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Anderson localization for Schrödinger operators with monotone potentials over circle homeomorphisms

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Abstract. In this paper, we prove pure point spectrum for a large class of Schrödinger operators over circle maps with conditions on the rotation number going beyond the Diophantine. More specifically, we develop the scheme to obtain pure point spectrum for Schrödinger operators with monotone bi-Lipschitz potentials over orientation-preserving circle homeomorphisms with Diophantine or weakly Liouville rotation number. The localization is uniform when the coupling constant is large enough.

1. Introduction

The spectral theory of quasiperiodic Schrödinger operators has been the subject of extensive study over the past several decades due to its deep origins in physics and the richness of its unusual mathematical features. The general setup of a quasiperiodic operator is given by a family of operators $H_{\lambda,f,T,x}$ acting on $\ell^2(\mathbb{Z})$, defined as

$$(H_{\lambda,f,T,x}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda f(T^n x)\psi(n), \tag{1.1}$$

where $x \in \mathbb{T}^1$, T is an irrational rotation on $\mathbb{T}^1$ defined by $Tx = R_\alpha x = x + \alpha$, with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $f : \mathbb{T}^1 \to \mathbb{R}$ is a potential function. Examples of such operators include $f(x) = \cos(x)$ for the almost Mathieu operator or $f(x) = \tan(x)$ for the Maryland model. One of the most interesting features of quasiperiodic operators is that their spectral type can often be fully characterized by the arithmetic properties of α (and/or x) in many situations, as demonstrated in works such as [8, 14]. Since R_α serves as a fundamental example of general circle homeomorphisms, a natural question arises: if T is not a rotation but a more general circle homeomorphism with rotation number α, can we still determine or get some information about the spectral type by the arithmetic properties of α?

As one can imagine, the answer may vary depending on properties of f, T, and α. The study of (1.1) for general circle diffeomorphisms T was initiated by [12, 22].

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In [12], the authors proved purely absolutely continuous spectrum for Hölder continuos f, $C^{1+\mathrm{BV}}$-smooth T, and super Liouville α. [22] further explored similar phenomena for circle diffeomorphisms with a critical point or break. On the other hand, for quasiperiodic (1.1), several recent papers have proved the opposite, i.e., pure point spectrum, under arithmetic properties of α that go beyond the Diophantine condition, e.g., [1,2,5,7,8,14–16,23–26]. In this paper, we add to this list by extending the results in [10], where the authors worked with an irrational rotation $T = R_\alpha$ with Diophantine α and potential f satisfying conditions $(\mathscr{F}1)$ and $(\mathscr{F}2)$ below. We work with the same conditions on f but consider a more general orientation-preserving circle homeomorphism T under the assumptions $(\mathscr{T}1)$.

- $(\mathscr{F}1)$ f is one-periodic on $\mathbb{R}$ and $f(0) = 0$, $f(1-0) := \lim_{x\to 1^-} f(x) = 1$.
- $(\mathscr{F}2)$ f is bi-Lipschitz monotone, i.e., there exist $\gamma_-, \gamma_+ > 0$ such that for all $0 \le x < y < 1$,

$$\gamma_-(y-x) \le f(y) - f(x) \le \gamma_+(y-x).$$

- $(\mathscr{T}1)$ Assume the invariant measure of T is denoted by ν and that

$$C_-\nu([x,y]) \le |x-y| \le C_+\nu([x,y]).$$

Under these conditions, we obtain results that are similar to the ones in [10]. In fact, in addition to extending to more general circle homeomorphisms, we also generalize the result by relaxing the Diophantine condition on α to both weakly Liouville and Diophantine cases. Specifically, we prove that the followng result.

Theorem 1 (Pure point spectrum). *For f satisfying $(\mathscr{F}1)$ and $(\mathscr{F}2)$ and T satisfying $(\mathscr{T}1)$ with weakly Liouville or Diophatine rotation number α, or more specifically, $0 \le \beta(\alpha) < \infty$, there is $C_0 = C_0(\gamma_\pm, C_\pm) = O\big(\frac{\gamma_- C_-}{\gamma_+ C_+}\big) > 0$ such that for all $\lambda > 0$, we have*

$$\sigma_c(H_{\lambda,f,T,x}) \cap \{E : \beta(\alpha) < C_0 L(E;\alpha)\} = \emptyset, \quad \textit{for all } x \in \mathbb{T}^1,$$

where $\beta(\alpha)$ and the Lyapunov exponent $L(E;\alpha)$ are defined in Section 2.

Remark 1. The theorem provides a meaningful statement for homeomorphisms with rotation number α when $\beta(\alpha)$ is small or zero, which corresponds to weakly Liouville or Diophantine α (see Section 2). In fact, the smaller $\beta(\alpha)$ is, the more "irrational" α is. For example, since we also proved positivity of Lyapunov exponent $L(E;\alpha) > 0$ for all irrational α in Corollary 3.3, when $\beta(\alpha) = 0$, this implies that $\sigma_c(H_{f,T,x}) = \emptyset$, i.e., the spectrum of $H_{f,T,x}$ is pure point.

Note that condition $(\mathscr{T}1)$ is equivalent to the existence of a bi-Lipschitz conjugacy between T and R_α, meaning that there exists a bi-Lipschitz function ϕ that is bounded from above and below such that $\phi \circ T = R_\alpha \circ \phi$. We acknowledge that if such a bi-Lipschitz conjugacy exists and α is Diophantine, our results follow directly from [10] by a change of variables: letting $y = \phi(x)$, we obtain $H_{f\circ\phi^{-1}, R_\alpha, y}$ and can apply known localization results. However, we choose to present the proof in the more general setting using the invariant measure, since the existence of a bi-Lipschitz (in fact, $C^{1+\varepsilon}$) conjugacy is currently only established for Diophantine α.

As a byproduct, we also establish Lipschitz continuity of the integrated density of states (IDS, see (2.5)) for all λ in Lemma 3.3, as well as the continuity and positivity of the Lyapunov exponent for large λ in Corollary 3.3. Together with our key Lemma 5.3, which is uniform in x, E, and α, these results allow us to achieve uniform localization of $H_{\lambda,f,T,x}$ (see Definition 3) for sufficiently large λ and "sufficiently irrational" α, i.e., $\beta(\alpha)$ sufficiently small (see Definition 1).

Theorem 2 (Uniform localization). *Let $C_0 = C_0(\gamma_\pm, C_\pm) > 0$. If $\lambda > \frac{4e}{\gamma_- C_-}$ and if α is weakly Liouville with $\beta(\alpha) < C_0 \ln\left(\frac{\lambda\gamma_- C_-}{4e}\right)$, then $H_{\lambda,f,T,x}$ has uniform localization for all x.*

We also remark that a somewhat different proof was developed in [19] for unbounded lower-Lipschitz monotone f and irrational rotation $T = R_\alpha$ with Diophantine α. The key idea is, that instead of controlling the change of eigenvalue functions horizontally (see Lemma 3.1), the author controls the change of counting function horizontally. We believe that the Lipschitz continuity of integrated density of states in our proof can be done through that of argument in [19] and the results here can be generalized to more general T with weakly Liouville α and unbounded f in similar ways to here. This will be explored in future work.

Finally, we mention several interesting works on the Schrödinger operator with monotonic potential during the submission process of this current paper, in higher dimensions [4,27] or for potentials with flat plateaus, and recent results [11,20,21].

Structure and key ideas. Under the assumption of ($\mathcal{T}$1) and allowing weakly Liouville α, we re-develop the proof following the method in [10] in the key step: we use the non-perturbative proof of localization, first developed in [18], together with a detailed analysis of the behavior of box eigenvalues. We provide the latter for general T in Section 3, which helps with building the large deviation estimates in Section 4. From the large deviation estimates, we get our key lemma on the uniform exponential decay of generalized eigenfunctions in x, α, E in Section 5. The main results follow immediately in Section 6.

The extension of our results from R_α to more general circle homeomorphisms T is based on the observation that the behavior of box eigenvalues is closely related to the distribution of orbits of T. While the orbits of T are not evenly distributed with respect to distance, they are evenly distributed with respect to the invariant measure, allowing us to obtain quantitative estimates of their distribution under the comparability assumption $(\mathcal{T}1)$. Appendix A provides the key statements that enable us to carry out this argument. The extension to weakly Liouville α, on the other hand, requires a more careful estimate of the decay of generalized eigenfunctions in Section 5.

2. Preliminaries

In this section, we will begin by discussing two fundamental concepts: continued fraction expansion and weakly Liouville numbers. Afterwards, we will introduce several fundamental properties of discrete Schrödinger operators, including the generalized eigenvalue and Schnol's theorem, the Green function and Poisson formula, transfer matrices and Lyapunov exponent, density of states measure, and the Thouless formula.

