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Blow-up of the critical norm for a supercritical semilinear heat equation

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Abstract. We consider the scaling critical Lebesgue norm of blow-up solutions to the semilinear heat equation $u_t = \Delta u + |u|^{p-1}u$ in an arbitrary $C^{2+\alpha}$ domain of $\mathbf{R}^n$. In the range $p > p_S := (n+2)/(n-2)$, we show that the critical norm must be unbounded near the blow-up time, where the type I blow-up condition is not imposed. The range $p > p_S$ is optimal in view of the existence of type II blow-up solutions with bounded critical norm for $p = p_S$.

Keywords: semilinear heat equation, blow-up, critical norm, ε-regularity.

Contents

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1. Introduction

1.1. Background

We study blow-up solutions of the following semilinear heat equation:

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $p > 1$, Ω is a domain in $\mathbf{R}^n$ with $n \geq 1$ and $u_0 \in L^q(\Omega)$ with $q \geq 1$. The boundary condition is not present if $\Omega = \mathbf{R}^n$.

The equation in (1.1) has attracted much attention as one of the simplest models for scaling invariant nonlinear parabolic equations. For each solution u, the rescaled function $u_\lambda(x,t) := \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t)$ $(\lambda > 0)$ also satisfies the equation. This implies that $L^{q_c}(\Omega)$ with

$$q_c := n(p-1)/2$$

is the scaling critical Lebesgue space for (1.1). The critical space plays a crucial role in well-posedness. If $u_0 \in L^{q_c}(\Omega)$ and $q_c > 1$, it is well-known [11,106,107] that there exists a unique classical L^{q_c}-solution u of (1.1) with maximal existence time $T \in (0,\infty]$. For the definition of the classical L^q-solution, see Remark 1.2. The solution u is smooth for $t \in (0,T)$, belongs to $C([0,T); L^{q_c}(\Omega)) \cap C((0,T); L^\infty(\Omega))$ and admits the following blow-up criterion: If $T < \infty$, then $\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. This leads to the critical norm blow-up problem.

Problem. If $T < \infty$, does the following property hold:

$$\lim_{t\to T} \|u(\cdot,t)\|_{L^{q_c}(\Omega)} = \infty\ ? \tag{CNB}$$

The problem is stated in Brezis and Cazenave [11, Open problem 7] and also a variant can be found in Quittner and Souplet [90, OP 2.1, Section 55] which asks about the existence of blow-up solutions with bounded L^{q_c} norm. Many sufficient conditions for (CNB) are known: see [10, 37, 43, 63, 64, 66, 76, 110] and [90, Section 16]. Recently, a significant progress was made by Mizoguchi and Souplet [76], who proved that (CNB) holds whenever the blow-up is type I. Here the blow-up of u is called *type I* if the blow-up rate is bounded by a spatially homogeneous solution up to the coefficient, that is, $\limsup_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. This rate is natural in view of the scaling. Note that the blow-up is called *type II* if it is not of type I. In the celebrated work [43] Giga and Kohn showed that the blow-up is always type I if Ω is convex and either $p < p_S$ and u is nonnegative, or $p < (3n+8)/(3n-8)$. Here p_S is the Sobolev critical exponent given by

$$p_S := \begin{cases} \dfrac{n+2}{n-2} & \text{for } n \geq 3, \\ \infty & \text{for } n = 1, 2. \end{cases}$$

Up to now, it has been proved that the blow-up is type I when either $p < p_S$ and Ω is convex [45,46], or $p < p_S$ and u is nonnegative [89]. Therefore, under such conditions, (CNB) holds for any blow-up solutions in the subcritical range $p < p_S$. For related results concerning sufficient conditions for type I blow-up, we refer to [17,18,20,38,45,46,53,63,65,68,75,84,88,89,109] and [90, Section 23].

The situation is different for $p \geq p_S$. Type II blow-up solutions were constructed in a number of papers; see Remark 1.7 for a brief review. In particular, the recent developments [23,26,27,47,59,91] provide type II blow-up solutions satisfying

$$\sup_{0<t<T} \|u(\cdot,t)\|_{L^{q_c}(\Omega)} < \infty$$

for $p = p_S$ and $3 \leq n \leq 5$; see [76, Section 4] for computations of the L^{q_c} norm of the solutions constructed in [23,91]. This demonstrates that type I assumption in [76] is indeed necessary if $p = p_S$ and $3 \leq n \leq 5$. For $p = p_S$ and $n \geq 6$, no counter-examples are known. Moreover, for $n \geq 7$, it was shown in [104] that (CNB) holds for interior blow-up solutions with $u \geq 0$ and either $\Omega = \mathbf{R}^n$ or Ω bounded. In contrast with the case $p = p_S$, all the known type II blow-up solutions do satisfy (CNB) in the range $p > p_S$. Taking the above results into account, we expect that (CNB) holds whenever $p > p_S$.

1.2. Main theorem

Our main result shows that, in the optimal range $p > p_S$, the critical norm of all finite time blow-up solutions must be unbounded without assuming nonnegativity, monotonicity, symmetry, convexity or the type of blow-up.

Theorem 1.1. *Let $n \geq 3$, $p > p_S$, Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 < \alpha < 1$ and u be a classical L^{q_c}-solution of (1.1) with $u_0 \in L^{q_c}(\Omega)$. If the maximal existence time $T > 0$ is finite, then*

$$\limsup_{t \to T} \|u(\cdot,t)\|_{L^{q_c}(\Omega)} = \infty.$$

The theorem immediately shows the nonexistence of blow-up solutions with bounded L^{q_c} norm, and so this resolves the open problem [90, OP 2.1, Section 55] for the supercritical case. We now give comments on the proof, and then we list remarks concerning the statement of this theorem and related results including other scaling invariant nonlinear evolution equations.

As in [35,102] for related equations, our proof of Theorem 1.1 consists of two parts: (i) the blow-up (rescaling and compactness) procedure and (ii) the analysis of the blow-up limit. However, several additional difficulties come from the differences in the nonlinear structure, in particular, the lack of coercivity of the energy and the absence of the derivative in the nonlinear term. Compared with the earlier work [76], there are also some novelties in the proof. In (i), a concentration theorem for the L^{q_c} norm near a blow-up point plays a crucial role in [76] for the nondegeneracy of the blow-up limit. Unfortunately, we could not rely on that concentration theorem due to the absence of the type I assumption. In order to circumvent this difficulty, we will prove a new ε-regularity theorem (Theorem 4.1), which guarantees the energy concentration near a blow-up point. Theorem 4.1 is motivated by similar ε-regularity theorems in [15]. In comparison, we do not need to assume that the solution is globally defined in time or has a certain lower bound of the energy. In (ii), unlike the case of [76], the smoothness of our blow-up limit is no longer clear even before the final time. To overcome this issue, we invoke a monotonicity estimate similar to [43], which plays a key role in identifying the blow-up limit. We note that the use of the ε-regularity theorem and of the monotonicity estimate is partially inspired by the related works [97, 102] for the harmonic map heat flow, but several modifications are needed for our problem as explained above.

Remark 1.2 (Classical L^q-solution). For $1 \le q \le \infty$, the definition of the classical L^q-solution is as follows (see [90, Definition 15.1]). Let $u_0 \in L^q(\Omega)$ and $T \in (0,\infty]$. We say that $u \in C([0,T); L^q(\Omega))$ is a *classical* L^q*-solution* of (1.1) if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T))$, $u(\cdot,0) = u_0$ and u is a classical solution of (1.1) for $t \in (0,T)$. If Ω is unbounded, we also impose $u \in L^\infty_{\mathrm{loc}}((0,T); L^\infty(\Omega))$. If $q = \infty$, then the condition $u \in C([0,T); L^q(\Omega))$ is replaced with $u \in C((0,T); L^\infty(\Omega))$ and $\lim_{t\to 0} \|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\infty(\Omega)} = 0$, where $e^{t\Delta}$ is the Dirichlet heat semigroup in Ω. We note that the maximal existence time can be defined for classical L^q-solutions; see [90, Proposition 16.1 (i, ii)] for the definition.

Remark 1.3 (Uniqueness). By [106, 107] and [11, Theorem 1], a unique classical $L^{q_c}(\Omega)$-solution of (1.1) exists for each $u_0 \in L^{q_c}(\Omega)$ with $q_c > 1$. Moreover, if we further assume $q_c > p$, unconditional uniqueness [11, Theorem 4] holds for mild solutions, that is, solutions of the corresponding integral equation

$$u(\cdot,t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}|u(\cdot,s)|^{p-1}u(\cdot,s)\,ds \qquad \text{for } t \in (0,T)$$

in $C([0,T); L^{q_c}(\Omega))$. Note that $q_c > p$ is equivalent to $p > n/(n-2)$ and is satisfied for $p > p_S$. Hence, under the assumption of Theorem 1.1, the condition $u_0 \in L^{q_c}(\Omega)$ implies the existence of a unique mild solution in $C([0,T); L^{q_c}(\Omega))$. This solution is also a classical L^{q_c}-solution, since $q_c > 1$.

Remark 1.4 (Other Lebesgue spaces). We recall the case $u_0 \in L^q(\Omega)$ with $q \ne q_c$. For $1 \le q < q_c$, there are results on the nonexistence and nonuniqueness of solutions; see [6, 13, 49, 107] and [84, Section 6]. By [38, Theorem 2.4], there exist solutions such that the maximal existence time T is finite and

$$\sup_{0<t<T} \|u(\cdot,t)\|_{L^q(\Omega)} < \infty.$$

For $q_c < q \le \infty$ with $q \ge 1$, it is known that (1.1) has a unique classical L^q-solution u (see [90, Theorem 15.2, Proposition 51.40] for instance). By [11, Corollary 13], it is also known that if $T < \infty$, then $\lim_{t\to T} \|u(\cdot,t)\|_{L^q(\Omega)} = \infty$. More precisely, the lower estimate of the blow-up rate

$$\|u(\cdot,t)\|_{L^q(\Omega)} \ge C(T-t)^{-\frac{n}{2}(\frac{1}{q_c}-\frac{1}{q})}$$

holds for some constant $C > 0$; see [108, Section 6] and [90, Remark 16.2 (iii), Proposition 23.1].

Remark 1.5 (Application to backward self-similar solutions). Let $\Omega = \mathbf{R}^n$. We say that the solution u is *backward self-similar* if it is of the form

$$u(x,t) = (T-t)^{-1/(p-1)} U(x/\sqrt{T-t})$$

for some $T > 0$ and some profile function $U \in C^2(\mathbf{R}^n)$. Theorem 1.1 immediately yields the following Liouville-type result for backward self-similar solutions u of (1.1) with $p > p_S$. It recovers [76, Corollary 2] in the case $p > p_S$.

Corollary 1.6. *Let $n \geq 3$, $p > p_S$ and u be a backward self-similar solution of* (1.1). *If the profile function U of u belongs to $L^{q_c}(\mathbf{R}^n)$, then $u \equiv 0$.*

Remark 1.7 (Brief review of type II blow-up). Type II blow-up solutions were first found by Herrero and Velázquez [50,51] in the Joseph–Lundgren supercritical range $p > p_{JL} := (n - 2\sqrt{n-1})/(n - 4 - 2\sqrt{n-1})$ $(> p_S)$ with $n \geq 11$. For a refined construction, see [72,77]. See also [65,73,74] for the Lepin supercritical range $p > p_L := (n-4)/(n-10)$ $(> p_{JL})$. The critical cases $p = p_{JL}$ and $p = p_L$ were handled in [92, 93]. In [78], the case where $p = p_S$, $n = 3$ and a suitably shrinking $\Omega = \Omega(t)$ was handled. Note that the above type II blow-up solutions are radially symmetric. In the range $p_S < p < p_{JL}$, it was proved in [63] that all radially symmetric blow-up solutions are of type I if either Ω is a ball, or $\Omega = \mathbf{R}^n$ under some assumption on intersection properties; see also [64,75]. For the existence of nonradial type II blow-up solutions, see [16, 19] for some $p > p_{JL}$, [25] for $p = p_2 := (n+1)/(n-3)$ and $n \geq 7$, and [21] for $p = p_{n-3} := 3$ and $5 \leq n \leq 7$. We note that p_2 and p_{n-3} are the so-called second critical exponent (after [22]) and $(n-3)$-th critical exponent (after [21]). They satisfy $p_S < p_2 < p_{JL}$ for $n \geq 4$ and $p_2 \leq p_{n-3}$ for $n \geq 5$, where $p_{JL} := \infty$ for $n \leq 10$. One of the reasons why such exponents appear is explained in [19, Section 1.4].

In the Sobolev critical case $p = p_S$, type II blow-up solutions were formally found by Filippas, Herrero and Velázquez [37] for $3 \leq n \leq 6$ and were rigorously constructed in [91] for $n = 4$. The recent development of the inner-outer gluing method refined the construction; see [26] for $n = 3$, [27,59] for $n = 4$, [23,47] for $n = 5$ and [48] for $n = 6$. On the other hand, it was proved in [17] for $n \geq 7$ that there is no type II blow-up solution if u_0 is close to the Aubin–Talenti function. Recently, it was also shown in [104] that all interior blow-up solutions are of type I provided that $n \geq 7$, $u \geq 0$ and either $\Omega = \mathbf{R}^n$ or Ω is bounded.

Remark 1.8 (Ancient solutions). We say that u is an *ancient solution* if $u_t - \Delta u = |u|^{p-1}u$ for $t \in (-\infty, T)$ with some $T < \infty$. Classification results for such solutions were obtained in [69] for $p < p_S$ and [83] for $p_S < p < p_{JL}$ and $p > p_L$; see also the references given there. In our context, as far as the authors know, the following question is open: Does there exist a nontrivial solution satisfying $\sup_{-\infty<t<T} \|u(\cdot,t)\|_{L^{q_c}(\Omega)} < \infty$ for $p > p_S$?

Remark 1.9 (Infinite time blow-up). Infinite time blow-up (or grow-up) solutions, that is, global-in-time solutions satisfying $\lim_{t\to\infty}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$, were constructed for $p\geq p_S$: see [40] for $p=p_S$, [86] for $p_S<p<p_{JL}$ and [85] for $p\geq p_{JL}$. For $p=p_S$, possible asymptotic behavior was conjectured by Fila and King [36]. Recently, the conjecture was confirmed by [24] for $n=3$, [105] for $n=4$ and [60] for $n=5$. Although this paper focuses on finite time blow-up solutions, it may also be interesting to study the behavior of the critical norm for infinite time blow-up solutions.

Remark 1.10 (Critical norm blow-up for the Navier–Stokes equations). Theorem 1.1 corresponds to the pioneering work of Escauriaza, Seregin and Šverák [35] for the three-dimensional Navier–Stokes equations. They proved the blow-up of the critical norm in the sense that if the maximal existence time T is finite, then

$$\limsup_{t\to T}\|u(\cdot,t)\|_{L^3}=\infty. \tag{1.2}$$

The limit superior condition was later improved to the limit condition in [95]. In the case of domains with boundary, the condition (1.2) was verified for the flat boundary [94] and for general cases [71]. These results were also refined to the limit condition for the flat case [8, 62] and for general cases [2]. On the other hand, the L^3 norm in (1.2) was further refined to the Lorentz norm [82] and the Besov norm [1, 41]. Actually, our norm in Theorem 1.1 can be replaced by the Lorentz norm $L^{q_c,r}$ with $r<\infty$, but we do not pursue this issue here. We also refer to [29, 30] for the critical norm blow-up in higher dimensions $n\geq 4$.

Recently, Tao [98] proved that if $T<\infty$, then

$$\limsup_{t\to T}\frac{\|u(\cdot,t)\|_{L^3}}{(\log\log\log(1/(T-t)))^c}=\infty$$

for some constant $c>0$. See [7, 81] for further developments in this subject. By analogy, it is expected that there is a general quantitative blow-up criterion for the semilinear heat equation (1.1) with $p>p_S$. This direction seems interesting and also challenging. Moreover, it remains an open problem whether the limit superior condition in Theorem 1.1 can be replaced with the limit condition. We note that the result of [76] for $p>p_S$ under the type I blow-up assumption is not a consequence of Theorem 1.1.

Remark 1.11 (Critical norm blow-up for other equations). Wang [102] studied the critical norm of the harmonic map heat flow between compact Riemannian manifolds without boundaries in the energy supercritical dimension $n\geq 3$. He showed that if the maximal existence time T is finite, then

$$\limsup_{t\to T}\|\nabla u(\cdot,t)\|_{L^n}=\infty.$$

One of the key ideas in the proof is the monotonicity formula of Struwe [97].

For nonlinear dispersive equations with power nonlinearities, there are also many works on the blow-up of the critical norm of the form

$$\limsup_{t\to T}\|u(\cdot,t)\|_{\dot{H}^{s_c}}=\infty.$$

Here s_c is the scaling critical exponent for each of the equations. Kenig and Merle [54] showed that blow-up solutions of the cubic defocusing Schrödinger equation in $\mathbf{R}^3$ must satisfy the above condition with $s_c = 1/2$. Their method is based on the concentration compactness procedure and the rigidity theorem. A similar method is applicable to a defocusing supercritical nonlinearity [56]. In the radial case, Merle and Raphaël [67] gave an explicit lower bound of the critical norm in some energy subcritical range. See also a recent result [12] on the lower bound of the critical norm in the radial supercritical case. Similar results were also obtained for the nonlinear wave equation, starting from [55]. For the focusing case, see [33, 34].

Remark 1.12 (Related results for supercritical elliptic equations). In the proof of Theorem 1.1, the blow-up limit $\overline{u}$ obtained is a weak solution of the semilinear heat equation and satisfies a monotonicity estimate. In addition, the singular set of $\overline{u}(\cdot, t)$ consists of finitely many points for each t. A similar situation occurs for the so-called stationary solutions of the semilinear elliptic equation $-\Delta u = |u|^{p-1}u$ for $p > p_S$; see [31, 79, 80, 103]. In this context, a weak solution u of the elliptic equation is called a stationary solution if u is a critical point of the corresponding energy functional with respect to domain variations.

1.3. Organization of the paper

This paper is organized as follows. In Section 2, we derive a key gradient estimate. In Section 3, we define a localized weighted energy and prove its quasi-monotonicity. In Section 4, we prove an ε-regularity theorem by analyzing the energy. In Section 5, we construct and examine a blow-up limit with the aid of ε-regularity, and then we prove Theorem 1.1. In Appendix A, we give regularity estimates used in Section 5. In Appendix B, we recall an Aubin–Lions type compactness result also used in Section 5.

1.4. Notation

For $x \in \mathbf{R}^n$, we often write $x = (x', x_n)$ with $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. Set $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n;\ x' \in \mathbf{R}^{n-1}, x_n > 0\}$. We denote by χ_A and $|A|$ the characteristic function and the Lebesgue measure of a measurable set A, respectively. For $r > 0$ and $(x, t) \in \mathbf{R}^{n+1}$, we write

$$\begin{aligned} &B_r(x) := \{y \in \mathbf{R}^n;\ |x-y| < r\}, && B_r := B_r(0), \\ &\Omega_r(x) := \Omega \cap B_r(x), && \Omega_r := \Omega_r(0), \\ &P_r(x,t) := B_r(x) \times (t - r^2, t), && P_r = P_r(0,0), \\ &Q_r(x,t) := \Omega_r(x) \times (t - r^2, t), && Q_r = Q_r(0,0). \end{aligned}$$

For $\rho > 0$ and $(x, t) \in \mathbf{R}^{n+1}$, we write

$$\begin{aligned} &B^+_\rho(x) := \mathbf{R}^n_+ \cap B_\rho(x), && B^+_\rho := \mathbf{R}^n_+ \cap B_\rho(0), \\ &Q^+_\rho(x,t) := B^+_\rho(x) \times (t - \rho^2, t), && Q^+_\rho := Q^+_\rho(0,0). \end{aligned}$$

We denote by $G_\Omega = G_\Omega(x,y,t)$ the Dirichlet heat kernel in Ω. Set

$$K_{(\tilde{x},\tilde{t})}(x,t) := (\tilde{t}-t)^{-\frac{n}{2}} e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}}$$

for $x, \tilde{x} \in \mathbf{R}^n$ and $t < \tilde{t}$. The critical exponents are defined by

$$p_S := \frac{n+2}{n-2}, \quad q_c := \frac{n(p-1)}{2}, \quad q_* := \frac{n(p-1)}{p+1}.$$

Note that each of the conditions $q_c > p+1$ and $q_* > 2$ is equivalent to $p > p_S$. In what follows, we always assume $n \geq 3$ and $p > p_S$.

2. Gradient estimate

Let $R > 0$ and Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 \in \overline{\Omega}$. As will be seen in Section 5, the proof of Theorem 1.1 is based on the study of the localized problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \Omega_R \times (-1,0), \\ u = 0 & \text{on } (\partial\Omega \cap B_R) \times (-1,0), \\ u \text{ is } C^{2,1} & \text{on } \overline{\Omega_R} \times (-1,0), \end{cases} \tag{2.1}$$

under the assumption that there exists $M > 0$ satisfying

$$\sup_{-1<t<0} \|u(\cdot,t)\|_{L^{q_c}(\Omega_R)} \leq M. \tag{2.2}$$

Here the boundary condition in (2.1) is ignored if $\partial\Omega \cap B_R = \emptyset$.

In this section, we show a gradient estimate in the Lorentz space $L^{q_*,\infty}$ with $q_* := n(p-1)/(p+1)$, which is our key tool to bound a weighted energy defined in Section 3. The method of estimating a Duhamel term from $|u|^{p-1}u$ is based on an idea due to Meyer [70, Theorem 18.1] (see also [99, Proposition 1.5]).

Proposition 2.1. *If u satisfies (2.1) and (2.2), then there exists a constant $C > 0$ depending on R such that*

$$\sup_{-3/4<t<0} \|\nabla u(\cdot,t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} \leq C(M + M^p).$$

Proof. In the spirit of [39, Proposition A.1], we derive a localized integral equation, and then we estimate each of the terms appearing. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \leq \phi \leq 1$ in $\mathbf{R}^n$, $\phi = 0$ in $\mathbf{R}^n \setminus B_{15R/16}$ and $\phi = 1$ in $B_{7R/8}$. Set $v(x,t) := u(x,t)\phi(x)$. Then v belongs to $C^{2,1}(\overline{\Omega} \times (-1,0))$ and satisfies

$$\begin{cases} v_t - \Delta v = \phi|u|^{p-1}u - 2\nabla\phi \cdot \nabla u - u\Delta\phi & \text{in } \Omega \times (-1,0), \\ v = 0 & \text{on } \partial\Omega \times (-1,0). \end{cases}$$

Thus,

$$\begin{aligned} u(x,t) &= \int_\Omega G_\Omega(x,y,t+7/8)\phi(y)u(y,-7/8)\,dy \\ &\quad + \int_{-7/8}^t \int_\Omega G_\Omega(x,y,t-s)\phi(y)|u(y,s)|^{p-1}u(y,s)\,dy\,ds \\ &\quad - \int_{-7/8}^t \int_\Omega G_\Omega(x,y,t-s)(2\nabla\phi\cdot\nabla u + u\Delta\phi)\,dy\,ds \end{aligned} \tag{2.3}$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$, where $G_\Omega = G_\Omega(x,y,t)$ is the Dirichlet heat kernel in Ω. Since $G_\Omega(x,y,t) = 0$ for $y \in \partial\Omega$ and $u\nabla\phi = 0$ on $\partial\Omega$, integrating by parts in the third term on the right-hand side yields

$$\begin{aligned} u(x,t) &= \int_\Omega G_\Omega(x,y,t+7/8)\phi(y)u(y,-7/8)\,dy \\ &\quad + \int_{-7/8}^t \int_\Omega G_\Omega(x,y,t-s)(\phi|u|^{p-1}u + u\Delta\phi)\,dy\,ds \\ &\quad + 2\int_{-7/8}^t \int_\Omega \nabla_y G_\Omega(x,y,t-s)\cdot\nabla\phi(y)u(y,s)\,dy\,ds \end{aligned} \tag{2.4}$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$.

Since Ω is $C^{2+\alpha}$, the following estimate holds (see e.g. [58, Theorem IV.16.3]): There exists a constant $C > 0$ such that

$$|\nabla_x^j G_\Omega(x,y,t)| \le CK_j(x-y,t) \quad (j = 0,1,2) \tag{2.5}$$

for $x, y \in \Omega$ and $0 < t < 1$, where

$$K_j(x,t) := t^{-\frac{n}{2}-\frac{j}{2}} e^{-\frac{|x|^2}{Ct}} \quad (j = 0,1,2). \tag{2.6}$$

Note that the constant in (2.5) and (2.6) depends only on n, Ω and the length of the time interval $(0,1)$. We prepare an estimate of $\partial_{x_i}\partial_{y_j}G_\Omega(x,y,t)$. By the semigroup property and (2.5), we have

$$\begin{aligned} |\partial_{x_i}\partial_{y_j}G_\Omega(x,y,t)| &= \left| \int_\Omega \partial_{x_i}G_\Omega(x,z,t/2)\partial_{y_j}G_\Omega(z,y,t/2)\,dz \right| \\ &\le C\int_\Omega K_1(x-z,t/2)K_1(z-y,t/2)\,dz \\ &\le Ct^{-1}\int_{\mathbf{R}^n} G(x-z,Ct/8)G(z-y,Ct/8)\,dz \\ &= Ct^{-1}G(x-y,Ct/4) \le CK_2(x-y,t), \end{aligned}$$

where $G(x,t) := (4\pi t)^{-n/2}e^{-|x|^2/(4t)}$ and we have changed the constant C in (2.6). Then by differentiating the integral equation (2.4) and using $K_1(x,t) \le K_2(x,t)$ for $x \in \Omega$ and

$0 < t < 1$, we see that

$$\begin{aligned} |\nabla u(x,t)| &\le C \int_{\mathbf{R}^n} K_1(x-y,t+7/8)|u(y,-7/8)|\chi_{\Omega_R}(y)\,dy \\ &\quad + C\int_{-7/8}^{t}\int_{\Omega_R} K_1(x-y,t-s)|u(y,s)|^p\,dy\,ds \\ &\quad + C\int_{-7/8}^{t}\int_{\mathbf{R}^n} K_2(x-y,t-s)|u|\chi_{\overline{\Omega_{15R/16}}\setminus\Omega_{7R/8}}\,dy\,ds \\ &=: CU_1(x,t) + CU_2(x,t) + CU_3(x,t) \end{aligned} \tag{2.7}$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$. Note that each U_i is defined for all $x \in \mathbf{R}^n$ and $-7/8 < t < 0$.

