J. Spectr. Theory (Online first)
DOI 10.4171/JST/528

On the spectrum of the Sturm–Liouville problem with arithmetically self-similar weight

Nikita Rastegaev

Abstract. We consider the spectral asymptotics of the Sturm–Liouville problem with an arithmetically self-similar singular weight. In previous papers, A. A. Vladimirov and I. A. Sheĭpak, as well as the author, relied on the spectral periodicity property, which places major constraints on the self-similarity parameters of the weight. In this study, a different approach to estimation of the eigenvalue counting function is presented. As a result, a significantly wider class of self-similar measures can be considered. The obtained asymptotics are applied to the problem of small ball deviations for the Green Gaussian processes.

1. Introduction

We generalize the results of [21, 31] on the spectral asymptotics of the problem

$$\begin{cases} -y'' = \lambda\mu y, \\ y'(0) = y'(1) = 0, \end{cases} \tag{1.1}$$

where the weight measure μ is a distributional derivative of a self-similar generalized Cantor-type function (in particular, μ is singular with respect to the Lebesgue measure).

Remark 1.1. It is well known that a change of the boundary conditions causes a rank-two perturbation of the quadratic form corresponding to the problem (1.1). It follows from the general variational theory (see [5, Section 10.3]) that the counting functions of the eigenvalues of boundary-value problems related to the same equation, but with different boundary conditions cannot differ by more than 2. Thus, the main term of spectral asymptotics does not depend on the boundary conditions.

Also, it follows from [14, Theorem 3.2] (see also [19, Lemma A.1] for a simple variational proof) that relatively compact perturbations of the operator (e.g., lower-order terms) do not affect the main term of the asymptotics given by (1.2) below.

Mathematics Subject Classification 2020: 34L20 (primary); 34B24 (secondary).
Keywords: spectral asymptotics, Sturm–Liouville equation, self-similar measure, singular measure, small ball deviations, Green Gaussian process.

Remark 1.2. Spectral asymptotics of the problem (1.1), aside from being interesting in itself, arise in the problem of small ball deviations of Green Gaussian processes in $L_2(\mu)$ (see [17]).

The problem of the asymptotic behavior of the eigenvalues for (1.1) goes back to the works of M. G. Kreĭn (see, e.g., [13]).

From [2] (see also [1, 3, 4, 6]), it follows that if the measure μ contains an absolutely continuous component, then its singular component does not influence the main term of the spectral asymptotic.

In the case of a purely singular measure μ, it follows from early works by M. G. Kreĭn that the counting function $N\colon (0, +\infty) \to \mathbb{N}$ of the eigenvalues of the problem (1.1) admits the estimate $o(\lambda^{\frac{1}{2}})$ instead of the usual asymptotics $N(\lambda) \sim C\lambda^{\frac{1}{2}}$ in the case of a measure containing a regular component (see, e.g., [10, 15], and also [6] for similar results for higher even-order operators and better lower bounds for eigenvalues for some special classes of measures). However, the exact order of eigenvalue growth for singular measures is not specified. An estimate of the growth order for higher-order multidimensional problems in terms of the Hausdorff dimension of the measure was obtained recently in [22].

The exact power exponent D of the counting function $N(\lambda)$ in the case of self-similar measure μ was established in [8]. See also earlier works [15, 26] for particular results, concerning the classical Cantor ladder, and [7, 11] and references therein for a generalization to the case of self-conformal, i.e., self-similar via non-affine contractions, measures.

It is shown in [24] that the eigenvalue counting function of (1.1) has the asymptotics

$$N(\lambda) = \lambda^D \cdot (s(\ln \lambda) + o(1)), \quad \lambda \to +\infty, \tag{1.2}$$

where $D \in \left(0, \frac{1}{2}\right)$ and $s(t), t \in \mathbb{R}$ is a periodic bounded function separated from 0. If the primitive of μ has a non-arithmetic type of self-similarity (see Definition 2.3 below), then the function s degenerates into constant. In the case of arithmetic self-similarity, it has a period T, which depends on the parameters of the self-similarity (see Definition 2.3 and (2.3) below).[1] See also [12] for a similar result on p.c.f. self-similar fractals, where s is shown to be right continuous, and [20] for the case of a more complicated graph-directed construction of self-similar measures.

In [29] the results of [24] are generalized for the case of indefinite weight. In the paper [17] the results of [24] are generalized to the case of an arbitrary even-order differential operator. In both works, s was shown to be continuous (in [29] for the first time). Also the following conjecture has been introduced.

[1] Since in this paper we mainly consider the arithmetic case, we often assume the function s has the argument $t \in [0, T]$ and extend it by periodicity.

Conjecture 1.3. *The function s is not constant for arbitrary non-constant weight with an arithmetically self-similar primitive.*

The paper [29] provides a computer-assisted proof of this conjecture in the simplest case, when the generalized primitive of weight μ is a classical Cantor ladder.

In [31], Conjecture 1.3 was confirmed for "even" ladders (see Definition 2.1 below). For such ladders the following theorem was proved.

Theorem 1.4. *The T-periodic continuous coefficient s from the asymptotic* (1.2) *satisfies the relation*

$$s(t) = e^{-Dt}\sigma(t) \quad \textit{for all } t \in [0, T], \tag{1.3}$$

where σ is a purely singular continuous non-decreasing function, that is, the primitive of a measure with no atoms, singular with respect to the Lebesgue measure.

Hence, the relation $s(t) \neq \text{const}$ follows immediately. This result is generalized in [28] to the case of fourth-order operators.

In the paper [21] the result of [31] is generalized to a wider class of ladders satisfying conditions (2.2).

The aim of this paper is to generalize Theorem 1.4 to the case of an arbitrary arithmetically self-similar ladder with non-empty intermediate intervals.

This paper has the following structure. Section 2 provides the necessary definitions of self-similar functions of generalized Cantor type, derives their properties, defines the classes of functions under consideration and states the main result. Section 3 establishes the auxiliary facts, concerning the spectral properties. In Section 4, Theorem 1.4 is proved for the suggested class of ladders. Finally, in Section 5 we provide the link between the results of this paper and the problem of small ball deviations of Gaussian processes.

2. Self-similar functions. Main result statement

Let $m \geqslant 2$, and let $\{I_k = [a_k, b_k]\}_{k=1}^m$ be sub-segments of $[0, 1]$, without interior intersection, i.e., $b_j \leqslant a_{j+1}$ for all $j = 1 \dots m-1$. Next, let positive values $\{\rho_i\}_{i=1}^m$ satisfy the relation $\sum_{k=1}^m \rho_k = 1$ and let $\{e_i\}_{i=1}^m$ be boolean values.

We define a family of affine mappings

$$S_i(t) = \begin{cases} a_i + (b_i - a_i)t, & e_i = 0, \\ b_i - (b_i - a_i)t, & e_i = 1, \end{cases}$$

contracting $[0, 1]$ onto I_i and changing the orientation when $e_i = 1$.

We define the operator $\mathcal{S}$ on the space $L_\infty(0,1)$ as follows:

$$\mathcal{S}(f) = \sum_{i=1}^{m} \big((e_i + (-1)^{e_i} f \circ S_i^{-1}) \cdot \chi_{I_i} + \chi_{\{x>b_i\}}\big)\rho_i,$$

where χ stands for the indicator function of a set. Thus, the graph of the function $\mathcal{S}(f)$ on each segment I_i is an appropriately shrunk graph of the function f. On any intermediate interval, $\mathcal{S}(f)$ is constant.

Proposition 2.1 (see, e.g., [23, Lemma 2.1]). *$\mathcal{S}$ is a contraction mapping on $L_\infty(0,1)$.*

Hence, by the Banach fixed-point theorem, there exists a (unique) function $\mathcal{C} \in L_\infty(0,1)$ such that $\mathcal{S}(\mathcal{C}) = \mathcal{C}$. Such a function $\mathcal{C}(t)$ will be called the *generalized Cantor ladder* with m steps.