Notations. For $x \in \mathbb{R}$, we use $|x|$ to denote the absolute value and

$$\|x\| = \inf_{n\in\mathbb{Z}} |x - n|$$

to denote the closest distance between $x \in \mathbb{R}$ and integers.

Continued fraction expansion and weakly Liouville number. Any number $\alpha \in [0, 1)$ can be written in the continued fraction expansion [28]:

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3+\cdots}}} := [a_1, a_2, a_3, \ldots].$$

with $a_k \in \mathbb{N}^+$. Let $\frac{p_n}{q_n} = [a_1, \ldots, a_n]$ denote the continued fraction approximants. They satisfy

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2},\ p_{-1} = 1, \quad p_0 = 0; \\ q_k &= a_k q_{k-1} + q_{k-2},\ q_{-1} = 0, \quad q_0 = 1. \end{aligned} \tag{2.1}$$

Definition 1 (Weakly Liouville). For $\alpha \in [0, 1)$, let

$$\beta(\alpha) = \limsup_{k\to\infty} \frac{\ln q_{k+1}}{q_k}.$$

We call α *weakly Liouville* if $0 < \beta(\alpha) < \infty$.

We mention that if α is Diophantine,[1] then $\beta(\alpha) = 0$.

For a detailed discussion on the next several definitions, please refer to [6, Chapters 9 and 10] and [3, Chapter VII].

[1] α is called *Diophantine* if there is $\kappa > 0$ and $\tau > 0$ such that $\|n\alpha\| > \frac{\kappa}{|n|^\tau}$ for all $n \neq 0$.

Generalized eigenfunction and Schnol's theorem. We say ψ is a *generalized eigenfunction* of an operator H with respect to a *generalized eigenvalue* E if ψ is polynomially bounded, i.e., $|\psi(n)| \leq C(1+|n|)^p$ for some $C > 0$, $p \in \mathbb{N}$ and $H\psi = E\psi$. Schnol's theorem states that the spectral measure of an operator H is supported by the set of its generalized eigenvalues.

According to Schnol's theorem, to prove that H has pure point spectrum, it is sufficient to show that all generalized eigenfunctions belong to ℓ^2. This is because if all generalized eigenfunctions and eigenvalues become eigenfunctions and eigenvalues, respectively, then the spectrum is pure point.

Green function and Poisson formula. Let $H_{[a,b]}(x)$ and $\widetilde{H}_{[a,b]}$ denote the restriction of $H_{\lambda,f,T,x}$ to $\ell^2([a,b])$ with Dirichlet and periodic boundary conditions, respectively. In particular, for the interval $[a,b] = [0, n-1]$, we use the simplified notations $H_n(x)$ and $\widetilde{H}_n(x)$. More specifically,

$$H_n(x) = \begin{pmatrix} \lambda f(x) & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \lambda f(T^{n-1}x) \end{pmatrix}_{n\times n},$$

$$\widetilde{H}_n(x) = \begin{pmatrix} \lambda f(x) & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & \lambda f(T^{n-1}x) \end{pmatrix}_{n\times n}.$$

Let $G_{x,E,[a,b]} = (H_{[a,b]}(x) - E)^{-1}$ denote the Green function and $G_{x,E,[a,b]}(m,n)$ be its (m,n)-entry. Denote $P_n(x,E) = \det(H_n(x) - E)$, and let $P_0(x,E) = 1$.

The *Poisson formula* provides a connection between the generalized eigenfunction and the Green function. Specifically, suppose $\psi(n)$ is a generalized eigenfunction of $H_{\lambda,f,T,x}$ with respect to generalized eigenvalue E, then for n in the interval $[a,b]$, we have the following formula:

$$\psi(n) = -G_{x,E,[a,b]}(a,n)\psi(a-1) - G_{x,E,[a,b]}(n,b)\psi(b+1). \tag{2.2}$$

Transfer matrix and Lyapunov exponent. Rewrite $H_{\lambda,f,T,x}\psi = E\psi$ into matrix form:

$$\begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = A_{n-1}(x,E)\begin{pmatrix} \psi_{n-1} \\ \psi_{n-2} \end{pmatrix} = A_{n-1}(x,E)\dots A_0(x,E)\begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix},$$

where

$$A_i(x,E) := \begin{pmatrix} E - \lambda f(T^i x) & -1 \\ 1 & 0 \end{pmatrix}.$$

We define the *n-step transfer matrix* by

$$M_n(x,E) := A_{n-1}(x,E)\dots A_0(x,E).$$

One can verify by induction that

$$M_n(x,E) := \begin{pmatrix} P_n(x,E) & -P_{n-1}(Tx,E) \\ P_{n-1}(x,E) & -P_{n-2}(Tx,E) \end{pmatrix}. \tag{2.3}$$

The *Lyapunov exponent* is defined to be

$$L(E) := \lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln \|M_n(x,E)\|\, d\nu(x). \tag{2.4}$$

Integrated density of states (IDS) and the Thouless formula. Next, we introduce the density of states measure and the Thouless formula, which connects the Lyapunov exponent of E with the density of states measure. The *integrated density of states* (IDS) is defined as follows:

$$N(E) := \lim_{n\to\infty} \frac{1}{n} \int_0^1 N_n(x,E)\, d\nu(x), \tag{2.5}$$

where $N_n(x,E) := \#\sigma(H_n(x)) \cap (-\infty, E]$.

Remark 2. We can define $\widetilde{P}_n(x,E)$ and $\widetilde{N}_n(x,E)$, analogous to $P_n(x,E)$ and $N_n(x,E)$ for H_n, respectively, for $\widetilde{H}_n(x)$.

Remark 3. Notice that $\widetilde{H}_n(x)$ is a rank-two perturbation of $H_n(x)$. Thus, we have $|\widetilde{N}_n(x,E) - N_n(x,E)| \le 2$. Thus, we can also define the IDS by

$$N(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \widetilde{N}_n(x,E)\, d\nu.$$

The function $N(E)$ is right-continuous, non-decreasing, and approaches zero as E approaches $-\infty$. Its derivative defines a unique probability measure, called the *density of states measure* $N(dE)$. The relation between the density of states measure $N(dE)$ and the Lyapunov exponents $L(E)$ is known as the *Thouless formula*. We state it here without proof, but refer the interested reader to [6] for more details:

$$L(E) = \int_{\mathbb{R}} \ln |E' - E|\, N(dE).$$

3. Positive Lyapunov exponent

In this section, we first establish some fundamental properties of box eigenvalue functions, which are the eigenvalues of $\widetilde{H}_n(x)$. We then derive estimates for the distance between these eigenvalue functions. Using these estimates, we obtain the Lipschitz continuity of the IDS $N(E)$ with respect to E and prove the positivity of the Lyapunov exponent $L(E)$ for large λ.

Recall that $\widetilde{H}_n(x)$ is the periodic restriction of $H_{\lambda,f,T,x}$ to $[0, n-1]$. Let $\tilde{\mu}_m(x)$, $0 \le m \le n-1$ be the eigenvalues of $\widetilde{H}_n(x)$ in increasing order. We refer to $\tilde{\mu}_m(x)$ as the *box eigenvalue functions*. Now, we establish some of their basic properties.

Proposition 3.1. *$\tilde{\mu}_i(x)$ have the following properties:*

(1) *$\tilde{\mu}_i(x)$ is 1-periodic, continuous on $[0,1)$ except at $\{T^{-j}(0)\}_{j=0}^{n-1}$. By rearranging these discontinuity points in increasing order, we denote them by $\{\beta_l\}_{l=0}^{n-1}$. We also denote $I_l := [\beta_l, \beta_{l+1})$.*

(2) *$\tilde{\mu}_i(x)$ is bi-Lipschitz continuous with respect to the invariant measure, and strictly increasing on each I_l. In fact,*

$$\lambda\gamma_- C_- \nu([x,y]) \le \tilde{\mu}_i(y) - \tilde{\mu}_i(x) \le \lambda\gamma_+ C_+ \nu([x,y]).$$

(3) *At each jump β_l, we have*

$$\tilde{\mu}_i(\beta_l - 0) \le \tilde{\mu}_{i+1}(\beta_l) \le \tilde{\mu}_{i+1}(\beta_l - 0), \quad 0 \le i \le n-2.$$

Remark 4. Because of (2) and (3) above, it is natural to define

$$\Lambda_i(x) := \sum_{l=0}^{n-1} \mu_{i+l}(x)\chi_{I_l}(x), \quad 0 \le i \le n-1$$

and extend it periodically from $[0,1)$ to $\mathbb{R}$. As a result, $\Lambda_j(x)$ is monotone increasing on $[\beta_{n-j+1} + N, \beta_{n-j+1} + N + 1)$ for any $N \in \mathbb{Z}$, and it inherits the properties of $\mu_j(x)$ on each I_l. In particular, $\Lambda_j(x)$ is also lower-Lipschitz with respect to invariant measure ν on $[\beta_{n-j+1} + N, \beta_{n-j+1} + N + 1)$:

$$\Lambda_i(y) - \Lambda_i(x) \ge \lambda\gamma_- C_- \nu([x,y]),$$

for $x < y$ and $x, y \in [\beta_{n-j+1} + N, \beta_{n-j+1} + N + 1)$.

