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2-nodal domain theorems for higher-dimensional circle bundles

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Abstract. We prove that the real parts of equivariant (but non-invariant) eigenfunctions of generic bundle metrics on non-trivial principal S^1 bundles over manifolds of any dimension have connected nodal sets and exactly 2 nodal domains. This generalizes earlier results of the authors in the 3-dimensional case. The failure of the results on for non-free S^1 actions is illustrated on even-dimensional spheres by one-parameter subgroups of rotations whose fixed point set consists of two antipodal points.

1. Introduction

In a recent article [4], the authors proved a novel type of connectivity theorem for nodal sets of eigenfunctions of certain Laplace operators on Riemannian 3-manifolds (X, G) which are non-trivial principal circle bundles $\pi: X \to M$ over (real) surfaces M. The metrics G are known as *bundle metrics, Kaluza–Klein metrics,* or *connection metrics.* (They are also called *Sasakian metrics;* see [2] for an early study of them.) They are induced by a choice of Riemannian metric g on M and a connection 1-form α on the non-trivial principal circle bundle $\pi: X \to M$; see Lemma 2.1. We denote the free circle S^1 action by r_θ and its infinitesimal generator by $Z = \frac{\partial}{\partial\theta}$. The connection determines an S^1 invariant splitting $TX = H \oplus V$ into horizontal and vertical sub-bundles, with $V = \mathbb{R}Z$, and G is defined so that $H \perp V$, so that $G|_H = \pi^* g$ and so that $G|_{\pi^{-1}(z)}$ is the standard metric on $S^1 = \mathbb{R}/\mathbb{Z}$ along the fibers. The horizontal Laplacian Δ_H commutes with the vertical Laplacian $\frac{\partial^2}{\partial\theta^2}$, so that there exists an orthonormal basis $\{\varphi_{m,j}\}$ of joint eigenfunctions of Δ_G and $\frac{\partial^2}{\partial\theta^2}$. We refer to such joint eigenfunctions as *equivariant eigenfunctions* since they transform by $\varphi_{m,j}(r_\theta x) = e^{im\theta}\varphi_{m,j}(x)$ under the S^1 action $r_\theta: S^1 \times X \to X$. They are called *invariant* when $m = 0$. It is proved in [4] that the nodal sets of the real parts of non-invariant equivariant eigenfunctions are connected for $m \neq 0$ and that they divide S^3 into exactly two nodal domains if 0 is a regular value of the eigenfunction;

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moreover, it is a generic property of the bundle metrics that all eigenfunctions are equivariant and that 0 is a regular value of them all. Until this result, sequences of eigenfunctions with a small number of nodal domains seemed to be rare, and it took ingenious arguments such as those of Lewy [6] and Stern [7] to construct them; but the result of [4] shows that the full orthonormal basis of eigenfunctions orthogonal to invariant functions has this property for generic bundle metrics.

1.1. Higher-dimensional circle bundles

The purpose of this addendum is to generalize the result to bundle Laplacians on non-trivial principal circle bundles over manifolds of any dimension. The generalization requires two main changes in the argument. First, our generic bundle metrics are of two kinds (i) general base metric perturbations, and (ii) perturbations of the connection 1-form as in [4]. The perturbations (i) bring in new types of equations. Secondly, the key last step that the single nodal component divides X into precisely two nodal domains requires a very different argument from [4]. In addition, we illustrate what goes wrong with the proofs if the S^1 action is not free by illustrating the failure of the result on even-dimensional spheres.

Bundle metrics on principal circle bundles can be defined in both a “top–down” and a “bottom–up” way. The top–down approach is to start with a principal S^1 bundle $\pi\colon X \to M$ and to construct a bundle metric G on it from a metric g on M and a connection 1-form α on the bundle (see Section 2). The S^1 action is generated by a Killing vector field $Z = \frac{\partial}{\partial\theta}$, which commutes with any bundle Laplacian Δ_G. The norm $|Z|_G$ is called the *lapse function;* it is constant along fibers but may vary with the fiber. For simplicity, when the S^1 action is free, we assume it is constant, in which case the fibers are geodesics. The bottom–up approach is to start with a suitable Riemannian manifold (M, g) and a complex line bundle $L \to M$, to endow L with a Hermitian metric h and to define X as the unit bundle $\partial D_h^* \subset L^*$ of the Hermitian metric in the dual line bundle. Given the circle bundle X and a character of S^1, one may define the associated complex line bundle L and Hermitian metric h so that $X = \partial D_h^*$. Although the bottom–up and top–down approaches are equivalent, it is natural to separate the two approaches since the lead to different perturbation formulae.

Definition 1.1. Let (X, G) be any Riemannian manifold with an S^1 action by isometries. Let $D_Z = \frac{1}{i}\frac{\partial}{\partial\theta}$ be the self-adjoint differential operator corresponding to the Killing vector field Z generating the S^1 action. By an *equivariant eigenfunction,* we mean a joint eigenfunction

$$\begin{cases} \Delta_G \varphi_{m,j} = -\lambda_{m,j}^2 \varphi_{m,j}, \\ D_Z \varphi_{m,j} = m \varphi_{m,j}. \end{cases}$$

Equivariant eigenfunctions are complex-valued and their real and imaginary parts are (non-equivariant) Δ_G-eigenfunctions. We are mainly interested in the nodal sets of the real part,

$$\mathcal{N}_{m,j} := \{\mathrm{Re}\, \varphi_{m,j} = 0\}. \tag{1.1}$$

It is equivalent to study the nodal set of the imaginary part.

Our main result pertains to nodal sets (1.1) for Laplacians of generic bundle metrics. To define "generic," we specify a class of metrics g on M (Section 4.1) and of connection 1-forms α (Section 4.1.1) in some Banach or Frechet space, and "generic" will refer to a residual subset of that space. Either we fix the base metric g and vary the connection, or we fix the connection and allow general variations of base metrics.

Theorem 1.2. *For generic data (g, α) of a bundle (Kaluza–Klein) metric G on a non-trivial principal circle bundle $\pi: X \to M$, the following facts hold.*

- 0 *is a regular value of all eigenfunctions of the bundle Laplacian Δ_G.*
- *All eigenvalues are simple. The eigenfunctions are joint eigenfunctions of $\frac{\partial^2}{\partial\theta^2}$ and of Δ_G.*
- *Except for pullback (invariant) eigenfunctions, the nodal set (1.1) is connected and there are exactly two nodal domains.*

The nodal domains of pullback eigenfunctions are easily seen to be the inverse image of the nodal domains in M, so the above theorem is best possible.

The proof of Theorem 1.2 splits up into two distinct parts. The first part is to prove that eigenfunctions of generic bundle metrics of the two basic types have the stated properties. The second part is to prove that zero sets (1.1) of real parts of equivariant eigenfunctions φ_m on X have the stated connectivity properties if 0 is a regular value.

Equivariant functions $\varphi_{m,j}$ on the circle bundle transforming by $e^{im\theta}$ correspond to sections of L^m. There is a canonical lift map $s \to \hat{s}$ taking sections of L^m to equivariant functions on X. Theorem 1.2 can therefore be restated in terms of real and imaginary parts of equivariant eigensections $s_{m,j}$ of L^m,

$$Z_{\mathbb{R}\hat{s}_{m,j}} := \{\mathrm{Re}\, \hat{s}_{m,j}(\zeta) = 0, \zeta \in \partial D_h^*\}. \tag{1.2}$$

See Section 3.1 for background. It is often easier to calculate with the associated eigensections, since connections on L^m are simpler to work with than horizontal derivatives.