For U_1, from $q_* < q_c$ and the same argument to prove the L^{q_c}-L^{q_c} estimate for the heat semigroup (see [42, Section 1.1.3] for instance), it follows that

$$\begin{aligned} \|U_1(\cdot,t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} &\le C\|U_1(\cdot,t)\|_{L^{q_c}(\mathbf{R}^n)} \\ &\le C(t+7/8)^{-1/2}\|u(\cdot,-7/8)\chi_{\Omega_R}\|_{L^{q_c}(\mathbf{R}^n)} \\ &\le C\|u(\cdot,-7/8)\|_{L^{q_c}(\Omega_R)} \le CM \end{aligned}$$

for $-3/4 < t < 0$. Then

$$\sup_{-3/4<t<0} \|U_1(\cdot,t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} \le CM. \tag{2.8}$$

We next consider U_3. Since $|x-y| \ge R/8$ for $x \in \Omega_{3R/4}$ and $y \in \overline{\Omega_{15R/16}} \setminus \Omega_{7R/8}$, we have

$$K_2(x-y,t-s)\chi_{\overline{\Omega_{15R/16}}\setminus\Omega_{7R/8}}(y) \le C\chi_{\Omega_R}(y)\sup_{s<t}(t-s)^{-\frac{n}{2}-1}e^{-\frac{R^2}{C(t-s)}} \le C\chi_{\Omega_R}(y)$$

for $x \in \Omega_{3R/4}$, $y \in \mathbf{R}^n$ and $-7/8 < s < t < 0$. Therefore,

$$U_3(x,t) \le C\int_{-7/8}^{t}\int_{\Omega_R}|u|\,dy\,ds \le CM$$

for $x \in \Omega_{3R/4}$ and $-7/8 < t < 0$. Thus,

$$\sup_{-3/4<t<0} \|U_3(\cdot,t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} \le CM. \tag{2.9}$$

We now estimate U_2. This part is a modification of [70, proof of Theorem 18.1]. Let $-7/8 < t < 0$. By a change of variables, we have

$$\begin{aligned} &U_2(x,t) \le \tilde{U}(x;t), \quad \tilde{U} = \tilde{U}(x;t) := \int_0^\infty S(x,s;t)\,ds, \\ &S = S(x,s;t) := \chi_{(0,t+7/8)}(s)\int_{\Omega_R} K_1(x-y,s)|u(y,t-s)|^p\,dy. \end{aligned}$$

For $\lambda > 0$ and $\tau > 0$, define $E := \{x \in \Omega_R;\ \tilde{U}(x) > \lambda\}$ and

$$\tilde{U}(x) = \left(\int_0^\tau + \int_\tau^\infty\right) S(x,s)\,ds =: \tilde{U}_1(x) + \tilde{U}_2(x).$$

We will estimate the Lebesgue measure $|E|$ of E. By the argument used to prove the $L^{q_c/p}$-L^∞ estimate for the heat semigroup, we have

$$\begin{aligned}\|S(\cdot,s)\|_{L^\infty(\Omega_R)} &\le Cs^{-\frac{np}{2q_c}-\frac{1}{2}}\chi_{(0,t+7/8)}(s)\big\||u(\cdot,t-s)|^p\big\|_{L^{q_c/p}(\Omega_R)}\\ &\le CM^p s^{-\frac{np}{2q_c}-\frac{1}{2}}\end{aligned}$$

for any $s > 0$, where $C > 0$ is independent of t. Then

$$\tilde{U}_2(x) \le \int_\tau^\infty S(x,s)\,ds \le CM^p\int_\tau^\infty s^{-\frac{np}{2q_c}-\frac{1}{2}}\,ds = C'M^p\tau^{-\frac{p+1}{2(p-1)}},$$

where $C' > 0$ is also independent of t. For $\lambda > 0$, we choose τ such that

$$C'M^p\tau^{-\frac{p+1}{2(p-1)}} = \lambda/2. \tag{2.10}$$

Then $\tilde{U}_2 \le \lambda/2$. By setting $E_1 := \{x \in \Omega_R;\ \tilde{U}_1(x) > \lambda/2\}$ and using $\tilde{U}_2 \le \lambda/2$ and $\tilde{U} = \tilde{U}_1 + \tilde{U}_2$, we see that $E \subset E_1$.

From the argument used to prove the $L^{q_c/p}$-$L^{q_c/p}$ estimate, it follows that

$$\begin{aligned}\|S(\cdot,s)\|_{L^{q_c/p,\infty}(\Omega_R)} &\le \|S(\cdot,s)\|_{L^{q_c/p}(\Omega_R)}\\ &\le Cs^{-1/2}\chi_{(0,t+7/8)}(s)\||u(\cdot,t-s)|^p\|_{L^{q_c/p}(\Omega_R)} \le CM^p s^{-1/2}\end{aligned}$$

for any $s > 0$. Thus,

$$\|\tilde{U}_1\|_{L^{q_c/p,\infty}(\Omega_R)} \le \int_0^\tau \|S(\cdot,s)\|_{L^{q_c/p,\infty}(\Omega_R)}ds \le CM^p\tau^{1/2}.$$

This together with the Hölder inequality for the Lorentz spaces (see [57, Proposition 2.1] for instance) shows that

$$\int_{E_1}\tilde{U}_1(x)\,dx \le C\|\chi_{E_1}\|_{L^{\frac{q_c}{q_c-p},1}(\Omega_R)}\|\tilde{U}_1\|_{L^{q_c/p,\infty}(\Omega_R)} \le CM^p|E_1|^{1-p/q_c}\tau^{1/2}.$$

On the other hand, the definition of E_1 gives $\int_{E_1}\tilde{U}_1(x)\,dx \ge (\lambda/2)|E_1|$. From $E \subset E_1$ and (2.10), we obtain

$$|E| \le |E_1| \le C\lambda^{-q_c/p}M^{q_c}\tau^{q_c/2p} = CM^{pq_*}\lambda^{-q_*},$$

and so $\lambda|\{x \in \Omega_R;\ \tilde{U}(x) > \lambda\}|^{1/q_*} \le CM^p$ for $\lambda > 0$. This implies $\|\tilde{U}(\cdot;t)\|_{L^{q_*,\infty}(\Omega_R)} \le CM^p$ for $-7/8 < t < 0$, where $C > 0$ is independent of t. Hence by the definition of $\tilde{U}$ and M^p, we see that

$$\sup_{-3/4<t<0}\|U_2(\cdot,t)\|_{L^{q_*,\infty}(\Omega_{3R/4})} \le \sup_{-3/4<t<0}\|\tilde{U}(\cdot;t)\|_{L^{q_*,\infty}(\Omega_R)} \le CM^p.$$

Combining this inequality and (2.7)–(2.9), we obtain the desired inequality. ■

3. Localized weighted energy

Let u be a solution of (2.1) satisfying the bound (2.2). In this section, we define a localized weighted energy of u analogously to Giga, Matsui and Sasayama [45, 46] and prove its quasi-monotonicity without assuming the convexity of Ω. Our computations to prove quasi-monotonicity are in the spirit of Chou and Du [14], but the details are different. From among the results in this section, we will refer only to Lemma 3.1 and Proposition 3.2 in the subsequent sections.

Let $n \geq 3$, $p > p_S$ and Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 \in \bar{\Omega}$. We fix $R > 0$ satisfying one of the following conditions:

$$\text{If } 0 \in \Omega, \text{ then } 0 < R < 1/2 \text{ is such that } \overline{B_R} \subset \Omega. \tag{3.1}$$

$$\begin{cases} \text{If } \partial\Omega \neq \emptyset \text{ and } 0 \in \partial\Omega, \text{ then } 0 < R < 1/2 \text{ is such that} \\ \text{there exists } f \in C_0^{2+\alpha}(\mathbf{R}^{n-1}) \text{ satisfying } f(0) = 0,\ \nabla' f(0) = 0, \\ \|\nabla' f\|_{L^\infty(\mathbf{R}^{n-1})} \leq 1/2 \text{ and } \Omega_R = \{x \in B_R;\ x_n > f(x')\} \\ \text{by relabeling and reorienting the coordinate axes if necessary.} \end{cases} \tag{3.2}$$

Here $\nabla' f$ is the gradient on $\mathbf{R}^{n-1}$. Note that the existence of f in (3.2) is guaranteed by the smoothness of Ω.

3.1. Definition and change of variables

We define a localized weighted energy E and show its boundedness by using Proposition 2.1. To obtain quasi-monotonicity, we locally straighten the boundary. After that, we introduce backward similarity variables and derive the corresponding representation of E.

Let $\varphi \in C^\infty([0,\infty))$ satisfy $\varphi(z) = 1$ for $0 \leq z \leq 1/2$, $0 < \varphi(z) < 1$ for $1/2 < z < 1$, $\varphi(z) = 0$ for $z \geq 1$ and $\varphi'(z) \leq 0$ for $z \geq 0$. For $x, \tilde{x} \in \mathbf{R}^n$, $\tilde{t} > t$ and $r > 0$, we set $\phi_r = \phi_{\tilde{x},r}(x) := \varphi(|x - \tilde{x}|/r)$ and

$$K = K_{(\tilde{x},\tilde{t})}(x,t) := (\tilde{t} - t)^{-\frac{n}{2}} e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}}.$$

For $\tilde{x} \in \overline{\Omega_{R/4}}$ and $-1 < t < \tilde{t} \leq 0$, define a localized weighted energy by

$$\begin{aligned} E(t) &= E_{(\tilde{x},\tilde{t})}(t; \phi_{\tilde{x},R/4}) \\ &:= (\tilde{t} - t)^{\frac{p+1}{p-1}} \int_{\Omega_R} \left(\frac{|\nabla u(x,t)|^2}{2} - \frac{|u(x,t)|^{p+1}}{p+1} + \frac{|u(x,t)|^2}{2(p-1)(\tilde{t}-t)} \right) \\ &\qquad\qquad \times K_{(\tilde{x},\tilde{t})}(x,t)\phi_{\tilde{x},R/4}^2(x)\,dx. \end{aligned} \tag{3.3}$$

Note that u is defined on $(\overline{\Omega_R} \cap B_R) \times (-1, 0)$, but we mainly consider the time interval $(-1/2, 0)$ to apply Proposition 2.1. Observe that

$$\operatorname{supp} \phi_{\tilde{x},R/4} \subset B_{R/2} \quad \text{for } \tilde{x} \in \overline{\Omega_{R/4}}. \tag{3.4}$$

The following lemma guarantees the boundedness of E.

Lemma 3.1. *There exists* $C > 0$ *such that*

$$|E_{(\tilde{x},\tilde{t})}(t;\phi_{\tilde{x},R/4})| \le C(M + M^p)^2$$

for any $\tilde{x} \in \overline{\Omega_{R/4}}$ *and* $-1/2 < t < \tilde{t} \le 0$.

Proof. From the Hölder inequality for Lorentz spaces (see [57, Proposition 2.1]), (3.4), Proposition 2.1 and a direct computation, it follows that

$$\begin{aligned}
&\int_{\Omega_R} |\nabla u|^2 K_{(\tilde{x},\tilde{t})}\phi^2_{\tilde{x},R/4}\,dx \\
&\qquad \le C(\tilde{t}-t)^{-n/2}\|\nabla u(\cdot,t)\|^2_{L^{q_*,\infty}(\Omega_{R/2})}\Big\|e^{-\frac{|\cdot-\tilde{x}|^2}{8(\tilde{t}-t)}}\Big\|^2_{L^{\frac{2q_*}{q_*-2},2}(\mathbf{R}^n)} \\
&\qquad \le C(M+M^p)^2(\tilde{t}-t)^{-\frac{p+1}{p-1}}
\end{aligned} \tag{3.5}$$

for $-1/2 < t < \tilde{t} \le 0$. The Hölder inequality and (2.2) show that

$$\begin{aligned}
\int_{\Omega_R} |u|^{p+1} K_{(\tilde{x},\tilde{t})}\phi^2_{\tilde{x},R/4}\,dx &\le (\tilde{t}-t)^{-n/2}\|u\|^{p+1}_{L^{q_c}(\Omega_{R/2})}\left(\int_{\mathbf{R}^n} e^{-\frac{|x-\tilde{x}|^2}{C(\tilde{t}-t)}}\,dx\right)^{1-\frac{p+1}{q_c}} \\
&\le CM^{p+1}(\tilde{t}-t)^{-\frac{p+1}{p-1}}, \\
\int_{\Omega_R} |u|^2 K_{(\tilde{x},\tilde{t})}\phi^2_{\tilde{x},R/4}\,dx &\le CM^2(\tilde{t}-t)^{-\frac{2}{p-1}},
\end{aligned}$$

for $-1/2 < t < \tilde{t} \le 0$. The lemma follows from the above estimates. ∎

We locally straighten the boundary. For (3.2), we define $C^{2+\alpha}$ maps $\Phi = (\Phi_1, \dots, \Phi_n)$ and $\Psi = (\Psi_1, \dots, \Psi_n)$ by

$$\begin{cases} \xi_i = x_i =: \Phi_i(x), \\ \xi_n = x_n - f(x') =: \Phi_n(x), \end{cases} \qquad \begin{cases} x_i = \xi_i =: \Psi_i(\xi), \\ x_n = \xi_n + f(\xi') =: \Psi_n(\xi), \end{cases}$$

for $i = 1, \dots, n-1$. To handle the case (3.1) in a unified way, we also set

$$\begin{cases} \xi_i = x_i =: \Phi_i(x), \\ \xi_n = x_n =: \Phi_n(x), \end{cases} \qquad \begin{cases} x_i = \xi_i =: \Psi_i(\xi), \\ x_n = \xi_n =: \Psi_n(\xi), \end{cases}$$

for (3.1). We note that the maps Φ and Ψ for (3.1) are identity maps and they are obtained by setting $f \equiv 0$ in the definitions of Φ and Ψ for (3.2).

We write $\Phi(x) = \xi$ and $\Psi(\xi) = x$. Set

$$\hat{u}(\xi,t) := u(\Psi(\xi),t). \tag{3.6}$$

Then direct computations show that

$$\begin{aligned}
\nabla_x u(x,t) &= \big(\partial_{\xi_1}\hat{u} - (\partial_{\xi_n}\hat{u})\partial_{\xi_1} f, \dots, \partial_{\xi_{n-1}}\hat{u} - (\partial_{\xi_n}\hat{u})\partial_{\xi_{n-1}} f, \partial_{\xi_n}\hat{u}\big) \\
&= (\nabla'\hat{u} - (\partial_{\xi_n}\hat{u})\nabla' f, \partial_{\xi_n}\hat{u})
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\Delta_x u(x,t) &= \Delta_\xi \hat u - 2\sum_{i=1}^{n-1}(\partial_{\xi_i}\partial_{\xi_n}\hat u)\partial_{\xi_i} f + \partial_{\xi_n}^2 \hat u \sum_{i=1}^{n-1}(\partial_{\xi_i} f)^2 - \partial_{\xi_n}\hat u \sum_{i=1}^{n-1}\partial_{\xi_i}^2 f\\ &= \Delta_\xi \hat u - 2\nabla'_\xi(\partial_{\xi_n}\hat u)\cdot \nabla'_\xi f + (\partial_{\xi_n}^2\hat u)|\nabla'_\xi f|^2 - (\partial_{\xi_n}\hat u)\Delta'_\xi f,\end{aligned}$$

where ∇' and Δ' are the gradient and Laplacian on $\mathbf{R}^{n-1}$ with respect to the first $n-1$ components, respectively. Note that the terms on the right-hand sides are evaluated at $(\xi,t)=(\Phi(x),t)$. Since u satisfies (2.1), we see that $\hat u$ satisfies

$$\begin{cases} \hat u_t - \hat A \hat u = |\hat u|^{p-1}\hat u & \text{in } \Phi(\Omega_R)\times(-1,0),\\ \hat u = 0 & \text{on } \Phi(\partial\Omega\cap B_R)\times(-1,0). \end{cases} \tag{3.8}$$

Here, abbreviating the subscripts ξ and ξ_n, we set

$$\hat A \hat u := \Delta \hat u - 2\nabla'(\partial_n \hat u)\cdot\nabla' f + (\partial_n^2 \hat u)|\nabla' f|^2 - (\partial_n \hat u)\Delta' f.$$

For $\tilde x \in \overline{\Omega_{R/4}}$, we write $\tilde\xi := \Phi(\tilde x)$. Note that $\tilde x = \Psi(\tilde \xi)$. We perform the change of variables $x=\Psi(\xi)$ in (3.3) by using the relation (3.7), the definition of K and the fact that the Jacobian determinant equals 1 from the definition of Ψ, that is, $dx = d\xi$. Then E satisfies

$$\begin{aligned} E(t) = (\tilde t - t)^{\frac{p+1}{p-1}-\frac{n}{2}} \int_{\Phi(\Omega_R)} &\left(\frac{|\widehat\nabla \hat u|^2}{2} - \frac{|\hat u|^{p+1}}{p+1} + \frac{\hat u^2}{2(p-1)(\tilde t - t)}\right)\\ &\times e^{-\frac{|\Psi(\xi)-\Psi(\tilde\xi)|^2}{4(\tilde t - t)}}\phi^2_{\Psi(\tilde\xi),R/4}(\Psi(\xi))\,d\xi, \end{aligned} \tag{3.9}$$

where $\widehat\nabla \hat u(\xi,t) := (\nabla'\hat u - (\partial_n \hat u)\nabla' f, \partial_n \hat u)$.

We introduce the backward similarity variables

$$\eta := \frac{\xi-\tilde\xi}{(\tilde t - t)^{1/2}}, \qquad \tau := -\log(\tilde t - t).$$

Then the rescaled functions are given by

$$w(\eta,\tau) := e^{-\frac{1}{p-1}\tau}\hat u(\tilde \xi + e^{-\tau/2}\eta, \tilde t - e^{-\tau}), \tag{3.10}$$

$$g(\eta',\tau) := e^{\tau/2} f(\tilde\xi' + e^{-\tau/2}\eta'). \tag{3.11}$$

Note that

$$\begin{aligned} &\xi = \tilde\xi + e^{-\tau/2}\eta, \quad \tilde t - t = e^{-\tau},\\ &\hat u(\xi,t) = (\tilde t - t)^{-\frac{1}{p-1}} w\big((\tilde t - t)^{-1/2}(\xi-\tilde\xi), -\log(\tilde t - t)\big). \end{aligned}$$

Since $\hat u$ satisfies (3.8), we see that w solves

$$\begin{cases} w_\tau + \dfrac{1}{2}\eta\cdot\nabla w + \dfrac{1}{p-1}w - Aw - |w|^{p-1}w = 0,\\ \qquad\qquad \eta\in\Omega(\tau), \tau\in(-\log(\tilde t + 1/2),\infty),\\ w = 0, \quad \eta \in e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde\xi), \end{cases} \tag{3.12}$$

where the time interval $(-\log(\tilde{t}+1),\infty)$ has been shortened to $(-\log(\tilde{t}+1/2),\infty)$ in order to be able to use Proposition 2.1 safely, and

$$Aw := \Delta w - 2\nabla'(\partial_n w)\cdot\nabla' g + (\partial_n^2 w)|\nabla' g|^2 - (\partial_n w)\Delta' g,$$

abbreviating the subscripts η and η_n. In addition,

$$\Omega(\tau) := \{\eta\in\mathbf{R}^n;\ \tilde{\xi}+e^{-\tau/2}\eta\in\Phi(\Omega_R)\} = e^{\tau/2}(\Phi(\Omega_R)-\tilde{\xi}).$$

By using the backward similarity variables, (3.9) can be written as

$$\begin{aligned} E(t) = \int_{\Omega(\tau)} \left(\frac{|\widehat{\nabla} w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right) \exp\left(-\frac{e^{\tau}}{4}|\Psi(\tilde{\xi}+e^{-\tau/2}\eta)-\Psi(\tilde{\xi})|^2\right) \\ \times\varphi^2\left(\frac{4}{R}|\Psi(\tilde{\xi}+e^{-\tau/2}\eta)-\Psi(\tilde{\xi})|\right)d\eta, \end{aligned}$$

where $\widehat{\nabla} w := (\nabla' w - (\partial_n w)\nabla' g, \partial_n w)$. We observe that

$$\begin{aligned} &|\Psi(\tilde{\xi}+e^{-\tau/2}\eta)-\Psi(\tilde{\xi})|^2 \\ &\qquad = e^{-\tau}|(\eta', \eta_n + g(\eta',\tau)-g(0,\tau))|^2 \\ &\qquad = e^{-\tau}\big(|\eta|^2 + 2(g(\eta',\tau)-g(0,\tau))\eta_n + (g(\eta',\tau)-g(0,\tau))^2\big). \end{aligned} \tag{3.13}$$

Then by setting

$$\mathcal{E}(\tau) := \int_{\Omega(\tau)}\left(\frac{|\widehat{\nabla} w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right)\rho\psi^2\,d\eta, \tag{3.14}$$

$$\begin{aligned} \rho = \rho(\eta,\tau) := \exp\big(-\tfrac{1}{4}(|\eta|^2 + 2(g(\eta',\tau)-g(0,\tau))\eta_n \\ + (g(\eta',\tau)-g(0,\tau))^2)\big), \end{aligned} \tag{3.15}$$

$$\psi = \psi(\eta,\tau) := \varphi\left(\frac{4}{R}e^{-\tau/2}|(\eta',\eta_n+g(\eta',\tau)-g(0,\tau))|\right), \tag{3.16}$$

we see that $E(t)=\mathcal{E}(\tau)$ with $\tau=-\log(\tilde{t}-t)$. By direct computations, we note that

$$\begin{cases} \partial_i\rho = -\frac{1}{2}(\eta_i + (\eta_n + g(\eta',\tau)-g(0,\tau))\partial_i g)\rho \quad (i=1,\dots,n-1), \\ \partial_i^2\rho = -\frac{1}{2}(1+(\partial_i g)^2 + (\eta_n+g(\eta',\tau)-g(0,\tau))\partial_i^2 g)\rho \\ \qquad + \frac{1}{4}(\eta_i^2 + 2(\eta_n+g(\eta',\tau)-g(0,\tau))\eta_i\partial_i g)\rho \\ \qquad + \frac{1}{4}(\eta_n+g(\eta',\tau)-g(0,\tau))^2(\partial_i g)^2\rho \quad (i=1,\dots,n-1), \\ \partial_n\rho = -\frac{1}{2}(\eta_n+g(\eta',\tau)-g(0,\tau))\rho, \\ \partial_n^2\rho = -\frac{1}{2}\rho + \frac{1}{4}(\eta_n+g(\eta',\tau)-g(0,\tau))^2\rho \end{cases} \tag{3.17}$$

and that

$$\rho_\tau = -\tfrac{1}{2}\partial_\tau(g(\eta',\tau)-g(0,\tau))(\eta_n+g(\eta',\tau)-g(0,\tau))\rho. \tag{3.18}$$

3.2. *Quasi-monotonicity*

We prove the quasi-monotonicity of E. This property plays a crucial role in the proof of the ε-regularity theorem and also in the blow-up analysis.

Proposition 3.2. *Fix $R > 0$ such that either (3.1) or (3.2) holds. Let u be a solution of (2.1) satisfying (2.2). Then there exists a constant $C > 0$ depending only on n, p, Ω and R such that*

$$\begin{aligned}
&E_{(\tilde{x},\tilde{t})}(t;\phi_{\tilde{x},R/4}) + \frac{1}{2}\int_{t'}^{t} (\tilde{t}-s)^{\frac{2}{p-1}-\frac{n}{2}-1} \\
&\quad\times \int_{\Phi(\Omega_R)} \left| \frac{\hat{u}}{p-1} + \frac{(\xi - \Phi(\tilde{x}))\cdot\nabla\hat{u}}{2} - (\tilde{t}-s)\hat{u}_s \right|^2 e^{-\frac{|\Psi(\xi)-\tilde{x}|^2}{4(\tilde{t}-s)}} \phi_{\tilde{x},R/4}^2(\Psi(\xi))\, d\xi\, ds \\
&\qquad\qquad \le E_{(\tilde{x},\tilde{t})}(t';\phi_{\tilde{x},R/4}) + C(M+M^p)^2(\tilde{t}-t')^{1/2} \qquad (3.19)
\end{aligned}$$

for any $\tilde{x} \in \overline{\Omega_{R/4}}$ and $-1/2 < t' < t < \tilde{t} \le 0$.

To prove this, we compute the derivative of $\mathcal{E}$.

Lemma 3.3. *The derivative of $\mathcal{E}(\tau)$ satisfies*

$$\frac{d}{d\tau}\mathcal{E}(\tau) = -\int w_\tau^2 \rho\psi^2 - \mathcal{B}(\tau) + \mathcal{R}(\tau), \qquad (3.20)$$

where

$$\mathcal{B}(\tau) := \frac{1}{4}\int_{e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})} (\partial_\nu w)^2\rho\psi^2(\nu\cdot\eta)(1-2\nu_n\partial_{\nu'}g + |\nabla' g|^2\nu_n^2)\, dS(\eta),$$

$\nu = (\nu', \nu_n)$ is the outward unit normal, $\partial_\nu w := \nabla w\cdot\nu$, $\partial_{\nu'}g := \nabla' g\cdot\nu'$, dS is the surface area element and

$$\begin{aligned}
\mathcal{R}(\tau) := &\int \frac{w^2}{p-1}\rho\psi\psi_\tau - \int \frac{2|w|^{p+1}}{p+1}\rho\psi\psi_\tau - 2\int w_\tau\rho\psi\nabla w\cdot\nabla\psi \\
&+ \int |\nabla w|^2\rho\psi\psi_\tau + 2\int w_\tau\rho\psi(\partial_n\psi)\nabla' w\cdot\nabla' g \\
&- 2\int(\partial_n w)\rho\psi\psi_\tau\nabla' w\cdot\nabla' g + 2\int w_\tau(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g \\
&- 2\int w_\tau(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2 + \int(\partial_n w)^2\rho\psi\psi_\tau|\nabla' g|^2 \\
&- \int(\partial_n w)\rho\psi^2\nabla' w\cdot\nabla' g_\tau + \int(\partial_n w)^2\rho\psi^2\nabla' g\cdot\nabla' g_\tau \\
&+ \frac{1}{2}\int w_\tau(\partial_n w)\rho\psi^2(g(\eta',\tau) - g(0,\tau) - \eta'\cdot\nabla' g) \\
&+ \int\left(\frac{|\widehat{\nabla} w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right)\rho_\tau\psi^2
\end{aligned}$$

with the abbreviation $\int\ldots = \int_{\Omega(\tau)}\ldots\, d\eta$.

Proof. In what follows, we will perform integration by parts several times, and so we need to know the boundary value of $w\psi$. We claim that

$$w\psi = 0 \quad \text{on } \partial\Omega(\tau). \tag{3.21}$$

To prove this, we note that

$$\partial\Omega(\tau) = e^{\tau/2}(\Phi(\partial\Omega \cap B_R) - \tilde{\xi}) \cup e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \tilde{\xi}).$$

For $\eta \in e^{\tau/2}(\Phi(\partial\Omega \cap B_R) - \tilde{\xi})$, by the boundary condition in (3.12), we obtain $w(\eta,\tau) = 0$. On the other hand, for $\eta \in e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \tilde{\xi})$, we have $\Phi(x) = \xi = e^{-\tau/2}\eta + \tilde{\xi} \in \Phi(\Omega \cap \partial B_R)$. Thus $x \in \Omega \cap \partial B_R$ and $|x| = R$. This together with $\tilde{x} \in \overline{\Omega_{R/4}}$ gives $|x - \tilde{x}| \geq 3R/4$. Therefore $\psi(\eta,\tau) = \varphi(4|x-\tilde{x}|/R) = 0$ for η in $e^{\tau/2}(\Phi(\Omega \cap \partial B_R) - \tilde{\xi})$. Hence (3.21) holds.

For simplicity, we write $\int \ldots = \int_{\Omega(\tau)} \ldots \, d\eta$ when no confusion can arise. By (3.21), we see that

$$\begin{aligned}
&\frac{d}{d\tau}\int \left(-\frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right)\rho\psi^2 \\
&\quad = \int \frac{ww_\tau}{p-1}\rho\psi^2 - \int ww_\tau|w|^{p-1}\rho\psi^2 + \int \left(-\frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right)\rho_\tau\psi^2 + \mathcal{R}_0,
\end{aligned} \tag{3.22}$$

where

$$\mathcal{R}_0 := \int \frac{w^2}{p-1}\rho\psi\psi_\tau - \int \frac{2|w|^{p+1}}{p+1}\rho\psi\psi_\tau. \tag{3.23}$$

On the other hand, by taking into account the computation

$$\begin{aligned}
|\widehat{\nabla} w|^2 &= |\nabla w - (\partial_n w \nabla' g, 0)|^2 \\
&= |\nabla w|^2 - 2(\partial_n w)\nabla' w \cdot \nabla' g + (\partial_n w)^2 |\nabla' g|^2,
\end{aligned} \tag{3.24}$$

we set

$$\begin{aligned}
\frac{d}{d\tau}\int \frac{|\widehat{\nabla} w|^2}{2}\rho\psi^2 &= \frac{1}{2}\frac{d}{d\tau}\int |\nabla w|^2\rho\psi^2 - \frac{d}{d\tau}\int (\partial_n w)\rho\psi^2\nabla' w \cdot \nabla' g \\
&\quad + \frac{1}{2}\frac{d}{d\tau}\int (\partial_n w)^2\rho\psi^2|\nabla' g|^2 \\
&=: \frac{1}{2}\frac{dI_1}{d\tau} - \frac{dI_2}{d\tau} + \frac{1}{2}\frac{dI_3}{d\tau}.
\end{aligned} \tag{3.25}$$

We compute the derivatives of I_1, I_2 and I_3 in the following way:

(1) If a term contains the derivative of ψ, keep the term as it is.