The function $\mathcal{C}(t)$ can be found as the uniform limit of the sequence $\mathcal{S}^k(f)$ for $f(t) \equiv t$, which allows us to assume that it is continuous. It is also easy to demonstrate that it is monotone, $\mathcal{C}(0) = 0$, and $\mathcal{C}(1) = 1$. The derivative of $\mathcal{C}(t)$ in the sense of distributions is a singular measure μ without atoms, self-similar in the sense of Hutchinson (see [9, 25]), i.e., it satisfies the relation

$$\mu(E) = \sum_{k=1}^{m} \rho_k \cdot \mu(S_k^{-1}(E \cap I_k))$$

for arbitrary measurable set E. More general constructions of self-similar functions are described in [23].[2]

Remark 2.2. Without loss of generality, we could assume that $a_1 = 0$ and $b_m = 1$; otherwise the measure could be stretched, which leads to the spectrum being multiplied by a constant.

Definition 2.3. The self-similarity is called *arithmetic* if the logarithms of the values $\rho_k(b_k - a_k)$ are commensurable. In other words,

$$\rho_i(b_i - a_i) = \tau^{k_i}, \quad i = 1, \dots, m,$$

for a certain constant τ and $k_i \in \mathbb{N}$, such that $\mathrm{GCD}(k_i, i = 1, \dots, m) = 1$.

We call the generalized Cantor ladder *even* if, for all $i = 2, \dots, m$,

$$\rho_i = \rho_1 = \frac{1}{m}, \quad b_i - a_i = b_1 - a_1, \quad a_i - b_{i-1} = a_2 - b_1 > 0. \tag{2.1}$$

That is the class of ladders considered in [31].

[2] In the case $m = 1$, a similar construction gives a so-called *degenerate self-similar measure* (see [30]).

In [21] the relation (1.3) is proved for arithmetically self-similar ladders with the following conditions:

$$k_i = k_1 = 1, \quad a_i - b_{i-1} > 0 \quad \text{for all } i = 2, \dots, m. \tag{2.2}$$

Let us state the main result of this paper.

Theorem 2.4. *Let the ladder be arithmetically self-similar, and let $a_i - b_{i-1} > 0$ for all $i = 2, \dots, m$. Then formula (1.3) holds.*

For the described class of ladders the power exponent D and the period of the function $s(t)$ in the asymptotics (1.2) are defined by the following relations, obtained in [24]:

$$\sum_{i=1}^{m} \tau^{k_i D} = 1, \quad T = -\ln \tau. \tag{2.3}$$

The relationship between D and the Hausdorff dimensions of μ has been established in [24] (see [17] for the higher-order case). The Hausdorff dimension α of the measure support supp μ is the unique solution of the equation

$$\sum_{i=1}^{m} (b_i - a_i)^{\alpha} = 1.$$

The Hausdorff dimension of the measure μ itself is

$$\beta = \frac{\sum\limits_{i=1}^{m} \rho_i \log \rho_i}{\sum\limits_{i=1}^{m} \rho_i \log(b_i - a_i)}.$$

The exponent D satisfies

$$D = \frac{\gamma}{1-\gamma}, \quad \beta \leqslant \gamma \leqslant \alpha,$$

and the inequalities are strict when $\beta < \alpha$. It is clear that $\alpha = \beta$ if and only if $\rho_i = (b_i - a_i)^{\alpha}$ for all $i = 1, \dots, m$ (e.g., in the case of the even ladder (2.1)), and in this case the exponent agrees with the result of [22] for the case $\ell = 1$, $N = 1$, because μ is Ahlfors regular of order $\alpha = \gamma$. When $\beta < \alpha$, the measure μ is not Ahlfors regular, therefore [22] provides only upper and lower bounds for D.

We also mention the paper [11], where the exponent D is introduced as the fixed point of the analytic function $\upsilon(t)$ solving

$$\sum_{i=1}^{m} \rho_i^t (b_i - a_i)^{\upsilon(t)} = 1$$

(notice that $\upsilon(0) = \alpha$, $\upsilon'(1) = -\beta$).

3. Auxiliary facts about the spectrum

We consider the formal boundary value problem on a segment $[a, b] \subset [0, 1]$:

$$\begin{cases} -y'' = \lambda\mu y & \text{in } [a, b], \\ y'(a) - \gamma_0 y(a) = y'(b) + \gamma_1 y(b) = 0. \end{cases} \tag{3.1}$$

We call the function $y \in W_2^1[a, b]$ its *generalized solution* if it satisfies the integral identity

$$\int_a^b y'\eta'\, dx + \gamma_0 y(a)\eta(a) + \gamma_1 y(b)\eta(b) = \lambda \int_a^b y\eta\, \mu(dx)$$

for any $\eta \in W_2^1[a, b]$. Substituting the functions $\eta \in \overset{\circ}{W}{}_2^1[a, b]$ into the integral identity, we establish that the derivative y' is a primitive of a singular signed measure without atoms $-\lambda\mu y$, whence y is continuously differentiable.

Hereinafter, a particular case of [27, Proposition 11] is required.

Proposition 3.1. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of eigenvalues of the boundary value problem (3.1) numbered in ascending order. Then, regardless of the choice of index $n \in \mathbb{N}$, the eigenvalue λ_n is simple, and the corresponding eigenfunction does not vanish on the boundary of the segment $[a, b]$ and has exactly n different zeroes within this segment.*

Let us denote by $\lambda_n([a, b])$, $n \geqslant 0$, the eigenvalues of the problem

$$\begin{cases} -y'' = \lambda\mu y & \text{in } [a, b], \\ y'(a) = y'(b) = 0, \end{cases}$$

and by

$$N(\lambda, [a, b]) = \#\{n \mid \lambda_n([a, b]) < \lambda\}$$

their counting function. Note that $\lambda_0([a, b]) = 0$.

The following relations follow from the self-similarity of the measure μ.

Lemma 3.2. *One has*

$$\lambda_n(I_i) = \tau^{-k_i}\lambda_n([0, 1]),$$
$$N(\lambda, I_i) = N(\tau^{k_i}\lambda, [0, 1]).$$

Proof. These two relations are equivalent. To prove the first one, let us consider the eigenfunction y_n corresponding to the eigenvalue $\lambda_n(I_i)$ and let us define function z on $[0, 1]$ as

$$z = y_n \circ S_i,$$

where S_i is an affine contraction defined in Section 2. It is clear that the function z satisfies the Neumann boundary conditions on $[0,1]$, and the following relation holds:

$$z'' = (y_n'' \circ S_i)\cdot(b_i-a_i)^2 = \lambda_n(I_i)(b_i-a_i)^2\cdot(\mu\circ S_i)\cdot(y_n\circ S_i).$$

Note also that

$$\mathcal{C}\circ S_i = \mathcal{S}(\mathcal{C})\circ S_i = \rho_i\cdot(e_i+(-1)^{e_i}\mathcal{C}) + \sum_{j=1}^{i-1}\rho_j,$$

whence, by differentiating, we obtain

$$\mu\circ S_i = \rho_i(b_i-a_i)^{-1}\mu,$$

Thus,

$$z'' = \lambda_n(I_i)\rho_i(b_i-a_i)\mu z = \lambda_n(I_i)\tau^{k_i}\mu z.$$

Thereby, the function z corresponds to the eigenvalue $\lambda_n(I_i)\tau^{k_i}$ of the Neumann problem on $[0,1]$ and has exactly n zeroes on it; thus, the proof is complete. ∎

We now prove the main statement of this section.

Theorem 3.3. *Let* $J_1=[c_1,d_1]$, $J_2=[c_2,d_2]$ *be subsegments of* $[0,1]$, *such that* $c_2-d_1\geqslant 0$, *and* $\mu|_{[d_1,c_2]}\equiv 0$. *Denote* $J:=[c_1,d_2]$. *Then the function*

$$F(\lambda) := N(\lambda,J)-N(\lambda,J_1)-N(\lambda,J_2) \tag{3.2}$$

has discontinuities $\lambda_n(J)$, $\lambda_n(J_1)$, $\lambda_n(J_2)$. *Further, the elements of* $\{\lambda_n(J)\}_{n=0}^\infty$ *and* $\{\lambda_n(J_1)\}_{n=0}^\infty\cup\{\lambda_n(J_2)\}_{n=0}^\infty$ *(the latter renumbered in non-decreasing order) are non-strictly interlacing*[3] *beginning with the element of the latter. Moreover,* F *changes its value from* 0 *to* -1 *at the points* $\{\lambda_n(J_1)\}_{n=0}^\infty\cup\{\lambda_n(J_2)\}_{n=0}^\infty$ *and from* -1 *to* 0 *at the points* $\{\lambda_n(J)\}_{n=0}^\infty$ *not included in* $\{\lambda_n(J_1)\}_{n=0}^\infty\cup\{\lambda_n(J_2)\}_{n=0}^\infty$.

