respectively, of the set of deformations of a drifting function) which preserves double eigenvalues. Moreover, we consider a family of perturbations of a domain in a Riemannian manifold and obtain Hadamard-type formulas for the eigenvalues of the η-Laplacian in this case. We also establish generic properties of eigenvalues in this context.

1. Introduction

In [4], Berger derived variation formulas for the eigenvalues of the Laplace–Beltrami operator with respect to a differentiable one-parameter family of Riemannian metrics $g(t)$ on a smooth manifold M. Such formulas are known as *Hadamard-type variation formulas.* In a seminal work, Uhlenbeck [15] proved results on generic properties of the eigenvalues and eigenfunctions of the Laplace–Beltrami operator Δ_g on a compact Riemannian manifold (M, g) without boundary. In order to prove her results on the genericity of the eigenvalues of Δ_g, she used the Thom transversality theorem.

Here we work on the Dirichlet problem for the η-Laplacian $L_g := \Delta_g - g(\nabla\eta, \nabla\cdot)$ on a compact Riemannian manifold (M, g). Our main tools are Hadamard-type variation formulas, where the differentiable function $\eta\colon M \to \mathbb{R}$ is known as *drifting*

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function. Such formulas are the optimal device to apply Teytel's approach [14]. The crucial step in Teytel's work has been to impose a condition that is closely related to the strong Arnol'd hypothesis [2] for double eigenvalues, but significantly easier to check. More precisely, let $\mathcal{M}^r$ denote the space of all C^r Riemannian metrics on M equipped with the C^r topology, for $2 \leq r < \infty$, and let Γ be the set of all $g \in \mathcal{M}^r$ such that the eigenvalues of L_g are all simple, so that each $g \in \Gamma$ can be obtained as a generic member of a differentiable family of self-adjoint operators $A(q)$ indexed by a parameter $q \in \mathcal{M}^r$. In this setting, we know a precise information about the complement of Γ as well as about the structure of the set of deformations of a Riemannian metric g which preserves double eigenvalues of L_g; see Gomes and Marrocos [7], or Teytel's paper for an abstract setting.

The use of Hadamard-type formulas appears in works such as Albert [1], El Soufi and Ilias [6], Henry [8], and Pereira [12], which are good references in the literature on this topic. These formulas have also been used by Gomes and Marrocos [11] to show a density theorem for a class of warping functions that make the spectrum of the Laplacian a warped-simple spectrum. As an application, they provided a partial answer to a question about the generic situation of the multiplicity of the eigenvalues of the Laplacian on principal bundles posed by Zeldich [17]. More importantly, they partially answered a more general question regarding the generic G-simple spectrum of the real Laplace–Beltrami operator on a G-manifold, which was formulated as number 42 in Yau's list of open problems (Yau, 1993).

The interest in understanding how the eigenvalues of a family of self-adjoint operators change emerges naturally in quantum mechanical systems. For instance, von Neumann and Wigner [16] studied the behavior of eigenvalues (energy values) in adiabatic processes. These energy values are eigenvalues of a Hermitian matrix H_{ij}, which can be finite or infinite dimensional. Assuming that $\{H_{ij}\}$ is a family of n-dimensional real Hermitian matrices depending on n^2 real parameters, they showed that the set of parameters for which H_{ij} has double eigenvalues has codimension 2 in $\mathbb{R}^{n^2}$. If one considers only a family of (not the set of all) self-adjoint operators, some transversality hypothesis is necessary to guarantee this type of result. Arnol'd [2] dealt with this question in more detail in the context of membrane vibration frequencies.

In this paper, we consider the Dirichlet problem for the η-Laplacian parametrized by three types of parameters: Riemannian metrics, drifting functions, and bounded domains in M. We derive generic properties of the eigenvalues of the η-Laplacian with respect to variation metrics and/or drifting functions. We also work with perturbations of a bounded domain Ω (given by diffeomorphisms) in a Riemannian manifold (M, g) and establish generic properties of eigenvalues with respect to these perturbations. For this, we consider a family of operators $\eta(t)$-Laplacian where the drifting function η depends on the parameter t, see equations (3.5) and (3.7). Besides, we consider a family $\{L_\eta\}$ of η-Laplacians parameterized by drifting functions η in order to obtain analogous results as in [7, Sections 5 and 6], see Theorem 4.

Before stating our theorems, we recall that a subset $\Gamma \subset \mathcal{M}^r$ is called residual if it contains a countable intersection of open dense sets. The property of metrics in Γ is called generic if it holds on a residual subset.

In the following, we assume all manifolds to be oriented and those that are compact are assumed to have a boundary.

Theorem 1. *Given a compact smooth manifold M^n, $n \geq 2$, there exists a residual subset $\Gamma \subset \mathcal{M}^r$, $2 \leq r < \infty$, such that for all $g \in \Gamma$ the eigenvalues of the Dirichlet problem for the η-Laplacian L_g are simple.*

Let Ω be a bounded domain in a Riemannian manifold (M, g), and $D^r(\Omega)$ (with the fixed C^r topology, $2 \leq r < \infty$) be the set of all $f : \Omega \to M$ which are C^r diffeomorphisms from Ω to $f(\Omega)$. It is known that this set is an affine manifold of a Banach space (see [5]). Then, we show that the following property is generic.

Theorem 2. *Given a Riemannian manifold (M^n, g), $n \geq 2$, define $\mathfrak{D} \subset D^r(\Omega)$, $2 \leq r < \infty$, to be the subset of $f : \Omega \to (M, g)$ such that all eigenvalues of the η-Laplacian L_g on $C_c^\infty(f(\Omega))$ (with Dirichlet boundary condition on $f(\Omega)$) are simple. Then $\mathfrak{D}$ is a residual subset.*

Let $\mathcal{F}^r$ (with the fixed C^r topology, $2 \leq r < \infty$) be the set of all C^r drifting functions η, and let us use the notation $L_\eta := \Delta_g - g(\nabla\eta, \nabla\cdot)$ just to emphasize that the parameter is η.

Theorem 3. *Given a compact Riemannian manifold (M^n, g), $n \geq 2$, there exists a residual subset $\mathcal{E} \subset \mathcal{F}^r$, $2 \leq r < \infty$, such that, for all $\eta \in \mathcal{E}$, the eigenvalues of the Dirichlet problem for L_η are simple.*

Now, we discuss an interesting case of the spectrum of L_η which stems from work of Teytel [14].