1.2. Non-free S^1 actions on even-dimensional spheres

Odd-dimensional spheres fit into the framework of bundle metrics on circle bundles over Kähler manifolds (namely $\mathbb{S}^{2m-1} \to \mathbb{CP}^m$), although the metrics are very non-generic: eigenvalues have high multiplicities, 0 is not a regular value of many eigenfunctions (spherical harmonics) of a given eigenvalue (or, degree), and the there may exist many nodal domains in the singular case. On the other hand, as in [5], the real part of a random equivariant spherical harmonic on $\mathbb{S}^{2m-1}$ does have a connected nodal set with exactly two nodal domains.

If we drop the assumption that $X \to M$ is a principal circle bundle and allow the S^1 action to have fixed points, then most of Theorem 1.2 fails. The simplest example where the conclusion of Theorem 1.2 is false is that of the S^1 action by rotations around an axis for the standard metric on S^2. The equivariant eigenfunctions are the usual spherical harmonics Y_N^m. The nodal sets of their real parts are connected but they have mN nodal domains. Moreover, 0 is not a regular value and they have mN singular points. We show that much of this is true for similar S^1 actions on spheres of any even dimension. The following is proved in Section 7.

Lemma 1.3. *Suppose that $X = \mathbb{S}^{2n}$ is an even-dimensional sphere and S^1 is a 1-parameter subgroup of* $\mathrm{SO}(2n+1)$ *with exactly two fixed points. Then,*

- *for any S^1-invariant metric G, the nodal sets of the real parts of the joint eigenfunctions of the S^1 action and Δ_G are connected;*
- 0 *is never a regular value for real parts of the joint eigenfunctions of the S^1 action and Δ_G;*
- *the set of singular points (critical points on the nodal set) of real parts of the equivariant eigenfunctions is exactly of co-dimension* 2.

1.3. Comments

To avoid duplicating material in [4], we only give a detailed presentation of the new steps in the generalization and refer to [4] for much of the background and references. However, for the sake of readability, we do state some overlapping calculations of Laplacians and their perturbations of connection 1-forms from [4]. As mentioned above, the generic metric data in higher dimensions is quite different from that in dimension two, and the calculations are quite different.

We refer the readers to [3], published shortly after the release of our current article. While there are overlapping results, it is important to note that the two works were conducted independently, and the proof methods used are different.

2. Geometric background

In this section we introduce the geometric data that goes into the construction of bundle (i.e., connection or Kaluza–Klein) metrics on a circle bundle, and give formulae for the associated Bochner connection Laplacians $\nabla^*\nabla$ and Kaluza–Klein Laplacians Δ_G associated to different types of bundle Laplacians.

2.1. Classification of S^1 bundles and bundle metrics

In this section we consider the general top–down approach to constructing bundle metrics for principal S^1 bundles $\pi\colon X \to M$. Let M be a compact manifold. Then there is a 1–1 correspondence between equivalence classes of circle bundles over M and elements of $H^2(M,\mathbb{Z})$. Given an integral closed 2-form ω on M there is a circle bundle $\pi\colon X \to M$ with connection form α such that $\pi^*\omega = d\alpha$. The cohomology class $c_1 = [\omega] \in H^2(M,\mathbb{Z})$ of the associated complex line bundle is its Chern class.

A connection 1-form α is an S^1-invariant 1-form satisfying $\alpha(Z) = 1$. The difference $\alpha_1 - \alpha_2$ of two connections is an S^1 invariant 1-form which annihilates Z and is therefore horizontal. Hence, it descends to M as a real-valued one form. It follows that the space of connection 1-forms on a fixed circle bundle is parameterized by $\Omega^1(M,\mathbb{R})$.

The curvature form of the S^1 bundle is the S^1 horizontal form defined by $\omega = d\alpha$. It descends to M as a closed 2-form. When ω is a symplectic form, the connection is called *fat* [9] and then (M,ω) is a symplectic manifold; in this case X is odd-dimensional.

Let $TX = VX \oplus HX$, where HX is the horizontal space for α. A Riemannian metric g together with α determines the bundle metric as long as $Z = \frac{\partial}{\partial\theta}$ is a geodesic. More generally, we can allow the *lapse function* $N(Z) = G(Z,Z)$ to be non-constant. We do not pursue this generalization but summarize the statements in the general setting.

Lemma 2.1. *A bundle metric on a principal circle bundle $\pi\colon X \to M$ is defined by the data (N,α,g) with*

$$G(N,\alpha,g) := N^2\alpha \otimes \alpha + \pi^* g$$

where α is the Killing invariant connection 1-form with $\alpha(Z) = 1$. The S^1 orbits are geodesics if and only if $N \equiv C$ for some $C > 0$.

The volume form of the metric is given by

$$dV_G = \alpha \wedge \pi^* dV_g.$$

We often assume $N \equiv 1$ and then write $G = G(\alpha,g)$.

2.2. Associated line bundles and Hermitian metrics

We assume that X is a principal S^1 bundle $\pi\colon X \to M$ with action $r_\theta\colon X \to X$ for $e^{i\theta} \in S^1$. Given any character $\chi_k(e^{i\theta}) = e^{ik\theta}$ one can form the associated complex line bundles $L^k := X \times_{\chi_k} \mathbb{C} \to M$.

Conversely, given a complex line bundle $L \to M$, circle bundles X can be constructed using Hermitian metrics h on L. Given h, we define

$$X_h := \partial D_h^* = \{(z, \lambda) \in L^* : |\lambda|_z = 1\}$$

as the boundary of the unit co-disk bundle in the dual line bundle L^*.

We fix a connection,

$$\nabla\colon C^\infty(X, L) \to C^\infty(X, L \otimes T^*X).$$

Given a connection ∇ on L and a vector field V on X, the covariant derivative of a section s is defined by $\nabla_V s = \langle \nabla s, V\rangle$. The curvature is the 2-form Ω^∇ defined by $\Omega^\nabla(V, W) = [\nabla_V, \nabla_W] - \nabla_{[V,W]}$. If e_L is a local frame and $\nabla e_L = \alpha \otimes e_L$, then $\Omega^\nabla = d\alpha$.

The value in introducing the line bundle is that S^1-equivariant functions transforming by χ_k on X correspond to sections of $L^k \to M$. More precisely, there is a natural lift-map isomorphism,

$$s_k \in L^2(M, L^k) \to \hat{s} \in L^2_k(\partial D_h^*), \quad \hat{s}_k(z, \lambda) := \lambda^{\otimes k}(s(z))$$

from sections of the k-th tensor power L^k of L to equivariant functions on ∂D_h^*. The natural $\mathbb{C}^*$ action on L^* restricts to an S^1 action on ∂D_h^* and $\hat{s}_k(r_\theta x) = e^{ik\theta}\hat{s}_k(x)$. It is evident that the zero set of $\hat{s}_k$ is a circle bundle over the zero set of s_k. However, we are most interested in the zero set of $\mathrm{Re}\, \hat{s}_k$.

3. Laplacians

Let $G = G(g, \alpha)$ be a bundle metric on a circle bundle $\pi\colon X \to M$ as in Lemma 2.1. The bundle Laplacian Δ_G has the decomposition

$$\Delta_G = \Delta_H + \Delta_V, \quad \Delta_V = \frac{\partial^2}{\partial\theta^2},$$

where Δ_V is the vertical Laplacian and where Δ_H is the horizontal Laplacian. The equation $\Delta_V = \frac{\partial^2}{\partial\theta^2}$ holds because we assume that lapse function N equals 1. The weight spaces are Δ_H-invariant, i.e., $\Delta_H\colon \mathcal{H}_m \to \mathcal{H}_m$.