Proof. (1) Note that the box eigenvalue functions $\tilde{\mu}_i(x)$ are roots of the characteristic polynomial $\tilde{P}_n(x,E)$. Therefore, each $\tilde{\mu}_i(x)$ is continuous with respect to the coefficients of $\tilde{P}_n(x,E)$, which are polynomials of $\{\lambda f(T^j x)\}_{j=0}^{q_k-1}$. Since $f(T^j x)$ is only discontinuous at $T^{-j}(x)$, $\tilde{\mu}_i(x)$ is only potentially discontinuous at $\{T^{-j}(x)\}_{j=0}^{q_k-1}$.

(2) Notice that, for $x < y$ in the same I_l, $\widetilde{H}(y) - \widetilde{H}(x)$ is a non-negative diagonal matrix. By Lidskii's theorem (see [13, Theorem 2]), we have

$$\begin{aligned}\lambda \min_j (f(T^j y) - f(T^j y)) &\le \mu_j(y) - \mu_j(x) \\ &\le \lambda \max_j (f(T^j y) - f(T^j x)).\end{aligned}$$

Notice that $(\mathcal{F}2)$ and $(\mathcal{T}1)$ implies that

$$f(T^j y) - f(T^j x) \le \gamma_+ C_+ \nu([T^j x, T^j y]),$$

and similarly, $f(T^j y) - f(T^j x) \ge \lambda\gamma_- C_- \nu([x, y])$.

(3) Notice that

$$\widetilde{H}_n(\beta_l - 0) - \widetilde{H}_n(\beta_l) = \lambda e_j \otimes e_j \tag{3.1}$$

where $0 \le j \le q_k - 1$ such that $T^{-(j-1)}(x) = \beta_l$. This leads to the second inequality since $\widetilde{H}_n(\beta_l - 0) - \widetilde{H}_n(\beta_l)$ is positive semi-definite. To derive the first inequality, by (3.1), let D be the matrix obtained by deleting the row j and column j from $\widetilde{H}_n(\beta_l - 0)$ or $\widetilde{H}_n(\beta_l)$. Let $\omega_1 \le \omega_2 \le \cdots \le \omega_{n-1}$ be the eigenvalues of D. By eigenvalue interlacing theorem,

$$\begin{aligned}&\tilde{\mu}_0(\beta_l - 0) \le \omega_1 \le \tilde{\mu}_1(\beta_l - 0) \le \omega_2 \le \cdots \le \omega_{n-1} \le \tilde{\mu}_{n-1}(\beta_l - 0),\\ &\tilde{\mu}_0(\beta_l) \le \omega_1 \le \tilde{\mu}_1(\beta_l) \le \omega_2 \le \cdots \le \omega_{n-1} \le \tilde{\mu}_{n-1}(\beta_l).\end{aligned}$$

Therefore, $\tilde{\mu}_m(\beta_l - 0) \le \omega_{m+1} \le \tilde{\mu}_{m+1}(\beta_l)$, for all $0 \le m \le n-2$. ■

Horizontal comparison. From now on, we fix α and consider $n = q_k$ since we will use the dynamical properties of the irrational circle map to compare box eigenvalue functions horizontally and vertically. The following lemma provides an upper bound control if we compare the box eigenvalue functions $\tilde{\mu}_i(x)$ and $\tilde{\mu}_i(T^r x)$ horizontally. Note that the estimate is uniform in r.

Lemma 3.1. *For any* $-q_k + 1 \le r \le q_k - 1$,

$$|\tilde{\mu}_i(x) - \tilde{\mu}_i(T^r x)| \le \frac{\lambda\gamma_+ C_+}{q_{k+1}}.$$

Proof. Define an $q_k \times q_k$ unitary matrix $S = [e_{q_k}, e_1, e_2, \ldots, e_{q_k - 1}]$ where $e_j \in \mathbb{R}^n$ are standard unit vectors. Then

$$\begin{aligned}S^r \widetilde{H}_{q_k}(x) S^{-r} &= \widetilde{H}_{q_k - r}(T^r x) \oplus \widetilde{H}_r(x),\\ \widetilde{H}_{q_k}(T^r x) &= \widetilde{H}_{q_k - r}(T^r x) \oplus \widetilde{H}_r(T^{q_k} x).\end{aligned}$$

By ($\mathscr{F}$2), ($\mathscr{T}$1), and Lemma A.2,

$$
\begin{aligned}
\|\widetilde{H}_r(x) - \widetilde{H}_r(T^{q_k}x)\| &\le \lambda \max_{0\le i\le r-1} |f(T^i x) - f(T^{q_k+i}x)| \\
&\le \lambda\gamma_+ |T^i x - T^{q_k+i}x| \\
&\le \lambda\gamma_+ C_+ \nu([T^i x, T^{q_k+i}x]) = \lambda\gamma_+ C_+ \nu([x, T^{q_k}x]) \\
&\le \frac{\lambda\gamma_+ C_+}{q_{k+1}}.
\end{aligned}
$$

The result follows from Lidskii's theorem. ∎

Corollary 3.2. *For any $x, y \in [0,1)$,*

$$
|\tilde{\mu}_i(x) - \tilde{\mu}_i(y)| \le \frac{\lambda\gamma_+ C_+}{q_{k+1}} + \frac{2\lambda\gamma_+ C_+}{q_k} \le \frac{3\lambda\gamma_+ C_+}{q_k}.
$$

Proof. First notice that for given $x \in [0,1)$, depending on which I_k it belongs to, there exists $-q_k + 1 \le \alpha \le 0$, such that each point in $\{T^r x\}_{r=\alpha}^{\alpha+q_k-1}$ precisely falls in one interval among $\{I_l\}_{l=0}^{q_k-1}$. Thus, there is $-q_k + 1 \le r \le q_k - 1$ such that $T^r x$ and y are in the same I_k. Then

$$
\begin{aligned}
|\tilde{\mu}_i(x) - \tilde{\mu}_i(y)| &\le |\tilde{\mu}_i(x) - \tilde{\mu}_i(T^r x)| + |\tilde{\mu}_i(T^r x) - \mu_i(y)| \\
&\le \frac{\lambda\gamma_+ C_+}{q_{k+1}} + \lambda\gamma_+ C_+ \nu([T^r x, y]) \\
&\le \frac{\lambda\gamma_+ C_+}{q_{k+1}} + \lambda\gamma_+ C_+ \Big(\frac{1}{q_k} + \frac{1}{q_{k+1}}\Big)
\end{aligned}
$$

where the first inequality follows from Lemma 3.1 and the second inequality follows from Lemma A.2. ∎

Vertical comparison. Now, we estimate the lower bound of vertical distance between eigenvalue functions. Unfortunately, the vertical distance between two closest eigenvalue functions $\tilde{\mu}_i(x)$ and $\tilde{\mu}_{i+1}(x)$ is not always positive. However, we can show that at most M eigenvalues can be very close to each other, others will be nicely separated from them.