(2) If a term contains spatial derivative(s) of w_τ, perform integration by parts to remove the spatial derivative(s).

(3) If a term does not contain w_τ but contains second order spatial derivatives of w, perform integration by parts to lower the order.

For I_1, integration by parts and (3.21) show that

$$\begin{aligned}\frac{1}{2}\frac{dI_1}{d\tau} &= -\frac{d}{d\tau}\left(\int (w\psi\nabla w\cdot\nabla\psi)\rho + \frac{1}{2}\int w\psi^2\nabla\cdot(\rho\nabla w)\right)\\ &= -\int (w\psi\nabla w\cdot\nabla\psi)_\tau\rho - \frac{1}{2}\int w_\tau\psi^2\nabla\cdot(\rho\nabla w)\\ &\quad -\int w\psi\psi_\tau\nabla\cdot(\rho\nabla w) - \frac{1}{2}\int w\psi^2\nabla\cdot(\rho\nabla w_\tau)\\ &\quad -\frac{1}{2}\int w\psi^2\nabla\cdot(\rho_\tau\nabla w) - \int w\rho_\tau\psi\nabla w\cdot\nabla\psi.\end{aligned}$$

Integrating by parts twice and using (3.21) yields

$$\begin{aligned}-\frac{1}{2}\int w\psi^2\nabla\cdot(\rho\nabla w_\tau) &= -\frac{1}{2}\int w_\tau\psi^2\nabla\cdot(\rho\nabla w) - \int w_\tau\rho\psi\nabla w\cdot\nabla\psi\\ &\quad + \int w\rho\psi\nabla w_\tau\cdot\nabla\psi + \frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau(\partial_\nu w)\rho\psi^2\,dS.\end{aligned}$$

In addition, we see that

$$-\frac{1}{2}\int w\psi^2\nabla\cdot(\rho_\tau\nabla w) = \frac{1}{2}\int |\nabla w|^2\rho_\tau\psi^2 + \int w\rho_\tau\psi\nabla w\cdot\nabla\psi.$$

The above computations imply that

$$\begin{aligned}\frac{1}{2}\frac{dI_1}{d\tau} &= -\int w_\tau\psi^2\nabla\cdot(\rho\nabla w) + \frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau\rho\psi^2\nabla w\cdot\nu\,dS\\ &\quad + \frac{1}{2}\int |\nabla w|^2\rho_\tau\psi^2 + \mathcal{R}_1,\end{aligned}\tag{3.26}$$

where

$$\begin{aligned}\mathcal{R}_1 := &-\int (w\psi\nabla w\cdot\nabla\psi)_\tau\rho - \int w\psi\psi_\tau\nabla\cdot(\rho\nabla w)\\ &-\int w_\tau\rho\psi\nabla w\cdot\nabla\psi + \int w\rho\psi\nabla w_\tau\cdot\nabla\psi.\end{aligned}$$

Expanding the first term and integrating by parts in the second term, we obtain

$$\mathcal{R}_1 = -2\int w_\tau\rho\psi\nabla w\cdot\nabla\psi + \int |\nabla w|^2\rho\psi\psi_\tau. \tag{3.27}$$

By the argument of [43, Proposition 2.1] with $\Phi(\partial\Omega\cap B_R)\subset\mathbf{R}^n_+$, we can see that

$$\frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau(\partial_\nu w)\rho\psi^2\,dS = -\frac{1}{4}\int_{e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})} (\partial_\nu w)^2\rho\psi^2(\nu\cdot\eta)\,dS. \tag{3.28}$$

Indeed, from the boundary conditions in (3.8) and (3.12), it follows that

$$0 = \hat{u}_t = (\tilde{t}-t)^{-\frac{1}{p-1}-1}\left(\nabla w\cdot\frac{y}{2} + w_\tau\right),$$

and so $w_\tau = -\nabla w\cdot(\eta/2)$ on $e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})$. By the boundary condition in (3.12), we also have $\nabla w = (\nabla w\cdot\nu)\nu = (\partial_\nu w)\nu$. Thus,

$$w_\tau = -\tfrac{1}{2}(\partial_\nu w)(\nu\cdot\eta), \quad \eta\in e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi}). \tag{3.29}$$

Recall that $\psi(\eta,\tau)=0$ for $\eta\in e^{\tau/2}(\Phi(\Omega\cap\partial B_R)-\tilde{\xi})$ by the proof of (3.21). Then (3.28) follows. For later use, we note that, on the boundary,

$$\nabla' w=(\partial_\nu w)\nu',\quad \partial_n w=(\partial_\nu w)\nu_n, \tag{3.30}$$

which follows from $\nabla w=(\partial_\nu w)\nu$.

We next consider I_2. Since

$$-I_2=\int w\psi^2\nabla'(\rho\partial_n w)\cdot\nabla' g+2\int w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g+\int w(\partial_n w)\rho\psi^2\Delta' g,$$

we have

$$\begin{aligned}-\frac{dI_2}{d\tau}&=\int w_\tau\psi^2\nabla'(\rho\partial_n w)\cdot\nabla' g+\int w_\tau(\partial_n w)\rho\psi^2\Delta' g\\&\quad+\int w\psi^2\nabla'(\rho\partial_n w_\tau)\cdot\nabla' g+\int w(\partial_n w_\tau)\rho\psi^2\Delta' g\\&\quad+\int w\psi^2\nabla'(\rho_\tau\partial_n w)\cdot\nabla' g+\int w(\partial_n w)\rho_\tau\psi^2\Delta' g\\&\quad+\int w\psi^2\nabla'(\rho\partial_n w)\cdot\nabla' g_\tau+\int w(\partial_n w)\rho\psi^2\Delta' g_\tau\\&\quad+2\int w\psi\psi_\tau\nabla'(\rho\partial_n w)\cdot\nabla' g+2\int (w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g)_\tau\\&\quad+2\int w(\partial_n w)\rho\psi\psi_\tau\Delta' g,\end{aligned}$$

where the 8th term on the right-hand side requires the 3rd derivative of $f\in C_0^{2+\alpha}(\mathbf{R}^{n-1})$. But the computations here and below can be justified by the standard approximation procedure. Again by integrating by parts twice, we can see that

$$\begin{aligned}&\int w\psi^2\nabla'(\rho\partial_n w_\tau)\cdot\nabla' g+\int w(\partial_n w_\tau)\rho\psi^2\Delta' g\\&\quad=\int w_\tau\psi^2\partial_n(\rho\nabla' w)\cdot\nabla' g-\int_{\partial\Omega(\tau)} w_\tau\rho\psi^2\nu_n\nabla' w\cdot\nabla' g\,dS\\&\quad\quad-2\int w(\partial_n w_\tau)\rho\psi\nabla'\psi\cdot\nabla' g+2\int w_\tau\rho\psi(\partial_n\psi)\nabla' w\cdot\nabla' g.\end{aligned}\tag{3.31}$$

Moreover, we have

$$\begin{aligned}&\int w\psi^2\nabla'(\rho_\tau\partial_n w)\cdot\nabla' g+\int w(\partial_n w)\rho_\tau\psi^2\Delta' g\\&\qquad+\int w\psi^2\nabla'(\rho\partial_n w)\cdot\nabla' g_\tau+\int w(\partial_n w)\rho\psi^2\Delta' g_\tau\\&\quad=-\int(\partial_n w)\rho_\tau\psi^2\nabla' w\cdot\nabla' g-2\int w(\partial_n w)\rho_\tau\psi\nabla'\psi\cdot\nabla' g\\&\quad\quad-\int(\partial_n w)\rho\psi^2\nabla' w\cdot\nabla' g_\tau-2\int w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g_\tau.\end{aligned}$$

These computations show that

$$
\begin{aligned}
-\frac{dI_2}{d\tau} &= \int w_\tau \psi^2 \nabla'(\rho\partial_n w)\cdot\nabla' g + \int w_\tau \psi^2 \partial_n(\rho\nabla' w)\cdot\nabla' g \\
&\quad + \int w_\tau(\partial_n w)\rho\psi^2\Delta' g - \int_{\partial\Omega(\tau)} w_\tau\rho\psi^2\nu_n\nabla' w\cdot\nabla' g\, dS \\
&\quad - \int (\partial_n w)\rho_\tau\psi^2\nabla' w\cdot\nabla' g + \mathcal{R}_2,
\end{aligned}
\tag{3.32}
$$

where

$$
\begin{aligned}
\mathcal{R}_2 := &-2\int w(\partial_n w_\tau)\rho\psi\nabla'\psi\cdot\nabla' g + 2\int w_\tau\rho\psi(\partial_n\psi)\nabla' w\cdot\nabla' g \\
&- 2\int w(\partial_n w)\rho_\tau\psi\nabla'\psi\cdot\nabla' g - \int(\partial_n w)\rho\psi^2\nabla' w\cdot\nabla' g_\tau \\
&- 2\int w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g_\tau + 2\int w\psi\psi_\tau\nabla'(\rho\partial_n w)\cdot\nabla' g \\
&+ 2\int (w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g)_\tau + 2\int w(\partial_n w)\rho\psi\psi_\tau\Delta' g.
\end{aligned}
$$

Integrating by parts in the sixth term and expanding the seventh term, we obtain

$$
\begin{aligned}
\mathcal{R}_2 = &\ 2\int w_\tau\rho\psi(\partial_n\psi)\nabla' w\cdot\nabla' g - \int(\partial_n w)\rho\psi^2\nabla' w\cdot\nabla' g_\tau \\
&- 2\int(\partial_n w)\rho\psi\psi_\tau\nabla' w\cdot\nabla' g + 2\int w_\tau(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g.
\end{aligned}
\tag{3.33}
$$

From the same computations as in the proof of (3.28), it follows that

$$
\begin{aligned}
&-\int_{\partial\Omega(\tau)} w_\tau\rho\psi^2\nu_n\nabla' w\cdot\nabla' g\, dS \\
&\qquad = \frac{1}{2}\int_{e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})} (\partial_\nu w)^2\rho\psi^2(\nu\cdot\eta)\nu_n\nabla' g\cdot\nu'\, dS.
\end{aligned}
\tag{3.34}
$$

Indeed, by (3.30), we have $\nabla' w\cdot\nabla' g = (\partial_\nu w)(\partial_{\nu'} g)$ on $e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})$. This together with (3.29) gives the above relation.

We examine I_3. Again by integration by parts, we have

$$
\begin{aligned}
\frac{1}{2}\frac{dI_3}{d\tau} &= -\frac{d}{d\tau}\left(\frac{1}{2}\int w\psi^2\partial_n(\rho\partial_n w)|\nabla' g|^2 + \int w(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2\right) \\
&= -\frac{1}{2}\int w_\tau\psi^2\partial_n(\rho\partial_n w)|\nabla' g|^2 - \frac{1}{2}\int w\psi^2\partial_n(\rho\partial_n w_\tau)|\nabla' g|^2 \\
&\quad - \frac{1}{2}\int w\psi^2\partial_n(\rho_\tau\partial_n w)|\nabla' g|^2 - \int w\psi^2\partial_n(\rho\partial_n w)\nabla' g\cdot\nabla' g_\tau \\
&\quad - \int w\psi\psi_\tau\partial_n(\rho\partial_n w)|\nabla' g|^2 - \int (w(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2)_\tau.
\end{aligned}
$$

In the same manner as in (3.31), we see that

$$\begin{aligned}
-\frac{1}{2}\int w_\tau\psi^2\partial_n(\rho\partial_n w)|\nabla' g|^2 - \frac{1}{2}\int w\psi^2\partial_n(\rho\partial_n w_\tau)|\nabla' g|^2 \\
= -\int w_\tau\psi^2\partial_n(\rho\partial_n w)|\nabla' g|^2 + \frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau(\partial_n w)\rho\psi^2|\nabla' g|^2\nu_n\,dS \\
-\int w_\tau(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2 + \int w(\partial_n w_\tau)\rho\psi(\partial_n\psi)|\nabla' g|^2
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2}\int w\psi^2\partial_n(\rho_\tau\partial_n w)|\nabla' g|^2 - \int w\psi^2\partial_n(\rho\partial_n w)\nabla' g\cdot\nabla' g_\tau \\
= \frac{1}{2}\int(\partial_n w)^2\rho_\tau\psi^2|\nabla' g|^2 + \int(\partial_n w)^2\rho\psi^2\nabla' g\cdot\nabla' g_\tau \\
+\int w(\partial_n w)\rho_\tau\psi(\partial_n\psi)|\nabla' g|^2 + 2\int w(\partial_n w)\rho\psi(\partial_n\psi)\nabla' g\cdot\nabla' g_\tau.
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2}\frac{dI_3}{d\tau} = -\int w_\tau\psi^2\partial_n(\rho\partial_n w)|\nabla' g|^2 + \frac{1}{2}\int(\partial_n w)^2\rho_\tau\psi^2|\nabla' g|^2 \\
+\frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau(\partial_n w)\rho\psi^2|\nabla' g|^2\nu_n\,dS + \mathcal{R}_3,
\end{aligned}\tag{3.35}$$

where

$$\begin{aligned}
\mathcal{R}_3 := &-\int w_\tau(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2 + \int w(\partial_n w_\tau)\rho\psi(\partial_n\psi)|\nabla' g|^2 \\
&+\int(\partial_n w)^2\rho\psi^2\nabla' g\cdot\nabla' g_\tau + \int w(\partial_n w)\rho_\tau\psi(\partial_n\psi)|\nabla' g|^2 \\
&+2\int w(\partial_n w)\rho\psi(\partial_n\psi)\nabla' g\cdot\nabla' g_\tau - \int w\psi\psi_\tau\partial_n(\rho\partial_n w)|\nabla' g|^2 \\
&-\int(w(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2)_\tau.
\end{aligned}$$

Integrating by parts in the sixth term and expanding the seventh term, we obtain

$$\begin{aligned}
\mathcal{R}_3 = -2\int w_\tau(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2 + \int(\partial_n w)^2\rho\psi^2\nabla' g\cdot\nabla' g_\tau \\
+\int(\partial_n w)^2\rho\psi\psi_\tau|\nabla' g|^2.
\end{aligned}\tag{3.36}$$

By (3.29) and (3.30), we see that

$$\begin{aligned}
\frac{1}{2}\int_{\partial\Omega(\tau)} w_\tau(\partial_n w)\rho\psi^2|\nabla' g|^2\nu_n\,dS \\
= -\frac{1}{4}\int_{e^{\tau/2}(\Phi(\partial\Omega\cap B_R)-\tilde{\xi})}(\partial_\nu w)^2\rho\psi^2(\nu\cdot\eta)\nu_n^2|\nabla' g|^2\,dS.
\end{aligned}\tag{3.37}$$

By combining (3.22), (3.25), (3.26), (3.28), (3.32), (3.34), (3.35) and (3.37), and then by (3.24), we obtain

$$\frac{d}{d\tau}\mathcal{E}(\tau) = \mathcal{J} - \mathcal{B} + \sum_{i=0}^{3}\mathcal{R}_i + \int \left(\frac{|\widehat{\nabla} w|^2}{2} - \frac{|w|^{p+1}}{p+1} + \frac{w^2}{2(p-1)}\right)\rho_\tau\psi^2, \tag{3.38}$$

where $\mathcal{B}$ is given in the statement of the lemma and

$$\mathcal{J} := \int w_\tau\psi^2\Big(\frac{w\rho}{p-1} - w|w|^{p-1}\rho - \nabla\cdot(\rho\nabla w) + \nabla'(\rho\partial_n w)\cdot\nabla' g$$
$$+ \partial_n(\rho\nabla' w)\cdot\nabla' g + \rho\partial_n w\Delta' g - \partial_n(\rho\partial_n w)|\nabla' g|^2\Big).$$

From (3.12) and $\nabla'(\partial_n w) = \partial_n\nabla' w$, it follows that

$$\mathcal{J} = \int w_\tau\psi^2\big(-w_\tau\rho - \tfrac{1}{2}\rho\eta\cdot\nabla w - \nabla\rho\cdot\nabla w + \partial_n\rho\nabla' w\cdot\nabla' g$$
$$+ \partial_n w\nabla'\rho\cdot\nabla' g - \partial_n w(\partial_n\rho)|\nabla' g|^2\big).$$

By using (3.17), we have

$$-\tfrac{1}{2}\rho\eta\cdot\nabla w - \nabla\rho\cdot\nabla w + \partial_n\rho\nabla' w\cdot\nabla' g = \tfrac{1}{2}(\partial_n w)\rho(g(\eta',\tau) - g(0,\tau)),$$
$$\partial_n w\nabla'\rho\cdot\nabla' g - \partial_n w(\partial_n\rho)|\nabla' g|^2 = -\tfrac{1}{2}(\partial_n w)\rho\eta'\cdot\nabla' g.$$

Thus,

$$\mathcal{J} = -\int w_\tau^2\rho\psi^2 + \frac{1}{2}\int w_\tau(\partial_n w)\rho\psi^2\big(g(\eta',\tau) - g(0,\tau) - \eta'\cdot\nabla' g\big).$$

Substituting this into (3.38) and combining (3.23), (3.27), (3.33) and (3.36) yields the desired equality. The proof is complete. ∎

We next estimate the terms $\mathcal{B}$ and $\mathcal{R}$ in (3.20).

Lemma 3.4. $\mathcal{B}(\tau) \geq 0$.

Proof. In (3.1), we have $\mathcal{B} = 0$, since the domain of integration is far from the boundary. In the case (3.2), since $\Phi(\partial\Omega\cap B_R) \subset \mathbf{R}^n_+$ and $\tilde{\xi}_n = \Phi_n(\tilde{x}) = \tilde{x}_n - f(\tilde{x}') \geq 0$ for $\tilde{x} \in \overline{\Omega_{R/4}}$ by the choice of f, we see that $\nu\cdot\eta \geq 0$ for $\eta \in e^{\tau/2}(\Phi(\partial\Omega\cap B_R) - \tilde{\xi})$. In addition, since $|\nu'| \leq 1$, we have

$$1 - 2\nu_n\partial_{\nu'} g + |\nabla' g|^2\nu_n^2 = (1 - (\partial_{\nu'} g)\nu_n)^2 + (|\nabla' g|^2 - (\partial_{\nu'} g)^2)\nu_n^2 \geq 0.$$

Hence the lemma follows. ∎

Lemma 3.5. *There exists $C > 0$ such that*

$$\mathcal{R}(\tau) \leq \frac{1}{2}\int w_\tau^2\psi^2\rho + C\tilde{\mathcal{R}}(\tau)$$

for $\tau = -\log(\tilde t - t)$ *with* $-1/2 < t < \tilde t \le 0$, *where*

$$\tilde{\mathcal{R}}(\tau) := \int (w^2 + |\nabla w|^2 + |w|^{p+1}) e^{-\frac{|\eta|^2}{32}} e^{-\tau/2} \chi_{[0,1]}\Big(\frac{4}{R}|\Psi(\tilde\xi + e^{-\tau/2}\eta) - \Psi(\tilde\xi)|\Big).$$

Proof. We only consider the case (3.2), since (3.1) is simpler. By (3.11) and the choice of f, we have

$$\begin{aligned} &\|\nabla' g(\cdot,\tau)\|_{L^\infty(\mathbf{R}^{n-1})} \le 1/2, \quad \|(\nabla')^2 g(\cdot,\tau)\|_{L^\infty(\mathbf{R}^{n-1})} \le Ce^{-\tau/2},\\ &\|\nabla' g_\tau(\cdot,\tau)\|_{L^\infty(\mathbf{R}^{n-1})} \le C|\eta'|e^{-\tau/2}, \end{aligned} \tag{3.39}$$

where $C > 0$ is independent of τ. By Cauchy's inequality and (3.39), we obtain

$$\begin{aligned} \mathcal{R} \le \frac12 \int w_\tau^2 \rho\psi^2 &+ C\int (w^2 + |\nabla w|^2 + |w|^{p+1})(\rho(|\nabla\psi|^2 + |\psi_\tau|) + |\rho_\tau|\psi^2)\\ &+ C\int |\nabla w|^2 \rho\psi^2(|\eta'|e^{-\tau/2} + |\eta'|^4 e^{-\tau}). \end{aligned}$$

From (3.15) and (3.39), it follows that

$$\rho(\eta,\tau) \le \exp\big(-\tfrac14\big(|\eta'|^2 + \tfrac12\eta_n^2 - (g(\eta',\tau) - g(0,\tau))^2\big)\big) \le e^{-|\eta|^2/8}. \tag{3.40}$$

By (3.18) and (3.39), and then by (3.40), we also have

$$|\rho_\tau| \le C|\eta'|^2 e^{-\tau/2}(|\eta_n| + |\eta'|)\rho \le Ce^{-|\eta|^2/16}e^{-\tau/2}. \tag{3.41}$$

These inequalities together with $\tau > -\log(\tilde t + 1/2) > 0$ show that

$$\begin{aligned} \mathcal{R} \le \frac12 &\int w_\tau^2 \rho\psi^2\\ &+ C\int (w^2 + |\nabla w|^2 + |w|^{p+1})(|\nabla\psi|^2 + |\psi_\tau| + \psi^2 e^{-\tau/2})e^{-|\eta|^2/16}. \end{aligned} \tag{3.42}$$

From (3.16) and (3.39), it follows that

$$\begin{aligned} |\nabla\psi| &= \frac{4e^{-\tau/2}}{R}\Big|\varphi'\Big(\frac4R e^{-\tau/2}|(\eta', \eta_n + g(\eta',\tau) - g(0,\tau))|\Big)\Big|\\ &\quad\times \frac{|(\eta' + (\eta_n + g(\eta') - g(0))\nabla' g, \eta_n + g(\eta') - g(0))|}{|(\eta', \eta_n + g(\eta') - g(0))|}\\ &\le Ce^{-\tau/2}\chi_{(1/2,1)}\Big(\frac4R e^{-\tau/2}|(\eta', \eta_n + g(\eta',\tau) - g(0,\tau))|\Big) \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} |\psi_\tau| &= \frac{4e^{-\tau/2}}{R}\Big|\varphi'\Big(\frac4R e^{-\tau/2}|(\eta', \eta_n + g(\eta',\tau) - g(0,\tau))|\Big)\Big|\\ &\quad\times\Big|-\frac12|(\eta', \eta_n + g(\eta') - g(0))| + \partial_\tau(|(\eta', \eta_n + g(\eta') - g(0))|)\Big|\\ &\le Ce^{-\tau/2}|\varphi'|\Big(\frac12(|\eta'|^2 + 2\eta_n^2 + 2(g(\eta') - g(0))^2) + |\eta'|^2 e^{-\tau/2}\Big)\\ &\le C|\eta|^2 e^{-\tau/2}\chi_{(1/2,1)}\Big(\frac4R e^{-\tau/2}|(\eta', \eta_n + g(\eta',\tau) - g(0,\tau))|\Big), \end{aligned}$$

where $g(\eta') := g(\eta', \tau)$ and $g(0) := g(0, \tau)$. Since $e^{-\tau/2}|(\eta', \eta_n + g(\eta', \tau) - g(0, \tau))| = |\Psi(\tilde{\xi} + e^{-\tau/2}\eta) - \Psi(\tilde{\xi})|$ by (3.13), we have

$$(|\nabla\psi|^2 + |\psi_\tau| + \psi^2 e^{-\tau/2})e^{-|\eta|^2/16} \le Ce^{-|\eta|^2/32}e^{-\tau/2}\chi_{[0,1]}\left(\frac{4}{R}|\Psi(\tilde{\xi} + e^{-\tau/2}\eta) - \Psi(\tilde{\xi})|\right). \tag{3.44}$$

The lemma follows from (3.44) and (3.42). ■

We are now in a position to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemmas 3.3–3.5, we see that

$$\mathcal{E}(\tau) + \frac{1}{2}\int_{\tau'}^{\tau}\int_{\Omega(\sigma)} w_\sigma^2 \rho\psi^2\, d\eta\, d\sigma \le \mathcal{E}(\tau') + C\int_{\tau'}^{\tau}\tilde{\mathcal{R}}(\sigma)\, d\sigma$$

for $\tau' = -\log(\tilde{t} - t')$ and $\tau = -\log(\tilde{t} - t)$ with $-1/2 < t' < t < \tilde{t} \le 0$. Note that this inequality holds for both (3.1) and (3.2). The change of variables and the same computations as in Lemma 3.1 yield

$$\begin{aligned}\int_{\tau'}^{\tau}\tilde{\mathcal{R}}(\sigma)\, d\sigma &= \int_{t'}^{t}(\tilde{t} - s)^{\frac{p+1}{p-1}-\frac{1}{2}}\int_{\Omega_R}\left(\frac{u^2}{\tilde{t} - s} + |\nabla u|^2 + |u|^{p+1}\right) \\ &\qquad\qquad \times (\tilde{t} - s)^{-n/2}e^{-\frac{|x-\tilde{x}|^2}{32(\tilde{t}-s)}}\, dx\, ds \\ &\le C(M + M^p)^2\int_{t'}^{t}(\tilde{t} - s)^{-1/2}\, ds \le C(M + M^p)^2(\tilde{t} - t')^{1/2}.\end{aligned} \tag{3.45}$$

Thus,

$$\mathcal{E}(\tau) + \frac{1}{2}\int_{\tau'}^{\tau}\int_{\Omega(\sigma)} w_\sigma^2\rho\psi^2\, d\eta\, d\sigma \le \mathcal{E}(\tau') + C(M + M^p)^2(\tilde{t} - t')^{1/2}. \tag{3.46}$$

On the other hand, since

$$w_\tau = (\tilde{t} - t)^{\frac{1}{p-1}}\left(-\frac{\hat{u}}{p-1} - \frac{(\xi - \tilde{\xi})\cdot\nabla\hat{u}}{2} + (\tilde{t} - t)\hat{u}_t\right),$$

by a change of variables we can see that

$$\begin{aligned}&\int_{\tau'}^{\tau}\int_{\Omega(\sigma)} w_\sigma^2\rho\psi^2\, d\eta\, d\sigma \\ &= \int_{t'}^{t}(\tilde{t} - s)^{\frac{2}{p-1}-\frac{n}{2}-1}\int_{\Phi(\Omega_R)}\left|\frac{\hat{u}}{p-1} + \frac{(\xi - \Phi(\tilde{x}))\cdot\nabla\hat{u}}{2} - (\tilde{t} - s)\hat{u}_s\right|^2 \\ &\qquad\qquad \times e^{-\frac{|\Psi(\xi)-\tilde{x}|^2}{4(\tilde{t}-s)}}\phi_{\tilde{x},R/4}^2(\Psi(\xi))\, d\xi\, ds.\end{aligned} \tag{3.47}$$

This together with (3.46) and $\mathcal{E}(\tau) = E(t)$ implies the desired inequality. ■

4. ε-regularity

The purpose of this section is to prove the following ε-regularity theorem:

Theorem 4.1. *Let $n \geq 3$, $p > p_S$ and Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 \in \overline{\Omega}$. Fix $0 < R < 1/2$ such that either* (3.1) *or* (3.2) *holds. Let u be a solution of* (2.1) *satisfying* (2.2)*. Then there exist constants ε_0, δ_0 and θ_0 with $0 < \varepsilon_0, \theta_0 < 1$ and $0 < \delta_0 < R$ depending only on n, p, M, Ω and R such that the following holds: If there exists $0 < \delta \leq \delta_0$ such that*

$$\delta^{\frac{4}{p-1}-n} \iint_{Q_\delta} (|\nabla u|^2 + |u|^{p+1})\, dx\, dt \leq \varepsilon_0, \tag{4.1}$$

then

$$\|u\|_{L^\infty(Q_{\theta_0 \delta})} \leq C(\theta_0 \delta)^{-\frac{2}{p-1}}. \tag{4.2}$$

Here C is a positive constant depending only on n, p, M, Ω and R and independent of ε_0, δ, δ_0 and θ_0.