3.1. Equivariant eigenfunctions and eigensections

Under the canonical identification using the lifting map of Section 3.1, $\Delta_H|_{\mathcal{H}_m}$ restricts to the Bochner connection Laplacians $D_m = \nabla_m^* \nabla_m$ on sections of L^m in the sense that $\mathcal{H}_m \cong L^2(X, L^m)$, $\Delta_H|_{\mathcal{H}_m} \cong D_m - m^2 I$. In the bottom–up approach, where X is defined as the unit bundle in L^* relative to a hermitian metric h, Bochner Laplacians are defined by $\nabla_{h,g}^* \nabla$ on $L^2(X, L)$ equipped with the data (g, h, J, J_L, ∇), where g is a metric on X, and ∇ is a connection on L. If e_L is a local frame for L, then

$$\widehat{\nabla_m^* \nabla_m (f e_L^m)} = \Delta_H (\widehat{f e_L^m}).$$

An equivariant eigenfunction $\varphi_{m,j}$ as in Definition 1.1 corresponds under the natural lifting map to an equivariant eigensection $f_{m,j} e_L^m$ of L^m in a local frame e_L^m. Let

$$\mathrm{Re}\, f_{m,j} = a_{m,j}(z), \quad \mathrm{Im}\, f_{m,j} = b_{m,j}(z).$$

Then,

$$f_{m,j}(z) e^{-im\theta} = (a_{m,j}(z) + i b_{m,j}(z))(\cos m\theta - i \sin m\theta),$$

so that, with $\varphi_{m,j} = u_{m,j} + i v_{m,j}$,

$$\begin{cases} u_{m,j} = a_{m,j} \cos m\theta + b_{m,j} \sin m\theta, \\ v_{m,j} = b_{m,j} \cos m\theta - a_{m,j} \sin m\theta. \end{cases} \tag{3.1}$$

3.2. Hilbert spaces of sections

In the top–down approach, the Hilbert space inner product on $L^2(X, dV_G)$ is with respect to the Riemannian volume form dV_G. In the bottom–up approach, we let $(L, h) \to M$ be a Hermitian line bundle (which we assume to be holomorphic in the Kähler setting). We thus have a pair of metrics, h resp. g (with Kähler form ω_φ) on L resp. TM. To each pair (h, g) of metrics, we associate Hilbert space inner products $\mathrm{Hilb}_m(h, g)$ on sections $s \in L^2_{m\varphi}(X, L^m)$ of the form

$$\|s\|_{h^m}^2 := \int_X |s(z)|_{h^m}^2 \, dV_g,$$

where $|s(z)|_{h^m}^2$ is the pointwise Hermitian norm-squared of the section s in the metric h^m. In a local frame e_L, we write

$$\|e_L\|_h^2 = e^{-\psi}.$$

In local coordinates z and the local frame e_L^m of L^m, we may write $s = f e_L^m$ and then

$$|s(z)|_{h^m}^2 = |f(z)|^2 e^{-m\psi(z)},$$

and

$$\|fe_L^m\|_{h^m}^2 := \int_X |f(z)|^2 e^{-m\psi} dV_g.$$

3.3. Quadratic forms

The horizontal Laplacian is the Laplacian $\Delta_H = d_H^* d_H$ where $d_H = \pi_H d$ is the horizontal part of the exterior derivative and where d_H^* is the adjoint with respect to the bundle metric G. Thus, for $F \in C^\infty(X)$, $\langle \Delta_H F, F\rangle = \int_X |d_H F|_G^2 dV_G$.

We now trivialize the bundle $\pi: X \to M$ by choosing a local *unitary* section $\mathfrak{u}: U \subset M \to X$ (which may be taken to be global on a set of full measure). We then write $x = r_\theta \mathfrak{u}(y)$, and use (y, θ) as local coordinates on X. In these coordinates $F(y,\theta) = f(y)e^{im\theta}$ for the induced local function $f(y) = \mathfrak{u}^* F$ on U. From a global viewpoint, f is a section of L^m. Then $df = \mathfrak{u}^* dF$ and $dF = (df)e^{im\theta} + imfe^{im\theta} d\theta$. Hence,

$$|dF - imF\alpha|_G^2 = |(df)e^{im\theta} + imfe^{im\theta}(d\theta - \alpha)|^2 = |df + imf(d\theta - \alpha)|^2$$

Further, we define the local connection 1-form $\mathfrak{u}^*(\alpha - d\theta)$ in the unitary frame $\mathfrak{u}$. Note that $(d\theta - \alpha)(Z) = 0$, so $(\alpha - d\theta)$ is a horizontal 1-form. We claim that on $\pi^{-1}(U), \alpha = d\theta + \pi^*\mathfrak{u}^*(\alpha - d\theta)$. We first check the equation at the points of $\mathfrak{u}(M)$. Since $\mathfrak{u} \circ \pi = \mathrm{Id}: \mathfrak{u}(M) \to \mathfrak{u}(M)$, $\alpha_{\mathfrak{u}(y)} = (d\theta + \pi^*\mathfrak{u}^*(\alpha - d\theta))|_{u(y)}$. To check it at a general point $r_\theta u(y)$ we note that both $d\theta$ and $\pi^*\mathfrak{u}^*(\alpha - d\theta)$ are S^1-invariant.

Before stating the next result, we summarize the notation and terminology.

Definition 3.1. Let $G = G(g, \alpha)$ be a bundle metric and let Δ_G be its Laplacian.

- An *equivariant* eigenfunction of Δ_G of *equivariant degre*e (or *weight*) m is a joint eigenfunction $\varphi_{m,j}: X \to \mathbb{C}$ transforming by $\varphi_{m,j}(r_\theta(x)) = e^{im\theta}\varphi_{m,j}(x)$.
- In a unitary frame $\mathfrak{u}: U \subset M \to X$, and the associated trivialization $X|_U = \mathfrak{u}(M) \times S^1$, we say that the function $f_{m,j}: U \to \mathbb{C}$ defined by $f_{m,j} = \mathfrak{u}^*\varphi_{m,j}$ is the *local expression* of the equivariant eigenfunction in the frame $\mathfrak{u}$. If we change the frame from $\mathfrak{u}$ to a local holomorphic frame e_L^m, then we obtain a *local expression* for the holomorphic section of L^m whose lift is $\varphi_{m,j}$.

Lemma 3.2. *Let $G = G(g,\alpha)$ be a bundle metric for a circle bundle $\pi: X \to M$ as in Lemma 2.1. Let $F \in H_m(X)$ and as above express $F(y,\theta) = f(y)e^{im\theta}$ in the local unitary frame $u: U \to X$ and local trivialization $X|_U \simeq U \times S^1$. Then, the quadratic form $Q_m(F) = \langle \Delta_H F, F\rangle$ on $F \in L^2_m(X, dV_G)$ equals $Q_m(F) = \langle L_m f, f\rangle_{L^2(M)}$, where*

$$L_m f = -\Delta_g f + m^2|\mathfrak{u}^*(\alpha - d\theta)|_g^2 f - imd_g^*(\mathfrak{u}^*(\alpha - d\theta))f. \tag{3.2}$$

Proof. Note that for any covector $\xi \in T_x^* X$, $\pi_H(\xi) = \xi - \langle \xi, Z^\flat \rangle Z^\flat = \xi - \xi(Z) Z^\flat \in H_x(X)$, where $Z^\flat$ is the 1-form dual to Z in the sense that $Z^\flat$ is vertical, $Z^\flat(Z) = 1$ and $Z^\flat|_{H(X)} = 0$. Clearly, $Z^\flat = \alpha$ so that

$$\pi_H(\xi) = \xi - \xi(Z)\alpha.$$

Hence,

$$\langle \Delta_H F, F \rangle = \int_X |dF - dF(Z)\alpha|_G^2 dV_G.$$

If $F \in H_m(X)$, then $dF(Z) = imF$ and we get

$$\begin{aligned} \langle \Delta_H F, F \rangle &= \int_X |dF - imF\alpha|_G^2 dV_G \\ &= \int_M \left(\int_{\pi^{-1}(y)} |dF - imF\alpha|_G^2 d\theta \right) dV_g(y). \end{aligned}$$