Theorem 4. *Let (M^n, g), $n \geq 2$, be a compact Riemannian manifold, and let $\mathcal{E} \subset \mathcal{F}^r$, $2 \leq r < \infty$, be a residual subset such that for all $\eta \in \mathcal{E}$, the eigenvalues of the Dirichlet problem for L_η are simple.*

(1) *The set $\mathcal{F}^r \backslash \mathcal{E}$ has meager codimension 2 in $\mathcal{F}^r$.*

(2) *Take $\eta_0 \in \mathcal{F}^r$, and let λ be an eigenvalue of the operator L_{η_0} of multiplicity 2. Then, in a neighborhood of η_0, the set of all $\eta \in \mathcal{F}^r$ such that L_η admits an eigenvalue $\lambda(\eta)$ of multiplicity 2 near λ form a submanifold of meager codimension 2 in $\mathcal{F}^r$.*

(3) *Consider the same setup as in* (2). *Then, in a neighborhood of* η_0, *the set of all* $\eta \in \mathcal{F}^r$ *which preserves double eigenvalues, i.e.,* L_η *admits an eigenvalue* $\lambda(\eta)$ *of multiplicity* 2 *such that* $\lambda(\eta) = \lambda(\eta_0)$, *form a submanifold of meager codimension* 3 *in* $\mathcal{F}^r$.

Remark 1. Theorem 4 is also true in the context of the family of bounded domains in a Riemannian manifold under the same setting as in Theorem 2. The proof follows the same steps as in the proofs of these theorems.

2. Preliminaries

Let us consider an oriented compact Riemannian manifold (M, g) with boundary ∂M and volume form dV. It is endowed with a weighted measure of the form $\mathrm{dm} = e^{-\eta} dV$, where $\eta\colon M \to \mathbb{R}$ is a differentiable function. Let d be the volume form induced on ∂M and $\mathrm{d}\,\mu = e^{-\eta}\mathrm{d}$ be the corresponding weighted measure on ∂M. We define the η-Laplacian by $L_g = \Delta_g - g(\nabla\eta, \nabla\cdot)$ which is essentially self-adjoint on $C_c^\infty(M)$. Observe that this allows us to use perturbation theory for linear operators [10]. To do this, we consider the set $\mathcal{M}^r$ of all $\mathcal{C}^r$ Riemannian metrics on M. Then every $g \in \mathcal{M}^r$ determines the sequence $0 = \mu_0(g) < \mu_1(g) \leq \mu_2(g) \leq \cdots \leq \mu_k(g) \leq \cdots$ of the eigenvalues of L_g counted with their multiplicities. We regard each eigenvalue $\mu_k(g)$ as a function of g in $\mathcal{M}^r$. Note that, in general, the functions $g \mapsto \mu_k(g)$ are continuous but not differentiable (see [10]). They are differentiable when μ_k is simple. With these notations, the divergence theorem remains valid under the form $\int_M L_g f \,\mathrm{dm} = \int_{\partial M} g(\nabla f, \nu)\,\mathrm{d}\,\mu$. Thus, the integration by parts formula is given by

$$\int_M \ell L_g f \,\mathrm{dm} = -\int_M g(\nabla\ell, \nabla f)\,\mathrm{dm} + \int_{\partial M} \ell g(\nabla f, \nu)\,\mathrm{d}\,\mu$$

for all $f, \ell \in C^\infty(M)$.

The inner product induced by g on the space of $(0,2)$–tensors on M is given by $\langle T, S\rangle = \mathrm{tr}\big(TS^*\big)$, where S^* denotes the adjoint tensor of S. Clearly, we get in local coordinates

$$\langle T, S\rangle = \sum_{i,j,k,l} g^{ik} g^{jl} T_{ij} S_{kl}.$$

Furthermore, we have $\Delta_g f = \langle\nabla^2 f, g\rangle$, where $\nabla^2 f = \nabla df$ is the Hessian of f. We also recall that each $(0,2)$–tensor T on (M, g) can be associated to a unique $(1,1)$–tensor by $g(T(Z), Y) := T(Z, Y)$ for all $Y, Z \in \mathfrak{X}(M)$. We shall slightly abuse notation here and will also write T for this $(1,1)$–tensor. So, we consider the $(0,1)$–tensor given by

$$(\mathrm{div}\, T)(v)(p) = \mathrm{tr}\big(w \mapsto (\nabla_w T)(v)(p)\big),$$

where $p \in M$ and $v \in T_pM$. Moreover, we can define a $(0,1)$–tensor $\operatorname{div}_\eta T$ putting $\operatorname{div}_\eta T := \operatorname{div} T - d\eta \circ T$.

Before proving our main results, we present the following one.

Lemma 1. *Let T be a symmetric $(0,2)$–tensor on a Riemannian manifold (M,g). Then*

$$\operatorname{div}_\eta(T(\phi Z)) = \phi\langle \operatorname{div}_\eta T, Z\rangle + \phi\langle \nabla Z, T\rangle + T(\nabla\phi, Z),$$

for each $Z \in \mathfrak{X}(M)$ and any differentiable function ϕ on M.

Proof. Let $\{e_1,\dots,e_n\}$ be a local orthonormal frame on (M,g). Using the properties of div_η and the symmetry of T, for each $Z \in \mathfrak{X}(M)$ and any differentiable function ϕ on M, we compute

$$\begin{aligned}\operatorname{div}_\eta(T(\phi Z)) &= \phi \operatorname{div}_\eta(T(Z)) + g(\nabla\phi, T(Z)) \\ &= \phi(\operatorname{div} T)(Z) + \phi\sum_i g(T(\nabla_{e_i} Z), e_i) \\ &\quad - \phi g(\nabla\eta, T(Z)) + T(\nabla\phi, Z) \\ &= \phi(\operatorname{div}_\eta T)(Z) + \phi\langle\nabla Z, T\rangle + T(\nabla\phi, Z).\end{aligned}$$

To complete the proof is sufficient to use the duality $(\operatorname{div}_\eta T)(Z) = \langle \operatorname{div}_\eta T, Z\rangle$. ∎

Let us observe that for every $X \in \mathfrak{X}(M)$ the operator $\operatorname{div}_\eta X = \operatorname{div} X - g(\nabla\eta, X)$ has the same properties of the operator $\operatorname{div} X$ as well as is valid

$$\int_M \operatorname{div}_\eta X \,\mathrm{dm} = \int_{\partial M} g(X,\nu)\,\mathrm{d}\mu.$$

3. Hadamard-type variation formulas

Consider a differentiable variation $g(t)$ of the metric g, so that $(M, g(t), dm_t)$ is a Riemannian manifold with a differentiable measure. Denoting by H the $(0,2)$–tensor given by $H_{ij} = \frac{d}{dt}\big|_{t=0} g_{ij}(t)$ and writing $h = \langle H, g\rangle$, we easily get $\frac{d}{dt}\big|_{t=0}\mathrm{dm}_t = \frac{1}{2}h\,\mathrm{dm}$. From now on, we shall write in local coordinates $f_i = \partial_i f$. We first prove the following lemma.