From among the contents of this section, we will use only Theorem 4.1, Remark 4.3 and Lemma 4.7 to prove our main result in Section 5.

Remark 4.2. Chou, Du and Zheng [15, Theorems 2, 2′] proved ε-regularity theorems for global-in-time solutions (or borderline solutions) of (2.1) in bounded convex domains. The theorems were applied to show the decay of borderline solutions as $t \to \infty$ (see [96] for an alternative approach). In [14, Theorem 2], the convexity assumption was removed (see also [32, Proposition 4.2]). By some nontrivial modifications of [14, 15], we show ε-regularity for local-in-time solutions.

We note that the proofs of the ε-regularity theorems in [14, 15] are based on a preliminary ε-regularity result [15, Lemma 3], where the time at which the regularity of the solution is considered should be in the interior of the time interval of existence of the solution. Thus, it seems difficult to apply their argument near the final time of existence of local-in-time solutions. To overcome this issue, we give a variant of [15, Lemma 3] with the aid of the results of Blatt and Struwe [9, Proposition 4.1] and Giga and Kohn [44, Theorems 2.1, 2.5] to require only the estimate of solutions shortly before the reference time; see Lemma 4.4.

Remark 4.3. Theorem 4.1 remains to hold for weak solutions satisfying an estimate of the form (4.9) in Lemma 4.7. In particular, we may apply it to the blow-up limit of certain rescaled solutions in Section 5.

In this section, unless otherwise stated, C denotes a constant depending only on n, p, M, Ω and R. Each C may have different values, even within the same line. We always assume either (3.1) or (3.2) and deal with both these cases in a unified way. In addition, we always assume that u is a solution of (2.1) satisfying (2.2).

We state a preliminary ε-regularity result.

Lemma 4.4. *There exist* $\varepsilon_1 > 0$ *and* $C > 0$ *depending only on* n, p *and* Ω *such that the following holds: If there exists* $0 < R_1 < R/4$ *such that*

$$(r/2)^{\frac{4}{p-1}-n}\int_{t_1-(r/2)^2}^{t_1-(r/4)^2}\int_{\Omega_{r/2}(x_1)}|u(x,t)|^{p+1}\,dx\,dt \le \varepsilon_1 \tag{4.3}$$

for any (x_1,t_1) *and* $r > 0$ *satisfying* $Q_r(x_1,t_1) \subset Q_{2R_1}$*, then*

$$\|u\|_{L^\infty(Q_{R_1/4})} \le CR_1^{-2/(p-1)}.$$

Proof. Let $\varepsilon_1 > 0$ be a constant to be chosen later. Throughout this proof, C depends only on n, p and Ω. Set $v(y,s) := R_1^{2/(p-1)}u(R_1 y, R_1^2 s)$. Since $R/R_1 > 4$ and $-1/R_1^2 < -16$, we can check that v satisfies $v_t = \Delta v + |v|^{p-1}v$ in Q_4', where $\Omega' := R_1^{-1}\Omega$, $Q_r'(x_1,t_1) := (\Omega' \cap B_r(x_1)) \times (t_1 - r^2, t_1)$ and $Q_r' := Q_r'(0,0)$. From (4.3) and a change of variables, it follows that if $Q_r'(x_1,t_1) \subset Q_2'$ and $(r/4)^2 \le -t_1$, then $Q_r'(x_1, t_1 + (r/4)^2) \subset Q_2'$ and

$$\begin{aligned}
&(r/2)^{\frac{4}{p-1}-n}\int_{t_1-(r/4)^2}^{t_1}\int_{\Omega'\cap B_{r/2}(x_1)}|v(y,s)|^{p+1}\,dy\,ds\\
&\quad\le (r/2)^{\frac{4}{p-1}-n}\int_{t_1+(r/4)^2-(r/2)^2}^{t_1+(r/4)^2-(r/4)^2}\int_{\Omega'\cap B_{r/2}(x_1)}|v(y,s)|^{p+1}\,dy\,ds \le \varepsilon_1.
\end{aligned} \tag{4.4}$$

Let $(\tilde{x},\tilde{t}) \in Q_{1/2}'$. Set $\lambda := (-\tilde{t})^{1/2}$ and $\Omega'' := \lambda^{-1}(\Omega' - \tilde{x})$. Then the rescaled function $\tilde{v}(x,t) := \lambda^{2/(p-1)}v(\lambda x + \tilde{x}, \lambda^2 t + \tilde{t})$ satisfies $\tilde{v}_t = \Delta\tilde{v} + |\tilde{v}|^{p-1}\tilde{v}$ in $(\Omega'' \cap B_{2/\lambda}(-\tilde{x}/\lambda)) \times (-4/\lambda^2, 0)$. If $B_{r/\lambda}((x_1 - \tilde{x})/\lambda) \subset B_2$ and $((t_1 - r^2)/\lambda^2, t_1/\lambda^2) \subset (-4,0)$, then $Q_r'(x_1, t_1 + \tilde{t}) \subset Q_2'$ and $(r/4)^2 \le -(t_1 + \tilde{t})$. Therefore, (4.4) shows that

$$\begin{aligned}
&(r/4\lambda)^{\frac{4}{p-1}-n}\int_{(t_1-(r/4)^2)/\lambda^2}^{t_1/\lambda^2}\int_{\Omega''\cap B_{r/4\lambda}((x_1-\tilde{x})/\lambda)}|\tilde{v}(x,t)|^{p+1}\,dx\,dt\\
&\qquad= (r/4)^{\frac{4}{p-1}-n}\int_{t_1+\tilde{t}-(r/4)^2}^{t_1+\tilde{t}}\int_{\Omega'\cap B_{r/4}(x_1)}|v(y,s)|^{p+1}\,dy\,ds \le C\varepsilon_1.
\end{aligned}$$

Replacing (x_1,t_1) and r with $(\lambda x_1 + \tilde{x}, \lambda^2 t_1)$ and λr, respectively, we see that

$$(r/4)^{\frac{4}{p-1}-n}\iint_{Q_{r/4}''(x_1,t_1)}|\tilde{v}(x,t)|^{p+1}\,dx\,dt \le C\varepsilon_1$$

for any $Q_r''(x_1,t_1) \subset Q_2''$, where $Q_r''(x_1,t_1) := (\Omega'' \cap B_r(x_1)) \times (t_1 - r^2, t_1)$ and $Q_r'' := Q_r''(0,0)$. Hence $\|\tilde{v}\|_{M^{p+1,\mu_c}(Q_{1/2}'')} \le C\varepsilon_1$ with $\mu_c := 2(p+1)/(p-1)$, where $\|\cdot\|_{M^{p+1,\mu_c}}$ is the parabolic Morrey norm.

Taking ε_1 sufficiently small, we may apply [9, Proposition 4.1] to deduce that $\|\tilde{v}\|_{L^\infty(Q_{1/4}'')} \le C\|\tilde{v}\|_{M^{p+1,\mu_c}(Q_{1/2}'')} \le C\varepsilon_1$. In particular, $\lambda^{2/(p-1)}|v(\lambda x + \tilde{x}, \lambda^2 t + \tilde{t})| \le C\varepsilon_1$ for $(x,t) \in Q_{1/4}''$. Letting $(x,t) \to (0,0)$ gives

$$|v(\tilde{x},\tilde{t})| \le C\varepsilon_1\lambda^{-\frac{2}{p-1}} = C\varepsilon_1(-\tilde{t})^{-\frac{1}{p-1}}$$

for $(\tilde{x},\tilde{t}) \in Q'_{1/2}$. Since Ω' is $C^{2+\alpha}$, by applying [44, Theorems 2.1, 2.5] with ε_1 replaced by a smaller constant if necessary, we obtain $\|v\|_{L^\infty(Q'_{1/4})} \le C$, and hence $|u(x,t)| \le CR_1^{-2/(p-1)}$ for $(x,t) \in Q_{R_1/4}$. The proof is complete. ■

In the rest of this section, we prove Theorem 4.1 by using Lemma 4.4 and estimates of E. First of all, note that we may replace δ with $A\delta$ in the assumption (4.1), where $A > 1$ is a large constant and $0 < \delta < 1$ is a small constant depending on A. More precisely, we may assume

$$\delta^{\frac{4}{p-1}-n} \iint_{Q_{A\delta}} (|\nabla u|^2 + |u|^{p+1})\, dx\, dt \le A^{n-\frac{4}{p-1}} \varepsilon_0. \tag{4.5}$$

Here we take the constants A, ε_0, δ and θ_0 with

$$A > 3, \quad 0 < \varepsilon_0 < 1, \quad 0 < \delta < \frac{R}{16A} < \frac{1}{16}, \quad 0 < \theta_0 < \frac{1}{32A}, \tag{4.6}$$

which will be specified later; see (4.17). Set

$$I_r = I_r(x_1,t_1) := (r/2)^{\frac{4}{p-1}-n} \int_{t_1-(r/2)^2}^{t_1-(r/4)^2} \int_{\Omega_{r/2}(x_1)} |u|^{p+1}\, dx\, dt.$$

In order to show Theorem 4.1, it suffices to check the following statement:

$$I_r(x_1,t_1) \le \varepsilon_1 \quad \text{for any } Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}, \tag{4.7}$$

where ε_1 is given in Lemma 4.4. Note that

$$0 < r < 8A\theta_0\delta < \frac{1}{4}\delta < \delta < \frac{R}{16} < \frac{1}{16}.$$

Indeed, once (4.7) is proved, Lemma 4.4 guarantees the desired L^∞ bound:

$$\|u\|_{L^\infty(Q_{A\theta_0\delta})} \le C(A\theta_0\delta)^{-2/(p-1)}.$$

Therefore our temporary task is to estimate I_r.

Proposition 4.5. *There exists a constant $C > 0$ depending only on n, p, M, Ω and R such that*

$$I_r(x_1,t_1) \le C\bar{h}(e^{-A^2/C} + A^{n-\frac{4}{p-1}}\varepsilon_0 + \delta)$$

for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$, where $\bar{h}(s) := s + s^{1/(p+1)}$ for $s \ge 0$.

We prove this proposition by means of several lemmas.

Lemma 4.6. *There exists $C > 0$ such that*

$$I_r \le C \int_{t_1-(r/2)^2}^{t_1-(r/4)^2} (t_1-s)^{\frac{2}{p-1}} \int_{\Omega_R} |u|^{p+1} K_{(x_1,t_1)}(x,s) \phi^2_{x_1,R/8}\, dx\, ds \tag{4.8}$$

for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$.

Proof. For $(x,s) \in B_{r/2}(x_1) \times (t_1-(r/2)^2, t_1-(r/4)^2)$, we have $|x-x_1| \le r/2 < R/32$ and $t_1 - r^2/4 < s < t_1 - r^2/16$. Thus,

$$\begin{aligned}K_{(x_1,t_1)}(x,s)\phi^2_{x_1,R/8}(x) &= (t_1-s)^{-n/2}e^{-\frac{|x-x_1|^2}{4(t_1-s)}}\varphi^2\left(\frac{8|x-x_1|}{R}\right)\\ &\ge (1/4)^{-n/2}r^{-n}e^{-1}\varphi^2(1/4) \ge Cr^{-n}.\end{aligned}$$

Since $(t_1-s)^{2/(p-1)} \ge 16^{-2/(p-1)}r^{4/(p-1)}$ for $t_1-r^2/4 < s < t_1 - r^2/16$, the lemma follows. ■

To estimate the right-hand side of (4.8), we prepare the following lemma by using Proposition 3.2.

Lemma 4.7. *There exists $C > 0$ such that*

$$\begin{aligned}&\int_{t'}^{t}(\tilde t - s)^{\frac{2}{p-1}}\int_{\Omega_R}|u(x,s)|^{p+1}K_{(\tilde x,\tilde t)}(x,s)\phi^2_{\tilde x,\frac{R}{4}}(x)\,dx\,ds\\ &\quad\le C\left(\log\frac{\tilde t - t'}{\tilde t - t}\right)^{1/2}\big(E_{(\tilde x,\tilde t)}(t';\phi_{\tilde x,R/4}) - E_{(\tilde x,\tilde t)}(t;\phi_{\tilde x,R/4}) + C(\tilde t - t')^{1/2}\big)^{1/2}\\ &\qquad + C_p\big(E_{(\tilde x,\tilde t)}(t';\phi_{\tilde x,R/4}) + C(\tilde t - t')^{1/2}\big)\log\frac{\tilde t - t'}{\tilde t - t} + C(\tilde t - t')^{1/2} \qquad (4.9)\end{aligned}$$

for any $\tilde x \in \overline{\Omega_{R/4}}$ and $-1/2 < t' < t < \tilde t \le 0$, where $C_p := 2(p+1)/(p-1)$.

Proof. Define w, $\mathcal{E}$, ρ and ψ by (3.10), (3.14), (3.15) and (3.16), respectively. Then the left-hand side of the desired inequality equals $\iint |w|^{p+1}\rho\psi^2$, where we write $\int \dots = \int_{\Omega(\tau)} \dots\, d\eta$ and $\iint \dots = \int_{\tau'}^{\tau}\int_{\Omega(\sigma)} \dots\, d\eta\, d\sigma$ unless otherwise stated. From (3.12) and (3.14), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int w^2\rho\psi^2 &= \int ww_\tau\rho\psi^2 + \frac{1}{2}\int w^2\rho_\tau\psi^2 + \int w^2\rho\psi\psi_\tau - 2\mathcal{E}(\tau) + 2\mathcal{E}(\tau)\\ &= -2\mathcal{E}(\tau) + \frac{p-1}{p+1}\int |w|^{p+1}\rho\psi^2 - \frac{1}{2}\int w\rho\psi^2\nabla w\cdot\eta + \int w\rho\psi^2\Delta w\\ &\quad - 2\int w\rho\psi^2\nabla'(\partial_n w)\cdot\nabla' g + \int w(\partial_n^2 w)\rho\psi^2|\nabla' g|^2\\ &\quad - \int w(\partial_n w)\rho\psi^2\Delta' g + \int |\nabla w|^2\rho\psi^2 - 2\int(\partial_n w)\rho\psi^2\nabla' w\cdot\nabla' g\\ &\quad + \int(\partial_n w)^2\rho\psi^2|\nabla' g|^2 + \frac{1}{2}\int w^2\rho_\tau\psi^2 + \int w^2\rho\psi\psi_\tau.\end{aligned}$$

Integrating by parts gives

$$\frac{1}{2}\frac{d}{d\tau}\int w^2\rho\psi^2 = -2\mathcal{E}(\tau) + \frac{p-1}{p+1}\int |w|^{p+1}\rho\psi^2 + \tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2,$$

where

$$\begin{aligned}\tilde{\mathcal{R}}_1 :=& -\frac{1}{2}\int w\rho\psi^2\nabla w\cdot\eta - \int w\psi^2\nabla w\cdot\nabla\rho + 2\int w(\partial_n w)\psi^2\nabla'\rho\cdot\nabla' g\\ &+\int w(\partial_n w)\rho\psi^2\Delta' g - \int w(\partial_n w)(\partial_n\rho)\psi^2|\nabla' g|^2 + \frac{1}{2}\int w^2\rho_\tau\psi^2,\\ \tilde{\mathcal{R}}_2 :=& -2\int w\rho\psi\nabla w\cdot\nabla\psi + 4\int w(\partial_n w)\rho\psi\nabla'\psi\cdot\nabla' g\\ &-2\int w(\partial_n w)\rho\psi(\partial_n\psi)|\nabla' g|^2 + \int w^2\rho\psi\psi_\tau.\end{aligned}$$

Since $w\nabla w = \nabla(w^2)/2$ and $w\partial_n w = \partial_n(w^2)/2$, integrating by parts again shows that

$$\begin{aligned}\tilde{\mathcal{R}}_1 =& \int w^2\psi^2\Big(\frac{\eta}{4}\cdot\nabla\rho + \frac{n}{4}\rho + \frac{1}{2}\Delta\rho - \partial_n(\nabla'\rho)\cdot\nabla' g\\ &-\frac{1}{2}(\partial_n\rho)\Delta' g + \frac{1}{2}(\partial_n^2\rho)|\nabla' g|^2\Big) + \frac{1}{2}\int w^2\rho\psi\nabla\psi\cdot\eta\\ &+\int w^2\psi\nabla\psi\cdot\nabla\rho - 2\int w^2\psi(\partial_n\psi)\nabla'\rho\cdot\nabla' g - \int w^2\rho\psi(\partial_n\psi)\Delta' g\\ &+\int w^2(\partial_n\rho)\psi(\partial_n\psi)|\nabla' g|^2 + \frac{1}{2}\int w^2\rho_\tau\psi^2.\end{aligned}$$

From (3.17), (3.39), (3.40) and direct computations, it follows that

$$\begin{aligned}\frac{\eta}{4}\cdot\nabla\rho + \frac{n}{4}\rho + \frac{1}{2}\Delta\rho - \partial_n(\nabla'\rho)\cdot\nabla' g - \frac{1}{2}(\partial_n\rho)\Delta' g + \frac{1}{2}(\partial_n^2\rho)|\nabla' g|^2&\\ = \frac{1}{8}(\eta_n + g(\eta') - g(0))(g(\eta') - g(0) - \eta'\cdot\nabla' g)\rho&\\ + \frac{1}{8}(\eta_n + g(\eta') - g(0))^2(1 - 2|\nabla' g|^2)\rho&\\ \ge -\frac{1}{8}|\eta_n + g(\eta') - g(0)||g(\eta') - g(0) - \eta'\cdot\nabla' g|\rho&\\ \ge -C|\eta|^3 e^{-\tau/2}e^{-|\eta|^2/8} \ge -Ce^{-\tau/2}e^{-|\eta|^2/16},&\end{aligned}$$

where $g(\eta') := g(\eta',\tau)$ and $g(0) := g(0,\tau)$. The remainder terms in $\tilde{\mathcal{R}}_1$ can be estimated by using (3.17), (3.39), (3.40) and (3.41). Then since $\tau > -\log(\tilde{t} + 1/2) > 0$, we obtain

$$\begin{aligned}\tilde{\mathcal{R}}_1 &\ge -C\int w^2\psi^2 e^{-\tau/2}e^{-|\eta|^2/16} - C\int w^2|\nabla\psi|e^{-|\eta|^2/16}(1 + e^{-\tau/2})\\ &\ge -C\int w^2(|\nabla\psi| + \psi^2 e^{-\tau/2})e^{-|\eta|^2/16}.\end{aligned}$$

On the other hand, by Cauchy's inequality, (3.39) and (3.40), we see that

$$\tilde{\mathcal{R}}_2 \ge -C\int (w^2 + |\nabla w|^2)(|\nabla\psi| + |\psi_\tau|)e^{-|\eta|^2/8},$$

and so by (3.44) with (3.43), we obtain

$$\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2 \geq -C \int (w^2 + |\nabla w|^2)(|\nabla \psi| + |\psi_\tau| + \psi^2 e^{-\tau/2}) e^{-|\eta|^2/16} \geq -C\tilde{\mathcal{R}},$$

where $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\tau)$ is given in Lemma 3.5.

From $\mathcal{E}(\tau) = E(t)$ and Proposition 3.2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int w^2\rho\psi^2 &\geq -2\mathcal{E}(\tau) + \frac{p-1}{p+1}\int |w|^{p+1}\rho\psi^2 - C\tilde{\mathcal{R}} \\ &\geq -2(E(t') + C(\tilde{t}-t')^{1/2}) + \frac{p-1}{p+1}\int |w|^{p+1}\rho\psi^2 - C\tilde{\mathcal{R}}.\end{aligned}$$

Set $C_p := 2(p+1)/(p-1)$. Then

$$\int |w|^{p+1}\rho\psi^2 \leq \frac{C_p}{4}\frac{d}{d\tau}\int w^2\rho\psi^2 + C_p(E(t') + C(\tilde{t}-t')^{1/2}) + C\tilde{\mathcal{R}}.$$

Integrating this inequality over $\sigma \in (\tau', \tau)$ with $\tau' = -\log(\tilde{t} - t')$ and $\tau = -\log(\tilde{t} - t)$, we have

$$\begin{aligned}\iint |w|^{p+1}\rho\psi^2 &\leq \frac{C_p}{4}(\mathcal{K}(\tau) - \mathcal{K}(\tau')) \\ &\quad + C_p(E(t') + C(\tilde{t}-t')^{1/2})\log\frac{\tilde{t}-t'}{\tilde{t}-t} + C\int_{\tau'}^{\tau}\tilde{\mathcal{R}}\,d\sigma, \qquad (4.10)\end{aligned}$$

where

$$\mathcal{K}(\tau) := \int w^2(\eta,\tau)\rho(\eta,\tau)\psi^2(\eta,\tau)\,d\eta.$$

We now estimate $|\mathcal{K}(\tau) - \mathcal{K}(\tau')|$. By (3.40) and (3.41), we have

$$\begin{aligned}|\mathcal{K}(\tau) - \mathcal{K}(\tau')| = \left|\int_{\tau'}^{\tau}\frac{d\mathcal{K}}{d\sigma}\,d\sigma\right| &= \left|\iint (2ww_\sigma\rho\psi^2 + w^2\rho_\sigma\psi^2 + 2w^2\rho\psi\psi_\sigma)\right| \\ &\leq 2\iint |w|\,|w_\sigma|\rho\psi^2 + C\int_{\tau'}^{\tau}\tilde{\mathcal{R}}(\sigma)\,d\sigma.\end{aligned}$$

The Hölder inequality and (3.46) with $\mathcal{E}(\tau) = E(t)$ yield

$$\begin{aligned}\iint |w|\,|w_\sigma|\rho\psi^2 &\leq \left(\iint w^2\rho\psi^2\right)^{1/2}\left(\iint w_\sigma^2\rho\psi^2\right)^{1/2} \\ &\leq \sqrt{2}\left(\iint w^2\rho\psi^2\right)^{1/2}\left(E(t') - E(t) + C(\tilde{t}-t')^{1/2}\right)^{1/2}.\end{aligned}$$

Computations similar to (3.45) give

$$\begin{aligned}\iint w^2\rho\psi^2 &\leq \int_{t'}^{t}(\tilde{t}-s)^{\frac{2}{p-1}-1}\int_{\Omega_R}|u|^2(\tilde{t}-s)^{-n/2}e^{-\frac{|x-\tilde{x}|^2}{8(\tilde{t}-s)}}\phi_{\tilde{x},R/4}^2\,dx\,ds \\ &\leq C\int_{t'}^{t}(\tilde{t}-s)^{-1}\,ds \leq C\log\frac{\tilde{t}-t'}{\tilde{t}-t}.\end{aligned}$$

Thus,

$$|\mathcal{K}(\tau)-\mathcal{K}(\tau')|\leq C\left(\log\frac{\tilde{t}-t'}{\tilde{t}-t}\right)^{1/2}\big(E(t')-E(t)+C(\tilde{t}-t')^{1/2}\big)^{1/2}+C\int_{\tau'}^{\tau}\tilde{\mathcal{R}}(\sigma)\,d\sigma.$$

The above estimates show that

$$\begin{aligned}\iint|w|^{p+1}\rho\psi^2 &\leq C\left(\log\frac{\tilde{t}-t'}{\tilde{t}-t}\right)^{1/2}(E(t')-E(t)+C(\tilde{t}-t')^{1/2})^{1/2}\\ &\quad+C_p(E(t')+C(\tilde{t}-t')^{1/2})\log\frac{\tilde{t}-t'}{\tilde{t}-t}+C\int_{\tau'}^{\tau}\tilde{\mathcal{R}}(\sigma)\,d\sigma.\end{aligned}$$

From (3.45), it follows that $\iint|w|^{p+1}\rho\psi^2\,d\eta\,d\sigma$ is bounded by the right-hand side of the desired inequality. Thus the lemma follows. ■

We now estimate the right-hand side of (4.8). Assume $Q_r(x_1,t_1)\subset Q_{8A\theta_0\delta}$. Then since $x_1\in\overline{\Omega_{R/4}}$ and $-1/2<t_1-r^2/4<t_1-r^2/16<t_1\leq 0$, we see that Lemma 4.7 (with $\phi_{x_1,R/4}$ replaced by $\phi_{x_1,R/8}$) gives

$$\begin{aligned}I_r &\leq C\big(E_{(x_1,t_1)}(t_1-r^2/4)-E_{(x_1,t_1)}(t_1-r^2/16)+Cr\big)^{1/2}\\ &\quad+C_p\big(E_{(x_1,t_1)}(t_1-r^2/4;\phi_{x_1,R/8})+Cr\big)\log 4+Cr\end{aligned}\tag{4.11}$$

for $Q_r(x_1,t_1)\subset Q_{8A\theta_0\delta}$. We will derive an upper bound and a lower bound of $E_{(x_1,t_1)}$.

Lemma 4.8. *There exists $C>0$ independent of t_1 such that*

$$E_{(x_1,t_1)}(t_1-r^2/4;\phi_{x_1,R/8})\leq E_{(x_1,t_1)}(-4\delta^2;\phi_{x_1,R/8})+C\delta$$

for $Q_r(x_1,t_1)\subset Q_{8A\theta_0\delta}$.

Proof. From $x_1\in\overline{\Omega_{R/4}}$, $-1/2<-4\delta^2<t_1-r^2/4<t_1\leq 0$ and Proposition 3.2 (with $\phi_{x_1,R/4}$ replaced by $\phi_{x_1,R/8}$), we see that

$$E_{(x_1,t_1)}(t_1-r^2/4;\phi_{x_1,R/8})\leq E_{(x_1,t_1)}(-4\delta^2;\phi_{x_1,R/8})+C(t_1+4\delta^2)^{1/2}.$$

This implies the desired inequality. ■

Lemma 4.9. *There exists $C>0$ independent of t_1 such that*

$$E_{(x_1,t_1)}(t_1-r^2/16;\phi_{x_1,R/8})\geq -C\delta$$

for $Q_r(x_1,t_1)\subset Q_{8A\theta_0\delta}$.

Proof. By (4.10) and (3.45), there exists a constant $C'>0$ such that

$$\begin{aligned}&\int_{\tau'}^{\tau}\int_{\Omega(\sigma)}|w|^{p+1}\rho\psi^2\,d\eta\,d\sigma\\ &\quad\leq\frac{C_p}{4}\mathcal{K}(\tau)+C_p(E_{(\tilde{x},\tilde{t})}(t')+C'(\tilde{t}-t')^{1/2})\log\frac{\tilde{t}-t'}{\tilde{t}-t}+C'(\tilde{t}-t')^{1/2}\end{aligned}\tag{4.12}$$

for any $\tilde{x} \in \overline{\Omega_{R/4}}$, $\tau' = -\log(\tilde{t} - t')$ and $\tau = -\log(\tilde{t} - t)$ with $-1/2 < t' < t < \tilde{t} \leq 0$. We claim that

$$E_{(\tilde{x},\tilde{t})}(t') + C'(\tilde{t} - t')^{1/2} \geq 0$$

for any $\tilde{x} \in \overline{\Omega_{R/4}}$ and $-1/2 < t' < \tilde{t} \leq 0$. Towards a contradiction, we suppose that there exist $\tilde{x}_0 \in \overline{\Omega_{R/4}}$ and $-1/2 < t_0' < \tilde{t}_0 \leq 0$ such that

$$E_{(\tilde{x}_0,\tilde{t}_0)}(t_0') + C'(\tilde{t}_0 - t_0')^{1/2} < 0.$$

Then by (4.12), we have

$$\begin{aligned}
&\int_{\tau_0'}^{\tau} \int_{\Omega(\sigma)} |w|^{p+1} \rho \psi^2 \, d\eta \, d\sigma \\
&\qquad \leq \frac{C_p}{4} \mathcal{K}(\tau) + C_p(E_{(\tilde{x}_0,\tilde{t}_0)}(t_0') + C'(\tilde{t}_0 - t_0')^{1/2}) \log \frac{\tilde{t}_0 - t_0'}{\tilde{t}_0 - t} + C'(\tilde{t}_0 - t_0')^{1/2}
\end{aligned}$$

for any $\tau = -\log(\tilde{t}_0 - t)$ with $t_0' < t < \tilde{t}_0$, where $\tau_0' := -\log(\tilde{t}_0 - t_0')$. Therefore, there exists $t_0' < t_* < \tilde{t}_0$ such that

$$\int_{\tau_0'}^{\tau} \int_{\Omega(\sigma)} |w|^{p+1} \rho \psi^2 \, d\eta \, d\sigma \leq \frac{C_p}{4} \int_{\Omega(\tau)} w^2 \rho \psi^2 \, d\eta - 1$$

for any $\tau = -\log(\tilde{t}_0 - t)$ with $t_* < t < \tilde{t}_0$. From the Hölder inequality and (3.40), it follows that

$$\int_{\tau_0'}^{\tau} \left(\int_{\Omega(\sigma)} w^2 \rho \psi^2 \, d\eta \right)^{\frac{p+1}{2}} d\sigma \leq C \int_{\Omega(\tau)} w^2 \rho \psi^2 \, d\eta - C^{-1}$$

for any $-\log(\tilde{t}_0 - t_*) < \tau < \infty$. This integral inequality contradicts the fact that τ varies from $-\log(\tilde{t}_0 - t_*)$ to ∞. Hence the claim holds. Since $x_1 \in \overline{\Omega_{R/4}}$ and $-1/2 < t_1 - (r^2/16) < t_1 \leq 0$, the claim yields the desired inequality. ■

By using the above lemmas, we can estimate I_r by using $E_{(x_1,t_1)}(-4\delta^2)$.