It follows that $|dF - imF\alpha|_G^2 = |df - imf \mathfrak{u}^*(\alpha - d\theta)|^2$ and

$$\begin{aligned} \langle \Delta_H F, F \rangle &= \int_M |df - imf \mathfrak{u}^*(\alpha - d\theta)|_g^2 dV_g(y) \\ &= \int_M \big(|df|_g^2 + im\langle df, f\mathfrak{u}^*(\alpha - d\theta)\rangle \\ &\quad - im\langle f\mathfrak{u}^*(\alpha - d\theta), df\rangle + m^2|f|^2|\mathfrak{u}^*(\alpha - d\theta)|_g^2 \big) dV_g \\ &= \langle L_m f, f \rangle_{L^2(M)}, \end{aligned}$$

where

$$\begin{aligned} L_m f \; &= -\Delta_g f + m^2|\mathfrak{u}^*(\alpha - d\theta)|_g^2 f + im\langle df, \mathfrak{u}^*(\alpha - d\theta)\rangle \\ &\quad - imd_g^*(f\mathfrak{u}^*(\alpha - d\theta)) \\ &= -\Delta_g f + m^2|\mathfrak{u}^*(\alpha - d\theta)|_g^2 f - imd_g^*(\mathfrak{u}^*(\alpha - d\theta)) f. \end{aligned}$$

In the last equation, we used

$$d_g^*(\varphi w) = g^*(d\varphi, w) + \varphi d_g^* w$$

for any 0-form φ and any 1-form w. ∎

4. Perturbation formulae

In this section, we prepare for the proof of the first two statements of Theorem 1.2 in the next section (See Theorem 5.1) by calculating variations of the relevant Laplacians and their associated quadratic forms. The main Laplacian is Δ_G, but only its horizontal part Δ_H varies as the data (g,α) is varied and so we concentrate on that. In turn, $\Delta_H = \bigoplus_m L_m$, where L_m is defined in (3.2); hence, it suffices to calculate variations of L_m under variations of the base metric or the connection. It is simpler to calculate variations of operators on sections of L^m than their lifts to X.

Under any variation (i.e., along a curve of data), we have by (3.2),

$$\dot{L}_m f = -\dot{\Delta}_g f + m^2 \dot{g}^* \langle \mathrm{u}^*(\alpha - d\theta), \mathrm{u}^*(\alpha - d\theta)\rangle f - im\dot{d}_g^*(\mathrm{u}^*(\alpha - d\theta)) f. \quad (4.1)$$

Here we use the notation $\dot{F} = \frac{d}{dt}\big|_{t=0} F_t$ for the infinitesimal deformation (or variation) of F along a curve.

Remark 4.1. It is sometimes notationally inconvenient to use the "dot" notation and we also use the notation δF for $\dot{F}$.

We now calculate how the various terms deform when g or α is deformed. When applying the results to prove generic properties, we need to move the derivatives on the metric or connection data onto the other factors; hence it is advantageous to evaluate the variation of the quadratic form associated to (4.1) rather than the operator $\dot{L}_m$ with derivatives applied to factors not being varied.

4.1. General metric perturbations

Lemma 4.2. *Let $g(t)$ be a 1-parameter deformation of g with $g(0) = g$ and $b(t)g^{-1} = g(t)^{-1}$. Then, for $f \in C^\infty(M)$ and $W \in L^1(M)$, we have*

$$\langle \dot{L}_m f, W\rangle_{L^2(g)} = \int_M (df + imf\eta, dW + imW\eta)_{bg^*} dV_g + \frac{1}{2}\int_M (df + imf\eta, W d\,\mathrm{Tr}(b))_{g^*} dV_g,$$

where $b = b'(0)$.

Proof. Denote by $L_m(t)$ the horizontal Laplacian corresponding to $g(t)$ with $L_m(0) = L_m$. Let $\eta = u^*(\alpha - d\theta)$. For any $f \in C^\infty(M)$ and $W \in L^1(M)$, we have

$$
\begin{aligned}
&\langle L_m(t) f, W\rangle_{L^2(g)} \\
&\quad = \langle L_m(t) f, W \sqrt{\det b(t)}\rangle_{L^2(g(t))} \\
&\quad = \int_M (df + imf\eta, d(W\sqrt{\det b(t)}) + imW\sqrt{\det b(t)}\eta)_{g^*(t)} dV_{g(t)} \\
&\quad = \int_M (df + imf\eta, dW + imW\eta)_{g^*(t)} dV_g \\
&\qquad + \int_M (df + imf\eta, Wd\sqrt{\det b(t)})_{g^*(t)} dV_{g(t)}.
\end{aligned}
$$

Here dW is the distribution derivative of $W \in L^1$. The second integral is

$$
\begin{aligned}
&\int_M (df + imf\eta, Wd\sqrt{\det b(t)})_{g^*(t)} dV_{g(t)} \\
&\quad = \int_M \Big(df + imf\eta, W\frac{d\sqrt{\det b(t)}}{\sqrt{\det b(t)}}\Big)_{g^*(t)} dV_g \\
&\quad = \int_M (df + imf\eta, Wd\log\sqrt{\det b(t)})_{g^*(t)} dV_g \\
&\quad = \frac{1}{2}\int_M (df + imf\eta, Wd\log\det b(t))_{g^*(t)} dV_g.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\langle \dot{L}_m f, W\rangle_{L^2(g)} \\
&= \frac{d}{dt}\langle L_m(t) f, W\rangle_{L^2(g)}\Big|_{t=0} \\
&= \int_M (df + imf\eta, dW + imW\eta)_{bg^*} dV_g \\
&\quad + \frac{1}{2}\int_M (df + imf\eta, Wd\log\det b(0))_{bg^*} dV_g \\
&\quad + \frac{1}{2}\int_M \Big(df + imf\eta, Wd\Big(\frac{d}{dt}\log\det b(t)\Big|_{t=0}\Big)\Big)_{g^*} dV_g \\
&= \int_M (df + imf\eta, dW + imW\eta)_{bg^*} dV_g \\
&\quad + \frac{1}{2}\int_M \Big(df + imf\eta, Wd\Big(\frac{d}{dt}\det b(t)\Big|_{t=0}\Big)\Big)_{g^*} dV_g,
\end{aligned}$$

where we used $\det b(0) = 1$ to eliminate the second term and simplify the third term. Now, because $b(0)$ is the identity matrix, we see that

$$\frac{d}{dt}\det b(t)\Big|_{t=0} = \mathrm{Tr}(b),$$

and so we have

$$\begin{aligned}
\langle \dot{L}_m f, W\rangle_{L^2(g)} = &\int_M (df + imf\eta, dW + imW\eta)_{bg^*} dV_g \\
&+ \frac{1}{2}\int_M (df + imf\eta, Wd\,\mathrm{Tr}(b))_{g^*} dV_g. \qquad \blacksquare
\end{aligned}$$

4.1.1. Varying the connection. We now vary α or equivalently its local expression $u^*(\alpha - d\theta)$ in the unitary frame u. The deformation equation is essentially the same as in [4].

Lemma 4.3. *The variation $\dot{L}_m$ under a variation of the connection 1-form α is given by*

$$\dot{L}_m f = (-2iG(df, \dot{\alpha}) + ifd_g^*\dot{\alpha} + 2G(\dot{\alpha}, \alpha) f)e_L.$$

This calculation is precisely the same as in [4], so only briefly review it:

$$\nabla^*\nabla(fe_L) = (-\Delta_g f - 2ig^*(df, \alpha) + ifd_g^*\alpha + g^*(\alpha, \alpha) f)e_L,$$

where $\Delta_g f$ is the scalar Laplace operator. Taking the variation with respect to α gives

$$\delta\nabla^*\nabla(fe_L) = (-2iG(df, \dot{\alpha}) + ifd_g^*\dot{\alpha} + 2G(\dot{\alpha}, \alpha) f)e_L.$$

5. Generic properties of eigenvalues and eigenfunctions of general prinicipal S^1 bundle Laplacians

In this section we prove the first two statements of Theorem 1.2, which we reformulate as Theorem 5.1. We prove the third statement about nodal sets in the next section.