Proof. Consider the quadratic form

$$Q_1(y,y) := \int_J |y'|^2\,dt, \quad \mathcal{D}(Q_1) = \{y\in W_2^1(J)\mid y \text{ is linear on } [d_1,c_2]\}.$$

We recall (see, e.g., [5, Section 10.2]) that the counting function $N(\lambda,J)$ could be expressed in terms of this quadratic form:

$$N(\lambda,J) = \sup\dim\left\{\mathcal{H}\subset\mathcal{D}(Q_1)\;\middle|\; Q_1(y,y)<\lambda\int_J y^2(t)\,\mu(dt) \text{ on } \mathcal{H}\right\}.$$

[3] Two sequences $\{\lambda_n^{(1)}\}_{n=0}^\infty$ and $\{\lambda_n^{(2)}\}_{n=0}^\infty$ numbered in non-decreasing order are called *non-strictly interlacing beginning with the second* if $\lambda_{n-1}^{(2)}\leqslant\lambda_{n-1}^{(1)}\leqslant\lambda_n^{(2)}\leqslant\lambda_n^{(1)}$ for all $n\in\mathbb{N}$.

Similarly, if we consider the quadratic form

$$Q_2(y,y) := \int_{J_1} |y'|^2\,dt + \int_{J_2} |y'|^2\,dt,$$
$$\mathcal{D}(Q_2) = \{y \in W_2^1(J) \mid y \text{ is linear on } [d_1, c_2]\},$$

then

$$N(\lambda, J_1) + N(\lambda, J_2) = \sup\dim\left\{\mathcal{H} \subset \mathcal{D}(Q_2) \,\middle|\, Q_2(y,y) < \lambda \int_J y^2(t)\,\mu(dt) \text{ on } \mathcal{H}\right\}.$$

We note that the quadratic forms differ by a positive term

$$Q_1(y,y) - Q_2(y,y) = \int_{d_1}^{c_2} |y'|\,dt,$$

and coincide on a space of codimension 1:

$$Q_1(y,y) = Q_2(y,y) \text{ on } \left\{y \in W_2^1(J) \mid y \text{ is constant on } [d_1, c_2]\right\}.$$

Thus,

$$-1 \leqslant F(\lambda) \leqslant 0. \tag{3.3}$$

Note now that if some point is a discontinuity of two terms on the right side of (3.2), then it is the discontinuity of the third term as well. For example, let $\lambda_n(J)$ be a discontinuity of $N(\lambda, J_1)$. Consider the eigenfunction y_n on J corresponding to $\lambda_n(J)$. Then $y_n|_{J_1}$ is an eigenfunction, corresponding to $\lambda_k(J_1)$ for some k, $y_n'(d_1) = 0$, thus $y_n'(c_2) = 0$ as well, and $y_n|_{J_2}$ is an eigenfunction of the Neumann problem on J_2, which means that $\lambda_n(J)$ is also a discontinuity point of $N(\lambda, J_2)$. Note also that, according to Proposition 3.1, every term changes exactly by 1 at every discontinuity point.

This implies that F decreases by 1 at all points of $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$ (at each point either only one term changes by 1, or all three do). By (3.3), it changes its value from 0 to -1. Obviously, it must change the value from -1 to 0 at all other discontinuities, which are the elements of $\{\lambda_n(J)\}_{n=0}^{\infty}$ not included in $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$. Moreover, no two points from the collection $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$ could go in a row without a point from $\{\lambda_n(J)\}_{n=0}^{\infty}$ between them. Similarly, two points form $\{\lambda_n(J)\}_{n=0}^{\infty}$ could not go in succession without a point from $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$ between them, which implies that the two collections non-strictly interlace. Since $F(0) = 0$, the interlacing starts with an element of $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$. ■

Remark 3.4. The proof of Theorem 3.3 does not require $c_2 - d_1 > 0$, so it does not use the restriction $a_i - b_{i-1} > 0$, thus it could be used even when the ladder has empty intermediate intervals. Moreover, the proof uses the eigenfunction oscillation properties (Proposition 3.1), but does not use the self-similarity of measure μ.

In the case of $m = 2$, $k_1 = k_2 = 1$, Theorem 3.3 and Lemma 3.2 imply that

$$N(\tau^{-1}\lambda_n) = 2N(\lambda_n)$$

and, respectively,

$$\tau\lambda_{2n} = \lambda_n.$$

This relation is called *spectral periodicity* in [21,28,31] and *renormalization property* after [12].

4. Proof of the main result

To prove Theorem 2.4 we need the following facts.

Proposition 4.1 ([31, Proposition 4.1.3]). *Let $f \in L_2[0,1]$ be a bounded non-decreasing non-constant function, let $\{f_n\}_{n=0}^{\infty}$ be a sequence of non-decreasing non-constant step functions, and let $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ be the sequence of discontinuity point sets of the functions f_n. Suppose also that the following asymptotic relation holds as $n \to \infty$:*

$$\#\mathfrak{A}_n \cdot \|f - f_n\|_{L_2[0,1]} = o(1).$$

Then the monotone function f is purely singular, i.e., it is the primitive of a measure, singular with respect to the Lebesgue measure.

Proposition 4.2 ([31, Proposition 5.2.1]). *Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of the eigenvalues of the boundary value problem*

$$\begin{cases} -y'' = \lambda\mu y & \text{in } [a,b], \\ y'(a) = y'(b) = 0, \end{cases}$$

numbered in ascending order. Let $\{\nu_n\}_{n=0}^{\infty}$ be a similar sequence corresponding to the boundary value problem

$$y'(a) - \gamma_0 y(a) = y'(b) + \gamma_1 y(b) = 0$$

for the same equation and some $\gamma_0, \gamma_1 \geqslant 0$. Then

$$\sum_{n=1}^{\infty} |\ln \nu_n - \ln \lambda_n| < +\infty.$$

Remark 4.3. This result could be rewritten as

$$1 < \prod_{n=1}^{\infty} \frac{\nu_n}{\lambda_n} = \exp \sum_{n=1}^{\infty} |\ln \nu_n - \ln \lambda_n| < +\infty.$$

For more general results about similar products of eigenvalue ratios, see [18].

Let the assumptions of Theorem 3.3 be fulfilled. Define F by the relation (3.2). Denote by $\{\tilde{\lambda}_n(J)\}_{n=0}^{\infty}$ the elements of the collection $\{\lambda_n(J_1)\}_{n=0}^{\infty} \cup \{\lambda_n(J_2)\}_{n=0}^{\infty}$ numbered in ascending order. By Theorem 3.3, we have

$$F(\lambda) = -1 \iff \lambda \in \bigcup_{n=0}^{\infty} (\tilde{\lambda}_n(J), \lambda_n(J)].$$

We recall that $\tilde{\lambda}_0(J) = \lambda_0(J) = \tilde{\lambda}_1(J) = 0$, but the rest of $\tilde{\lambda}_n(J)$ and $\lambda_n(J)$ are greater than zero, and we will now prove that the set $\{\ln \lambda : F(\lambda) = -1\}$ has finite measure, i.e.,

$$\left| \bigcup_{n=2}^{\infty} (\ln \tilde{\lambda}_n(J), \ln \lambda_n(J)] \right| < +\infty.$$

Theorem 4.4. *Let the assumptions of Theorem 3.3 be fulfilled and let $c_2 - d_1 > 0$. Then*

$$\sum_{n=2}^{\infty} |\ln \lambda_n(J) - \ln \tilde{\lambda}_n(J)| < +\infty.$$

Proof. Denote by $\nu_n^{(1)}$ the eigenvalues of the problem

$$\begin{cases} -y'' = \nu \mu y & \text{in } J_1, \\ y'(c_1) = y'(d_1) + \dfrac{2}{c_2 - d_1} \cdot y(d_1) = 0, \end{cases}$$

and by $\nu_n^{(2)}$ — the eigenvalues of the problem

$$\begin{cases} -y'' = \nu \mu y & \text{in } J_2, \\ y'(c_2) - \dfrac{2}{c_2 - d_1} \cdot y(c_2) = y'(d_2) = 0. \end{cases}$$