Lemma 2. *Let (M,g) be a Riemannian manifold and $g(t)$ be a differentiable variation of the metric g. Then, for all $f \in C_c^\infty(M)$, we have*

$$L'f = \Big\langle \frac{1}{2}dh - \operatorname{div}_\eta H, df\Big\rangle - \langle H, \nabla^2 f\rangle,$$

where $L' := \frac{d}{dt}\big|_{t=0} L_{g(t)}$.

Proof. Since $\langle df, d\ell\rangle = g^{ij}(t) f_i \ell_j$, for $\ell \in C_c^\infty(M)$, and

$$\frac{d}{dt}\Big|_{t=0} g^{ij}(t) = -g^{ik} g^{js} H_{ks},$$

we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\langle df, d\ell\rangle &= -g^{ik} g^{js} H_{ks} f_i \ell_j = -H(g^{ik} f_i \partial_k, g^{js}\ell_j \partial_s)\\ &= -H(\nabla f, \nabla \ell).\end{aligned} \tag{3.1}$$

By integration by parts formula, we get

$$\int_M \ell L_{g(t)} f \,\mathrm{dm}_t = -\int_M \langle d\ell, df\rangle \,\mathrm{dm}_t\,.$$

Hence, from equation (3.1), we have at $t = 0$

$$\int_M \ell L' f \,\mathrm{dm} + \frac{1}{2}\int_M \ell h L f \,\mathrm{dm} = \int_M H(\nabla f, \nabla \ell)\,\mathrm{dm} - \frac{1}{2}\int_M h\langle d\ell, df\rangle \,\mathrm{dm}\,. \tag{3.2}$$

Applying Lemma 1 for $T = H$, $\phi = \ell$ and $Z = \nabla f$ we have

$$\mathrm{div}_\eta(H(\ell\nabla f)) = \ell\langle \mathrm{div}_\eta H, df\rangle + \ell\langle H, \nabla^2 f\rangle + H(\nabla f, \nabla\ell). \tag{3.3}$$

Moreover,

$$\mathrm{div}_\eta(\ell h\nabla f) = \ell h L f + \ell\langle dh, df\rangle + h\langle d\ell, df\rangle. \tag{3.4}$$

Hence, plugging (3.3) and (3.4) into (3.2), we obtain

$$\int_M \ell L' f \,\mathrm{dm} = \int_M \ell\Big(\frac{1}{2}\langle dh, df\rangle - \langle \mathrm{div}_\eta H, df\rangle - \langle H, \nabla^2 f\rangle\Big)\,\mathrm{dm},$$

which concludes the proof of the lemma. ∎

Next, we consider a differentiable function $\eta\colon I\times M \to \mathbb{R}$ and write for simplicity $\dot\eta = \frac{d}{dt}\big|_{t=0}\eta(t)$. For all $f \in C^\infty(M)$, we define

$$\overline{L}_t f := \Delta_t f - g(t)(\nabla\eta(t), \nabla f). \tag{3.5}$$

Thus,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\overline{L}_t f &= \Delta' f - \Big(\frac{d}{dt}\Big|_{t=0} g^{ij}(t)\Big)\eta_i f_j - g^{ij}\partial_i \frac{d}{dt}\Big|_{t=0}\eta(t) f_j\\ &= L' f - \langle\nabla\dot\eta, \nabla f\rangle.\end{aligned}$$

The next result extends Berger's [4, Lemma 3.15] to the η-Laplacian. Firstly, we note that given an eigenvalue $\lambda(g_0)$ of $\overline{L}_{g_0}$ with multiplicity $m(\lambda(g_0))$, there are a positive number $\epsilon_{\lambda(g_0),g_0}$ and a neighborhood $\mathcal{V}_\epsilon$ in $\mathcal{M}^r$, $2 \le r < \infty$, such that for all $g \in \mathcal{V}_\epsilon$ one has

$$\sum_{\{|\lambda-\lambda(g_0)|<\epsilon_{\lambda(g_0),g_0}\}\cap \operatorname{spec}(\overline{L}_g)} m(\lambda) = m(\lambda(g_0)). \tag{3.6}$$

Indeed, equation (3.6) is a consequence of the continuity of a finite system of eigenvalues, see [10, Section 5, Chapter 4]. In this setting, we prove the following generic result.

Proposition 1. *Let (M, g) be a compact Riemannian manifold. Consider a real analytic one-parameter family of Riemannian metrics $g(t)$ on M with $g = g(0)$. If λ is an eigenvalue of multiplicity $m > 1$ for the η-Laplacian L_g, then there exist $\varepsilon > 0$, scalars $\lambda_i(t)$ $(i = 1, \dots, m)$, and functions $\phi_i(t)$ varying analytically in t such that for all $|t| < \varepsilon$ the following relations hold:*

(1) *$L_{g(t)}\phi_i(t) = \lambda_i(t)\phi_i(t)$;*

(2) *$\lambda_i(0) = \lambda$;*

(3) *$\{\phi_i(t)\}$ is orthonormal in $L^2(M, \mathrm{dm}_t)$.*

Proof. First, let us consider an extension $g(z)$ of $g(t)$ to a domain D_0 of the complex plane $\mathbb{C}$. So, we consider the operator

$$L_{g(z)}: \mathcal{C}^\infty(M;\mathbb{C}) \to \mathcal{C}^\infty(M;\mathbb{C}),$$

which in a local coordinate system is given by

$$L_{g(z)} f = g^{ij}(z)\Big(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma^k_{ij}(z)\frac{\partial f}{\partial x_k} - \frac{\partial \eta}{\partial x_i}\frac{\partial f}{\partial x_j}\Big)$$

for all $f \in \mathcal{C}^\infty(M;\mathbb{C})$, with

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}\Big(\frac{\partial g_{i\ell}}{\partial x_j} + \frac{\partial g_{j\ell}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell}\Big).$$