Lemma 4.10. *There exists $C > 0$ such that*

$$I_r \leq C h(E_{(x_1,t_1)}(-4\delta^2; \phi_{x_1,R/8}) + C\delta)$$

for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$, where $h(s) := s + s^{1/2}$ for $s \geq 0$.

Proof. By combining (4.11) and Lemmas 4.8 and 4.9, we see that

$$\begin{aligned}
I_r &\leq C(E_{(x_1,t_1)}(-4\delta^2) + C\delta + Cr)^{1/2} \\
&\qquad + C_p(E_{(x_1,t_1)}(-4\delta^2) + C\delta + Cr) \log 4 + Cr \\
&\leq C h(E_{(x_1,t_1)}(-4\delta^2) + C(\delta + r))
\end{aligned}$$

for $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$, where $h(s) := s + s^{1/2}$ for $s \geq 0$. By using $r < \delta$, we obtain the desired inequality for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$. ■

We have estimated the right-hand side of (4.8), and then obtained Lemma 4.10. To prove Proposition 4.5, we will estimate $E_{(x_1,t_1)}(-4\delta^2)$ by using the functional

$$\begin{aligned} J_0(t) &= J_0(t; x_1, t_1, R) \\ &:= \int_{\Omega_R} (|\nabla u(x,t)|^2 + |u(x,t)|^{p+1}) K_{(x_1,t_1)}(x,t) \phi^2_{x_1,R/8}(x)\, dx. \end{aligned}$$

Lemma 4.11. *There exists* $C > 0$ *such that*

$$E_{(x_1,t_1)}(-4\delta^2; \phi_{x_1,R/8}) \le C\tilde{h}(\delta^{\frac{4}{p-1}+2} J_0(t)) + C\delta$$

for any $-9\delta^2 \le t \le -4\delta^2$ *and* $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$, *where* $\tilde{h}(s) := s + s^{2/(p+1)}$ *for* $s \ge 0$.

Proof. We first claim that

$$E_{(x_1,t_1)}(-4\delta^2) \le E_{(x_1,t_1)}(t) + C\delta$$

for $-9\delta^2 \le t \le -4\delta^2$. This inequality clearly holds for $t = -4\delta^2$, and hence it suffices to consider the case $-9\delta^2 \le t < -4\delta^2$. Since $x_1 \in \bar{\Omega} \cap B_{R/4}$ and $-1/2 < -9\delta^2 \le t < -4\delta^2 < t_1 \le 0$, Proposition 3.2 yields the claim. Indeed,

$$E_{(x_1,t_1)}(-4\delta^2) \le E_{(x_1,t_1)}(t) + C(t_1 - t)^{1/2} \le E_{(x_1,t_1)}(t) + C\delta.$$

From the definition of E, it follows that

$$\begin{aligned} E_{(x_1,t_1)}(t) &\le \frac{1}{2}(t_1 - t)^{\frac{p+1}{p-1}} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx \\ &\quad + (t_1 - t)^{\frac{2}{p-1}} \int_{\Omega_R} |u|^2 K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx \\ &\le C\delta^{\frac{2(p+1)}{p-1}} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx \\ &\quad + C\delta^{\frac{4}{p-1}} \int_{\Omega_R} |u|^2 K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx. \end{aligned}$$

By the Hölder inequality, and since $\phi_{x_1,R/8} \le 1$ and $\int_{\mathbf{R}^n} K\, dx = (4\pi)^{n/2}$, we have

$$\begin{aligned} E_{(x_1,t_1)}(t) &\le C\delta^{\frac{4}{p-1}+2} \int_{\Omega_R} |\nabla u|^2 K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx \\ &\quad + C\left(\delta^{\frac{4}{p-1}+2} \int_{\Omega_R} |u|^{p+1} K_{(x_1,t_1)} \phi^{p+1}_{x_1,R/8}\, dx\right)^{\frac{2}{p+1}} \\ &\le C\tilde{h}\left(\delta^{\frac{4}{p-1}+2} \int_{\Omega_R} (|\nabla u|^2 + |u|^{p+1}) K_{(x_1,t_1)} \phi^2_{x_1,R/8}\, dx\right), \end{aligned}$$

where $\tilde{h}(s) := s + s^{2/(p+1)}$ for $s \ge 0$. Hence the desired inequality follows. ∎

We next estimate $J_0(t; x_1, t_1, R)$ uniformly in x_1 and t_1.

Lemma 4.12. *There exists* $C > 0$ *such that*

$$J_0(t; x_1, t_1, R) \le C J_0(t; 0, 20\delta^2, 2R)$$

for any $-9\delta^2 \le t \le -4\delta^2$ *and* $Q_r(x_1, t_1) \subset Q_{8A\theta_0\delta}$.

Proof. By the definition of J_0, it suffices to show that

$$K_{(x_1,t_1)}(x,t) \le C K_{(0,20\delta^2)}(x,t), \tag{4.13}$$

$$\phi^2_{x_1,R/8}(x) \le C\phi^2_{0,R/4}(x), \tag{4.14}$$

for $x \in \Omega_R$ and $-9\delta^2 \le t \le -4\delta^2$. Since $(x_1, t_1) \in Q_{8A\theta_0\delta}$ and $A\theta_0 < 1/32$, we have

$$\begin{aligned}\frac{K_{(x_1,t_1)}(x,t)}{K_{(0,20\delta^2)}(x,t)} &= \left(\frac{20\delta^2 - t}{t_1 - t}\right)^{n/2} \exp\left(-\frac{|x-x_1|^2}{4(t_1-t)} + \frac{|x|^2}{4(20\delta^2 - t)}\right)\\ &\le \left(\frac{20\delta^2 - t}{-(8A\theta_0\delta)^2 - t}\right)^{n/2} \exp\left(-\frac{|x-x_1|^2}{36\delta^2} + \frac{|x|^2}{96\delta^2}\right)\\ &\le C \exp\left(-\frac{|x-x_1|^2}{36\delta^2} + \frac{|x|^2}{96\delta^2}\right).\end{aligned}$$

If $|x| \ge 10\delta$, then

$$|x - x_1| \ge |x| - |x_1| \ge 10\delta - 8A\theta_0\delta \ge 10\delta - \delta = 9\delta,$$

and so

$$\frac{|x|}{|x-x_1|} \le \frac{|x-x_1| + |x_1|}{|x-x_1|} \le 1 + \frac{8A\theta_0\delta}{9\delta} \le \frac{10}{9}.$$

Hence, if $|x| \ge 10\delta$, then

$$\frac{K_{(x_1,t_1)}(x,t)}{K_{(0,20\delta^2)}(x,t)} \le C \exp\left(-\frac{|x-x_1|^2}{36\delta^2} + \frac{25|x-x_1|^2}{24\cdot 81\delta^2}\right) \le C.$$

On the other hand, if $|x| < 10\delta$, then

$$\frac{K_{(x_1,t_1)}(x,t)}{K_{(0,20\delta^2)}(x,t)} \le C \exp\left(-\frac{|x-x_1|^2}{36\delta^2} + \frac{25}{24}\right) \le Ce^{\frac{25}{24}}.$$

Therefore, we obtain (4.13).

To check (4.14), we prove $\varphi(8|x-x_1|/R) \le C\varphi(4|x|/R)$. If $|x - x_1| \le R/8$, we have

$$|x| \le |x - x_1| + |x_1| \le R/8 + 8A\theta_0\delta \le \tfrac{3}{16}R.$$

Then by the choice of φ in the first part of Section 3.1, we see that $\varphi(4|x|/R) \ge \varphi(3/4) > 0$ for $|x - x_1| \le R/8$. Thus,

$$\varphi(8|x-x_1|/R) \le 1 \le \frac{\varphi(4|x|/R)}{\varphi(3/4)}.$$

If $|x - x_1| \ge R/8$, we see that $\varphi(8|x-x_1|/R) = 0 \le \varphi(4|x|/R)$. Hence (4.14) holds. ■

By Lemmas 4.11 and 4.12, to obtain a bound of $E_{(x_1,t_1)}(-4\delta^2)$, it suffices to estimate $J_0(t; 0, 20\delta^2, 2R)$ at some $t = \hat{t}$.

Lemma 4.13. *There exists* $-9\delta^2 \le \hat{t} \le -4\delta^2$ *such that*

$$\delta^{\frac{4}{p-1}+2} J_0(\hat{t}; 0, 20\delta^2, 2R) \le CA^{n-\frac{4}{p-1}}\varepsilon_0 + Ce^{-A^2/C}.$$

Proof. Due to the mean value theorem, it suffices to prove

$$\delta^{\frac{4}{p-1}} \int_{-9\delta^2}^{-4\delta^2} J_0(s; 0, 20\delta^2, 2R)\, ds \le CA^{n-\frac{4}{p-1}}\varepsilon_0 + Ce^{-A^2/C}.$$

To see this, we show that

$$\begin{aligned} J_1 &:= \delta^{\frac{4}{p-1}} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_R \cap B_{A\delta}} (|\nabla u|^2 + |u|^{p+1}) K_{(0,20\delta^2)} \phi_{0,R/4}^2 \, dx\, ds \\ &\le CA^{n-\frac{4}{p-1}}\varepsilon_0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} J_2 &:= \delta^{\frac{4}{p-1}} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_R \setminus B_{A\delta}} (|\nabla u|^2 + |u|^{p+1}) K_{(0,20\delta^2)} \phi_{0,R/4}^2 \, dx\, ds \\ &\le Ce^{-A^2/C}. \end{aligned} \tag{4.16}$$

As for (4.15), we recall our assumption (4.5), that is,

$$\delta^{\frac{4}{p-1}-n} \int_{-(A\delta)^2}^{0} \int_{\Omega_{A\delta}} (|\nabla u|^2 + |u|^{p+1})\, dx\, ds \le A^{n-\frac{4}{p-1}}\varepsilon_0.$$

Since $A > 3$, we have

$$\delta^{\frac{4}{p-1}-n} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_{A\delta}} (|\nabla u|^2 + |u|^{p+1})\, dx\, ds \le A^{n-\frac{4}{p-1}}\varepsilon_0.$$

Therefore

$$\begin{aligned} J_1 &\le \delta^{\frac{4}{p-1}} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_{A\delta}} (|\nabla u|^2 + |u|^{p+1})(20\delta^2 - s)^{-n/2} e^{-\frac{|x|^2}{4(20\delta^2 - s)}} \, dx\, ds \\ &\le C\delta^{\frac{4}{p-1}-n} \int_{-9\delta^2}^{-4\delta^2} \int_{\Omega_{A\delta}} (|\nabla u|^2 + |u|^{p+1})\, dx\, ds \le CA^{n-\frac{4}{p-1}}\varepsilon_0, \end{aligned}$$

which proves (4.15).

We next show (4.16). For $x \in \Omega_R \setminus B_{A\delta}$ and $-9\delta^2 \le s \le -4\delta^2$, we see that

$$\frac{K_{(0,20\delta^2)}(x,s)}{K_{(0,21\delta^2)}(x,s)} = \left(\frac{21\delta^2 - s}{20\delta^2 - s}\right)^{n/2} \exp\left(-\frac{|x|^2}{4} \frac{\delta^2}{(20\delta^2 - s)(21\delta^2 - s)}\right) \le Ce^{-A^2/C}.$$

This together with the same computations as in Lemma 3.1 yields

$$\begin{aligned} J_2 &\le Ce^{-A^2/C} \int_{-9\delta^2}^{-4\delta^2} \delta^{\frac{4}{p-1}} \int_{\Omega_R \setminus B_{A\delta}} (|\nabla u|^2 + |u|^{p+1}) K_{(0,21\delta^2)} \phi_{0,R/4}^2 \, dx\, ds \\ &\le Ce^{-A^2/C} \delta^{\frac{4}{p-1}} \int_{-9\delta^2}^{-4\delta^2} (21\delta^2 - s)^{-\frac{p+1}{p-1}} \, ds \le Ce^{-A^2/C}, \end{aligned}$$

and so (4.16) follows. As stated at the beginning of the proof, this proves the lemma. ∎

Proof of Proposition 4.5. By combining Lemmas 4.11–4.13, we obtain

$$E_{(x_1,t_1)}(-4\delta^2;\phi_{x_1,R/8}) \le C\tilde{h}(A^{n-\frac{4}{p-1}}\varepsilon_0 + e^{-A^2/C}) + C\delta$$

for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$. This together with Lemma 4.10 shows

$$I_r \le Ch\big(C\tilde{h}(A^{n-\frac{4}{p-1}}\varepsilon_0 + e^{-A^2/C}) + C\delta\big)$$

for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$. Since $h \circ \tilde{h} + h \le C\overline{h}$ with $\overline{h}(s) := s + s^{1/(p+1)}$ for $s \ge 0$, the desired inequality follows. ■

Proof of Theorem 4.1. As explained just before Proposition 4.5, it suffices to prove that the statement (4.7) holds under the assumption (4.5). Let A, ε_0, δ_0 and θ_0 satisfy (4.6) with δ replaced by δ_0. Let $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta_0}$. Then by Proposition 4.5, we have

$$I_r(x_1,t_1) \le C\overline{h}(e^{-A^2/C}) + C\overline{h}(A^{n-\frac{4}{p-1}}\varepsilon_0) + C\overline{h}(\delta_0),$$

where $C > 0$ is a constant depending only on n, p, M, Ω and R. By taking (4.6) into account, we choose constants A, ε_0, δ_0 and θ_0 satisfying the following conditions:

$$\begin{cases} A > 3 & \text{with } C\overline{h}(e^{-A^2/C}) < \varepsilon_1/3, \\ 0 < \varepsilon_0 < 1 & \text{with } C\overline{h}(A^{n-\frac{4}{p-1}}\varepsilon_0) < \varepsilon_1/3, \\ 0 < \delta_0 < \dfrac{R}{16A} & \text{with } C\overline{h}(\delta_0) < \varepsilon_1/3, \\ 0 < \theta_0 < \dfrac{1}{32A}. & \end{cases} \tag{4.17}$$

Here ε_1 is given in Lemma 4.4. Note that ε_0, δ_0 and θ_0 can be chosen independently.

Finally, we assume that there exists $0 < \delta \le \delta_0$ such that (4.5) holds. Then we obtain $I_r(x_1,t_1) \le \varepsilon_1$ for any $Q_r(x_1,t_1) \subset Q_{8A\theta_0\delta}$. Hence the statement (4.7) holds under the assumption (4.5), and so Lemma 4.4 shows that $\|u\|_{L^\infty(Q_{A\theta_0\delta})} \le C(A\theta_0\delta)^{-2/(p-1)}$. Then we can conclude that the original assumption (4.1) implies (4.2). The proof of Theorem 4.1 is complete. ■

5. Proof of main theorem

We first prove a localized version of Theorem 1.1. At the level of strategy, the proof is based on [35] and [101, Theorem 9.8] for the Navier–Stokes equations, and [61, Chapter 8] and [97, 102] for the harmonic map heat flow. Indeed, we use compactness, backward uniqueness and unique continuation. However, the analysis in each step seems to be more involved.

Theorem 5.1 (Localized statement). *Let $n \ge 3$, $p > p_S$, Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ and u be a classical L^{q_c}-solution of* (1.1) *with $u_0 \in L^{q_c}(\Omega)$. If the maximal existence time $T > 0$ is finite and u has a blow-up point $a \in \overline{\Omega}$, then for any $r > 0$,*

$$\limsup_{t\to T} \|u(\cdot,t)\|_{L^{q_c}(\Omega_r(a))} = \infty.$$

Note that if Ω is bounded, then Theorem 1.1 immediately follows from this theorem. The unbounded case will be considered in Section 5.4.

5.1. *Existence of blow-up limit*

To prove Theorem 5.1, we set up some notation, and then we define rescaled solutions and take the limit. Let u be a classical L^{q_c}-solution of (1.1). Assume that $a \in \overline{\Omega}$ is a blow-up point of u. By translation and scaling, we may assume that $a = 0$ and u satisfies

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \Omega_{R_1} \times (-1, 0), \\ u = 0 & \text{on } (\partial\Omega \cap B_{R_1}) \times (-1, 0), \end{cases}$$

for some $0 < R_1 < 1$. Here we assume that $t = 0$ is the blow-up time. Suppose, contrary to Theorem 5.1, that

$$\sup_{-1<t<0} \|u(\cdot, t)\|_{L^{q_c}(\Omega_{R_2})} \le M \tag{5.1}$$

for some $M > 0$ and $0 < R_2 < 1$. Fix $0 < R < \min\{R_1, R_2\}$ so small that (3.1) and (3.2) hold. Note that u satisfies (2.1) and (2.2).

From the parabolic regularity estimates in Lemma A.2, it follows that

$$\|u\|_{W^{1,l}(-1/4,0;L^r(\Omega_{R/2}))} \le C(M + M^p)$$

for $1 \le l < \infty$ and $1 \le r \le q_c/p$. Hence after a redefinition on a zero set in the time interval, we may assume $u \in C([-1/4, 0]; L^r(\Omega_{R/2}))$. This together with the uniform bound (2.2) gives

$$u \in C_{\mathrm{weak}}([-1/4, 0]; L^{q_c}(\Omega_{R/2})). \tag{5.2}$$

By the contraposition of Theorem 4.1, there exist $\varepsilon_0 > 0$ and $0 < \delta_k < 1/2$ with $\delta_k \to 0$ as $k \to \infty$ such that

$$\delta_k^{\frac{4}{p-1}-n} \iint_{Q_{\delta_k}} (|\nabla u|^2 + |u|^{p+1})\, dx\, dt > \varepsilon_0. \tag{5.3}$$

To take a blow-up limit of u, we define rescaled solutions and derive the corresponding equations. In the case (3.1), we define

$$u_k(x, t) := \delta_k^{\frac{2}{p-1}} u(\delta_k x, \delta_k^2 t).$$

Then

$$\begin{cases} (u_k)_t - \Delta u_k = |u_k|^{p-1}u_k & \text{in } \delta_k^{-1}\Omega_R \times (-\delta_k^{-2}, 0), \\ u_k = 0 & \text{on } \delta_k^{-1}(\partial\Omega \cap B_R) \times (-\delta_k^{-2}, 0). \end{cases}$$

In the case (3.2), we define

$$u_k(x, t) := \delta_k^{\frac{2}{p-1}} \hat{u}(\delta_k x, \delta_k^2 t), \quad f_k(x') := \delta_k^{-1} f(\delta_k x'), \tag{5.4}$$

where $\hat{u}$ is defined by (3.6). Then u_k satisfies

$$\begin{cases} (u_k)_t - A_k u_k = |u_k|^{p-1}u_k & \text{in } \delta_k^{-1}\Phi(\Omega_R) \times (-\delta_k^{-2}, 0), \\ u_k = 0 & \text{on } \delta_k^{-1}\Phi(\partial\Omega \cap B_R) \times (-\delta_k^{-2}, 0), \end{cases} \tag{5.5}$$

where

$$A_k u_k(x,t) := \Delta u_k - 2\nabla'(\partial_{x_n} u_k)\cdot\nabla' f_k + (\partial_{x_n}^2 u_k)|\nabla' f_k|^2 - (\partial_{x_n} u_k)\Delta' f_k.$$

In what follows, we take a limit of u_k and study its properties. Since the case (3.1) is easier to handle than (3.2), we focus on (3.2). For $\rho > 0$, we set $B_\rho^+ := \mathbf{R}_+^n \cap B_\rho$. To take a limit of u_k, we give estimates of rescaled solutions uniformly for $k \ge k_\rho$. Here $k_\rho \ge 1$ is taken so that $0 < \delta_k < 1/2$, $\Psi(B_{3\rho\delta_k}^+) \subset \Omega_{R/2}$ and $\Psi(B_{3\rho\delta_k}) \subset B_{R/2}$ for all $k \ge k_\rho$. Since $\nabla' f(0) = 0$ and $f \in C_0^{2+\alpha}(\mathbf{R}^{n-1})$, and $\delta_k \to 0$ as $k \to \infty$, we have

$$|\nabla' f_k(x')| = |\nabla' f(\delta_k x')| \to 0, \tag{5.6}$$

$$\|\Delta' f_k\|_{L^\infty(\mathbf{R}^{n-1})} \le \delta_k \|\Delta' f\|_{L^\infty(\mathbf{R}^{n-1})} \to 0. \tag{5.7}$$

We first give uniform estimates of u_k and ∇u_k.

Lemma 5.2. *Assume* (3.2). *Let $\rho > 0$. Then there exists $C > 0$ such that the rescaled functions u_k satisfy*

$$\sup_{-1<t<0} \|u_k(\cdot,t)\|_{L^{q_c}(B_\rho^+)} \le M,$$

$$\sup_{-1<t<0} \|\nabla u_k(\cdot,t)\|_{L^{q_*,\infty}(B_\rho^+)} \le C(M + M^p),$$

for all $k \ge k_\rho$, where C depends on R and is independent of ρ and k.

Proof. By a change of variables and the choice of k_ρ, we can easily prove the inequality for u_k. From the computation

$$\begin{aligned}|\nabla u_k(x,t)| &= \delta_k^{\frac{2}{p-1}+1} |\nabla u(\Psi(\delta_k x), \delta_k^2 t) + \partial_{x_n} u(\Psi(\delta_k x), \delta_k^2 t)\nabla' f(\delta_k x')| \\ &\le \delta_k^{\frac{2}{p-1}+1}(1 + \|\nabla' f\|_{L^\infty(\mathbf{R}^{n-1})})|\nabla u(\Psi(\delta_k x), \delta_k^2 t)|\end{aligned}$$

and Proposition 2.1, it follows that

$$\begin{aligned}\|\nabla u_k(\cdot,t)\|_{L^{q_*,\infty}(B_\rho^+)} &\le (1 + \|\nabla' f\|_{L^\infty(\mathbf{R}^{n-1})})\|\nabla u(\cdot, \delta_k^2 t)\|_{L^{q_*,\infty}(\Omega_{R/2})} \\ &\le C(M + M^p)\end{aligned}$$

for $-1 < t < 0$. Thus the lemma follows. ■

We next give a uniform parabolic regularity estimate.

Lemma 5.3. *Assume* (3.2). *Let $1 \le l < \infty$, $1 \le r \le q_c/p$ and $\rho > 0$. Then there exists $C > 0$ such that the rescaled functions u_k satisfy*

$$\|(u_k)_t\|_{L^l(-1,0;L^r(B_\rho^+))} + \|\nabla^2 u_k\|_{L^l(-1,0;L^r(B_\rho^+))} \le C(M + M^p)$$

for all $k \ge k_\rho$, where C depends on ρ and R and is independent of k.

Proof. By direct computations, we have

$$\begin{aligned}&\|(u_k)_t\|_{L^l(-1,0;L^r(B_\rho^+))} + \|\nabla^2 u_k\|_{L^l(-1,0;L^r(B_\rho^+))}\\ &\qquad = \delta_k^{\frac{2}{p-1}+2-\frac{n}{r}-\frac{2}{l}}\big(\|\hat{u}_t\|_{L^l(-\delta_k^2,0;L^r(B^+_{\rho\delta_k}))} + \|\nabla^2\hat{u}\|_{L^l(-\delta_k^2,0;L^r(B^+_{\rho\delta_k}))}\big).\end{aligned}$$

This together with Lemma A.1 proves the lemma. ■

To end this subsection, we show the existence of a blow-up limit. Let $\rho > 0$. To avoid technicalities due to the corner of ∂B_ρ^+, we consider a smooth domain $\mathcal{B}_\rho$ satisfying $B^+_{\rho/2} \subset \mathcal{B}_\rho \subset B^+_\rho$.

Lemma 5.4. *Assume* (3.2). *There exist a subsequence still denoted by u_k and a function $\bar{u}$ defined on $\mathbf{R}^n_+ \times [-1,0]$ satisfying the following statements:*

(i) *$u_k \to \bar{u}$ strongly in $L^\infty(-1,0;W^{1,2}(\mathcal{B}_\rho))$ for each $\rho > 0$ as $k \to \infty$.*

(ii) *$u_k \to \bar{u}$ strongly in $L^\infty(-1,0;L^{p+1}(\mathcal{B}_\rho))$ for each $\rho > 0$ as $k \to \infty$.*

(iii) $\|\bar{u}\|_{L^\infty(-1,0;L^{q_c}(\mathbf{R}^n_+))} \le M$.

(iv) $\|\nabla\bar{u}\|_{L^\infty(-1,0;L^{q_*,\infty}(\mathbf{R}^n_+))} \le C(M + M^p)$.

Here M is the constant in (5.1) *and $C > 0$ is a constant depending on R.*

Proof. Let $1 < r < \min\{2, q_c/p\}$. As a consequence of an Aubin–Lions type compactness result (see Lemma B.2), we have

$$W^{1,5}(-1,0;L^r(\mathcal{B}_\rho)) \cap L^5(-1,0;W^{2,r}(\mathcal{B}_\rho)) \hookrightarrow C([-1,0];W^{1,r}(\mathcal{B}_\rho))$$

and this embedding is compact. Therefore, the uniform bounds in Lemma 5.3 show that there exists a subsequence still denoted by $\{u_k\}$ with $u_k \to \bar{u}$ in $C([-1,0];W^{1,r}(\mathcal{B}_\rho))$ as $k \to \infty$. By Lemma 5.2, the rescaled functions u_k also satisfy the uniform bounds

$$\begin{aligned}&\sup_{-1<t<0}\|u_k(\cdot,t)\|_{L^{q_c}(\mathcal{B}_\rho)} \le M,\\ &\sup_{-1<t<0}\|\nabla u_k(\cdot,t)\|_{L^{q_*,\infty}(\mathcal{B}_\rho)} \le C(M+M^p),\end{aligned} \tag{5.8}$$

for all $k \ge k_\rho$, where C depends on R and is independent of ρ and k. Hence the standard interpolations give

$$\begin{aligned}\|u_k - \bar{u}\|_{L^\infty(-1,0;L_x^{p+1})} &\le \|u_k-\bar{u}\|^{\theta_1}_{L^\infty(-1,0;L_x^r)}\|u_k-\bar{u}\|^{1-\theta_1}_{L^\infty(-1,0;L_x^{q_c})}\\ &\le CM\|u_k-\bar{u}\|^{\theta_1}_{L^\infty(-1,0;L_x^r)},\\ \|\nabla(u_k-\bar{u})\|_{L^\infty(-1,0;L_x^2)} &\le \|\nabla(u_k-\bar{u})\|^{\theta_2}_{L^\infty(-1,0;L_x^r)}\|\nabla(u_k-\bar{u})\|^{1-\theta_2}_{L^\infty(-1,0;L_x^{q_*,\infty})}\\ &\le C(M+M^p)\|\nabla(u_k-\bar{u})\|^{\theta_2}_{L^\infty(-1,0;L_x^r)},\end{aligned}$$

where $L_x^r := L^r(\mathcal{B}_\rho)$ and $L_x^{q_*,\infty} := L^{q_*,\infty}(\mathcal{B}_\rho)$. These imply (i) and (ii). By the lower semicontinuity of the weak limit, we see that

$$\|\bar{u}(\cdot,t)\|_{L^{q_c}(\mathcal{B}_\rho)} \le \liminf_{k\to\infty}\|u_k(\cdot,t)\|_{L^{q_c}(\mathcal{B}_\rho)} \le M$$

for $-1 < t < 0$. Thus, letting $\rho \to \infty$ gives (iii). Since the constant C in (5.8) is independent of ρ, we also obtain (iv). The proof is complete. ∎

Remark 5.5. By Lemma 5.3, up to a subsequence still denoted by u_k, we conclude that $(u_k)_t \to \overline{u}_t$ weakly in $L^r(\mathscr{B}_\rho \times (-1,0))$ for each $\rho > 0$ as $k \to \infty$.