Let us fix an eigenfunction y_n corresponding to $\lambda_n(J)$ and consider its restrictions on segments J_1 and J_2. Since $\mu|_{[d_1,c_2]} \equiv 0$, the function $y_n|_{[d_1,c_2]}$ is linear, which means that

$$\frac{y_n(c_2)}{y_n'(c_2)} - \frac{y_n(d_1)}{y_n'(d_1)} = c_2 - d_1.$$

This implies

$$\frac{y_n(c_2)}{y_n'(c_2)} \geqslant \frac{c_2 - d_1}{2} \quad \text{or} \quad -\frac{y_n(d_1)}{y_n'(d_1)} \geqslant \frac{c_2 - d_1}{2},$$

which means that one of the following estimates holds:

$$0 \leqslant -\frac{y_n'(d_1)}{y_n(d_1)} \leqslant \frac{2}{c_2 - d_1}, \tag{4.1}$$

or

$$0 \leqslant \frac{y_n'(c_2)}{y_n(c_2)} \leqslant \frac{2}{c_2 - d_1}.$$

Note that $\lambda_n(J)$ is an eigenvalue of the problem

$$\begin{cases} -y'' = \lambda \mu y \quad \text{in } J_1, \\ y'(c_1) = y'(d_1) + \gamma \cdot y(d_1) = 0, \end{cases}$$

with $\gamma = -\frac{y_n'(d_1)}{y_n(d_1)}$. Its number (in the sequence of eigenvalues of this problem numbered in increasing order) is the same as the number of zeroes of y_n inside J_1. Note also that $\lambda_k(J_1)$ is the eigenvalue of the same problem with $\gamma = 0$ for any $k \in \mathbb{N}$, and $\nu_k^{(1)}$ is an eigenvalue of the same problem with $\gamma = \frac{2}{c_2 - d_1}$.

By the variational principle, this implies that, if (4.1) holds, then

$$\lambda_k(J_1) \leqslant \lambda_n(J) \leqslant \nu_k^{(1)},$$

where k is the number of zeroes of y_n inside J_1. Otherwise, a similar argument shows that

$$\lambda_k(J_2) \leqslant \lambda_n(J) \leqslant \nu_k^{(2)},$$

where k is the number of zeroes of y_n inside J_2. Further, by Theorem 3.3, the collections $\{\tilde{\lambda}_n(J)\}_{n=0}^{\infty}$ and $\{\lambda_n(J)\}_{n=0}^{\infty}$ non-strictly interlace starting with $\tilde{\lambda}_0(J)$. Therefore, we have $\tilde{\lambda}_n(J) \leqslant \lambda_n(J) \leqslant \tilde{\lambda}_{n+1}(J)$. Since all $\lambda_k(J_{1,2})$ belong to the set $\{\tilde{\lambda}_n(J)\}_{n=0}^{\infty}$, the relation $\lambda_k(J_{1,2}) \leqslant \lambda_n(J)$ implies $\lambda_k(J_{1,2}) \leqslant \tilde{\lambda}_n(J)$. Thus, we obtain, for every n, that, if (4.1) holds, then there exists k such that

$$\lambda_k(J_1) \leqslant \tilde{\lambda}_n(J) \leqslant \lambda_n(J) \leqslant \nu_k^{(1)};$$

otherwise, there exists k such that

$$\lambda_k(J_2) \leqslant \tilde{\lambda}_n(J) \leqslant \lambda_n(J) \leqslant \nu_k^{(2)}.$$

Thus, each of the non-intersecting intervals $(\tilde{\lambda}_n(J), \lambda_n(J)]$ is contained in the union

$$\Big(\bigcup_{k=0}^{\infty} [\lambda_k(J_1), \nu_k^{(1)}]\Big) \cup \Big(\bigcup_{k=0}^{\infty} [\lambda_k(J_2), \nu_k^{(2)}]\Big),$$

which implies

$$\bigcup_{n=0}^{\infty} (\tilde{\lambda}_n(J), \lambda_n(J)] \subset \Big(\bigcup_{k=0}^{\infty} [\lambda_k(J_1), \nu_k^{(1)}]\Big) \cup \Big(\bigcup_{k=0}^{\infty} [\lambda_k(J_2), \nu_k^{(2)}]\Big), \tag{4.2}$$

and if we discard the segments corresponding to $k = 0$ in the right part of (4.2), then we will only need to discard a finite number of intervals in the left part; in other words, there exists a number n_0 such that

$$\bigcup_{n=n_0}^{\infty} (\tilde{\lambda}_n(J), \lambda_n(J)] \subset \Big(\bigcup_{k=1}^{\infty} [\lambda_k(J_1), \nu_k^{(1)}] \Big) \cup \Big(\bigcup_{k=1}^{\infty} [\lambda_k(J_2), \nu_k^{(2)}] \Big),$$

and thus

$$\begin{aligned} &\sum_{n=n_0}^{\infty} |\ln \lambda_n(J) - \ln \tilde{\lambda}_n(J)| \\ &\leqslant \sum_{k=1}^{\infty} |\ln \nu_k^{(1)} - \ln \lambda_k(J_1)| + \sum_{k=1}^{\infty} |\ln \nu_k^{(2)} - \ln \lambda_k(J_2)|. \end{aligned} \tag{4.3}$$

From (4.3), using Proposition 4.2 we obtain the estimate

$$\begin{aligned} \sum_{n=2}^{\infty} |\ln \lambda_n(J) - \ln \tilde{\lambda}_n(J)| \leqslant &\sum_{n=2}^{n_0-1} |\ln \lambda_n(J) - \ln \tilde{\lambda}_n(J)| \\ &+ \sum_{k=1}^{\infty} |\ln \nu_k^{(1)} - \ln \lambda_k(J_1)| \\ &+ \sum_{k=1}^{\infty} |\ln \nu_k^{(2)} - \ln \lambda_k(J_2)| < +\infty. \end{aligned}$$

∎

Proof of Theorem 2.4. By (1.2), we have

$$N(\lambda) = \lambda^D (s(\ln \lambda) + \varepsilon(\lambda)),$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$. For arbitrary $k \in \mathbb{N}$, we obtain

$$N(\tau^{-k}\lambda) = \tau^{-kD} \lambda^D (s(\ln \lambda) + \varepsilon(\tau^{-k}\lambda)),$$

whence

$$s(\ln \lambda)\lambda^D = \lim_{k\to\infty} \tau^{kD} N(\tau^{-k}\lambda)$$

uniformly on any segment. Denote

$$\sigma(t) := s(t)e^{Dt} = \lim_{k\to\infty} \tau^{kD} N(\tau^{-k}e^t), \quad t \in [0, T].$$

We need to define a more suitable sequence of step functions approximating σ.