Now, we observe that the domain $D = H^2(M) \cap H^1_0(M)$ of the operator $L_{g(z)}$ is independent of z, since M is compact, any two metrics are equivalent. Besides, the application $z \mapsto L_{g(z)} f$ is holomorphic for $z \in D_0$ and for every $f \in D$. Thus, $L_{g(z)}$ is a holomorphic family of type (A) in [10]. Now, we need to prove that the operator $L_{g(z)}$ is self-adjoint with fixed inner product. For this purpose, for each t, we can construct an isometry

$$P: L^2(M, \mathrm{dm}) \to L^2(M, \mathrm{dm}_t)$$

taking, for each u, $P(u) = \frac{\sqrt[4]{\det(g_{ij}(0))}}{\sqrt[4]{\det(g_{ij}(t))}}u$. In fact,

$$\int_M P(u)P(v)\,\mathrm{dm}_t = \int_M \frac{\sqrt{\det(g_{ij}(0))}}{\sqrt{\det(g_{ij}(t))}} uv\,\mathrm{dm}_t = \int_M uv\,\mathrm{dm}\,.$$

Thus, the operator $\widetilde{L}_t := P^{-1} \circ L_t \circ P$ will have the same eigenvalues of

$$L_t \colon H^2(M, \mathrm{dm}_t) \cap H_0^1(M, \mathrm{dm}_t) \to L^2(M, \mathrm{dm}_t).$$

But the compactness of M implies that $\widetilde{L}_t$ is self-adjoint, since

$$\begin{aligned}\int_M v\widetilde{L}_t u\,\mathrm{dm} &\overset{\text{(isom.)}}{=} \int_M P(v)L_tP(u)\,\mathrm{dm}_t = \int_M P(u)L_tP(v)\,\mathrm{dm}_t \\ &\overset{\text{(isom.)}}{=} \int_M P^{-1}P(u)P^{-1}L_tP(v)\,\mathrm{dm} = \int_M u\widetilde{L}_t v\,\mathrm{dm}\,.\end{aligned}$$

Under these conditions, we can apply a theorem due to Rellich [13] or in Kato's [10, Theorem 3.9] to obtain the result of the proposition. ∎

We observe that the same result of Proposition 1 holds for the operator $\overline{L}_t$ defined in (3.5). Now, we will derive the first Hadamard-type variation formula which generalizes substantially one of Berger's formulas [4].

Proposition 2. *Let (M, g) be a compact Riemannian manifold, $g(t)$ be a differentiable variation of the metric g, $\phi_i(t) \in C^\infty(M)$ be a differentiable family of functions, and $\lambda(t)$ be a differentiable family of real numbers such that $\lambda_i(0) = \lambda$ for each $i = 1, \ldots, m$ and, for all t,*

$$\begin{cases} -\overline{L}_t\phi_i(t) = \lambda_i(t)\phi_i(t) & \text{in } M, \\ \phi_i(t) = 0 & \text{on } \partial M, \end{cases}$$

with $\langle \phi_i(t), \phi_j(t)\rangle_{L^2(M,\mathrm{dm}_t)} = \delta_{ij}$. Then the derivative of $t \mapsto (\lambda_i(t) + \lambda_j(t))\delta_{ij}$ is given by

$$(\lambda_i + \lambda_j)'\delta_{ij} = \int_M \Big\langle \frac{1}{2}L(\phi_i\phi_j)g - 2d\phi_i \otimes d\phi_j, H\Big\rangle\,\mathrm{dm} + \int_M \langle \nabla\dot{\eta}, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,. \tag{3.7}$$

Proof. We begin by proving the case when η does not depend on t. Differentiating the equation $-L_{g(t)}\phi_i(t) = \lambda_i(t)\phi_i(t)$, we have at $t = 0$, $-L'\phi_i - L\phi_i' = \lambda_i'\phi_i + \lambda_i\phi_i'$, so

$$-\int_M(\phi_jL'\phi_i + \phi_jL\phi_i')\,\mathrm{dm} = \int_M(\lambda_i'\phi_i\phi_j + \lambda_j\phi_j\phi_i')\,\mathrm{dm} = \lambda_i'\int_M\phi_j\phi_i\,\mathrm{dm} - \int_M\phi_i'L\phi_j\,\mathrm{dm}\,.$$

By integration by parts and the fact that $\phi_i = 0$ on ∂M, we obtain

$$\lambda_i' \delta_{ij} = -\int_M \phi_j L' \phi_i \,\mathrm{dm}\,.$$

Thus, writing $s_{ij} = (\lambda_i' + \lambda_j')$, we deduce from Lemma 2

$$\begin{aligned} -s_{ij}\delta_{ij} &= \int_M \phi_j L'\phi_i \,\mathrm{dm} + \int_M \phi_i L'\phi_j \,\mathrm{dm} \\ &= \int_M \left(\left\langle \frac{1}{2} dh - \operatorname{div}_\eta H, \phi_j d\phi_i + \phi_i d\phi_j \right\rangle - \langle H, \phi_j \nabla^2 \phi_i + \phi_i \nabla^2 \phi_j \rangle \right) \mathrm{dm}, \end{aligned}$$

and then

$$\begin{aligned} -s_{ij}\delta_{ij} = & \int_M \left\langle \frac{1}{2} dh, d(\phi_i\phi_j) \right\rangle \mathrm{dm} - \int_M \phi_j \left(\langle \operatorname{div}_\eta H, d\phi_i \rangle + \langle H, \nabla^2 \phi_i \rangle \right) \mathrm{dm} \\ & - \int_M \phi_i \left(\langle \operatorname{div}_\eta H, d\phi_j \rangle + \langle H, \nabla^2 \phi_j \rangle \right) \mathrm{dm}\,. \end{aligned}$$

Next, we use Lemma 1 and again the integration by parts formula to get

$$-s_{ij}\delta_{ij} = -\int_M \frac{h}{2} L(\phi_i\phi_j)\,\mathrm{dm} + 2\int_M H(\nabla\phi_i, \nabla\phi_j)\,\mathrm{dm},$$

or equivalently

$$s_{ij}\delta_{ij} = \int_M \left\langle \frac{1}{2} L(\phi_i\phi_j) g - 2 d\phi_i \otimes d\phi_j, H \right\rangle \mathrm{dm}\,.$$

In the general case, we differentiate the equation $-\overline{L}_t \phi_i(t) = \lambda_i(t)\phi_i(t)$ at $t = 0$ to obtain

$$-\overline{L}'\phi_i - L\phi_i' = \lambda_i'\phi_i + \lambda_i \phi_i'.$$

So, $-L'\phi_i - L\phi_i' = \lambda_i'\phi_i + \lambda_j \phi_i' - \langle \nabla\dot{\eta}, \nabla\phi_i \rangle$. Thus, we have that

$$\lambda_i'\delta_{ij} = -\int_M \phi_j L'\phi_i \,\mathrm{dm} + \int_M \phi_j \langle \nabla\dot{\eta}, \nabla\phi_i \rangle \,\mathrm{dm}\,.$$

A calculation analogous to the one above completes the proof. ■

Now, we show how to extend for the η-Laplacian a result by El Soufi and Ilias [6, Corollary 2.1].