Theorem 5.1. *For generic bundle metrics $G = G(g, \alpha)$ with g, α generic data in one of the ways listed below, all of the eigenvalues of Δ_G are simple and all of the equivariant eigenfunctions have 0 as a regular value. Equivalently, for every m, the spectrum of the operator L_m (3.2) is simple, the spectra of L_m and $L_{m'}$ are disjoint if $m \neq m'$, and all of its eigensections have zero as a regular value. Moreover, if we lift sections to equivariant eigenfunctions φ, then $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ have zero as a regular value.*

The generic admissible data is of the following kinds:

(i) *we fix α and vary g among all metrics on M;*

(ii) *we fix g and vary α among connection 1-forms on $\pi: X \to M$.*

As in [4] we prove Theorem 5.1 using the approach of Uhlenbeck [8]. We review this method before giving the proof.

5.1. Review of Uhlenbeck's transversality and genericity results

K. Uhlenbeck has given a general set-up for proving that the spectra of generic elliptic operators are simple and that the eigenfunctions have 0 as a regular value [8]. The proofs are based on the Thom–Smale infinite-dimensional transversality theorem rather than on the Kato–Rellich perturbation theory as in [1].We adapted her approach to S^1 bundle Laplacians over surfaces in [4]. For the sake of completeness, we briefly review the proof that for generic metrics on compact C^rRiemannian manifolds, all eigenvalues are simple and all eigenfunctions have 0 as a regular value.

One considers a family L_b of elliptic operators depending on data b in a Banach space B. One mainly needs that B is a Baire space, i.e., that residual subsets are dense. As in [4], we define B to be a C^k space of data (g, α) needed to define a bundle metric as in Lemma 2.1. The candidate eigenfunctions are denoted by u.

As in [8, Section 2], we study the map $\Phi: C^k(X) \times \mathbb{R} \times B \to \mathbb{R}$ given by

$$\Phi(u, \lambda, b) = (L_b + \lambda)u. \tag{5.1}$$

The zeros of the map correspond to Laplace-type operators L_b together with their eigenvalues λ and eigenfunctions u. This map is sufficient to prove generic simplicity of eigenvalues. To prove the more difficult fact that 0 is a regular value of all eigenfunctions, Uhlenbeck introduces the additional maps

- $Q := \{(u, \lambda, g) \in C^k(X) \times \mathbb{R}_+ \times B),\ \Phi(u, \lambda, b) = 0\}$,
- $\alpha\colon Q \times M \to \mathbb{C}, \alpha(u, \lambda, b, x) = u(x)$.

We use the following "abstract genericity" result of [8, Theorem 1] and [8, Lemmas 2.7 and 2.8].

Theorem 5.2. *Assume that Φ (5.1) is C^k and has zero as a regular value. Then the eigenspaces of L_b are one-dimensional. If, additionally, $\alpha\colon Q \times M \to \mathbb{C}$ has zero as a regular value, then*

$$\{b \in B : \textit{the eigenfunctions of } L_b \textit{ have } 0 \textit{ as a regular value}\} \textit{ is residual in } B.$$

The next two propositions give a tool to verify the first hypothesis of Theorem 5.2 (see [8, Propositions 2.4 and 2.10]).

Proposition 5.3. *The following facts hold.*

(1) *If at points of $\Phi^{-1}(0)$, $D_2\Phi\colon T_b B \to H^p_{k-2}(X) \subset H^q_{-1}(X)$ has dense image in $H^q_{-1}(M), 1 < q < \infty$, then 0 is a regular value of Φ.*

(2) *Let $J = \operatorname{im} D_2\Phi$ and assume that for $W \in L^1(M)$ and $W \in C^2(M - \{y\})$, the property*

$$\int_M W(x) j(x) d\mu_x = 0$$

for all $j \in J$ implies $W = 0$. Then Φ is C^k and has zero as a regular value.

The second statement follows from [8, Lemma 2.7]. Let $\pi\colon Q \to B$ be a C^k Fredholm map of index 0. Then if $f\colon Q \times X \to Y$ is a C^k map for k sufficiently large and if f is transverse to Y' then $\{b \in B : f_b := f|_{\pi^{-1}(b)}$ is transverse to $Y'\}$ is residual in B. Let

$$\alpha\colon f^{-1}(Y') \to B$$

be

$$\alpha\colon f^{-1}(Y') \subset Q \to B.$$

Lemma 5.4. *The eigenfunctions of L_b have zero as a regular value if b is a regular value of π and if 0 is a regular value of $\alpha|_{\pi^{-1}(b)} \times M := \alpha_b$.*

In the following, we use that

$$T_{u,\lambda,b} Q = \left\{(v, \eta, s) \in H^{1,0}(X) \times \mathbb{C} \times T_b B : \int_X uv dV_g = 0, (L_b + \lambda)v + \eta u + D_2\varphi s = 0\right\}.$$

and that the image of $D_2\Phi$ is given by

$$J = \text{Image } D_2\Phi_{(u,\lambda,b)} = \{\dot{\Delta}u : \dot{\Delta} \text{ is a variation of } \Delta \text{ along a curve of metrics}\}.$$

We often write

$$v = \dot{u}, \eta = \dot{\lambda}, D_2(\Phi)s = \lambda\dot{\Delta}u, \quad (\Delta + \lambda)\dot{u} + (\dot{\Delta} + \dot{\lambda})u = 0.$$

Further, let $D_1\alpha$ denote the derivative of α along Q. Then,

$$D_{(u,\lambda,b)}\alpha(v, 0, c, 0) = v(x) = \dot{u}(x).$$

5.2. Generic simplicity of eigenvalues

In this section, we sketch the proof that the eigenvalues of Δ_G are of multiplicity one for generic bundle metrics; the details are similar to those in [4] once the variations are calculated and we do not repeat the overlapping arguments. According to Theorem 5.2, we need to prove that the map Φ (5.1) is C^k and has zero as a regular value. By Proposition 5.3 (b), it suffices to show that $W \in L^1(M)$ and $W \in C^2(M - \{y\})$, the property $\int_M W(x)j(x)d\mu_x = 0$ for all $j \in J$ implies $W = 0$. In our problem, $j = \text{Im}D_2\Phi$ is given by $j = \dot{\Delta}_G u$ where $(\Delta_G + \lambda)u = 0$ for some λ, and where $\dot{\Delta}_G$ arises from variations from (i) general perturbations of metrics on M, or (ii) the space of connection 1-forms α on $\pi: X \to M$. In the calculations, we use that $\Delta_G = \bigoplus_m L_m$ where L_m operates on equivariant functions. Not only is the spectrum of each L_m (3.2) simple, but the spectra of L_m and $L_{m'}$ are disjoint sets if $m \neq m'$ for generic bundle metrics G. This last step is proved in [4, Lemma 4.9]; the same proofs works in the present higher-dimensional setting and will not be repeated here.