Lemma 3.2. *Given $\gamma_\pm$, λ and $C_\pm$. For any $\varepsilon > 0$, there is a $j_0 = j_0(\varepsilon) = \frac{2\gamma_+ C_+}{\varepsilon\gamma_- C_-}$, such that for any i, j, q_k satisfying $j \ge j_0$ and $0 \le i < i + j \le q_k - 1$, we have*

$$
|\tilde{\mu}_{i+j}(x) - \tilde{\mu}_i(x)| \ge \lambda\gamma_- C_-(1-\varepsilon)\frac{j}{q_k} =: d_0(\varepsilon)\frac{j}{q_k}.
$$

Proof. First notice that, given x, there exists $-q_k + 1 \le \alpha \le 0$, such that each point in $\{T^r x\}_{r=\alpha}^{\alpha+q_k-1}$ falls in precisely one interval among $\{I_l\}_{l=0}^{q_k-1}$. Then for any $\alpha \le r$, $r' \le \alpha + q_k - 1$,

$$\begin{aligned}|\tilde{\mu}_{i+j}(x) - \tilde{\mu}_i(x)| &\ge |\tilde{\mu}_{i+j}(T^r x) - \tilde{\mu}_i(T^{r'} x)| - |\tilde{\mu}_{i+j}(x) - \tilde{\mu}_{i+j}(T^r x)| \\ &\quad - |\tilde{\mu}_i(T^{r'} x) - \tilde{\mu}_i(x)| \\ &\ge |\tilde{\mu}_{i+j}(T^r x) - \tilde{\mu}_i(T^{r'} x)| - \frac{2\lambda\gamma_+ C_+}{q_{k+1}} \\ &\ge \sup_{r,r'} |\tilde{\mu}_{i+j}(T^r x) - \tilde{\mu}_i(T^{r'} x)| - \frac{2\lambda\gamma_+ C_+}{q_{k+1}}.\end{aligned}$$

In particular, we can pick r, r' such that $(T^{r'} x, \tilde{\mu}_i(T^{r'} x))$ and $(T^r x, \tilde{\mu}_{i+j}(T^r x))$ are on the graph of the same Λ_m, defined in Remark 4. Put such pairs of (r, r') together and denote the set by S_j. Then $[T^r x, T^{r'} x]$ includes j out of q_k subintervals created by the partition $\{T^i x\}_{i=\alpha}^{\alpha+q_k-1}$ on $[0, 1)$, where each intervals have the same invariant measure. Thus, by pigeonhole principle,[2]

$$\sup_{r,r'} |\tilde{\mu}_{i+j}(T^r x) - \tilde{\mu}_i(T^{r'} x)| \ge \lambda\gamma_- C_- \sup_{r,r' \in S_j} \nu([T^r x, T^{r'} x]) \ge \lambda\gamma_- C_- \frac{j}{q_k}.$$

Thus,

$$\begin{aligned}|\tilde{\mu}_{i+j}(x) - \tilde{\mu}_i(x)| &\ge \lambda\gamma_- C_- \frac{j}{q_k} - \frac{2\lambda\gamma_+ C_+}{q_{k+1}} \ge \lambda\gamma_- C_- \frac{j}{q_k}\Big(1 - \frac{2\gamma_+ C_+}{\gamma_- C_-} \frac{q_k}{q_{k+1} j}\Big) \\ &\ge \lambda\gamma_- C_- (1 - \varepsilon_0) \frac{j}{q_k}\end{aligned}$$

when $j \ge j_0 := \frac{2\gamma_+ C_+}{\varepsilon_0 \gamma_- C_-}$. ■

Lipschitz continuity of IDS. Recall that $\widetilde{N}_n(x, E) = \#\sigma(\widetilde{H}_n(x)) \cap (-\infty, E]$ and

$$N(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \widetilde{N}_n(x, E)\, d\nu(x).$$

[2] In fact, we could bound $|\tilde{\mu}_{i+j}(T^r x) - \tilde{\mu}_i(T^{r'} x)|$, the distance between eigenvalue functions, directly by Lemma A.2 without taking the supremum or referring to the pigeonhole principle. However, the authors choose to prove it this way both because it is more interesting, and because it reveals the uniformity in x in the vertical comparison of eigenvalue functions. It implies that vertical differences of eigenvalue functions at any x are uniformly controlled by the largest vertical differences among all Λ_m. This observation can be useful in dealing with singular ν where certain I_k's are too small or the case when f is flat at some I_k's.

Now, we can derive Lipschitz continuity of $\widetilde{N}_{q_k}(x, E)$ and $N(E)$ from vertical distance of $\tilde{\mu}_i(x)$.

Lemma 3.3. *Given $\lambda, \gamma_+, \gamma_-, E, E' \in \mathbb{R}$, we have*

$$|N(E) - N(E')| \leq \frac{|E - E'|}{\lambda \gamma_- C_-}.$$

Proof. Fix E and E'. For any $\varepsilon > 0$, we see from Lemma 3.2 that any interval of length $d_0(\varepsilon)\frac{j_0}{q_k}$ contains at most j_0 eigenvalues for q_k large enough. This allows us to estimate the number of eigenvalues between E and E':

$$\begin{aligned}|\widetilde{N}_{q_k}(E, x) - \widetilde{N}_{q_k}(E', x)| &\leq \Big(\frac{q_k|E - E'|}{d_0(\varepsilon) j_0} + 1\Big) j_0 \\ &= \frac{q_k|E - E'|}{d_0(\varepsilon)}\Big(1 + \frac{j_0 d_0(\varepsilon)}{q_k|E - E'|}\Big)\end{aligned}$$

for any x. Let $k \to \infty$, we get

$$\begin{aligned}|N(E) - N(E')| &\leq \liminf_{k\to\infty} \frac{|E - E'|}{d_0(\varepsilon)}\Big(1 + \frac{j_0 d_0(\varepsilon)}{q_k|E - E'|}\Big) \\ &= \frac{|E - E'|}{d_0(\varepsilon)} = \frac{|E - E'|}{\lambda\gamma_- C_-(1-\varepsilon)}.\end{aligned}$$

Since this inequality is true for all ε, the result follows. ■

Positivity of Lyapunov exponent. This is a corollary of Lemma 3.3 which is also useful in the later proof of uniform localization.

Corollary 3.3. *The Lyapunov exponent $L(E)$ of $H_{\lambda,f,T,x}$ is continuous in E and $L(E)$ admits a lower bound*

$$L(E) \geq \max\Big\{0, \ln\Big(\frac{\lambda\gamma_- C_-}{2e}\Big)\Big\}. \tag{3.2}$$

Therefore, $L(E)$ is uniformly positive if $\lambda > \frac{2e}{\gamma_- C_-}$.

Proof. By Lemma 3.3, $dN(E)$ is absolutely continuous with respect to dE and the Radon-Nikodym derivative $\frac{dN(E)}{dE} \leq \frac{1}{\lambda\gamma_- C_-} := \frac{1}{d}$, for a.e. E. Thus, by the Thouless formula,

$$\begin{aligned}L(E) &= \int_{\mathbb{R}} \ln|E' - E| dN(E') = \int_{\mathbb{R}} (\ln|E' - E|)\frac{dN(E')}{dE'} dE' \\ &\geq \int_{E-\frac{d}{2}}^{E+\frac{d}{2}} \frac{1}{d}\cdot \ln|E' - E|\, dE' = \frac{2}{d}\int_0^{\frac{d}{2}} \ln|E'|\, dE' = \ln\frac{d}{2e} = \ln\frac{\lambda\gamma_- C_-}{2e},\end{aligned}$$

where the first inequality follows from monotonicity of ln function and boundedness of $\frac{dN(E')}{dE'}$. Finally, notice that $L(E) \geq 0$ follows from the definition. Thus, we get (3.2). ∎

4. Large deviation theorem

In this section, we provide two essential ingredients for the proof of localization. Lemma 4.1 provides an upper bound of $P_n(x, E)$ while Theorem 3 provides the large deviation estimate which is central of the non-perturbative proofs of localization, as introduced in [18]. The first is a result that can be directly adapted from [17, Lemma 3.5]. It holds for arbitrary $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and arbitrary piecewise potentials.

Lemma 4.1. *For any $\kappa > 0$ and $E \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $n > N$*

$$|P_n(x, E)| \leq e^{n(L(E)+\kappa)} \quad \text{for all } x \in [0, 1).$$

Moreover, N can be chosen to be uniform in $E \in I$ as long as $L(E)$ is continuous on interval I.

Proof. This was proved in [17, Lemma 3.5] for irrational rotation $T = R_\alpha$. The same method applies to a general circle diffeomorphism under the assumption $(\mathcal{T}1)$. ∎

Theorem 3 (Large deviation theorem). *Fix E such that $L(E) > 0$. There exists $C_0 = C_0(\gamma_\pm, C_\pm) > 0$ such that for any $0 < \delta < L(E)$, there is k_0 such that for any $k \geq k_0$,*

$$\nu\left\{x \in [0, 1) : \frac{1}{q_k} \ln |P_{q_k}(x, E)| < L(E) - \delta\right\} < e^{-C_0 \delta q_k}$$

Moreover, the set on the left-hand side is composed of at most q_k many intervals.

Proof. Recall that

$$P_{q_k}(x, E) := \det(H_{q_k}(x) - E) = \prod_{i=0}^{q_k-1} (\mu_i(x) - E).$$

Denote for convenience

$$f_{q_k}(x) := \frac{1}{q_k} \ln |P_{q_k}(x; E)| = \frac{1}{q_k} \sum_{i=0}^{q_k-1} \ln |\mu_i(x) - E|.$$

Notice that $\mu_i(x)$ is monotone and $f_{q_k}(x) = -\infty$ at $\{x : \mu_i(x) = E \text{ for some } i\}$. Thus, "large deviation" happens near $\{x : \mu_i(x) = E \text{ for some } i\}$. The aim is to estimate how large this set can be without rising $f_{q_k}(x)$ too high. The idea is since $\mu_i(x)$

are well separated, only the closest (to E) several $\mu_i(x)$ contribute the most to the negativity of $f_{q_k}(x)$, the rest are nicely controlled.