Proposition 3. *Let (M,g) be a Riemannian manifold, $\Omega \subset M$ be a bounded domain, $f_t\colon \Omega \to (M,g)$ be an analytic family of diffeomorphisms from Ω to $\Omega_t = f_t(\Omega)$ such that f_0 is the identity map, and λ be an eigenvalue of multiplicity $m > 1$. Then there exist an analytic family of m functions $\phi_i(t) \in C^\infty(\Omega_t)$ with $\langle \phi_i(t), \phi_j(t)\rangle_{L^2(\Omega_t,\mathrm{dm})} = \delta_{ij}$ and real numbers $\lambda_i(t)$ with $\lambda_i(0) = \lambda$, such that, for all t and $i = 1, \ldots, m$, they are solutions for the Dirichlet problem*

$$\begin{cases} -L\phi_i(t) = \lambda_i(t)\phi_i(t) & \text{in } \Omega_t, \\ \phi_i(t) = 0 & \text{on } \partial\Omega_t. \end{cases}$$

Moreover, the derivative of the curve $t \mapsto (\lambda_i(t) + \lambda_j(t))\delta_{ij}$ is given by

$$(\lambda_i + \lambda_j)'\delta_{ij} = -2\int_{\partial\Omega} \langle V, \nu\rangle \frac{\partial \phi_i}{\partial \nu}\frac{\partial \phi_j}{\partial \nu}\, \mathrm{d}\mu, \tag{3.8}$$

where $\langle V, \nu\rangle = g(V,\nu)$ and $V = \frac{d}{dt}\big|_{t=0} f_t$.

Proof. We consider the family of metrics $g(t) = f_t^* g$ on Ω. So, we can apply Proposition 1 for $\overline{L}_t$. Considering

$$\overline{L}_t(\phi_i(t)\circ f_t) := \Delta_t(\phi_i(t)\circ f_t) - g(t)(\nabla(\eta\circ f_t), \nabla(\phi_i(t)\circ f_t)),$$

we obtain

$$\overline{L}_t(\phi_i(t)\circ f_t)(p) = -\lambda_i(t)\phi_i(t)\circ f_t(p).$$

For $\bar{\phi}_i(t) = \phi_i(t)\circ f_t$, for all t we have $\langle \bar{\phi}_i(t), \bar{\phi}_j(t)\rangle_{L^2(\Omega,\mathrm{dm}_t)} = \delta_{ij}$ and

$$\begin{cases} -\overline{L}_t\bar{\phi}_i(t) = \lambda_i(t)\bar{\phi}_i(t) & \text{in } \Omega \\ \bar{\phi}_i(t) = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\phi_i \circ f_0 = \phi_i$ and $\eta(t) = \eta\circ f_t$, we have by Proposition 2

$$s_{ij}\delta_{ij} = \int_\Omega \frac{h}{2}L(\phi_i\phi_j)\,\mathrm{dm} - 2\int_\Omega H(\nabla\phi_i, \nabla\phi_j)\,\mathrm{dm} + \int_\Omega \langle \nabla\dot{\eta}, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}.$$

Recall that $H = \frac{d}{dt}\big|_{t=0} f_t^* g = \mathscr{L}_V g$, where $V = \frac{d}{dt}\big|_{t=0} f_t$. Then

$$\begin{aligned} s_{ij}\delta_{ij} =& \int_\Omega \frac{1}{2}L(\phi_i\phi_j)\langle g, H\rangle\,\mathrm{dm} - 2\int_\Omega \Big(\frac{d}{dt}\Big|_{t=0} f_t^* g\Big)(\nabla\phi_i, \nabla\phi_j)\,\mathrm{dm} \\ &+ \int_\Omega \langle \nabla\dot{\eta}, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm} \end{aligned}$$

$$= \int_\Omega L(\phi_i\phi_j)\,\mathrm{div}\, V\,\mathrm{dm} - 2\int_\Omega \langle \nabla_{\nabla\phi_i} V, \nabla\phi_j\rangle\,\mathrm{dm}$$
$$- 2\int_\Omega \langle \nabla_{\nabla\phi_j} V, \nabla\phi_i\rangle\,\mathrm{dm} + \int_\Omega \langle \nabla\dot\eta, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,.$$

But,

$$\langle \nabla_{\nabla\phi_i} V, \nabla\phi_j\rangle = \mathrm{div}_\eta(\langle V, \nabla\phi_j\rangle\nabla\phi_i) + \lambda\langle V, \nabla\phi_j\rangle\phi_i - \nabla^2\phi_j(V, \nabla\phi_i).$$

Since $\lambda = \lambda_i(0) = \lambda_j(0)$ and $\frac{s_{ij}}{2}\delta_{ij} = a_{ij}$, we have

$$\begin{aligned} a_{ij} = &-\lambda\int_\Omega \phi_i\phi_j\,\mathrm{div}\,V\,\mathrm{dm} + \int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle\,\mathrm{div}\,V\,\mathrm{dm} - \int_\Omega \mathrm{div}_\eta(\langle V, \nabla\phi_j\rangle\nabla\phi_i)\,\mathrm{dm} \\ &-\lambda\int_\Omega \langle V, \nabla\phi_j\rangle\phi_i\,\mathrm{dm} + \int_\Omega \nabla^2\phi_j(V, \nabla\phi_i)\,\mathrm{dm} - \int_\Omega \mathrm{div}_\eta(\langle V, \nabla\phi_i\rangle\nabla\phi_j)\,\mathrm{dm} \\ &-\lambda\int_\Omega \langle V, \nabla\phi_i\rangle\phi_j\,\mathrm{dm} + \int_\Omega \nabla^2\phi_i(V, \nabla\phi_j)\,\mathrm{dm} + \frac{1}{2}\int_\Omega \langle\nabla\dot\eta, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm} \\ = &-\lambda\int_\Omega \big(\phi_i\phi_j\,\mathrm{div}\,V + \langle V, \nabla(\phi_i\phi_j)\rangle\big)\,\mathrm{dm} - \int_{\partial\Omega} \langle V, \nabla\phi_j\rangle\langle\nabla\phi_i, \nu\rangle\,\mathrm{d}\mu \\ &-\int_{\partial\Omega} \langle V, \nabla\phi_i\rangle\langle\nabla\phi_j, \nu\rangle\,\mathrm{d}\mu + \int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle\,\mathrm{div}\,V\,\mathrm{dm} + \int_\Omega \nabla^2\phi_j(V, \nabla\phi_i)\,\mathrm{dm} \\ &+\int_\Omega \nabla^2\phi_i(V, \nabla\phi_j)\,\mathrm{dm} + \frac{1}{2}\int_\Omega \langle\nabla\dot\eta, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,. \end{aligned}$$