5.3. Generic base metrics

The following Lemma proves that the criterion in Proposition 5.3

Lemma 5.5. *Let f be an eigenfunction of L_m with the non-zero eigenvalue $\lambda \neq 0$. Let $g(t)$ be a 1-parameter deformation of g with $g(0) = g$ and $b(t)g^{-1} = g(t)^{-1}$. Then the image of $\dot{\Delta}f$ is dense.*

Proof. We continue the perturbation calculation of Lemma 4.2. In local coordinates, let

$$df + imf\eta = a_i dx_i \quad dW + imW\eta = w_i dx_i,$$

and let

$$d_g^*(\overline{W}(df + imf\eta)) = F.$$

Then

$$\langle \dot{L}_m f, W\rangle_{L^2(g)} = 0$$

is equivalent to

$$\int_M a_i b^{ij} g^{jk} w_k + \frac{F}{2} \operatorname{Tr}(b) dV_g = 0. \tag{5.2}$$

We now assume for contradiction that $\dot{L}_m f$ does not have dense image, and W is a non-zero section of L^m which is orthogonal to $\dot{L}_m f$ under any variation. For (5.2) to hold for all b, we must have

$$a_i g^{jk} w_k + a_j g^{ik} w_k = 0$$

for all fixed $i \neq j$, and

$$a_i g^{ik} w_k = -\frac{F}{2}$$

for all fixed i. In particular, we have

$$(a_i dx_i) \wedge (g^{jk} w_k dx_j) = -\frac{F}{2}(dx_i \vee dx_i).$$

Observe that the left-hand side of the equation has rank at most 2, while the right-hand side has rank n if $F \neq 0$, or 0 if $F = 0$. Because $n \geq 3$, we have

$$F = 0$$

and

$$(a_i dx_i) \wedge (g^{jk} w_k dx_j) = 0$$

on M. Because g is clearly invertible, this implies that the support of

$$df + imf\eta \text{ and } dW + imW\eta$$

are disjoint, and

$$d_g^*(\overline{W}(df + imf\eta)) = 0$$

on M. Now, we use the assumption that f is an eigenfunction of L_m, i.e.,

$$\lambda f = L_m f = d_g^*(df + imf\eta) - im(df + imf\eta, \eta)_{g^*} \tag{5.3}$$

Then we have

$$\begin{aligned}
0 &= d_g^*(\overline{W}(df + imf\eta)) \\
&= -(df + imf\eta, dW)_{g^*} + \overline{W} d_g^*(df + imf\eta) \\
&= (df + imf\eta, imW\eta)_{g^*} + \overline{W}(\lambda f + im(df + imf\eta, \eta)_{g^*}) \\
&= \lambda \overline{W} f,
\end{aligned}$$

where we used $dW + imW\eta = 0$ on the support of $df + imf\eta$ and (5.3) in the third equation. This forces $W = 0$ on an open set, and the Lemma follows from [8, Proposition 2.10]. ■

5.3.1. Generic connection 1-forms

Lemma 5.6. *For generic connection* 1*-forms* α*, the spectrum of* Δ_G *is of multiplicity one.*

Proof. The variation of Δ_G induced by varying α is calculated Section 4.1.1. Assume for purposes of contradiction of Proposition 5.3, that there exists $W \in L^1(M)$ and $W \in C^2(M - \{y\})$ with the property $\int_M W(x)\dot{\Delta}_G u d\mu_x = 0$ some Δ-eigensection $u = fe_L^m$ and all variations. We need to prove that $W = 0$. The argument is almost identical to that in [4], so we only briefly summarize it.

We prove this by representing equivariant functions on X as sections of the associated bundle, and write sections relative to a local frame e_L. Exactly as in [4], if the image is not dense, there exists $W = Fe_L \neq 0$ so that (by Lemma 4.1.1),

$$\int_X \big(-2iG(df,\dot{\alpha}) + ifd_g^*\dot{\alpha} + 2G(\dot{\alpha},\alpha)f\big)\bar{F}e^{-\psi}dV_g = 0$$

$$\iff \int_X \big((-2iG(df,\dot{\alpha}) + 2G(\dot{\alpha},\alpha)f)\bar{F} + iG(\dot{\alpha}, d(f\bar{F}))\big)e^{-\psi}dV_g = 0,$$

for all $\dot{\alpha} \in \Omega^1(X)$. Here, we integrated d_g^* by parts to remove it from $\dot{\alpha}$. As in [4], this boils down to

$$(-2idf + 2\alpha f)\bar{F} + id(f\bar{F}) = 0 \iff (-idf + 2\alpha f)\bar{F} + ifd\bar{F} = 0$$
$$\implies d\alpha = 0.$$

on a dense open set; but a generic C^k 1-form does not satisfy $d\alpha = 0$. ■

5.4. For generic bundle metrics the eigenfunctions have regular nodal sets

To prove this statement we again use Theorem 5.2. To establish the hypothesis, we now use Lemma 5.4.

Let

$$Q := \{(u,\lambda,b) \in C^k(X) \times \mathbb{R}_+ \times B : \Phi(u,\lambda,b) = 0\},$$

and let

$$\alpha\colon Q \times M \to \mathbb{C}, \quad \alpha(u,\lambda,b,x) = u(x).$$

To complete the proof of Theorem 5.1 it suffices to recall the following Proposition, proved in [4, Proposition 4.8].

Proposition 5.7. *For each m, $D_1\alpha_m$ is surjective to $\mathbb{C}$. We need to show that for each $x \in M$,*

$$\alpha_m \colon Q \times \{x\} \to \mathbb{C}, \quad \alpha(u, \lambda, g, x) = u(x)$$

has $0 \in \mathbb{C}$ as a regular value, i.e., that

$$D_1\alpha(\cdot, x) \colon T_{u,\lambda,b}(Q) \to \mathbb{C}, \quad D_1\alpha(\cdot, x)_{(u,\lambda,g)}(\delta u(x), 0, c, 0) = \delta u(x)$$

is surjective to $\mathbb{C}$, where D_1 is the differential along Q with $x \in M$ held fixed.

The proof is very similar to that in [4, Proposition 4.8], so we only sketch enough of it to ensure that the argument used there applies to our new types of deformations.

The proof is based on general features of the Green's function

$$G_{m,\lambda} \colon [\ker(D_m + \lambda)]^\perp \to [\ker(D_m + \lambda)]^\perp,$$

for, i.e., the Schwartz kernel of the resolvent $(D_m(g) + \lambda)^{-1}$ on the space where it is well defined.

We need to show that, for each $x \in M$,

$$\alpha_m \colon Q \times \{x\} \to \mathbb{C}, \quad \alpha(u, \lambda, g, x) = u(x)$$

has $0 \in \mathbb{C}$ as a regular value, i.e., that

$$D_1\alpha(\cdot, x) \colon T_{u,\lambda,b}(Q) \to \mathbb{C}, \quad D_1\alpha(\cdot, x)_{(u,\lambda,g)}(\delta u(x), 0, c, 0) = \delta u(x)$$

is surjective to $\mathbb{C}$, where D_1 is the differential along Q with $x \in M$ held fixed.

As in the proof of [4, Proposition 4.8], $D_1\alpha$ is surjective to $\mathbb{C}$ unless for all $j \perp \ker(D_m(g) + \lambda)$, either the real or imaginary parts of

$$G_{m,\lambda}(j)(x) = \int_M G_{m,\lambda}(x, y) j(y) dV(y)$$

vanishes for every such j. Since $j = [D_m(g) + \lambda]f$ where $\int f = 0$ we would get the absurd conclusion that

$$f(x) = 0, \quad \text{for all } f \perp \ker(D_m(g) + \lambda).$$

This is impossible, concluding the proof.