To do so, we split eigenvalues $\tilde{\mu}_j(x)$ into three clusters: $\mathcal{K}^+$ above E, $\mathcal{K}^0$ around E, and $\mathcal{K}^-$ below E. Notice that, by $|N_{q_k}(x;E) - \widetilde{N}_{q_k}(x;E)| \leq 2$ and Lemma 3.2, we can make sure that

(1) the cluster of eigenvalues above E, denoted by $\mu_i^+(x)$, $i = 1, 2, \ldots$ in an increasing order with $\mu_i^+(x) \geq E + \frac{id_0}{q_k}$;
(2) the cluster of eigenvalues below E, denoted by $\mu_i^-(x)$, $i = 1, 2, \ldots$ in an decreasing order with $\mu_i^-(x) \leq E - \frac{id_0}{q_k}$;
(3) the cluster of the rest of eigenvalues, denoted by $\mu_i^0(x)$, with the number of eigenvalues in this cluster does not exceed some N_0 uniform in E.

For example, this can be achieved by considering the closest $2j_0 + 4$ eigenvalues $\mu_i(x)$ to E to be in the third cluster and every eigenvalue above/below them to be in the first/second cluster. Here j_0 is to guarantee the lower and upper bound estimates above and $4 = 2 \times 2$ is due to $|N_{q_k}(x;E) - \widetilde{N}_{q_k}(x;E)| \leq 2$. In fact, we can do the same thing for $\tilde{\mu}_i(x)$, then we just need to pick the closest $2j_0$ eigenvalues instead of $2j_0 + 4$.

Now, decompose P_{q_k}, $\widetilde{P}_{q_k}$ correspondingly,

$$P_{q_k}(x;E) = P_{q_k}^+(x;E)P_{q_k}^0(x;E)P_{q_k}^-(x;E),$$
$$\widetilde{P}_{q_k}(x;E) = \widetilde{P}_{q_k}^+(x;E)\widetilde{P}_{q_k}^0(x;E)\widetilde{P}_{q_k}^-(x;E),$$

where

$$P_{q_k}^*(x;E) = \prod_{\mu_i^* \in \mathcal{K}^*} \mu_i^*(x) - E,$$

where $* \in \{+, -, 0\}$.

Claim 1. *Let $a, b > 0$,*

$$\sum_{j=1}^{n} [\ln(aj + b) - \ln(aj)] \leq \sum_{j=1}^{n} \ln\Big(1 + \frac{b}{aj}\Big) \leq \sum_{j=1}^{n} \frac{b}{aj} \leq \frac{b}{a} \ln(n+1).$$

By Corollary 3.2 and the claim, we have for any $x, y \in [0, 1)$,

$$\begin{aligned} \big|\ln|\widetilde{P}_{q_k}^{\pm}(x;E)| - \ln|\widetilde{P}_{q_k}^{\pm}(y;E)|\big| &\leq \sum_{i=1}^{q_k} \Big[\ln\Big(\frac{jd_0}{q_k} + \frac{3\lambda\gamma_+ C_+}{q_k}\Big) - \ln\Big(\frac{jd_0}{q_k}\Big)\Big] \\ &\leq C \ln q_k. \end{aligned}$$

Here we considered all maximum potential perturbation of all $\mu_i(x)$ at the maximum potential place $\{\frac{jd_0}{q_k}\}_{j=1}^{q_k}$. There might be extra terms of $\mu_i^{\pm}(x)$ that do not pair to

$\mu_i^{\pm}(y)$ but since there are only finitely many terms and they are bounded, the result is still true with a modification of C. For the same reason, the inequality holds for $P_{q_k}(x; E)$ as well.

Thus, there is $L_{q_k}(E)$ such that

$$L_{q_k}(E) \leq \frac{1}{q_k} \ln |P_{q_k}^{+}(x; E) P_{q_k}^{-}(x; E)| \leq L_{q_k}(E) + C \frac{\ln q_k}{q_k}, \tag{4.1a}$$

$$L_{q_k}(E) \leq \frac{1}{q_k} \ln |\widetilde{P}_{q_k}^{+}(x; E) \widetilde{P}_{q_k}^{-}(x; E)| \leq L_{q_k}(E) + C \frac{\ln q_k}{q_k}, \tag{4.1b}$$

Claim 2. *There is $C_0 = C_0(j_0) = C_0(\gamma_{\pm}, C_{\pm})$ such that for k large enough, for $\delta > 0$ small enough,*

$$\nu\{x \in [0, 1) : \frac{1}{q_k} \ln |P_{q_k}(x, E)| < L_{q_k}(E) - \delta\} < e^{-C_0 \delta q_k}.$$

Proof. If x is such that $\frac{1}{q_k} \ln |P_{q_k}(x; E)| \leq L_{q_k} - \delta$, then $\frac{1}{q_k} \ln |P_{q_k}^{0}(x; E)| \leq -\delta$. Since there are at most N_0 eigenvalues in $\mathcal{K}^0(x)$, thus there is some l such that

$$\frac{1}{q_k} \ln |\mu_l(x) - E| \leq -\delta / N_0 \quad \Rightarrow |\mu_l(x) - E| \leq e^{-\delta q_k / N_0}. \tag{4.2}$$

Among all $x \in [0, 1)$, there are at most q_k intervals of x such that some $\mu_l(x) - E$ satisfies (4.2). In fact, there are at most q_k intersections of $\overline{\text{graph}(\Lambda_j)}$ and $[0, 1) \times \{E\}$. Since each Λ_j is monotone, (4.2) is only possible for x near such intersections. By Proposition 3.1, $\mu_i(x)$ are lower-Lipschitz with respect to invariant measure. Thus, for q_k large enough,

$$\begin{aligned} \nu\{x : \text{there is } l \text{ such that } |\mu_l(x) - E| \leq e^{-\delta q_k / N_0}\} &\leq q_k \frac{e^{-\delta q_k / N_0}}{\lambda \gamma_- C_-} \\ &\leq e^{-\frac{\delta q_k}{2 N_0}} \leq e^{-C_0 \delta q_k}. \end{aligned}$$

■

Thus, we have proved the result with $L_{q_k}(E)$ instead of $L(E)$. Now, we need the last component of the proof.

Claim 3. *For any $\varepsilon > 0$, $L(E) \leq L_{q_k}(E) + \varepsilon$ uniform in E when q_k is large enough.*

Proof. In fact, since the operator is bounded, $\ln |P_{q_k}^{0}(x; E)| \leq N_0 C_1$. Together with (4.1), we get

$$\frac{1}{q_k} \ln |P_{q_k}(x; E)| \leq L_{q_k}(E) + C \frac{\ln q_k}{q_k} + \frac{N_0 C_1}{q_k} \leq L_{q_k}(E) + \varepsilon \quad \text{for all } x \in [0, 1) \tag{4.3}$$

uniformly in E when q_k is large enough (depending on $\lambda, \gamma_\pm, C_\pm$). The same holds for $\widetilde{P}_{q_k}(x; E)$.

On the other hand, by Lemma 4.1, for any $\varepsilon > 0$, $\frac{1}{n} \ln |P_n(x; E)| \le L(E) + \varepsilon$ eventually. While by definition of Lyapunov exponent (2.4), $L(E)$ is the limiting averaging of $\frac{1}{n} \ln \|M_n(x; E)\|$. But M_n and P_n are connected by (2.3). Thus, we see that on a set of measures at least $\frac{1}{4}$, the following is true for either $n = q_k, q_k - 1$, or $q_k - 2$:

$$\frac{1}{n} \ln |P_n(x; E)| \ge L(E) - \varepsilon. \tag{4.4}$$

If $n = q_k$, combining (4.4) and (4.3) gives us what we want. Otherwise, we first notice by row expansion of determinant, we have

$$P_n(x; E) + P_{n-2}(x; E) = (\lambda f(T^{n-1}x) - E) P_{n-1}(x; E) \tag{4.5}$$

$$\widetilde{P}_n(x; E) + 2(-1)^n = P_n(x; E) - P_{n-2}(Tx; E). \tag{4.6}$$

Then, when $n = q_k - 1$, by (4.5), either P_{q_k} or P_{q_k-2} satisfies (4.4), so we can combine it with (4.3) to derive the result. If $n = q_k - 2$, by (4.6), we have either P_{q_k} or $\widetilde{P}_{q_k}$ satisfies (4.4). For the former case, we get the result. For the latter, combining (4.4) and (4.3) with $\widetilde{P}_{q_k}$ instead of $P_{q_k}(x; E)$. The claim follows. ∎

Now, the result follows immediately: for any $\delta > 0$, apply Claim 3 to get $L(E) \le L_{q_k}(E) + \delta/2$ eventually so that

$$\left\{x \in [0,1) : \frac{1}{n} \ln |P_n(x; E)| \le L(E) - \delta\right\} \subset \left\{x \in [0,1) : \frac{1}{n} \ln |P_n(x; E)| \le L_{q_k}(E) - \frac{\delta}{2}\right\}.$$

Then the result follows from Claim 2. ∎

5. Exponential decay of eigenfunctions

We prove our key Lemma 5.3, which provides uniform exponential decay of generalized eigenfunction in x, E, α. To do so, we introduce some definitions and prove a typical "either or" argument in the proof of localization in Lemma 5.2.