As $\phi_i = 0$ on $\partial\Omega$, we have $\nabla\phi_i = \langle\nabla\phi_i, \nu\rangle\nu = \frac{\partial\phi_i}{\partial\nu}\nu$ on $\partial\Omega$. Moreover,

$$\begin{aligned} &\mathrm{div}_\eta(\langle\nabla\phi_i, \nabla\phi_j\rangle V) + \langle\nabla\phi_i, \nabla\phi_j\rangle\langle\nabla\eta, V\rangle \\ &\quad = \mathrm{div}(\langle\nabla\phi_i, \nabla\phi_j\rangle V) \\ &\quad = \langle\nabla\phi_i, \nabla\phi_j\rangle\,\mathrm{div}\,V + \langle\nabla\langle\nabla\phi_i, \nabla\phi_j\rangle, V\rangle \\ &\quad = \langle\nabla\phi_i, \nabla\phi_j\rangle\,\mathrm{div}\,V + \nabla^2\phi_j(V, \nabla\phi_i) + \nabla^2\phi_i(V, \nabla\phi_j). \end{aligned}$$

So,

$$\begin{aligned} a_{ij} = &-\lambda\int_\Omega \mathrm{div}(\phi_i\phi_j V)\,\mathrm{dm} - 2\int_{\partial\Omega} \langle V, \nu\rangle\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\,\mathrm{d}\mu + \int_\Omega \mathrm{div}_\eta(\langle\nabla\phi_i, \nabla\phi_j\rangle V)\,\mathrm{dm} \\ &+\int_\Omega \langle\nabla\phi_i, \nabla\phi_j\rangle\langle\nabla\eta, V\rangle\,\mathrm{dm} + \frac{1}{2}\int_\Omega \langle\nabla\dot\eta, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,. \end{aligned}$$

It follows that

$$\begin{aligned} a_{ij} &= -\lambda\int_{\Omega}\operatorname{div}(\phi_i\phi_j V)\,\mathrm{dm} - 2\int_{\partial\Omega}\langle V,\nu\rangle\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\,\mathrm{d}\mu + \int_{\partial\Omega}\langle V,\nu\rangle\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\,\mathrm{d}\mu \\ &\quad + \int_{\Omega}\langle\nabla\phi_i,\nabla\phi_j\rangle\langle\nabla\eta,V\rangle\,\mathrm{dm} + \frac{1}{2}\int_{\Omega}\langle\nabla\dot{\eta},\nabla(\phi_i\phi_j)\rangle\,\mathrm{dm} \\ &= -\int_{\partial\Omega}\langle V,\nu\rangle\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\,\mathrm{d}\mu - \lambda\int_{\Omega}\operatorname{div}(\phi_i\phi_j V)\,\mathrm{dm} \\ &\quad + \int_{\Omega}\langle\nabla\phi_i,\nabla\phi_j\rangle\langle\nabla\eta,V\rangle\,\mathrm{dm} + \frac{1}{2}\int_{\Omega}\langle\nabla\dot{\eta},\nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,. \end{aligned}$$

On the other hand,

$$0 = \int_{\Omega}\operatorname{div}_{\eta}(\phi_i\phi_j V)\,\mathrm{dm} = \int_{\Omega}\operatorname{div}(\phi_i\phi_j V)\,\mathrm{dm} - \int_{\Omega}\phi_i\phi_j\langle\nabla\eta,V\rangle\,\mathrm{dm}\,.$$

Hence,

$$\begin{aligned} a_{ij} &= -\int_{\partial\Omega}\langle V,\nu\rangle\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\,\mathrm{d}\mu + \int_{\Omega}\big(\langle\nabla\phi_i,\nabla\phi_j\rangle - \lambda\phi_i\phi_j\big)\langle\nabla\eta,V\rangle\,\mathrm{dm} \\ &\quad + \frac{1}{2}\int_{\Omega}\langle\nabla\dot{\eta},\nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}\,. \end{aligned} \tag{3.9}$$

Since $\eta(t,p) = \eta\circ f(t,p)$ we have

$$\begin{aligned} \dot{\eta} &= \frac{d}{dt}\Big|_{t=0}\eta(t,p) = \frac{d}{dt}\Big|_{t=0}(\eta\circ f)(t,p) \\ &= d\eta\Big|_{f(0,p)}\cdot\frac{d}{dt}\Big|_{t=0}f_t(p) = d\eta|_p(V) = \langle\nabla\eta,V\rangle. \end{aligned}$$

Next, we use that $\lambda_i(0) = \lambda_j(0) = \lambda$, $L(\phi_i\phi_j) = \phi_i L\phi_j + \phi_j L\phi_i + 2\langle\nabla\phi_i,\nabla\phi_j\rangle$ and the integration by parts formula to calculate

$$\begin{aligned} \frac{1}{2}\int_{\Omega}\langle\nabla\dot{\eta},\nabla(\phi_i\phi_j)\rangle\,\mathrm{dm} &= -\frac{1}{2}\int_{\Omega}\dot{\eta}L(\phi_i\phi_j)\,\mathrm{dm} + \frac{1}{2}\int_{\partial\Omega}\dot{\eta}\langle\nu,\nabla(\phi_i\phi_j)\rangle\,\mathrm{d}\mu \\ &= \int_{\Omega}\langle\nabla\eta,V\rangle\big(\lambda\phi_i\phi_j - \langle\nabla\phi_i,\nabla\phi_j\rangle\big)\,\mathrm{dm}\,. \end{aligned}$$

This computation tells us that the last two terms in (3.9) cancel each other, which concludes the proof of the proposition. ∎

4. Applications

In this section, we concentrate on the applications of the Hadamard-type formulas. We first prove the following.