5.2. *Blow-up analysis*

We continue to study the case (3.2); for (3.1), see Remark 5.12. By using Lemma 5.4, we show several properties of the blow-up limit $\overline{u}$. For $\tilde{x} \in \overline{\mathbf{R}^n_+}$ and $-1 < t < \tilde{t} \le 0$, define a global weighted energy by

$$\overline{E}_{(\tilde{x},\tilde{t})}(t) := (\tilde{t}-t)^{\frac{p+1}{p-1}} \int_{\mathbf{R}^n_+} \left(\frac{|\nabla \overline{u}|^2}{2} - \frac{|\overline{u}|^{p+1}}{p+1} + \frac{\overline{u}^2}{2(p-1)(\tilde{t}-t)} \right) K_{(\tilde{x},\tilde{t})}(x,t)\, dx.$$

Note that the same computations as in Lemma 3.1 together with Lemma 5.4 (iii, iv) yield

$$|\overline{E}_{(\tilde{x},\tilde{t})}(t)| \le C(M + M^p)^2 \tag{5.9}$$

for any $\tilde{x} \in \overline{\mathbf{R}^n_+}$ and $-1 < t < \tilde{t} \le 0$.

We first show that $\overline{E}$ is a scaling limit of E.

Lemma 5.6. *Assume* (3.2). *Then, for each* $\tilde{x} \in \overline{\mathbf{R}^n_+}$ *and* $-1 < t < \tilde{t} \le 0$,

$$E_{(\delta_k \tilde{x}, \delta_k^2 \tilde{t})}(\delta_k^2 t; \phi_{\delta_k \tilde{x}, R/4}) \to \overline{E}_{(\tilde{x},\tilde{t})}(t) \quad \textit{as } k \to \infty.$$

Proof. Fix $\tilde{x} \in \overline{\mathbf{R}^n_+}$ and $-1 < t < \tilde{t} \le 0$. We take k so large that $\tilde{x} \in \delta_k^{-1} B_{R/4}$ and $-(1/2)\delta_k^{-2} < t < 0$. By the definition of E in (3.3) and a change of variables, we have

$$\begin{aligned} &E_{(\delta_k \tilde{x}, \delta_k^2 \tilde{t})}(\delta_k^2 t; \phi_{\delta_k \tilde{x}, R/4}) \\ &\quad = (\tilde{t}-t)^{\frac{p+1}{p-1} - \frac{n}{2}} \int_{\delta_k^{-1}\Phi(\Omega_R)} \left(\frac{|\widehat{\nabla} u_k|^2}{2} - \frac{|u_k|^{p+1}}{p+1} + \frac{|u_k(x,t)|^2}{2(p-1)(\tilde{t}-t)} \right) \\ &\qquad\qquad \times \exp\left(-\frac{|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2}{4(\tilde{t}-t)} \right) \varphi^2\left(\frac{4}{R} |\Psi(\delta_k x) - \delta_k \tilde{x}| \right) dx, \end{aligned}$$

where $\widehat{\nabla} u_k := (\nabla' u_k - (\partial_n u_k)\nabla' f_k, \partial_n u_k)$ and $f_k(x') = \delta_k^{-1} f(\delta_k x')$ (see (5.4)). In what follows, we focus on the convergence of the first term, since the other terms can be handled in the same manner. Namely, we prove that

$$\begin{aligned} &\int_{\delta_k^{-1}\Phi(\Omega_R)} |\widehat{\nabla} u_k|^2 e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2}{4(\tilde{t}-t)}} \varphi_k^2(x)\, dx - \int_{\mathbf{R}^n_+} |\nabla \overline{u}|^2 e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}}\, dx \\ &\qquad\qquad =: J_1^k + J_2^k + J_3^k + J_4^k \to 0 \end{aligned} \tag{5.10}$$

as $k \to \infty$, where $\varphi_k(x) := \varphi(4\delta_k |\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|/R)$ and

$$J_1^k := \int_{\delta_k^{-1}\Phi(\Omega_R)} (|\widehat{\nabla} u_k|^2 - |\nabla \overline{u}|^2) e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2}{4(\tilde{t}-t)}} \varphi_k^2\, dx,$$

$$
\begin{aligned}
J_2^k &:= \int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla \overline{u}|^2 e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x)-\tilde{x}|^2}{4(\tilde{t}-t)}} (\varphi_k^2 - 1)\, dx, \\
J_3^k &:= \int_{\delta_k^{-1}\Phi(\Omega_R)} |\nabla \overline{u}|^2 \bigl(e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x)-\tilde{x}|^2}{4(\tilde{t}-t)}} - e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}}\bigr)\, dx, \\
J_4^k &:= -\int_{\mathbf{R}_+^n \setminus \delta_k^{-1}\Phi(\Omega_R)} |\nabla \overline{u}|^2 e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}}\, dx.
\end{aligned}
$$

Note that the resulting convergence in (5.10) is pointwise for $\tilde{x}$, t and $\tilde{t}$, but this does not cause any problems for the proof of this lemma.

Let us estimate J_1^k. For $R_1' > 0$, we estimate

$$
\begin{aligned}
|J_1^k| &\le \Bigl(\int_{B_{R_1'}} + \int_{\mathbf{R}_+^n \setminus B_{R_1'}}\Bigr) \bigl| |\widehat{\nabla} u_k|^2 - |\nabla \overline{u}|^2 \bigr| e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x)-\tilde{x}|^2}{4(\tilde{t}-t)}} \varphi_k^2 \chi_{\delta_k^{-1}\Phi(\Omega_R)}\, dx \\
&=: J_{\mathrm{in}}^k + J_{\mathrm{out}}^k.
\end{aligned}
$$

We first consider J_{out}^k. From (3.2) and $|f(\delta_k x')| \le 2^{-1}\delta_k |x'|$, it follows that

$$
|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2 = |x + (0, \delta_k^{-1} f(\delta_k x')) - \tilde{x}|^2 \ge \tfrac{1}{8}|x|^2 - |\tilde{x}|^2. \tag{5.11}
$$

Thus,

$$
\begin{aligned}
J_{\mathrm{out}}^k &\le e^{\frac{|\tilde{x}|^2}{8(\tilde{t}-t)}} e^{-\frac{(R_1')^2}{64(\tilde{t}-t)}} \int_{\mathbf{R}_+^n} (|\widehat{\nabla} u_k|^2 + |\nabla \overline{u}|^2) e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x)-\tilde{x}|^2}{8(\tilde{t}-t)}} \varphi_k^2 \chi_{\delta_k^{-1}\Phi(\Omega_R)}\, dx \\
&\le e^{\frac{|\tilde{x}|^2}{8(\tilde{t}-t)}} e^{-\frac{(R_1')^2}{64}} \int_{\delta_k^{-1}\Phi(\Omega_R)} |\widehat{\nabla} u_k|^2 e^{-\frac{|\Psi(\delta_k x)-\delta_k \tilde{x}|^2}{8\delta_k^2(\tilde{t}-t)}} \varphi_k^2\, dx \\
&\quad + e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} e^{-\frac{(R_1')^2}{64}} \int_{\mathbf{R}_+^n} |\nabla \overline{u}|^2 e^{-\frac{|x|^2}{512(\tilde{t}-t)}}\, dx
\end{aligned}
$$

for $-1 < t < \tilde{t} \le 0$. The second term on the right-hand side can be estimated by computations similar to (3.5) with the aid of Lemma 5.4 (iv). As for the first term, we go back to the original variables. Then since $\operatorname{supp} \varphi^2(4|\cdot - \delta_k \tilde{x}|/R) \subset B_{R/4}(\delta_k \tilde{x})$, $\tilde{x} \in \delta_k^{-1} B_{R/4}$ and $-(1/2)\delta_k^{-2} < t < \tilde{t} \le 0$, by computations similar to (3.5) with (5.1) we see that

$$
\begin{aligned}
&\int_{\delta_k^{-1}\Phi(\Omega_R)} |\widehat{\nabla} u_k(x,t)|^2 e^{-\frac{|\Psi(\delta_k x)-\delta_k \tilde{x}|^2}{8\delta_k^2(\tilde{t}-t)}} \varphi_k^2(x)\, dx \\
&\quad = \delta_k^{\frac{2(p+1)}{p-1}-n} \int_{\Omega_R} |\nabla u(y, \delta_k^2 t)|^2 e^{-\frac{|y-\delta_k \tilde{x}|^2}{8\delta_k^2(\tilde{t}-t)}} \varphi^2\Bigl(\frac{4}{R}|y - \delta_k \tilde{x}|\Bigr)\, dy \\
&\quad \le \delta_k^{\frac{2(p+1)}{p-1}-n} \int_{\Omega_{R/2}} |\nabla u(y, \delta_k^2 t)|^2 e^{-\frac{|y-\delta_k \tilde{x}|^2}{8\delta_k^2(\tilde{t}-t)}}\, dy \le C(M + M^p)^2 (\tilde{t} - t)^{\frac{n}{2} - \frac{p+1}{p-1}},
\end{aligned}
$$

where $C > 0$ is independent of k. Thus,

$$J_{\mathrm{out}}^k \le C e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} (\tilde{t}-t)^{\frac{n}{2}-\frac{p+1}{p-1}} e^{-\frac{(R_1')^2}{64}}.$$

We now examine J_{in}^k. By using $\widehat{\nabla} u_k = (\nabla' u_k - (\partial_n u_k)\nabla' f_k, \partial_n u_k)$ and $f_k(x') = \delta_k^{-1} f(\delta_k x')$, we have

$$\begin{aligned}
&|\widehat{\nabla} u_k| = |\nabla u_k - ((\partial_n u_k)\nabla' f(\delta_k x'), 0)| \le (1 + |\nabla' f(\delta_k x')|)|\nabla u_k|,\\
&|\widehat{\nabla} u_k - \nabla \bar{u}|^2 \le C|\nabla u_k - \nabla \bar{u}|^2 + C|\nabla u_k|^2 |\nabla' f(\delta_k x')|^2.
\end{aligned}$$

These estimates together with the Hölder inequality show that

$$\begin{aligned}
J_{\mathrm{in}}^k &\le \int_{B_{R_1'}} |\widehat{\nabla} u_k + \nabla \bar{u}|\,|\widehat{\nabla} u_k - \nabla \bar{u}| \chi_{\delta_k^{-1}\Phi(\Omega_R)}\, dx\\
&\le \left(\int_{B_{R_1'}} \big((1 + |\nabla' f(\delta_k x')|)|\nabla u_k| + |\nabla \bar{u}|\big)^2 \chi\, dx \right)^{1/2}\\
&\quad \times \left(C \int_{B_{R_1'}} |\nabla u_k - \nabla \bar{u}|^2 \chi\, dx + C \int_{B_{R_1'}} |\nabla u_k|^2 |\nabla' f(\delta_k x')|^2 \chi\, dx \right)^{1/2},
\end{aligned}$$

where $\chi := \chi_{\delta_k^{-1}\Phi(\Omega_R)}$. Since $f \in C_0^{2+\alpha}(\mathbf{R}^{n-1})$, by (5.8) and Lemma 5.4 (iv) the first integral on the right-hand side is bounded by a constant. In addition, Lemma 5.4 (i) guarantees that the second integral converges to 0 as $k \to \infty$. Therefore, from $\nabla' f(0) = 0$ and the Lebesgue dominated convergence theorem, it follows that

$$\limsup_{k\to\infty} J_{\mathrm{in}}^k \le C \limsup_{k\to\infty} \left(\int_{B_{R_1'}} |\nabla u_k|^2 |\nabla' f(\delta_k x')|^2 \chi_{\delta_k^{-1}\Phi(\Omega_R)}\, dx \right)^{1/2} = 0.$$

Hence the above estimates for J_{in}^k and J_{out}^k show that

$$\limsup_{k\to\infty} |J_1^k| \le C e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} (\tilde{t}-t)^{\frac{n}{2}-\frac{p+1}{p-1}} e^{-(R_1')^2/64} \to 0$$

as $R_1' \to \infty$, and so $\lim_{k\to\infty} J_1^k = 0$.

We now consider J_2^k. For $R_2' > 0$, we note that

$$|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}| \le |x| + \tfrac{1}{2}|x'| + |\tilde{x}| \le \tfrac{3}{2}R_2' + |\tilde{x}|$$

if $x \in B_{R_2'}$. This together with (5.11), $0 \le \varphi_k \le 1$ and $\varphi' \le 0$ shows that

$$\begin{aligned}
\limsup_{k\to\infty} |J_2^k| &\le \limsup_{k\to\infty} \left(1 - \varphi^2 \left(\frac{4\delta_k}{R} \left(\frac{3}{2} R_2' + |\tilde{x}| \right) \right) \right) \int_{B_{R_2'}^+} |\nabla \bar{u}|^2\, dx\\
&\quad + e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} \int_{\mathbf{R}_+^n \setminus B_{R_2'}^+} |\nabla \bar{u}|^2 e^{-\frac{|x|^2}{32(\tilde{t}-t)}}\, dx
\end{aligned}$$

for $-1 < t < \tilde{t} \le 0$. Hence the inner part converges to 0 as $k \to \infty$, and then letting $R_2' \to \infty$ yields $\lim_{k\to\infty} J_2^k = 0$.

As for J_3^k, by (5.11), we have

$$|J_3^k| \le \int_{B^+_{R_3'}} |\nabla \overline{u}|^2 \Big| e^{-\frac{|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2}{4(\tilde{t}-t)}} - e^{-\frac{|x-\tilde{x}|^2}{4(\tilde{t}-t)}} \Big| \, dx$$
$$+ \int_{\mathbf{R}^n_+ \setminus B^+_{R_3'}} |\nabla \overline{u}|^2 \Big(e^{-\frac{|x|^2}{32(\tilde{t}-t)}} e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} + e^{-\frac{|x|^2}{8(\tilde{t}-t)}} e^{\frac{|\tilde{x}|^2}{4(\tilde{t}-t)}} \Big) \, dx.$$

The outer part can be handled in the same way as above. We can also check that the inner part converges to 0 as $k \to \infty$ because

$$\delta_k^{-1}\Psi(\delta_k x) = x + (0, \delta_k^{-1} f(\delta_k x')) = x + (0, \nabla' f(\theta_{x'}\delta_k x')) \to x$$

for some $0 \le \theta_{x'} \le 1$ as $k \to \infty$. Therefore $\lim_{k\to\infty} J_3^k = 0$. By Lemma 5.4 (iv), we can easily see that $\lim_{k\to\infty} J_4^k = 0$. Hence we obtain (5.10). ∎

We next estimate an integral concerning $|\overline{u}|^{p+1}$ by using $\overline{E}_{(\tilde{x},\tilde{t})}$. More precisely, we prove an analog of Lemma 4.7 for the blow-up limit $\overline{u}$.

Lemma 5.7. *Assume* (3.2). *Then there exists* $C > 0$ *such that*

$$\int_{t'}^{t} (\tilde{t}-s)^{\frac{p+1}{p-1}} \int_{\mathbf{R}^n_+} |\overline{u}|^{p+1} K_{(\tilde{x},\tilde{t})}(x,s) \, dx \, ds$$
$$\le C \Big(\log \frac{\tilde{t}-t'}{\tilde{t}-t} \Big)^{1/2} \big(\overline{E}_{(\tilde{x},\tilde{t})}(t') - \overline{E}_{(\tilde{x},\tilde{t})}(t) + C(\tilde{t}-t')^{1/2} \big)^{1/2}$$
$$+ C_p \big(\overline{E}_{(\tilde{x},\tilde{t})}(t') + C(\tilde{t}-t')^{1/2} \big) \log \frac{\tilde{t}-t'}{\tilde{t}-t} + C(\tilde{t}-t')^{1/2}$$

for $\tilde{x} \in \overline{\mathbf{R}^n_+}$ *and* $-1 < t' < t < \tilde{t} \le 0$. *Here* $C_p := 2(p+1)/(p-1)$ *and* $h(s) := s + s^{1/2}$ *for* $s \ge 0$.

Proof. Define $\tilde{\varphi} \in C^\infty([0,\infty))$ so that $\tilde{\varphi}' \le 0$, $0 \le \tilde{\varphi} \le 1$, $\tilde{\varphi}(z) = 0$ for $z > 2$ and $\tilde{\varphi}(z) = 1$ for $0 \le z < 1$. For $\tilde{R} > 1$, set $\tilde{\varphi}_{\tilde{R}}(z) := \tilde{\varphi}(z/\tilde{R})$. For $\tilde{x} \in \overline{\mathbf{R}^n_+}$ and $-1 < t < \tilde{t} \le 0$, we define

$$E_k(t) := (\tilde{t}-t)^{\frac{p+1}{p-1}-\frac{n}{2}} \int_{\delta_k^{-1}\Phi(\Omega_R)} \Big(\frac{|\widehat{\nabla} u_k(\xi,t)|^2}{2} - \frac{|u_k|^{p+1}}{p+1} + \frac{(\tilde{t}-t)^{-1}|u_k|^2}{2(p-1)} \Big)$$
$$\times e^{-\frac{|\Psi_k(\xi)-\Psi_k(\tilde{\xi}_k)|^2}{4(\tilde{t}-t)}} \varphi^2\Big(\frac{4\delta_k}{R} |\Psi_k(\xi) - \Psi_k(\tilde{\xi}_k)| \Big) \tilde{\varphi}^2_{\tilde{R}}(|\xi|) \, d\xi,$$

where $\Psi_k(\xi) := \delta_k^{-1}\Psi(\delta_k\xi)$, $\Phi_k(x) := \delta_k^{-1}\Phi(\delta_k x)$ and $\tilde{\xi}_k := \Phi_k(\tilde{x})$. We note that $\Psi_k(\tilde{\xi}_k) = \tilde{x}$ and the right-hand side of this definition coincides with $E_{(\delta_k\tilde{x},\delta_k^2\tilde{t})}(\delta_k^2 t; \phi_{\delta_k\tilde{x},R/4})$ if we replace $\tilde{\varphi}_{\tilde{R}}$ with 1.

By using the backward similarity variables $\eta := (\xi - \tilde{\xi}_k)/(\tilde{t} - t)^{1/2}$ and $\tau := -\log(\tilde{t} - t)$, we define

$$
\begin{aligned}
w_k(\eta, \tau) &:= e^{-\frac{1}{p-1}\tau} u_k(\tilde{\xi}_k + e^{-\tau/2}\eta, \tilde{t} - e^{-\tau}),\\
g_k(\eta', \tau) &:= e^{\tau/2} f_k(\tilde{\xi}_k' + e^{-\tau/2}\eta').
\end{aligned}
$$

Then w_k satisfies

$$
\begin{cases}
(w_k)_\tau + \dfrac{1}{2}\eta \cdot \nabla w_k + \dfrac{1}{p-1} w_k - A_k w_k - |w_k|^{p-1} w_k = 0,\\
\qquad\qquad\qquad \eta \in \Omega_k(\tau) := e^{\tau/2}(\Phi_k(\delta_k^{-1}\Omega_R) - \tilde{x}), \tau \in (-\log(\tilde{t} + 1), \infty),\\
w_k = 0, \quad \eta \in e^{\tau/2}(\Phi_k(\delta_k^{-1}(\partial\Omega \cap B_R)) - \tilde{\xi}_k),
\end{cases}
$$

where $A_k w_k := \Delta w_k - 2\nabla'(\partial_n w_k) \cdot \nabla' g_k + (\partial_n^2 w_k)|\nabla' g_k|^2 - (\partial_n w_k)\Delta' g_k$. Setting $\widehat{\nabla} w_k := (\nabla' w_k - (\partial_n w_k)\nabla' g_k, \partial_n w_k)$ and

$$
\begin{aligned}
&\mathcal{E}_k(\tau) := \int_{\Omega_k(\tau)} \left(\frac{|\widehat{\nabla} w_k|^2}{2} - \frac{|w_k|^{p+1}}{p+1} + \frac{w_k^2}{2(p-1)} \right) \rho_k \psi_k^2 \, d\eta,\\
&\rho_k = \rho_k(\eta, \tau) := \exp\bigl(-\tfrac{1}{4}(|\eta|^2 + 2(g_k(\eta', \tau) - g_k(0, \tau))\eta_n + (g_k(\eta', \tau) - g_k(0, \tau))^2)\bigr),\\
&\psi_k = \psi_k(\eta, \tau) := \varphi\left(\frac{4|\Psi_k(\tilde{\xi}_k + e^{-\tau/2}\eta) - \Psi_k(\tilde{\xi}_k)|}{R} \right) \tilde{\varphi}_{\tilde{R}}(|\tilde{\xi}_k + e^{-\tau/2}\eta|),
\end{aligned}
$$

we can check that $E_k(t) = \mathcal{E}_k(\tau)$ with $\tau = -\log(\tilde{t} - t)$. In addition, we can observe that the above situation is almost the same as in Section 3, except for the appearance of $\tilde{\varphi}_{\tilde{R}}$.

By the choice of $\tilde{\varphi}_{\tilde{R}}$ and $\tilde{R} > 1$, we have

$$
\begin{aligned}
&|\nabla(\tilde{\varphi}_{\tilde{R}}(|\tilde{\xi}_k + e^{-\tau/2}\eta|))| \le C|\tilde{\varphi}_{\tilde{R}}'(|\tilde{\xi}_k + e^{-\tau/2}\eta|)|e^{-\tau/2} \le Ce^{-\tau/2},\\
&|\partial_\tau(\tilde{\varphi}_{\tilde{R}}(|\tilde{\xi}_k + e^{-\tau/2}\eta|))| \le C|\tilde{\varphi}_{\tilde{R}}'(|\tilde{\xi}_k + e^{-\tau/2}\eta|)|\,|\eta|e^{-\tau/2} \le C|\eta|e^{-\tau/2},
\end{aligned}
$$

where $C > 0$ is independent of k and $\tilde{R}$. Then the same computations as in Section 3 show that there exists a constant $C > 0$ independent of k and $\tilde{R}$ satisfying

$$
\mathcal{E}_k(\tau) + \frac{1}{2}\int_{\tau'}^{\tau} \int_{\Omega_k(\sigma)} (w_k)_\sigma^2 \rho_k \psi_k^2 \, d\eta \, d\sigma \le \mathcal{E}_k(\tau') + C(\tilde{t} - t')^{1/2}.
$$

Therefore, the same computations as in Lemma 4.7 yield

$$
\begin{aligned}
\int_{t'}^{t} (\tilde{t} - s)^{\frac{p+1}{p-1} - \frac{n}{2}} \int_{\delta_k^{-1}\Phi(\Omega_R)} |u_k|^{p+1} \exp\left(-\frac{|\delta_k^{-1}\Psi(\delta_k x) - \tilde{x}|^2}{4(\tilde{t} - s)} \right)&\\
\times \varphi^2\left(\frac{4}{R}|\Psi(\delta_k x) - \delta_k \tilde{x}| \right) \tilde{\varphi}_{\tilde{R}}^2(|x|) \, dx \, ds&\\
\le C\left(\log \frac{\tilde{t} - t'}{\tilde{t} - t} \right)^{1/2} (E_k(t') - E_k(t) + C(\tilde{t} - t')^{1/2})^{1/2}&\\
+ C_p(E_k(t') + C(\tilde{t} - t')^{1/2}) \log \frac{\tilde{t} - t'}{\tilde{t} - t} + C(\tilde{t} - t')^{1/2},&
\end{aligned}
$$

where $C_p := 2(p+1)/(p-1)$ and $C > 0$ is independent of k and $\tilde R$. By Lemma 5.4 (ii) and the same argument as in Lemma 5.6, letting $k \to \infty$, we have

$$\begin{aligned}\int_{t'}^{t} (\tilde t - s)^{\frac{p+1}{p-1}} \int_{\mathbf{R}^n_+} |\bar u|^{p+1} K_{(\tilde x,\tilde t)}(x,s)\tilde\varphi_{\tilde R}^2(|x|)\,dx\,ds \\ \le C\left(\log\frac{\tilde t - t'}{\tilde t - t}\right)^{1/2} \left(\bar E_{\tilde\varphi_{\tilde R}}(t') - \bar E_{\tilde\varphi_{\tilde R}}(t) + C(\tilde t - t')^{1/2}\right)^{1/2} \\ + C_p\left(\bar E_{\tilde\varphi_{\tilde R}}(t') + C(\tilde t - t')^{1/2}\right)\log\frac{\tilde t - t'}{\tilde t - t} + C(\tilde t - t')^{1/2},\end{aligned}$$

where

$$\bar E_{\tilde\varphi_{\tilde R}}(t) := (\tilde t - t)^{\frac{p+1}{p-1}} \int_{\mathbf{R}^n_+} \left(\frac{|\nabla \bar u|^2}{2} - \frac{|\bar u|^{p+1}}{p+1} + \frac{\bar u^2}{2(p-1)(\tilde t - t)}\right) K_{(\tilde x,\tilde t)}\tilde\varphi_{\tilde R}^2\,dx.$$

Letting $\tilde R \to \infty$ and applying the monotone convergence theorem to the left-hand side and the dominated convergence theorem to the right-hand side, we obtain the desired inequality. ■

Remark 5.8. As stated in Remark 4.3, ε-regularity (Theorem 4.1) is also valid for $\bar u$, since $\bar u$ satisfies the analog of Lemma 4.7.

We next prove a monotonicity estimate for $\bar E_{(0,0)}$.

Lemma 5.9. *Assume* (3.2). *Then there exists* $C > 0$ *such that*

$$\begin{aligned}\bar E_{(0,0)}(t') - \bar E_{(0,0)}(t) \ge C^{-1}\frac{(-t)^{4/(p-1)}}{(-t')^{2p/(p-1)}} \\ \times \left(\int_{t'}^{t}\int_{\mathbf{R}^n_+}\left(\frac{\bar u}{p-1} + \frac{x\cdot\nabla\bar u}{2} + s\bar u_s\right)K_{(0,0)}(x,s)\zeta(x,s)\,dx\,ds\right)^2\end{aligned}$$

for any $-1 < t' < t < 0$ *and* $\zeta \in C_0^\infty(\mathbf{R}^n_+ \times (-1,0))$ *with* $|\zeta| \le 1$.