Definition 2 (Regular point). We say a point $n \in \mathbb{Z}$ is (x, c, q_k)-*regular* if there is an interval $[a, b]$ with

$$n \in [a, b], \quad b = a + q_k - 1, \quad |a - n| \ge \frac{q_k}{5}, \quad |n - b| \ge \frac{q_k}{5}, \tag{5.1}$$

such that

$$|G_{x,E,[a,b]}(a,n)| \le e^{-c|n-a|} \quad \text{and} \quad |G_{x,E,[a,b]}(n,b)| \le e^{-c|n-b|}.$$

Otherwise we say n is (x,c,q_k)-*singular.*

Lemma 5.1. *Fix δ, E such that $0<\delta<L(E)$. For q_k large enough, for any x, if n is $(x,L(E)-\delta,q_k)$-singular, then for any $a\in\left[n-\lfloor\frac{3q_k}{4}\rfloor, n-\lfloor\frac{q_k}{4}\rfloor\right]$,*

$$|P_{q_k}(T^a x)| \le e^{q_k(L(E)-\delta/10)}. \tag{5.2}$$

Furthermore, let $N_k=\lfloor\frac{3q_k}{4}\rfloor-\lfloor\frac{q_k}{4}\rfloor+1$ denote the number of such a, then

$$\frac{q_k+1}{2} \le N_k \le \frac{q_k+3}{2}.$$

Proof. Since n is $(x,L(E)-\delta,q_k)$-singular, for any $[a,b]$ satisfying (5.1), in particular, for any $a\in\left[n-\lfloor\frac{3q_k}{4}\rfloor, n-\lfloor\frac{q_k}{4}\rfloor\right]$, $b=a+q_k-1$, we have

$$\begin{cases} \text{either} & |G_{x,E,[a,b]}(a,m)| \ge e^{-(L(E)-\delta)(m-a)}, \\ \text{or} & |G_{x,E,[a,b]}(m,b)| \ge e^{-(L(E)-\delta)(b-m)}. \end{cases} \tag{5.3}$$

Notice that

$$\begin{cases} |G_{x,E,[a,b]}(a,m)| = \dfrac{|P_{b-m}(T^{m+1}x)|}{|P_{q_k}(T^a x)|}, \\ |G_{x,E,[a,b]}(m,b)| = \dfrac{|P_{m-a}(T^a x)|}{|P_{q_k}(T^a x)|}. \end{cases} \tag{5.4}$$

Now, we consider the first case in (5.3) for simplicity. The other case is similar. By Lemma 4.1, we have when q_k is large enough

$$|P_{b-m}(T^{m+1}x)| \le e^{(L(E)+\delta/10)(b-m)}. \tag{5.5}$$

By (5.3),(5.4), and (5.5), we see that

$$\begin{aligned} |P_{q_k}(T^a x)| &\le e^{(L(E)+\frac{\delta}{10})(b-m)+(L(E)-\delta)(m-a)} \\ &\le e^{L(E)(b-a)+\frac{\delta}{10}(b-m)-\delta(m-a)} \\ &\le e^{L(E)q_k+\frac{\delta}{10}q_k-\delta\frac{q_k}{5}} \le e^{(L(E)-\frac{\delta}{10})q_k}, \end{aligned}$$

Thus, we proved (5.2). The bound of N_k follows from direct computation when $q_k\equiv 0,1,2,3 \pmod 4$. ∎

In other words, there are many "large deviation points" near each singular point. This fact, together with the large deviation estimates in Theorem 3 and appropriate weakly Liouville assumption (Definition 1), leads to the repelling of two singular points. In fact, we prove below that two $(x, L(E)-\delta, q_k)$ singular points are at least "$q_{k+1} - q_k/2$" away from each other.

Lemma 5.2 (Either or argument). *Let C_0 be as in Theorem 3. Assume α and E satisfy $\beta(\alpha) < C_0 L(E)$. For any $\frac{\beta(\alpha)}{C_0} < \delta < L(E)$, we have that for q_k large enough, and for any $\frac{q_k+1}{2} < |n-m| \le q_{k+1} - 1 - \frac{q_k+1}{2}$, either m or n is $(x, L(E)-\delta, q_k)$-regular for any x.*

Proof. Without loss of generality, assume $n > m$. For any $\delta < L(E)$, assume both m and n are $(x, L(E)-\delta, q_k)$-singular. By Lemma 5.1, we have

$$|P_{q_k}(T^a x)| \le e^{(L(E)-\delta/10)q_k}$$

for any $a \in \left[m - \lfloor \frac{3q_k}{4} \rfloor, m - \lfloor \frac{q_k}{4} \rfloor\right] \cup \left[n - \lfloor \frac{3q_k}{4} \rfloor, n - \lfloor \frac{q_k}{4} \rfloor\right]$. Notice further that

$$\begin{aligned} n - \left\lfloor \frac{3q_k}{4} \right\rfloor - \left(m - \left\lfloor \frac{q_k}{4} \right\rfloor\right) &= n - m - N_k + 1 \\ &> \frac{q_k+1}{2} - \frac{q_k+3}{2} + 1 = 0. \end{aligned}$$

Thus, the two intervals of a have no intersection. Overall there are $2N_k \ge q_k + 1$ many possible a such that $|P_{q_k}(T^a x)| \le e^{(L(E)-\delta/10)q_k}$. By Theorem 3 and pigeonhole principle, there are $i, j \in \left[m - \lfloor \frac{3q_k}{4} \rfloor, m - \lfloor \frac{q_k}{4} \rfloor\right] \cup \left[n - \lfloor \frac{3q_k}{4} \rfloor, n - \lfloor \frac{q_k}{4} \rfloor\right]$ such that

$$\nu([T^i x, T^j x]) \le e^{-C_0 \delta q_k}.$$

Notice that

$$|i-j| \le n - \left\lfloor \frac{q_k}{4} \right\rfloor - \left(m - \left\lfloor \frac{3q_k}{4} \right\rfloor\right) = n - m + N_k - 1 \le q_{k+1} - 1.$$

By Lemma A.1 and (A.2), we have

$$e^{-C_0 \delta q_k} \ge \nu([T^i x, T^j x]) \ge \nu([x, T^{q_k} x]) \ge \frac{1}{q_{k+1}}.$$

This implies that

$$C_0 \delta < \frac{\ln q_{k+1}}{q_k} \implies C_0 \delta \implies C_0 \delta \le \limsup \frac{\ln q_{k+1}}{q_k} = \beta(\alpha).$$

which leads to a contradiction with the assumption. ∎

Lemma 5.3. *Let C_0 be as in Theorem 3. If (x, E, α) satisfy*

($\mathcal{E}1$) *E is a generalized eigenvalue of $H_{\lambda,f,T,x}$,*

($\mathcal{E}2$) *$\beta(\alpha) < C_0 L(E)$,*

then E is an eigenvalue with exponentially decaying eigenfunction. Denote the normalized eigenfunction by ψ with $\|\psi\|_\infty = 1$.

Furthermore, for any $\varepsilon > 0$, there is a $C = C(\varepsilon)$, uniform in all x, E, α which satisfy ($\mathcal{E}1$) and ($\mathcal{E}2$) above, such that for any pair of eigenvalue E and normalized eigenvector ψ, there is $n_0 = n_0(E)$ such that

$$|\psi(n)| \le C(\varepsilon) e^{-\frac{1}{10}(L(E) - \frac{\beta(\alpha)}{C_0} - \varepsilon)|n - n_0|}. \tag{5.6}$$

Proof. Take any $\frac{\beta(\alpha)}{C_0} < \delta < L(E)$. Let ψ be a generalized eigenfunction of $H_{\lambda,f,T,x}$ with respect to E. Thus, $|\psi(n)| \le C_1(1 + |n|)^p$ where $C_1 = C_1(E, x, \alpha)$. We first prove ψ decay exponentially so that E is an eigenvalue, then we prove the decay is uniform in the sense of (5.6).