Proposition 4. *Let (M, g_0) be a compact Riemannian manifold and λ be an eigenvalue of L_{g_0} for the Dirichlet problem with multiplicity $m > 1$. Take the positive number ϵ_{λ,g_0} and the neighborhood $\mathcal{V}_\epsilon$ of g_0 in $\mathcal{M}^r$ as in (3.6). Then, for each open neighborhood $\mathcal{U} \subset \mathcal{V}_\epsilon$, there is $g \in \mathcal{U}$ such that all eigenvalues $\lambda(g)$ of L_g with $|\lambda(g) - \lambda| < \epsilon_{\lambda,g_0}$ are simple.*

Proof. We argue by contradiction. Suppose that there is an open neighborhood $\mathcal{U} \subset \mathcal{V}_\epsilon$ of g_0 such that for all $g \in \mathcal{U}$ the eigenvalue $\lambda(g)$ of L_g with $|\lambda(g) - \lambda| < \epsilon_{\lambda,g_0}$ has multiplicity $m > 1$. In this case, for any symmetric $(0, 2)$–tensor T on (M, g) the perturbation $g(t) = g + tT$ fails to split the eigenvalue λ. The eigenvalue curves $\lambda(t)$ satisfy

$$\begin{cases} -L_{g(t)}\phi_i(t) = \lambda(t)\phi_i(t) & \text{in } M, \\ \phi_i(t) = 0 & \text{on } \partial M. \end{cases}$$

Since $H = \frac{d}{dt}g(t) = T$ and $L = L_g$, by Proposition 2 we have

$$\lambda'\delta_{ij} = \int_M \Big\langle \frac{1}{4}L(\phi_i\phi_j)g - d\phi_i \otimes d\phi_j, T\Big\rangle \,\mathrm{dm}\,.$$

Now, considering the symmetrization tensor $S_{ij} = \frac{d\phi_i \otimes d\phi_j + d\phi_j \otimes d\phi_i}{2}$ and using the fact that

$$\langle d\phi_i \otimes d\phi_j, T\rangle = \langle d\phi_j \otimes d\phi_i, T\rangle,$$

we deduce the identity

$$\lambda'\delta_{ij} = \int_M \Big\langle \frac{1}{4}L(\phi_i\phi_j)g - S_{ij}, T\Big\rangle \,\mathrm{dm}\,.$$

If $i \neq j$, we have

$$\frac{1}{4}L(\phi_i\phi_j)g = S_{ij}. \tag{4.1}$$

Furthermore, taking the trace in equation (4.1), we have

$$\begin{aligned} g(\nabla\phi_i, \nabla\phi_j) &= \frac{n}{4}L(\phi_i\phi_j) = \frac{n}{4}\big(\phi_i L\phi_j + \phi_j L\phi_i + 2g(\nabla\phi_i, \nabla\phi_j)\big) \\ &= \frac{n}{2}(-\lambda\phi_i\phi_j + g(\nabla\phi_i, \nabla\phi_j)). \end{aligned} \tag{4.2}$$

For $n \neq 2$ we can write

$$\frac{n\lambda}{n-2}\phi_i\phi_j = g(\nabla\phi_i, \nabla\phi_j).$$

Fixing $p \in M$, we consider an integral curve α in M such that $\alpha(0) = p$ and $\alpha'(s) = \nabla\phi_i(\alpha(s))$. Defining $\beta(s) := \phi_j(\alpha(s))$, we compute

$$\begin{aligned}\beta'(s) =&\langle\nabla\phi_j(\alpha(s)), \alpha'(s)\rangle = g(\nabla\phi_j, \nabla\phi_i)(\alpha(s)) = \frac{n\lambda}{n-2}\phi_i\phi_j(\alpha(s))\\ =&\frac{n\lambda}{n-2}\phi_i(\alpha(s))\beta(s),\end{aligned}$$

which is a contradiction, since M is compact. For the case $n = 2$, we have from equation (4.2) that $\phi_i\phi_j = 0$. Then, it follows from the principle of the unique continuation [9] that at least one of the eigenfunctions vanishes, which is again a contradiction. Therefore, we completed the proof of the proposition. ■

Proposition 5. *Let (M, g) be a Riemannian manifold and let Ω a bounded domain in M. Let λ be an eigenvalue of L_g for the Dirichlet problem with multiplicity $m > 1$. Take the positive number $\epsilon_{\lambda,\Omega}$ and the neighborhood $\mathcal{V}_\epsilon$ of the identity in $D^r(\Omega)$ as in (3.6). Then, for any open neighborhood $\mathcal{U} \subset \mathcal{V}_\epsilon$, there is a diffeomorphism f such that all eigenvalues $\lambda(f)$ with $|\lambda(f) - \lambda| < \epsilon_{\lambda,\Omega}$ are simple.*

Proof. We also argue by contradiction. Suppose that there is an open neighborhood $\mathcal{U} \subset \mathcal{V}_\epsilon$ of the identity such that for all $f \in \mathcal{U}$ the eigenvalue $\lambda(f)$ of L_g with $|\lambda(f) - \lambda| < \epsilon_{\lambda,\Omega}$ has multiplicity $m > 1$. Then, it follows from (3.8) that $\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu} = 0$ on $\partial\Omega$. In this way, we have either $\frac{\partial\phi_i}{\partial\nu} = 0$ or $\frac{\partial\phi_j}{\partial\nu} = 0$ in some open set U of $\partial\Omega$. If $\frac{\partial\phi_i}{\partial\nu} = 0$ in U, since $\phi_i = 0$ on $\partial\Omega$, it follows from the unique continuation principle [9] that $\phi_i = 0$ on Ω, which is a contradiction. ■

Proof of Theorem 1. Let $\mathcal{C}_m$ be the set of metrics in $\mathcal{M}^r$ such that the first m eigenvalues of L_g are simple. It is known that if these eigenvalues depend continuously on the metric (see [3]), then for each m the set $\mathcal{C}_m$ is open in $\mathcal{M}^r$. On the other hand, by Proposition 4 the set $\mathcal{C}_m$ is dense in $\mathcal{M}^r$. Since $\mathcal{M}^r$ is a complete metric space in the C^r topology, the set $\Gamma = \bigcap_{m=1}^{\infty}\mathcal{C}_m$ is dense. ■

Proof of Theorem 2. Since $D^r(\Omega)$ is an affine manifold in a Banach space, similar arguments to those above allow us to obtain Theorem 2. ■