Proof. Let $-1 < t' < t < 0$. We take k so large that $-\frac{1}{2}\delta_k^{-2} < t' < t < 0$. From (3.19) and a change of variables, it follows that

$$E_{(0,0)}(\delta_k^2 t;\phi_{0,R/4}) + \tfrac{1}{2}\mathcal{J} \le E_{(0,0)}(\delta_k^2 t';\phi_{0,R/4}) + C\delta_k(-t')^{1/2}, \tag{5.12}$$

where

$$\begin{aligned}\mathcal{J} := \int_{t'}^{t}(-s)^{\frac{2}{p-1}-\frac{n}{2}-1}\int_{\delta_k^{-1}\Phi(\Omega_R)}\left|\frac{u_k}{p-1} + \frac{x\cdot\nabla u_k}{2} + s(u_k)_s\right|^2 \\ \times\exp\left(-\frac{|\delta_k^{-1}\Psi(\delta_k x)|^2}{4(-s)}\right)\varphi^2\left(\frac{4}{R}|\Psi(\delta_k x)|\right)dx\,ds.\end{aligned}$$

Let $\zeta \in C_0^\infty(\mathbf{R}^n_+ \times (-1,0))$ satisfy $|\zeta| \le 1$. Then

$$\mathcal{J} \ge \iint_{\delta_k^{-1}\Phi(\Omega_R)\times(t',t)} |F(u_k)|^2\zeta^2(x,s)\,d\mu_k(x,s).$$

Here $F(u_k)$ and μ_k are defined by

$$\begin{aligned} F(u_k) &:= \frac{u_k}{p-1} + \frac{x\cdot\nabla u_k}{2} + s(u_k)_s, \\ d\mu_k(x,s) &:= (-s)^{\frac{2}{p-1}-1} K_{(0,0)}(\delta_k^{-1}\Psi(\delta_k x), s)\varphi^2\left(\frac{4}{R}|\Psi(\delta_k x)|\right) dx\,ds. \end{aligned}$$

By (5.11) with $\tilde{x} = 0$ and $\varphi \le 1$, we compute that

$$\mu_k(\delta_k^{-1}\Phi(\Omega_R)\times(t',t)) \le C\int_{t'}^{t}(-s)^{\frac{2}{p-1}-1}\int_{\mathbf{R}^n} K_{(0,0)}\left(\frac{x}{\sqrt{8}}, s\right) dx\,ds \le C(-t')^{\frac{2}{p-1}},$$

where C depends only on n and p and is independent of k. Then by Jensen's inequality,

$$\begin{aligned} \mathcal{J} &\ge \frac{1}{\mu_k(\delta_k^{-1}\Phi(\Omega_R)\times(t',t))}\left(\iint_{\delta_k^{-1}\Phi(\Omega_R)\times(t',t)} F(u_k)\zeta\,d\mu_k(x,s)\right)^2 \\ &\ge C^{-1}(-t')^{-\frac{2}{p-1}}\left(\iint_{\delta_k^{-1}\Phi(\Omega_R)\times(t',t)} F(u_k)\zeta\,d\mu_k(x,s)\right)^2. \end{aligned}$$

From this and (5.12), it follows that

$$\begin{aligned} E_{(0,0)}(\delta_k^2 t;\phi_{0,R/4}) + C^{-1}(-t')^{-\frac{2}{p-1}}\left(\iint_{\delta_k^{-1}\Phi(\Omega_R)\times(t',t)} F(u_k)\zeta\,d\mu_k\right)^2 \\ \le E_{(0,0)}(\delta_k^2 t';\phi_{0,R/4}) + C\delta_k(-t')^{1/2}. \end{aligned}$$

From Remark 5.5, Lemma 5.4 and computations similar to the derivation of (5.10), it follows that

$$\iint_{\delta_k^{-1}\Phi(\Omega_R)\times(t',t)} F(u_k)\zeta\,d\mu_k(x,s) \to \int_{t'}^{t}(-s)^{\frac{2}{p-1}-1}\int_{\mathbf{R}^n_+} F(\bar{u})K_{(0,0)}\zeta\,dx\,ds$$

as $k\to\infty$. This together with Lemma 5.6 and letting $k\to\infty$ implies that

$$\begin{aligned} \bar{E}_{(0,0)}(t') - \bar{E}_{(0,0)}(t) &\ge C^{-1}(-t')^{-\frac{2}{p-1}}\left(\int_{t'}^{t}(-s)^{\frac{2}{p-1}-1}\int_{\mathbf{R}^n_+} F(\bar{u})K_{(0,0)}\zeta\,dx\,ds\right)^2 \\ &\ge C^{-1}(-t')^{-\frac{2p}{p-1}}(-t)^{\frac{4}{p-1}}\left(\int_{t'}^{t}\int_{\mathbf{R}^n_+} F(\bar{u})K_{(0,0)}\zeta\,dx\,ds\right)^2. \end{aligned}$$

Then the desired inequality follows. ■

The monotonicity estimate gives the following equality based on the argument of [97, Theorem 8.1].

Lemma 5.10. *Assume* (3.2). *Then*

$$\frac{\bar{u}(x,t)}{p-1} + \frac{x\cdot\nabla\bar{u}}{2} + t\bar{u}_t = 0 \qquad a.e.\ in\ \mathbf{R}^n_+\times(-1,0).$$

Proof. For $0 < r < 1/2$, we set

$$\Phi(r) := \int_{-4r^2}^{-r^2} \overline{E}_{(0,0)}(t)\, \frac{dt}{-t}.$$

We note that (5.9) guarantees that Φ is well-defined. By Lemmas 5.6 and 3.1, the dominated convergence theorem and a change of variables, we have

$$\Phi(r) = \lim_{k\to\infty} \int_{-4r^2}^{-r^2} E_{(0,0)}(\delta_k^2 t)\, \frac{dt}{-t} = \lim_{k\to\infty} \int_{-4\delta_k^2 r^2}^{-\delta_k^2 r^2} E_{(0,0)}(s)\, \frac{ds}{-s}.$$

We claim that $\Phi(r)$ is independent of r. To show this, we set

$$H(\tilde r) := \int_{-4\tilde r^2}^{-\tilde r^2} E_{(0,0)}(s)\, \frac{ds}{-s} = \int_{-4}^{-1} E_{(0,0)}(\tilde r^2 \lambda)\, \frac{d\lambda}{-\lambda}$$

for $0 < \tilde r < 1/2$ and check that $\lim_{\tilde r\to 0} H(\tilde r)$ exists. Let $\varepsilon > 0$. Since Lemma 3.1 yields the boundedness of H, we see that $H_\varepsilon := \inf_{0<\tilde r<\varepsilon} H(\tilde r)$ is finite. Then there exists $0 < \tilde r_\varepsilon < \varepsilon$ such that $H(\tilde r_\varepsilon) \le H_\varepsilon + \varepsilon$. From Proposition 3.2 and the negativity of λ, it follows that

$$\begin{aligned}\limsup_{\tilde r\to 0} H(\tilde r) &\le \limsup_{\tilde r\to 0} \int_{-4}^{-1} \left(E_{(0,0)}(\tilde r_\varepsilon^2 \lambda) + C(M + M^p)^2(-\tilde r_\varepsilon^2 \lambda)\right) \frac{d\lambda}{-\lambda} \\ &= H(\tilde r_\varepsilon) + 3C(M + M^p)^2 \tilde r_\varepsilon^2 \le H_\varepsilon + \varepsilon + 3C(M + M^p)^2 \varepsilon^2.\end{aligned}$$

Thus, letting $\varepsilon \to 0$ gives $\limsup_{\tilde r\to 0} H(\tilde r) \le \liminf_{\tilde r\to 0} H(\tilde r)$. Hence the limit of $H(\tilde r)$ exists, and so $\Phi(r) = \lim_{\tilde r\to 0} H(\tilde r)$. Then the claim follows.

From the claim and Lemma 5.9, it follows that

$$\begin{aligned}0 &= \int_{r_1}^{r_2} \Phi'(s)\, ds = \int_{r_1}^{r_2} \frac{2}{s}(\overline{E}_{(0,0)}(-4s^2) - \overline{E}_{(0,0)}(-s^2))\, ds \\ &\ge C^{-1} \int_{r_1}^{r_2} s^{\frac{9-5p}{p-1}} \left(\int_{-4s^2}^{-s^2} \int_{\mathbf{R}^n_+} \left(\frac{\overline{u}}{p-1} + \frac{x\cdot\nabla\overline{u}}{2} + t\overline{u}_t \right) K_{(0,0)} \zeta \, dx\, dt \right)^2 ds\end{aligned}$$

for any $0 < r_1 < r_2 < 1/2$ and $\zeta \in C_0^\infty(\mathbf{R}^n_+ \times (-1, 0))$ with $|\zeta| \le 1$. Thus,

$$\int_{-4s^2}^{-s^2} \int_{\mathbf{R}^n_+} \left(\frac{\overline{u}}{p-1} + \frac{x\cdot\nabla\overline{u}}{2} + t\overline{u}_t \right) K_{(0,0)}(x,t)\zeta(x,t)\, dx\, dt = 0$$

for a.e. $s \in (0, 1/2)$. Hence by the fundamental lemma of the calculus of variations, the lemma follows. ∎

To end this subsection, we prove a partial regularity result for $\overline{u}$ by using ε-regularity. For $t \in (-1, 0)$, define the singular set of $\overline{u}(\cdot, t)$ by

$$\Sigma(t) := \{x \in \mathbf{R}^n_+;\ \overline{u}(\cdot, t) \notin L^\infty(B_\rho^+(x)) \text{ for all } \rho > 0\}.$$

By modifying the argument of Wang [102, Lemma 3.3], we show that $\Sigma(t)$ consists of at most finitely many points.

Lemma 5.11. *Assume* (3.2)*. Then the singular set* $\Sigma(t)$ *consists of finitely many points for each* $t \in (-1/4, 0)$*. More precisely, there exists a constant* $C > 0$ *depending on* R *such that*

$$\mathrm{card}(\Sigma(t)) \le C(M + M^p)^{q_c} \varepsilon_0^{-q_c/2}$$

for each $t \in (-1/4, 0)$*, where* card *denotes cardinality. Here* ε_0 *is given in Theorem* 4.1 (*see also Remark* 5.8) *and* C *is independent of* t *and* M*.*

Proof. Let $t_0 \in (0, -1/4)$. Obviously, it suffices to consider the case $\Sigma(t_0) \ne \emptyset$. Let $x_0 \in \Sigma(t_0)$. We examine the L^{q_c} norm of $\overline{u}$ near (x_0, t_0). From the contraposition of Theorem 4.1 (see also Remark 5.8), it follows that

$$\varepsilon_0 < (\rho/2)^{\frac{4}{p-1}-n} \iint_{Q^+_{\rho/2}(x_0,t_0)} (|\nabla \overline{u}|^2 + |\overline{u}|^{p+1})\, dx\, dt \quad \text{for } 0 < \rho \le 2\delta_0.$$

Lemma A.4 with translation gives

$$\iint_{Q^+_{\rho/2}(x_0,t_0)} |\nabla \overline{u}|^2\, dx\, dt \le C\rho^{n-\frac{4}{p-1}-\frac{4}{q_c}} (1 + M^{p-1})^2 \|\overline{u}\|^2_{L^{q_c}(Q^+_\rho(x_0,t_0))},$$

where $C > 0$ is independent of ρ. The Hölder inequality also gives

$$\begin{aligned}
&\iint_{Q^+_{\rho/2}(x_0,t_0)} |\overline{u}|^{p+1}\, dx\, dt \\
&\quad \le C\rho^{(n+2)(1-\frac{p+1}{q_c})} \left(\iint_{Q^+_{\rho/2}(x_0,t_0)} |\overline{u}|^{q_c}\, dx\, dt \right)^{\frac{p-1}{q_c}} \|\overline{u}\|^2_{L^{q_c}(Q^+_\rho(x_0,t_0))} \\
&\quad \le C\rho^{(n+2)(1-\frac{p+1}{q_c})+\frac{2(p-1)}{q_c}} M^{p-1} \|\overline{u}\|^2_{L^{q_c}(Q^+_\rho(x_0,t_0))}.
\end{aligned}$$

These estimates imply that

$$\varepsilon_0 \le C(1 + M^{p-1})^2 \rho^{-4/q_c} \|\overline{u}\|^2_{L^{q_c}(Q^+_\rho(x_0,t_0))}.$$

Hence there exists a constant $C > 0$ depending on R and independent of ρ, t_0 and M such that

$$\varepsilon_0^{q_c/2} \le C(1 + M^{p-1})^{q_c} \rho^{-2} \int_{t_0-\rho^2}^{t_0} \int_{B^+_\rho(x_0)} |\overline{u}|^{q_c}\, dx\, dt$$

for $0 < \rho \le 2\delta_0$.

Let S be any finite subset of $\Sigma(t_0)$. We write $S = \{x_1, \ldots, x_N\}$ and choose $0 < \rho_0 \le \delta_0$ $(< 1/2)$ such that $\{B^+_{\rho_0}(x_i)\}_{i=1}^N$ is pairwise disjoint. Then

$$\varepsilon_0^{q_c/2} \le C(1 + M^{p-1})^{q_c} \rho_0^{-2} \int_{t_0-\rho_0^2}^{t_0} \int_{B^+_{\rho_0}(x_i)} |\overline{u}|^{q_c}\, dx\, dt$$

for each $1 \le i \le N$, and so also

$$\begin{aligned} N\varepsilon_0^{q_c/2} &\le C(1+M^{p-1})^{q_c}\rho_0^{-2}\int_{t_0-\rho_0^2}^{t_0}\left(\sum_{i=1}^{N}\int_{B^+_{\rho_0}(x_i)}|\overline{u}|^{q_c}\,dx\right)dt \\ &\le C(1+M^{p-1})^{q_c}\rho_0^{-2}\int_{t_0-\rho_0^2}^{t_0}\int_{\mathbf{R}^n_+}|\overline{u}|^{q_c}\,dx\,dt \le C(M+M^p)^{q_c}, \end{aligned}$$

where $C>0$ is independent of ρ_0, t_0, M and N. Hence $\operatorname{card}(S) \le C(M+M^p)^{q_c}\varepsilon_0^{-q_c/2}$. Note that the constant $C(M+M^p)^{q_c}\varepsilon_0^{-q_c/2}$ does not depend on the choice of S. Thus, the number of elements in any subset of $\Sigma(t_0)$ cannot exceed the constant, regardless of whether $\Sigma(t_0)$ contains an accumulation point or not. Therefore we can conclude that $\operatorname{card}(\Sigma(t_0)) \le C(M+M^p)^{q_c}\varepsilon_0^{-q_c/2}$. The proof is complete. ■

Remark 5.12. In the case (3.1), Lemma 5.4 also holds with $\mathcal{B}_\rho$ and $\mathbf{R}^n_+$ replaced by B_ρ and $\mathbf{R}^n$, respectively. In particular, the blow-up limit $\overline{u}$ exists. The analogs to ε-regularity (Theorem 4.1), the equality in Lemma 5.10 and partial regularity (Lemma 5.11) also hold for $\overline{u}$ in the case (3.1).

5.3. *Proof of localized statement*

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. We show this theorem by contradiction under the condition assumed in Section 5.1. We focus on the case (3.2), since (3.1) is easier. From the lower bound (5.3) together with (3.6), (3.7) and (5.4), it follows that

$$\begin{aligned} \varepsilon_0 &< \delta_k^{\frac{4}{p-1}-n}\int_{-\delta_k^2}^{0}\int_{\Phi(\Omega_{\delta_k})}(|\widehat{\nabla}\hat{u}|^2+|\hat{u}|^{p+1})\,d\xi\,dt \\ &= \int_{-1}^{0}\int_{\delta_k^{-1}\Phi(\Omega_{\delta_k})}(|\widehat{\nabla}u_k|^2+|u_k|^{p+1})\,dx\,dt, \end{aligned}$$

where $\widehat{\nabla}\hat{u} := (\nabla'\hat{u}-(\partial_n\hat{u})\nabla' f, \partial_n\hat{u})$ and $\widehat{\nabla}u_k := (\nabla' u_k - (\partial_n u_k)\nabla' f_k, \partial_n u_k)$. By using $\|\nabla' f\|_{L^\infty(\mathbf{R}^{n-1})} \le 1/2$, we have $\delta_k^{-1}\Phi(\Omega_{\delta_k}) \subset B^+_{\sqrt{2}}$. Hence by (5.6) and Lemma 5.4, we see that

$$\int_{-1}^{0}\int_{B^+_{\sqrt{2}}}(|\nabla\overline{u}|^2+|\overline{u}|^{p+1})\,dx\,dt > \varepsilon_0. \tag{5.13}$$

We will show that $\overline{u} \equiv 0$ a.e. in $\mathbf{R}^n_+ \times (-1,0)$, contrary to (5.13).

By Lemma 5.10, it follows that

$$\frac{d}{d\lambda}(\lambda^{\frac{2}{p-1}}\overline{u}(\lambda x, \lambda^2 t)) = 2\lambda^{\frac{2}{p-1}-1}\left(\frac{\overline{u}(y,s)}{p-1}+\frac{y\cdot\nabla_y\overline{u}}{2}+s\overline{u}_s\right) = 0$$

for $y=\lambda x$, $s=\lambda^2 t$ and $0<\lambda<1/\sqrt{-t}$ in the weak sense. Hence $\overline{u}$ is self-similar, and so there exists a profile function $\overline{U} \in L^{q_c}(\mathbf{R}^n_+)$ such that

$$\overline{u}(x,t) = (-t)^{-\frac{1}{p-1}}\overline{U}(z), \quad z := x/\sqrt{-t},$$

where $\bar{U}$ is a weak solution of the equation

$$\Delta\bar{U} - \frac{1}{2}z\cdot\nabla\bar{U} - \frac{1}{p-1}\bar{U} + |\bar{U}|^{p-1}\bar{U} = 0, \quad z \in \mathbf{R}^n_+. \tag{5.14}$$

We note that the method for proving self-similarity can also be found in [61, Lemma 8.5.3] and [97, 102].

We claim that there exist constants $\tilde{R}, C > 1$ satisfying

$$\begin{cases} \bar{u}(\cdot,0) = 0 & \text{a.e. in } \mathbf{R}^n_+, \\ |\bar{u}| \le C & \text{a.e. in } (\mathbf{R}^n_+ \setminus B_{\tilde{R}}) \times [-1/9, 0]. \end{cases} \tag{5.15}$$

Recall from the proof of Lemma 5.4 that $u_k \to \bar{u}$ in $C([-1,0]; W^{1,r}(\mathcal{B}_\rho))$ as $k \to \infty$ for any $\rho > 0$ with some $1 < r < \min\{2, q_c/p\}$. Then, for $x_0 \in \mathbf{R}^n_+$ and $\delta > 0$, we have

$$\int_{B_1^+(x_0)} |\bar{u}(x,0)|^r\,dx \le 2^{r-1}\delta + 2^{r-1}\int_{B_1^+(x_0)} |u_k(x,0)|^r\,dx$$

for sufficiently large k. By returning to the original variables, we also have

$$\begin{aligned} \int_{B_1^+(x_0)} |u_k(x,0)|^r\,dx &= \delta_k^{\frac{2r}{p-1}-n}\int_{\Psi(B^+_{\delta_k}(\delta_k x_0))} |u(x,0)|^r\,dx \\ &\le \delta_k^{\frac{2r}{p-1}-n}\int_{\Psi(B^+_{(1+|x_0|)\delta_k}(0))} |u(x,0)|^r\,dx \\ &\le C(1+|x_0|)^{n-\frac{2r}{p-1}}\|u(\cdot,0)\|^r_{L^{q_c}(\Psi(B^+_{(1+|x_0|)\delta_k}(0)))} \to 0 \end{aligned}$$

as $k \to \infty$, where u belongs to $C_{\mathrm{weak}}([-1/4,0]; L^{q_c}(\Omega_{R/2}))$ in our situation (see (5.2)). Hence we obtain the equality on the first line of (5.15).

We now prove the inequality in (5.15). Let $\tilde{\varepsilon} > 0$. We claim that there exists a constant $\tilde{R} > 1$ depending on $\tilde{\varepsilon}$ and M satisfying

$$\iint_{Q^+_{1/2}(x_1,0)} (|\nabla\bar{u}|^2 + |\bar{u}|^{p+1})\,dx\,dt \le \tilde{\varepsilon} \tag{5.16}$$

for any $x_1 \in \mathbf{R}^n_+ \setminus B^+_{\tilde{R}}(0)$, where $Q^+_{1/2}(x_1,0) = B^+_{1/2}(x_1) \times (-1/4, 0)$. Indeed, for $\tilde{\varepsilon}' > 0$, by Lemma 5.4 (iii) we can choose $\tilde{R}' > 1$ so large that

$$\int_{-1}^0\int_{\mathbf{R}^n_+\setminus B_{\tilde{R}'-1}(0)} |\bar{u}|^{q_c}\,dx\,dt \le \tilde{\varepsilon}'.$$

From the Hölder inequality, it follows that

$$\iint_{Q_1^+(x_1,0)} |\bar{u}|^{p+1}\,dx\,dt \le C\left(\iint_{Q_1^+(x_1,0)} |\bar{u}|^{q_c}\,dx\,dt\right)^{\frac{p+1}{q_c}} \le C(\tilde{\varepsilon}')^{\frac{p+1}{q_c}}$$

for any $x_1 \in \mathbf{R}^n_+ \setminus B^+_{\tilde{R}'}(0)$. On the other hand, Lemma A.4 with $\rho = 1$ yields

$$\iint_{Q^+_{1/2}(x_1,0)} |\nabla \bar{u}|^2 \, dx \, dt \le C(1 + M^{p-1})^2 \left(\iint_{Q^+_1(x_1,0)} |\bar{u}|^{q_c} \, dx \, dt \right)^{2/q_c}$$
$$\le C(1 + M^{p-1})^2 (\tilde{\varepsilon}')^{2/q_c}.$$

Thus choosing $\tilde{\varepsilon}'$ small gives (5.16), and so the claim follows. Hence by translation and Theorem 4.1 (see also Remark 5.8), we see that $|\bar{u}| \le C\delta_0^{-2/(p-1)}$ in $B^+_{\delta_0}(x_1) \times [-1/9, 0]$ for any $x_1 \in \mathbf{R}^n_+ \setminus B^+_{\tilde{R}}(0)$, where C and δ_0 are constants independent of x_1. This proves the inequality in (5.15).

From (5.15) and the backward uniqueness theorem [35, Theorem 5.1], it follows that

$$\bar{u} \equiv 0 \quad \text{on } (\mathbf{R}^n_+ \setminus B_{\tilde{R}}) \times [-1/9, 0].$$

In particular,

$$\bar{u}(x, -1/9) = 9^{-1/(p-1)} \bar{U}(3x) = 0 \quad \text{for } x \in \mathbf{R}^n_+ \setminus B^+_{\tilde{R}}.$$

Since $\bar{U}$ satisfies (5.14) and is smooth except on a finite set by Lemma 5.11, the unique continuation theorem for elliptic equations in a connected set (see [5] for instance) implies that $\bar{U} \equiv 0$ in $\mathbf{R}^n_+$. This contradicts (5.13), and hence the proof of the localized statement is complete. ∎

5.4. *Completion of proof*

Proof of Theorem 1.1. If Ω is bounded, then Theorem 1.1 immediately follows from Theorem 5.1. In what follows, we consider the case where Ω is unbounded. To obtain a contradiction, suppose that

$$\sup_{0<t<T} \|u(\cdot, t)\|_{L^{q_c}(\Omega)} \le M$$

for some $M > 0$. Let $\varepsilon > 0$ and $a \in \Omega$. Then by the same argument as in the derivation of (5.16), there exists a constant $\tilde{R} > 0$ depending on ε, a and M satisfying

$$\int_{T/2}^{T} \int_{\Omega_{1/2}(\tilde{x})} (|u|^{p+1} + |\nabla u|^2) \, dx \, dt \le \varepsilon$$

for any $\tilde{x} \in \Omega \setminus B_{\tilde{R}}(a)$. Therefore, in the same way as in the proof of (5.15), we see that u is bounded on $(\Omega \setminus B_{\tilde{R}'}(a)) \times (T/3, T)$ for some $\tilde{R}' > 0$. This implies that there exists at least one blow-up point $a' \in \overline{\Omega_{\tilde{R}'}(a)}$. Hence Theorem 5.1 shows that

$$\limsup_{t \to T} \|u(\cdot, t)\|_{L^{q_c}(\Omega_r(a'))} = \infty$$

for any $r > 0$, a contradiction. The proof of Theorem 1.1 is complete. ∎

Appendix A. Regularity estimates

We give some parabolic regularity estimates for solutions of (2.1) and a gradient estimate for the blow-up limit obtained in Lemma 5.4. Let $n \geq 3$, $p > p_S$ and Ω be any $C^{2+\alpha}$ domain in $\mathbf{R}^n$ with $0 \in \overline{\Omega}$. Fix $R > 0$ such that either (3.1) or (3.2) holds. Let u satisfy (2.1) and (2.2). Define $\hat{u}$ by (3.6). For $\rho > 0$, we take $0 < \delta' < 1/2$ such that $\Psi(B^+_{3\rho\delta'}) \subset \Omega_{R/2}$ and $\Psi(B_{3\rho\delta'}) \subset B_{R/2}$. We first give parabolic regularity estimates. Note that we mainly focus on the case (3.2), since (3.1) is easier.

Lemma A.1. *Assume* (3.2)*. Let* $1 \leq l < \infty$ *and* $1 \leq r \leq q_c/p$*. Then there exists a constant* $C > 0$ *such that*

$$\|\hat{u}_t\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} + \|\nabla^2 \hat{u}\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} \leq C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{p}{q_c})}(M + M^p)$$

for any $0 < \delta < \delta'$*, where* C *depends on* R *and* ρ *and is independent of* δ*.*

Proof. By (3.6), we have

$$|\nabla^2 \hat{u}(x,t)| \leq C\big(|(\nabla u)(\Psi(x),t)| + |(\nabla^2 u)(\Psi(x),t)|\big),$$

where $C > 0$ depends on $\|\nabla f\|_{L^\infty(\mathbf{R}^{n-1})}$ and $\|\nabla^2 f\|_{L^\infty(\mathbf{R}^{n-1})}$. Note that the choice of δ' guarantees $\Psi(B^+_{\rho\delta}) \subset \Omega_{3R/4}$, and so Proposition 2.1 is applicable in $\Psi(B^+_{\rho\delta})$. Then from $r \leq q_c/p < q_*$, the Hölder inequality in Lorentz spaces (see [57, Proposition 2.1] for instance) and Proposition 2.1, we see that

$$\begin{aligned} &\|\nabla u(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} \\ &\qquad \leq C\delta^{n(\frac{1}{r}-\frac{1}{q_*})}\|\nabla u(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^{q_*,\infty}(B^+_{\rho\delta}))} \\ &\qquad \leq C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{1}{q_*})}(M + M^p) \leq C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{p}{q_c})}(M + M^p). \end{aligned} \tag{A.1}$$

We now estimate $\|\nabla^2 u(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))}$. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \leq \phi \leq 1$ in $\mathbf{R}^n$, $\phi = 0$ in $\mathbf{R}^n \setminus B_{3\rho}$ and $\phi = 1$ in $B_{2\rho}$. Set $\tilde{\phi}(x) := \phi(\delta^{-1}\Phi(x))$ and $v(x,t) := u(x,t)\tilde{\phi}(x)$. We consider a $C^{2+\alpha}$ domain $\mathcal{D}$ satisfying $\Psi(B^+_{3\rho\delta'}) \subset \mathcal{D} \subset \Omega_R$ to avoid technicalities due to the corner of $\partial B^+_{3\rho\delta'}$. Then v satisfies

$$\begin{cases} v_t - \Delta v = \tilde{\phi}|u|^{p-1}u - 2\nabla\tilde{\phi}\cdot\nabla u - u\Delta\tilde{\phi} & \text{in } \mathcal{D}\times(-1,0), \\ v = 0 & \text{on } \partial\mathcal{D}\times(-1,0). \end{cases}$$

By the same computation as in the derivation of (2.3), we have

$$\begin{aligned} u(x,t) = &\int_{\mathcal{D}} G_{\mathcal{D}}(x,y,t+2\delta^2)\tilde{\phi}(y)u(y,-2\delta^2)\,dy \\ &+ \int_{-2\delta^2}^{t}\int_{\mathcal{D}} G_{\mathcal{D}}(x,y,t-s)\tilde{\phi}|u|^{p-1}u\,dy\,ds \\ &- \int_{-2\delta^2}^{t}\int_{\mathcal{D}} G_{\mathcal{D}}(x,y,t-s)(2\nabla\tilde{\phi}\cdot\nabla u + u\Delta\tilde{\phi})\,dy\,ds \end{aligned}$$

for $x \in \Psi(B^+_{\rho\delta})$ and $-2\delta^2 < t < 0$, and so

$$\begin{aligned}|\nabla^2 u(x,t)| \le C &\int_{\mathbf{R}^n} K_2(x-y, t+2\delta^2)|u(y,-2\delta^2)|\chi_{\Omega_R \cap \Psi(B_{3\rho\delta})}(y)\,dy \\ &+ \left|\nabla^2 \int_{-2\delta^2}^{t} \int_{\mathcal{D}} G_{\mathcal{D}}(x,y,t-s)\tilde{\phi}|u|^{p-1}u\,dy\,ds\right| \\ &+ C\int_{-2\delta^2}^{t}\int_{\mathbf{R}^n} K_2\left(\frac{|u|}{\delta^2} + \frac{|\nabla u|}{\delta}\right)\chi_{\Omega_R \cap \Psi(\overline{B_{3\rho\delta}} \setminus B_{2\rho\delta})}\,dy\,ds \\ =:\ & CV_1(x,t) + |\nabla^2 V_2(x,t)| + CV_3(x,t)\end{aligned}$$

for $x \in \Psi(B^+_{\rho\delta})$ and $-2\delta^2 < t < 0$, where K_2 is defined by (2.6).