The case of $m = 2$. Let us first consider for clarity the case of $m = 2$. Without loss of generality, consider $k_1 \leqslant k_2$, where the $k_{1,2}$ are introduced in Definition 2.3. Denote

$$f_j(t) := C\tau^{jD} \sum_{i=0}^{k_2-1} C_i N(\tau^{-i-j} e^t), \quad t \in [0, T].$$

We wish to choose the coefficients C_i in such a way that

$$f_{j+1}(t) - f_j(t) = C\tau^{jD}\big(N(\lambda) - N(\tau^{k_1}\lambda) - N(\tau^{k_2}\lambda)\big), \tag{4.4}$$

where $\lambda = \tau^{-k_2-j} e^t$. By direct calculations, using (2.3), it is easy to demonstrate that the wanted coefficient values are

$$C_i = \begin{cases} \tau^{-(k_2-i)D} & k_2 - i = 1, \ldots, k_1, \\ \tau^{-(k_2-i)D} \cdot (1 - \tau^{k_1 D}) & k_2 - i = k_1 + 1, \ldots, k_2. \end{cases}$$

The second line is redundant when $k_1 = k_2$, since it corresponds to no indexes i. Note that

$$\lim_{j\to+\infty} f_j(t) = C \sum_{i=0}^{k_2-1} C_i \tau^{-iD} \lim_{j\to+\infty} \tau^{(i+j)D} N(\tau^{-i-j} e^t) = \sigma(t) \cdot C \sum_{i=0}^{k_2-1} C_i \tau^{-iD},$$

so, if we assign

$$C := \Big(\sum_{i=0}^{k_2-1} C_i \tau^{-iD} \Big)^{-1},$$

then we have

$$\sigma(t) = \lim_{j\to+\infty} f_j(t).$$

It is clear that the f_j are non-decreasing by definition, therefore σ is non-decreasing as their limit. Note, also, that by Lemma 3.2 we have

$$N(\lambda) - N(\tau^{k_1}\lambda) - N(\tau^{k_2}\lambda) = N(\lambda, [0, 1]) - N(\lambda, I_1) - N(\lambda, I_2); \tag{4.5}$$

thus, using Theorem 3.3 we obtain from (4.4) the following estimate:

$$|f_{j+1}(t) - f_j(t)| \leqslant C\tau^{jD}. \tag{4.6}$$

Further, since f_j is a sum of k_2 terms, each having no more than $N(\tau^{-k_2-j})$ discontinuity points, the number of discontinuities of f_j could be estimated as

$$\#\mathfrak{A}_j \leqslant k_2 N(\tau^{-k_2-j}) \leqslant \widetilde{C}\tau^{-jD}. \tag{4.7}$$

All that is left in order to use Proposition 4.1 is to prove the estimate

$$\operatorname{mes}\{t \in [0, T] \mid f_j(t) \neq f_{j+1}(t)\} = o(1), \quad j \to \infty. \tag{4.8}$$

Denote by $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ the elements of the collection $\{\lambda_n(I_1)\} \cup \{\lambda_n(I_2)\}$ numbered in ascending order. Since λ_n and $\tilde{\lambda}_n$ non-strictly interlace starting with $\tilde{\lambda}_0$, we have $\tilde{\lambda}_n \leqslant \lambda_n$ for all $n \geqslant 0$. Considering (4.4) and (4.5), in order to prove (4.8), we need to estimate the Lebesgue measure of the set of t for which

$$N(\lambda, [0, 1]) - N(\lambda, I_1) - N(\lambda, I_2) \neq 0,$$

where $\lambda = \tau^{-k_2-j} e^t$, $t \in [0, T]$. That is only true when

$$\lambda \in \bigcup_{n=0}^{\infty} [\tilde{\lambda}_n, \lambda_n],$$

and, considering $\lambda = \tau^{-k_2-j} e^t$ and $t \in [0, T]$, we obtain

$$(k_2 + j)T + t \in \Big(\bigcup_{n=0}^{\infty} [\ln \tilde{\lambda}_n, \ln \lambda_n] \Big) \cap [(k_2 + j)T, (k_2 + j + 1)T].$$

The measure of the union

$$\Big| \bigcup_{n=2}^{\infty} [\ln \tilde{\lambda}_n, \ln \lambda_n] \Big| = \sum_{n=2}^{\infty} |\ln \lambda_n - \ln \tilde{\lambda}_n|$$

is bounded by Theorem 4.4, which means that the measure of the intersection of this union with the segments $[(k_2 + j)T, (k_2 + j + 1)T]$, which move to infinity as $j \to +\infty$, tends to zero, which proves the estimate (4.8).

From (4.6) and (4.8) we obtain

$$\| f_{j+1} - f_j \|_{L_2[0,T]} = o(\tau^{jD}),$$

whence

$$\|\sigma - f_j\|_{L_2[0,T]} = o(\tau^{jD}).$$

Using this estimate and (4.7), we obtain

$$\#\mathfrak{A}_n \cdot \|\sigma - f_n\|_{L_2[0,T]} = o(1),$$

which allows us to use Proposition 4.1 for the function σ, concluding the proof of the theorem for this case.

General case. Let $\{\kappa_i\}_{i=1}^p$ be the elements of the set $\{k_i\}_{i=1}^m$ numbered in ascending order without duplication, and let $\{l_i\}_{i=1}^p$ be their multiplicities (the number of segments I_n corresponding to each value $k_n = \kappa_i$). Similarly to the case of $m = 2$, we define the functions and the constants

$$f_j(t) := C\tau^{jD} \sum_{i=0}^{\kappa_p - 1} C_i N(\tau^{-i-j} e^t), \quad t \in [0, T],$$

$$C_i = \begin{cases} \tau^{-(\kappa_p - i)D} & \kappa_p - i = 1, \dots, \kappa_1, \\ \tau^{-(\kappa_p - i)D} \cdot (1 - l_1 \tau^{\kappa_1 D}) & \kappa_p - i = \kappa_1 + 1, \dots, \kappa_2, \\ \tau^{-(\kappa_p - i)D} \cdot (1 - l_1 \tau^{\kappa_1 D} - l_2 \tau^{\kappa_2 D}) & \kappa_p - i = \kappa_2 + 1, \dots, \kappa_3, \\ \vdots & \\ \tau^{-(\kappa_p - i)D} \cdot \Big(1 - \sum_{j=1}^{p-1} l_j \tau^{\kappa_j D}\Big) & \kappa_p - i = \kappa_{p-1} + 1, \dots, \kappa_p, \end{cases}$$

$$C = \Big(\sum_{i=0}^{\kappa_p - 1} C_i \tau^{-iD}\Big)^{-1}.$$

From (2.3), we have

$$\sum_{i=1}^{p} l_i \tau^{\kappa_i D} = 1,$$

whence

$$1 - \sum_{i=1}^{r} l_i \tau^{\kappa_i D} > 0, \quad r = 1, \dots, p - 1.$$

This means that the coefficients C_i are positive, and C is well defined. Also, similarly to the previous case, such a choice of coefficients C_i gives us the relation

$$f_{j+1}(t) - f_j(t) = C\tau^{jD}\Big(N(\lambda) - \sum_{i=1}^{p} l_i N(\tau^{\kappa_i}\lambda)\Big),$$

where $\lambda = \tau^{-\kappa_p - j} e^t$, and the choice of C gives us the relation

$$\sigma(t) = \lim_{j \to +\infty} f_j(t).$$

It is clear that the f_j are non-decreasing by definition, therefore σ is non-decreasing as their limit. Using Lemma 3.2, we obtain

$$\begin{aligned} f_{j+1}(t) - f_j(t) &= C\tau^{jD}\Big(N(\lambda, [0, 1]) - \sum_{i=1}^{m} N(\lambda, I_i)\Big) \\ &= C\tau^{jD} \sum_{i=1}^{m-1} \big(N(\lambda, [a_i, 1]) - N(\lambda, I_i) - N(\lambda, [a_{i+1}, 1])\big). \end{aligned}$$

Considering Theorem 3.3, it is easy to see that

$$|f_{j+1}(t) - f_j(t)| \leqslant C(m-1)\tau^{jD},$$

and the number of discontinuities of f_j could be estimated as

$$\#\mathfrak{A}_j \leqslant \kappa_p N(\tau^{-\kappa_p - j}) \leqslant \widetilde{C}\tau^{-jD}.$$

All we need to use Proposition 4.1 is to prove the estimate

$$\mathrm{mes}\{t \in [0,T] \mid f_j(t) \neq f_{j+1}(t)\} = o(1), \quad j \to \infty,$$

which follows directly from Theorem 4.4, same as in the case of $m = 2$. ■

5. Small ball deviations of Gaussian processes

This section is dedicated to the problem of L_2-small ball deviations of Gaussian processes. Problems of this type are studied intensively in the last decades, largely due to the spectral approach (see the recent review [16]). Consider a Gaussian process $X(t)$, $t \in [0,1]$, with zero mean, and denote by $G_X(t,s) = EX(t)X(s)$, $t,s \in [0,1]$, its covariance function. Let μ be a measure on $[0,1]$. Denote

$$\|X\|_\mu = \left(\int_0^1 X^2(t)\,\mu(dt)\right)^{1/2}.$$

We call an asymptotics of $\ln \mathbf{P}\{\|X\|_\mu \leqslant \varepsilon\}$ as $\varepsilon \to 0$ a *logarithmic L_2-small ball asymptotics*. It turns out that, under some mild conditions, this asymptotics is determined by the leading term of the spectral asymptotics of the integral operator

$$\Lambda y(t) = \int_0^1 G_X(s,t)y(s)\,\mu(ds), \quad t \in [0,1]. \tag{5.1}$$

If G_X is the Green function for a boundary value problem for the Sturm–Liouville equation,[4] then the eigenvalues Λ_n of (5.1) and the eigenvalues λ_n of the corresponding problem are the inverse values of each other, $\Lambda_n^{-1} = \lambda_n$. Therefore, the asymptotics (1.2) considered in Remark 1.1 gives us the asymptotics

$$\Lambda_n = \frac{\varphi(\ln n)}{n^{1/D}}(1 + o(1)), \quad n \to +\infty, \tag{5.2}$$