Proof of Theorem 3. The proof follows from the analogous steps for the variation of metrics case. We shall present a brief sketch of the last claim. Indeed, the main tool is to show a proposition analogous to Proposition 4 for the η-variation case. For this, first note that from equation (3.7) we get $(\lambda_i + \lambda_j)'\delta_{ij} = \int_M\langle\nabla\dot{\eta}, \nabla(\phi_i\phi_j)\rangle\,\mathrm{dm}$. Second, by using the integration by parts formula, we obtain $2\lambda'\delta_{ij} = -\int_M \dot\eta L_\eta(\phi_i\phi_j)\,\mathrm{dm}$, since $\lambda_i = \lambda_j = \lambda$. Now, we argue as in the proof of the Proposition 4 to get a contradiction, namely, the integral $\int_M \dot\eta L_\eta(\phi_i\phi_j)\,\mathrm{dm} = 0$ for all $\dot\eta \in \mathcal{F}^r$. This is equivalent to $g(\nabla\phi_i, \nabla\phi_j) - \lambda\phi_i\phi_j = 0$, but nontrivial eigenfunctions cannot satisfy it. Finally, we can proceed as in the proof of Theorem 1, and this completes our sketch. ∎

Proof of Theorem 4. Following Teytel's approach as in Gomes and Marrocos [7, Section 5], we define the linear functionals

$$f_{ij}(\dot\eta) = \int_M \phi_i L'_\eta \phi_j \,\mathrm{dm},$$

where $\dot\eta \in \mathcal{F}^r$ and $L'_\eta = \frac{d}{dt}|_{t=0} L_{\eta(t)} = g(\nabla\dot\eta, \nabla\cdot)$.

In order to prove (1) and (2), it is enough to verify that there exist two orthonormal eigenfunctions ϕ_1 and ϕ_2 associated to λ such that the functionals $f_{11} - f_{22}$ and f_{12} are linearly independent, see [7, Remark 2] for details. However, we prove a stronger condition. Namely, f_{11}, f_{12}, f_{22} are linearly independent, so that we can apply the implicit function theorem as in the proof of [14, Theorem 1.1] to get (3) as well.

First of all, we use the integration by parts formula to obtain

$$f_{ij}(\dot\eta) = -\int_M \dot\eta L_\eta(\phi_i\phi_j)\,\mathrm{dm}.$$

So,

$$0 = \alpha f_{11} + \beta f_{12} + \gamma f_{22} = \int_M \dot\eta(\alpha L_\eta(\phi_1^2) + \beta L_\eta(\phi_1\phi_2) + \gamma L_\eta(\phi_2^2))\,\mathrm{dm}.$$

Whence, we conclude that

$$\alpha(|\nabla\phi_1|^2 - \lambda\phi_1^2) + \beta g((\nabla\phi_1, \nabla\phi_2) - \lambda\phi_1\phi_2) + \gamma(|\nabla\phi_2|^2 - \lambda\phi_2^2) = 0.$$

Now, we can proceed as in [7, Section 5.1] to complete our proof. ∎

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References

[1] J. H. Albert, Generic properties of eigenfunctions of elliptic partial differential operators. *Trans. Amer. Math. Soc.* **238** (1978), 341–354 Zbl 0379.35023 MR 0471000

[2] V. I. Arnol'd, Modes and quasimodes. *Funkcional. Anal. i Priložen.* **6** (1972), no. 2, 12–20 (in Russian) MR 0297274. English translation: *Funct. Anal. Its Appl.* **6** (1972), 94–101 Zbl 0251.70012

[3] S. Bando and H. Urakawa, Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds. *Tohoku Math. J. (2)* **35** (1983), no. 2, 155–172 Zbl 0534.58038 MR 0699924

[4] M. Berger, Sur les premières valeurs propres des variétés riemanniennes. *Compositio Math.* **26** (1973), 129–149 Zbl 0257.53048 MR 0316913

[5] M. C. Delfour and J.-P. Zolésio, *Shapes and geometries*. Adv. Des. Control 4, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001 Zbl 1002.49029 MR 1855817

[6] A. El Soufi and S. Ilias, Domain deformations and eigenvalues of the Dirichlet Laplacian in a Riemannian manifold. *Illinois J. Math.* **51** (2007), no. 2, 645–666 Zbl 1124.49035 MR 2342681

[7] J. N. V. Gomes and M. A. M. Marrocos, On eigenvalue generic properties of the Laplace-Neumann operator. *J. Geom. Phys.* **135** (2019), 21–31 Zbl 06997112 MR 3872620

[8] D. Henry, *Perturbation of the boundary in boundary-value problems of partial differential equations*. London Math. Soc. Lecture Note Ser. 318, Cambridge University Press, Cambridge, 2005 Zbl 1170.35300 MR 2160744

[9] L. Hörmander, *Linear partial differential operators*. Die Grundlehren Math. Wiss. 116, Springer, New York, 1969 Zbl 0175.39201 MR 0248435

[10] T. Kato, *A short introduction to perturbation theory for linear operators*. Springer, Berlin etc., 1982 Zbl 0493.47008 MR 0678094

[11] M. A. M. Marrocos and J. N. V. Gomes, Generic spectrum of warped products and G-manifolds. *J. Geom. Anal.* **29** (2019), no. 4, 3124–3134 Zbl 07125749 MR 4015431

[12] M. C. Pereira, Generic simplicity of eigenvalues for a Dirichlet problem of the bilaplacian operator. *Electron. J. Differential Equations* (2004), article no. 114 Zbl 1134.35302 MR 2108885

[13] F. Rellich, Störungstheorie der Spektralzerlegung. V. *Math. Ann.* **118** (1942), 462–484 Zbl 0027.22702 MR 0010791

[14] M. Teytel, How rare are multiple eigenvalues? *Comm. Pure Appl. Math.* **52** (1999), no. 8, 917–934 Zbl 0942.47012 MR 1686977

[15] K. Uhlenbeck, Generic properties of eigenfunctions. *Amer. J. Math.* **98** (1976), no. 4, 1059–1078 Zbl 0355.58017 MR 0464332

[16] J. von Neumann and E. P. Wigner, Über das Verhalten von Eigenwerten bei adiabatischen Prozessen. *Z. Phys. A* **30** (1929), 467–470 JFM 55.0520.05

[17] S. Zelditch, On the generic spectrum of a Riemannian cover. *Ann. Inst. Fourier (Grenoble)* **40** (1990), no. 2, 407–442 Zbl 0722.58044 MR 1070834

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