Without loss of generality, assume that $\psi(0) \ne 0$. By (2.2), 0 is eventually $(x, L(E) - \delta, q_k)$-singular. By Lemma 5.2, we have for q_k large enough, any $n \in \left(\frac{q_k+1}{2}, q_{k+1} - 1 - \frac{q_k+1}{2}\right] := (A_k, B_k]$ is $(L(E) - \delta, q_k)$-regular. Notice further that $A_{k+1} \le B_k$ since $q_{k+1} \ge q_k + 4$ for $k \ge 4$. Thus, eventually for any n, there is k such that $n \in (A_k, A_{k+1}]$. We derive exponential decay by considering two cases separately.

(1) If $n \in (A_k, q_k]$, n is $(L(E) - \delta, q_k)$-regular, by (2.2), we have for arbitrarily small $\varepsilon > 0$, eventually

$$|\psi(n)| \le C_1 e^{-(L(E)-\delta)q_k/5}(1 + 3n)^p \le e^{-(L(E)-\delta-\varepsilon)n/5}. \tag{5.7}$$

(2) If $n \in [q_k + 1, A_{k+1}]$, then it is easy to check that $|n - B_k| \ge |n - A_k| \ge \frac{n}{2}$. By (2.2), we have

$$|\psi(n)| \le 2e^{-(L(E)-\delta)q_k/5}|\psi(n_1)|$$

where $n_1 = a - 1$ or $b + 1$ for suitable $[a, b]$ satisfying (5.1). As long as $n_1 \in (A_k, B_k]$, where n_1 would be $(x, L(E) - \delta, q_k)$ regular, then we can apply (2.2) again to $\psi(n_1)$. We can repeat this process to get $\psi(n_2)$, $\psi(n_3), \ldots,$ as long as n_i stays in $(A_k, B_k]$. Since $|n - B_k| \ge |n - A_k| \ge \frac{n}{2}$ while $|n_i - n_{i+1}| \le q_k$, thus we can at least do

$$J \ge \frac{|n - A_k|}{q_k} \ge \frac{n}{2q_k}$$

many times. Then we get

$$\begin{aligned}|\psi(n)| &\le 2^J e^{-(L(E)-\delta)q_k J/5}|\psi(n_J)| \le e^{-(L(E)-\delta-\frac{5}{q_k})\frac{n}{10}}|\psi(n_J)| \\ &\le C_1 e^{-(L(E)-\delta-\frac{5}{q_k})n/10}(1 + 3n)^p \le C_1 e^{-(L(E)-\delta-\varepsilon)n/10}.\end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8) gives us the first half of the theorem. Now, since $\psi \in \ell^2$, we can normalize it so that $\|\psi\|_\infty = 1$.

The key point of the second half is the uniformity in x, E, α. Take $n_0 = \min\{n : \psi(n) = 1\} > -\infty$ to be the leftmost maximum point of ψ. By (2.2), we see that the maximum point n_0 is always $(x, L(E)-\delta, q_k)$-singular for all q_k. Thus, n is $(x, L(E)-\delta, q_k)$-regular if $A_k < |n-n_0| \le B_k$. We can now repeat the estimates (5.7) and (5.8) above with the new, uniform (in x, E, α) improvement that $|\psi(n_i)| \le 1$ instead of $|\psi(n_i)| \le C_1(x, E, \alpha)(1+n_i)^p$, where we get

$$\begin{cases} |\psi(n)| \le e^{-(L(E)-\delta)\frac{|n-n_0|}{5}}, & n \in (A_k, q_k], \\ |\psi(n)| \le e^{-(L(E)-\delta-\frac{5}{q_k})\frac{|n-n_0|}{10}}, & n \in (q_k, A_{k+1}]. \end{cases}$$

Since $\frac{\beta(\alpha)}{C_0} < \delta < L(E)$ is arbitrary and $\frac{5}{q_k}$ is arbitrarily small once q_k is large enough uniformly in x, E, α. Thus, (5.6) follows. ■

6. Localization results

Now, we prove our main results. Both of them follow directly from Lemma 5.3.

Proof of Theorem 1. Recall that by Schnol's theorem, spectral measure is supported on the set of generalized eigenvalues (see [3, Chapter VII]. Fix λ and x, the theorem follows directly from Lemma 5.3). ■

Definition 3 (Uniform localization). An operator H exhibits *uniform localization* if there exists C, c such that for any pair of eigenvalue and eigenfunction E, ψ, there exists $n_0 = n_0(E)$ such that

$$|\psi(n)| \le Ce^{-c|n-n_0|}.$$

Proof of Theorem 2. By Corollary 3.3, $0 < \ln\left(\frac{\lambda\gamma_- C_-}{4e}\right) \le L(E)$ for all E. It follows that $\beta(\alpha) < C_0 L(E)$. Thus, Lemma 5.3 applies to all x, all E and those α which satisfy our assumption. By taking $\varepsilon = \frac{1}{2}\ln\left(\frac{\lambda\hat{C}\gamma_-}{4\eta e}\right)$ in Lemma 5.3, we get uniform localization. ■

A. Orbital analysis

It is well known that the irrational rotation on the 1D-torus, $R_\alpha(x) = x + \alpha$, has the best-approximation property, cf. [28],

$$\|q_k\alpha\| \le \|n\alpha\| \quad \text{for all } 1 \le n < q_{k+1}\alpha$$

with estimates

$$\frac{1}{2q_{k+1}} \le \|q_k\alpha\| \le \frac{1}{q_{k+1}},$$

where q_k is defined in (2.1). Furthermore, the orbits of R_α is also well understood; we cite [10, Propositions 4.1 and 4.2] here.

Proposition A.1. *Let $k \ge 1$. The points $\{j\alpha\}$, $j = 0, 1, 2, \ldots, q_k - 1$ splits $[0, 1)$ into q_{k-1} "large" gaps of length $\|(q_k - q_{k-1})\alpha\|$ and $q_k - q_{k-1}$ "small" gaps of length $\|q_{k-1}\alpha\|$. Furthermore, we have the estimates*

$$\frac{1}{q_k} - \frac{q_{k-1}}{q_k q_{k+1}} \le \|q_{k-1}\alpha\| \le \frac{1}{q_k},$$
$$\frac{1}{q_k} \le \|(q_k - q_{k-1})\alpha\| \le \frac{1}{q_k} + \frac{1}{q_{k+1}}.$$

For a general measure-preserving circle homeomorphism T with rotation number α, such kind of best approximate properties and orbital analysis holds when we replace the distance function $\|\cdot\|$ by the invariant measure ν.

Lemma A.1 (Best approximation). *For any $x \in \mathbb{T}^1$ and $k \in \mathbb{N}$,*

$$\nu([x, T^i x]) \ge \nu([x, T^{q_k} x]),$$

where $0 \le i < q_{k+1}$.

Proof. Note that Lemma A.1 holds when the invariant measure is the Lebesgue measure – in other words when the map T is the irrational rotation.

For a general measure-preserving circle homeomorphism, this inequality holds since it is equivalent to the irrational rotation case. In fact, the Poincaré classification theorem [9, Theorem 4.3.20] guarantees the existence of the topological conjugacy h with a rotation R_α, and h is also the distribution function for the unique invariant measure ν. Hence, for any $x \in \mathbb{T}^1$ and $i \in \mathbb{N}$, we have

$$\nu([x, T^i x]) = |h(T^i x) - h(x)| = |R_\alpha^i(h(x)) - h(x)| = \|i\alpha\|. \qquad \blacksquare$$

Lemma A.2. *Fix x, Let $k \ge 1$. The points $\{T^j x\}_{j=0}^{q_k-1}$ split interval $[0, 1)$ into q_{k-1} "large" gaps of invariant measure $\nu([T^{q_k} x, T^{q_{k-1}} x]) = \nu([x, T^{q_k - q_{k-1}} x])$, and $q_k - q_{k-1}$ "small" gaps of invariant measure $\nu([x, T^{q_{k-1}} x])$. Furthermore, we have the estimates*

$$\frac{1}{q_k} - \frac{q_{k-1}}{q_k q_{k+1}} \le \nu([x, T^{q_{k-1}} x]) \le \frac{1}{q_k},$$
$$\frac{1}{q_k} \le \nu([x, T^{q_k - q_{k-1}} x]) \le \frac{1}{q_k} + \frac{1}{q_{k+1}}.$$

To prove the lemma, let us first introduce the dynamical partition on the circle by following the convention in [12]. For each $k \in \mathbb{N}$, let I_k be the interval between x and $T^{q_k}x$. It can be verified by induction in k that the following collection of intervals forms a *k-th dynamical partition* of $\mathbb{T}^1$:

$$\begin{aligned}\mathcal{P}_k(x) &:= \{I_k, T(I_k), \dots, T^{q_{k-1}-1}(I_k)\} \cup \{I_{k-1}, T(I_{k-1}), \dots, T^{q_k-1}(I_{k-1})\} \\ &:= \mathcal{S}_k \cup \mathcal{L}_k.\end{aligned}$$