First, we estimate V_1. Since $\|\nabla' f\|_{L^\infty(\mathbf{R}^{n-1})} \le 1/2$ in (3.2), we have

$$|\Psi(x) - \Psi(y)|^2 \ge |x' - y'|^2 + \tfrac{1}{2}(x_n - y_n)^2 - (f(x') - f(y'))^2 \ge \tfrac{1}{4}|x-y|^2.$$

This together with a change of variables shows that

$$\begin{aligned}V_1(\Psi(x),t) &= \int_{\mathbf{R}^n} K_2(\Psi(x) - \Psi(y), t+2\delta^2)|u(\Psi(y),-2\delta^2)|\chi_{\Phi(\Omega_R)\cap B_{3\rho\delta}}\,dy \\ &\le \int_{\mathbf{R}^n} K_2((x-y)/2, t+2\delta^2)|u(\Psi(y),-2\delta^2)|\chi_{\Phi(\Omega_R)\cap B_{3\rho\delta}}\,dy. \qquad \text{(A.2)}\end{aligned}$$

Then Young's inequality gives

$$\begin{aligned}\|V_1(\Psi(\cdot),t)\|_{L^{q_c}(B^+_{\rho\delta})} &\le C(t+2\delta^2)^{-1}\|u(\Psi(\cdot),-2\delta^2)\|_{L^{q_c}(\Phi(\Omega_R)\cap\Psi(B_{3\rho\delta}))} \\ &\le C\delta^{-2}M\end{aligned}$$

for $-\delta^2 < t < 0$. Therefore, the Hölder inequality shows that

$$\begin{aligned}&\|V_1(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} \\ &\quad \le C\delta^{n(\frac{1}{r}-\frac{1}{q_c})}\|V_1(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^{q_c}(B^+_{\rho\delta}))} \le C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{p}{q_c})}M. \qquad \text{(A.3)}\end{aligned}$$

Let us next estimate $\nabla^2 V_2$. From the Hölder inequality, the change of variables $y = \Psi(x)$ with $dy = dx$ and the choice of $\mathcal{D}$, it follows that

$$\begin{aligned}\|\nabla^2 V_2(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} &\le C\delta^{n(\frac{1}{r}-\frac{p}{q_c})}\|\nabla^2 V_2(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^{q_c/p}(B^+_{\rho\delta}))} \\ &\le C\delta^{n(\frac{1}{r}-\frac{p}{q_c})}\|\nabla^2 V_2\|_{L^l(-\delta^2,0;L^{q_c/p}(\mathcal{D}))}\end{aligned}$$

We observe that V_2 is a solution of

$$\begin{cases}(V_2)_t - \Delta V_2 = \tilde{\phi}|u|^{p-1}u & \text{in } \mathcal{D} \times (-2\delta^2, 0), \\ V_2 = 0 & \text{on } \partial\mathcal{D} \times (-2\delta^2, 0), \\ V_2(\cdot, -2\delta^2) = 0 & \text{in } \mathcal{D}.\end{cases}$$

Since $\mathcal{D}$ is a bounded $C^{2+\alpha}$ domain, it is also a uniformly regular domain of class C^2. Therefore, we can apply the maximal regularity results for inhomogeneous heat equations (see [90, Remark 51.5] and [28, Theorem 7.11] for instance), and so

$$\begin{aligned}\|\nabla^2 V_2\|_{L^l(-2\delta^2,0;L^{q_c/p}(\mathcal{D}))} &\le C\|\tilde{\phi}|u|^{p-1}u\|_{L^l(-2\delta^2,0;L^{q_c/p}(\mathcal{D}))}\\ &\le C\|u\|^p_{L^{pl}(-2\delta^2,0;L^{q_c}(\Omega_R))} \le C\delta^{2/l}M^p.\end{aligned}$$

Thus,

$$\|\nabla^2 V_2(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} \le C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{p}{q_c})}M^p. \tag{A.4}$$

Finally, we consider V_3. Again in the same way as in (A.2), we have

$$\begin{aligned}V_3(\Psi(x),t) \le \int_{-2\delta^2}^{t}\int_{\mathbf{R}^n} &K_2((x-y)/2,t-s)\\ &\times(\delta^{-2}|u|+\delta^{-1}|\nabla u|)(\Psi(y),s)\chi_{\Phi(\Omega_R)\cap(\overline{B_{3\rho\delta}}\setminus B_{2\rho\delta})}\,dy\,ds\end{aligned}$$

for $x \in B^+_{\rho\delta}$ and $-2\delta^2 < t < 0$. We observe that there exists $C > 0$ satisfying

$$K_2((x-y)/2,t-s) \le C\delta^{-n-2}$$

for $x \in B^+_{\rho\delta}$, $y \in \overline{B_{3\rho\delta}} \setminus B_{2\rho\delta}$ and $-2\delta^2 < s < t < 0$. Then by a change of variables, the Hölder inequality, $\Omega_R \cap \Psi(B_{3\rho\delta}) \subset \Omega_{3R/4}$ and Proposition 2.1, we see that

$$\begin{aligned}V_3(\Psi(x),t) &\le C\delta^{-n-2}\int_{-2\delta^2}^{t}\int_{\Phi(\Omega_R)\cap B_{3\rho\delta}}(\delta^{-2}|u(\Psi(y),s)|+\delta^{-1}|\nabla u(\Psi(y),s)|)\,dy\,ds\\ &\le C\delta^{-n-2}(M\delta^{n(1-\frac{1}{q_c})}+(M+M^p)\delta^{n(1-\frac{1}{q_*})+1}) \le C\delta^{-\frac{np}{q_c}}(M+M^p)\end{aligned}$$

for $x \in B^+_{\rho\delta}$ and $-2\delta^2 < t < 0$. Hence we obtain

$$\|V_3(\Psi(\cdot),\cdot)\|_{L^l(-\delta^2,0;L^r(B^+_{\rho\delta}))} \le C\delta^{\frac{2}{l}+n(\frac{1}{r}-\frac{p}{q_c})}(M+M^p).$$

By combining this inequality, (A.1), (A.3) and (A.4), we obtain the desired estimate for $\nabla^2\hat{u}$. The estimate for $\hat{u}_t$ can be obtained by using the equation in (3.8). ■

Lemma A.2. *Assume either* (3.1) *or* (3.2)*. Let* $1 \le l < \infty$ *and* $1 \le r \le q_c/p$*. Then there exists a constant* $C > 0$ *depending on* R *such that*

$$\|u_t\|_{L^l(-1/4,0;L^r(\Omega_{R/2}))} + \|\nabla^2 u\|_{L^l(-1/4,0;L^r(\Omega_{R/2}))} \le C(M+M^p).$$

Proof. By easy modifications of Lemma A.1, we can see that

$$\|\nabla^2 u\|_{L^l(-1/4,0;L^r(\Omega_{R/2}))} \le C(M+M^p).$$

Then by the equation in (2.1), we obtain the desired inequality. ■

Let us next show a gradient estimate for the blow-up limit $\overline{u}$ obtained in Lemma 5.4. To estimate $\nabla\overline{u}$, we derive a localized integral equation for $\overline{u}$.

Lemma A.3. *Assume* (3.2). *Let* $\phi \in C_0^\infty(\mathbf{R}^n)$ *satisfy* $0 \le \phi \le 1$ *in* $\mathbf{R}^n$, $\phi = 0$ *in* $\mathbf{R}^n \setminus B_{3/5}$ *and* $\phi = 1$ *in* $B_{4/5}$. *Set* $\phi_\rho(x) := \phi(x/\rho)$ *for* $0 < \rho < 1$. *Then* $\overline{u}$ *satisfies*

$$\begin{aligned}\overline{u}(x,t) &= \int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,\rho^2/4)\phi_\rho(y)\overline{u}(y,t-\rho^2/4)\,dy\\&\quad+\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,t-s)(\phi_\rho|\overline{u}|^{p-1}\overline{u}+\overline{u}\Delta\phi_\rho)\,dy\,ds\\&\quad+2\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} \nabla_y G_{\mathbf{R}^n_+}(x,y,t-s)\cdot\nabla\phi_\rho(y)\overline{u}(y,s)\,dy\,ds\end{aligned}$$

for a.e. $x \in B^+_{\rho/2}$ *and* $-\rho^2/4 < t < 0$.

Proof. Let us convert our problem to one in $\mathbf{R}^n_+$. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ satisfy $0 \le \psi \le 1$ in $\mathbf{R}^n$, $\psi = 0$ in $\mathbf{R}^n \setminus B_{4R/5}$ and $\psi = 1$ in $B_{3R/5}$. Set $\psi_k(x) := \psi(\Psi(\delta_k x))$ and $v_k := \phi_\rho\psi_k u_k$. Note that $\psi_k = 0$ in $\mathbf{R}^n \setminus \delta_k^{-1}\Phi(B_{4R/5})$ and $\psi_k = 1$ in $\delta_k^{-1}\Phi(B_{3R/5})$. Then by (5.5), we see that

$$\begin{cases}(v_k)_t - \Delta v_k = \phi_\rho\psi_k|u_k|^{p-1}u_k - \psi_k u_k\Delta\phi_\rho - 2\psi_k\nabla\phi_\rho\cdot\nabla u_k + \mathcal{R}_k\\ \hfill \text{in } \mathbf{R}^n_+ \times (-\delta_k^{-2},0),\\ v_k = 0 \quad \text{on } \partial\mathbf{R}^n_+ \times (-\delta_k^{-2},0),\end{cases}$$

where

$$\begin{aligned}\mathcal{R}_k := {}&\phi_\rho\psi_k\big(-2\nabla'(\partial_{x_n}u_k)\cdot\nabla' f_k + (\partial^2_{x_n}u_k)|\nabla' f_k|^2 - (\partial_{x_n}u_k)\Delta' f_k\big)\\&-(2u_k\nabla\psi_k\cdot\nabla\phi_\rho + \phi_\rho u_k\Delta\psi_k + 2\phi_\rho\nabla\psi_k\cdot\nabla u_k).\end{aligned}$$

Thus,

$$\begin{aligned}u_k(x,t) &= \int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,\rho^2/4)\rho_\rho(y)\psi_k(y)u_k(y,t-\rho^2/4)\,dy\\&\quad+\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,t-s)\phi_\rho\psi_k|u_k|^{p-1}u_k\,dy\,ds\\&\quad-\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(\psi_k u_k\Delta\phi_\rho + 2\psi_k\nabla\phi_\rho\cdot\nabla u_k + \mathcal{R}_k)\,dy\,ds\\&=: V_1^k(x,t) + V_2^k(x,t) + V_3^k(x,t)\end{aligned}$$

for $x \in B^+_{\rho/2}$, $-\rho^2/4 < t < 0$ and $k \ge k_\rho$, where k_ρ is given by the first part of Section 5.1. Lemma 5.4 (ii) shows that $u_k(\cdot,t)$ converges to $\overline{u}(\cdot,t)$ in $L^1(B^+_{\rho/2})$ for each $-\rho^2/4 < t < 0$ as $k \to \infty$.

We show that the right-hand side of the integral equation for a subsequence of u_k still denoted by u_k converges to the one in the following integral equation:

$$\begin{aligned}\overline{u}(x,t) &= \int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,\rho^2/4)\phi_\rho(y)\overline{u}(y,t-\rho^2/4)\,dy \\ &\quad + \int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,t-s)\phi_\rho|\overline{u}|^{p-1}\overline{u}\,dy\,ds \\ &\quad - \int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(x,y,t-s)(\overline{u}\Delta\phi_\rho + 2\nabla\phi_\rho\cdot\nabla\overline{u})\,dy\,ds \\ &=: \overline{V}_1(x,t) + \overline{V}_2(x,t) + \overline{V}_3(x,t) \end{aligned} \tag{A.5}$$

for a.e. $x \in B^+_{\rho/2}$ and $-\rho^2/4 < t < 0$. For V^k_1, from (2.5), $0 \le \psi_k \le 1$, the Hölder inequality and Lemma 5.4 (ii, iii), it follows that

$$\begin{aligned}\|V^k_1(\cdot,t) - \overline{V}_1(\cdot,t)\|_{L^1(B^+_{\rho/2})} &\le C\rho^{-n}\int_{B^+_{\rho/2}}\int_{\mathbf{R}^n_+}\phi_\rho|\psi_k u_k - \overline{u}|\,dy\,dx \\ &\le C\|u_k(\cdot,t-\rho^2/4) - \overline{u}(\cdot,t-\rho^2/4)\|_{L^1(B^+_\rho)} \\ &\quad + C\|(\psi_k - 1)\overline{u}(\cdot,t-\rho^2/4)\|_{L^1(B^+_\rho)} \to 0\end{aligned}$$

as $k \to \infty$. For V^k_2, by computations similar to that for V^k_1 with Young's inequality, we see that, as $k \to \infty$,

$$\begin{aligned}\|V^k_2(\cdot,t) - \overline{V}_2(\cdot,t)\|_{L^1(B^+_{\rho/2})} &\le C\int_{t-\rho^2/4}^{t}\|\psi_k|u_k|^{p-1}u_k - |\overline{u}|^{p-1}\overline{u}\|_{L^1(B^+_\rho)}\,ds \\ &\le C\int_{t-\rho^2/4}^{t}\big\|(|u_k|^{p-1} + |\overline{u}|^{p-1})|u_k - \overline{u}|\big\|_{L^1(B^+_\rho)}\,ds \\ &\quad + C\int_{t-\rho^2/4}^{t}\|(\psi_k - 1)|\overline{u}|^p\|_{L^1(B^+_\rho)}\,ds \to 0.\end{aligned}$$

For V^k_3, we focus on the most subtle term $\phi_\rho\psi_k\nabla'(\partial_{x_n}u_k)\cdot\nabla' f_k$ in $\mathcal{R}_k$, that is, we prove that

$$\tilde{V}^k_3(\cdot,t) := \int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} G_{\mathbf{R}^n_+}(\cdot,y,t-s)\phi_\rho\psi_k\nabla'(\partial_{x_n}u_k)\cdot\nabla' f_k\,dy\,ds \to 0$$

in $L^1(B^+_{\rho/2})$ for each $-\rho^2/4 < t < 0$ as $k \to \infty$. By integration by parts and (2.5), it follows that

$$\begin{aligned}|\tilde{V}^k_3(x,t)| &\le C\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} K_1(x-y,t-s)\phi_\rho\psi_k|\partial_{x_n}u_k|\,|\nabla' f_k|\,dy\,ds \\ &\quad + C\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} K_0|\nabla'(\phi_\rho\psi_k)|\,|\partial_{x_n}u_k|\,|\nabla' f_k|\,dy\,ds \\ &\quad + C\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+} K_0\phi_\rho\psi_k|\partial_{x_n}u_k|\,|\Delta' f_k|\,dy\,ds.\end{aligned}$$

By Young's inequality, $|\nabla\phi_\rho| \le C$, $|\nabla\psi_k| \le C\delta_k \le C$, Hölder's inequality and $\int_{t-\rho^2/4}^{t}(t-s)^{-(1/2)\cdot(4/3)}\,ds \le C$, we have

$$\|\tilde{V}_3^k(\cdot,t)\|_{L^1(B^+_{\rho/2})} \le C\left(\int_{t-\rho^2/4}^{t}\big\||\nabla u_k(\cdot,s)|\,|\nabla' f_k|\big\|^4_{L^1(B^+_\rho)}\,ds\right)^{1/4} + C\int_{t-\rho^2/4}^{t}\big\||\nabla u_k(\cdot,s)|(|\nabla' f_k| + |\Delta' f_k|)\big\|_{L^1(B^+_\rho)}\,ds.$$

From the Hölder inequality for Lorentz spaces, (5.6)–(5.8) and the Lebesgue dominated convergence theorem, it follows that

$$\|\tilde{V}_3^k(\cdot,t)\|_{L^1(B^+_{\rho/2})} \le C(M+M^p)\big\||\nabla' f_k| + |\Delta' f_k|\big\|_{L^{\frac{q_*}{q_*-1}}(B^+_\rho)} \to 0$$

for each $-\rho^2/4 < t < 0$ as $k \to \infty$. The other terms in V_3^k can be handled more easily. Hence we obtain (A.5). This implies the desired equality. ∎

We now give a gradient estimate for $\overline{u}$.

Lemma A.4. *Assume* (3.2). *Then there exists a constant* $C > 0$ *independent of* ρ *such that*

$$\|\nabla\overline{u}\|_{L^2(Q^+_{\rho/2})} \le C\rho^{\frac{n}{2}-\frac{2}{p-1}-\frac{2}{q_c}}(1+M^{p-1})\|\overline{u}\|_{L^{q_c}(Q^+_\rho)}$$

for any $0 < \rho \le 1$.

Proof. By Lemma A.3, $\nabla\phi_\rho(x) = \rho^{-1}\nabla\phi(x/\rho)$, $\Delta\phi_\rho(x) = \rho^{-2}\Delta\phi(x/\rho)$ and similar computations to (2.7), there exists $C > 0$ independent of ρ such that

$$\begin{aligned}
|\nabla\overline{u}| &\le C\int_{B^+_\rho} K_1(x-y,\rho^2/4)|\overline{u}(y,t-\rho^2/4)|\,dy \\
&\quad + C\int_{t-\rho^2/4}^{t}\int_{B^+_\rho} K_1(x-y,t-s)|\overline{u}(y,s)|^p\,dy\,ds \\
&\quad + C\int_{t-\rho^2/4}^{t}\int_{\mathbf{R}^n_+}\left(\frac{K_1}{\rho^2}+\frac{K_2}{\rho}\right)|\overline{u}|\chi_{\overline{B_{4\rho/5}}\setminus B_{3\rho/5}}\,dy\,ds \\
&=: CW_1 + CW_2 + C\rho^{-2}W_3 + C\rho^{-1}W_4
\end{aligned}$$

for a.e. $x \in B^+_{\rho/2}$ and $-\rho^2/4 < t < 0$, where K_1 and K_2 are given by (2.6).

By the Hölder inequality, we have

$$\begin{aligned}
W_1(x,t) &\le \|\overline{u}(\cdot,t-\rho^2/4)\|_{L^{q_c}(B^+_\rho)}\left(\int_{B^+_\rho} K_1(x-y,\rho^2/4)^{\frac{q_c}{q_c-1}}\,dy\right)^{1-1/q_c} \\
&\le C\rho^{-1-\frac{n}{q_c}}\|\overline{u}(\cdot,t-\rho^2/4)\|_{L^{q_c}(B^+_\rho)}, \\
\|W_1\|_{L^2(Q^+_{\rho/2})} &\le C\rho^{-\frac{n}{q_c}+\frac{n}{2}-\frac{2}{q_c}}\left(\int_{-\rho^2/4}^{0}\int_{L^{q_c}(B^+_\rho)}|\overline{u}(x,t-\rho^2/4)|^{q_c}\,dx\,dt\right)^{1/q_c} \\
&\le C\rho^{\frac{n}{2}-\frac{2}{p-1}-\frac{2}{q_c}}\|\overline{u}\|_{L^{q_c}(Q^+_\rho)}.
\end{aligned}$$

We now estimate W_2. We consider the cases $q_c/p \geq 2$ and $q_c/p < 2$. If $q_c/p \geq 2$, then the Hölder inequality gives

$$\|W_2\|_{L^2(Q^+_{\rho/2})} \leq C\rho^{\frac{n+2}{2}(1-\frac{2p}{q_c})}\|W_2\|_{L^{q_c/p}(Q^+_{\rho/2})}.$$

From the argument used to prove the $L^{q_c/p}$-$L^{q_c/p}$ estimate, it follows that

$$\begin{aligned}\|W_2(\cdot,t)\|_{L^{q_c/p}(B^+_{\rho/2})} &\leq C\int_{t-\rho^2/4}^{t}(t-s)^{-1/2}\big\||\overline{u}(\cdot,s)|^p\chi_{B^+_\rho}\big\|_{L^{q_c/p}(\mathbf{R}^n)}\,ds\\ &= C\int_0^{\rho^2/4}\tau^{-1/2}\|\overline{u}(\cdot,t-\tau)\|^p_{L^{q_c}(B^+_\rho)}\,d\tau.\end{aligned}$$

Thus, as $(-\tau-\rho^2/4,-\tau)\subset(-\rho^2,0)$ for $0<\tau<\rho^2/4$, we have

$$\begin{aligned}\|W_2\|_{L^{q_c/p}(Q^+_{\rho/2})} &\leq C\int_0^{\rho^2/4}\tau^{-1/2}\bigg(\int_{-\rho^2/4}^{0}\int_{B^+_\rho}|\overline{u}(\cdot,t-\tau)|^{q_c}\,dx\,dt\bigg)^{p/q_c}d\tau\\ &\leq C\int_0^{\rho^2/4}\tau^{-1/2}\bigg(\int_{-\rho^2}^{0}\int_{B^+_\rho}|\overline{u}(\cdot,s)|^{q_c}\,dx\,ds\bigg)^{p/q_c}d\tau\\ &\leq C\rho^{1+2(p-1)/q_c}M^{p-1}\|\overline{u}\|_{L^{q_c}(Q^+_\rho)},\end{aligned}$$

and so

$$\|W_2\|_{L^2(Q^+_{\rho/2})} \leq C\rho^{\frac{n}{2}-\frac{2}{p-1}-\frac{2}{q_c}}M^{p-1}\|\overline{u}\|_{L^{q_c}(Q^+_\rho)}.$$

If $q_c/p < 2$, then the argument used to prove the $L^{q_c/p}$-L^2 estimate yields

$$\begin{aligned}\|W_2(\cdot,t)\|_{L^2(B^+_\rho)} &\leq C\int_{t-\rho^2/4}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{p}{q_c}-\frac{1}{2})}\|\,|\overline{u}(\cdot,s)|^p\chi_{B^+_\rho}\|_{L^{q_c/p}(\mathbf{R}^n)}\,ds\\ &\leq C\int_{\mathbf{R}}|t-s|^{-1+\gamma}\|\overline{u}(\cdot,s)\|^p_{L^{q_c}(B^+_\rho)}\chi_{(-\rho^2,0)}(s)\,ds\\ &\leq CM^{p-1}\int_{\mathbf{R}}|t-s|^{-1+\gamma}\|\overline{u}(\cdot,s)\|_{L^{q_c}(B^+_\rho)}\chi_{(-\rho^2,0)}\,ds,\end{aligned}$$

where

$$\gamma := \frac{1}{2}-\frac{n}{2}\bigg(\frac{p}{q_c}-\frac{1}{2}\bigg).$$

Note that $0<\gamma<1/2$ since $p>p_S$ and $q_c/p<2$. From the Hardy–Littlewood–Sobolev inequality and the Hölder inequality with $2/(2\gamma+1)<q_c$, it follows that

$$\begin{aligned}\|W_2\|_{L^2(Q^+_{\rho/2})} &= \big\|\|W_2(\cdot,t)\|_{L^2(B^+_\rho)}\chi_{(-\rho^2/4,0)}\big\|_{L^2(\mathbf{R})}\\ &\leq CM^{p-1}\big\|\|\overline{u}(\cdot,t)\|_{L^{q_c}(B^+_\rho)}\chi_{(-\rho^2,0)}\big\|_{L^{\frac{2}{2\gamma+1}}(\mathbf{R})}\\ &\leq C\rho^{\frac{n}{2}-\frac{2}{p-1}-\frac{2}{q_c}}M^{p-1}\|\overline{u}\|_{L^{q_c}(Q^+_\rho)}.\end{aligned}$$

We now consider W_3 and W_4. By the definitions of K_1 and K_2 in (2.6), there exists a constant $C > 0$ independent of ρ such that

$$\rho^{-2}K_1(x-y,t)+\rho^{-1}K_2(x-y,t)\le C\rho^{-n-3}$$

for $x \in B_{\rho/2}$, $y \in \overline{B_{4\rho/5}} \setminus B_{3\rho/5}$ and $t > 0$. Hence by the Hölder inequality, we have

$$\begin{aligned}&\rho^{-2}|W_3(x,t)|+\rho^{-1}|W_4(x,t)|\le C\rho^{-n-3+(n+2)(1-\frac{1}{qc})}\|\bar{u}\|_{L^{qc}(Q^+_\rho)},\\&\rho^{-2}\|W_3\|_{L^2(Q^+_{\rho/2})}+\rho^{-1}\|W_4\|_{L^2(Q^+_{\rho/2})}\le C\rho^{\frac{n}{2}-\frac{2}{p-1}-\frac{2}{qc}}\|\bar{u}\|_{L^{qc}(Q^+_\rho)}.\end{aligned}$$

The above estimates yield the desired inequality. ■

Remark A.5. Regularity estimates similar to the above are known for semilinear elliptic equations; see [52] and the references given there for recent developments.

Appendix B. Compactness results

We recall an Aubin–Lions type compactness result from [90, Proposition 51.3] and [87, Proposition 2.1]. See also [3] and [4, Sections 2.7, 2.8] for more general statements. Note that a pair (E_0, E_1) of Banach spaces is called an *interpolation couple* if there exists a locally convex space E such that $E_0, E_1 \hookrightarrow E$.

Proposition B.1. *Let (E_0, E_1) be an interpolation couple. Assume that E_1 is compactly embedded in E_0. Let $1 \le p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $s < 1-\theta$. Then*

$$W^{1,p_0}(0,T;E_0)\cap L^{p_1}(0,T;E_1)\hookrightarrow W^{s,p_\theta}(0,T;(E_0,E_1)_{\theta,p_\theta})$$

and this embedding is compact.

We give a consequence of this result in a form which is used to prove Lemma 5.4.

Lemma B.2. *Let $1 < r < \infty$ and let $\mathcal{B}$ be a smooth bounded domain in $\mathbf{R}^n$. Then*

$$W^{1,5}(-1,0;L^r(\mathcal{B}))\cap L^5(-1,0;W^{2,r}(\mathcal{B}))\hookrightarrow C([-1,0];W^{1,r}(\mathcal{B}))$$

and this embedding is compact.

Proof. We write $L^r_x := L^r(\mathcal{B})$, $W^{2,r}_x := W^{2,r}(\mathcal{B})$ and so on. Since $W^{2,r}_x \hookrightarrow L^r_x$ is compact, Proposition B.1 with $p_0 = p_1 = 5$, $\theta = 2/3$ and $s = 1/4$ gives

$$W^{1,5}(-1,0;L^r_x)\cap L^5(-1,0;W^{2,r}_x)\hookrightarrow W^{1/4,5}(-1,0;(L^r_x,W^{2,r}_x)_{2/3,5}).$$

By [100, p. 327], we have $(L^r_x, W^{2,r}_x)_{2/3,5} = B^{4/3}_{r,5,x} \hookrightarrow W^{1,r}_x$. Then the Sobolev embedding in time yields

$$W^{1/4,5}(-1,0;(L^r_x,W^{2,r}_x)_{2/3,5})\hookrightarrow C([-1,0];W^{1,r}_x),$$

and hence the lemma follows. ■

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