6. Proof of connectivity and smoothness of nodal sets and the 2 nodal domain theorem for equivariant eigenfunctions with regular nodal set

In this section, we prove the main result on connectivity. The proof is entirely different, and much simpler than, the proof of the analogous statement in [4] and in particular does not use a local study of eigenfunctions as in [4, Section 5]. Recall from (1.2) that

$$Z_{\mathrm{Re}\,\hat{s}} := \{\mathrm{Re}\,\hat{s}(\zeta) = 0, \zeta \in \partial D_h^*\}.$$

Proposition 6.1. *If $\hat{s} \in H_m(X)$ is an equivariant eigenfunction, then natural projection $\pi\colon Z_{\mathrm{Re}\,\hat{s}} \to M$ is an $2m$-fold covering map on the complement of the complex zero set $\{z : f_{m,j}(z)e_L^m = 0\} \subset M$.*

Proof. The proof is almost the same as in [4] and consists of a sequence of Lemmas. In the notation of Section 3.1, we denote by $\mathcal{Z}_{f_{m,j}}$ the zero set of the eigensection $f_{m,j}e_L^m$ on M:

$$\mathcal{Z}_{f_{m,j}} = \{z \in M : f_{m,j}(z) = 0\}.$$

We denote by $\mathcal{Z}_{\varphi_{m,j}}$ the nodal set of the (complex-valued) equivariant eigenfunction $\varphi_{m,j}$ on X. Under the natural projection $\pi\colon X \to M$,

$$\mathcal{Z}_{\varphi_{m,j}} = \pi^{-1}\mathcal{Z}_{f_{m,j}}.$$ ∎

Lemma 6.2. *For $z \in M$ such that $f_{m,j}(z) \neq 0$, there exist $2m$ distinct solutions v of $\mathrm{Re}\, f_{m,j}e_L^m(v) = 0$ with $v \in L_z^*X$.*

Proof. We trivialize L_z^* using the dual frame $e_L^* \sim \mathbb{C}$ and use polar coordinates (r, φ) on $\mathbb{C}$. Since the equations are homogeneous, we set $r = 1$ and identify $v = (\cos\theta, \sin\theta)$. In the notation of (3.1), the equation for a nodal point is

$$(a_{m,j}\mathfrak{c}_m - b_{m,j}\mathfrak{s}_m)(\cos\theta, \sin\theta) = 0.$$

Here $\mathfrak{c}_m = \mathrm{Re}(\cos\theta + i\sin\theta)^m = \cos m\theta$, and the equation is

$$a_{m,j}(z)\cos m\theta - b_{m,j}(z)\sin m\theta = 0 \iff \tan m\theta = \frac{a_{m,j}}{b_{m,j}},$$

where $a_{m,j}, b_{m,j} \in \mathbb{R}$ and where we assume with no loss of generality that $b_{m,j} \neq 0$. For $0 \leq \theta < 2\pi$, we have $0 \leq m\theta < 2m\pi$, and so there are exactly $2m$ choices of θ. ∎

The following is an immediate consequence of Lemma 6.2.

Lemma 6.3. *If 0 is a regular value, then the nodal set $\mathcal{N}_{m,j} \subset X$ of $\mathrm{Re}\,\varphi_{m,j}$ is a singular $2m$-fold cover of M with blow-down singularities over points where $f_{m,j}(z)e_L^m = 0$.*

Indeed, the $2m$ zeros of $\operatorname{Re}\omega_{m,j}(v) = 0$ in $S_z X$ give $2m$ points on the fiber $\pi^{-1}(z)$ in P_h. Since locally there exist $2m$ smooth determinations of the zeros, the nodal set is a covering map away from the singular points. This completes the proof of Proposition 6.1.

We prove the connectivity statement of Theorem 1.2.

Proposition 6.4. *If there exists z_0 such that $f_{m,j}(z_0)e_L^m = 0$, then the nodal set of* $\operatorname{Re}\varphi_{m,j}$ *is connected.*

Proof. Let $\emptyset \neq \Sigma \subset M$ be the zero set of $f_{m,j}e_L^m$. By Proposition 6.1, $\mathcal{N}_{m,j}\backslash(\mathcal{N}_{m,j} \cap \pi^{-1}(\Sigma)) \to M\backslash\Sigma$ is an $2m$-sheeted cover. Because M is connected, any point on M is connected to a point in Σ. Therefore, any point on $N_{m,j}$ is connected to a lift of a point in Σ. Because the lift of a point in Σ is a circle, this proves the connectivity of any two points in the lift of a point in $M\backslash\Sigma$. Now, again by the connectivity of M, for any given $p, q \in M$, we see that any lift of a point p is connected to some lift of point q. This concludes that any lift of p is connected to any lift of q, which shows that $\mathcal{N}_{m,j}$ is connected. ∎

We now prove that the nodal set is smooth.

Lemma 6.5. *If* 0 *is a regular value of $f_{m,j}$, then $\mathcal{N}_{m,j} \subset X$ is a smooth submanifold of X.*

Proof. Assume that $f_{m,j}(z_0) = 0$ is a regular zero. Then

$$Df_{m,j}(z_0)\colon T_{z_0}(M) \to \mathbb{C}$$

is a surjection. Now, for any $0 \leq \theta < 2\pi$,

$$Du_{m,j}(z_0,\theta)\colon T_{z_0,\theta}(X) \to \mathbb{C}$$

restricted to the vectors tangent to M is given by

$$\cos m\theta Da_{m,j}(z_0) - \sin m\theta Db_{m,j}(z_0), \tag{6.1}$$

which is the same as multiplying $Df_{m,j}(z_0)$ by $e^{im\theta}$ and then taking the real part. Since the image of $Df_{m,j}(z_0)$ is entire $\mathbb{C}$, it is clear that (6.1) surjects onto $\mathbb{R}$. This implies that $Du_{m,j}(z_0,\theta)$ is surjective for any θ, and therefore $\mathcal{N}_{m,j} \subset SX$ is a smooth submanifold, by Proposition 6.1. ∎

Finally, we complete the proof of Theorem 1.2. Note that the existence of z_0 such that $f_{m,j}(z_0) = 0$ follows from the non-triviality of the principal bundle.

Proposition 6.6. *If* 0 *is a regular value of $\varphi_{m,j}$ and there exists z_0 such that one has $f_{m,j}(z_0) = 0$, then $SX\backslash\mathcal{N}_{m,j}$ has exactly two connected components.*

Proof. By Proposition 6.4, if 0 is a regular value then $\mathcal{N}_{m,j}$ is a connected, embedded hypersurface. To prove that it separates X into two connected components, note first that any connected embedded hypersurface of X may either separate X into two components, or $X \backslash \mathcal{N}_{m,j}$ is connected. Since $\operatorname{Re} \varphi_{m,j}$ is a non-constant eigenfunction of the Laplacian, it integrates to 0, and so there are at least two non-empty nodal domains one in $\{\operatorname{Re} \varphi_{m,j} > 0\}$ and the other one in $\{\operatorname{Re} \varphi_{m,j} < 0\}$. Therefore, $\mathcal{N}_{m,j}$ separates X into exactly two connected components. ■

7. Even-dimensional spheres with effective S^1 actions: Proof of Lemma 1.3

The results above do not apply to even-dimensional spheres, or any manifold with an effective S^1 action whose Euler characteristic is non-zero, since then Z has zeros and S^1 cannot act freely. An obvious question is, what can be said of nodal sets of real parts of equivariant eigenfunctions in this case? For a non-free S^1 actions by isometries of a Riemannian manifold (X, G), the lapse function $|Z|$ is never constant in this case and of course equals zero at the fixed points. It is still the case that $[\Delta_G, D_Z] = 0$ and there still exists an orthonormal basis $\{\varphi_{m,j}\}$ of joint (equivariant) eigenfunctions. There are many possible types of examples which depend, for instance, on the nature of the fixed point set of S^1.

The simple reason why nodal sets are singular in even dimensions is the existence of fixed points of the S^1 action and that, at the fixed points p, every equivariant eigenfunction with $m \neq 0$ must vanish, since $\varphi_{m,j}(p) = \varphi_{m,j}(r_\theta p) = e^{im\theta} \varphi_{m,j}(p)$. Moreover, each fixed point is a critical point since $d\varphi_{m,j}(p) = d\varphi_{m,j}(p) \circ Dr_\theta = e^{im\theta} d\varphi_{m,j}(p)$ where $D_p r_\theta \colon T_p X \to T_p X$. Thus, each fixed point is a singular point of $\varphi_{m,j}$ (a critical nodal point), i.e., 0 is not a regular value of any equivariant eigenfunction or of $\operatorname{Re} \varphi_{m,j}$ and obviously Theorem 1.2 does not hold for any bundle metric in this case.