[4] This class of Gaussian processes includes many classical processes, e.g., the Wiener process, the Brownian bridge, and the Ornstein–Uhlenbeck process.

where the TD-periodic function φ satisfies

$$\varphi(x) = \Big(s\Big(\frac{x}{D} + \ln(\varphi(x))\Big)\Big)^{1/D}. \tag{5.3}$$

The transformation of the asymptotics (5.2) into the L_2-small ball asymptotics was derived in [17].

Proposition 5.1 ([17, Theorem 4.2]). *Let $X(t)$ be a Gaussian process. If the eigenvalues of* (5.1) *satisfy* (5.2)*, then*

$$\ln \mathbf{P}\{\|X\|_\mu \leqslant \varepsilon\} \sim -\varepsilon^{-\frac{2D}{1-D}} \zeta\Big(\ln\Big(\frac{1}{\varepsilon}\Big)\Big), \tag{5.4}$$

where ζ is a $\frac{T(1-D)}{2}$-periodic function bounded and separated from zero.

The function ζ is related to the function φ – and therefore to the function s – in a very indirect and convoluted way. Some of the steps in this connection involve integral transformations of periodic functions, which, in theory, could have led to the loss of periodicity. In this section, we follow and analyze each step of the proof of [17, Theorem 4.2] to show that, when s is not constant, ζ cannot be constant as well.

Theorem 5.2. *For a fixed self-similar measure μ, the component ζ of the small ball asymptotics* (5.4) *degenerates into constant if and only if the component s of the spectral asymptotics* (1.2) *degenerates into constant.*

Proof. It follows from the proof of [17, Theorem 4.2] that

$$\ln \mathbf{P}\{\|X\|^2_\mu \leqslant r\} \sim -(u(r))^D \cdot \eta(\ln(u(r))), \quad r \to 0, \tag{5.5}$$

where $u(r)$ is an arbitrary function satisfying

$$r \sim u^{D-1}\theta(\ln(u)), \quad u \to \infty, \tag{5.6}$$

the periodic functions η and θ in (5.5) and (5.6) are defined by the formulae

$$\eta(\ln(u)) = \int_0^\infty F\Big(\frac{\varphi(D\ln u + \ln z)}{z^{1/D}}\Big)\,dz, \tag{5.7}$$

$$\theta(\ln(u)) = \int_0^\infty F_1\Big(\frac{\varphi(D\ln u + \ln z)}{z^{1/D}}\Big)\,dz, \tag{5.8}$$

where

$$F(x) = \frac{1}{2}\ln(1+2x) - \frac{x}{1+2x}, \quad F_1(x) = \frac{x}{1+2x}.$$

and φ is a periodic function defined in (5.3).

To reiterate, we now have periodic functions ζ, η, θ, φ, and s interconnected via various relations, and we aim to show the equivalence between their conditions for degenerating into constant. Let us break it down into simpler steps:

$$\zeta = \text{const} \overset{\text{Step 1}}{\Longleftrightarrow} \eta\theta^{\frac{D}{1-D}} = \text{const} \overset{\text{Step 2}}{\Longleftrightarrow} \eta = \text{const and } \theta = \text{const}$$

$$\overset{\text{Steps 3,4}}{\Longleftrightarrow} \varphi = \text{const} \overset{\text{Step 5}}{\Longleftrightarrow} s = \text{const}.$$

Step 1. *The function ζ is constant if and only if the function $\eta\theta^{\frac{D}{1-D}}$ is constant.* If we substitute u from (5.6) into (5.5), then we obtain

$$\ln \mathbf{P}\{\|X\|_\mu^2 \leqslant r\} \sim -r^{-\frac{D}{1-D}} \cdot \eta(\ln(u(r)))\theta^{\frac{D}{1-D}}(\ln(u(r))), \quad r \to 0.$$

Thus, the function ζ in (5.4) has the asymptotics

$$\zeta(\ln(1/\varepsilon)) \sim \eta(\ln(u(\varepsilon^2)))\theta^{\frac{D}{1-D}}(\ln(u(\varepsilon^2))), \quad \varepsilon \to 0.$$

Since $u(r)$ could be arbitrary within the given asymptotics (5.6), we assume, without loss of generality, that it is continuous. Therefore, its image contains all values in some vicinity of infinity. We conclude that the function ζ is constant if and only if the function $\eta\theta^{\frac{D}{1-D}}$ is constant.

Step 2. *The function $\eta\theta^{\frac{D}{1-D}}$ is constant if and only if both θ and η are constant.* It is clear that, if θ and η are constant, then the product $\eta\theta^{\frac{D}{1-D}}$ is constant as well. To prove the other implication, we change the variable $x = u^D z$ in (5.7) and (5.8) to obtain

$$\eta(\ln u)u^D = \int_0^\infty F(u\psi(x))\,dx, \quad \theta(\ln u)u^D = \int_0^\infty F_1(u\psi(x))\,dx, \tag{5.9}$$

where $\psi(x) = \frac{\varphi(\ln(x))}{x^{1/D}}$. Differentiating these formulae and taking into account the relation

$$(F(x) + F_1(x))' \cdot x = F_1(x),$$

we arrive at

$$(\eta(\ln u)u^D + \theta(\ln u)u^D)' \cdot u = \theta(\ln u)u^D,$$

thus

$$\eta'(x) + D\eta(x) + \theta'(x) - (1-D)\theta(x) = 0.$$

If $\eta\theta^{\frac{D}{1-D}} \equiv c$ here, then minimal and maximal values of θ on the period satisfy

$$cD\theta^{-\frac{D}{1-D}} - (1-D)\theta = 0;$$

therefore they coincide and the claim follows.

Step 3. *If the function φ is constant, then both functions η and θ are constant.* This follows trivially from their definitions (5.7) and (5.8).

Step 4. *If the function θ is constant, then the function φ is constant.* This statement is proved in Lemma 5.3 below.

Step 5. *The function φ is constant if and only if the function s is constant.* The function φ is defined by the relation (5.3). Therefore, φ and s are constant simultaneously. ■

Lemma 5.3. *Let θ be a T-periodic function defined by* (5.8) *with a TD-periodic function φ. If θ is constant, then function φ is also constant.*

Proof. Let $\theta \equiv c$ be constant. From (5.9), we obtain

$$cu^D = \int_0^\infty F_1(u\psi(x))\,dx. \tag{5.10}$$

We note the following properties of the function $\psi(x) = \varphi(\ln(x))/x^{1/D}$.

(1) From the periodicity of φ, we obtain

$$\psi(xe^{-DT}) = \psi(x)e^T. \tag{5.11}$$

(2) We claim that ψ is a monotone decreasing function. Indeed, if it is not, then there exist $x_1 > x_2$, such that $\psi(x_1) > \psi(x_2)$. However, for any $k \in \mathbb{N}$ formulae (5.2) and (5.11) imply

$$e^{-kT}(\psi(x_1) - \psi(x_2)) = \psi(x_1e^{kDT}) - \psi(x_2e^{kDT}) = \lambda_{n_1} - \lambda_{n_2} + o(e^{-kT}),$$

where $n_{1,2} = \lfloor x_{1,2}e^{kDT}\rfloor$ are the integer parts. Thus, for sufficiently large k, it would be possible to find two eigenvalues $\lambda_{n_1} > \lambda_{n_2}$, $n_1 > n_2$, which is impossible.