That is, they are disjoint except for the endpoints, and the union covers the whole circle. Notice that intervals in $\mathcal{S}_k$ all have smaller invariant measure $\nu(I_k) < \nu(I_{k-1})$ than intervals in $\mathcal{L}_k$, thus we call them "short" and "long" intervals correspondingly. One can check by induction on k that each "long" interval $T^j(I_{k-1})$ in k-th dynamical partition is divided into a_{k+1} "long" intervals and one "short" interval in $(k+1^{th})$-th dynamical partition. More specifically,

$$\begin{aligned}&T^j(I_{k-1}) \in \mathcal{L}_k \\ &\quad \Longrightarrow T^{j+q_{k-1}}(I_k), T^{j+q_{k-1}+q_k}(I_k), \dots, T^{j+q_{k-1}+(a_{k+1}-1)q_k}(I_k) \in \mathcal{L}_{k+1} \\ &\quad\quad \text{and } T^j(I_{k+1}) \in \mathcal{S}_{k+1}.\end{aligned}$$

This allows us to estimate the "large" and "small" gaps[3] in Lemma A.2 now.

Proof of Lemma A.2. Since ν is the invariant measure of T, for dynamical partition $\mathcal{P}_{k+1}(x)$, we have

$$1 = \sum_{i=0}^{q_{k+1}-1} \nu(T^i(I_k)) + \sum_{j=0}^{q_k-1} \nu(T^j(I_{k+1})) = q_{k+1}\nu(I_k) + q_k\nu(I_{k+1}). \tag{A.1}$$

By (A.1), we get

$$\nu(I_k) = \frac{1 - q_k\nu(I_{k+1})}{q_{k+1}} \le \frac{1}{q_{k+1}}.$$

Moreover, since (A.1) holds for any k, we also get $\nu(I_{k+1}) \le \frac{1}{q_{k+2}}$. So,

$$\nu(I_k) \ge \frac{1}{q_{k+1}} - \frac{q_k}{q_{k+1}q_{k+2}} \ge \frac{1}{2q_{k+1}}.$$

The last inequality follows from the recurrence relation (2.1) and $a_k \ge 1$:

$$q_{k+2} = a_{k+2}q_{k+1} + q_k \ge 2q_k.$$

By the comparability between ν and the Lebesgue measure on a circle ($\mathcal{T}1$), the claim follows. ■

[3]Notice that the partition in Lemma A.2 is different from dynamical partition, "long" and "short" intervals are also different concepts from "large" and "small" gaps.

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References

[1] A. Avila, J. You, and Q. Zhou, Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.* **166** (2017), no. 14, 2697–2718 Zbl 1503.47041 MR 3707287

[2] J. Béllissard, R. Lima, and E. Scoppola, Localization in v-dimensional incommensurate structures. *Comm. Math. Phys.* **88** (1983), no. 4, 465–477 Zbl 0543.35018 MR 0702564

[3] J. M. Berezans'kiĭ, *Expansions in eigenfunctions of selfadjoint operators*. Transl. Math. Monogr. 17, American Mathematical Society, Providence, RI, 1968 Zbl 0157.16601 MR 0222718

[4] H. Cao, Y. Shi, and Z. Zhang, Localization for Lipschitz monotone quasi-periodic Schrödinger operators on $\mathbb{Z}^d$ via Rellich functions analysis. 2024, arXiv:2407.01970v1

[5] W. Craig, Pure point spectrum for discrete almost periodic Schrödinger operators. *Comm. Math. Phys.* **88** (1983), no. 1, 113–131 Zbl 0525.35076 MR 0691202

[6] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Study edn., Texts Monogr. Phys., Springer, Berlin, 1987 Zbl 0619.47005 MR 0883643

[7] L. Ge, J. You, and X. Zhao, The arithmetic version of the frequency transition conjecture: new proof and generalization. *Peking Math. J.* **5** (2022), no. 2, 349–364 Zbl 1513.81054 MR 4492656

[8] R. Han, S. Jitomirskaya, and F. Yang, Anti-resonances and sharp analysis of Maryland localization for all parameters. 2022, arXiv:2205.04021v1

[9] B, Hasselblatt and A, Katok, *A first course in dynamics. With a panorama of recent developments*. Cambridge University Press, Cambridge and New York, 2003 Zbl 1027.37001 MR 1995704

[10] S. Jitomirskaya and I. Kachkovskiy, All couplings localization for quasiperiodic operators with monotone potentials. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 3, 777–795 Zbl 1514.47056 MR 3908765

[11] S. Jitomirskaya and I. Kachkovskiy, Sharp arithmetic localization for quasiperiodic operators with monotone potentials. arXiv:2407.00703v1

[12] S. Jitomirskaya and S. Kocić, Spectral theory of Schrödinger operators over circle diffeomorphisms. *Int. Math. Res. Not. IMRN* (2022), no. 13, 9810–9829 Zbl 1501.37038 MR 4447138

[13] S. Jitomirskaya, L. Konstantinov, and I. Krasovsky, On the spectrum of critical almost Mathieu operators in the rational case. *J. Spectr. Theory* **12** (2022), no. 1, 11–21 Zbl 1501.47051 MR 4404805

[14] S. Jitomirskaya and W. Liu, Arithmetic spectral transitions for the Maryland model. *Comm. Pure Appl. Math.* **70** (2017), no. 6, 1025–1051 Zbl 06734215 MR 3639318

[15] S. Jitomirskaya and W. Liu, Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math. (2)* **187** (2018), no. 3, 721–776 Zbl 1470.47025 MR 3779957

[16] S. Jitomirskaya, W. Liu, and S. Zhang, Arithmetic spectral transitions: a competition between hyperbolicity and the arithmetics of small denominators. In *Harmonic analysis and applications*, pp. 35–71, IAS/Park City Math. Ser. 27, Amer. Math. Soc., Providence, RI, 2020 Zbl 1495.47057 MR 4249625

[17] S. Jitomirskaya and R. Mavi, Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials. *Int. Math. Res. Not. IMRN* (2017), no. 1, 96–120 Zbl 1405.35175 MR 3632099

[18] S. Y. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)* **150** (1999), no. 3, 1159–1175 MR 1740982

[19] I. Kachkovskiy, Localization for quasiperiodic operators with unbounded monotone potentials. *J. Funct. Anal.* **277** (2019), no. 10, 3467–3490 Zbl 1481.47043 MR 4001077

[20] I. Kachkovskiy, L. Parnovski, and R. Shterenberg, Convergence of perturbation series for unbounded monotone quasiperiodic operators. *Adv. Math.* **409** (2022), no. part B, article no. 108647 Zbl 1500.39014 MR 4481139

[21] I. Kachkovskiy, L. Parnovski, and R. Shterenberg, On gaps in the spectra of quasiperiodic Schrödinger operators with discontinuous monotone potentials. 2024, arXiv:2407.00705v2

[22] S. Kocić, Singular continuous phase for Schrödinger operators over circle maps. *Math. Ann.* **389** (2024), no. 2, 1545–1573 MR 4745745

[23] W. Liu, Almost Mathieu operators with completely resonant phases. *Ergodic Theory Dynam. Systems* **40** (2020), no. 7, 1875–1893 Zbl 1446.37029 MR 4108908

[24] W. Liu, Distributions of resonances of supercritical quasi-periodic operators. *Int. Math. Res. Not. IMRN* (2024), no. 1, 197–233 Zbl 07795381 MR 4686650

[25] W. Liu and X. Yuan, Anderson localization for the completely resonant phases. *J. Funct. Anal.* **268** (2015), no. 3, 732–747 Zbl 1333.47025 MR 3292353

[26] J. Pöschel, *https://doi.org/10.1007/bf01211953*Examples of discrete Schrödinger operators with pure point spectrum. *Comm. Math. Phys.* **88** (1983), no. 4, 447–463 Zbl 0532.35070 MR 0702563

[27] Y. Shi, Localization for almost-periodic operators with power-law long-range hopping: a Nash–Moser iteration type reducibility approach. *Comm. Math. Phys.* **402** (2023), no. 2, 1765–1806 Zbl 1530.81083 MR 4627331

[28] A. J. van der Poorten, Notes on continued fractions and recurrence sequences. In *Number theory and cryptography (Sydney, 1989)*, pp. 86–97, London Math. Soc. Lecture Note Ser. 154, Cambridge Univ. Press, Cambridge, 1990 MR 1055401

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