We now consider in more detail the case of even-dimensional spheres $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ and assume that the S^1 action is a one-parameter subgroup of $\mathrm{SO}(2n+1)$ and that it has precisely two fixed points on $\mathbb{S}^{2n}$, which we assume to be $\pm e_{2n+1}$ where e_j is the standard basis. We denote the N-th degree equivariant spherical harmonics by φ_N^m. These form a $\sim N^{d-2}$-dimensional subspace and to uniquely specify the harmonic we could use an orthonormal basis of joint eigenfunctions of the Laplacian and a Cartan subgroup. But we stick to this simple but ambiguous notation, since the claims hold for any element of the subspace.

Let $G = \mathrm{SO}(2n+1)$ and let G_p denote the isotropy group of $p \in \mathbb{S}^{2n}$. If $p = e_{2n+1}$, then the G_p induces a derived action on $T_{e_{2n+1}} \mathbb{S}^{2n}$. We define S^1 to be a one-parameter subgroup of G_p which has precisely two fixed points $p, -p$ (in the sense that $S^1_p = S^1$ or $Z(p) = 0$). For instance, we can let S^1 be the direct sum of 2×2 blocks $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ plus a 1×1 block with entry 1 corresponding to e_{2n+1}. The S^1 action preserves $\mathbb{R}^{2n} := (\mathbb{R}e_{2n+1})^\perp$ and therefore the equatorial subsphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$. The quotient $\mathbb{S}^{2n+1}/S^1$ is not a smooth manifold due to the fixed points, but the standard metric is bundle-like with respect to the natural projection $\pi\colon \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/S^1$ since the map $\pi\colon \mathbb{S}^{2n+1}\backslash\{p,-p\} \to (\mathbb{S}^{2n+1}\backslash\{p,-p\})/S^1$ is a smooth projection. It also preserves all "latitude spheres" defined as level sets of $x_{2n+1}\colon \mathbb{S}^{2n} \to \mathbb{R}$ and acts freely on each. It follows that one has $\mathbb{S}^{2n}\backslash\{p,-p\} \simeq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{S}^{2n-1}$ and the quotient by the S^1 action is given by the diffeomorphism $\mathbb{S}^{2n}\backslash\{p,-p\}/S^1 \simeq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (\mathbb{S}^{2n-1}/S^1)$. We may therefore consider equivariant eigenfunctions in a similar spirit to the case of odd-dimensional spheres. A key difference is that Z is not of constant norm, so the bundle metric necessarily has a non-constant lapse function N depending on (and only on) x_{2n+1}; it is given by $N(x_{2n+1}) = \sin r(x_{2n+1})$ where r is the distance of the latitude sphere of height x_{2n+1} to p.

The free S^1 action on $\mathbb{S}^{2n}\backslash\{p,-p\}$ determines a complex line bundle $L^m \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (\mathbb{S}^{2n-1}/S^1)$ associated to characters χ_m and as in the bundle case, the equivariant eigenfunctions of $\Delta_{\mathbb{S}^{2n}}$ correspond to complex eigensections of this line bundle for the induced operators L_m (3.2). Note that $(\mathbb{S}^{2n-1}/S^1) \simeq \mathbb{CP}^{n-1}$.

We now prove Lemma 1.3.

Proof. For $n > 1$, on $\mathbb{S}^{2n}$ one has a map $q\colon \mathbb{S}^{2m}\backslash\{\pm e_{2n+1}\} \to \mathbb{S}^{2n-1}$ where $\mathbb{S}^{2n-1}$ is the equatorial sphere, obtained by following the orthogonal geodesics to $\mathbb{S}^{2n-1}$ to the poles. The second map is again $x_{2n+1}\colon \mathbb{S}^{2n} \to (-1,1)$. Together we have a 1-parameter family of latitude spheres and a $2n-1$-parameter family of orthogonal longitude lines. There is a third map $\mathbb{S}^{2n}\backslash\{p,-p\}/S^1 \simeq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (\mathbb{S}^{2n-1}/S^1)$ to the orbit space. Since x_{2n+1} is constant on S^1 orbits, the third map and second map coincide when $n = 2$.

A unitary section $u\colon \mathbb{S}^{2n}\backslash\{p,-p\}/S^1 \to \mathbb{S}^{2n}$ is an inverse of the map $\mathbb{S}^{2n} \to \mathbb{S}^{2n}\backslash\{p,-p\}/S^1$ giving a cross section to the S^1 action. When $n = 2$, a section is given by a meridian line. This would be a section of the $\mathrm{SO}(2n-1)$ action stabilizing the poles. Now, it is given by a family $(2n-2)$-dimensional family of meridian lines which is the flowout of the transversal to the S^1 orbit in $S^*_p\mathbb{S}^{2n}$. An equivariant function has the form $\varphi^m_N(ue^{i\theta}) = e^{im\theta}\varphi^m_N(u)$. The real part is $\operatorname{Re}\varphi^m_N = u^m_N(u)\cos m\theta$, where we assume that $u^m_N(u)$ is real valued If we consider the $2m$ zeros $\theta_{m,j}$ of $\cos m\theta$, we get the union $\bigcup_{j=1}^{2m}\{\theta_{m,j}\} \times \mathbb{S}^{2n}\backslash\{p,-p\}/S^1$, disjoint union of $2m$ hypersurfaces. When $m \neq 0$ the submanifolds meet at the poles. There also exist S^1 invariant transverse components of the nodal set coming from the factor $\varphi^m_N(u)$.

In the notation of Section 3.1, the S^1 fibers of the zero set of φ^m_N over the complex zero set $\{f^m_N = 0\} \subset \mathbb{S}^{2n}\backslash\{p,-p\}/S^1$ of the sections of L^m intersect the set

$\bigcup_{j=1}^{2m}\{\theta_{m,j}\} \times \mathbb{S}^{2n} \backslash \{p, -p\}/S^1$. For $n \geq 2$, f_m^N is complex valued, and this set is of real codimension 3.

The nodal set of φ_N^m is $S^1 \times \{f_N^m = 0\}$ and it obviously intersects $\bigcup_{j=1}^{2m}\{\theta_{m,j}\} \times \mathbb{S}^{2n} \backslash \{p, -p\}/S^1$ in the set $\bigcup_{j=1}^{2m}\{\theta_{m,j}\} \times \{f_N^m = 0\}$.

This completes the proof of Lemma 1.3. ∎

The most familiar case $n = 1$ is very special and we pause to contrast it with the case $n > 1$. In this case, the quotient is simply $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The equivariant eigenfunctions form the standard basis $Y_N^m(\theta, \varphi) = \sqrt{2N+1} e^{im\theta} P_N^m(\cos\varphi)$. Here, P_N^m are the standard associated Legendre polynomials and (3.2) is the mth Legendre operator. The real part of Y_N^m is $\sqrt{2N+1} P_N^m(\cos\varphi) \cos m\theta$. Its nodal set is the union of the nodal sets $\{P_N^m = 0\} \cup \{\cos m\theta = 0\}$, and it has a singular set of mN points given by the Cartesian product of the zero set of $\cos m\varphi$ and the zero set of $P_N^m(\cos\varphi)$. The complex line bundle L_m is the product bundle $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{C}$. In the constant frame, the eigensections are all real-valued and are the Legendre factors $P_N^m(\cos\varphi)$. The inverse image of their zero sets thus has real codimension one rather than 2. Over the complement of $\{P_N^m = 0\}$, the zero set of $\operatorname{Re} Y_N^m$ is still an m-fold cover of the base, indeed consists of the m zeros of $\cos m\theta$ on each latitude circle and the real zero set consists of m longitude geodesics through these zeros. The special features are that the complex zero set of Y_N^m is a union of N latitude circles rather than being of real codimension 2 and that the m longitude circles all meet at the poles. The nodal set is connected but has Nm singular points and Nm nodal domains.

Notes added in proof. The second author passed away before the paper got published. All edits made thereafter are minor edits in order to improve the readability and make the paper self-contained.

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