On the other hand, a direct computation gives

$$\int_0^\infty F_1(u\phi(x))\,dx = cu^D, \tag{5.12}$$

where

$$\phi(x) = c_1/x^{1/D}, \qquad c_1 = \left(\frac{1}{c}\int_0^\infty F_1(x^{-1/D})\,dx\right)^{-1/D}.$$

We change variables in the integral identities (5.10) and (5.12) to $w = -\ln(u\psi(x))$ and $w = -\ln(u\phi(x))$ respectively, subtract the resulting relations and arrive at

$$\int_{-\infty}^{+\infty} F_1(e^{-w})\Big(\frac{1}{\psi'(\psi^{-1}(e^{v-w}))} - \frac{1}{\phi'(\phi^{-1}(e^{v-w}))}\Big)e^{v-w}\,dw = 0 \tag{5.13}$$

for all $v \in \mathbb{R}$, where $v = -\ln u$.

The integral in (5.13) has the structure of a convolution. Namely, we multiply (5.13) by e^{Dv} and rewrite it as

$$(g_1 * g_2)(v) = 0 \quad \text{for all } v \in \mathbb{R}, \tag{5.14}$$

where

$$g_1(v) = F_1(e^{-v})e^{Dv}, \qquad g_2(v) = \Big(\frac{1}{\psi'(\psi^{-1}(e^{v}))} - \frac{1}{\phi'(\phi^{-1}(e^{v}))}\Big)e^{(1+D)v}.$$

We aim to demonstrate that $g_2 \equiv 0$ using the Fourier transform argument.

Step 1. We show that g_1 and g_2 have valid Fourier transforms. By the definition of F_1, it is easy to see that

$$g_1(v) \sim e^{(D-1)v}, \quad v \to +\infty, \qquad g_1(v) \sim \frac{1}{2}e^{Dv}, \quad v \to -\infty,$$

thus $g_1 \in L_1(\mathbb{R})$, and its Fourier transform is a continuous function on $\mathbb{R}$.

Further, we show that g_2 is a T-periodic function. Note that, by the definition of ϕ, we have

$$\phi'(\phi^{-1}(1/u)) = -\frac{u^{-D-1}}{Dc_1^D}. \tag{5.15}$$

Therefore, the second term of g_2 is a constant:

$$-\frac{e^{(1+D)v}}{\phi'(\phi^{-1}(e^{v}))} = Dc_1^D.$$

For the first term of g_2, we obtain, for all $a \in \mathbb{R}$

$$\int_a^{a+T} \frac{e^{(1+D)v}}{\psi'(\psi^{-1}(e^{v}))}\,dv \overset{(a)}{=} \int_{\psi^{-1}(e^a)}^{\psi^{-1}(e^a e^T)} (\psi(z))^D dz \overset{(b)}{=} \int_{\psi^{-1}(e^a)}^{\psi^{-1}(e^a)e^{-TD}} \frac{(\varphi(\ln z))^D}{z} dz$$
$$\overset{(c)}{=} \int_{\ln\psi^{-1}(e^a)}^{\ln\psi^{-1}(e^a)-TD} \varphi(\xi)d\xi = \text{const}.$$

Equality (a) is obtained by the change of variables $z = \psi^{-1}(e^v)$. Equality (b) uses the property (5.11) for the upper limit and the definition of ψ for the integrand. Finally, in (c) we change the variable to $\xi = \ln z$ and obtain an integral of a periodic function φ over its period TD. Thus, g_2 is a T-periodic function and its Fourier transform (in the sense of distributions) is a linear combination of delta-functions at points $\frac{kT}{2\pi}$ for $k \in \mathbb{Z}$.

Step 2. We show that the Fourier transform of g_1 is never zero:

$$\hat{g}_1(\omega) = \int_{-\infty}^{+\infty} F_1(e^{-v}) e^{Dv - iv\omega} dv \neq 0 \quad \text{for all } \omega \in \mathbb{R}.$$

For $\omega > 0$ we calculate this integral using the residue theorem for a semicircular contour in the lower imaginary half-plane. The ingularities of the function $F_1(e^{-v})$ are simple poles at points $\xi_k = \ln 2 + i\pi(2k-1)$ for all $k \in \mathbb{Z}$. We calculate the residues using de L'Hôpital's rule

$$\operatorname{Res}\Big(\frac{f}{h}, a\Big) = \frac{f(a)}{h'(a)},$$

if $f(a) \neq 0$, $h(a) = 0$, $h'(a) \neq 0$. In our case,

$$\operatorname{Res}(F_1(e^{-v})e^{v(D-i\omega)}, \xi_k) = -\frac{1}{2} e^{\xi_k(D-i\omega)} = -\frac{1}{2} e^{(D-i\omega)(\ln 2 - i\pi)} (e^{2\pi i(D-i\omega)})^k.$$

Summing all residues in the lower half-plane we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} F_1(e^{-v}) e^{v(D-1-i\omega)} dv &= 2\pi i \sum_{k=0}^{\infty} \frac{1}{2} e^{(D-i\omega)(\ln 2 - i\pi)} (e^{2\pi i(D-i\omega)})^{-k} \\ &= \frac{\pi i e^{(D-i\omega)(\ln 2 - i\pi)}}{1 - e^{-2\pi i(D-i\omega)}} \neq 0. \end{aligned}$$

For $\omega < 0$

$$\hat{g}_1(\omega) = \overline{\hat{g}_1(-\omega)} = \frac{-\pi i e^{(D-i\omega)(\ln 2 + i\pi)}}{1 - e^{2\pi i(D-i\omega)}} \neq 0.$$

Finally,

$$\hat{g}_1(0) = \frac{-\pi i e^{D(\ln 2 + i\pi)}}{1 - e^{2\pi i D}} = \frac{\pi 2^{D-1}}{\sin(\pi D)} \neq 0.$$

Step 3. We are now ready to apply the Fourier transform to the relation (5.14):

$$\hat{g}_1(\omega) \cdot \hat{g}_2(\omega) = 0 \quad \text{for all } \omega \in \mathbb{R}.$$

Since $\hat{g}_1(\omega) \neq 0$, we conclude that $\hat{g}_2 \equiv 0$ and thus

$$u^{-1-D} g_2(\ln u) = \frac{1}{\psi'(\psi^{-1}(u))} - \frac{1}{\phi'(\phi^{-1}(u))} = 0 \quad \text{for all } u > 0.$$

By (5.15), it immediately follows that

$$\psi'(x) = -\frac{\psi^{D+1}(x)}{D c_1^D} \quad \text{for all } x > 0.$$

This differential equation has a family of solutions

$$\psi(x) = \frac{c_1}{(x + c_2)^{1/D}},$$

and since $\psi(x) x^{1/D} = \varphi(\ln(x))$ is a periodic function of logarithm, we obtain $\varphi \equiv c_1$, and the lemma is proved. ■

Remark 5.4. It is evident from (5.9) that both η and θ are smooth functions. Thus, the function ζ in (5.4) is smooth even though the function s in (1.2) is not.

Acknowledgments. The author thanks A. I. Nazarov for the statement of the problem and constant encouragement, D. D. Cherkashin for valuable remarks, and A. A. Vladimirov for the remark that allowed him to greatly simplify the proof of Theorem 3.3.

Funding. This work was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no.075-15-2022-289 date 06/04/2022).

References

[1] M. Š. Birman and V. V. Borzov, The asymptotic behavior of the discrete spectrum of certain singular differential operators. In *Problems of mathematical physics, No. 5: Spectral theory*, pp. 24–38, Izdatel'stvo Leningradskogo Universiteta, Leningrad, 1971; English transl., *Spectral Theory,* pp. 19–30. Topics in Mathematical Physics 5, Springer, Boston, 1972 Zbl 0299.35073 MR 0301568

[2] M. Š. Birman and M. Z. Solomjak, Asymptotics of the spectrum of weakly polar integral operators. *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 1142–1158; English transl., *Math. USSR, Izv.* **4** (1970), no. 5, 1151–1168 Zbl 0261.47027 MR 0279638

[3] M. Š. Birman and M. Z. Solomjak, Spectral asymptotics of nonsmooth elliptic operators. I. *Trudy Moskov. Mat. Obšč.* **27** (1972), 3–52; English transl., *Trans. Mosc. Math. Soc.* **27** (1972), 1–52 (1975) Zbl 0251.35075 Zbl 0296.35065 (transl.) MR 0364898

[4] M. Š. Birman and M. Z. Solomjak, Spectral asymptotics of nonsmooth elliptic operators. II. *Trudy Moskov. Mat. Obšč.* **28** (1973), 3–34; English transl., *Trans. Mosc. Math. Soc.* **28** (1973), 1–32 (1975) Zbl 0281.35079 Zbl 0296.35066 (transl.) MR 0364898

[5] M. Š. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*. 2nd edn. Lan' publishers, St-Peterbg, 2010

[6] V. V. Borzov, The quantitative characteristics of singular measures. In *Problems of Mathematical Physics, No. 4: Spectral Theory. Wave Processes*, pp. 42–47. Izdatel'stvo Leningradskogo Universiteta, Leningrad, 1970; English tranls. in *Spectral Theory and Wave Processes,* pp. 37–42. Topics in Mathematical Physics 4, Springer, Boston, 1971 MR 0281860

[7] U. R. Freiberg and N. V. Rastegaev, On spectral asymptotics of the Sturm–Liouville problem with self-conformal singular weight with strong bounded distortion property. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **477** (2018), 129–135; English transl., *J. Math. Sci., New York*, **244** (2020), 1010–1014 Zbl 1453.34036 MR 3904060

[8] T. Fujita, A fractional dimension, self-similarity and a generalized diffusion operator. In *Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)*, pp. 83–90, Academic Press, Boston, MA, 1987 Zbl 0652.60084 MR 0933819

[9] J. E. Hutchinson, Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), no. 5, 713–747 Zbl 0598.28011 MR 0625600

[10] I. S. Kac and M. G. Kreĭn, Criteria for the discreteness of the spectrum of a singular string. *Izv. Vysš. Učebn. Zaved. Matematika* **1958** (1958), no. 2(3), 136–153 Zbl 1469.34111 MR 0139804

[11] M. Kesseböhmer and A. Niemann, Spectral asymptotics of Kreĭn–Feller operators for weak Gibbs measures on self-conformal fractals with overlaps. *Adv. Math.* **403** (2022), article no. 108384 Zbl 1490.35240 MR 4406930

[12] J. Kigami and M. L. Lapidus, Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals. *Comm. Math. Phys.* **158** (1993), no. 1, 93–125 Zbl 0806.35130 MR 1243717

[13] M. G. Kreĭn, Determination of the density of a nonhomogeneous symmetric cord by its frequency spectrum. *Doklady Akad. Nauk SSSR (N.S.)* **76** (1951), 345–348 Zbl 0042.09502 MR 0042045

[14] A. S. Markus and V. I. Matsaev, Comparison theorems for spectra of linear operators and spectral asymptotics. *Trudy Moskov. Mat. Obshch.* **45** (1982), 133–181; English transl., *Trans. Mosc. Math. Soc.* (1984), no. 1, 139-187 Zbl 0532.47012 Zbl 0557.47009 (transl.) MR 0704630

[15] H. P. McKean, Jr. and D. B. Ray, Spectral distribution of a differential operator. *Duke Math. J.* **29** (1962), 281–292 Zbl 0114.04902 MR 0146444

[16] A. Nazarov and Y. Petrova, L_2-small ball asymptotics for Gaussian random functions: a survey. *Probab. Surv.* **20** (2023), 608–663 Zbl 1517.60041 MR 4605340

[17] A. I. Nazarov, Logarithmic asymptotics of small deviations for some Gaussian processes in the L_2-norm with respect to a self-similar measure. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **311** (2004), 190–213, 301; English transl., *J. Math. Sci., New York* **133** (2006), no. 3, 1314–1327 Zbl 1076.60030 MR 2092208

[18] A. I. Nazarov, On a family of transformations of Gaussian random functions. *Teor. Veroyatn. Primen.* **54** (2009), no. 2, 209–225; English transl. *Theory Probab. Appl.* **54** (2010), no. 2, 203–216 Zbl 1214.60011 MR 2761552

[19] A. I. Nazarov and I. A. Sheĭpak, Degenerate self-similar measures, spectral asymptotics and small deviations of Gaussian processes. *Bull. Lond. Math. Soc.* **44** (2012), no. 1, 12–24 Zbl 1244.60040 MR 2881320

[20] S.-M. Ngai and Y. Xie, Spectral asymptotics of Laplacians related to one-dimensional graph-directed self-similar measures with overlaps. *Ark. Mat.* **58** (2020), no. 2, 393–435 Zbl 1452.35125 MR 4176086

[21] N. V. Rastegaev, On spectral asymptotics of the Neumann problem for the Sturm–Liouville equation with self-similar weight of generalized Cantor type. *Zap. Nauchn. Semin. POMI* **425** (2014), 86–98; English transl., *J. Math. Sci. (N.Y.)* **210** (2015), no. 6, 814–821 Zbl 1334.34186 MR 3407794

[22] G. Rozenblum and G. Tashchiyan, Eigenvalues of the Birman–Schwinger operator for singular measures: the noncritical case. *J. Funct. Anal.* **283** (2022), no. 12, article no. 109704 Zbl 07605362 MR 4484835

[23] I. A. Sheĭpak, On the construction and some properties of self-similar functions in the spaces $L_p[0, 1]$. *Mat. Zametki* **81** (2007), no. 6, 924–938; English transl., *Math. Notes* **81** (2007), 827–839 Zbl 1132.28006 MR 2349108

[24] M. Solomyak and E. Verbitsky, On a spectral problem related to self-similar measures. *Bull. London Math. Soc.* **27** (1995), no. 3, 242–248 Zbl 0823.34071 MR 1328700

[25] H. Triebel, *Fractals and spectra*. Mod. Birkhauser Class., Birkhäuser, Basel, 2011 MR 2732650

[26] T. Uno and I. Hong, Some consideration of asymptotic distribution of eigenvalues for the equation $d^2u/dx^2 + \lambda\rho(x)u = 0$. *Jpn. J. Math.* **29** (1959), 152–164 Zbl 0098.06002 MR 0118891

[27] A. A. Vladimirov, On the oscillation theory of the Sturm–Liouville problem with singular coefficients. *Zh. Vychisl. Mat. Mat. Fiz.* **49** (2009), no. 9, 1609–1621; English transl., *Comput. Math. Math. Phys.* **49** (2009), 1535–1546 Zbl 1199.34136 MR 2603297

[28] A. A. Vladimirov, An oscillation method in a problem on the spectrum of a fourth-order differential operator with self-similar weight. *Algebra i Analiz* **27** (2015), no. 2, 83–95; English transl., *St. Petersbg. Math. J.* **27** (2016), no. 2, 237–244 Zbl 1343.34193 MR 3444462

[29] A. A. Vladimirov and I. A. Sheĭpak, Self-similar functions in the space $L_2[0, 1]$ and the Sturm–Liouville problem with a singular indefinite weight. *Mat. Sb.* **197** (2006), no. 11, 13–30; English transl., *Sb. Math.* **197** (2006), no. 11, 1569–1586 Zbl 1177.34039 MR 2437086

[30] A. A. Vladimirov and I. A. Sheĭpak, Asymptotics of the eigenvalues of the Sturm–Liouville problem with discrete self-similar weight. *Mat. Zametki* **88** (2010), no. 5, 662–672; English transl., *Math. Notes* **88** (2010), no. 5, 637–646 Zbl 1235.34228 MR 2868390

[31] A. A. Vladimirov and I. A. Sheĭpak, On the Neumann problem for the Sturm–Liouville equation with Cantor-type self-similar weight. *Funktsional. Anal. i Prilozhen.* **47** (2013), no. 4, 18–29; English transl., *Funct. Anal. Appl.* **47** (2013), no. 4, 261–270 Zbl 1310.34038 MR 3185121

Received 28 May 2023; revised 12 May 2024.

Nikita Rastegaev
St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, 27 Fontanka, 191023 St Petersburg; Leonhard Euler International Mathematical Institute, St. Petersburg State University, 7/9 Universitetskaya nab., 199034 St. Petersburg, Russia; rastmusician@gmail.com